

# *Real Analysis*

*Measure Theory, Integration, & Hilbert Spaces*<sup>1</sup>

–TW–

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<sup>1</sup>参考书籍：

《*Real Analysis – Measure Theory, Integration, & Hilbert Spaces*》— Elias M. Stein

《*Real Analysis – Modern Techniques and Their Applications*》— Gerald B. Folland

# 序

天道几何，万品流形先自守；  
变分无限，孤心测度有同伦。

2024 年 4 月 13 日

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# 目录

第一章	<i>Measure Theory</i>	1
1.1	<i>Preliminaries</i> . . . . .	1
1.2	<i>The Exterior Measure</i> . . . . .	4
1.3	<i>Measurable sets and the Lebesgue measure</i> . . . . .	9
1.3.1	<i>Measurable sets</i> . . . . .	9
1.3.2	<i>Lebesgue measure</i> . . . . .	13
1.4	<i><math>\sigma</math> – algebras and Borel sets</i> . . . . .	17
1.4.1	<i><math>\sigma</math> – algebra</i> . . . . .	17
1.4.2	<i>Borel sets</i> . . . . .	18
1.5	<i>Non – measurable sets</i> . . . . .	19
第二章	<i>Measurable Functions</i>	22
2.1	<i>Measurable Functions</i> . . . . .	22
2.2	<i>Measurable functions are nearly simple</i> . . . . .	27
第三章	<i>Integration Theory</i>	33
3.1	<i>The Lebesgue integral</i> . . . . .	33
3.1.1	<i>Simple functions</i> . . . . .	33
3.1.2	<i>Non – negative measurable functions</i> . . . . .	39
3.1.3	<i>General case</i> . . . . .	49

# 第一章 *Measure Theory*

## 1.1 Preliminaries

定义 1.1.1. A (closed) **rectangle**  $R$  in  $\mathbb{R}^d$  is given by of  $d$  one-dimensional closed and bounded intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d] \quad (1.1)$$

where  $a_j \leq b_j$  are real numbers,  $j = 1, 2, \cdots, d$ . In other word, we have

$$R = \{(x_1, \cdots, x_d) \in \mathbb{R}^d \mid a_j \leq x_j \leq b_j, \forall j = 1 \sim d\} \quad (1.2)$$

The **volume** of  $R$  is

$$|R| = (b_1 - a_1) \cdots (b_d - a_d) \quad (1.3)$$

An **open** rectangle is the product of open intervals, and **the interior of the rectangle**  $R$  is

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d) \quad (1.4)$$

Also, a **cube** is a rectangle for which  $b_1 - a_1 = \cdots = b_d - a_d$ .

定义 1.1.2. A union of rectangles is said to be **almost disjoint** if the interiors of them are disjoint.

引理 1.1.1. If a rectangle is the almost disjoint union of finitely many rectangles, say  $R = \bigcup_{k=1}^N R_k$ , then

$$|R| = \sum_{k=1}^N |R_k| \quad (1.5)$$

注. 本质上即指的是对于方体的任意的垂直划分可转化为“十字形”划分.

引理 1.1.2. If  $R, R_1, \dots, R_N$  are rectangles, and  $R \subset \bigcup_{k=1}^N R_k$ , then

$$|R| \leq \sum_{k=1}^N |R_k| \quad (1.6)$$

注. 此即对 Lemma 1.1.1 的 slight modification, 即各方体之间不一定再为 almost disjoint.

Now we can give a description of the strcture of open sets in terms of cubes. Begin with the case of  $\mathbb{R}$ .

定理 1.1.3. Every open subset  $O$  of  $\mathbb{R}$  can be written uniquely as countable union of disjoint open intervals.

证明. For each  $x \in O$ , let  $I_x$  be the largest open interval containing  $x$  and contained in  $O$ .

Step 1 : Construct  $I_x$ :

$O$  is open  $\Rightarrow x$  is contained in some small open interval contained in  $O$ .

Let

$$a_x = \inf\{a < x \mid (a, x) \subset O\} \quad (1.7)$$

$$b_x = \sup\{b > x \mid (x, b) \subset O\} \quad (1.8)$$

Let  $I_x = (a_x, b_x)$ , then  $O = \bigcup_{x \in O} I_x$ .

Step 2 : Suppose  $I_x \cap I_y \neq \emptyset$ .

$$I_x \cup I_y \text{ is an open interval s. t. } \begin{cases} x \in I_x \cup I_y \\ I_x \cup I_y \subset O \end{cases}$$

Since  $I_x$  is maximal,  $I_x \cup I_y \subset I_x$ . Similarly,  $I_x \cup I_y \subset I_y$ .

$$\Rightarrow I_x = I_y$$

$$\Rightarrow \text{if } I_x \neq I_y, \text{ then } I_x \cap I_y = \emptyset.$$

$$\Rightarrow Z = \{I_x\}_{x \in O} \text{ is a disjoint famliy of sets.}$$

Step 3 : Since every  $I_x$  contains at least a  $a_x \in \mathbb{Q}$ , construct a map  $f$

$$f : Z \longrightarrow \mathbb{Q} \quad (1.9)$$

$$I_x \longmapsto a_x \quad (1.10)$$

$$f \text{ is an injective. } \Rightarrow \{I_x\}_{x \in O} \text{ is countable. } \Rightarrow O = \bigcup_{j=1}^{\infty} (a_j, b_j).$$

□

**定理 1.1.4.** Every open set  $O$  of  $\mathbb{R}^d$ ,  $d \geq 1$ , can be written as a countable union of almost disjoint closed cubes.

**证明.** Let

$$Q_k := \text{grid of } 2^{-k}\mathbb{Z}^d, \quad k \geq 0 \quad (1.11)$$

$$\underline{A}(O, k) := \{Q \in Q_k \mid Q \subset O\} \quad (1.12)$$

$$\overline{A}(O, k) := \{Q \in Q_k \mid Q \cap O \neq \emptyset\} \quad (1.13)$$

Since  $\forall Q \in \underline{A}(O, k)$ ,  $\exists q \in Q^\circ$ , s. t.  $q \in \mathbb{Q}^d$ ,

According to the Axiom of Choice,  $\exists$  the map  $f_k : \underline{A}(O, k) \longrightarrow \mathbb{Q}^d$ , which is an injection.

Hence  $\underline{A}(O, k)$  is countable.

Let

$$\underline{A}(O) := \bigcup_{k=1}^{\infty} (\underline{A}(O, k) \setminus \underline{A}(O, k-1)) \cup \underline{A}(O, 0) \quad (1.14)$$

Then  $\underline{A}(O)$  is also countable. Similarly define  $\overline{A}(O)$ .

