Real Analysis

Measure Theory, Integration, & Hilbert Spaces[1]

−TW−

2024 年 5 月 8 日

[1]参考书籍：

《*Real Analysis − −Measure Theroy, Integration, & Hilbert Spaces*》— *Elias M. Stein*

《*Real Analysis − −Modern Techniques and Their Applications*》— *Gerald B. Folland*

《实变函数论 (第三版)》— 周民强

序

天道几何，万品流形先自守；

变分无限，孤心测度有同伦。

2024 年 5 月 8 日

长夜伴浪破晓梦，梦晓破浪伴夜长

目录

第一章 *Measure Theory*

1.1 *Preliminaries*

定义 **1.1.1.** A (closed) **rectangle** R in $\mathbb{R}^d$ is given by of d one-dimensional closed and bounded intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d] \tag{1.1}$$

where $a_j \le b_j$ are real numbers , $j = 1, 2, \cdots, d$. In other word , we have

$$R = \{(x_1, \cdots, x_d) \in \mathbb{R}^d \mid a_j \le x_j \le b_j,\ \forall j = 1 \sim d\} \tag{1.2}$$

The **volume** of R is

$$|R| = (b_1 - a_1) \cdots (b_d - a_d) \tag{1.3}$$

An **open** rectangle is the product of open intervals , and **the interior of the rectangle** R is

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d) \tag{1.4}$$

Also , a **cube** is a rectangle for which $b_1 - a_1 = \cdots = b_d - a_d$.

定义 **1.1.2.** A union of rectangles is said to be **almost disjoint** if the interiors of them are disjoint.

引理 **1.1.1.** If a rectangle is the almost disjoint union of finitely many rectangles , say $R = \bigcup\limits_{k=1}^{N} R_k$, then

$$|R| = \sum_{k=1}^{N} |R_k| \tag{1.5}$$

注. 本质上即指的是对于方体的任意的垂直划分可转化为“十字形”划分.

引理 1.1.2. If $R, R_1, \cdots, R_N$ are rectangles , and $R \subset \bigcup_{k=1}^{U} R_k$, then

$$|R| \leq \sum_{k=1}^{N} |R_k| \tag{1.6}$$

注. 此即对 *Lemma* 1.1.1 的 slight modification , 即各方体之间不一定再为 almost disjoint.

Now we can give a description of the strcture of open sets in terms of cubes. Begin with the case of $\mathbb{R}$.

定理 1.1.3. Every open subset $\mathcal{O}$ of $\mathbb{R}$ can be written uniquely as countable union of disjoint open intervals.

证明. For each $x \in \mathcal{O}$, let I_x be the largest open interval containing x and contained in $\mathcal{O}$.

Step 1 : Construct I_x:

$\mathcal{O}$ is open $\Rightarrow$ x is contained in some small open interval contained in $\mathcal{O}$.

Let

$$a_x = inf\{a < x \mid (a, x) \subset \mathcal{O}\} \tag{1.7}$$

$$b_x = sup\{b > x \mid (x, b) \subset \mathcal{O}\} \tag{1.8}$$

Let $I_x = (a_x, b_x)$, then $\mathcal{O} = \bigcup_{x \in \mathcal{O}} I_x$.

Step 2 : Suppose $I_x \cap I_y \neq \varnothing$.

$I_x \cup I_y$ is an open interval s. t. $\begin{cases} x \in I_x \cup I_y \\ I_x \cup I_y \subset \mathcal{O} \end{cases}$

Since I_x is maximal , $I_x \cup I_y \subset I_x$. Similarly , $I_x \cup I_y \subset I_y$.

$\Rightarrow \ I_x = I_y$

$\Rightarrow \ if\ I_x \neq I_y,\ then\ I_x \cap I_y = \varnothing.$

$\Rightarrow Z = \{I_x\}_{x \in \mathcal{O}}$ is a disjoint famliy of sets.

Step 3 : Since every I_x contains at least a $a_x \in \mathbb{Q}$, construct a map f

$$f : Z \longrightarrow \mathbb{Q} \tag{1.9}$$

$$I_x \longmapsto a_x \tag{1.10}$$

f is an injective. $\Rightarrow$ $\{I_x\}_{x \in \mathcal{O}}$ is countable. $\Rightarrow$ $\mathcal{O} = \bigcup_{j=1}^{\infty} (a_j, b_j)$.

□

定理 1.1.4. Every open set $\mathcal{O}$ of $\mathbb{R}^d$, $d \geq 1$, can be written as a countable union of almost disjoint closed cubes.

证明. Let

$$\mathcal{Q}_k := grid\ of\ 2^{-k}\mathbb{Z}^d,\ k \geq 0 \tag{1.11}$$

$$\underline{A}(\mathcal{O}, k) := \{Q \in \mathcal{Q}_k \mid Q \subset \mathcal{O}\} \tag{1.12}$$

$$\overline{A}(\mathcal{O}, k) := \{Q \in \mathcal{Q}_k \mid Q \cap \mathcal{O} \neq \emptyset\} \tag{1.13}$$

Since $\forall Q \in \underline{A}(\mathcal{O}, k),\ \exists q \in Q^{\circ},\ \text{s. t.}\ q \in \mathbb{Q}^d,$

According to the Axiom of Choice , $\exists$ the map $f_k : \underline{A}(\mathcal{O}, k) \longrightarrow \mathbb{Q}^d$, which is an injection.

Hence $\underline{A}(\mathcal{O}, k)$ is countable.

Let

$$\underline{A}(\mathcal{O}) := \bigcup_{k=1}^{\infty} (\underline{A}(\mathcal{O}, k) \setminus \underline{A}(\mathcal{O}, k-1)) \cup \underline{A}(\mathcal{O}, 0) \tag{1.14}$$

Then $\underline{A}(\mathcal{O})$ is also countable. Similarly define $\overline{A}(\mathcal{O})$.

$\forall x \in \mathcal{O}$, let $\delta_x := inf\{|y - x| \mid y \notin \mathcal{O}\}$. Since $\mathcal{O}$ is open , $\Rightarrow\ \ \delta_x > 0$.

$$\exists N_x \in \mathbb{N},\ \text{s. t.}\ 2^{-k}\sqrt{d} \leq \frac{\delta_x}{2} < \delta_x, \forall k \geq N_x \tag{1.15}$$

$$\Rightarrow \forall Q \in \overline{A}(\mathcal{O}, N_x),\ \text{s. t.}\ |s - t| \leq 2^{-N_x}\sqrt{d} < \delta_x, \forall s, t \in Q \tag{1.16}$$

$$\Rightarrow Since\ \mathcal{O} \subset \overline{A}(\mathcal{O}),\ \exists Q_x \in \overline{A}(\mathcal{O}, N_x) \subset \overline{A}(\mathcal{O}),\ \text{s. t.}\ x \in Q_x \tag{1.17}$$

$$\Rightarrow x \in Q_x \subset \mathcal{O} \tag{1.18}$$

$$\Rightarrow x \in Q_x \in \underline{A}(\mathcal{O}, N_x) \subset \underline{A}(\mathcal{O}) \tag{1.19}$$

$$\Rightarrow \mathcal{O} \subset \underline{A}(\mathcal{O}) \tag{1.20}$$

Obviously $\underline{A}(\mathcal{O}) \subset \mathcal{O}$, so

$$\mathcal{O} = \underline{A}(\mathcal{O}) = \bigcup_{k=1}^{\infty} (\underline{A}(\mathcal{O}, k) \setminus \underline{A}(\mathcal{O}, k-1)) \cup \underline{A}(\mathcal{O}, 0) \tag{1.21}$$

which is a countable union of almost disjoint closed cubes. □

1.2 *The Exterior Measure*

Definition The exterior measure attempts to describe the volume of a set E by approximating it from the outside.

Loosely speaking , the exterior measure m_* assigns to **any subset of** $\mathbb{R}^d$ a first notion of size.

定义 **1.2.1.** If E is a subset of $\mathbb{R}^d$, the **exterior measure** of E is

$$m_*(E) := \inf\left\{\sum_{j=1}^{\infty}\left|Q_j\right| \;\middle|\; E \subset \bigcup_{j=1}^{\infty} Q_j,\ Q_j \textit{ is a closed cube}\right\} \tag{1.22}$$

注. • **Well definition:** $\forall E \subset \mathbb{R}^d,\ E \subset \bigcup\limits_{n=1}^{\infty} Q_n,\ Q_n = [-n, n]^d \subset \mathbb{R}^d$, which means m_* can be defined on every subset of $\mathbb{R}^d$.

- It is immediate from the definition that:

  For every $\epsilon > 0$, there exists a covering $E \subset \bigcup\limits_{j=1}^{\infty} Q_j$, s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \tag{1.23}$$

- It is important to note that it would **not suffice** to allow **finite sums** in the definition of $m_*(E)$. If one considered only coverings of E by finite unions of cubes , the quantity is **in general larger** than $m_*(E)$.

  (In fact , it is defined as the **outer Jordan content** $J_*(E)$.)

  例 **1.2.1.** Consider the set $\mathbb{Q} \cap [0, 1]$.

  – For the outer Jordan content , since it's obvious that $J_*(\overline{E}) = J_*(E),\ \forall E \subset \mathbb{R}^d$,

    $J_*(\mathbb{Q} \cap [0, 1]) = J_*(\overline{\mathbb{Q} \cap [0, 1]}) = J_*([0, 1]) = 1$

  – For the exterior measure , since $\mathbb{Q} \cap [0, 1]$ is countable , let $\mathbb{Q} \cap [0, 1] = \{x_1, x_2, \cdots\}$.

    Since for all $\epsilon > 0$,

$$\mathbb{Q} \cap [0, 1] \subset \bigcup_{j=1}^{\infty} [x_j - \frac{\epsilon}{2^j}, x_j + \frac{\epsilon}{2^j}] \tag{1.24}$$

    Hence $m_*(\mathbb{Q} \cap [0, 1]) \leq \epsilon$. For ϵ is arbitrary , $m_*(\mathbb{Q} \cap [0, 1]) = 0$.

Examples Let's check that whether the exterior measure matches our intuitive idea of volume.

Example 1. **The exterior measure of a point is zero.**

证明. It's clear that a point is a cube with $a_j = b_j, \forall j = 1 \sim d$ and which covers itself. □

Example 2. **The exterior measure of a closed cube is equal to its volume.**

证明.

- Let $Q \subset \mathbb{R}^d$ be a closed cube. Since $Q \subset Q$, $m_*(Q) \leq |Q|$.
- Suppose $Q \subset \bigcup_{j=1}^{\infty} Q_j$ by closed cubes. For fixed $\epsilon > 0$, $\forall j \in \mathbb{N}$, choose an open cube S_j ,

$$\text{s. t.} \begin{cases} S_j \supset Q_j \\ \left|S_j\right| = (1+\epsilon)\left|Q_j\right| \end{cases} \tag{1.25}$$

Then $Q \subset \bigcup_{j=1}^{\infty} S_j$. Since Q is compact , $\exists S_1, \cdots, S_n \in \{S_j\}_{j=1}^{\infty}$, s. t. $Q \subset \bigcup_{j=1}^{n} S_j$.
Therefore , according to Lemma 1.1.2

$$|Q| \leq \sum_{j=1}^{n} \left|S_j\right| = (1+\epsilon) \sum_{j=1}^{n} \left|Q_j\right| \leq (1+\epsilon) \sum_{j=1}^{\infty} \left|Q_j\right| \tag{1.26}$$

For $\epsilon > 0$ is arbitrary , we get

$$|Q| \leq \sum_{j=1}^{\infty} \left|Q_j\right| \tag{1.27}$$

$$|Q| \leq \inf \sum_{j=1}^{\infty} \left|Q_j\right| = m_*(Q) \tag{1.28}$$

□

Example 3. **If Q is an open cube , then $m_*(Q) = |Q|$.**

证明.

- Since $Q \subset \overline{Q}$, $m_*(Q) \leq \left|\overline{Q}\right| = |Q|$.
- We note that for all closed cubes Q_0 contained in Q , then $m_*(Q_0) = |Q_0| \leq m_*(Q)$.

  For fixed $\epsilon > 0$ which is suffice small , choose a closed cube Q_0 contained in Q with a volume $|Q_0| = (1-\epsilon)|Q|$, then we have

$$|Q_0| = (1-\epsilon)|Q| \leq m_*(Q) \tag{1.29}$$

For ϵ is arbitrary , $|Q| \leq m_*(Q)$.

□

Example 4. **The exterior measure of a rectangle R is equal to its volume.**

Example 5. $m_*(\mathbb{R}^d) = \infty$.

证明. Since any covering of $\mathbb{R}^d$ is also a covering of any cube $Q \subset \mathbb{R}^d$, $m_*(\mathbb{R}^d) \geq m_*(Q)$
$\forall N > 0,\ \exists Q \subset \mathbb{R}^d,$ s. t. $|Q| > N$, so $m_*(\mathbb{R}^d) = \infty$. □

Properties

Observation 1. (*Monotonicity*)

If $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$.

Observation 2. (*Countable sub − additivity*)

If $E \subset \bigcup_{j=1}^{\infty} E_j$, then $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$.

证明. For a fixed $\epsilon > 0$, for all E_j , there exists a covering $\{Q_{j_k}\}_{k=1}^{\infty},\ E \subset \bigcup_{k=1}^{\infty} Q_{j_k},$ s. t.

$$\sum_{k=1}^{\infty} m_*(Q_{j_k}) \leq m_*(E_j) + \frac{\epsilon}{2^j} \tag{1.30}$$

Since $E \subset \bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} Q_{j_k}$, $\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} Q_{j_k}$ covers E , then

$$m_*(E) \leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} m_*(Q_{j_k}) \leq \sum_{j=1}^{\infty} m_*(E_j) + \epsilon \tag{1.31}$$

Since ϵ is arbitrary , $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$ □

Observation 3. If $E \subset \mathbb{R}^d$, then $m_*(E) = \inf\{m_*(O) \mid E \subset O,\ O \textit{ is an open set}\}$.

证明.

- By monotonicity , $m_*(E) \leq m_*(O)$, for all O covers E. Then take the infimum.

- For a fixed $\epsilon > 0$, $\exists$ *covering* $E \subset \bigcup_{j=1}^{\infty} Q_j$, s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \frac{\epsilon}{2} \tag{1.32}$$

For all Q_j , choose an open set $\widetilde{Q_j}$ containing Q_j with a volume $\left|\widetilde{Q_j}\right| \leq \left|Q_j\right| + \frac{\epsilon}{2^{j+1}}$.
Let $O = \bigcup_{j=1}^{\infty} \widetilde{Q_j}$, then by Observation 2 ,

$$m_*(O) \leq \sum_{j=1}^{\infty} m_*(\widetilde{Q_j}) = \sum_{j=1}^{\infty} \left|\widetilde{Q_j}\right| \leq \sum_{j=1}^{\infty} \left|Q_j\right| + \frac{\epsilon}{2} \leq m_*(E) + \epsilon \tag{1.33}$$

Since ϵ is arbitrary , $m_*(O) \leq m_*(E)$, so $\inf m_*(O) \leq m_*(E)$.

□

Observation 4. If $E = E_1 \cup E_2$, and $d(E_1, E_2) > 0$, then

$$m_*(E) = m_*(E_1) + m_*(E_2) \tag{1.34}$$

证明. For a fixed $\epsilon > 0$, $\exists$ a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$, s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \tag{1.35}$$

Subdevide the cubes Q_j and assume that $diam(Q_j) <= \frac{d(E_1,E_2)}{3}$. Then each Q_j can intersect at most one of the two sets E_1 or E_2. Devide $\{Q_j\}_{j=1}^{\infty}$ into two subsets $\{Q_j\}_{j\in J_1}$, $\{Q_j\}_{j\in J2}$, s. t.

$$E_1 \subset \bigcup_{j\in J_1} Q_j,\ E_2 \subset \bigcup_{j\in J_2} Q_j \tag{1.36}$$

J_1 and J_2 are both countable. $J_1 \cap J_2 = \varnothing$. Then

$$m_*(E_1) \leq \sum_{j\in J_1} m_*(Q_j),\ m_*(E_2) \leq \sum_{j\in J_2} m_*(Q_j) \tag{1.37}$$

Therefore

$$m_*(E_1) + m_*(E_2) \leq \sum_{j\in J_1} m_*(Q_j) + \sum_{j\in J_2} m_*(Q_j) \leq \sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \tag{1.38}$$

Since ϵ is arbitrary , $m_*(E_1) + m_*(E_2) \leq m_*(E)$. □

Observation 5. If a set E is the countable union of almost disjoint cubes $E = \bigcup_{j=1}^{\infty} Q_j$, then

$$m_*(E) = \sum_{j=1}^{\infty} \left| Q_j \right| \tag{1.39}$$

证明. For a fixed $\epsilon > 0$, for all Q_j , choose a closed cube $\widetilde{Q_j}$ strictly contained in Q_j with its volume $\left|\widetilde{Q_j}\right| \geq \left|Q_j\right| - \frac{\epsilon}{2^j}$. Then for every $N \in \mathbb{N}$, the cubes $\widetilde{Q_1}, \cdots, \widetilde{Q_N}$ are disjoint with a finite distance from one another. By Observation 4 ,

$$m_*(\bigcup_{j=1}^{N} \widetilde{Q_j}) = \sum_{i=1}^{N} \left|\widetilde{Q_j}\right| \geq \sum_{j=1}^{N} \left|Q_j\right| - \epsilon \tag{1.40}$$

Since $\bigcup_{j=1}^{\infty} \widetilde{Q_j} \subset E$, we conclude that for every N

$$m_*(E) \geq \sum_{j=1}^{N} \left|Q_j\right| - \epsilon \tag{1.41}$$

Let $N \to \infty$, we deduce

$$m_*(E) \geq \sum_{j=1}^{\infty} \left|Q_j\right| - \epsilon \tag{1.42}$$

Since ϵ is arbitrary , $\sum_{j=1}^{\infty} \left|Q_j\right| \leq m_*(E)$. □

1.3 *Measurable sets and the Lebesgue measure*

1.3.1 *Measurable sets*

Definition

定义 1.3.1. A subset E of $\mathbb{R}^d$ is *(Lebesgue) measurable*, if for any $\epsilon > 0$ there exists an open set O with $E \subset O$ and $m_*(O \backslash E) \leq \epsilon$.

If E is measurable, we define its *(Lebesgue) measurable* $m(E)$ by $m(E) = m_*(E)$.

注.

- 可用映射的观点来理解外测度 m_* 与测度 m 的关系 (*Folland*). 即

$$m_* : \mathcal{P}(\mathbb{R}^d) \longrightarrow \overline{\mathbb{R}}_+ = [0, +\infty] \tag{1.43}$$

$$m : \mathcal{M} \longrightarrow \overline{\mathbb{R}}_+ = [0, +\infty] \tag{1.44}$$

$$m = m_*\Big|_{\mathcal{M}} \tag{1.45}$$

其中 $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^d)$ 为 $\mathbb{R}^d$ 中所有 (*Lebesgue*) *measurable sets* 构成的集合.

- 类比于抽象代数中各代数结构的性质，比如群 (*group*) 对加法 / 乘法封闭，我们下面探讨集合族 $\mathcal{M}$ 对于可数个集合的运算 (*countable unions*, *countable intersections*, *complement*) 是否封闭. 即通过此引出代数结构 *σ − algebra*.

Properties 下面开始探讨 (*Lebesgue*) *measure* 的部分性质.

Property 1. Every open set in $\mathbb{R}^d$ is measurable.

Property 2. If $m_*(E) = 0$, then E is measurable.

证明. By Observation 3 in §1.2, for a fixed $\epsilon > 0$, $\exists E \subset O$ open, s. t.

$$m_*(O) \leq m_*(E) + \epsilon = \epsilon \tag{1.46}$$

Since $O \backslash E \subset O$, then $m_*(O \backslash E) \leq m_*(O) \leq \epsilon$. □

Property 3. Let $\{E_j\}_{j=1}^{\infty}$ be a family of measurable sets, then $\bigcup_{j=1}^{\infty} E_j$ is measurable.

注. 即说明集合族 $\mathcal{M}$ 对 *countable unions* 封闭.

证明. Since E_j is measurable, for a fixed $\epsilon > 0$, $\exists E_j \subset O_j$ open, s. t.

$$m_*(O_j \backslash E_j) \leq \frac{\epsilon}{2^j} \tag{1.47}$$

Let $O = \bigcup_{j=1}^{\infty} O_j \underset{open}{\subset} \mathbb{R}^d$, then

$$O \backslash \bigcup_{j=1}^{\infty} E_j = \left(\bigcup_{j=1}^{\infty} O_j\right) \cap \left(\bigcap_{j=1}^{\infty} E_j^c\right) \tag{1.48}$$

$$= \bigcup_{j=1}^{\infty}\left(O_j \cap \left(\bigcap_{k=1}^{\infty} E_k^c\right)\right) \subset \bigcup_{j=1}^{\infty}\left(O_j \cap E_j^c\right) = \bigcup_{j=1}^{\infty}\left(O_j \backslash E_j\right) \tag{1.49}$$

Therefore

$$m_*\left(O \backslash \bigcup_{j=1}^{\infty} E_j\right) \leq m_*\left(\bigcup_{j=1}^{\infty}\left(O_j \backslash E_j\right)\right) \leq \sum_{j=1}^{\infty} m_*\left(O_j \backslash E_j\right) \leq \epsilon \tag{1.50}$$

So $\bigcup_{j=1}^{\infty} E_j$ is measurable. □

Property 4. Closed sets are measurable.

为了证明该性质，先证明如下的分离定理.

引理 1.3.1. If F is closed, K is compact, and $K \cap F = \varnothing$, then $d(F, K) > 0$.

证明. 反证法.Suppose $d(F, K) = 0$, then for any fixed $n \in \mathbb{N}$, $\exists x_n \in F, y_n \in K$, s. t.

$$|x_n - y_n| \leq \frac{1}{n} \tag{1.51}$$

Since K is compact, $\{y_n\}_{n=1}^{\infty}$ is bounded. Then there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$, s. t.

$$y_{n_k} \to y_0 \in K, \ \ as \ k \to \infty \tag{1.52}$$

Since $\left|x_{n_k} - y_{n_k}\right| \leq \frac{1}{n_k}$, then

$$\left|x_{n_k} - y_0\right| \leq \left|x_{n_k} - y_{n_k}\right| + \left|y_{n_k} - y_0\right| \to 0, \ \ as \ k \to \infty \tag{1.53}$$

So $x_{n_k} \to y_0 \in F$, $y_0 \in F \cap K \neq \varnothing$ 矛盾. □

下面证明 Property 4.

证明.

- Suppose F is bounded, then F is compact.

  By Observation 3 in §1.2, for a fixed $\epsilon > 0$, $\exists F \subset O$ open, s. t.

$$m_*(O) \leq m_*(F) + \epsilon \tag{1.54}$$

  Since F is closed, $O \backslash F = O \cap F^c$ is open. By Thm1.1.4, $\exists \{Q_j\}_{j=1}^{\infty}$, s. t.

$$O \backslash F = \bigcup_{j=1}^{\infty} Q_j \tag{1.55}$$

  For a fixed $N \in \mathbb{N}$, let $K = \bigcup_{j=1}^{N} Q_j$, then K is compact. By Lemma1.3.1, $d(K, F) > 0$. Since $K \cup F \subset O$, by Observation 4 in §1.2,

$$m_*(K) + m_*(F) = m_*(K \cup F) \leq m_*(O) \tag{1.56}$$

  So for each fixed $N \in \mathbb{N}$,

$$\sum_{j=1}^{N} \left| Q_j \right| = m_*(K) \leq m_*(O) - m_*(F) \leq \epsilon \tag{1.57}$$

  Let $N \to \infty$, we get

$$m_*(O \backslash F) = \sum_{j=1}^{\infty} \left| Q_j \right| \leq \epsilon \tag{1.58}$$

  Therefore, F is measurable.

- For the general situation, since $\mathbb{R}^d = \bigcup_{j=1}^{\infty} B_j$, then

$$F = F \cap \mathbb{R}^d = \bigcup_{j=1}^{\infty} \left(F \cap B_j \right) \tag{1.59}$$

  Since B_k is compact and F is closed, then $F \cap B_j$ is compact.

  Due to the previous proof, $F \cap B_j$ is measurable. By Property 3 in §1.3.1,

$$F = \bigcup_{j=1}^{\infty} \left(F \cap B_j \right) \textit{ is measurable.} \tag{1.60}$$

□

Property 5. If E is measurable, then E^c is measurable.

注. 即说明集合族 $\mathcal{M}$ 对集合的补运算 *complement* 封闭.

证明. Since E is measurable, then for all fixed $n \in \mathbb{N}$, $\exists E \subset O_n$ open, s. t. $m_*(O_n \backslash E) \leq \frac{1}{n}$.
Let $S = \bigcup\limits_{j=1}^{\infty} O_j^c \subset E^c$. Since O_j^c is closed, O_j^c is measurable. Then S is measurable.
Since

$$E^c \backslash S = E^c \cap \left(\bigcap_{j=1}^{\infty} O_j\right) = \bigcap_{j=1}^{\infty} \left(E^c \cap O_j\right) \subset E^c \cap O_n = O_n \backslash E, \ \forall n \in \mathbb{N} \tag{1.61}$$

Then, $m_*(E^c \backslash S) \leq m_*(O_n \backslash E) \leq \frac{1}{n}$, $\forall n \in \mathbb{N}$. So $E^c \backslash S$ is measurable.
Therefore, $E^c = (E^c \backslash S) \cup S$ is measurable. □

Property 6. If $\{E_j\}_{j=1}^{\infty}$ is a family of measurable sets, then $\bigcap\limits_{j=1}^{\infty} E_j$ is measurable.

注. 即说明集合族 $\mathcal{M}$ 对 *countableintersections* 封闭.

证明. Since

$$\bigcap_{j=1}^{\infty} E_j = \left(\bigcup_{j=1}^{\infty} E_j^c\right)^c \tag{1.62}$$

Then, E_j^c is measurable and so $\bigcap\limits_{j=1}^{\infty} E_j$ is measurable. □

综上，本节介绍了 (*Lebesgue*) *measurable sets* 的性质，并且证明了 *Lebesgue measurable sets* 构成的集合族 $\mathcal{M}$ 对 *countable unions, countable intersections, complement* 运算封闭. 从而 $(\mathcal{M}, \cup, \cap, complement)$ 构成代数结构，即为后续介绍的 $\sigma - algebra$.

1.3.2 *Lebesgue measure*

下面着重来介绍一下 *Lebesgue measure* 的 *properties*.

可数可加性 首先便是可数可加性 *countable additivity*.

定理 1.3.2. If $E_1, E_2, \cdots$ are disjoint measurable sets, then

$$m(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} m(E_j) \tag{1.63}$$

证明. Since $m(\bigcup\limits_{j=1}^{\infty} E_j) \le \sum\limits_{j=1}^{\infty} m(E_j)$ always holds, we then proof the reverse inequality.

- Suppose that E_j is bounded.

  Since E_j^c is measurable, for any fixed $\epsilon > 0$, there exists an closed subset $F_j \subset E_j$, s. t.

$$m(E_j \backslash F_j) \le \frac{\epsilon}{2^j} \tag{1.64}$$

  Since E_j is bounded, F_j is compact.