$\forall x \in O$ , let  $\delta_x := \inf\{|y - x| \mid y \notin O\}$ . Since  $O$  is open,  $\Rightarrow \delta_x > 0$ .

$$\exists N_x \in \mathbb{N}, \text{ s. t. } 2^{-k} \sqrt{d} \leq \frac{\delta_x}{2} < \delta_x, \forall k \geq N_x \quad (1.15)$$

$$\Rightarrow \forall Q \in \overline{A}(O, N_x), \text{ s. t. } |s - t| \leq 2^{-N_x} \sqrt{d} < \delta_x, \forall s, t \in Q \quad (1.16)$$

$$\Rightarrow \text{Since } O \subset \overline{A}(O), \exists Q_x \in \overline{A}(O, N_x) \subset \overline{A}(O), \text{ s. t. } x \in Q_x \quad (1.17)$$

$$\Rightarrow x \in Q_x \subset O \quad (1.18)$$

$$\Rightarrow x \in Q_x \in \underline{A}(O, N_x) \subset \underline{A}(O) \quad (1.19)$$

$$\Rightarrow O \subset \underline{A}(O) \quad (1.20)$$

Obviously  $\underline{A}(O) \subset O$ , so

$$O = \underline{A}(O) = \bigcup_{k=1}^{\infty} (\underline{A}(O, k) \setminus \underline{A}(O, k-1)) \cup \underline{A}(O, 0) \quad (1.21)$$

which is a countable union of almost disjoint closed cubes. □

## 1.2 The Exterior Measure

*Definition* The exterior measure attempts to describe the volume of a set  $E$  by approximating it from the outside.

Loosely speaking, the exterior measure  $m_*$  assigns to **any subset of  $\mathbb{R}^d$**  a first notion of size.

定义 1.2.1. If  $E$  is a subset of  $\mathbb{R}^d$ , the exterior measure of  $E$  is

$$m_*(E) := \inf \left\{ \sum_{j=1}^{\infty} |Q_j| \mid E \subset \bigcup_{j=1}^{\infty} Q_j, Q_j \text{ is a closed cube} \right\} \quad (1.22)$$

注. • **Well definition:**  $\forall E \subset \mathbb{R}^d$ ,  $E \subset \bigcup_{n=1}^{\infty} Q_n$ ,  $Q_n = [-n, n]^d \subset \mathbb{R}^d$ , which means  $m_*$  can be defined on every subset of  $\mathbb{R}^d$ .

- It is immediate from the definition that:

For every  $\epsilon > 0$ , there exists a covering  $E \subset \bigcup_{j=1}^{\infty} Q_j$ , s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \quad (1.23)$$

- It is important to note that it would **not suffice** to allow **finite sums** in the definition of  $m_*(E)$ . If one considered only coverings of  $E$  by finite unions of cubes, the quantity is **in general larger** than  $m_*(E)$ .

(In fact, it is defined as the **outer Jordan content**  $J_*(E)$ .)

例 1.2.1. Consider the set  $\mathbb{Q} \cap [0, 1]$ .

- For the outer Jordan content, since it's obvious that  $J_*(\overline{E}) = J_*(E)$ ,  $\forall E \subset \mathbb{R}^d$ ,

$$J_*(\mathbb{Q} \cap [0, 1]) = J_*(\overline{\mathbb{Q} \cap [0, 1]}) = J_*([0, 1]) = 1$$

- For the exterior measure, since  $\mathbb{Q} \cap [0, 1]$  is countable, let  $\mathbb{Q} \cap [0, 1] = \{x_1, x_2, \dots\}$ .

Since for all  $\epsilon > 0$ ,

$$\mathbb{Q} \cap [0, 1] \subset \bigcup_{j=1}^{\infty} [x_j - \frac{\epsilon}{2^j}, x_j + \frac{\epsilon}{2^j}] \quad (1.24)$$

Hence  $m_*(\mathbb{Q} \cap [0, 1]) \leq \epsilon$ . For  $\epsilon$  is arbitrary,  $m_*(\mathbb{Q} \cap [0, 1]) = 0$ .

*Examples* Let's check that whether the exterior measure matches our intuitive idea of volume.

**Example 1. The exterior measure of a point is zero.**

证明. It's clear that a point is a cube with  $a_j = b_j, \forall j = 1 \sim d$  and which covers itself.  $\square$

**Example 2. The exterior measure of a closed cube is equal to its volume.**

证明.

- Let  $Q \subset \mathbb{R}^d$  be a closed cube. Since  $Q \subset Q$ ,  $m_*(Q) \leq |Q|$ .
- Suppose  $Q \subset \bigcup_{j=1}^{\infty} Q_j$  by closed cubes. For fixed  $\epsilon > 0$ ,  $\forall j \in \mathbb{N}$ , choose an open cube  $S_j$ ,

$$\text{s. t. } \begin{cases} S_j \supset Q_j \\ |S_j| = (1 + \epsilon) |Q_j| \end{cases} \quad (1.25)$$

Then  $Q \subset \bigcup_{j=1}^{\infty} S_j$ . Since  $Q$  is compact,  $\exists S_1, \dots, S_n \in \{S_j\}_{j=1}^{\infty}$ , s. t.  $Q \subset \bigcup_{j=1}^n S_j$ .

Therefore, according to Lemma 1.1.2

$$|Q| \leq \sum_{j=1}^n |S_j| = (1 + \epsilon) \sum_{j=1}^n |Q_j| \leq (1 + \epsilon) \sum_{j=1}^{\infty} |Q_j| \quad (1.26)$$

For  $\epsilon > 0$  is arbitrary, we get

$$|Q| \leq \sum_{j=1}^{\infty} |Q_j| \quad (1.27)$$

$$|Q| \leq \inf \sum_{j=1}^{\infty} |Q_j| = m_*(Q) \quad (1.28)$$

$\square$

**Example 3. If  $Q$  is an open cube, then  $m_*(Q) = |Q|$ .**

证明.

- Since  $Q \subset \overline{Q}$ ,  $m_*(Q) \leq |\overline{Q}| = |Q|$ .
- We note that for all closed cubes  $Q_0$  contained in  $Q$ , then  $m_*(Q_0) = |Q_0| \leq m_*(Q)$ .

For fixed  $\epsilon > 0$  which is suffice small, choose a closed cube  $Q_0$  contained in  $Q$  with a volume  $|Q_0| = (1 - \epsilon) |Q|$ , then we have

$$|Q_0| = (1 - \epsilon) |Q| \leq m_*(Q) \quad (1.29)$$

For  $\epsilon$  is arbitrary,  $|Q| \leq m_*(Q)$ .

$\square$



**Example 4. The exterior measure of a rectangle  $R$  is equal to its volume.**

**Example 5.**  $m_*(\mathbb{R}^d) = \infty$ .

**证明.** Since any covering of  $\mathbb{R}^d$  is also a covering of any cube  $Q \subset \mathbb{R}^d$ ,  $m_*(\mathbb{R}^d) \geq m_*(Q)$

$\forall N > 0$ ,  $\exists Q \subset \mathbb{R}^d$ , s. t.  $|Q| > N$ , so  $m_*(\mathbb{R}^d) = \infty$ . □

### Properties

*Observation 1. (Monotonicity)*

If  $E_1 \subset E_2$ , then  $m_*(E_1) \leq m_*(E_2)$ .