  Let $K = \bigcup\limits_{j=1}^{N} F_j$ be a disjoint union of compact sets for some fixed N, then

$$K \subset \bigcup_{j=1}^{\infty} E_j \tag{1.65}$$

$$m(K) = \sum_{j=1}^{N} m(F_j) \le m(\bigcup_{j=1}^{\infty} E_j) \tag{1.66}$$

  Since

$$m(E_j) \le m(E_j \backslash F_j) + m(F_j) \le m(F_j) + \frac{\epsilon}{2^j} \tag{1.67}$$

  Therefore

$$\sum_{j=1}^{N} m(E_j) - \epsilon \le \sum_{j=1}^{N} m(F_j) \le m(\bigcup_{j=1}^{\infty} E_j) \tag{1.68}$$

  Let $N \to \infty$, for ϵ is arbitrary, we get

$$\sum_{j=1}^{\infty} m(E_j) \le m(\bigcup_{j=1}^{\infty} E_j) \tag{1.69}$$

- In the general case, we choose the sequence of cubes $\{Q_k\}_{k=1}^{\infty}$, $Q_k = [-k, k]^d \subset \mathbb{R}^d$.

  Let $S_1 = Q_1$, $S_k = Q_k - Q_{k-1}$, $\forall k \geq 2$. Then $\{S_k\}_{k=1}^{\infty}$ are disjoint and bounded.

  Since $\{S_k\}_{k=1}^{\infty}$ covers $\mathbb{R}^d$,

$$E_j = \bigcup_{k=1}^{\infty} (E_j \cap S_k) \tag{1.70}$$

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} (E_j \cap S_k) \tag{1.71}$$

  Since $E_j \cap S_k$ is bounded and disjoint, by the previous case,

$$m(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} m(E_j \cap S_k) = \sum_{j=1}^{\infty} m(E_j) \tag{1.72}$$

□

单调连续性　下面我们可以给出单调可测集合列的连续性.*continuity from below/above*

定理 1.3.3. Let $E_1, E_2, \cdots$ be measurable sets in $\mathbb{R}^d$.

(i) If $E_k \nearrow E$, then $m(E) = \lim_{n\to\infty} m(E_n)$.

(ii) If $E_k \searrow E$ and $m(E_1) < \infty$, then $m(E) = \lim_{n\to\infty} m(E_n)$.

注.
- 事实上即可写为

$$m(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} m(E_n) \tag{1.73}$$

  即单调可测集合列**可交换极限与测度顺序**.

- (ii) 中条件 $m(E_1)$ finite 不可省略，下面给出一个反例.

  例 1.3.1. If $E_n = (n, +\infty)$, then $m(E_n) = \infty$ and $E = \bigcap_{j=1}^{\infty} E_j = \varnothing$. So

$$m(E) = m(\lim_{n\to\infty} E_j) = 0, \quad \lim_{n\to\infty} m(E_j) = \infty \tag{1.74}$$

证明.

(i) Let $S_1 = E_1$, $S_k = E_k - E_{k-1}$, $\forall k \geq 2$. Then $\{S_k\}_{k=1}^{\infty}$ are disjoint and measurable.

  Since $E = \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} S_k$, by Thm1.3.2,

$$m(E) = \sum_{k=1}^{\infty} m(S_k) = \lim_{N\to\infty} \sum_{k=1}^{N} m(S_k) = \lim_{N\to\infty} m(\bigcup_{k=1}^{N} S_k) = \lim_{N\to\infty} m(E_N) \tag{1.75}$$

(ii) Let $S_1 = E_1$, $S_k = E_k - E_{k+1}$, $\forall k \geq 2$. Then $\{S_k\}_{k=1}^{\infty}$ are disjoint and measurable.
Since $E_1 = E \cup \left(\bigcup\limits_{k=1}^{\infty} S_k\right)$, then

$$m(E_1) = m(E) + \sum_{k=1}^{\infty} m(S_k) = m(E) + \lim_{N\to\infty} m(\bigcup_{k=1}^{N} S_k) = m(E) + \lim_{N\to\infty} m(E_1 - E_N) \tag{1.76}$$

For $E_1 = (E_1 - E_N) \sqcup E_N$ is a disjoint union,

$$m(E_1 - E_N) = m(E_1) - m(E_N) \tag{1.77}$$

Thus

$$m(E_1) = m(E) + \lim_{N\to\infty} m(E_1 - E_N) = m(E) + m(E_1) - \lim_{N\to\infty} m(E_N) \tag{1.78}$$

$$m(E) = \lim_{N\to\infty} m(E_N) \tag{1.79}$$

□

Geometric insight of measurable sets 最后我们来给出 (*Lebesgue*) *measurable sets* 的几何性质 (与开集、闭集、紧集等之间的关系).

定理 1.3.4. Lebesgue 测度的正则性.
Suppose $E \subset \mathbb{R}^d$ is measurable, then $\forall \epsilon > 0$:

(i) $\exists$ open $O \supset E$ with $m(O \backslash E) \leq \epsilon$.

(ii) $\exists$ closed $F \subset E$ with $m(E \backslash F) \leq \epsilon$.

(iii) If $m(E) < \infty$, $\exists$ compact $K \subset E$ with $m(E \backslash K) \leq \epsilon$.

(iv) If $m(E) < \infty$, $\exists F = \bigcup\limits_{j=1}^{N} Q_j$, $\{Q_j\}_{j=1}^{\infty}$ are closed cubes, s. t. $m(E \triangle F) \leq \epsilon$.

证明.

(i) It's just the definition of measurability.

(ii) Since E_j^c is measurable, $\exists$ open $O_j \supset E_j^c$, s. t.

$$m(O_j \backslash E_j^c) \leq \epsilon \tag{1.80}$$

Since $O_j^c \subset E_j$ is closed and $E_j \backslash O_j^c = O_j \backslash E_j^c$, let $F = O_j^c$ closed, then

$$m(E_j \backslash F) = m(O_j \backslash E_j^c) \leq \epsilon \tag{1.81}$$

(iii) By (ii), $\exists$ closed $F \subset E$, s. t. $m(E \backslash F) \leq \frac{\epsilon}{2}$.

Let B_n denote the closed ball centered at the origin of radius n, then B_n is compact.

$$F = \bigcup_{j=1}^{\infty} (F \cap B_k) \tag{1.82}$$

Let $K_n = \bigcup_{k=1}^{n} (F \cap B_k)$, then K_n is compact and $K_n \nearrow F \Rightarrow E \backslash K_n \nearrow E \backslash F$.
Since $m(E \backslash K_1) \leq m(E)$ is finite, by Thm1.3.3(ii)

$$\lim_{n \to \infty} m(E \backslash K_n) = m(E \backslash F) \tag{1.83}$$

As for $\epsilon > 0$, $\exists N \in \mathbb{N}$, s. t. for all $n \geq N$

$$|m(E \backslash K_n) - m(E \backslash F)| \leq \frac{\epsilon}{2} \tag{1.84}$$

$$m(E \backslash K_n) \leq m(E \backslash F) + \frac{\epsilon}{2} \leq \epsilon \tag{1.85}$$

Therefore, $m(E \backslash K_N) \leq \epsilon$, where $K_N \subset E$ is compact.

(iv) $\exists$ open $O \supset E$, s. t. $m(O \backslash E) \leq \frac{\epsilon}{2}$. By Thm1.1.4, $\exists \{Q_j\}_{j=1}^{\infty}$, s. t.

$$E \subset O = \bigcup_{j=1}^{\infty} Q_j \tag{1.86}$$

So

$$m(O) = \sum_{j=1}^{\infty} \left|Q_j\right| \leq m(O \backslash E) + m(E) \leq \frac{\epsilon}{2} + m(E) \tag{1.87}$$

Since $m(E)$ is finite, $\sum_{j=1}^{\infty} \left|Q_j\right|$ converges. Then $\exists N \in \mathbb{N}$, s. t.

$$\sum_{j=N+1}^{\infty} \left|Q_j\right| \leq \frac{\epsilon}{2} \tag{1.88}$$

Let $F = \bigcup_{j=1}^{N} Q_j$. Since $E \triangle F = (E \backslash F) \sqcup (F \cap E)$, then

$$m(E \triangle F) = m(E \backslash F) + m(F \backslash E) \tag{1.89}$$

$$\leq m\Big(\bigcup_{j=N+1}^{\infty} Q_j\Big) + m\Big(\bigcup_{j=1}^{\infty} Q_j \backslash E\Big) \tag{1.90}$$

$$= \sum_{j=N+1}^{\infty} \left|Q_j\right| + \sum_{j=1}^{\infty} \left|Q_j\right| - m(E) \tag{1.91}$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{1.92}$$

$\square$

1.4 $\sigma - algebras\ and\ Borel\ sets$

1.4.1 $\sigma - algebra$

首先给出 $\mathbb{R}^d$ 中 *algebra* 的定义.

定义 1.4.1. Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^d)$. $\mathcal{A}$ is called an *algebra* if

(1) If $A_1, \cdots, A_n \in \mathcal{A}$, then $\bigcup_{j=1}^{n} A_j \in \mathcal{A}$.

(2) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

注. 容易证明，若 $\mathcal{A}$ 为 $\mathbb{R}^d$ 中 *algebra*，则其对 *finite intersections* 也封闭，同时 $\varnothing, \mathbb{R}^d \in \mathcal{A}$.

下面给出 $\mathbb{R}^d$ 中 $\sigma - algebra$ 的定义.(将 *algebra* 中的 *finite* 条件加强为 *countable*)

定义 1.4.2. Let $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^d)$. $\mathcal{M}$ is a $\sigma - algebra$ if

(1) If $A_1, A_2, \cdots \in \mathcal{M}$, then $\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$.

(2) If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$.

注. 容易证明 $\mathcal{M}$ 对 *countable intersections* 同样封闭，$\varnothing, \mathbb{R}^d \in \mathcal{M}$.

例 1.4.1. All Lebesgue measurable sets forms a $\sigma - algebra$ $\mathcal{M}$.

类比线性空间、拓扑空间中 (拓扑) 基的概念，下面给出**生成** $\sigma - algebra$ 的概念.

定义 1.4.3. Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^d)$, then the $\sigma - algebra$ **generated by $\mathcal{A}$** is the smallest $\sigma - algebra$ containing $\mathcal{A}$.

注. 即为 the intersection of all $\sigma - algebras$ containing $\mathcal{A}$，这也说明了对于任一给定的集族 $\mathcal{A}$，其生成的 $\sigma - algebra$ 必存在且唯一.

1.4.2 *Borel sets*

下面给出 *Borel* $\sigma - algebra$ 及 *Borel sets* 的定义.

定义 1.4.4. The *Borel* $\sigma - algebra$ is the $\sigma - algebra$ generated by all open sets in $\mathbb{R}^d$, denoted by $\mathcal{B}_{\mathbb{R}^d}$.

Elements of this $\sigma - algebra$ are called *Borel sets*.

注. 事实上，*Borel* $\sigma - algebra$ 为 Lebesgue countable sets 的一个真子集，后续会利用 Cantor 集证明.

为了方便研究 *Borel* $\sigma - algebra$ 的结构，我们把其中较为复杂 (非平凡) 的元素单独拎出来并称为 G_δ, F_σ.

定义 1.4.5. 1. The countable intersections of open sets are called G_δ sets.

2. The countable unions of closed sets are called F_σ sets.

下面我们可给出 $\mathcal{B}_{\mathbb{R}^d}$ 与 Lebesgue 可测集 $\mathcal{L}$ 之间的关系.($\mathcal{L}$ 只比 $\mathcal{B}_{\mathbb{R}^d}$ 多了一些零测集)

定理 1.4.1. Lebesgue 测度的正规性.

$E \subset \mathbb{R}^d$ is $\mathcal{L} - measurable$

(i) if and only if $E = G_\delta \backslash N_1$, for some G_δ, $m(N_1) = 0$.

(ii) if and only if $E = F_\sigma \backslash N_2$, for some F_σ, $m(N_2) = 0$.

证明. Clearly E is measurable whenever it satisfies either (i) or (ii).

(i) Since E is measurable, $\exists$ open sets $O_n \supset E$, s. t.

$$m(O_n \backslash E) \le \frac{1}{n} \tag{1.93}$$

Let $O = \bigcap_{j=1}^{\infty} O_j$, then

$$m(O \backslash E) \le \frac{1}{n}, \ \forall n \in \mathbb{N} \tag{1.94}$$

Let $n \to \infty$, we get $m(O \backslash E) = 0$. Let $G_\delta = O,\ N_1 = O \backslash E$. Then $E = G_\delta \backslash N_1$.

(ii) Similarly, we can easily proof it by Thm1.3.4(ii).

□

1.5 Non − measurable sets

在这一节我们将介绍 $\mathbb{R}$ 上一个经典的不可测集 *Vitali set*，并说明 $\mathbb{R}$ 上每个正测度集都有不可测子集.

Vitali set Let $x, y \in [0, 1]$. Write $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$.

$\Rightarrow$ 容易验证 $\sim$ 为 an equivalence relation.

$\Rightarrow$ $\sim$ partions [0 , 1]. 记 [0 , 1] 上等价类为 ε_a，则

$$[0, 1] = \bigsqcup_{a} \varepsilon_a, \ \{\varepsilon_a\}_a \ \textit{are disjoint} \tag{1.95}$$

$\Rightarrow$ By **the Axiom of Choice**, we can choose exactly one element x_a from each ε_a.

$\Rightarrow$ Let $\mathcal{N} = \{x_a\}_a$. Then $\mathcal{N}$ is the Vitali set.

定理 1.5.1. $\mathcal{N}$ is not measurable.

证明. Assume that $\mathcal{N}$ is measurable. Let $\{r_k\}_{k=1}^{\infty}$ be an enumeration of $\mathbb{Q} \cap [-1, 1]$. Define

$$\mathcal{N}_k := N + r_k = \{x_a + r_k\}_a \tag{1.96}$$

Then we shall proof that $\{\mathcal{N}_k\}_{k=1}^{\infty}$ are disjoint, and $[0, 1] \subset \bigcup_{k=1}^{\infty} \mathcal{N}_k \subset [-1, 2]$.

- If $\mathcal{N}_k \cap \mathcal{N}_m \neq \varnothing$, then $\exists x_a, x_\beta \in \mathcal{N}$, $r_k, r_m \in \mathbb{Q} \cap [-1, 1]$, s. t.

$$x_a + r_k = x_\beta + r_m \tag{1.97}$$

  Then $x_a - x_\beta = r_m - r_k \in \mathbb{Q} \Rightarrow x_a \sim x_\beta \Rightarrow x_a, x_\beta \in \varepsilon_a$ or $x_a, x_\beta \in \varepsilon_\beta \Rightarrow x_a = x_\beta$ and $r_k = r_m$. Therefore, $\mathcal{N}_k = \mathcal{N}_m$.

- Since $r_k \in [-1, 1]$, $\mathcal{N}_k \in [-1, 2]$, $\forall k$. Therefore,

$$\bigcup_{k=1}^{\infty} \mathcal{N}_k \subset [-1, 2] \tag{1.98}$$

- $\forall x \in [0, 1]$. Since $\{\varepsilon_a\}_a$ partions $[0, 1]$, there exists a_0, s. t.

$$x \in \varepsilon_{a_0}, \ \ x \sim x_{a_0} \tag{1.99}$$

  which means $x - x_{a_0} \in \mathbb{Q} \cap [-1, 1]$. Then $\exists k_0 \in \mathbb{N}$, s. t.

$$x - x_{a_0} = r_{k_0} \ \ \Rightarrow \ \ x \in \mathcal{N}_{k_0} \tag{1.100}$$

Therefore,

$$[0,1] \subset \bigcup_{k=1}^{\infty} \mathcal{N}_k \tag{1.101}$$

Since $\{\mathcal{N}_k\}_{k=1}^{\infty}$ are disjoint, we get

$$m([0,1]) \leq \sum_{k=1}^{\infty} m(\mathcal{N}_k) \leq m([-1,2]) \tag{1.102}$$

Since $\mathcal{N}_k$ is a translate of $\mathcal{N}$, we have $m(\mathcal{N}) = m(\mathcal{N}_k)$ for each k. Then

$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}) \leq 3 \;\Rightarrow\; \textit{Neither } m(\mathcal{N}) = 0 \textit{ nor } m(\mathcal{N}) > 0 \textit{ is possible.} \tag{1.103}$$

Therefore, it's a contradiction. $\mathcal{N}$ is non-measurable. □

正测度集必有不可测子集 下面要证明一个结论，即 $\mathbb{R}$ 上任一正测度集必有不可测子集. 这实际上为书[1]Exercises of Chapter 1 的第 32 题 (b).

命题 1.5.1. Let $\mathcal{N}$ denote the non-measurable subset of $[0,1]$ constructed in Thm1.5.1.

(a) If E is a measurable subset of $\mathcal{N}$, then $m(E) = 0$.

(b) If $G \subset \mathbb{R}$ with $m_*(G) > 0$, then there exists a subset of G is non-measurable.

证明.

(a) Note $\mathcal{N} = \{x_\alpha\}_{\alpha\in\mathcal{A}}$, then $E = \{x_\beta\}_{\beta\in\mathcal{B}\subset\mathcal{A}}$. Similarly, we can proof

$$\bigcup_{k=1}^{\infty} E_k \subset [-1,2] \tag{1.104}$$

Since $\{E_k\}_{k=1}^{\infty}$ are disjoint, and E_k is a translate of E, we get

$$\sum_{k=1}^{\infty} m(E) \leq 3 \;\Rightarrow\; m(E) = 0 \tag{1.105}$$

(b) Let $\mathbb{Q} = \{r_k\}_{k=1}^{\infty}$, $\mathcal{N}_k = \mathcal{N} + r_k$, then

$$\mathbb{R} = \bigsqcup_{k=1}^{\infty} \mathcal{N}_K \tag{1.106}$$

[1]参考书籍：《*Real Analysis – –Measure Theroy, Integration, & Hilbert Spaces*》— *Elias M. Stein*

Suppose G is measurable. Then

$$G = G \cap \mathbb{R} = \bigsqcup_{k=1}^{\infty} (G \cap \mathcal{N}_k) \tag{1.107}$$

If $G \cap \mathcal{N}_k$ is measurable, then $G \cap \mathcal{N}_k \subset \mathcal{N}_k$ is a subset of a non-measurable set $\mathcal{N}_k$.
By the previous (a), we get

$$m(G \cap \mathcal{N}_k) = 0 \tag{1.108}$$

Therefore, there exists $k_0 \in \mathbb{N}$, s. t. $G \cap \mathcal{N}_{k_0} \subset G$ is a non-measurable subset of G.
(otherwise $m(G) = 0$ contradicts)

□

第二章　*Measurable Functions*

2.1　*Measurable Functions*

定义　下面给出 $\mathbb{R}^d$ 上**可测函数**的定义.(注意值域为**扩充实数系** $\overline{\mathbb{R}}$)

定义 2.1.1. A function defined on a measurable subset $E \subset \mathbb{R}^d$ is **measurable** if for all $a \in \mathbb{R}$,

$$f^{-1}([-\infty, a)) = \{x \in E \mid f(x) < a\} \tag{2.1}$$

is measurable.

注.　• $f^{-1}([-\infty, a))$ 常简记作 $\{f < a\}$.

• 下面给出几条等价定义.

(1) $\{f < a\}$ is measurable. $\Leftrightarrow$ $\{f \leq a\}$ is measurable.

(2) $\Leftrightarrow$ $\{f > a\}$ is measurable $\Leftrightarrow$ $\{f \geq a\}$ is measurable.

(3) If f is finite-valued, then

$$f \textit{ is measurable} \;\Leftrightarrow\; \{a < f < b\} \textit{ is measurable}, \;\forall a, b \in \mathbb{R} \tag{2.2}$$

证明.

(1) Since the collection of measurable sets is closed under countable intersections and unions,

$$\{f \leq a\} = \bigcap_{n=1}^{\infty} \{f < a + \frac{1}{n}\} \tag{2.3}$$

$$\{f < a\} = \bigcup_{n=1}^{\infty} \{f \leq a - \frac{1}{n}\} \tag{2.4}$$

Therefore, $\{f < a\}$ is measurable. $\Leftrightarrow$ $\{f \leq a\}$ is measurable.

(2) Since the collection of measurable sets is closed under complements, easily proof by (1).

(3) Since f is finite-valued,

$$\{f < a\} = \bigcup_{n=1}^{\infty} \{-n < f < a\} \tag{2.5}$$

$$\{a < f < b\} = \{f > a\} \cap \{f < b\} \tag{2.6}$$

Therefore, by (2), f is measurable $\Leftrightarrow$ $\{a < f < b\}$ is measurable.

□

Property 下面给出可测函数的一些性质.

Property 1. Let $-\infty < f(x) < +\infty$ (finite-valued), then

$$f \text{ is measurable } \Leftrightarrow f^{-1}(O) \text{ is measurable } \forall \text{open set } O \tag{2.7}$$

$$\Leftrightarrow f^{-1}(F) \text{ is measurable } \forall \text{closed set } F \tag{2.8}$$

证明. $\forall O \underset{open}{\subset} \mathbb{R}$, there exists $\{(a_n, b_n)\}_{n=1}^{\infty}$, s. t.

$$O = \bigcup_{n=1}^{\infty} (a_n, b_n) \tag{2.9}$$

Then

$$f^{-1}(O) = f^{-1}(\bigcup_{n=1}^{\infty} (a_n, b_n)) = \bigcup_{n=1}^{\infty} f^{-1}((a_n, b_n)) \tag{2.10}$$

Since f is finite-valued and measurable, then $f^{-1}(a_n, b_n)$ is measurable.

Therefore, $f^{-1}(O)$ is measurable. □

Property 2. $\{continuous\ functions\} \subset \{measurable\ functions\}$

(a) (a)If f is continuous on $\mathbb{R}^d$, then f is measurable.

(b) If f is measurable, finite-valued and Φ is continuous on $\mathbb{R}$, then $\Phi \circ f$ is measurable.

证明.

(a) Since f is continuous, $\forall O \underset{open}{\subset} \mathbb{R}$, $f^{-1}(O) \underset{open}{\subset} \mathbb{R}^d$. By Property 1, f is measurable.

(b) $\forall O \underset{open}{\subset} \mathbb{R}$. Since Φ is continuous, then $\Phi^{-1}(O)$ is open.

Since f is finite-valued and measurable, then $(\Phi \circ f)^{-1}(O) = f^{-1}(\Phi^{-1}(O))$ is open.

Therefore, by Property 1, $\Phi \circ f$ is measurable.

$\square$

Property 3. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions. Then

$$\sup_n f_n(x),\ \inf_n f_n(x),\ \limsup_{n\to\infty} f_n(x),\ \liminf_{n\to\infty} f_n(x) \tag{2.11}$$

are measurable.

注. 类比数列的上下极限，此处

$$\limsup_{n\to\infty} f_n(x) := \lim_{k\to\infty} \sup_{n\geq k} \{f_n(x)\} = \inf_k \sup_{n\geq k} \{f_n(x)\} \tag{2.12}$$

$$\liminf_{n\to\infty} f_n(x) := \lim_{k\to\infty} \inf_{n\geq k} \{f_n(x)\} = \sup_k \inf_{n\geq k} \{f_n(x)\} \tag{2.13}$$

证明. Since

$$\{x \mid \sup_n f_n(x) > a\} = \bigcup_{n=1}^{\infty} \{x \mid f_n(x) > a\} \tag{2.14}$$

$$\{x \mid \inf_n f_n(x) < a\} = \bigcup_{n=1}^{\infty} \{x \mid f_n(x) < a\} \tag{2.15}$$

Then $\sup_n f_n(x)$, $\inf_n f_n(x)$ is measurable.

Since $\sup_{n\geq k} f_n(x)$, $\inf_{n\geq k} f_n(x)$ are measurable, by the previous conclusion, then

$$\limsup_{n\to\infty} f_n(x) = \inf_k \sup_{n\geq k} \{f_n(x)\} \tag{2.16}$$

$$\liminf_{n\to\infty} f_n(x) = \sup_k \inf_{n\geq k} \{f_n(x)\} \tag{2.17}$$

are measurable. $\square$

Property 4. If $\{f_n\}_{n=1}^{\infty}$ is a collection of measurable functions and

$$\lim_{n\to\infty} f_n(x) = f(x) \tag{2.18}$$

then f is measurable.

注. • 与数列上下极限相同，

$$\lim_{n\to\infty} f_n(x) = f(x) \ \Leftrightarrow\ \limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x) = f(x) \tag{2.19}$$

• 此 Property 即说明可测函数列对极限运算封闭. 注意到连续函数列对极限运算并不具备封闭性.(下面给出经典范例)

例 2.1.1.

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases} \tag{2.20}$$

证明. Since $\{f_n\}_{n=1}^{\infty}$ are measurable, $f(x) = \limsup_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x)$, then according to Property 3, f is measurable. □

Property 5. If f and g are measurable, then

(i) f^k, $k \in \mathbb{N}$ are measurable.

(ii) $f+g$ and fg are measurable if both f and g are finite-valued.

证明.

(i) Since

$$\{f^k > a\} = \{f > a^{\frac{1}{k}}\},\ \forall k \text{ is odd} \tag{2.21}$$

$$\{f^k > a\} = \{f > a^{\frac{1}{k}}\} \cup \{f < -a^{\frac{1}{k}}\},\ \forall k \text{ is even and } a > 0 \tag{2.22}$$

Therefore, f^k, $k \in \mathbb{N}$ are measurable.

(ii) Since[1]

$$\{f+g > a\} = \bigcup_{r\in\mathbb{Q}} \{f > a-r\} \cap \{g > r\} \tag{2.23}$$

[1]即必 $\exists r \in \mathbb{Q}$, s. t.$\{f+g > a\} \supset \{f > a-r\} \cap \{g > r\}$. (另一侧包含关系 $\subset$ 显然易证)
(反证. $\forall r \in \mathbb{Q}$ 上式不成立，则对于 $r = 0 \in \mathbb{Q}$, $\exists x_0$, s. t.$f(x_0) > a$, $g(x_0) > 0$, 且 $f(x_0) + g(x_0) \le a$, 矛盾.)

then $f+g$ is measurable.

By the previous results in (i) and (ii), since

$$fg = \frac{1}{4}[(f+g)^2 - (f-g)^2] \tag{2.24}$$

Therefore, fg is also measurable.

$\square$

下面给出数学分析中曾介绍过的**几乎处处**的定义.

定义 2.1.2. A property or statement is said to hold **almost everywhere (a.e.)** if it is true except on a set of measure zero.

例 2.1.2.

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases} \tag{2.25}$$

We say f is continuous a.e. on $[0,1]$ since $D(f) = \{1\}$ has measure zero.

下面说明几乎处处相等可保持函数可测性.

命题 2.1.1. If f is measurble and $f = g$ a.e. , then g is measurable.

证明. Since f is measurable and

$$g = (g-f) + f \tag{2.26}$$

then we shall proof that $g-f$ is measurable.