*Observation 2. (Countable sub – additivity)*

If  $E \subset \bigcup_{j=1}^{\infty} E_j$ , then  $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$ .

**证明.** For a fixed  $\epsilon > 0$ , for all  $E_j$ , there exists a covering  $\{Q_{jk}\}_{k=1}^{\infty}$ ,  $E \subset \bigcup_{k=1}^{\infty} Q_{jk}$ , s. t.

$$\sum_{k=1}^{\infty} m_*(Q_{jk}) \leq m_*(E_j) + \frac{\epsilon}{2^j} \quad (1.30)$$

Since  $E \subset \bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} Q_{jk}$ ,  $\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} Q_{jk}$  covers  $E$ , then

$$m_*(E) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} m_*(Q_{jk}) \leq \sum_{j=1}^{\infty} m_*(E_j) + \epsilon \quad (1.31)$$

Since  $\epsilon$  is arbitrary,  $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$  □

*Observation 3. If  $E \subset \mathbb{R}^d$ , then  $m_*(E) = \inf \{m_*(O) \mid E \subset O, O \text{ is an open set}\}$ .*

**证明.**

- By monotonicity,  $m_*(E) \leq m_*(O)$ , for all  $O$  covers  $E$ . Then take the infimum.

- For a fixed  $\epsilon > 0$ ,  $\exists$  covering  $E \subset \bigcup_{j=1}^{\infty} Q_j$ , s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \frac{\epsilon}{2} \quad (1.32)$$

For all  $Q_j$ , choose an open set  $\tilde{Q}_j$  containing  $Q_j$  with a volume  $|\tilde{Q}_j| \leq |Q_j| + \frac{\epsilon}{2^{j+1}}$ .

Let  $O = \bigcup_{j=1}^{\infty} \tilde{Q}_j$ , then by Observation 2,

$$m_*(O) \leq \sum_{j=1}^{\infty} m_*(\tilde{Q}_j) = \sum_{j=1}^{\infty} |\tilde{Q}_j| \leq \sum_{j=1}^{\infty} |Q_j| + \frac{\epsilon}{2} \leq m_*(E) + \epsilon \quad (1.33)$$

Since  $\epsilon$  is arbitrary,  $m_*(O) \leq m_*(E)$ , so  $\inf m_*(O) \leq m_*(E)$ .

□

Observation 4.

If  $E = E_1 \cup E_2$ , and  $d(E_1, E_2) > 0$ , then

$$m_*(E) = m_*(E_1) + m_*(E_2) \quad (1.34)$$

证明. For a fixed  $\epsilon > 0$ ,  $\exists$  a covering  $E \subset \bigcup_{j=1}^{\infty} Q_j$ , s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \quad (1.35)$$

Subdivide the cubes  $Q_j$  and assume that  $\text{diam}(Q_j) < \frac{d(E_1, E_2)}{3}$ . Then each  $Q_j$  can intersect at most one of the two sets  $E_1$  or  $E_2$ . Devide  $\{Q_j\}_{j=1}^{\infty}$  into two subsets  $\{Q_j\}_{j \in J_1}$ ,  $\{Q_j\}_{j \in J_2}$ , s. t.

$$E_1 \subset \bigcup_{j \in J_1} Q_j, \quad E_2 \subset \bigcup_{j \in J_2} Q_j \quad (1.36)$$

$J_1$  and  $J_2$  are both countable.  $J_1 \cap J_2 = \emptyset$ . Then

$$m_*(E_1) \leq \sum_{j \in J_1} m_*(Q_j), \quad m_*(E_2) \leq \sum_{j \in J_2} m_*(Q_j) \quad (1.37)$$

Therefore

$$m_*(E_1) + m_*(E_2) \leq \sum_{j \in J_1} m_*(Q_j) + \sum_{j \in J_2} m_*(Q_j) \leq \sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \quad (1.38)$$

Since  $\epsilon$  is arbitrary,  $m_*(E_1) + m_*(E_2) \leq m_*(E)$ .

□

*Observation 5.* If a set  $E$  is the countable union of almost disjoint cubes  $E = \bigcup_{j=1}^{\infty} Q_j$ , then

$$m_*(E) = \sum_{j=1}^{\infty} |Q_j| \quad (1.39)$$

**证明.** For a fixed  $\epsilon > 0$ , for all  $Q_j$ , choose a closed cube  $\widetilde{Q}_j$  strictly contained in  $Q_j$  with its volume  $|\widetilde{Q}_j| \geq |Q_j| - \frac{\epsilon}{2^j}$ . Then for every  $N \in \mathbb{N}$ , the cubes  $\widetilde{Q}_1, \dots, \widetilde{Q}_N$  are disjoint with a finite distance from one another. By Observation 4,

$$m_*\left(\bigcup_{j=1}^N \widetilde{Q}_j\right) = \sum_{i=1}^N |\widetilde{Q}_i| \geq \sum_{j=1}^N |Q_j| - \epsilon \quad (1.40)$$

Since  $\bigcup_{j=1}^{\infty} \widetilde{Q}_j \subset E$ , we conclude that for every  $N$

$$m_*(E) \geq \sum_{j=1}^N |Q_j| - \epsilon \quad (1.41)$$

Let  $N \rightarrow \infty$ , we deduce

$$m_*(E) \geq \sum_{j=1}^{\infty} |Q_j| - \epsilon \quad (1.42)$$

Since  $\epsilon$  is arbitrary,  $\sum_{j=1}^{\infty} |Q_j| \leq m_*(E)$ . □

## 1.3 Measurable sets and the Lebesgue measure

### 1.3.1 Measurable sets

#### Definition

**定义 1.3.1.** A subset  $E$  of  $\mathbb{R}^d$  is (Lebesgue) measurable, if for any  $\epsilon > 0$  there exists an open set  $O$  with  $E \subset O$  and  $m_*(O \setminus E) \leq \epsilon$ .

If  $E$  is measurable, we define its (Lebesgue) measurable  $m(E)$  by  $m(E) = m_*(E)$ .

**注.** • 可用映射的观点来理解外测度  $m_*$  与测度  $m$  的关系 (Folland). 即

$$m_* : \mathcal{P}(\mathbb{R}^d) \longrightarrow \overline{\mathbb{R}}_+ = [0, +\infty] \quad (1.43)$$

$$m : \mathcal{M} \longrightarrow \overline{\mathbb{R}}_+ = [0, +\infty] \quad (1.44)$$

$$m = m_* \Big|_{\mathcal{M}} \quad (1.45)$$

其中  $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^d)$  为  $\mathbb{R}^d$  中所有 (Lebesgue) measurable sets 构成的集合.