Let $A := \{x \mid g(x) - f(x) \neq 0\}$, then $m(A) = 0$. We get

$$\forall a \geq 0, (g-f)^{-1}((-\infty, a]) = (\mathbb{R}^d \backslash A) \cup N,\ \ where\ N \subset A \tag{2.27}$$

Since $m(A) = 0$, then N is measurable and $m(N) = 0$. So $(g-f)^{-1}((-\infty, a])$ is measurable. Therefore, $g-f$ is measurable. Then g is measurable. $\square$

2.2 *Measurable functions are nearly simple*

本节来介绍一个非常重要的定理. 即可测函数可由简单函数逼近.

特征函数 下面先来介绍**特征函数**的定义.

定义 2.2.1. If $E \subset \mathbb{R}$, the **characteristic / indicator function** $\chi_E/\mathbb{1}_E$ of E is defined by

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases} \tag{2.28}$$

下面给出可测集与其对应特征函数的关系.

命题 2.2.1. χ_E is measurable $\Leftrightarrow$ E is measurable

证明. Since

$$\chi_E^{-1}((-\infty, a]) = \begin{cases} \varnothing, & a < 0 \\ E^c, & 0 \le a < 1 \\ \mathbb{R}^d, & a \ge 1 \end{cases} \tag{2.29}$$

Then E is measurable $\Rightarrow$ χ_E is measurable.

χ_E is measurable $\Rightarrow$ $\chi_E^{-1}((-\infty, a]) = E^c$ is measurable. $\Rightarrow$ E is measurable. □

下面给出特征函数的基本性质.

命题 2.2.2. [Property].

(1) If $A \cap B = \varnothing$, then

$$\chi_{A\cup B} = \max\{\chi_A, \chi_B\} = \chi_A + \chi_B \tag{2.30}$$

(2) $\chi_{A\cap B} = \min\{\chi_A, \chi_B\} = \chi_A \cdot \chi_B$.

Simple functions 对特征函数做线性组合，即可得到**简单函数**.

定义 2.2.2. A **simple function** on $\mathbb{R}^d$ is a finite linear combination

$$f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x) \tag{2.31}$$

where each E_j is measurable and $m(E_j) < \infty$.

注. 此处定义中并未要求 $\{E_j\}_{j=1}^{n}$ disjoint. 而事实上这便引出了下面介绍的**标准形式**.

下面的命题说明了每个简单函数都可写为**标准形式** ($\{E_j\}_{j=1}^{n}$ disjoint).

命题 2.2.3. Every simple function f has a **standard representaion**

$$f = \sum_{k=1}^{N} a_k \chi_{E_k}, \ \textit{where } \{E_j\}_{k=1}^{N} \textit{ are disjoint} \tag{2.32}$$

证明. Suppose $f = \sum\limits_{k=1}^{N} b_k \chi_{E_k}$, $\{E_j\}_{k=1}^{N}$ may not be disjoint.
Since $\{E_j\}_{k=1}^{N}$ is finite, the number of elements of range f is also finite. Suppose

$$\textit{range } f = \{a_1, \cdots, a_M\} \tag{2.33}$$

Then let $F_k = f^{-1}(\{a_k\})$, then $\{F_k\}_{k=1}^{M}$ are disjoint. Therefore, we get the standard representation

$$f = \sum_{k=1}^{M} a_k \chi_{F_k} \tag{2.34}$$

□

简单函数逼近可测函数 下面给出一个定理，说明任一可测函数可由简单函数列逼近.

定理 2.2.1. Suppose $f : \mathbb{R}^d \longrightarrow [-\infty, \infty]$ is measurable.
Then there exists a sequence $\{\varphi_n\}$ of simple functions, s. t.

$$0 \leq |\varphi_1| \leq |\varphi_2| \leq \cdots \leq |f| \tag{2.35}$$

$$\lim_{k\to\infty} \varphi_k(x) = f(x),\ for\ all\ x \tag{2.36}$$

and $\varphi_k \to f$ uniformly on any set on which f is bounded.

证明. 下面从两方面分类讨论，即非负函数 & 变号函数，f 有界 & 无界.

(1) 非负函数 $f : \mathbb{R}^d \longrightarrow [0, \infty]$.

1° f is bounded. Assume $|f(x)| \leq M$.

Let[2]

$$E_n^k = f^{-1}((\frac{k}{2^n}, \frac{k+1}{2^n}]), k = 0, \cdots, N_n \tag{2.37}$$

$$\varphi_n(x) = \frac{k}{2^n},\ if\ x \in E_n^k \tag{2.38}$$

Then

$$\varphi_n(x) = \sum_{k=0}^{N_n} \frac{k}{2^n} \chi_{E_n^k}(x) \tag{2.39}$$

Therefore[3]

$$|\varphi_n(x) - f(x)| \leq \frac{1}{2^n} \to 0\ (independent\ of\ x) \tag{2.40}$$

$\Rightarrow \varphi_n \to f$ uniformly.

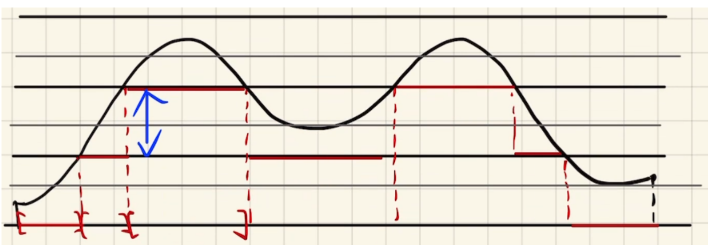

图 2.1: 对 f 值域进行分划

[2] E_n^k 表示第 n 次对值域进行分划后产生的第 k 个值域区间，其中 $\frac{N_n+1}{2^n} \geq M$.

[3] $|\varphi_n(x) - f(x)|$ 小于等于第 n 次分划后两个相邻值域区间的步长值，即 $\frac{1}{2^n}$.

2° f is unbounded. (<u>idea</u>: truncation，将 f 截断为一列有界函数列，并逐点收敛于 f)

Let

$$f_k(x) = \begin{cases} f(x), & \text{if } f(x) \le k \\ k, & \text{if } f(x) > k \end{cases} \tag{2.41}$$

Then $f_k(x) \to f(x)$, $\forall x \in \mathbb{R}^d$.

Since f_k is bounded, by the previous result in 1°,

For each k, $\exists$ a sequence of simple functions $\{\psi_{kn}\}_{n=1}^{\infty}$, s. t.

$$\psi_{kn}(x) \to f_k(x), \ \forall x \tag{2.42}$$

So we get

$$\begin{array}{cccccc} \psi_{11} & \psi_{12} & \psi_{13} & \cdots & \to & f_1 \\ \psi_{21} & \psi_{22} & \psi_{23} & \cdots & \to & f_2 \\ \psi_{31} & \psi_{32} & \psi_{33} & \cdots & \to & f_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \downarrow \\ & & & & & f \end{array} \tag{2.43}$$

From the previous results in 1°, we get

$$|\psi_{kn}(x) - f_k(x)| \le \frac{1}{2^n} \tag{2.44}$$

Let $n = k$, then $|\psi_{kk}(x) - f_k(x)| \le \frac{1}{2^k}$. Let $\varphi_k = \psi_{kk}$, then

$$|\varphi_k(x) - f(x)| \le |\varphi_k(x) - f_k(x)| + |f_k(x) - f(x)| \tag{2.45}$$

Since $f_k(x) \to f(x)$, we get $\varphi_k(x) \to f(x)$, $\forall x$, where $\{\varphi_k = \psi_{kk}\}_{k=1}^{\infty}$ are simple functions.

(2) 变号函数 $f : \mathbb{R}^d \longrightarrow [-\infty, \infty]$.

We denote that

$$f^+(x) := \max\{f(x), 0\} \tag{2.46}$$

$$f^-(x) := \max\{-f(x), 0\} \tag{2.47}$$

By the previous results in (1), there exist sequences of simple functions $\{\varphi_k\}_{k=1}^{\infty}, \{\psi_k\}_{k=1}^{\infty}$, s. t.

$$\varphi_k \to f^+ \ \textit{and} \ \psi_k \to f^- \ \textit{pointwisely} \tag{2.48}$$

We can observe that $f = f^+ - f^-$ and $|f| = f^+ - f^-$.

Let $\phi_k(x) = \varphi_k(x) - \psi_k(x)$, then ϕ_k is a simple function with $\phi_k \to f$ pointwisely.

□

阶梯函数逼近可测函数 在证明了可测函数可由简单函数逼近后，我们更进一步，来说明可测函数可由更加简单的**阶梯函数**来逼近.

先给出**阶梯函数**的定义.

定义 2.2.3. A **step function** is a finite sum

$$f=\sum_{k=1}^{N}a_k\chi_{R_k},\ where\ R_k\ is\ a\ rectangle \tag{2.49}$$

注. 阶梯函数 & 简单函数的区别在于，简单函数是作用于有限个**可测集** E_k，而阶梯函数是作用于有限个**矩形** R_k.

下面的定理说明了 measurable functions are almost step functions.

定理 2.2.2. Suppose f is measurable on $\mathbb{R}^d$. Then there exists a sequence of step functions $\{\psi_k\}_{k=1}^{\infty}$, s. t.

$$\lim_{k\to\infty}\psi_k(x)=f(x),\ a.e.\ x \tag{2.50}$$

注. 首先介绍函数列收敛点集的几种不同的等价表述：

$$\{x\mid \forall\epsilon>0,\exists N\in\mathbb{N},\forall n\geq N,\ |f_n(x)-f(x)|<\epsilon\} \tag{2.51}$$

$$\Leftrightarrow\{x\mid \forall n\in\mathbb{N},\exists N\in\mathbb{N},\forall k\geq N,\ |f_k(x)-f(x)|<\frac{1}{n}\} \tag{2.52}$$

$$\Leftrightarrow\bigcap_{n=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{k=N}^{\infty}\{x\mid\ |f_k(x)-f(x)|<\frac{1}{n}\} \tag{2.53}$$

从而可以得到函数列发散点集 (Negation)：

$$\{x\mid \exists n\in\mathbb{N},\forall N\in\mathbb{N},\exists k\geq N,\ |f_k(x)-f(x)|\geq\frac{1}{n}\} \tag{2.54}$$

$$\Leftrightarrow\bigcup_{n=1}^{\infty}\bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty}\{x\mid\ |f_k(x)-f(x)|\geq\frac{1}{n}\} \tag{2.55}$$

$$\Leftrightarrow\bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty}\{x\mid\ f_k(x)\neq f(x)\} \tag{2.56}$$

证明. (证明思路：先用阶梯函数逼近简单函数，再用简单函数逼近可测函数.)

It suffices to show that χ_E can be approxiamted by step functions, for any measurable set E.

According to Thm1.3.4 (iv)

Let $f=\chi_E$, then $\forall\epsilon>0$, $\exists$ cubes $\bigcup_{j=1}^{N}Q_j$, s. t.

$$m(E\bigtriangleup\bigcup_{j=1}^{N}Q_j)\leq\epsilon \tag{2.57}$$

By considering the grid formed by extending the sides of these cubes, there exists almost disjoint rectangles $\{\widetilde{R}_j\}_{j=1}^{M}$, s. t.

$$\bigcup_{j=1}^{N}Q_j=\bigcup_{j=1}^{M}\widetilde{R}_j \tag{2.58}$$

By taking ranctangles R_j contained in $\widetilde{R}_j$, we can find a collection of disjoint rectangles $\{R_j\}_{j=1}^{M}$, s. t.

$$m(E\bigtriangleup\bigsqcup_{j=1}^{M}R_j)\leq 2\epsilon \tag{2.59}$$

For every $k\in\mathbb{N}$, there exists disjoint rectangles $\{R_j\}_{j=1}^{M}$, s. t.

$$m(E\bigtriangleup\bigsqcup_{j=1}^{M}R_j)\leq\frac{1}{2^{k+1}} \tag{2.60}$$

There also exists a step function ψ_k

$$\psi_k(x):=\chi_{\bigcup_{j=1}^{M}R_j}(x)=\sum_{j=1}^{M}\chi_{R_j}(x) \tag{2.61}$$

Let

$$E_k:=\{x\mid f_k(x)\neq f(x)\} \tag{2.62}$$

Since $E_k\subset E\bigtriangleup\bigsqcup_{j=1}^{M}R_j$, then $m(E_k)\leq\frac{1}{2^k}$. Let[4]

$$F_j=\bigcup_{j=k+1}^{\infty}E_j,\ \ F=\bigcap_{k=1}^{\infty}F_k \tag{2.63}$$

Then $\psi_k(x)\to f(x)$, $\forall x\in F^c$. Since

$$m(F)\leq m(F_k),\ \ \forall k\in\mathbb{N} \tag{2.64}$$

$$m(F_k)=m(\bigcup_{j=k+1}^{\infty}E_j)\leq\sum_{j=k+1}^{\infty}m(E_j)\leq\frac{1}{2^k} \tag{2.65}$$

Therefore, $m(F)=0$. $\lim\limits_{k\to\infty}\psi_k(x)=f(x)$, $a.e.\ x$. □

[4]根据注中式 (2.56)，F 即为函数列 $\{\psi_k\}_{k=1}^{\infty}$ 的发散点集，从而 $\psi_k(x)\to f(x)$ 在 F^c 上收敛.

第三章 *Integration Theory*

3.1 *The Lebesgue integral*

Lebesgue Integral 的构造可以分为三步，分别为构造下列函数的积分:

1. **Simple functions**
2. **Non-negative measurable functions**

$$\int f := \sup\{\int \varphi \mid \varphi \text{ simple},\ 0 \le \varphi \le f\} \tag{3.1}$$

3. **General case**

$$f = f^+ - f^- \tag{3.2}$$

$$\int f := \int f^+ - \int f^- \tag{3.3}$$

3.1.1 *Simple functions*

定义 下面先给出非负简单函数在**标准形式**下的**积分**定义.

定义 3.1.1. If φ is a non-negative simple function with **standard representation**

$$\varphi(x) = \sum_{k=1}^{M} a_k \chi_{E_k}(x) \tag{3.4}$$

We define the **Lebesgue integral** of φ by

$$\int_{\mathbb{R}^d} \varphi(x)dx = \sum_{k=1}^{M} a_k m(E_k) \tag{3.5}$$

If E is a measurable subset of $\mathbb{R}^d$ with finite measure, then

$$\varphi(x)\chi_E(x) = \sum_{k=1}^{M} a_k \chi_{E_k}(x)\chi_E(x) = \sum_{k=1}^{M} a_k \chi_{E_k \cap E}(x) \tag{3.6}$$

is also a simple function, and define

$$\int_E \varphi(x)dx = \int_{\mathbb{R}^d} \varphi(x)\chi_E(x)dx \tag{3.7}$$

注. • 此处仅对**标准形式**定义了积分. 事实上，此处定义的积分与简单函数的表达形式无关 (即**Property 1.**).

• 关于记号，当测度非常明确时，大多数情况下可简写，如

$$\int_E \varphi(x)dx \Rightarrow \int_E \varphi \tag{3.8}$$

$$\int_{\mathbb{R}^d} \varphi(x)dx \Rightarrow \int \varphi \tag{3.9}$$

当为了强调我们选择了何种测度 μ 时，还可用以下的记号：

$$\int_E \varphi(x)d\mu(x) \tag{3.10}$$

Property 下面给出简单函数积分的性质.

Property 1. **Independence of the representation**.

If $\varphi = \sum\limits_{k=1}^{N} a_k \chi_{E_k}$ is any representation of φ, then

$$\int \varphi = \sum_{k=1}^{N} a_k m(E_k) \tag{3.11}$$

在证明这个性质之前，先来证明一条引理.(书[1]Exercises Of Chapter 2 的第 1 题)

引理 3.1.1. Given a collection of sets $\{F_k\}_{k=1}^{n}$, there exists another collection $\{\widetilde{F_j}\}_{j=1}^{N}$ with $N = 2^n - 1$, so that

$$(i). \quad \bigcup_{k=1}^{n} F_k = \bigcup_{j=1}^{N} \widetilde{F_j} \tag{3.12}$$

$$(ii). \quad \{\widetilde{F_j}\}_{j=1}^{N} \text{ are disjoint} \tag{3.13}$$

$$(iii). \quad F_k = \bigcup_{\widetilde{F_j} \subset F_k} \widetilde{F_j} \tag{3.14}$$

证明. Consider the collection

$$\mathcal{F} := \{\bigcup_{k=1}^{n} G_k - \bigcap_{k=1}^{n} F_k^c \;\Big|\; G_k \text{ denotes } F_k \text{ or } F_k^c\} \tag{3.15}$$

□

[1]参考书籍：《*Real Analysis – –Measure Theroy, Integration, & Hilbert Spaces*》— *Elias M. Stein*

下面来证明原命题.

证明. According to Lemma 3.1.1, there exists another decompositon of $\bigcup_{k=1}^{N} E_k$, i.e.

$$\bigcup_{j=1}^{M} \widetilde{E}_j = \bigcup_{k=1}^{N} E_k \tag{3.16}$$

where $\{\widetilde{E}_j\}_{j=1}^{M}$ are disjoint, and for each $1 \le k \le M$,

$$E_k = \bigcup_{\widetilde{E}_j \subset E_k} \widetilde{E}_j \tag{3.17}$$

Let

$$\widetilde{a}_j := \sum_{\widetilde{E}_j \subset E_k} a_k \tag{3.18}$$

Then clearly

$$\varphi = \sum_{j=1}^{M} \widetilde{a}_j \chi_{\widetilde{E}_j} \tag{3.19}$$

Since $\{\widetilde{E}_j\}_{j=1}^{M}$ are disjoint, we get

$$\int \varphi = \sum_{j=1}^{M} \widetilde{a}_j m(\widetilde{E}_j) = \sum_{j=1}^{M} \sum_{\widetilde{E}_j \subset E_k} a_k m(\widetilde{E}_j) = \sum_{k=1}^{N} a_k m(E_k) \tag{3.20}$$

□

Property 2. **Linearity**.

If φ and ψ are non-negative simple, and $a, b \ge 0$, then

$$\int (a\varphi + b\psi) = a \int \varphi + b \int \psi \tag{3.21}$$

证明. 下面分为两步来证明.

(a) $\forall c \ge 0$, $\int c\varphi = c \int \varphi$.

Suppose $\varphi = \sum_{k=1}^{M} a_k \chi_{E_k}$, where $\{E_k\}_{k=1}^{M}$ are disjoint. Then

$$c\varphi = \sum_{k=1}^{M} c a_k \chi_{E_j} \tag{3.22}$$

is also a non-negative simple function. Therefore,

$$\int c\varphi = \sum_{k=1}^{M} c a_k m(E_k) = c \sum_{k=1}^{M} a_k m(E_k) = c \int \varphi \tag{3.23}$$

(b) $\int(\varphi+\psi)=\int\varphi+\int\psi$.

Suppose

$$\varphi=\sum_{k=1}^{M}a_k\chi_{E_k},\quad \psi=\sum_{j=1}^{N}b_j\chi_{F_j} \tag{3.24}$$

where both $\{E_k\}_{k=1}^{M}$ and $\{F_j\}_{j=1}^{N}$ are disjoint and $\mathbb{R}^d=\bigcup_{k=1}^{M}E_k=\bigcup_{j=1}^{N}F_j$. Since

$$E_k=E_k\cap\mathbb{R}^d=E_k\cap\bigsqcup_{j=1}^{N}F_j=\bigsqcup_{j=1}^{N}(E_k\cap F_j) \tag{3.25}$$

Then

$$\varphi=\sum_{k=1}^{M}a_k\chi_{E_k}=\sum_{k=1}^{M}a_k\chi_{\bigsqcup_{j=1}^{N}(E_k\cap F_j)}=\sum_{k=1}^{M}\sum_{j=1}^{N}a_k\chi_{E_k\cap F_j} \tag{3.26}$$

Similarly

$$\psi=\sum_{j=1}^{N}b_j\chi_{F_j}=\sum_{j=1}^{N}b_k\chi_{\bigsqcup_{k=1}^{M}(E_k\cap F_j)}=\sum_{j=1}^{N}\sum_{k=1}^{M}b_k\chi_{E_k\cap F_j} \tag{3.27}$$

Therefore

$$\varphi+\psi=\sum_{j,k}(a_k+b_j)\chi_{E_k\cap F_j} \tag{3.28}$$

$$\int(\varphi+\psi)=\sum_{j,k}(a_k+b_j)m(E_k\cap F_j) \tag{3.29}$$

$$=\sum_{j,k}a_k m(E_k\cap F_j)+\sum_{j,k}b_j m(E_k\cap F_j) \tag{3.30}$$

$$=\int\varphi+\int\psi \tag{3.31}$$

□

Property 3. **Monotonicity**.

If $\varphi\le\psi$ are non-negative and simple, then

$$\int\varphi\le\int\psi \tag{3.32}$$

证明. Suppose

$$\varphi=\sum_{k=1}^{M}a_k\chi_{E_k},\quad \psi=\sum_{j=1}^{N}b_j\chi_{F_j} \tag{3.33}$$

where both $\{E_k\}_{k=1}^M$ and $\{F_j\}_{j=1}^N$ are disjoint. Similar to the proof in Property 2, we get

$$\psi - \varphi = \sum_{j,k} (b_j - a_k)\chi_{E_k \cap F_j} \tag{3.34}$$

Since $\varphi(x) \leq \psi(x)$, $\forall x \in \mathbb{R}^d$, then $\psi - \varphi$ is non-negative and simple. Therefore,

$$\int (\psi - \varphi) = \sum_{j,k} (b_j - a_k) m(E_k \cap F_j) \geq 0 \ \Rightarrow \ \int \varphi \leq \int \psi \tag{3.35}$$

□

Property 4. Additivity.

If $\{E_k\}_{k=1}^\infty$ are disjoint subsets of $\mathbb{R}^d$ with finite measure, then

$$\int_{\bigcup_{k=1}^\infty E_k} \varphi = \sum_{k=1}^\infty \int_{E_k} \varphi \tag{3.36}$$

注. 首先回顾 *abstract measure* 的定义.

定义 3.1.2. Let X be a set and let $\mathcal{M}$ be a $\sigma - algebra$ on X.

A **measure** on $\mathcal{M}$ is a function $\mu : \mathcal{M} \longrightarrow [0, \infty]$, s. t.

(i) $\mu(\varnothing) = 0$.

(ii) If $\{E_j\}_{j=1}^\infty \subset \mathcal{M}$ are disjoint, then

$$\mu(\bigcup_{j=1}^\infty E_j) = \sum_{j=1}^\infty \mu(E_j) \tag{3.37}$$

回到我们积分的性质上来. 下面我们将说明，对于任一给定的非负简单函数 φ，将 φ 在任一可测集 A 上的积分看作 *Lebesgue* $\sigma - algebra$ $\mathcal{L}$ 上的映射，则该映射为定义在 $\mathcal{L}$ 上的测度.(从而 Property 4. 作为测度的必要条件自然成立)

命题 3.1.1. For any fixed non-negative and simple function φ, the map

$$\mu : \mathcal{L} \longrightarrow [0, \infty] \tag{3.38}$$

$$A \longmapsto \int_A \varphi \tag{3.39}$$

is a measure on $\mathcal{L}$.

证明. Suppose $\{A_j\}_{j=1}^{\infty} \subset \mathcal{L}$ are disjoint, and

$$\varphi = \sum_{k=1}^{M} a_k \chi_{E_k}, \ \ where \ \{E_k\}_{k=1}^{M} \ are \ disjoint \tag{3.40}$$

Let $A = \bigcup_{j=1}^{\infty} A_j$, then

$$\int_{\bigcup_{j=1}^{\infty} A_j} \varphi = \int_A \varphi = \int \varphi \chi_A = \int (\sum_{k=1}^{M} a_k \chi_{E_k \cap A}) \tag{3.41}$$

$$= \sum_{k=1}^{M} a_k m(E_k \cap A) \tag{3.42}$$

$$= \sum_{k=1}^{M} a_k m(E_k \cap (\bigcup_{j=1}^{\infty} A_j)) \tag{3.43}$$

$$= \sum_{k=1}^{M} a_k m(\bigsqcup_{j=1}^{\infty} (E_k \cap A_j)) \tag{3.44}$$

$$= \sum_{k=1}^{M} a_k \sum_{j=1}^{\infty} m(E_k \cap A_j) \tag{3.45}$$

$$= \sum_{k=1}^{M} \sum_{j=1}^{\infty} a_k m(E_k \cap A_j) \tag{3.46}$$

Since positive series always converges in $[0, \infty]$, then

$$\int_A \varphi = \sum_{k=1}^{M} \sum_{j=1}^{\infty} a_k m(E_k \cap A_j) = \sum_{j=1}^{\infty} \sum_{k=1}^{M} a_k m(E_k \cap A_j) = \sum_{j=1}^{\infty} \int_{A_j} \varphi \tag{3.47}$$

Therefore, **the integral on any non-negative simple function is accually a measure on $\mathcal{L}$**. □

3.1.2 *Non – negative measurable functions*

为了讨论的方便，先给出**非负可测函数**的一个记号.

$$\mathcal{M}^+ := \{all\ non-negative\ measurable\ functions\} \tag{3.48}$$

定义 下面给出非负可测函数的积分的定义.

定义 3.1.3. For $f \in \mathcal{M}^+$, we define

$$\int f(x)dx := \sup\{\int \varphi(x)dx \mid 0 \le \varphi \le f,\ \varphi\ simple\} \tag{3.49}$$

注. 此处对 Non-negative measurable function 积分的定义兼容定义 3.1.1 中对 Non-negative simple function 积分的定义，具体表现为：$\forall \varphi_0$ non-negative and simple,

$$\sup\{\int \varphi(x)dx \mid 0 \le \varphi \le \varphi_0,\ \varphi\ simple\} = \int \varphi_0(x)dx \tag{3.50}$$

性质 下面来验证定义 3.1.3 中定义的积分满足几条基本性质.

Property 1. **Monotonicity**.

Let $f, g \in \mathcal{M}^+$. Then

$$\int f \le \int g\ \ if\ \ f \le g \tag{3.51}$$

证明. Let

$$A = \{\varphi\ simple \mid 0 \le \varphi \le f\} \tag{3.52}$$

$$B = \{\psi\ simple \mid 0 \le \psi \le g\} \tag{3.53}$$

Then for all $\varphi \in A$, $0 \le \varphi \le f \le g \Rightarrow \varphi \in B \Rightarrow A \subset B$. Since

$$\int f = \sup_{\varphi \in A}\{\int \varphi\},\ \int g = \sup_{\psi \in B}\{\int \psi\} \tag{3.54}$$

Therefore

$$\int f \le \int g \tag{3.55}$$

□

Property 2. 齐次性.

Let $f \in \mathcal{M}^+$. If $c \geq 0$, then

$$\int cf = c\int f \tag{3.56}$$

证明. Assume $c > 0$. Then

$$\int cf = \sup\{\int \varphi \mid 0 \leq \varphi \leq cf,\ \varphi \textit{ simple}\} \tag{3.57}$$

$$= \sup\{\int \varphi \mid 0 \leq \frac{\varphi}{c} \leq f,\ \varphi \textit{ simple}\} \tag{3.58}$$

$$\overset{\psi=\frac{\varphi}{c}}{=} \sup\{\int c\psi \mid 0 \leq \psi \leq f,\ \psi \textit{ simple}\} \tag{3.59}$$

$$= c\sup\{\int \psi \mid 0 \leq \psi \leq f,\ \psi \textit{ simple}\} \tag{3.60}$$

$$= c\int f \tag{3.61}$$

□

单调收敛定理 下面我们正式迈入实分析的**"大门"**, 介绍第一个**收敛定理**.