- 类比于抽象代数中各代数结构的性质, 比如群 (group) 对加法 / 乘法封闭, 我们下面探讨集合族  $\mathcal{M}$  对于可数个集合的运算 (countable unions, countable intersections, complement) 是否封闭. 即通过此引出代数结构  $\sigma$ -algebra.

**Properties** 下面开始探讨 (Lebesgue) measure 的部分性质.

Property 1. Every open set in  $\mathbb{R}^d$  is measurable.

Property 2. If  $m_*(E) = 0$ , then  $E$  is measurable.

**证明.** By Observation 3 in §1.2, for a fixed  $\epsilon > 0$ ,  $\exists E \subset O$  open, s. t.

$$m_*(O) \leq m_*(E) + \epsilon = \epsilon \quad (1.46)$$

Since  $O \setminus E \subset O$ , then  $m_*(O \setminus E) \leq m_*(O) \leq \epsilon$ . □

Property 3. Let  $\{E_j\}_{j=1}^{\infty}$  be a family of measurable sets, then  $\bigcup_{j=1}^{\infty} E_j$  is measurable.

**注.** 即说明集合族  $\mathcal{M}$  对 *countable unions* 封闭.

**证明.** Since  $E_j$  is measurable, for a fixed  $\epsilon > 0$ ,  $\exists E_j \subset O_j$  open, s. t.

$$m_*(O_j \setminus E_j) \leq \frac{\epsilon}{2^j} \quad (1.47)$$

Let  $O = \bigcup_{j=1}^{\infty} O_j \subset \mathbb{R}^d$ , then

$$O \setminus \bigcup_{j=1}^{\infty} E_j = \left( \bigcup_{j=1}^{\infty} O_j \right) \cap \left( \bigcap_{j=1}^{\infty} E_j^c \right) \quad (1.48)$$

$$= \bigcup_{j=1}^{\infty} \left( O_j \cap \left( \bigcap_{k=1}^{\infty} E_k^c \right) \right) \subset \bigcup_{j=1}^{\infty} (O_j \cap E_j^c) = \bigcup_{j=1}^{\infty} (O_j \setminus E_j) \quad (1.49)$$

Therefore

$$m_* \left( O \setminus \bigcup_{j=1}^{\infty} E_j \right) \leq m_* \left( \bigcup_{j=1}^{\infty} (O_j \setminus E_j) \right) \leq \sum_{j=1}^{\infty} m_*(O_j \setminus E_j) \leq \epsilon \quad (1.50)$$

So  $\bigcup_{j=1}^{\infty} E_j$  is measurable. □

Property 4. Closed sets are measurable.

为了证明该性质，先证明如下的分离定理.

**引理 1.3.1.** If  $F$  is closed,  $K$  is compact, and  $K \cap F = \emptyset$ , then  $d(F, K) > 0$ .

**证明.** 反证法. Suppose  $d(F, K) = 0$ , then for any fixed  $n \in \mathbb{N}$ ,  $\exists x_n \in F, y_n \in K$ , s. t.

$$|x_n - y_n| \leq \frac{1}{n} \quad (1.51)$$

Since  $K$  is compact,  $\{y_n\}_{n=1}^{\infty}$  is bounded. Then there exists a subsequence  $\{y_{n_k}\}_{k=1}^{\infty}$ , s. t.

$$y_{n_k} \rightarrow y_0 \in K, \text{ as } k \rightarrow \infty \quad (1.52)$$

Since  $|x_{n_k} - y_{n_k}| \leq \frac{1}{n_k}$ , then

$$|x_{n_k} - y_0| \leq |x_{n_k} - y_{n_k}| + |y_{n_k} - y_0| \rightarrow 0, \text{ as } k \rightarrow \infty \quad (1.53)$$

So  $x_{n_k} \rightarrow y_0 \in F, y_0 \in F \cap K \neq \emptyset$  矛盾. □

下面证明 Property 4.

证明.

- Suppose  $F$  is bounded, then  $F$  is compact.

By Observation 3 in §1.2, for a fixed  $\epsilon > 0$ ,  $\exists F \subset O$  open, s. t.

$$m_*(O) \leq m_*(F) + \epsilon \quad (1.54)$$

Since  $F$  is closed,  $O \setminus F = O \cap F^c$  is open. By Thm1.1.4,  $\exists \{Q_j\}_{j=1}^\infty$ , s. t.

$$O \setminus F = \bigcup_{j=1}^\infty Q_j \quad (1.55)$$

For a fixed  $N \in \mathbb{N}$ , let  $K = \bigcup_{j=1}^N Q_j$ , then  $K$  is compact. By Lemma1.3.1,  $d(K, F) > 0$ .

Since  $K \cup F \subset O$ , by Observation 4 in §1.2,

$$m_*(K) + m_*(F) = m_*(K \cup F) \leq m_*(O) \quad (1.56)$$

So for each fixed  $N \in \mathbb{N}$ ,

$$\sum_{j=1}^N |Q_j| = m_*(K) \leq m_*(O) - m_*(F) \leq \epsilon \quad (1.57)$$

Let  $N \rightarrow \infty$ , we get

$$m_*(O \setminus F) = \sum_{j=1}^\infty |Q_j| \leq \epsilon \quad (1.58)$$

Therefore,  $F$  is measurable.

- For the general situation, since  $\mathbb{R}^d = \bigcup_{j=1}^\infty B_j$ , then

$$F = F \cap \mathbb{R}^d = \bigcup_{j=1}^\infty (F \cap B_j) \quad (1.59)$$

Since  $B_k$  is compact and  $F$  is closed, then  $F \cap B_j$  is compact.

Due to the previous proof,  $F \cap B_j$  is measurable. By Property 3 in §1.3.1,

$$F = \bigcup_{j=1}^\infty (F \cap B_j) \text{ is measurable.} \quad (1.60)$$

□

Property 5. If  $E$  is measurable, then  $E^c$  is measurable.

**注.** 即说明集合族  $\mathcal{M}$  对集合的补运算 *complement* 封闭.

**证明.** Since  $E$  is measurable, then for all fixed  $n \in \mathbb{N}$ ,  $\exists E \subset O_n$  open, s. t.  $m_*(O_n \setminus E) \leq \frac{1}{n}$ .

Let  $S = \bigcup_{j=1}^{\infty} O_j^c \subset E^c$ . Since  $O_j^c$  is closed,  $O_j^c$  is measurable. Then  $S$  is measurable.

Since

$$E^c \setminus S = E^c \cap \left( \bigcap_{j=1}^{\infty} O_j \right) = \bigcap_{j=1}^{\infty} (E^c \cap O_j) \subset E^c \cap O_n = O_n \setminus E, \quad \forall n \in \mathbb{N} \quad (1.61)$$

Then,  $m_*(E^c \setminus S) \leq m_*(O_n \setminus E) \leq \frac{1}{n}$ ,  $\forall n \in \mathbb{N}$ . So  $E^c \setminus S$  is measurable.