定理 3.1.2. The Monotone Convergence Theorem.

If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+, f_j \leq f_{j+1}$ for all j, and $\lim\limits_{n\to\infty} f_n = f$, then

$$\int f = \lim_{n\to\infty}\int f_n \tag{3.62}$$

注.
- 此即为**"单调收敛定理"**，这个定理说明了对于**单调递增**的**非负可测**函数列，其积分与极限可交换次序. 具体表现为

$$\int f = \int \lim_{n\to\infty} f_n = \lim_{n\to\infty}\int f_n \tag{3.63}$$

- 该定理还说明了，我们可以给出非负可测函数的另一个更自然的等价定义，即用非负简单函数列的积分逼近非负可测函数的积分.

定义 3.1.4. For $f \in \mathcal{M}^+$, we can also define

$$\int f := \lim_{n\to\infty}\int \varphi_n \tag{3.64}$$

where $\varphi_n \to f$ and $0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq f$ by Thm 2.2.1.

并且该定理说明了该积分定义的**唯一性**及 **well-defined**.

在证明定理前，先来证明一个引理 (将定理 1.3.3 (i) 拓展到一般的抽象测度上).

引理 3.1.3. Let X be a set, $\mathcal{M}$ be a $\sigma-algebra$ on X, $\mu:\mathcal{M}\longrightarrow[0,\infty]$ be a measure on $\mathcal{M}$. If $\{E_n\}_{n=1}^{\infty}\subset\mathcal{M}$, $E_n\nearrow E$, then

$$\lim_{n\to\infty}\mu(E_n)=\mu(E) \tag{3.65}$$

证明. 证明过程与 Thm 1.3.3 完全一致 (仅用到了测度的可数可加性).
Let $S_1=E_1$, $S_k=E_k-E_{k-1}$, $\forall k\geq 2$. Then $\{S_k\}_{k=1}^{\infty}\subset\mathcal{M}$ are disjoint.
Since $E=\bigcup\limits_{k=1}^{\infty}E_k=\bigcup\limits_{k=1}^{\infty}S_k$, then

$$\mu(E)=\mu(\bigsqcup_{k=1}^{\infty}S_k)=\sum_{k=1}^{\infty}\mu(S_k)=\lim_{n\to\infty}\sum_{k=1}^{n}\mu(S_k)=\lim_{n\to\infty}\mu(\bigsqcup_{k=1}^{n}S_k)=\lim_{n\to\infty}\mu(E_n) \tag{3.66}$$

□

下面证明原定理.

证明.

- $\lim\limits_{n\to\infty}\int f_n\leq\int f$.

  Since $f_n\leq f$, $\forall n$, then

  $$\int f_n\leq\int f,\ \forall n \tag{3.67}$$

  Since $\{\int f_n\}_{n=1}^{\infty}$ always converges in $[0,\infty]$, then let $n\to\infty$, we get

  $$\lim_{n\to\infty}\int f_n\leq\int f \tag{3.68}$$

- $\lim\limits_{n\to\infty}\int f_n\geq\int f$.

  Fix $0<a<1$, for any $0\leq\varphi\leq f$ simple, let

  $$E_n=\{x\mid f_n(x)\geq a\varphi(x)\} \tag{3.69}$$

  Then since $\forall x\in E_n$, we have $f_{n+1}(x)\geq f_n(x)\geq a\varphi(x)\Rightarrow x\in E_{n+1}\Rightarrow E_n\subset E_{n+1}$.

  Then $E_n\nearrow$. Since

  $$\int_{\mathbb{R}^d}f_n\geq\int_{E_n}f_n\geq\int_{E_n}a\varphi,\ \forall n \tag{3.70}$$

  Let $n\to\infty$, we get

  $$\lim_{n\to\infty}\int_{\mathbb{R}^d}f_n\geq\lim_{n\to\infty}\int_{E_n}a\varphi \tag{3.71}$$

Then we have to calculate $\lim_{n\to\infty}\int_{E_n} a\varphi$:

- Since $a\varphi$ is non-negative and simple, by Prop 3.1.1, the map

$$\mu : \mathcal{L} \longrightarrow [0, \infty] \tag{3.72}$$

$$E \longmapsto \int_E a\varphi \tag{3.73}$$

is a measure on the collection of Lebesgue measurable sets $\mathcal{L}$. (将积分视作测度)

Since $\{E_n\}_{n=1}^{\infty} \subset \mathcal{L}$ and $E_n \nearrow$, by Lemma 3.1.3, we get

$$\lim_{n\to\infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n) \tag{3.74}$$

i.e.

$$\lim_{n\to\infty} \int_{E_n} a\varphi = \int_{\bigcup_{n=1}^{\infty} E_n} a\varphi \tag{3.75}$$

For all $x \in \mathbb{R}^d$, since $a\varphi(x) < f(x)$ and $f_n \to f$, there exists $N_x \in \mathbb{N}$, s. t.

$$f_n(x) \geq a\varphi(x), \ \forall n \geq N_x \tag{3.76}$$

which indicates $x \in E_{N_x}$ for some N_x. Therefore

$$\bigcup_{n=1}^{\infty} E_n = \mathbb{R}^d \Rightarrow \lim_{n\to\infty} \int_{E_n} a\varphi = \int_{\bigcup_{n=1}^{\infty} E_n} a\varphi = \int_{\mathbb{R}^d} a\varphi \tag{3.77}$$

Therefore, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \geq \lim_{n\to\infty} \int_{E_n} a\varphi = \int_{\mathbb{R}^d} a\varphi \tag{3.78}$$

Let $a \to 1$, then

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \geq \int_{\mathbb{R}^d} \varphi \tag{3.79}$$

Since φ is arbitratry, taking the supremum over φ, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \geq \sup\{\int_{\mathbb{R}^d} \varphi \mid 0 \leq \varphi \leq f, \ \varphi \ simple\} = \int f \tag{3.80}$$

□

函数项级数的可数可加性　接下来我们将给出**单调收敛定理**在**函数项级数**上的表达形式，它说明了对于**非负可测函数项级数**，其**积分与求和可交换次序**.

在此之前，先来证明有限项的情况.

(此也可视作非负可测函数积分的**Property** **线性性**的一部分.)

命题 3.1.2. Linearity.

If $f, g \in \mathcal{M}^+$, then

$$\int (f+g) = \int f + \int g \tag{3.81}$$

证明. By Thm 2.2.1 and Thm 3.1.2, there exists sequences of non-negative and simple functions $\{\varphi_n\}_{n=1}^{\infty}$ and $\{\psi_n\}_{n=1}^{\infty}$, $\varphi_n \to f$ and $\psi_n \to g$, s. t.

$$\int f = \lim_{n\to\infty} \int \varphi_n, \quad \int g = \lim_{n\to\infty} \int \psi_n \tag{3.82}$$

Since $\varphi_n + \psi_n$ is still non-negative and simple, then

By the Linearity of integral on non-negative and simple functions, (**Property 2.** in §3.1.1)

$$\int (\varphi_n + \psi_n) = \int \varphi_n + \int \psi_n \tag{3.83}$$

Let $n \to \infty$, by Thm 3.1.2, we get (**极限与积分交换次序**)

$$\int (f+g) = \int f + \int g \tag{3.84}$$

□

根据 Prop 3.1.2，由归纳法，容易得到其对任意**有限项**函数项级数都成立.

下面给出函数项级数上的**单调收敛定理**.

定理 3.1.4. Monotone Convergence Theorem (MCT , series version).
If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+$ and $f = \sum_{n=1}^{\infty} f_n$, then

$$\int f = \sum_{n=1}^{\infty} \int f_n \tag{3.85}$$

注. 该定理说明了对于**非负可测函数项级数**，其**积分与求和可交换次序**.

证明. Let $F_n = \sum_{k=1}^{n} f_k$, then $F_n \nearrow \sum_{k=1}^{\infty} f_k = f$. By **MCT** (Thm 3.1.2),

$$\lim_{n\to\infty} \int F_n = \int f \tag{3.86}$$

i.e.

$$\lim_{n\to\infty} \int \sum_{k=1}^{n} f_k = \int f \tag{3.87}$$

By the **Linearity** of integral on non-negative functions (Prop 3.1.2),

$$\lim_{n\to\infty} \int \sum_{k=1}^{n} f_k = \lim_{n\to\infty} \sum_{k=1}^{n} \int f_k = \sum_{k=1}^{\infty} \int f_k = \int f \tag{3.88}$$

□

积分的唯一性 在实分析中，我们并不关心零测集上的各种性质，进而常常忽略函数在零测集上的情况. 在给出**单调收敛定理**的更一般版本前，我们先来给出**几乎处处**意义下，函数**积分的唯一性**.

下面的命题说明了，若两个非负可测函数**几乎处处**相等，则其积分相等.

命题 3.1.3. Uniqueness.

If $f \in \mathcal{M}^+$, then

$$\int f = 0 \ \Leftrightarrow \ f = 0 \ a.e. \tag{3.89}$$

注. 根据该命题，对于任意非负可测函数 f, g

$$\int f = \int g \ \Leftrightarrow \ \int (f - g) = 0 \ \Leftrightarrow \ f - g = 0 \ a.e. \ \Leftrightarrow \ f = g \ a.e. \tag{3.90}$$

证明.

- 充分性 "$\Leftarrow$"：If $f = 0$ a.e.

  $\forall 0 \le \varphi \le f$ simple, $\varphi = 0$ a.e. . Let $E = \{x \mid \varphi(x) = 0\}$, then $m(E^c) = 0$.

$$\int \varphi = \int_E \varphi + \int_{E^c} \varphi = 0 + 0 = 0 \tag{3.91}$$

  Taking the supremum of φ, we get

$$\int f = \sup\{\int \varphi \mid 0 \le \varphi \le f, \ \varphi \ simple\} = 0 \tag{3.92}$$

- 必要性 "$\Rightarrow$"：If $\int f = 0$, let

$$E_n := \{x \mid f(x) > \frac{1}{n}\} \tag{3.93}$$

  Then

$$\bigcup_{n=1}^{\infty} E_n = \{x \mid f(x) > 0\} = \{f \neq 0\} \tag{3.94}$$

  Suppose $m(\bigcup\limits_{n=1}^{\infty} E_n) > 0$, then there exists $N \in \mathbb{N}$, s. t. $m(E_N) > 0$. Then

$$\int f \ge \int_{E_N} f > \frac{1}{N} m(E_N) > 0 \tag{3.95}$$

  which is a contradiction to $\int f = 0$.

  Therefore, $m(\bigcup\limits_{n=1}^{\infty} E_n) = m(\{f \neq 0\}) = 0, f = 0$ a.e.

□

“几乎处处”版 MCT 根据积分的唯一性 (命题 3.1.3)，下面说明在**“几乎处处收敛”**条件下，**单调收敛定理**成立 (积分与极限仍可交换次序).

推论 3.1.5. a.e. MCT.
If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+, f \in \mathcal{M}^+, f_n \nearrow f$ a.e. , then

$$\int f = \lim_{n\to\infty} \int f_n \tag{3.96}$$

证明. Let $f_n \nearrow f$ on E, then $m(E^c) = 0$ and $f_n - f_n\chi_E = 0$ a.e.
By Prop 3.1.3, we get

$$\int f_n = \int f_n\chi_E \tag{3.97}$$

Since $f_n\chi_E \nearrow f\chi_E$, then by **MCT** (Thm 3.1.2, 单调收敛定理)

$$\lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \int f_n\chi_E = \int f\chi_E = \int_E f \tag{3.98}$$

Since $m(E^c) = 0$, then

$$\int f = \int_E f = \lim_{n\to\infty} \int f_n \tag{3.99}$$

($\forall 0 \le \varphi \le f$ simple, $\int \varphi = \int_E \varphi + \int_{E_c} \varphi = \int_E \varphi$. Taking the supremum of $\varphi \Rightarrow \int f = \sup\{\int \varphi\} = \int_E f$)

□

Fatou's Lemma 我们首先来考虑一个问题，若我们将**单调收敛定理 (MCT)** 中的**“单调”**条件去掉，结论是否仍然成立 (积分与极限是否仍可交换次序)？即

$$\text{Suppose } f_n \to f \text{ a.e. , do we have } \int f_n \to \int f?$$

事实上答案为 absolutely no. 下面给出一个反例.

例 3.1.1. Consider $f_n = n\chi_{(0,\frac{1}{n})}$. Then $f_n \to 0$ a.e. on $[0, 1]$. However,

$$\int f_n = n \cdot \frac{1}{n} = 1, \ \forall n \in \mathbb{N} \not\to 0 \tag{3.100}$$

事实上，将**“单调收敛”**条件整个去除，我们将得到如下的更一般的 **Fatou's Lemma**.

定理 3.1.6. Fatou's Lemma.

If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+$, then

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n \tag{3.101}$$

注. • 回顾函数列下极限的定义.

$$\liminf_{n\to\infty} f_n = \lim_{n\to\infty} \left(\inf_{k\geq n} f_k\right) \tag{3.102}$$

即对定义域上每一点 x，取数列 $\{f_n(x)\}_{n=1}^{\infty}$ 的下极限，再将所有的 x 所对应的下极限拼成一个函数，即定义为函数列 $\{f_n\}_{n=1}^{\infty}$ 的下极限.

(上式右侧作用在固定的 x 上，即为数列 $\{f_n(x)\}_{n=1}^{\infty}$ 下极限的定义.)

• **Fatou's Lemma** 告诉我们，对于任意一列非负可测函数列，其**函数列的下极限的积分**，要小于每个函数积分后得到的**积分数列的下极限**.

证明. Since

$$\liminf_{n\to\infty} f_n = \lim_{n\to\infty} (\inf_{k\geq n} f_k) \tag{3.103}$$

Let $g_n = \inf_{k\geq n} f_k$, then $g_n \nearrow \lim_{n\to\infty} g_n$. By **MCT** (Thm 3.1.2, 单调收敛定理),

$$\int \lim_{n\to\infty} g_n = \lim_{n\to\infty} \int g_n \tag{3.104}$$

i.e.

$$\int \liminf_{n\to\infty} f_k = \lim_{n\to\infty} (\int \inf_{k\geq n} f_k) \tag{3.105}$$

For each n, since $\inf_{k\geq n} f_k \leq f_j$, $\forall j \geq n$, then

$$\int \inf_{k\geq n} f_k \leq \int f_j, \ \forall j \geq n \tag{3.106}$$

Taking the infimum of $\{\int f_j\}_{j=n}^{\infty}$, then

$$\int \inf_{k\geq n} f_k \leq \inf_{j\geq n} \int f_j, \ \forall n \in \mathbb{N} \tag{3.107}$$

For n is arbitrary, let $n \to \infty$, we get

$$\lim_{n\to\infty} (\int \inf_{k\geq n} f_k) \leq \lim_{n\to\infty} (\inf_{k\geq n} \int f_k) = \liminf_{n\to\infty} \int f_n \tag{3.108}$$

Therefore

$$\int \liminf_{n\to\infty} f_k = \lim_{n\to\infty} (\int \inf_{k\geq n} f_k) \leq \liminf_{n\to\infty} \int f_n \tag{3.109}$$

□

3.1.3 *General case*

可积函数 跟 Riemann 积分类似，对于 Lebesgue 积分，我们也有**可积函数**的概念.

下面先让我们回到**非负可测**函数，定义非负可测函数中**可积**的概念.

定义 3.1.5. For $f \in \mathcal{M}^+$, if

$$\int f < \infty \tag{3.110}$$

Then we say f is **Lebesgue integrable** or simply **integrable**.

下面扩展到一般的可测函数，给出其 **Lebesgue 积分**及**可积**的定义.

定义 3.1.6. For any f measurable on $\mathbb{R}^d$

$$f^+(x) := \max\{f(x), 0\}\ ,\ f^-(x) := \max\{-f(x), 0\} \tag{3.111}$$

If at least one of $\int f^+$ and $\int f^-$ is finite, we define the **integral of f**

$$\int f := \int f^+ - \int f^- \tag{3.112}$$

We say that f is **(Lebesgue) integrable** if $|f|$ is integrable.

注.
- 注意到

$$f = f^+ - f^- \tag{3.113}$$
$$|f| = f^+ + f^- \tag{3.114}$$

- 根据定义，对于任意可测函数 f，

$$f \text{ integrable } \Leftrightarrow |f| \text{ integrable } \Leftrightarrow \int |f| = \int f^+ + \int f^- < \infty \tag{3.115}$$
$$\Leftrightarrow f^+ \text{ and } f^- \text{ integrable} \tag{3.116}$$

即 f 可积 $\Leftrightarrow \int f^+$ 和 $\int f^-$ 均有界.

性质 下面我们将说明，定义在任一集合 X 上的**实可积函数**构成的空间 $\mathcal{L}^1$ 为**线性空间**，以及 $f \in \mathcal{L}^1$ 时的一些性质.

在此之前，先给出上述定义的一般的可测函数的积分的**基本性质**.

命题 3.1.4. Suppose $f, g \in \mathcal{L}$, then

1. **Linearity**: $\int (af + bg) = a \int f + b \int g$.

2. **Finite Additivity**:

$$\int_{\bigsqcup_{j=1}^n A_j} f = \sum_{j=1}^n \int_{A_j} f \tag{3.117}$$

   where $\{A_j\}_{j=1}^n$ are disjoint.

3. **Monotonicity**: If $f \le g$, then $\int f \le \int g$.

4. **Triangle inequality**: $\left|\int f\right| \le \int |f|$.

证明.

2. : We shall show that $\int_{\bigsqcup_{j=1}^n A_j} f^+ = \sum_{j=1}^n \int_{A_j} f^+$ and $\int_{\bigsqcup_{j=1}^n A_j} f^- = \sum_{j=1}^n \int_{A_j} f^-$.
   By **Thm 2.2.1**, there exists simple $\varphi_n \nearrow f^+$, then by **MCT (Thm 3.1.2, 单调收敛定理)**,

$$\int_{\bigsqcup_{j=1}^n A_j} f^+ = \lim_{n\to\infty} \int_{\bigsqcup_{j=1}^n A_j} \varphi_n \tag{3.118}$$

   Since φ_n are simple, by the **countable additivity (简单函数的可数可加性)**, we have

$$\int_{\bigsqcup_{j=1}^n A_j} f^+ = \lim_{n\to\infty} \int_{\bigsqcup_{j=1}^n A_j} \varphi_n = \lim_{n\to\infty} \sum_{j=1}^n \int_{A_j} \varphi_n = \sum_{j=1}^n \lim_{n\to\infty} \int_{A_j} \varphi_n \tag{3.119}$$

$$\overset{\text{MCT}}{=} \sum_{j=1}^n \int_{A_j} f^+ \tag{3.120}$$

4. 根据实数域上的三角不等式，we have

$$\left|\int f\right| = \left|\int f^+ - \int f^-\right| \le \left|\int f^+\right| + \left|\int f^-\right| = \int f^+ + \int f^- = \int |f| \tag{3.121}$$

$\square$

现在我们便可以来说明，定义在任一集合 X 上的**实可积函数**构成的空间 $\mathcal{L}^1$ 为**线性空间**.

命题 3.1.5. The set of integrable real-valued functions on X is a real vector space.

证明. $\forall f, g \in \mathcal{L}^1$, if $a \in \mathbb{R}$,

$$\int |f+g| \leq \int (|f|+|g|) = \int |f| + \int |g| < \infty \tag{3.122}$$

$$\int |af| = |a| \int |f| < \infty \tag{3.123}$$

Therefore, $f+g, af \in \mathcal{L}^1$. $\Rightarrow$ $\mathcal{L}^1$ is a real vector space. □

对于可积函数，我们往往是在整个 $\mathbb{R}^d$ 空间上讨论其可积性，类比 **Riemann** 可积函数，合理地猜测其在 $\mathbb{R}^d$ 平面上 **“较远”** 的地方的积分值应当较小. 这就是下面我们要给出的 $\mathcal{L}^1$ 可积函数的性质.

命题 3.1.6. Suppose $f \in \mathcal{L}^1(\mathbb{R}^d)$. Then $\forall \epsilon > 0$

(i) $\exists$ a set of finite measure B such that

$$\int_{B^c} |f| < \epsilon$$

(ii) [**Absolutely Continuity**].

$\exists \delta > 0$ such that

$$\int_E |f| < \epsilon, \ \forall m(E) < \delta$$

注.

- (i) 和 (ii) 共同说明了，若 $f \in \mathcal{L}^1(\mathbb{R}^d)$，则 f 的积分主要集中在一个**有限测度**区域内，且在很小的区域内 f 的积分值趋于零.

- (ii) 本质为测度的**绝对连续性 (正测度关于正测度的绝对连续性)**. 此处令正测度

$$\mu : \mathcal{L} \longrightarrow [0, \infty] \tag{3.124}$$

$$E \longmapsto \mu(E) = \int_E |f| \tag{3.125}$$

则命题 (ii) 可表示为：$\forall \epsilon > 0, \exists \delta > 0$, s. t.

$$\mu(E) < \epsilon, \ \forall m(E) < \delta$$

证明.

(i) : 对定义域做截断.

Suppose $f \geq 0$. Let $B_n = B(0, n), f_n = f\chi_{B_n}$, then $f_n \nearrow f$.

By **MCT (Thm 3.1.2, 单调收敛定理)**,

$$\lim_{n\to\infty} \int f_n = \int f \tag{3.126}$$

Then $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s. t.

$$\left|\int f - \int f_N\right| = \int f - \int f_N = \int f(1 - \chi_{B_N}) = \int f\chi_{B_N^c} = \int_{B_N^c} f < \epsilon \tag{3.127}$$

Therefore, let $B = B_N = B(0, N)$, the desired result follows.

(ii) : 同样是做截断. 不过此处是对 f 的取值做截断.

Let $B_n = \{x \in \mathbb{R}^d \mid f(x) \leq n\}, f_n = f\chi_{B_n}$. Then $f_n \nearrow f, f_n \leq n$.

同 (i), By **MCT (Thm 3.1.2, 单调收敛定理)**,

$$\lim_{n\to\infty} \int f_n = \int f \tag{3.128}$$

$\forall \epsilon > 0, \exists N \in \mathbb{N}$, s. t.

$$\left|\int f - \int f_N\right| = \int (f - f_N) < \frac{\epsilon}{2} \tag{3.129}$$

Pick $\delta > 0$, s. t. $N\delta < \frac{\epsilon}{2}$. Then for all $m(E) < \delta$,

$$\int_E f = \int_E (f - f_N) + \int_E f_N \leq \int (f - f_N) + N \cdot m(E) \tag{3.130}$$

$$< \frac{\epsilon}{2} + N\delta \tag{3.131}$$

$$< \epsilon \tag{3.132}$$

□

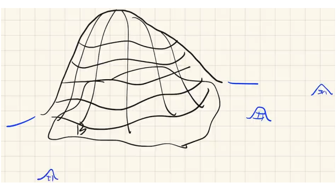

图 3.1: Prop 3.1.6 (i)

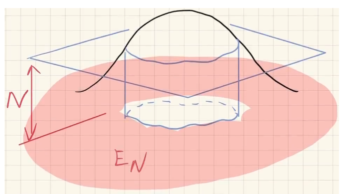

图 3.2: Prop 3.1.6 (ii)

3.1.4 *The Dominated Convergence Theorem*

下面我们来介绍实分析中最最有用的定理——

<u>**控制收敛定理 (The Dominated Convergence Theorem)**</u>.

在 **Riemann 积分**中，对于函数列**交换极限与积分的次序**的条件太过于奇怪与繁琐，而在 **Lebesgue 积分**中，**控制收敛定理**则很完美地解决了这一问题. 它对于**交换极限与积分的次序**的条件十分简洁. 下面便来介绍这一定理.

定理 3.1.7. The Dominated Convergence Theorem (DCT).
Suppose $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+, f_n \to f$ a.e.. If $|f_n| \le g$, where $g \in \mathcal{L}^1(\mathbb{R}^d)$, then

$$\int |f_n - f| \to 0,\ n \to \infty \tag{3.133}$$

and consequently

$$\int f_n \to \int f,\ n \to \infty \tag{3.134}$$

证明. 分别对 $g + f_n$ 和 $g - f_n$ 利用 **Fatou's Lemma (Thm 3.1.6)** 即可得证.

- Since $g + f_n \ge 0$, then by **Fatou's Lemma (Thm 3.1.6)**,

$$\int \liminf_{n\to\infty} (g + f_n) \le \liminf_{n\to\infty} \int (g + f_n) \tag{3.135}$$

  Since $f_n \to f$, we have

$$\int g + \int f \le \int g + \liminf_{n\to\infty} \int f_n \tag{3.136}$$

$$\int f \le \liminf_{n\to\infty} \int f_n \tag{3.137}$$

- Since $g - f_n \ge 0$, then by **Fatou's Lemma (Thm 3.1.6)**,

$$\int \liminf_{n\to\infty} (g - f_n) \le \liminf_{n\to\infty} \int (g - f_n) \tag{3.138}$$

$$\int g - \int f \le \int g + \liminf_{n\to\infty} (-\int f_n) \tag{3.139}$$

$$= \int g - \limsup_{n\to\infty} \int f_n \tag{3.140}$$

  Then

$$\int f \ge \limsup_{n\to\infty} \int f_n \tag{3.141}$$

Therefore

$$\limsup_{n\to\infty} \int f_n \leq \int f \leq \liminf_{n\to\infty} \int f_n \tag{3.142}$$

which means $\lim_{n\to\infty} \int f_n$ exists, and

$$\lim_{n\to\infty} \int f_n = \int f \tag{3.143}$$

□

3.1.5 *Complex – Valued Functions*

下面我们将**实值函数**上的 **Lebesgue 积分**推广至**复值函数**.

先来规定一些记号：

- Let $f : \mathbb{R}^d \to \mathbb{C}$, write $f(x) = u(x) + iv(x)$.

下面给出复值函数**可测**以及**可积**的定义.

定义 3.1.7. Suppose $f : \mathbb{R}^d \to \mathbb{C}, f = u + iv$, then we say

- f is **measurable** if u and v are both measurable.
- f is **Lebesgue integrable** if $|f|$ is Lebesgue integrable.

注. 事实上，根据此处定义，f 可积 $\Leftrightarrow$ u and v 都可积.

证明.

- f is integrable $\Rightarrow \int \sqrt{u^2+v^2} < \infty \Rightarrow \int |u|, \int |v| \le \int \sqrt{u^2+v^2} < \infty \Rightarrow u$ and v 可积.
- u and v 可积 $\Rightarrow \int |u|, \int |v| < \infty \Rightarrow \int \sqrt{u^2+v^2} \le \int |u| + \int |v| < \infty \Rightarrow f$ 可积.

□

下面对**命题 3.1.5** 的结论进行推广，即由**复值可积函数**构成的空间为**线性空间**.

命题 3.1.7. $\mathcal{L}^1(\mathbb{R}^d, \mathbb{C})$ is a vector space.

证明. Trivial. □

3.2 $\mathcal{L}^1$ 空间的完备性

引入 在讲 **Riemann** 积分时，我们称 **Riemann 可积函数**构成的空间是**不完备的 (not complete)**. 在提及**完备**这个概念之前，我们需要先引入衡量“距离”的工具，即**范数**和**度量**.

3.2.1 范数，度量

下面给出**范数**和**度量**的严格定义.

定义 3.2.1. Let X be a vector space over $\mathbb{F}$, a **norm** is a function:

$$X \longrightarrow \mathbb{R}_{\geq 0} \tag{3.144}$$

$$f \longmapsto \|f\| \tag{3.145}$$

satisfying the following properties:

(i) $\|f\| \geq 0, \forall f \in X.$ $\quad (\|f\| = 0 \Leftrightarrow f = 0 \ a.e.)$

(ii) $\|af\| = |a|\,\|f\|, \forall a \in \mathbb{F}, f \in X.$

(iii) $\|f + g| \leq \|f\| + \|g\|, \forall f, g \in X.$

注.

- (i) 中的“$\|f\| = 0 \Leftrightarrow f = 0 \ a.e.$”的“**a.e.**”是对于 X 取函数空间时的条件，在实分析的取等条件中基本为默认叙述，在后续定义中往往省略. 在对 $\mathcal{L}^1$ 空间的定义 (**定义 3.2.4**) 中可以看到其合理性.
- **范数**实际上是对 $\mathbb{R}^n$ 空间中**“与原点之间的距离”**这一概念的推广. 将函数视作向量，则其范数即为到原点的距离，即**模长**.
- 若一个**线性空间** X 上配备了一个**范数**，则称其为赋范向量空间 (赋范线性空间).

将函数视作向量，就有其**到原点的距离为范数**. 但若是想要衡量**任意两个函数之间的距离**，则需要引入下面**度量**的概念.

定义 3.2.2. A **metric** on X is a map

$$d : X \times X \longrightarrow \mathbb{R}_{\geq 0} \tag{3.146}$$

$$(x, y) \longmapsto d(x, y) \tag{3.147}$$

satisfying

(i) $d(x, y) \geq 0, \forall x, y \in X.$ $\qquad (d(x, y) = 0 \iff x = y)$

(ii) $d(x, y) = d(y, x), \forall x, y \in X.$

(iii) $d(x, y) + d(y, z) \geq d(x, z), \forall x, y, z \in X.$

注.

- 若 X 为函数空间，则 (i) 中 “$d(x, y) = 0$” 等价条件默认为 “$x = y$ $a.e.$” .
- **度量**可看作将两个函数 (向量) 的起点均平移至原点后，其两个终点之间的**距离**.

3.2.2 *The Space* $\mathcal{L}^1(\mathbb{R}^d)$

范数 下面先在所有 **Lebesgue 可积函数**构成的空间上定义**范数**.

定义 3.2.3. For any integrable function f on $\mathbb{R}^d$, we define the **norm** of f,

$$\|f\| = \int_{\mathbb{R}^d} |f| \, dx \tag{3.148}$$

注.

- 由**命题 3.1.3**可知，此处 $\|f\| = 0 \iff f = 0$ $a.e.$
- 容易证明，如此定义的**范数**满足范数应当满足的三条公理. (**定义 3.2.1**)

Space $\mathcal{L}^1(\mathbb{R}^d)$ 由于**定义 3.2.3**中“$\|f\| = 0 \Leftrightarrow f = 0\ a.e.$”，而我们对零测集上的函数性质并不关心，因而引出了如下关于 $\mathcal{L}^1$ 空间的定义.

定义 3.2.4. 我们在所有 **Lebesgue 可积函数**构成的空间上定义一个等价关系“$\sim$”：

$$f \sim g \ \Leftrightarrow \ f = g\ a.e.$$

$\mathcal{L}^1(\mathbb{R}^d)$ is **the space of equivalences classes** of integrable functions.

注. 由定义可知，$\mathcal{L}^1(\mathbb{R}^d)$ 空间中的元素实际上为**函数的等价类 (集合)**

$$[f] = \{g\ integrable \mid g \sim f\}$$

而在实际中，我们还是习惯性地当作单独的函数进行运算，这在**几乎处处**的意义下时等价的.

度量 下面我们说明，根据**定义 3.2.3** 中所定义的**范数**可诱导出 $\mathcal{L}^1(\mathbb{R}^d)$ 上的一个**度量**.

命题 3.2.1.

$$d : \mathcal{L}^1(\mathbb{R}^d) \times \mathcal{L}^1(\mathbb{R}^d) \longrightarrow \mathbb{R}_{\geq 0} \tag{3.149}$$

$$(f, g) \longmapsto d(f, g) := \|f - g\| \tag{3.150}$$

defines a **metric** on $\mathcal{L}^1(\mathbb{R}^d)$.

证明. 下面即来逐一验证**定义 3.2.2** 中的三条公理.

- 根据范数的非负性，$\forall f, g \in \mathcal{L}^1(\mathbb{R}^d)$，$d(f, g) = \|f - g\| \geq 0$.

$$d(f, g) = 0 \ \Leftrightarrow \ f - g = 0\ a.e. \ \Leftrightarrow \ f = g\ in\ \mathcal{L}^1(\mathbb{R}^d)$$

- 可交换性. $\forall f, g \in \mathcal{L}^1(\mathbb{R}^d)$,

$$d(f, g) = |f - g\| = \int_{\mathbb{R}^d} |f - g| = \int_{\mathbb{R}^d} |g - f| = \|g - f\| = d(g, f) \tag{3.151}$$

- 根据范数的三角不等式，$\forall f, g, h \in \mathcal{L}^1(\mathbb{R}^d)$,

$$d(f, g) + d(g, h) = \|f - g\| + \|g - h\| \geq \|(f - g) + (g - h)\| = \|f - h\| = d(f, h)$$

$\square$

3.2.3 $\mathcal{L}^1$ 空间的完备性

定义 在得到了**范数**、**度量**的定义后，我们下面给出**完备空间**的定义.

定义 3.2.5. A **metric space** X is **complete** if every Cauchy Sequence $\{x_k\}_{k=1}^{\infty}$ has a limit in X.

注.
- **完备空间**即指空间中的任一柯西列都有**收敛到自身**的极限.
- 下面给出一个**不完备的度量空间**的例子.

  例 3.2.1. 取一维实数域 $\mathbb{R}$ 的子空间 $(0,1) \subset \mathbb{R}$，考虑其上的 Cauchy Sequence $\{\frac{1}{n}\}_{n=2}^{\infty} \subset (0,1)$.

  由于 $\frac{1}{n} \to 0 \notin (0,1)$，因此度量空间 $(0,1)$ 不完备.

$\mathcal{L}^1$ 空间的完备性 下面我们将给出本小节最重要的结论，即 $\mathcal{L}^1$ 空间的**完备性**，这也是其比 **Riemann 可积函数**所构成的空间的优越性之所在.

定理 3.2.1. (Riesz - Fischer).

$$\mathcal{L}^1 \text{ is complete in its metric.}$$

证明. Let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{L}^1(\mathbb{R}^d)$ be a Cauchy Sequence in $\mathcal{L}^1$, then

$$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}, \forall n, m \geq N(\epsilon), \text{ s. t. } \|f_n - f_m\| \leq \epsilon$$

Tacking $\epsilon = 2^{-k}$, then $\exists N(2^{-k}) \geq N^{2^{-(k-1)}}$, s. t. for $n_k = N(2^{-k}), n_{k+1} = N(2^{-(k+1)})$,

$$\|f_{n_k} - f_{n_{k+1}}\| \leq 2^{-k}$$

下面分为三步进行证明.

- 构建 $f(x)$ 并利用 $g(x)$ 证明 $f \in \mathcal{L}^1$，证明子列 $\{f_{n_j}\}_{j=1}^{\infty}$ 收敛到 f.

  Let

$$f = f_{n_1} + \sum_{j=1}^{\infty} (f_{n_{j+1}} - f_{n_j}) \tag{3.152}$$

$$g = |f_{n_1}| + \sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}| \tag{3.153}$$

Then by **MCT (Thm 3.1.2, 控制收敛定理)**

$$\int g = \int |f_{n_1}| + \int \sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}| = \int |f_{n_1}| + \sum_{j=1}^{\infty} \int |f_{n_{j+1}} - f_{n_j}| \tag{3.154}$$

$$= \int |f_{n_1}| + \sum_{j=1}^{\infty} \|f_{n_{j+1}} - f_{n_j}\| \tag{3.155}$$

$$\leq \int |f_{n_1}| + \sum_{j=1}^{\infty} 2^{-j} < \infty \tag{3.156}$$

Therefore g is integrable, $g \in \mathcal{L}^1$. Since $|f| \leq g$, then $\int |f| < \infty$. f is integrable.

Let

$$S_k = f_{n_1} + \sum_{j=1}^{k} (f_{n_{j+1}} - f_{n_j}) = f_{n_{k+1}}, \ \ k = 1, 2, \cdots \tag{3.157}$$

$$f \text{ is integrable } \Rightarrow f < \infty \ a.e. \ \Rightarrow \ S_k \text{ converges a.e. } \Rightarrow \ S_k = f_{n_{k+1}} \to f \ a.e.$$

So we find

$$f_{n_k} \to f \ a.e.$$

- 将逐点收敛性转化为 $\mathcal{L}^1$ 收敛性，即证 $\|f - f_{n_k}\| \to 0$.

  We note that

$$|f - f_{n_k}| = \left| \left(f_{n_1} + \sum_{j=1}^{\infty} (f_{n_{j+1}} - f_{n_j}) \right) - \left(f_{n_1} + \sum_{j=1}^{k-1} (f_{n_{j+1}} - f_{n_j}) \right) \right| \tag{3.158}$$

$$= \left| \sum_{j=k}^{\infty} (f_{n_{j+1}} - f_{n_j}) \right| \leq g \tag{3.159}$$

By **DCT (Thm 3.1.7, 控制收敛定理)**, since $|f - f_{n_k}| \to 0 \ a.e.$, $|f - f_{n_k}| \leq g$, g integrable,

$$\lim_{k\to\infty} \|f - f_{n_k}\| = \lim_{k\to\infty} \int |f - f_{n_k}| \overset{\textbf{DCT}}{=} \int \lim_{k\to\infty} |f - f_{n_k}| = 0 \tag{3.160}$$

Therefore, $\|f - f_{n_k}\| \to 0$. 即 f_{n_k} 依 $\mathcal{L}^1$ 范数收敛到 f.

- 利用子列 $\{f_{n_k}\}_{k=1}^{\infty}$ 作为“桥梁”，证明 f_n 依 $\mathcal{L}^1$ 范数收敛到 f，即 $\|f_n - f\| \to 0$.
  $\forall \epsilon > 0$，由于 $\{f_n\}_{n=1}^{\infty}$ 为 $\mathcal{L}^1$ 中 Cauchy Sequence, 因此 $\exists N \in \mathbb{N}$, s. t.
  $$\|f_n - f_m\| < \frac{\epsilon}{2}, \ \forall n, m > N$$
  Since $\|f_{n_k} - f\| \to 0$, then for $\epsilon > 0$, pick $n_k > N$ which s. t.
  $$\|f_{n_k} - f\| < \frac{\epsilon}{2}$$
  Then
  $$\|f_n - f\| \leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\| < \epsilon, \ \forall n > n_k > N \tag{3.161}$$
  Therefore $\|f_n \to f\| \to 0$ with $f \in \mathcal{L}^1$. $\mathcal{L}^1$ is complete in its metric.

□

根据上述定理的证明过程，可以得到下面的推论.

推论 3.2.2. If $\{f_n\}_{n=1}^{\infty}$ converges to f in $\mathcal{L}^1$, then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that

$$f_{n_k}(x) \to f(x) \ a.e.$$

注. 即在**依 $\mathcal{L}^1$ 范数收敛**的函数序列中，总存在**“几乎处处收敛”**意义的子列.

3.2.4 $\mathcal{L}^1$ 的稠密子空间

下面说明 $\mathcal{L}^1$ 空间中以下的函数集合是**稠密的**.

定理 3.2.3. The following families of functions are dense in $\mathcal{L}^1(\mathbb{R}^d)$:

(i) The simple functions.

(ii) The step functions.

(iii) The continuous functions of compact support.

证明. 详情可见视频Urysohn 引理与 $\mathcal{L}^1$ 的稠密子空间. □

3.3 *Lebesgue* 积分的平移不变性

首先给出**平移算符**及**函数平移**的符号表达.

定义 3.3.1. The **translation** by a vector h on $\mathbb{R}^d$ is denoted by the map $\tau_h : x \mapsto x - h$. If f is a function defined on $\mathbb{R}^d$, the **translation** of f by $h \in \mathbb{R}^d$ is the function f_h, defined by

$$f_h(x) = (f \circ \tau_h)(x) = f(x - h)$$

下面给出 **Lebesgue 积分的平移不变性**.

定理 3.3.1. If $f \in \mathcal{L}^1(\mathbb{R}^d)$, then $\forall h \in \mathbb{R}^d, f_h \in \mathcal{L}^1(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} f(x-h)dx = \int_{\mathbb{R}^d} f(x)dx \tag{3.162}$$

证明. 下面按 *Lebesgue* 积分的构造过程来证明，即**特征函数** ⇒ **简单函数** ⇒ **非负可测**.

- **Characteristic Function.**

  Suppose $f = \chi_E$, where $E \subset \mathbb{R}^d$ is measurable. Then

$$f_h(x) = f(x-h) = \chi_E(x-h) = \begin{cases} 1, \mathit{if}\ \ x-h \in E \\ 0, \mathit{if}\ \ x-h \notin E \end{cases} = \begin{cases} 1, \mathit{if}\ \ x \in E+h = E_h \\ 0, \mathit{if}\ \ x \in (E+h)^c = E_h^c \end{cases} \tag{3.163}$$

  根据 *Lebesgue* 测度的**平移不变性**，

$$\int_{\mathbb{R}^d} f_h = m(E_h) = m(E) = \int_{\mathbb{R}^d} f \tag{3.164}$$

- **Simple Function.**

  $\forall \varphi = \sum\limits_{k=1}^{n} a_k \chi_{E_k}$ simple, by the **linearity of integration**,

$$\int_{\mathbb{R}^d} \varphi_h = \sum_{k=1}^{n} \int_{\mathbb{R}^d} \chi_{(E_k)_h} = \sum_{k=1}^{n} \int_{\mathbb{R}^d} \chi_{E_k} = \int_{\mathbb{R}^d} \varphi \tag{3.165}$$

- **Non-negative Function**.

  $\forall f$ non-negative, $\exists\{\varphi_n\}_{n=1}^{\infty}$ simple, s. t. $\varphi \nearrow f$ and $\varphi \geq 0$. Then by **MCT (Thm 3.1.2)**,

$$\int_{\mathbb{R}^d} \varphi_n \to \int_{\mathbb{R}^d} f \ as \ n \to \infty \tag{3.166}$$

  Since $(\varphi_n)_h \nearrow f_h$ and $\int \varphi_n = \int (\varphi_n)_h$, then by **MCT (Thm 3.1.2)**,

$$\int_{\mathbb{R}^d} \varphi_n = \int_{\mathbb{R}^d} (\varphi_n)_h \to \int_{\mathbb{R}^d} f_h \ as \ n \to \infty \tag{3.167}$$

  Therefore

$$\int_{\mathbb{R}^d} f_h = \int_{\mathbb{R}^d} f \tag{3.168}$$

- **General Case**.

  $\forall f \in \mathcal{L}^1(\mathbb{R}^d), f = f^+ - f^-$, where f^+ and f^- are non-negative.

  Then by the **linearity of integration**,

$$\int_{\mathbb{R}^d} f_h = \int_{\mathbb{R}^d} f_h^+ - \int_{\mathbb{R}^d} f_h^- = \int_{\mathbb{R}^d} f^+ - \int_{\mathbb{R}^d} f^- = \int_{\mathbb{R}_d} f \tag{3.169}$$

□

3.4 *Lebesgue* 可积函数的 $\mathcal{L}^1$ 连续性

引入 Recall 数学分析中**连续的等价定义**:

$$f \text{ is continous at } x \Leftrightarrow f(x) - f(x-h) \to 0 \text{ as } h \to 0 \tag{3.170}$$

$$\Leftrightarrow |f_h(x) - f(x)| \to 0 \text{ as } h \to 0 \tag{3.171}$$

即可大致视作 **Riemann 可积函数关于 2-范数的连续性**.

Lebesgue 可积函数的 $\mathcal{L}^1$ 连续性 在 $\mathcal{L}^1$ 空间中，**Lebesgue 可积函数**也有类似的**关于 $\mathcal{L}^1$ 范数的连续性**. 这就是下面的定理.

定理 3.4.1. Suppose $f \in \mathcal{L}^1(\mathbb{R}^d)$, then

$$\|f_h - f\|_{\mathcal{L}^1} \to 0 \text{ as } h \to 0 \tag{3.172}$$

证明. 详见视频积分的平移不变性与可积函数的 $\mathcal{L}^1$ 连续性.
其中需要用到如下的引理.

引理 3.4.2. [2] If $f \in C_c(\mathbb{R}^d)$, then f is uniformly continuous.

注. $f \in C_c(\mathbb{R}^d)$ 表示 f 为具有紧支集的连续函数.

□

[2]此为书:《*Real Analysis – –Modern Techniques and Their Applications*》— *Gerald B. Folland* **P238 Lemma 8.4**

3.5 *Fubini* 定理

为了讨论的方便，下面先给出函数及集合的**切片**的定义.

定义 3.5.1. If f is a function on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, the **slice** of f w.r.t. $y \in \mathbb{R}^{d_2}$ is the function

$$\begin{aligned} f^y : \mathbb{R}^{d_1} &\longrightarrow \overline{\mathbb{R}} \qquad (3.173) \\ x &\longmapsto f(x, y) \qquad (3.174) \end{aligned}$$

Similarly, the **slice** of f for a fixed $x \in \mathbb{R}^{d_1}$ is

$$\begin{aligned} f_x : \mathbb{R}^{d_2} &\longrightarrow \overline{\mathbb{R}} \qquad (3.175) \\ y &\longmapsto f(x, y) \qquad (3.176) \end{aligned}$$

Let $E \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, we define its **slices** by

$$E^y := \{x \in \mathbb{R}^{d_1} \mid (x, y) \in E\}, \ E_x := \{y \in \mathbb{R}^{d_2} \mid (x, y) \in E\} \qquad (3.177)$$

下面给出 **Fubini 定理**.

定理 3.5.1. Fubini.

Suppose $f(x, y)$ is **integrable** on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for a.e. $y \in \mathbb{R}^{d_2}$:

(i) The slice f^y is integrable on $\mathbb{R}^{d_1}$.

(ii) The function $y \mapsto \int_{\mathbb{R}^{d_1}} f^y(x)dx$ is integrable on $\mathbb{R}^{d_2}$.

(iii) Moreover,

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f^y(x)dx \right) dy = \int_{\mathbb{R}^d} f \qquad (3.178)$$

3.5.1 *Fubini* 定理的证明

证明. Let $\mathcal{F} = \{f \in \mathcal{L}^1(\mathbb{R}^d) \mid f \ \textit{satisfies}\ (i) \sim (iii)\}$. It suffices to show that $\mathcal{L}^1(\mathbb{R}^d) \subset \mathcal{F}$.
下面仍按照构造 *Lebesgue* 积分的顺序思路进行证明，即**特征函数** $\Rightarrow$ **简单函数** $\Rightarrow$ **非负可测**.

(其中**特征函数**部分 (**Step 3 ~ 5**) 最为复杂繁琐，后续的证明则是水到渠成)

在此之前，还要先证明 $\mathcal{F}$ 对**函数的线性组合**及**单调函数列的极限**封闭.

- **Step 1: Any finite linear combination of functions in $\mathcal{F}$ also belongs to $\mathcal{F}$.**
  Suppose $\{f_k\}_{k=1}^N \subset \mathcal{F}$. By the condition, $\forall k, \exists A_k \subset \mathbb{R}^{d_2}, m(A_k) = 0$, s. t.

$$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in A_k^c.$$

  Let $A = \bigcup\limits_{k=1}^N A_k$, then $m(A) = 0$ and

$$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in A^c, \forall k.$$

  下面对定理结论逐条验证. By the linearity of integration, $\forall a_k \in \mathbb{R}$,

$$(\sum_{k=1}^N a_k f_k)^y = \sum_{k=1}^N a_k f_k^y \textit{ is integrable on } \mathbb{R}^{d_1} \tag{3.179}$$

$$\int_{\mathbb{R}^{d_1}} \sum_{k=1}^N (a_k f_k)^y(x)dx = \sum_{k=1}^N a_k \int_{\mathbb{R}^{d_1}} f_k^y(x)dx \textit{ is integrable on } \mathbb{R}^{d_2} \tag{3.180}$$

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \sum_{k=1}^N (a_k f_k)^y(x)dx \right) dy = \sum_{k=1}^N a_k \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f_k^y(x)dx \right) dy \tag{3.181}$$

$$= \sum_{k=1}^N a_k \int_{\mathbb{R}^d} f_k \tag{3.182}$$

$$= \int_{\mathbb{R}^d} \sum_{k=1}^N a_k f_k \tag{3.183}$$

$$\tag{3.184}$$

  Therefore, $\sum\limits_{k=1}^N a_k f_k \in \mathcal{F}, \forall a_k \in \mathbb{R}$.

- **Step 2:** $\mathcal{F}$ 对单调函数列的极限封闭，即 $\forall \{f_k\}_{k=1}^{\infty}$，$f_k \nearrow f$, f **integrable** $\Rightarrow$ $f \in \mathcal{F}$.

  Suppose $f_k \geq 0$. By the condition, $\forall k$, $\exists A_k \subset \mathbb{R}^d$, $m(A_k) = 0$, s. t.

  $$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in A_k^c.$$

  Let $A = \bigcup\limits_{k=1}^{\infty} A_k$, then $m(A) = 0$ and

  $$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in A^c, \forall k.$$

  Since $f_k^y(x) \nearrow f^y(x)$, by **MCT (Thm 3.1.2)**

  $$\int_{\mathbb{R}^{d_1}} f_k^y(x)dx \to \int_{\mathbb{R}^{d_1}} f^y(x)dx \text{ as } k \to \infty \tag{3.185}$$

  Let

  $$g_k(y) = \int_{\mathbb{R}^{d_1}} f_k^y(x)dx,\ g(y) = \int_{\mathbb{R}^{d_1}} f^y(x)dx \tag{3.186}$$

  Then we have $g_k(y) \nearrow g(y)$ and $g_k \geq 0$. By **MCT (Thm 3.1.2)**

  $$\int_{\mathbb{R}^{d_2}} g_k(y)dy \to \int_{\mathbb{R}^{d_2}} g(y)dy \text{ as } k \to \infty \tag{3.187}$$

  i.e.

  $$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f_k^y(x)dx\right) dy \to \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f^y(x)dx\right) dy \tag{3.188}$$

  By the condition (iii), we have

  $$\int_{\mathbb{R}^d} f_k \to \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f^y(x)dx\right) dy \tag{3.189}$$

  Since $f_k \nearrow f, f_k \geq 0$, by **MCT (Thm 3.1.2)**

  $$\int_{\mathbb{R}^d} f_k \to \int_{\mathbb{R}^d} f \tag{3.190}$$

  Therefore

  $$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f^y(x)dx\right) dy = \int_{\mathbb{R}^{d_2}} g(y)dy \tag{3.191}$$

  Since f is integrable, $g(y) = \int f^y(x)dx$ is integrable on $\mathbb{R}^{d_2}$ $\Rightarrow$ $\int g < \infty$.

  Then we have

  $$g(y) = \int_{\mathbb{R}^{d_1}} f^y(x)dx < \infty \text{ for a.e. } y \in \mathbb{R}^{d_2} \tag{3.192}$$

  Therefore $f^y(x)$ is integrable for a.e. $y \in \mathbb{R}^{d_2}$.

  Then $f \in \mathcal{F}$.

- **Step 3: Any characteristic function of a set E of type G_δ with finite measure belongs to $\mathcal{F}$.**

下面对 E 进行讨论，分 $a \sim e$ 五种情况来证明：

(a) $E \subset \mathbb{R}^d$ **is a bounded open cube**.

Suppose $E = Q_1 \times Q_2$, where $Q_1 \subset \mathbb{R}^{d_1}$ and $Q_2 \subset \mathbb{R}^{d_2}$ are open cubes.

$\forall y \in \mathbb{R}^{d_2}$, $\chi_E(x, y)$ is measurable in x, and integrable with

$$g(y) = \int_{\mathbb{R}^{d_1}} \chi_E(x, y)dx = \begin{cases} |Q_1|, & if\ y \in Q_2 \\ 0, & if\ y \notin Q_2 \end{cases} = |Q_1|\, \chi_{Q_2}(y) \tag{3.193}$$

Since $g(y) = |Q_1|\, \chi_{Q_2}(y)$ is measurable and integrable with

$$\int_{\mathbb{R}^{d_2}} g(y)dy = |Q_1|\,|Q_2| = |E| = \int_{\mathbb{R}^d} \chi_E \tag{3.194}$$

i.e.

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \chi_E(x, y)dx \right) dy = \int_{\mathbb{R}^d} \chi_E \tag{3.195}$$

Therefore, $\chi_E \in \mathcal{F}$.

(b) $E \subset \mathbb{R}^d$ **is a subset of the boundary of some closed cube**.

Since $m(E) = 0$, we have

$$\int_{\mathbb{R}^d} \chi_E = m(E) = 0 \tag{3.196}$$

After an investigation of various possibilities, we note that (此处细节证明暂且留疑)

$$\forall \text{ a.e. } y \in \mathbb{R}^{d_2},\ E^y = \{x \in \mathbb{R}^{d_1} \mid (x, y) \in E\} \text{ has measure 0 in } \mathbb{R}^{d_1}.$$

Then $\forall$ a.e. $y \in \mathbb{R}^{d_2}$, $\chi_E^y(x)$ is integrable on $\mathbb{R}^{d_1}$ with

$$g(y) = \int_{\mathbb{R}^{d_1}} \chi_E^y(x)dx = \int_{\mathbb{R}^{d_1}} \chi_E(x, y)dx = 0,\ \forall\ a.e.\ y \in \mathbb{R}^{d_2} \tag{3.197}$$

So $g(y)$ is integrable on $\mathbb{R}^{d_2}$ with

$$\int_{\mathbb{R}^{d_2}} g(y)dy = 0 = \int_{\mathbb{R}^d} \chi_E \tag{3.198}$$

i.e.

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \chi_E(x, y)dx \right) dy = \int_{\mathbb{R}^d} \chi_E \tag{3.199}$$

Therefore, $\chi_E \in \mathcal{F}$.

(c) $E \subset \mathbb{R}^d$ **is a finite union of almost disjoint closed cubes**.

Suppose $E = \bigcup\limits_{k=1}^{N} Q_k$, where $\{Q_k^\circ\}_{k=1}^{N}$ are disjoint.

Let $A_k = Q_k - Q_k^\circ$ be the boundary of closed cube Q_k. Then $\chi_{A_k} \in \mathcal{F}$. (by **Step 3 (b)**)

χ_E is a linear combination of χ_{Q_k} and χ_{A_k}, $k = 1 \sim N$.