Therefore,  $E^c = (E^c \setminus S) \cup S$  is measurable.  $\square$

Property 6. If  $\{E_j\}_{j=1}^{\infty}$  is a family of measurable sets, then  $\bigcap_{j=1}^{\infty} E_j$  is measurable.

**注.** 即说明集合族  $\mathcal{M}$  对 *countable intersections* 封闭.

**证明.** Since

$$\bigcap_{j=1}^{\infty} E_j = \left( \bigcup_{j=1}^{\infty} E_j^c \right)^c \quad (1.62)$$

Then,  $E_j^c$  is measurable and so  $\bigcap_{j=1}^{\infty} E_j$  is measurable.  $\square$

综上, 本节介绍了 (*Lebesgue measurable sets*) 的性质, 并且证明了 *Lebesgue measurable sets* 构成的集合族  $\mathcal{M}$  对 *countable unions*, *countable intersections*, *complement* 运算封闭. 从而  $(\mathcal{M}, \cup, \cap, \text{complement})$  构成代数结构, 即为后续介绍的  *$\sigma$ -algebra*.

### 1.3.2 Lebesgue measure

下面着重来介绍一下 *Lebesgue measure* 的 *properties*.

可数可加性 首先便是可数可加性 *countable additivity*.

定理 1.3.2. If  $E_1, E_2, \dots$  are disjoint measurable sets, then

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} m(E_j) \quad (1.63)$$

证明. Since  $m\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m(E_j)$  always holds, we then proof the reverse inequality.

- Suppose that  $E_j$  is bounded.

Since  $E_j^c$  is measurable, for any fixed  $\epsilon > 0$ , there exists an closed subset  $F_j \subset E_j$ , s. t.

$$m(E_j \setminus F_j) \leq \frac{\epsilon}{2^j} \quad (1.64)$$

Since  $E_j$  is bounded,  $F_j$  is compact.

Let  $K = \bigcup_{j=1}^N F_j$  be a disjoint union of compact sets for some fixed  $N$ , then

$$K \subset \bigcup_{j=1}^{\infty} E_j \quad (1.65)$$

$$m(K) = \sum_{j=1}^N m(F_j) \leq m\left(\bigcup_{j=1}^{\infty} E_j\right) \quad (1.66)$$

Since

$$m(E_j) \leq m(E_j \setminus F_j) + m(F_j) \leq m(F_j) + \frac{\epsilon}{2^j} \quad (1.67)$$

Therefore

$$\sum_{j=1}^N m(E_j) - \epsilon \leq \sum_{j=1}^N m(F_j) \leq m\left(\bigcup_{j=1}^{\infty} E_j\right) \quad (1.68)$$

Let  $N \rightarrow \infty$ , for  $\epsilon$  is arbitrary, we get

$$\sum_{j=1}^{\infty} m(E_j) \leq m\left(\bigcup_{j=1}^{\infty} E_j\right) \quad (1.69)$$



- In the general case, we choose the sequence of cubes  $\{Q_k\}_{k=1}^\infty$ ,  $Q_k = [-k, k]^d \subset \mathbb{R}^d$ .

Let  $S_1 = Q_1$ ,  $S_k = Q_k - Q_{k-1}$ ,  $\forall k \geq 2$ . Then  $\{S_k\}_{k=1}^\infty$  are disjoint and bounded.

Since  $\{S_k\}_{k=1}^\infty$  covers  $\mathbb{R}^d$ ,

$$E_j = \bigcup_{k=1}^\infty (E_j \cap S_k) \quad (1.70)$$

$$\bigcup_{j=1}^\infty E_j = \bigcup_{j=1}^\infty \bigcup_{k=1}^\infty (E_j \cap S_k) \quad (1.71)$$

Since  $E_j \cap S_k$  is bounded and disjoint, by the previous case,

$$m\left(\bigcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty \sum_{k=1}^\infty m(E_j \cap S_k) = \sum_{j=1}^\infty m(E_j) \quad (1.72)$$

□

**单调连续性** 下面我们可以给出单调可测集合列的连续性. *continuity from below/above*

**定理 1.3.3.** Let  $E_1, E_2, \dots$  be measurable sets in  $\mathbb{R}^d$ .

- (i) If  $E_k \nearrow E$ , then  $m(E) = \lim_{n \rightarrow \infty} m(E_n)$ .
- (ii) If  $E_k \searrow E$  and  $m(E_1) < \infty$ , then  $m(E) = \lim_{n \rightarrow \infty} m(E_n)$ .

**注.** • 事实上即可写为

$$m(\lim_{n \rightarrow \infty} E_n) = \lim_{n \rightarrow \infty} m(E_n) \quad (1.73)$$

即单调可测集合列可交换极限与测度顺序.

- (ii) 中条件  $m(E_1)$  finite 不可省略, 下面给出一个反例.

**例 1.3.1.** If  $E_n = (n, +\infty)$ , then  $m(E_n) = \infty$  and  $E = \bigcap_{j=1}^\infty E_j = \emptyset$ . So

$$m(E) = m(\lim_{n \rightarrow \infty} E_j) = 0, \quad \lim_{n \rightarrow \infty} m(E_j) = \infty \quad (1.74)$$

**证明.**

- (i) Let  $S_1 = E_1$ ,  $S_k = E_k - E_{k-1}$ ,  $\forall k \geq 2$ . Then  $\{S_k\}_{k=1}^\infty$  are disjoint and measurable.

Since  $E = \bigcup_{k=1}^\infty E_k = \bigcup_{k=1}^\infty S_k$ , by Thm 1.3.2,

$$m(E) = \sum_{k=1}^\infty m(S_k) = \lim_{N \rightarrow \infty} \sum_{k=1}^N m(S_k) = \lim_{N \rightarrow \infty} m\left(\bigcup_{k=1}^N S_k\right) = \lim_{N \rightarrow \infty} m(E_N) \quad (1.75)$$

(ii) Let  $S_1 = E_1$ ,  $S_k = E_k - E_{k+1}$ ,  $\forall k \geq 2$ . Then  $\{S_k\}_{k=1}^\infty$  are disjoint and measurable.

Since  $E_1 = E \cup \left( \bigcup_{k=1}^\infty S_k \right)$ , then

$$m(E_1) = m(E) + \sum_{k=1}^\infty m(S_k) = m(E) + \lim_{N \rightarrow \infty} m\left(\bigcup_{k=1}^N S_k\right) = m(E) + \lim_{N \rightarrow \infty} m(E_1 - E_N) \quad (1.76)$$

For  $E_1 = (E_1 - E_N) \sqcup E_N$  is a disjoint union,

$$m(E_1 - E_N) = m(E_1) - m(E_N) \quad (1.77)$$

Thus

$$m(E_1) = m(E) + \lim_{N \rightarrow \infty} m(E_1 - E_N) = m(E) + m(E_1) - \lim_{N \rightarrow \infty} m(E_N) \quad (1.78)$$

$$m(E) = \lim_{N \rightarrow \infty} m(E_N) \quad (1.79)$$

□

*Geometric insight of measurable sets* 最后我们来给出 (Lebesgue) measurable sets 的几何性质 (与开集、闭集、紧集等之间的关系).