Since $\chi_{Q_k}, \chi_{A_k} \in \mathcal{F}$, $k = 1 \sim N$, then by **Step 1**, $\chi_E \in \mathcal{F}$.

(d) $E \subset \mathbb{R}^d$ **is open and of finite measure**.

Since $E \subset \mathbb{R}^d$ is open, by **Thm 1.1.4**, $\exists$ alomst disjoint closed cubes $\{Q_k\}_{k=1}^{\infty}$, s. t.

$$E = \bigcup_{k=1}^{\infty} Q_k, \ \ where\ \{Q_k^\circ\}_{k=1}^{\infty}\ are\ disjoint \tag{3.200}$$

Let

$$f_k = \chi_{\bigcup_{j=1}^{k} Q_j} \tag{3.201}$$

Then by **Step 3 (c)**, $f_k \in \mathcal{F}$, and $f_k \nearrow f = \chi_E, f_k \geq 0$. By **Step 2**, we have $f = \chi_E \in \mathcal{F}$.

(e) $E \subset \mathbb{R}^d$ **is a** G_δ **of finite measure**.

By the **definition of** G_δ **(Def 1.4.5)**,

$$E = \bigcap_{k=1}^{\infty} \widetilde{Q_k}, \ \ where\ \widetilde{Q_k} \underset{open}{\subset} \mathbb{R}^d \tag{3.202}$$

Since E has finite measure, $\exists \widetilde{O_0} \subset \mathbb{R}^d$ open, s. t. $E \subset \widetilde{O_0}$.

Let

$$O_k = O_0 \cap \bigcap_{j=1}^{k} \widetilde{O_j} \tag{3.203}$$

Then $O_1 \supset O_2 \supset \cdots$ and $E = \bigcap\limits_{k=1}^{\infty} O_k$. Let $f_k = \chi_{O_k}$, then $f_k \in \mathcal{F}$. (By **Step 3 (d)**)

Since $f_k \searrow f = \chi_E, f_k \in \mathcal{F}$, then by **Step 2**, $f = \chi_E \in \mathcal{F}$.

- **Step 4: If $E \subset \mathbb{R}^d$ has measure 0, then $\chi_E \in \mathcal{F}$.**

  By **Thm 1.4.1**, $\exists$ a set $G \subset \mathbb{R}^d$ of type G_δ with $E = G \backslash N$, where $m(N) = 0$. Then

$$E \subset G,\ m(G) = m(E) + m(G \backslash E) = 0.$$

  By **Step 3**, $\chi_G \in \mathcal{F}$, then

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \chi_G(x, y) dx \right) dy = \int_{\mathbb{R}^d} \chi_G = 0 \tag{3.204}$$

  Then

$$\int_{\mathbb{R}^{d_1}} \chi_G(x, y) dx = 0 \ for \ a.e. \ y \in \mathbb{R}^{d_2} \tag{3.205}$$

  Since

$$\int_{\mathbb{R}^{d_1}} \chi_G(x, y) dx = \int_{\mathbb{R}^{d_1}} \chi_G^y(x) dx = \int_{\mathbb{R}^{d_1}} \chi_{G^y}(x) dx = m(G^y) \tag{3.206}$$

  Therefore

$$G^y = \{x \in \mathbb{R}^{d_1} \mid (x, y) \in G\} \ has \ measure \ 0 \ for \ a.e. \ y \in \mathbb{R}^{d_2} \tag{3.207}$$

  Since $E^y \subset G^y$, then E^y has measure 0 for a.e. $y \in \mathbb{R}^{d_2}$.

$$\Rightarrow \int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx = m(E^y) = 0 \ for \ a.e. \ y \in \mathbb{R}^{d_2} \tag{3.208}$$

$$\Rightarrow \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx \right) dy = 0 = \int_{\mathbb{R}^d} \chi_E \tag{3.209}$$

$$\Rightarrow \chi_E \in \mathcal{F} \tag{3.210}$$

- **Step 5: If E is any measurable subset of $\mathbb{R}^d$ with finite measure, then χ_E belongs to $\mathcal{F}$.**

  By **Thm 1.4.1**, $\exists$ a finite measure G of type G_δ with $E \subset G$ and $m(G - E) = 0$.

  Since $\chi_E = \chi_G - \chi_{G-E}$, by **Step 4**, $\chi_G, \chi_{G-E} \in \mathcal{F}$, then by **Step 1**, $\chi_E \in \mathcal{F}$.

- **Step 6: If f is integrable, then $f \in \mathcal{F}$.**

  不妨 Suppose f non-negative. By **Step 1 and Step 5**, $\forall \varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ simple, $\varphi \in \mathcal{F}$.
  By **Thm 2.2.1**, $\exists \{\varphi_k\}_{k=1}^{\infty}$ simple, $\varphi_k \nearrow f$, $\varphi_k \geq 0$. Then by **Step 2**, $f \in \mathcal{F}$.

Therefore,

$$\mathcal{L}^1(\mathbb{R}^d) \subset \mathcal{F}$$

$\square$

3.5.2 *Fubini* 定理的应用

Tonelli 定理 下面给出一个 **Fubini 定理**的延伸形式，就是 **Tonelli 定理**，常与 **Fubini 定理**一起使用，用于判断函数的可积性.

定理 3.5.2. Tonelli.

Suppose $f(x,y)$ is a **non-negative measurable** function on $\mathbb{R}^{d_1}\times\mathbb{R}^{d_2}$. Then for a.e. $y\in\mathbb{R}^{d_2}$:

(i) The slice f^y is **measurable** on $\mathbb{R}^{d_1}$.

(ii) The function defined by $y\mapsto\int_{\mathbb{R}^{d_1}}f^y(x)dx$ is **measurable** on $\mathbb{R}^{d_2}$.

(iii) Moreover,

$$\int_{\mathbb{R}^{d_2}}\left(\int_{\mathbb{R}^{d_1}}f(x,y)dx\right)dy=\int_{\mathbb{R}^d}f\in[0,\infty] \tag{3.211}$$

注. 在尚未知晓 f 的可积性时，可先用 **Tonelli 定理**计算 $|f|$ 的可积性，从而得到 f 的可积性，再去考虑使用 **Fubini 定理**.

证明. Consider the **truncations**

$$f_k(x,y)=\begin{cases}f(x,y),\ \ if\ \ |(x,y)|<k\ and\ f(x,y)<k\\ 0,\ \ otherwise\end{cases} \tag{3.212}$$

Since

$$\int_{\mathbb{R}^d}f_k\leq k^{d+1}<\infty \tag{3.213}$$

f_k is integrable for all k. Then by **Fubini (Thm 3.5.1)**, $\exists E_k\subset\mathbb{R}^{d_2}$, $m(E_k)=0$, s. t.

$$f_k^y(x)\text{ is integrable on }\mathbb{R}^{d_1},\ \forall y\in E_k^c.$$

$$g_k(y)=\int_{\mathbb{R}^{d_1}}f_k^y(x)dx\ is\ integrable\ on\ \mathbb{R}^{d_2}\ a.e. \tag{3.214}$$

$$\int_{\mathbb{R}^{d_2}}\left(\int_{\mathbb{R}^{d_1}}f_k(x,y)dx\right)dy=\int_{\mathbb{R}^d}f_k \tag{3.215}$$

下面开始验证 f 满足定理中的各条结论.

- Let $E=\bigcup\limits_{k=1}^{\infty}E_k$, then $m(E)=0$ and

  $$f_k^y(x)\text{ is integrable on }\mathbb{R}^{d_1},\ \forall y\in E^c,\forall k.$$

  $\forall y\in E^c$, since $f_k^y(x)\nearrow f^y(x)$, $f_k^y(x)$ integrable on $\mathbb{R}^{d_1}$, specifically measurable

  Then $\forall y\in E^c$ $f^y(x)$ is measurable. i.e. f^y measurable for a.e. $y\in\mathbb{R}^{d_2}$.

- Since $f_k^y(x) \nearrow f^k(x)$, $\forall y \in E^c$, then by **MCT (Thm 3.1.2)**

$$g_k(y) = \int_{\mathbb{R}^{d_1}} f_k^y(x)dx \nearrow \int_{\mathbb{R}^{d_1}} f^y(x)dx = g(y) \; for \; a.e. \; y \in \mathbb{R}^{d_2} \tag{3.216}$$

  Since $g_k(y)$ is integrable on $\mathbb{R}^{d_2}$ a.e., specifically measurable,

  Then $g(y)$ is measurable on $\mathbb{R}^{d_2}$ a.e.

- Since $g_k(y) \nearrow g(y)$, $\forall$ a.e. $y \in \mathbb{R}^{d_2}$, then by **MCT (Thm 3.1.2)**

$$\int_{\mathbb{R}^{d_2}} g_k(y)dy \to \int_{\mathbb{R}^{d_2}} g(y)dy \tag{3.217}$$

  i.e.

$$\int_{\mathbb{R}^d} f_k = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f_k(x,y)dx \right) dy \to \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)dx \right) dy \tag{3.218}$$

  Since $f_k \nearrow f$, by **MCT (Thm 3.1.2)**

$$\int_{\mathbb{R}^d} f_k \to \int_{\mathbb{R}^d} f \tag{3.219}$$

  Therefore

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)dx \right) dy = \int_{\mathbb{R}^d} f \in [0,\infty] \tag{3.220}$$

□

乘积测度　下面给出**乘积测度**在 **Lebesgue** 测度下的一些表现性质. 具体证明可见书[3]P82~ 85，基本都是 Trivial 的.

推论 3.5.3. If E is a measurable set in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, then for a.e. $y \in \mathbb{R}^{d_2}$, the slice

$$E^y = \{x \in \mathbb{R}^{d_1} \mid (x, y) \in E\} \text{ is a measurable subset of } \mathbb{R}^{d_1}.$$

Moreover, $m(E^y)$ is a measurable function of y and

$$m(E) = \int_{\mathbb{R}^{d_2}} m(E^y)dy \tag{3.221}$$

注.
- 该命题为 **Tonelli 定理 (Thm 3.5.2)** 的推论，考虑 $f = \chi_E$ 即可轻松得证.
- 该推论说明了对于任一可测集 E，**其切片** E^y **都是几乎处处可测的**.

有了**推论 3.5.3**，我们自然会去思考一般情况下其逆命题是否成立，即

$$E_y \text{ measurable for a.e. } y \in \mathbb{R}^{d_2} \;\; \Rightarrow \;\; E \subset \mathbb{R}^d \text{ measurable ?}$$

然而答案是**否定的**. 下面给出一个反例.

例 3.5.1. Let $\mathcal{N}$ denote a non-measurable subset $\mathbb{R}$ (**正测度集必有不可测子集, Prop 1.5.1**). Then define

$$E = [0, 1] \cap \mathcal{N} \subset \mathbb{R} \times \mathbb{R}$$

We see that

$$E^y = \begin{cases} [0, 1], & \text{if } y \in \mathcal{N} \\ 0, & \text{if } y \notin \mathcal{N} \end{cases}$$

Thus E^y is measurable for every $y \in \mathbb{R}$. However, if E is measurable, then by **Cor 3.5.3**,

$$E_x = \{y \in \mathbb{R} \mid (x, y) \in E\} = \begin{cases} \mathcal{N}, & \text{if } x \in [0, 1] \\ 0, & \text{if } x \notin [0, 1] \end{cases}$$

which is a contradiction for $\mathcal{N}$ is non-measurable.

---

[3] 《*Real Analysis – –Measure Theroy, Integration, & Hilbert Spaces*》 — *Elias M. Stein*

下面对**推论 3.5.3**进行一定程度的推广，得到如下命题.

命题 3.5.1. If $E = E_1 \times E_2$ is a measurable subset of $\mathbb{R}^d$, and $m_*(E_2) > 0$, then E_1 is measurable.

而我们接下来将说明，若两个集合均可测，则他们的 **Descartes 积也是可测集**. 而这事实上就是抽象测度中**乘积测度**的定义. 在此之前，先来说明一个证明时需要用到的引理.

引理 3.5.4. If $E_1 \subset \mathbb{R}^{d_1}$ and $E_2 \subset \mathbb{R}^{d_2}$, then

$$m_*(E_1 \times E_2) \leq m_*(E_1)m_*(E_2)$$

下面便给出**乘积测度**在 **Lebesgue 测度**下的定义.

命题 3.5.2. Suppose E_1 and E_2 are measurable subsets of $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively. Then $E = E_1 \times E_2$ is a measurable subset of $\mathbb{R}^d$. Moreover,

$$m(E) = m(E_1)m(E_2)$$

几何联系　在 **Riemann 积分**中我们都熟知积分 $\int f$ 即代表 f 下方所围成区域的 **"体积"**. 而下面我们将说明，在 **Lebesgue 积分**中，积分与**几何直观**之间的联系. (Stein P85~ 86)

在此之前先给出一个命题，此为**命题 3.5.2**的推论.

推论 3.5.5. Suppose f is a measurable function on $\mathbb{R}^{d_1}$. Then the function $\widetilde{f}$ defined by

$$\widetilde{f}(x, y) = f(x)$$

is measurable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.

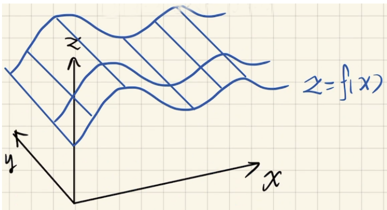

图 3.3: Prop 3.5.5

下面给出 **Lebesgue 积分**与**几何直观**之间的联系.

推论 3.5.6. Suppose $f(x)$ is a non-negative function on $\mathbb{R}^d$, and let

$$\mathcal{A} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} \mid 0 \le y \le f(x)\}$$

Then:

(i) f is measurable on $\mathbb{R}^d$ iff $\mathcal{A}$ is measurable on $\mathbb{R}^{d+1}$.

1. If the conditions in (i) hold, then

$$\int_{\mathbb{R}^d} f = m(\mathcal{A}) \tag{3.222}$$

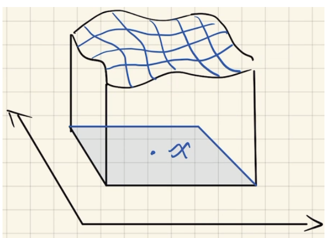

图 3.4: Prop 3.5.6

3.6 *Lebesgue* 积分与 *Riemann* 积分的联系

下面我们将说明，**Lebesgue 积分**可视作 **Riemann 积分**的延拓，它很好地囊括了 **Riemann 积分**的定义.

在此之前，先来给出 **MCT (Thm 3.1.2, 单调收敛定理)** 在**单调递减函数列**上的表述.

定理 3.6.1. Monotone Convergence Theorem (decreasing).
Let $\{f_n\}_{n=1}^{\infty}$ be non-negative, $f_n \searrow f$, $\int f_1 < \infty$, then

$$\lim_{n\to\infty} \int f_n = \int f \tag{3.223}$$

证明. Let $g_n = f_1 - f_n$, $n \in \mathbb{N}$. Then $g_n \geq 0$ and $g_n \nearrow g = f_1 - f$. By **MCT (Thm 3.1.2)**

$$\int g_n \to \int g \ as \ n \to \infty \tag{3.224}$$

i.e.

$$\int (f_1 - f_n) \to \int (f_1 - f) \ as \ n \to \infty \tag{3.225}$$

Therefore

$$\int f_n \to \int f \tag{3.226}$$

□

下面说明 **Riemann 可积函数的积分**即为其 **Lebesgue 积分**.

定理 3.6.2. Suppose f is Riemann integrable, then

$$\int_{[a,b]}^{\mathcal{R}} f(x)dx = \int_{[a,b]}^{\mathcal{L}} f(x)dx \tag{3.227}$$

证明. 详细证明可见书[4]§4.4 或视频 Lebesgue 积分与 Riemann 积分的联系. □

[4]《实变函数论 (第三版)》——周民强

3.7 *Lebesgue* 积分的伸缩变换

下面我们给出 **Lebesgue 积分**的伸缩变换公式. 这实质上为一般的**抽象测度**的**变量替换公式**在 **Lebesgue 测度**下的特例，而此处我们的证明方法为 **Lebesgue 测度**下的方法，依赖于 $\mathbb{R}^d$ 中的几何直观，较为枯燥繁琐，不具有一般性. 在后续学习**抽象测度**时会给出一般性的方法论.

命题 3.7.1. Lebesgue 积分的伸缩变换公式.

- $m(\delta E) = |\delta|\, m(E)$, $\delta \in \mathbb{R}$, $E \subset \mathbb{R}$.
- $\int f(x)dx = |\delta| \int f(\delta x)dx$, $\delta \in \mathbb{R}, f \in \mathcal{L}^1(\mathbb{R})$.
- $\int f(x)dx = \delta_1 \cdots \delta_d \int f(\delta x)dx$, $\delta \in \mathbb{R}^d$, $\delta_j > 0, f \in \mathcal{L}^1(\mathbb{R}^d)$.
- $m(\delta E) = \delta_1 \cdots \delta_d m(E)$, $\delta_j > 0$, $E \subset \mathbb{R}^d$.

证明. 可见视频 积分的伸缩变换 或参考书[5]P73~ 74. □

[5]《*Real Analysis – –Measure Theroy, Integration, & Hilbert Spaces*》— *Elias M. Stein*

3.8 *Littlewood* 三原则

Motivation 尽管我们建立起了围绕 **Lebesgus** 测度为中心的新的理论体系，但我们仍应当重视其与数学分析中概念的联系. 而 **Littlewood** 便总结归纳出了这样三条 principles:

(i) Every (measurable) set is **nearly** a finite union of intervals.

(ii) Every function (of class $\mathcal{L}^{\hat{n}}$) is **nearly** continuous.

(iii) Every convergent sequence is **nearly** uniformly convergent.

不难发现其叙述显得并不太严谨，其中的 **nearly** 一词需要我们给予严格的数学定义.

Littlewood 三原则告诉了我们**可测函数与连续函数之间的联系**，包括**收敛函数列与一致收敛的关系**. 其中第一条原则即为**定理 1.3.4 (iv)**.

下面我们从后往前依次给出第三、二条原则，即 **Egorov 定理**与 **Lusin 定理**. 这在**抽象测度**中仍然起着重要作用.

3.8.1 *Egorov* 定理

关于 **Littlewood 三原则**中的 (iii)，实际上在数学分析中已不陌生. 下面给出一个经典例子.

例 3.8.1. Consider the sequence $f_n(x) = x^n$, $x \in [0, 1]$. Then f_n converges on $[0, 1]$ to f:

$$f(x) = \begin{cases} 0, & \mathit{if} \ \ x \in [0, 1) \\ 1, & \mathit{if} \ \ x = 1 \end{cases}$$

So $f_n \to f$ but not uniformly on $[0, 1]$.

However, if we consider the closed interval $[0, 1 - \epsilon]$ or any closed interval $[a, b]$ except 1, then

$$f_n \rightrightarrows f \text{ uniformly on } [0, 1 - \epsilon] \text{ or } [a, b].$$

which implies **"convergent sequence is nearly uniformly convergent"**.

下面给出 **Egorov 定理**的表述.

定理 3.8.1. Egorov (Almost Uniform Convergence, 近一致收敛).
Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions on a measurable set A with $m(A) < \infty$, and $f_k(x) \to f(x)$ on A a.e. Given $\epsilon > 0$, we can find a set $E \subset A$ s. t.

$$m(E) < \epsilon \text{ and } f_k \rightrightarrows f \text{ uniformly on } E^c.$$

注.

- 此处若将 **Lebesgue 测度 m** 换为一般的**抽象测度** μ，即可得到**抽象测度**下的 **Egorov 定理**. (可见书[6]**P62 Thm 2.33**)

- 在证明定理前，先回顾一下函数列**收敛点集** & **发散点集**的表述.
  - **收敛点集**.

$$x \in \text{收敛点集} \Leftrightarrow \forall \epsilon > 0, \exists N, \forall n \geq N, |f_n(x) - f(x)| < \epsilon \tag{3.228}$$

$$\overset{\text{离散}}{\Leftrightarrow} \forall k \in \mathbb{N}, \exists N, \forall n \geq N, |f_n(x) - f(x)| < \frac{1}{k} \tag{3.229}$$

$$\Rightarrow C(f) = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ x \mid |f_n(x) - f(x)| < \frac{1}{k} \right\} \tag{3.230}$$

  - **发散点集**.

$$x \in \text{发散点集} \Leftrightarrow \exists \epsilon > 0, \forall N, \exists n \geq N, |f_n(x) - f(x)| \geq \epsilon \tag{3.231}$$

$$\overset{\text{离散}}{\Leftrightarrow} \exists k \in \mathbb{N}, \forall N, \exists n \geq N, |f_n(x) - f(x)| \geq \frac{1}{k} \tag{3.232}$$

$$\Rightarrow D(f) = \bigcup_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \left\{ x \mid |f_n(x) - f(x)| \geq \frac{1}{k} \right\} \tag{3.233}$$

[6]《*Real Analysis – –Modern Techniques and Their Applications*》— *Gerald B. Folland*

证明. Let

$$E_{m}(k)=\bigcup_{n=m}^{\infty}\left\{x \mid |f_n(x)-f(x)|\geq \frac{1}{k}\right\} \tag{3.234}$$

Then $E_m(k)\searrow$ in m.

Since $f_k(x)\to f(x)$ on A a.e., then $m(D(f))=0$. Since

$$\bigcap_{m=1}^{\infty}E_m(k)\subset D(f)=\bigcup_{k=1}^{\infty}\left(\bigcap_{m=1}^{\infty}E_m(k)\right) \tag{3.235}$$

Then $m\left(\bigcap\limits_{m=1}^{\infty}E_m(k)\right)=0$. Then by **Thm 1.3.3**,

$$m\left(\bigcap_{m=1}^{\infty}E_m(k)\right)=m\left(\lim_{N\to\infty}\bigcap_{m=1}^{N}E_m(k)\right)=\lim_{N\to\infty}m\left(\bigcap_{m=1}^{N}E_m(k)\right)=0 \tag{3.236}$$

即得到数列 $\left\{\bigcap_{m=1}^{N}E_m(k)\right\}_{N=1}^{\infty}$ 极限为 0. Then for any fixed $\epsilon>0$, $\exists N_k\in\mathbb{N}$, s. t.

$$m\left(\bigcap_{m=1}^{N_k}E_m(k)\right)=m\left(E_{N_k}(k)\right)<\frac{\epsilon}{2^k} \tag{3.237}$$

Let $E=\bigcup\limits_{k=1}^{\infty}E_{N_k}(k)$, then $m(E)<\epsilon$ and

$$E^c=\bigcap_{k=1}^{\infty}E_{N_k}^c(k)=\bigcap_{k=1}^{\infty}\bigcap_{n=N_k}^{\infty}\left\{x \mid |f_n(x)-f(x)|<\frac{1}{k}\right\} \tag{3.238}$$

Then we get for a fixed $k_0\in\mathbb{N}$, $\exists N_{k_0}$, $\forall n\geq N_{k_0}$, s. t.

$$|f_n(x)-f(x)|<\tfrac{1}{k_0} \text{ for all } x\in E^c.$$

Therefore, $f_n\rightrightarrows f$ uniformly on E^c with $m(E)<\epsilon$. $\square$

3.8.2 *Lusin* 定理

下面给出 **Littlewood 三原则**中的第 (ii) 点，**可测函数 nearly 连续**，即 **Lusin 定理**.

定理 3.8.2. Lusin.

Suppose $f : E \to \mathbb{R}$ is measurable and finite-valued on E with $m(E) < \infty$. Then for every $\epsilon > 0$, there exists a compact set $F \subset E$, s. t.

$$m(F^c) < \epsilon \text{ and } f|_F \text{ is continuous.}$$

证明. 可见书[7]**P34 Thm 4.5** 或视频 可测函数与连续函数的联系. □

[7]《*Real Analysis – –Measure Theroy, Integration, & Hilbert Spaces*》— *Elias M. Stein*

第四章　*Differentiation and Integration*

Motivation　在 **Riemann 积分**的框架下，我们知道**积分**和**微分**可以视作一对互逆的运算. 而在这一章，我们将在全新的 **Lebesgue 测度**的框架下重新审视**积分**和**微分**之间的关系.

下面先来描述一下想要解决的问题.

- Let $f \in \mathcal{L}^1(\mathbb{R}^d)$. 对于**变上限积分** $F(x) = \int_a^x f(y)dy$，我们知道根据 **Riemann 积分下的微积分基本定理**，对 F 求导就会回到被积函数 f 本身. 那么我们就会好奇：

  - 在 **Lebesgue 积分**的框架下，这个结论是否还成立？
  - 如果成立的话，又对哪些 x 成立呢？

  此时回顾求导的定义，即对于差商 (此处改写为更具一般性的符号 $I = (x, x+h)$)

  $$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\int_x^{x+h} f(y)dy = \frac{1}{|I|}\int_I f(y)dy \tag{4.1}$$

  对差商中的增量 $h \to 0$，即得到**导数**的定义. 那么我们的问题就转化为了

  $$\lim_{\substack{|I|\to 0\\ I\ni x}} \frac{1}{|I|}\int_I f(y)dy = f(x) \quad \text{holds for which } x? \tag{4.2}$$

  更一般地，将上述问题从一维实直线 $\mathbb{R}$ 推广至 $\mathbb{R}^d$ 空间上，将区间 I 用开球 B 替换，得

  $$\lim_{\substack{m(B)\to 0\\ B\ni x}} \frac{1}{m(B)}\int_B f(y)dy = f(x) \quad \text{holds for which } x? \tag{4.3}$$

注.
  - 此处看似是随着开球 B 的测度减小，$x \in B$ 在跟着 B “跑”，但实际上则相反:

    对于每个固定的 x，让包含着 x 的球 $B \ni x$ 不断减小其测度，最后取极限

    而这也就是此处极限条件写为 “$B \ni x$” 而非 “$x \in B$” 的原因，逻辑更清晰.

  - 事实上该结论对于**几乎处处的** x 都成立 (若 f **Lebesgue 可积**)，这就是后面要讲的 **Lebesgue 微分定理**.

4.1 *Hardy – Littlewood* 极大函数 (非球心)

定义 下面我们给出 **Hardy-Littlewood 极大函数**的定义.

定义 4.1.1. If $f \in \mathcal{L}^1(\mathbb{R}^d)$, we define its **maximal function** Mf by

$$Mf(x) = \sup_{B \ni x} \frac{1}{m(B)} \int_B |f(y)|\, dy \tag{4.4}$$

注. 我们目前并不知道球面测度的具体数值与计算方法，但事实上我们也并不需要知道其具体数值，具体表现在:
设 $\mathbb{R}^d$ 中单位球 $B(0,1)$ 的测度为 $m(B(0,1)) = v_d$. $\forall B(x,r) \subset \mathbb{R}^d$，根据 **Lebesgue 测度的平移不变性和伸缩变换公式 (Prop 3.7.1)**

$$B(0,r) = rB(0,1) \;\; \Rightarrow \;\; m(B(x,r)) = m(B(0,r)) = r^d m(B(0,1)) = r^d v_d$$

性质 下面来说明 **Hardy-Littlewood 极大函数**的三条性质.

命题 4.1.1. Suppose $f \in \mathcal{L}^1(\mathbb{R}^d)$. Then:

(i) Mf is measurable.

(ii) $Mf(x) < \infty$ for a.e. x.

(iii) **weak-type inequality**.
Mf satisfies

$$m\left(\{x \in \mathbb{R}^d \mid Mf(x) > \alpha\}\right) \leq \frac{A}{\alpha} \|f\|_{\mathcal{L}^1}, \;\; \forall \alpha > 0 \tag{4.5}$$

where $A = 3^d$.

证明.

(i) Let $E_\alpha = \{x \in \mathbb{R}^d \mid Mf(x) > \alpha\}$. 下面证明 E_α open.
$\forall x \in E_\alpha$, by the **definition of Mf (Def 4.1.1)**, $\exists B_x \ni x$, s. t.

$$\frac{1}{m(B_x)} \int_{B_x} |f| > \alpha \tag{4.6}$$

Then $\forall y \in B_x$, B_x is also an open ball containing y, so we have $y \in E_\alpha$. i.e. $B_x \subset E_\alpha$.
Therefore, E_α is open, specifically measurable for all α. Then Mf is measurable.

(ii) 下面说明 (iii) $\Rightarrow$ (ii):

Let $E_a = \{x \in \mathbb{R}^d \mid Mf(x) > a\}$. Then $E_n \searrow E = \{x \in \mathbb{R}^d \mid Mf(x) = \infty\}$.

Since $f \in \mathcal{L}^1(\mathbb{R}^d)$, $\|f\|_{\mathcal{L}^1}$ is finite. Then by **(iii)**, $m(E_1) < \infty$.

Then by **Thm 1.3.3**,

$$m(E) = \lim_{n\to\infty} m(E_n) \leq \lim_{n\to\infty} \frac{A}{n} \|f\|_{\mathcal{L}^1} = 0 \tag{4.7}$$

Therefore $m(E) = 0$. i.e. $Mf(x) < \infty$ for a.e. x.

(iii) 在证明 (iii) 之前，先来介绍 **Vitali 覆盖引理**.

引理 4.1.1. Vitali Covering Lemma (Elementary Version).

Suppose $\mathcal{B} = \{B_1, B_2, \cdots, B_N\}$, $B_i \subset \mathbb{R}^d$ are open balls, then there is a disjoint subcollection $B_{i_1}, \cdots, B_{i_k}$ that satisfies

$$m\left(\bigcup_{l=1}^{N} B_l\right) \leq 3^d \sum_{j=1}^{k} m(B_{i_j}) \tag{4.8}$$

注. 这是 **Vitali 覆盖引理的初等版本 (有限版本)**，更一般的版本是对**一列**球结论成立.

证明. 详见视频(非球心)Hardy-Littlewood 极大函数 23:10 (类似**贪心算法**的迭代步骤) □

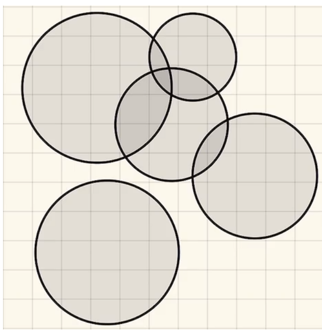

图 4.1: Lemma 4.1.1

下面继续来证明 (iii):

Fix $\alpha > 0$, $\forall x \in B_\alpha$, $\exists$ open ball B_x, s. t.

$$\frac{1}{m(B_x)} \int_{B_x} |f| > \alpha \tag{4.9}$$

So we have $E_\alpha \subset \bigcup_{x \in E_\alpha} B_x$.

Since E_α is measurable (by **(i)**), then by **Thm 1.3.4 (Lebesgue 测度的内正则性)**,

$\forall \epsilon > 0$, $\exists$ compact $K_\epsilon \subset E_\alpha$, s. t.

$$m(E_\alpha \backslash K_\epsilon) \leq \epsilon$$

i.e.

$$m(E_\alpha) - m(K_\epsilon) \leq \epsilon$$

Since K_ϵ is compact, $K_\epsilon \subset \bigcup_{x \in K_\epsilon} B_x$, there exists a subcollection $B_{x_1}, \cdots, B_{x_N}$, s. t.

$$K_\epsilon \subset \bigcup_{l=1}^{N} B_{x_l}$$

Then by **Vitali Covering Lemma (Lemma 4.1.1)**, there exists a subcollection $B_{x_{i_1}}, \cdots, B_{x_{i_k}}$, s. t.

$$m\left(\bigcup_{l=1}^{N} B_{x_l}\right) \leq 3^d \sum_{j=1}^{k} m(B_{x_{i_j}})$$

Therefore

$$m(K_\epsilon) \leq m\left(\bigcup_{l=1}^{N} B_{x_l}\right) \leq 3^d \sum_{j=1}^{k} m(B_{x_{i_j}}) \tag{4.10}$$

$$= \frac{3^d}{\alpha} \sum_{j=1}^{k} \alpha \cdot m(B_{x_{i_j}}) \tag{4.11}$$

$$\leq \frac{3^d}{\alpha} \int_{\bigcup_{j=1}^{k} B_{x_{i_j}}} |f| \tag{4.12}$$

$$\leq \frac{3^d}{\alpha} \int_{\mathbb{R}^d} |f| \tag{4.13}$$

$$= \frac{3^d}{\alpha} \|f\|_{\mathcal{L}^1} \tag{4.14}$$

Then

$$m(E_\alpha) \leq m(K_\epsilon) + \epsilon \leq \frac{A}{\alpha} \|f\|_{\mathcal{L}^1} + \epsilon \tag{4.15}$$

where $A = 3^d$, $\epsilon > 0$.

Since ϵ is arbitrary, let $\epsilon \to 0$, we have

$$m(E_\alpha) \leq \frac{A}{\alpha} \|f\|_{\mathcal{L}^1}, \ A = 3^d, \ \forall \alpha > 0 \tag{4.16}$$

□

4.2 *Lebesgue* 微分定理 (非球心)

在这一节我们将利用 **Hardy-Littlewood 极大函数**来证明 **Lebesgue 微分定理**.

4.2.1 *Chebyshev's Inequality*

在此之前，我们先来证明一个非常有用的不等式，即**切比雪夫不等式**.

定理 4.2.1. Chebyshev's Inequality.
If $g \in \mathcal{L}^1(\mathbb{R}^d)$, then

$$m\left(\{x \in \mathbb{R}^d \mid |g(x)| > \alpha\}\right) \leq \frac{1}{\alpha} \|g\|_{\mathcal{L}^1}, \ \forall \alpha > 0 \tag{4.17}$$

证明. Let $E_\alpha = \{x \in \mathbb{R}^d \mid |g(x)| > \alpha\}$. Then

$$\|g\|_{\mathcal{L}^1} = \int_{\mathbb{R}^d} |g| \geq \int_{E_\alpha} |g| \geq \int_{E_\alpha} \alpha = \alpha \cdot m(E_\alpha) \tag{4.18}$$

□

4.2.2 *The Lebesgue Differentiation Theorem*

下面我们就来给出 **Lebesgue 微分定理**.

定理 4.2.2. If $f \in \mathcal{L}^1(\mathbb{R}^d)$, then

$$\lim_{\substack{m(B)\to 0\\ B\ni x}} \frac{1}{m(B)} \int_B f(y)dy = f(x) \ \ for\ a.e.\ x \tag{4.19}$$

注. • **Lebesgue 微分定理**说明了对于**几乎处处的** x，当包含 x 的球体 B 的测度趋于 0 时，f 在球体 B 上积分的平均值就会收敛到 $f(x)$.

• 定理左侧实际上是关于集合 B 的函数的一个极限过程，用 $\epsilon-\delta$ 语言叙述如下：
$\forall \epsilon > 0, \exists \delta > 0$, s. t. for all $B \ni x$ and $m(B) < \delta$, we have

$$\left| \frac{1}{m(B)} \int_B f(y)dy - f(x) \right| \leq \epsilon \tag{4.20}$$

• 要证明该定理，首先需要说明等式左侧**极限的存在性**，但这并不好说明. 为了跳过说明其存在性的问题，我们需要引入类似 **"上极限"** 的函数，即:
If suffices to show

$$\lim_{\delta\to 0} \sup_{\substack{m(B)<\delta\\ B\ni x}} \left| \frac{1}{m(B)} \int_B f(y)dy - f(x) \right| = 0 \ \ for\ a.e.\ x \tag{4.21}$$

由于极限内的函数随着 δ 递减而单调递减，又存在下界 0，因此在 $\delta = 0$ 处必存在右极限. 这样就跳过了原极限是否存在的问题.

• 事实上此处极限 **"怪异"** 的本质原因在于开球 B 的选取的任意性，若将其定义为以 x 为球心，r 为半径的球，则可直接令 $r \to 0$ 变为正常的函数极限，即

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)dy = f(x) \tag{4.22}$$

在下一节我们会从 **Hardy-Littlewood 极大函数**开始，以此方法重新说明 **Lebesgue 微分定理**.

证明. Let

$$E_a = \left\{ x \in \mathbb{R}^d \mid \lim_{\delta\to 0} \sup_{\substack{m(B)<\delta \\ B\ni x}} \left| \frac{1}{m(B)} \int_B f(y)dy - f(x) \right| > 2a \right\} \tag{4.23}$$

Then we **WTS (want to show)**:

$$m(E_a) = 0, \forall a \geq 0$$

Fix $a \geq 0$. By **Thm 3.2.3**, $C_c(\mathbb{R}^d)$ is dense in $\mathcal{L}^1(\mathbb{R}^d)$ (有紧支集的连续函数), then $\forall \epsilon > 0, \exists g \in C_c(\mathbb{R}^d)$, s. t.

$$\|f - g\|_{\mathcal{L}^1} < \epsilon$$

Since g is uniformly continuous, then $\exists \delta > 0$, s. t.

$$\left| \frac{1}{m(B)} \int_B g(y)dy - g(x) \right| \leq \frac{1}{m(B)} \int_B |g(y) - g(x)|\, dy < \frac{1}{m(B)} \int_B \epsilon dy = \epsilon \tag{4.24}$$

for all $B \ni x$ and $m(B) < \delta$.

下面对 $m(E_a)$ 进行估计. $\forall x \in E_a$,

$$\left| \frac{1}{m(B)} \int_B f(y)dy - f(x) \right| \leq \left| \frac{1}{m(B)} \int_B (f(y) - g(y))dy \right| + \left| \frac{1}{m(B)} \int_B g(y)dy - g(x) \right| + |g(x) - f(x)| \tag{4.25}$$

对上述不等式中的开球 $B \ni x$ 取上确界 sup，得

$$\sup_{B\ni x} \left| \frac{1}{m(B)} \int_B f(y)dy - f(x) \right| \leq \sup_{B\ni x} \left| \frac{1}{m(B)} \int_B (f(y) - g(y))dy \right| + \sup_{B\ni x} \left| \frac{1}{m(B)} \int_B g(y)dy - g(x) \right| + |g(x) - f(x)| \tag{4.26}$$

再令 $m(B) \to 0$，由于根据式 **(4.24)**，

$$\lim_{\delta\to 0} \sup_{\substack{m(B)<\delta \\ B\ni x}} \left| \frac{1}{m(B)} \int_B g(y)dy - g(x) \right| = 0 \tag{4.27}$$

因此

$$\lim_{\delta\to 0} \sup_{\substack{m(B)<\delta \\ B\ni x}} \left| \frac{1}{m(B)} \int_B f(y)dy - f(x) \right| \leq \lim_{\delta\to 0} \sup_{\substack{m(B)<\delta \\ B\ni x}} \left| \frac{1}{m(B)} \int_B (f(y) - g(y))dy \right| + 0 + |g(x) - f(x)| \tag{4.28}$$

下面对**红色部分**进行估计. 根据对 δ 的单调性可知,

$$\lim_{\delta\to 0}\sup_{\substack{m(B)<\delta\\ B\ni x}}\left|\frac{1}{m(B)}\int_B (f(y)-g(y))dy\right| \le \sup_{B\ni x}\frac{1}{m(B)}\int_B |f(y)-g(y)|\,dy = M(f-g)(x) \tag{4.29}$$

又因为对于 $\forall x\in E_a$,

$$\lim_{\delta\to 0}\sup_{\substack{m(B)<\delta\\ B\ni x}}\left|\frac{1}{m(B)}\int_B f(y)dy - f(x)\right| > 2a \tag{4.30}$$

所以

$$M(f-g)(x) + |g(x)-f(x)| > 2a \tag{4.31}$$

$$\Rightarrow M(f-g)(x) > a \;\; or \;\; |g(x)-f(x)| > a \tag{4.32}$$

$$\Rightarrow E_a \subset \{x\in\mathbb{R}^d \mid M(f-g)(x) > a\} \cup \{x\in\mathbb{R}^d \mid |g-f| > a\} \tag{4.33}$$

下面分别来估计 the **purple** one 和 the **orange** one 的测度.

- 由于 $|f-g|\in\mathcal{L}^1\mathbb{R}^d$, 因此根据 **Chebyshev's Inequality (Thm 4.2.1)**,

$$m\left(\{x\in\mathbb{R}^d \mid |g-f| > a\}\right) \le \frac{1}{a}\,\|f-g\|_{\mathcal{L}^1} \tag{4.34}$$

- 根据 **Hardy-Littlewood 极大函数的 weak-type inequality (Prop 4.1.1 (iii))**,

$$m\left(\{x\in\mathbb{R}^d \mid M(f-g)(x) > a\}\right) \le \frac{A}{a}\,\|f-g\|_{\mathcal{L}^1} \tag{4.35}$$

从而根据 $\|f-g\|_{\mathcal{L}^1} < \epsilon$,

$$m(E_a) \le m\left(\{x\in\mathbb{R}^d \mid M(f-g)(x) > a\}\right) + m\left(\{x\in\mathbb{R}^d \mid |g-f| > a\}\right) \tag{4.36}$$

$$\le \frac{A+1}{a}\,\|f-g\|_{\mathcal{L}^1} \tag{4.37}$$

$$< \frac{A+1}{a}\epsilon,\;\; \forall a\ge 0 \tag{4.38}$$

Since $\epsilon > 0$ is arbitrary, let $\epsilon\to 0$, we get

$$m(E_a) = 0,\;\; \forall a\ge 0 \tag{4.39}$$

□

4.3 *Hardy – Littlewood* 极大函数 & *Lebesgue* 微分定理 (球心)

4.3.1 *Hardy – Littlewood* 极大函数

本节的重点是给出 **Hardy-Littlewood** 极大函数 **(centered)** 的定义并证明其**连续性**.

Premilinaries 在此之前，先来给出一些**记号与命题**.

We define 开球 & 球面

$$B(x,r) := \{y \in \mathbb{R}^d \mid |y-x| < r\}$$
$$S(x,r) := \{y \in \mathbb{R}^d \mid |y-x| = r\}$$

下面说明球面为零测集.

命题 4.3.1. $m(S(x,r)) = 0, \forall x \in \mathbb{R}^d, r \geq 0.$

证明. 根据 **Lebesgue** 测度的平移不变性和伸缩变换公式 **(Prop 3.7.1)**, it suffices to show

$$m(S(0,1)) = 0$$

反证法. Suppose $m(S(0,1)) > 0$. By **Prop 3.7.1**,

$$m(rS(0,1)) = r^d m(S(0,1)) \geq m(S(0,1)), \forall r \geq 1.$$

Consider the compact set $\{x \in \mathbb{R}^d \mid 1 \leq |x| \leq 2\}$. We have

$$\bigcup_{k=1}^{\infty} S(0, 1+\frac{1}{k}) = \bigcup_{k=1}^{\infty} (1+\frac{1}{k}) S(0,1) \subset \{x \in \mathbb{R}^d \mid 1 \leq |x| \leq 2\} \tag{4.40}$$

However

$$m\left(\bigcup_{k=1}^{\infty} S(0, 1+\frac{1}{k})\right) = \sum_{k=1}^{\infty} m\left(S(0, 1+\frac{1}{k})\right) \geq \sum_{k=1}^{\infty} m\left(S(0,1)\right) = \infty \tag{4.41}$$

which is a contradiction for $\{1 \leq |x| \leq 2\}$ is compact. □

下面我们给出当球心收敛时，开球的特征函数的收敛性.

命题 4.3.2. $\forall (x_j, r_j) \to (x, r)$ on $\mathbb{R}^d \times \mathbb{R}$,

$$\chi_{B(x_j,r_j)}(y) \to \chi_{B(x,r)}(y) \text{ on } \mathbb{R}^d \backslash S(x, r)$$

注. 该命题在开球 $B(x_j, r_j)$ 的边界，即 $S(x, r)$ 上不一定成立. 下面给出一个反例.

例 4.3.1. In $\mathbb{R}$, take $x_j = 0$, $r_j = 1 + \frac{1}{j+1}$. Then $(x_j, r_j) \to (0, 1)$.

$$\chi_{B(x_j,r_j)} = \chi_{(-1-\frac{1}{j+1}, 1+\frac{1}{j+1})} \to \chi_{[-1,1]}$$

and $\chi_{[-1,1]}(x) \neq \chi_{(-1,1)}(x)$, for $x = -1$ or 1.

证明. 下面分别对 $|y| < r$ 与 $|y - x| > r$ 两种情况进行讨论.

(i) $|y - x| < r$. WTS

$$\left|x_j - y\right| < r_j, \forall j > N \text{ for some } N$$

Since $\left|x_j - y\right| < \left|x_j - x\right| + |x - y|$, it suffices to show

$$\left|x_j - x\right| + |x - y| < r_j$$

Suppose $|x - y| = r - \epsilon$, $\epsilon > 0$. Then

$$\Leftrightarrow \left|x_j - x\right| + r - \epsilon < r_j$$
$$\Leftrightarrow \left|x_j - x\right| + r - r_j < \epsilon$$

It suffices to show

$$\left|x_j - x\right| + \left|r_j - r\right| < \epsilon$$

Since $(x_j, r_j) \to (x, r)$, then $\exists N \in \mathbb{N}$, s. t.

$$\left|x_j - x\right| < \tfrac{\epsilon}{3}, \left|r_j - r\right| < \tfrac{\epsilon}{3}, \forall j > N$$

Then $\left|x_j - y\right| < r_j$, $\forall j > N$.

(ii) $|y - x| > r$. 同理 Suppose $|x - y| = r + \epsilon$, $\epsilon > 0$. WTS $\left|x_j - y\right| > r_j$, $\forall j > N$ for some N.

It suffices to show

$$\left|x_j - y\right| > |x - y| - \left|x_j - x\right| = r + \epsilon - \left|x_j - x\right| > r_j$$
$$\Leftrightarrow \left|x_j - x\right| + r_j - r < \epsilon$$

□

Average Value of f on B(x , r) 在说明 f 的**连续性**之前，先来说明去掉 sup 的函数的**连续性**.

定义 4.3.1. We define the **average value of f on B(x , r)**

$$A_r f(x) = \frac{1}{m(B(x,r))}\int_{B(x,r)} f(y)dy \tag{4.42}$$

下面给出**局部可积**的概念.

定义 4.3.2. A measurable function $f : \mathbb{R}^d \to \mathbb{C}$ is called **locally integrable** if

$$\int_K |f| < \infty \text{ for every bounded measurable set } K \subset \mathbb{R}^d \tag{4.43}$$

We write $f \in \mathcal{L}^1_{loc}(\mathbb{R}^d)$.

例 4.3.2. $f(x) = e^x$ is locally integrable but not integrable on $\mathbb{R}$.

下面给出 $A_r f(x)$ 的**连续性**.

引理 4.3.1. If $f \in \mathcal{L}^1_{loc}(\mathbb{R}^d)$, then

$$A_r f(x) \text{ is jointly continuous in } r \text{ and } x\ (r > 0,\ x \in \mathbb{R}^d).$$

证明. 下面分为两步进行证明.

- $\int_{B(x,r)} f(y)dy$ is continuous.

$$\int_{B(0,r)} f(y)dy = \int_{\mathbb{R}^d} f(y)\chi_{B(x,r)}(y)dy \tag{4.44}$$

Fix $(x, r) \in \mathbb{R}^d \times \mathbb{R}$, $\forall (x_j, r_j) \to (x, r)$, by **Prop 4.3.2**,

$$\chi_{B(x_j,r_j)}(y) \to \chi_{B(x,r)}(y) \text{ on } \mathbb{R}^d \backslash S(x, r)$$

Since by **Prop 4.3.1**, $m(S(x, r)) = 0$, we have

$$\chi_{B(x_j,r_j)}(y) \to \chi_{B(x,r)}(y) \text{ for a.e. } y$$
$$\Rightarrow f(y)\chi_{B(x_j,r_j)}(y) \to f(y)\chi_{B(x,r)}(y) \text{ for a.e. } y$$

Since $(x_j, r_j) \to (x, r)$, $\exists N$, s. t. $B(x_j, r_j) \subset B(x, 100r)$, $\forall j > N$. Then

$$\left|f(y)\chi_{B(x_j,r_j)}(y)\right| \leq \left|f(y)\chi_{B(x,100r)}(y)\right| \in \mathcal{L}^1 \text{ for a.e. } y$$

Therefore, by **DCT (Thm 3.1.7, 控制收敛定理)**

$$\int_{\mathbb{R}^d} f(y)\chi_{B(x_j,r_j)}(y)dy \to \int_{\mathbb{R}^d} f(y)\chi_{B(x,r)}(y)dy \tag{4.45}$$

i.e.

$$\int_{B(x_j,r_j)} f(y)dy \to \int_{B(x,r)} f(y)dy, \ \forall (x_j, r_j) \to (x, r) \tag{4.46}$$

Then by **Heine** 归结原理, $\int_{B(x,r)} f(y)dy$ is continuous.

- $A_r f(x)$ is continuous. Since $m(B(x, r)) = r^d m(B(0, 1)) = r^d v_d$, then

$$A_r f(x) = v_d^{-1} r^{-d} \int_{B(x,r)} f(y)dy \text{ is continuous} \tag{4.47}$$

□

Hardy-Littlewood 极大函数 下面先给出 **Hardy-Littlewood 极大函数 (球心)** 的定义.

定义 4.3.3. If $f \in \mathcal{L}^1_{loc}$, we define its **Hardy-Littlewood maximal function** Hf by

$$Hf(x) = \sup_{r>0} A_r|f|(x) = \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)|\, dy \tag{4.48}$$

下面说明 **Hardy-Littlewood 极大函数**的连续性.

推论 4.3.2. Hf is continuous.

证明. $\forall (a,\infty) \subset \mathbb{R}$, by **Lemma 4.3.1**, $A_r|f|$ is continuous, then

$$(Hf)^{-1}((a,\infty)) = \bigcup_{r>0} (A_r|f|)^{-1}((a,\infty)) \text{ is open}, \ \forall a \in \mathbb{R} \tag{4.49}$$

Therefore Hf is continuous. □

此版本的 **Hardy-Littlewood 极大函数**同样有 **weak-type inequality**.

命题 4.3.3. weak-type inequality.
If $f \in \mathcal{L}^1$, then

$$m\left(\{x \in \mathbb{R}^d \mid Hf(x) > a\}\right) \le \frac{A}{a} \|f\|_{\mathcal{L}^1}, \ \forall a > 0 \tag{4.50}$$

where $A = 3^d$.

证明. 与命题 **4.1.1 (iii)** 证明类似. □

4.3.2 *Lebesgue* 微分定理

函数的上极限 首先来回顾一下**函数的上极限**的定义.

定义 4.3.4. $\forall$ 函数 $f : E \subset \mathbb{R}^d \to \mathbb{R}$，$x_0$ 为 E 的聚点，定义 f 在 x_0 的上极限 为

$$\limsup_{x\to x_0} f(x) = \lim_{\delta\to 0^+} \sup_{0<|x-x_0|<\delta} f(x) \tag{4.51}$$

同理可定义函数 f 在 x_0 点的下极限.

注. 不难证明，该定义与通常数学分析书[1]上的定义等价，即

$$\limsup_{x\to x_0} f(x) = \sup\{l \in \mathbb{R} \mid \exists\{x_n\}_{n=1}^{\infty} \subset E,\ x_n \to x_0,\ \text{s.t.} f(x_n) \to l\} \tag{4.52}$$

下面利用**函数的上极限**给出**函数极限**的**等价定义**.

命题 4.3.4. $\forall$ 函数 $f : E \subset \mathbb{R}^d \to \mathbb{R}$，$x_0$ 为 E 的聚点，则

$$\lim_{x\to x_0} f(x) = c \iff \limsup_{x\to x_0} |f(x) - c| = 0$$

证明. 此处证明采用数学分析中的定义更方便. 根据 **Heine 归结原理**,

$$\lim_{x\to x_0} f(x) = c \iff \forall\{x_j\}_{j=1}^{\infty},\ x_j \to x_0,\ \text{s.t.} f(x_j) \to c \tag{4.53}$$

$$\iff E = \{l \in \mathbb{R} \mid \exists\{x_n\}_{n=1}^{\infty} \subset E,\ x_n \to x_0,\ \text{s.t.} |f(x_n) - c| \to l\} = \{0\} \tag{4.54}$$

$$\iff \limsup_{x\to x_0} |f(x) - c| = \sup E = 0 \tag{4.55}$$

□

[1]此处参考书籍：《数学分析教程 (上册) (第一版)》——常庚哲、史济怀编 § 2.11 定义 2.22

Lebesgue 微分定理 下面给出 **Lebesgue 微分定理 (球心)**.

定理 4.3.3. Lebesgue Differentiation Theorem.
If $f \in \mathcal{L}^1_{loc}$, then

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)dy = f(x) \text{ for a.e. } x \in \mathbb{R}^d \tag{4.56}$$

证明. Since $f \in \mathcal{L}^1_{loc}$, then for all $x \in \mathbb{R}^d$, $\exists N$, s. t. $|x| < N$ and

$$A_r f(x) = \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)dy \ \text{ depends only on } f(y) \text{ for } |y| \le N+1. \tag{4.57}$$

$$(\text{with } r \le 1 \text{ and } |x| < N)$$

Then we can replace f with $f\chi_{B(0,N+1)} \in \mathcal{L}^1$. 于是我们不妨设 $f \in \mathcal{L}^1$. 根据 **Prop 4.3.4**, 要证

$$\lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y)dy = \lim_{r\to 0} A_r f(x) = f(x) \text{ for a.e. } x \in \mathbb{R}^d \tag{4.58}$$

即证

$$\limsup_{r\to 0} |A_r f(x) - f(x)| = 0 \text{ for a.e. } x \in \mathbb{R}^d \tag{4.59}$$

下面来估计 $\limsup_{r\to 0} |A_r f(x) - f(x)|$.
Fix $\epsilon > 0$, 根据 **Thm 3.2.3**, $\exists g \in C_c(\mathbb{R}^d)$, s. t.

$$\|f - g\|_{\mathcal{L}^1} < \epsilon$$

Since $A_r g(x)$ is continuous (by **Lemma 4.3.1**), we have

$$|A_r g(x) - g(x)| = |A_r g(x) - A_0 g(x)| < \epsilon \text{ for all small } r.$$

Then

$$\limsup_{r\to 0} |A_r f(x) - f(x)| = \limsup_{r\to 0} |A_r(f-g)(x) + (A_r g - g)(x) + (g - f)(x)| \tag{4.60}$$

$$\le \lim_{\delta\to 0^+} \sup_{0<r<\delta} |A_r(f-g)(x)| + 0 + |g(x) - f(x)| \tag{4.61}$$

Since

$$\sup_{0<r<\delta} |A_r(f-g)(x)| \leq \sup_{r>0} |A_r(f-g)(x)| = \sup_{r>0} \left| \frac{1}{m(B(x,r))} \int_{B(x,r)} f(y) - g(y) dy \right| \tag{4.62}$$

$$\leq \sup_{r>0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y) - g(y)|\, dy \tag{4.63}$$

$$= H(f-g)(x), \ \forall \delta > 0 \tag{4.64}$$

Let $\delta \to 0^+$, we have

$$\limsup_{r\to 0} |A_r f(x) - f(x)| \leq \lim_{\delta \to 0^+} \sup_{0<r<\delta} |A_r(f-g)(x)| + |g(x) - f(x)| \tag{4.65}$$

$$\leq H(f-g)(x) + |g(x) - f(x)| \tag{4.66}$$

后续证明与 **Thm 4.2.2**一致，即定义 E_a，并分别对 $H(f-g)(x)$ 与 $|g-f|$ 运用 **weak-type inequality (Prop 4.3.3)** 与 **Chebyshev's Inequality (Thm 4.2.1)**，即可证明 $m(E_a) = 0, \forall a \geq 0$. 从而得证. □

4.4 有界变差函数

引入 在 **Riemann 积分**中，我们知道对于一阶连续可微函数，我们有**微积分基本定理**

$$F(b) - F(a) = \int_a^b F^{'}(x)dx \tag{4.67}$$

而对于 **Lebesgue 积分**，我们也想要得到该命题成立的条件，且最好为**充要条件**. 可以举例证明，仅仅 F 连续并不能保证 F 可导 (可见视频 a continuous but nowhere differentiable function). 同时仅仅要求 F 导数存在也可能出现 $F^{'}$ 不可积的情况，如下反例.