**定理 1.3.4.** Suppose  $E \subset \mathbb{R}^d$  is measurable, then  $\forall \epsilon > 0$  :

- (i)  $\exists$  open  $O \supset E$  with  $m(O \setminus E) \leq \epsilon$ .
- (ii)  $\exists$  closed  $F \subset E$  with  $m(E \setminus F) \leq \epsilon$ .
- (iii) If  $m(E) < \infty$ ,  $\exists$  compact  $K \subset E$  with  $m(E \setminus K) \leq \epsilon$ .
- (iv) If  $m(E) < \infty$ ,  $\exists F = \bigcup_{j=1}^N Q_j$ ,  $\{Q_j\}_{j=1}^\infty$  are closed cubes, s. t.  $m(E \Delta F) \leq \epsilon$ .

**证明.**

- (i) It's just the definition of measurability.
- (ii) Since  $E_j^c$  is measurable,  $\exists$  open  $O_j \supset E_j^c$ , s. t.

$$m(O_j \setminus E_j^c) \leq \epsilon \quad (1.80)$$

Since  $O_j^c \subset E_j$  is closed and  $E_j \setminus O_j^c = O_j \setminus E_j^c$ , let  $F = O_j^c$  closed, then

$$m(E_j \setminus F) = m(O_j \setminus E_j^c) \leq \epsilon \quad (1.81)$$

(iii) By (ii),  $\exists$  closed  $F \subset E$ , s. t.  $m(E \setminus F) \leq \frac{\epsilon}{2}$ .

Let  $B_n$  denote the closed ball centered at the origin of radius  $n$ , then  $B_n$  is compact.

$$F = \bigcup_{j=1}^{\infty} (F \cap B_k) \quad (1.82)$$

Let  $K_n = \bigcup_{k=1}^n (F \cap B_k)$ , then  $K_n$  is compact and  $K_n \nearrow F \Rightarrow E \setminus K_n \nearrow E \setminus F$ .

Since  $m(E \setminus K_1) \leq m(E)$  is finite, by Thm1.3.3(ii)

$$\lim_{n \rightarrow \infty} m(E \setminus K_n) = m(E \setminus F) \quad (1.83)$$

As for  $\epsilon > 0$ ,  $\exists N \in \mathbb{N}$ , s. t. for all  $n \geq N$

$$|m(E \setminus K_n) - m(E \setminus F)| \leq \frac{\epsilon}{2} \quad (1.84)$$

$$m(E \setminus K_n) \leq m(E \setminus F) + \frac{\epsilon}{2} \leq \epsilon \quad (1.85)$$

Therefore,  $m(E \setminus K_N) \leq \epsilon$ , where  $K_N \subset E$  is compact.

(iv)  $\exists$  open  $O \supset E$ , s. t.  $m(O \setminus E) \leq \frac{\epsilon}{2}$ . By Thm1.1.4,  $\exists \{Q_j\}_{j=1}^{\infty}$ , s. t.

$$E \subset O = \bigcup_{j=1}^{\infty} Q_j \quad (1.86)$$

So

$$m(O) = \sum_{j=1}^{\infty} |Q_j| \leq m(O \setminus E) + m(E) \leq \frac{\epsilon}{2} + m(E) \quad (1.87)$$

Since  $m(E)$  is finite,  $\sum_{j=1}^{\infty} |Q_j|$  converges. Then  $\exists N \in \mathbb{N}$ , s. t.

$$\sum_{j=N+1}^{\infty} |Q_j| \leq \frac{\epsilon}{2} \quad (1.88)$$

Let  $F = \bigcup_{j=1}^N Q_j$ . Since  $E \Delta F = (E \setminus F) \sqcup (F \cap E)$ , then

$$m(E \Delta F) = m(E \setminus F) + m(F \setminus E) \quad (1.89)$$

$$\leq m\left(\bigcup_{j=N+1}^{\infty} Q_j\right) + m\left(\bigcup_{j=1}^{\infty} Q_j \setminus E\right) \quad (1.90)$$

$$= \sum_{j=N+1}^{\infty} |Q_j| + \sum_{j=1}^{\infty} |Q_j| - m(E) \quad (1.91)$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (1.92)$$

□

## 1.4 $\sigma$ – algebras and Borel sets

### 1.4.1 $\sigma$ – algebra

首先给出  $\mathbb{R}^d$  中 *algebra* 的定义.

**定义 1.4.1.** Let  $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^d)$ .  $\mathcal{A}$  is called an algebra if

- (1) If  $A_1, \dots, A_n \in \mathcal{A}$ , then  $\bigcup_{j=1}^n A_j \in \mathcal{A}$ .
- (2) If  $A \in \mathcal{A}$ , then  $A^c \in \mathcal{A}$ .

**注.** 容易证明, 若  $\mathcal{A}$  为  $\mathbb{R}^d$  中 *algebra*, 则其对 *finite intersections* 也封闭, 同时  $\emptyset, \mathbb{R}^d \in \mathcal{A}$ .

下面给出  $\mathbb{R}^d$  中  $\sigma$  – *algebra* 的定义.(将 *algebra* 中的 *finite* 条件加强为 *countable*)

**定义 1.4.2.** Let  $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^d)$ .  $\mathcal{M}$  is a  $\sigma$  – algebra if

- (1) If  $A_1, A_2, \dots \in \mathcal{M}$ , then  $\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$ .
- (2) If  $A \in \mathcal{M}$ , then  $A^c \in \mathcal{M}$ .

**注.** 容易证明  $\mathcal{M}$  对 *countable intersections* 同样封闭,  $\emptyset, \mathbb{R}^d \in \mathcal{M}$ .

**例 1.4.1.** All Lebesgue measurable sets forms a  $\sigma$  – algebra  $\mathcal{M}$ .

类比线性空间、拓扑空间中 (拓扑) 基的概念, 下面给出 **生成  $\sigma$  – algebra** 的概念.

**定义 1.4.3.** Let  $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^d)$ , then the  $\sigma$  – algebra generated by  $\mathcal{A}$  is the smallest  $\sigma$  – algebra containing  $\mathcal{A}$ .

**注.** 即为 the intersection of all  $\sigma$  – *algebras* containing  $\mathcal{A}$ , 这也说明了对于任一给定的集族  $\mathcal{A}$ , 其生成的  $\sigma$  – *algebra* 必存在且唯一.

## 1.4.2 Borel sets

下面给出 *Borel  $\sigma$ -algebra* 及 *Borel sets* 的定义.

**定义 1.4.4.** The Borel  $\sigma$ -algebra is the  $\sigma$ -algebra generated by all open sets in  $\mathbb{R}^d$ , denoted by  $\mathcal{B}_{\mathbb{R}^d}$ .

Elements of this  $\sigma$ -algebra are called Borel sets.

**注.** 事实上, *Borel  $\sigma$ -algebra* 为 Lebesgue countable sets 的一个真子集, 后续会利用 Cantor 集证明.

为了方便研究 *Borel  $\sigma$ -algebra* 的结构, 我们把其中较为复杂 (非平凡) 的元素单独拎出来并称为  $G_\delta, F_\sigma$ .

**定义 1.4.5.** 1. The countable intersections of open sets are called  $G_\delta$  sets.

2. The countable unions of closed sets are called  $F_\sigma$  sets.

下面我们可给出  $\mathcal{B}_{\mathbb{R}^d}$  与 Lebesgue 可测集  $\mathcal{L}$  之间的关系. ( $\mathcal{L}$  只比  $\mathcal{B}_{\mathbb{R}^d}$  多了一些零测集)

**定理 1.4.1.**  $E \subset \mathbb{R}^d$  is  $\mathcal{L}$ -measurable

(i) if and only if  $E = G_\delta \setminus N_1$ , for some  $G_\delta$ ,  $m(N_1) = 0$ .

(ii) if and only if  $E = F_\sigma \setminus N_2$ , for some  $F_\sigma$ ,  $m(N_2) = 0$ .

**证明.** Clearly  $E$  is measurable whenever it satisfies either (i) or (ii).

(i) Since  $E$  is measurable,  $\exists$  open sets  $O_n \supset E$ , s. t.

$$m(O_n \setminus E) \leq \frac{1}{n} \quad (1.93)$$

Let  $O = \bigcap_{j=1}^{\infty} O_j$ , then

$$m(O \setminus E) \leq \frac{1}{n}, \quad \forall n \in \mathbb{N} \quad (1.94)$$

Let  $n \rightarrow \infty$ , we get  $m(O \setminus E) = 0$ . Let  $G_\delta = O$ ,  $N_1 = O \setminus E$ . Then  $E = G_\delta \setminus N_1$ .

(ii) Similarly, we can easily proof it by Thm1.3.4(ii).

□

## 1.5 Non – measurable sets

在这一节我们将介绍  $\mathbb{R}$  上一个经典的不可测集 *Vitali set*, 并说明  $\mathbb{R}$  上每个正测度集都有不可测子集.

**Vitali set** Let  $x, y \in [0, 1]$ . Write  $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$ .

$\Rightarrow$  容易验证  $\sim$  为 an equivalence relation.

$\Rightarrow \sim$  partitions  $[0, 1]$ . 记  $[0, 1]$  上等价类为  $\varepsilon_a$ , 则

$$[0, 1] = \bigsqcup_a \varepsilon_a, \{ \varepsilon_a \}_a \text{ are disjoint} \quad (1.95)$$

$\Rightarrow$  By **the Axiom of Choice**, we can choose exactly one element  $x_a$  from each  $\varepsilon_a$ .

$\Rightarrow$  Let  $\mathcal{N} = \{x_a\}_a$ . Then  $\mathcal{N}$  is the Vitali set.

**定理 1.5.1.**  $\mathcal{N}$  is not measurable.

**证明.** Assume that  $\mathcal{N}$  is measurable. Let  $\{r_k\}_{k=1}^\infty$  be an enumeration of  $\mathbb{Q} \cap [-1, 1]$ .

Define

$$\mathcal{N}_k := \mathcal{N} + r_k = \{x_a + r_k\}_a \quad (1.96)$$

Then we shall proof that  $\{\mathcal{N}_k\}_{k=1}^\infty$  are disjoint, and  $[0, 1] \subset \bigcup_{k=1}^\infty \mathcal{N}_k \subset [-1, 2]$ .

- If  $\mathcal{N}_k \cap \mathcal{N}_m \neq \emptyset$ , then  $\exists x_a, x_\beta \in \mathcal{N}, r_k, r_m \in \mathbb{Q} \cap [-1, 1]$ , s. t.

$$x_a + r_k = x_\beta + r_m \quad (1.97)$$

Then  $x_a - x_\beta = r_m - r_k \in \mathbb{Q} \Rightarrow x_a \sim x_\beta \Rightarrow x_a, x_\beta \in \varepsilon_a$  or  $x_a, x_\beta \in \varepsilon_\beta \Rightarrow x_a = x_\beta$  and  $r_k = r_m$ .

Therefore,  $\mathcal{N}_k = \mathcal{N}_m$ .

- Since  $r_k \in [-1, 1], \mathcal{N}_k \in [-1, 2], \forall k$ . Therefore,

$$\bigcup_{k=1}^\infty \mathcal{N}_k \subset [-1, 2] \quad (1.98)$$

- $\forall x \in [0, 1]$ . Since  $\{\varepsilon_a\}_a$  partitions  $[0, 1]$ , there exists  $a_0$ , s. t.

$$x \in \varepsilon_{a_0}, x \sim x_{a_0} \quad (1.99)$$

which means  $x - x_{a_0} \in \mathbb{Q} \cap [-1, 1]$ . Then  $\exists k_0 \in \mathbb{N}$ , s. t.

$$x - x_{a_0} = r_{k_0} \Rightarrow x \in \mathcal{N}_{k_0} \quad (1.100)$$

Therefore,

$$[0, 1] \subset \bigcup_{k=1}^{\infty} \mathcal{N}_k \quad (1.101)$$

Since  $\{\mathcal{N}_k\}_{k=1}^{\infty}$  are disjoint, we get

$$m([0, 1]) \leq \sum_{k=1}^{\infty} m(\mathcal{N}_k) \leq m([-1, 2]) \quad (1.102)$$

Since  $\mathcal{N}_k$  is a translate of  $\mathcal{N}$ , we have  $m(\mathcal{N}) = m(\mathcal{N}_k)$  for each  $k$ . Then

$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}) \leq 3 \Rightarrow \text{Neither } m(\mathcal{N}) = 0 \text{ nor } m(\mathcal{N}) > 0 \text{ is possible.} \quad (1.103)$$

Therefore, it's a contradiction.  $\mathcal{N}$  is non-measurable.  $\square$

**正测度集必有不可测子集** 下面要证明一个结论, 即  $\mathbb{R}$  上任一正测度集必有不可测子集. 这实际上为书<sup>1</sup>Exercises of Chapter 1 的第 32 题 (b).

**命题 1.5.1.** Let  $\mathcal{N}$  denote the non-measurable subset of  $[0, 1]$  constructed in Thm1.5.1.

(a) If  $E$  is a measurable subset of  $\mathcal{N}$ , then  $m(E) = 0$ .

(b) If  $G \subset \mathbb{R}$  with  $m_*(G) > 0$ , then there exists a subset of  $G$  is non-measurable.