例 4.4.1. **(书[2] P147 Ex 12).**
Consider the function $F(x) = x^2 \sin \frac{1}{x^2}$, $x \neq 0$, with $F(0) = 0$. Show that $F^{'}(x)$ exists for every x, but $F^{'}$ is not integrable on $[-1, 1]$.

证明. 详细证明可见视频 微积分基本定理: 问题引入. □

为了解决上述问题，我们在这一节将引入一种新类型的函数，叫做**有界变差函数**.

[2]《*Real Analysis – –Measure Theroy, Integration, & Hilbert Spaces*》— *Elias M. Stein*

4.4.1 有界变差函数的概念

引入 为了便于理解，我们将**有界变差函数与平面上的曲线**相联系. 首先来回顾数学分析中有关**平面上的曲线**的相关概念.

定义 4.4.1. Let γ be a **parametrized curve** in the plane given by $z(t) = (x(t), y(t))$, where $a \leq t \leq b$. Here $x(t)$ and $y(t)$ are continuous real-valued functions on $[a, b]$.

The curve is **rectifiable** if $\exists M < \infty$, s. t. for any partition $a = t_0 < t_1 < \cdots < t_N = b$ of $[a, b]$,

$$\sum_{j=1}^{N} \left| z(t_j) - z(t_{j-1}) \right| \leq M \tag{4.68}$$

The **Length** $L(\gamma)$ of the curve is defined as

$$L(\gamma) = \sup_{all\ partitions} \sum_{j=1}^{N} \left| z(t_j) - z(t_{j-1}) \right| \tag{4.69}$$

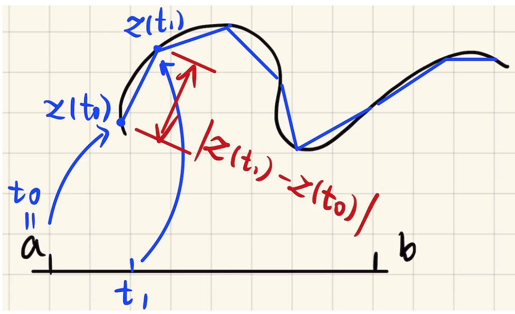

图 4.2: rectifiable curve

注. 为了表述的方便，后续我们常将 z 的值域视作复平面，即 $z(t) = x(t) + iy(t)$.

在定义了曲线**可求长**的概念后，我们自然要问，在什么情况下**曲线可求长**？即

What condition on $x(t)$ and $y(t)$ guarantees rectifiability of γ?

为了解决这个问题，下面我们给出**有界变差函数**的定义，并会在后续给出这个问题的**充要条件**，即

$$\gamma \text{ rectifiable} \iff x(t), y(t) \text{ 均为\textbf{有界变差函数}}.$$

定义 下面给出**有界变差函数**的定义.

定义 4.4.2. Suppose $F : [a, b] \to \mathbb{C}$, and $\mathcal{P}$ is a partition $a = t_0 < t_1 < \cdots < t_N = b$. The **variation** of F on this partition $\mathcal{P}$ is defined by

$$V_{\mathcal{P}}(F) = \sum_{j=1}^{N} \left|F(t_j) - F(t_{j-1})\right| \tag{4.70}$$

F is said to be of **bounded variation (BV)** if $V_{\mathcal{P}}(F)$ is bounded over all partitions.

注. **有界变差函数**不要求函数**连续**，而在考虑**平面曲线**时默认函数**连续**.

下面就能够来回答 **"引入"** 中提到的问题，即**平面曲线可求长的充要条件**.

定理 4.4.1. A curve parametrized by $F(t) = x(t) + iy(t)$, $a \le t \le b$, is rectifiable $\Leftrightarrow$ both $x(t)$ and $y(t)$ are of BV.

证明. $\forall$ partition $\mathcal{P}$: $a = t_0 < t_1 < \cdots < t_N = b$, we have

$$V_{\mathcal{P}}(F) = \sum_{j=1}^{N} \left|F(t_j) - F(t_{j-1})\right| \tag{4.71}$$

Since $|a + bi| \le |a| + |b| \le 2\,|a + bi|$,

$$\left|F(t_j) - F(t_{j-1})\right| = \left|x(t_j) - x(t_{j-1}) + i\left(y(t_j) - y(t_{j-1})\right)\right| \tag{4.72}$$

Then

- $\Leftarrow$: $\left|F(t_j) - F(t_{j-1})\right| \le \left|x(t_j) - x(t_{j-1})\right| + \left|y(t_j) - y(t_{j-1})\right|$, then $\exists M < \infty$, s. t.

$$\begin{aligned} V_{\mathcal{P}}(F) &= \sum_{j=1}^{N} \left|F(t_j) - F(t_{j-1})\right| &\quad (4.73)\\ &= \sum_{j=1}^{N} \left|x(t_j) - x(t_{j-1})\right| + \sum_{j=1}^{N} \left|y(t_j) - y(t_{j-1})\right| &\quad (4.74)\\ &\le 2M, \ \forall partition\ \mathcal{P} &\quad (4.75) \end{aligned}$$

  Therefore, the curve $F(t) = x(t) + iy(t)$ is rectifiable.

- $\Rightarrow$: $\left|x(t_j) - x(t_{j-1})\right| + \left|y(t_j) - y(t_{j-1})\right| \le 2\left|F(t_j) - F(t_{j-1})\right|$, then $\exists M < \infty$, s. t.

$$\sum_{j=1}^{N} \left|x(t_j) - x(t_{j-1})\right| + \sum_{j=1}^{N} \left|y(t_j) - y(t_{j-1})\right| \le 2\sum_{j=1}^{N} \left|F(t_j) - F(t_{j-1})\right| = 2V_{\mathcal{P}}(F) \le 2M \tag{4.76}$$

  Therefore, both $x(t)$ and $y(t)$ are of BV.

□

例子 下面来给出一些**有界变差函数**的例子.

例 4.4.2. • x, x^2 is of BV on $[a, b]$, $\forall [a, b] \subset \mathbb{R}$.

证明. $\forall$ partition $\mathcal{P}$: $a = x_0 < x_1 < \cdots < x_N = b$, since x is strictly increasing, then

$$V_{\mathcal{P}}(x) = \sum_{j=1}^{N} \left| x_j - x_{j-1} \right| = \sum_{j=1}^{N} x_j - x_{j-1} = b - a < \infty \tag{4.77}$$

Also for x^2,

$$V_{\mathcal{P}}(x^2) = \sum_{j=1}^{N} \left| x_j^2 - x_{j-1}^2 \right| = \sum_{j=1}^{N} \left| x_j + x_{j-1} \right| \left| x_j - x_{j-1} \right| \leq 2b \sum_{j=1}^{N} \left| x_j - x_{j-1} \right| = 2b(b - a) < \infty \tag{4.78}$$

Therefore, both x and x^2 are of BV on $[a, b]$, $\forall [a, b] \subset \mathbb{R}$. □

• If F is real-valued, monotonic, and bounded, then F is of BV.

证明. $\forall$ partition $\mathcal{P}$: $a = t_0 < t_1 < \cdots < t_N = b$.

Since F is bounded, $\exists M < \infty$, s. t. $|F| \leq M$. 不妨设 F 单调递增,

$$V_{\mathcal{P}}(F) = \sum_{j=1}^{N} \left| F(t_j) - F(t_{j-1}) \right| = F(b) - F(a) \leq 2M, \ \forall partition\ \mathcal{P} \tag{4.79}$$

So F is of BV. □

- **(书[3] P147 Ex 11).**

  If $a, b > 0$, let

  $$f(x) = \begin{cases} x^a \sin(x^{-b}), \ 0 \le x \le 1 \\ 0, \ x = 0 \end{cases} \tag{4.80}$$

  Then

  $$f \text{ is of BV in } [0,1] \ \Leftrightarrow \ a > b$$

证明.

- 先来考虑简单情形，即 $f(x) = \sin\frac{1}{x}$.
  根据直觉，随着 $x \to 0^+$，$\sin\frac{1}{x}$ 的震荡越剧烈，当分划足够密时，其变差中应当会出现各项为 1 的无穷级数，从而发散. 下面取一个特殊分划进行证明.
  对于 $\forall$ 奇数 $k \in \mathbb{N}$, 取

  $$\tfrac{1}{x_k} = 2k\pi + \tfrac{\pi}{2}, \tfrac{1}{x_{k+1}} = 2k\pi + \pi$$

  于是

  $$V_{\mathcal{P}}(f) = \sum_{j=1}^{N} \left|\sin\frac{1}{x_j} - \sin\frac{1}{x_{j-1}}\right| = \sum_{j=1}^{N} 1 = N, \ \textit{which is related to } \mathcal{P} \tag{4.81}$$

  故 $\forall M < \infty$，当分划 $\mathcal{P}$ 足够密时，$V_{\mathcal{P}}(f) > M$. 故 f 非 BV.

- 对于一般情况，下面先讨论一种特殊分划 $\mathcal{P}$. 即 **(Monotonic Partition, 单调划分)**

  $$[\frac{1}{x_{4k+1}^b}, \frac{1}{x_{4k+2}^b}] = [2k\pi + \frac{\pi}{2}, 2k\pi + \pi], \ [\frac{1}{x_{4k+3}^b}, \frac{1}{x_{4k+4}^b}] = [2k\pi + \frac{3\pi}{2}, 2k\pi + 2\pi] \tag{4.82}$$

  则以上述第一种的 $[x_{4k+1}, x_{4k+2}]$ 举例，

  $$\left|x_{k+1}^a \sin\frac{1}{x_{k+1}^b} - x_k^a \sin\frac{1}{x_k^b}\right| = \left|x_{k+1}^a \sin\frac{1}{x_{k+1}^b}\right| = \frac{1}{(2k\pi + \frac{\pi}{2})^{\frac{a}{b}}} = O(\frac{1}{k^{\frac{a}{b}}}) \tag{4.83}$$

  同理对于 $[x_{4k+2}, x_{4k+3}], [x_{4k+3}, x_{4k+4}], [x_{4k+4}, x_{4k+5}]$，均可得到

  $$\left|x_{k+1}^a \sin\frac{1}{x_{k+1}^b} - x_k^a \sin\frac{1}{x_k^b}\right| = O(\frac{1}{k^{\frac{a}{b}}}) \tag{4.84}$$

[3]《*Real Analysis – –Measure Theroy, Integration, & Hilbert Spaces*》— *Elias M. Stein*

于是

$$\sum_{k=0}^{N}\left|x_{k+1}^{a}\sin\frac{1}{x_{k+1}^{b}}-x_{k}^{a}\sin\frac{1}{x_{k}^{b}}\right|=O\left(\sum_{k=1}^{N}\frac{1}{k^{\frac{a}{b}}}\right) \tag{4.85}$$

根据 **p-级数** $\sum\limits_{n}^{\infty}\frac{1}{n^p}$ 的收敛性 (收敛 $\Leftrightarrow p>1$) 可得，

$$\sum_{k=0}^{N}\left|x_{k+1}^{a}\sin\frac{1}{x_{k+1}^{b}}-x_{k}^{a}\sin\frac{1}{x_{k}^{b}}\right|<\infty \tag{4.86}$$

$$\Leftrightarrow\ \frac{a}{b}>1\ \Leftrightarrow\ a>b \tag{4.87}$$

由于对于任一分划 $\mathcal{P}$，有：

“加密分割，变差不减”

因此对于 $\forall$ 分割，我们可以在其中加入如上节点，其变差不减，但因为上述节点中，$\sin\frac{1}{x^b}$ 在每个区间 $[x_k,x_{k+1}]$ 均单调，所以在各区间 $[x_k,x_{k+1}]$ 中的变差可直接去除绝对值，并得到

$$V_{\mathcal{P}}(f)=O\left(\sum_{k=1}^{N}\frac{1}{k^{\frac{a}{b}}}\right) \tag{4.88}$$

于是

$$f \text{ is of BV}\ \Leftrightarrow\ \tfrac{a}{b}>1\ \Leftrightarrow\ a>b$$

$\square$

4.4.2 有界变差函数的刻画

介绍 本小节将给出**有界变差函数**的一个刻画，即

任一有界变差函数可差分为两个有界递增函数之差.

同时还将研究函数的**全变差**的**性质**. 而这一切都是为了后续研究**微积分基本定理**做准备.

定义 回顾函数的**变差**的概念 (**Def 4.4.2**). 在此基础上，我们下面给出**全变差**的定义.

定义 4.4.3. Suppose $f : [a, b] \to \mathbb{C}$. The **total variation** of f on $[a, x]$ $(a \le x \le b)$ is defined by

$$V_f([a, x]) = \sup_{all\ partitions} \sum_{j=1}^{n} \left| f(t_j) - f(t_{j-1}) \right| \tag{4.89}$$

In particular, if f is real-valued, i.e. $f : [a, b] \to \mathbb{R}$. Then the **positive variation** of f on $[a, x]$ is

$$P_f([a, x]) = \sup_{all\ partitions} \sum_{(+)} f(t_j) - f(t_{j-1}) \tag{4.90}$$

Also the **negative variation** of f on $[a, x]$ is

$$N_f([a, x]) = \sup_{all\ partitions} \sum_{(-)} -\left[f(t_j) - f(t_{j-1}) \right] \tag{4.91}$$

注. • **全变差**对**任一复值函数**均可定义，而**正变差**和**负变差**则只对**实值函数**有定义. 在后续的讨论中基本默认 f 为**实值函数**.

- 下面对定义中的符号 (+) 和 (−) 进行说明，即

$$(+) \coloneqq \{ j \mid f(t_j) \ge f(t_{j-1}) \} \tag{4.92}$$

$$(-) \coloneqq \{ j \mid f(t_j) \le f(t_{j-1}) \} \tag{4.93}$$

- 常常将**全变差** $V_f([a, b])$ 简记为 $V_a^b(f)$.

有界变差函数的刻画 在刻画有界变差函数之前，先来给出一个引理. 它说明了对于**实值有界变差函数** f，其**全变差与正、负变差之间的关系**，以及 f **与正、负变差的关系**.

引理 4.4.2. Suppose f is real-valued and of BV on $[a, b]$. Then for all $x \in [a, b]$, we have

$$f(x) - f(a) = P_f([a, x]) - N_f([a, x])$$

and

$$V_f([a, x]) = P_f([a, x]) + N_f([a, x])$$

证明.

- $f(x) - f(a) = P_f([a, x]) - N_f([a, x])$:

  $\forall \epsilon > 0$, $\exists$ a partition $\mathcal{P}$: $a = t_0 < t_1 < \cdots < t_N = b$, s. t.

$$\left|P_f - \sum_{(+)} f(t_j) - f(t_{j-1})\right| \le \epsilon \text{ and } \left|N_f - \sum_{(-)} -\left[f(t_j) - f(t_{j-1})\right]\right| \le \epsilon \tag{4.94}$$

  Then

$$-\epsilon + \sum_{(+)} f(t_j) - f(t_{j-1}) \le P_f \le \sum_{(+)} f(t_j) - f(t_{j-1}) + \epsilon \tag{4.95}$$

$$-\epsilon + \sum_{(-)} -\left[f(t_j) - f(t_{j-1})\right] \le N_f \le \sum_{(-)} -\left[f(t_j) - f(t_{j-1})\right] + \epsilon \tag{4.96}$$

  Since

$$f(x) - f(a) = \left(\sum_{(+)} f(t_j) - f(t_{j-1})\right) - \left(\sum_{(-)} -\left[f(t_j) - f(t_{j-1})\right]\right) \tag{4.97}$$

  Then

$$P_f - N_f \in [f(x) - f(a) - 2\epsilon, f(x) - f(a) + 2\epsilon] \tag{4.98}$$

$$\Rightarrow \left|(P_f - N_f) - (f(x) - f(a))\right| \le 2\epsilon, \ \forall \epsilon > 0 \tag{4.99}$$

  Since ϵ is arbitrary, letting $\epsilon \to 0$, we get $f(x) - f(a) = P_f - N_f$.

- $V_f([a,x]) = P_f([a,x]) + N_f([a,x])$:

  $\forall$ partition $\mathcal{P}$: $a = t_0 < t_1 < \cdots < t_N = b$, s. t.

$$V_{\mathcal{P}}(f) = \sum_{j=1}^{N} \left| f(t_j) - f(t_{j-1}) \right| = \left(\sum_{(+)} f(t_j) - f(t_{j-1}) \right) + \left(\sum_{(-)} - \left[f(t_j) - f(t_{j-1}) \right] \right) \tag{4.100}$$

  - $V_f([a,x]) \le P_f([a,x]) + N_f([a,x])$:

    分别对右侧两项取上确界 through all partitions, we have

$$\sum_{j=1}^{N} \left| f(t_j) - f(t_{j-1}) \right| \le P_f([a,x]) + N_f([a,x]) \tag{4.101}$$

    再对左侧取上确界 through all partitions, then

$$V_f([a,x]) \le P_f([a,x]) + N_f([a,x]) \tag{4.102}$$

  - $P_f([a,x]) + N_f([a,x]) \le V_f([a,x])$:

    Similarly, 先对左侧取上确界，再对右侧分别取上确界，得到

$$P_f([a,x]) + N_f([a,x]) \le V_f([a,x]) \tag{4.103}$$

  综上，$V_f([a,x]) = P_f([a,x]) + N_f([a,x])$.

$\square$

下面我们说明，任一有界变差函数可差分为两个有界递增函数之差.

定理 4.4.3. A real-valued function $f : [a, b] \to \mathbb{R}$ on $[a, b]$ is of BV

$$\Leftrightarrow \; f \text{ is the difference of two increasing bounded functions.}$$

证明.

- $\Leftarrow$: Suppose $f = f_1 - f_2$, where f_j is increasing and bounded on $[a, b]$, $j = 1, 2$.
  Then by **Example 4.4.2**, f_j is of BV on $[a, b]$, $j = 1, 2$.
  $\forall$ partition $\mathcal{P}$: $a = t_0 < t_1 < \cdots < t_N = b$, since

$$\left|f(t_j) - f(t_{j-1})\right| \le \left|f_1(t_j) - f_1(t_{j-1})\right| + \left|f_2(t_j) - f_2(t_{j-1})\right|, \ \forall j - 1 \sim N \tag{4.104}$$

  Then since both f_1 and f_2 are of BV, $\exists M < \infty$, s. t.

$$\sum_{j=1}^{N} \left|f(t_j) - f(t_{j-1})\right| \le \sum_{j=1}^{N} \left|f_1(t_j) - f_1(t_{j-1})\right| + \sum_{j=1}^{N} \left|f_2(t_j) - f_2(t_{j-1})\right| \le 2M \tag{4.105}$$

  Therefore, f is of BV on $[a, b]$.

- $\Rightarrow$: By **Lemma 4.4.2**, $f(x) - f(a) = P_f([a, x]) - N_f([a, x])$, $\forall x \in [a, b]$.
  It's trivial to show that $P_f([a, x])$ is increasing in x.
  Similarly, we get $N_f([a, x])$ is increasing in x. Therefore

$$f(x) = (P_f([a, x]) + f(a)) - N_f([a, x])$$

  where $P_f([a, x]) + f(a)$ and $N_f([a, x])$ are increasing and bounded.

□

4.4.3 有界变差函数的全变差的性质

在本小节的最后，我们来讨论一下**实值有界变差函数的全变差的性质**.

命题 4.4.1. Let $f \in BV([a,b])$ and be real-valued. Then

(i) $\forall c \in (a,b)$, $V_f([a,b]) = V_f([a,c]) + V_f([c,b])$.

(ii) $V_f([a,x])$ and $U(x) = V_f([a,x]) - f(x)$ are both increasing in x on $[a,b]$.

(iii) $V_f([a,x])$ is continuous at $x_0 \Leftrightarrow f$ is continuous at x_0.

证明.

(i) 不难证明, $\forall c \in (a,b)$,

$$V_f([a,b]) = \sup\left[\sum_{j=1}^{k}\left|f(t_j) - f(t_{j-1})\right| + \sum_{j=k+1}^{N}\left|f(t_j) - f(t_{j-1})\right|\right] \tag{4.106}$$

The sup is taken over all partitions with $a = t_0 < t_1 < \cdots < t_k = c < \cdots < t_N = b$.

Since

$$\sum_{j=0}^{N}\left|f(t_j) - f(t_{j-1})\right| = \sum_{j=1}^{k}\left|f(t_j) - f(t_{j-1})\right| + \sum_{j=k+1}^{N}\left|f(t_j) - f(t_{j-1})\right| \tag{4.107}$$

Then 先对左侧取上确界 over all partitions on $[a,b]$, we have

$$V_f([a,b]) \geq \sum_{j=1}^{k}\left|f(t_j) - f(t_{j-1})\right| + \sum_{j=k+1}^{N}\left|f(t_j) - f(t_{j-1})\right| \tag{4.108}$$

再对左侧两项分别取上确界 over all partitions on $[a,c]$ and $[c,b]$, we have

$$V_f([a,b]) \geq V_f([a,c]) + V_f([c,b]) \tag{4.109}$$

Similarly, 改变两侧取上确界次序，可得

$$V_f([a,b]) \leq V_f([a,c]) + V_f([c,b]) \tag{4.110}$$

Therefore, $V_f([a,b]) = V_f([a,c]) + V_f([c,b])$, $\forall c \in (a,b)$.

(ii) Since $P_f([a,x])$ and $N_f([a,x])$ are both increasing, then by **Lemma 4.4.2**,

$$V_f([a,x]) = P_f([a,x]) + N_f([a,x]) \text{ is increasing in } x \text{ on } [a,b].$$

$\forall x \geq y$, we have

$$V(x) - V(y) = V_f([a,x]) - V_f([a,y]) \tag{4.111}$$

$$= V_y^x(f) \geq |f(x) - f(y)| \geq f(x) - f(y) \tag{4.112}$$

Therefore $V(x) - f(x) \geq V(y) - f(y)$, $\forall x \geq y$. i.e.

$$U(x) \geq U(y), \forall x \geq y$$

(iii) • $\Rightarrow$: 根据 **(ii)** 的证明过程 (式 (4.111)), we get

$$|V(x) - V(y)| \geq V(x) - V(y) \geq |f(x) - f(y)|, \forall x, y \in [a,b]$$

Suppose $V(x)$ is continuous at x_0, then $\forall \epsilon > 0$, $\exists \delta > 0$, s. t.

$$|f(x) - f(x_0)| \leq |V(x) - V(x_0)| \leq \epsilon, \forall\, |x - x_0| < \delta$$

Then $f(x)$ is continuous at x_0.

• $\Leftarrow$: Suppose $f(x)$ is continuous at x_0. Fix $\epsilon > 0$, then $\exists \delta_0 > 0$, s. t.

$$|f(x_0 + h) - f(x)| < \tfrac{\epsilon}{2}, \forall\, |h| < \delta_0$$

It suffices to show

$$|V(x_0 + h) - V(x)| < \epsilon, \forall \text{ small } h$$

考虑全变差 $V_a^{x_0}(f)$ and $V_{x_0}^b(f)$, for fixed $\epsilon > 0$, $\exists$ partitions

$$a = t_0 < t_1 < \cdots < t_n = x_0 \tag{4.113}$$

$$x_0 = s_0 < s_1 < \cdots < s_m = b \tag{4.114}$$

s. t.

$$\left| V_f([a, x_0]) - \sum_{j=1}^{n} \left| f(t_j) - f(t_{j-1}) \right| \right| < \frac{\epsilon}{2} \tag{4.115}$$

$$\left| V_f([x_0, b]) - \sum_{j=1}^{m} \left| f(s_j) - f(s_{j-1}) \right| \right| < \frac{\epsilon}{2} \tag{4.116}$$

Now update h with $|h| < h_0 = \min\{\delta_0, x_0 - t_{n-1}, s_1 - x_0\}$.
下面先对 $h > 0$ 的情况讨论.

$$
\begin{aligned}
V(x_0+h) - V(x_0) &= V_{x_0}^b(f) - V_{x_0+h}^b(f) && (4.117)\\
&\le \sum_{j=1}^{m} \left| f(t_j) - f(t_{j-1}) \right| + \frac{\epsilon}{2} - V_{x_0+h}^b(f) && (4.118)\\
&= |f(s_0) - f(s_1)| + \sum_{j=2}^{m} \left| f(t_j) - f(t_{j-1}) \right| + \frac{\epsilon}{2} - V_{x_0+h}^b(f) && (4.119)\\
&\le |f(x_0+h) - f(x_0)| + |f(s_1) - f(x_0+h)| + \sum_{j=2}^{m} \left| f(t_j) - f(t_{j-1}) \right| + \frac{\epsilon}{2} - V_{x_0+h}^b(f) && (4.120)
\end{aligned}
$$

Since $x_0 + h < s_1 < \cdots < s_m = b$ is a partition of $[x_0 + h, b]$, then

$$
|f(s_1) - f(x_0+h)| + \sum_{j=2}^{m} \left| f(t_j) - f(t_{j-1}) \right| \le V_{x_0+h}^b(f) \tag{4.121}
$$

Therefore

$$
\begin{aligned}
V(x_0+h) - V(x_0) &\le |f(x_0+h) - f(x_0)| + \frac{\epsilon}{2} && (4.122)\\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \ \forall\, |h| < h_0 && (4.123)
\end{aligned}
$$

Similarly, 对于 $h < 0$ 的情况，我们可同样估计 $V(x_0) - V(x_0+h) = V_a^{x_0}(f) - V_a^{x_0+h}(f)$, 从而得出结论.
综上, $V(x)$ is continuous at x_0.

□