**证明.**

(a) Note  $\mathcal{N} = \{x_\alpha\}_{\alpha \in \mathcal{A}}$ , then  $E = \{x_\beta\}_{\beta \in \mathcal{B} \subset \mathcal{A}}$ . Similarly, we can proof

$$\bigcup_{k=1}^{\infty} E_k \subset [-1, 2] \quad (1.104)$$

Since  $\{E_k\}_{k=1}^{\infty}$  are disjoint, and  $E_k$  is a translate of  $E$ , we get

$$\sum_{k=1}^{\infty} m(E) \leq 3 \Rightarrow m(E) = 0 \quad (1.105)$$

(b) Let  $\mathcal{Q} = \{r_k\}_{k=1}^{\infty}$ ,  $\mathcal{N}_k = \mathcal{N} + r_k$ , then

$$\mathbb{R} = \bigcup_{k=1}^{\infty} \mathcal{N}_k \quad (1.106)$$

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<sup>1</sup>参考书籍: 《Real Analysis – Measure Theory, Integration, & Hilbert Spaces》— Elias M. Stein

Suppose  $G$  is measurable. Then

$$G = G \cap \mathbb{R} = \bigcup_{k=1}^{\infty} (G \cap \mathcal{N}_k) \quad (1.107)$$

If  $G \cap \mathcal{N}_k$  is measurable, then  $G \cap \mathcal{N}_k \subset \mathcal{N}_k$  is a subset of a non-measurable set  $\mathcal{N}_k$ .

By the previous (a), we get

$$m(G \cap \mathcal{N}_k) = 0 \quad (1.108)$$

Therefore, there exists  $k_0 \in \mathbb{N}$ , s. t.  $G \cap \mathcal{N}_{k_0} \subset G$  is a non-measurable subset of  $G$ .

(otherwise  $m(G) = 0$  contradicts)

□



## 第二章 Measurable Functions

### 2.1 Measurable Functions

定义 下面给出  $\mathbb{R}^d$  上可测函数的定义.(注意值域为扩充实数系  $\bar{\mathbb{R}}$ )

定义 2.1.1. A function defined on a measurable subset  $E \subset \mathbb{R}^d$  is measurable if for all  $a \in \mathbb{R}$ ,

$$f^{-1}([-\infty, a)) = \{x \in E \mid f(x) < a\} \quad (2.1)$$

is measurable.

注. •  $f^{-1}([-\infty, a))$  常简记作  $\{f < a\}$ .

• 下面给出几条等价定义.

(1)  $\{f < a\}$  is measurable.  $\Leftrightarrow \{f \leq a\}$  is measurable.

(2)  $\Leftrightarrow \{f > a\}$  is measurable  $\Leftrightarrow \{f \geq a\}$  is measurable.

(3) If  $f$  is finite-valued, then

$$f \text{ is measurable} \Leftrightarrow \{a < f < b\} \text{ is measurable, } \forall a, b \in \mathbb{R} \quad (2.2)$$

证明.

(1) Since the collection of measurable sets is closed under countable intersections and unions,

$$\{f \leq a\} = \bigcap_{n=1}^{\infty} \{f < a + \frac{1}{n}\} \quad (2.3)$$

$$\{f < a\} = \bigcup_{n=1}^{\infty} \{f \leq a - \frac{1}{n}\} \quad (2.4)$$

Therefore,  $\{f < a\}$  is measurable.  $\Leftrightarrow \{f \leq a\}$  is measurable.

(2) Since the collection of measurable sets is closed under complements, easily proof by (1).

(3) Since  $f$  is finite-valued,

$$\{f < a\} = \bigcup_{n=1}^{\infty} \{-n < f < a\} \quad (2.5)$$

$$\{a < f < b\} = \{f > a\} \cap \{f < b\} \quad (2.6)$$

Therefore, by (2),  $f$  is measurable  $\Leftrightarrow \{a < f < b\}$  is measurable.

□

**Property** 下面给出可测函数的一些性质.

**Property 1.** Let  $-\infty < f(x) < +\infty$  (finite-valued), then

$$f \text{ is measurable} \Leftrightarrow f^{-1}(O) \text{ is measurable } \forall \text{ open set } O \quad (2.7)$$

$$\Leftrightarrow f^{-1}(F) \text{ is measurable } \forall \text{ closed set } F \quad (2.8)$$

**证明.**  $\forall O \subset \mathbb{R}$ , there exists  $\{(a_n, b_n)\}_{n=1}^{\infty}$ , s. t.

$$O = \bigcup_{n=1}^{\infty} (a_n, b_n) \quad (2.9)$$

Then

$$f^{-1}(O) = f^{-1}\left(\bigcup_{n=1}^{\infty} (a_n, b_n)\right) = \bigcup_{n=1}^{\infty} f^{-1}((a_n, b_n)) \quad (2.10)$$

Since  $f$  is finite-valued and measurable, then  $f^{-1}(a_n, b_n)$  is measurable.

Therefore,  $f^{-1}(O)$  is measurable.

□

**Property 2.**  $\{\text{continuous functions}\} \subset \{\text{measurable functions}\}$

(a) If  $f$  is continuous on  $\mathbb{R}^d$ , then  $f$  is measurable.

(b) If  $f$  is measurable, finite-valued and  $\Phi$  is continuous on  $\mathbb{R}$ , then  $\Phi \circ f$  is measurable.

**证明.**

(a) Since  $f$  is continuous,  $\forall O \subset \mathbb{R}$ ,  $f^{-1}(O) \subset \mathbb{R}^d$ . By Property 1,  $f$  is measurable.

(b)  $\forall O \subset_{\text{open}} \mathbb{R}$ . Since  $\Phi$  is continuous, then  $\Phi^{-1}(O)$  is open.

Since  $f$  is finite-valued and measurable, then  $(\Phi \circ f)^{-1}(O) = f^{-1}(\Phi^{-1}(O))$  is open.

Therefore, by Property 1,  $\Phi \circ f$  is measurable.

□

**Property 3.** Suppose  $\{f_n\}_{n=1}^{\infty}$  is a sequence of measurable functions. Then

$$\sup_n f_n(x), \inf_n f_n(x), \limsup_{n \rightarrow \infty} f_n(x), \liminf_{n \rightarrow \infty} f_n(x) \quad (2.11)$$

are measurable.

**注.** 类比数列的上下极限, 此处

$$\limsup_{n \rightarrow \infty} f_n(x) := \lim_{k \rightarrow \infty} \sup_{n \geq k} \{f_n(x)\} = \inf_k \sup_{n \geq k} \{f_n(x)\} \quad (2.12)$$

$$\liminf_{n \rightarrow \infty} f_n(x) := \lim_{k \rightarrow \infty} \inf_{n \geq k} \{f_n(x)\} = \sup_k \inf_{n \geq k} \{f_n(x)\} \quad (2.13)$$

**证明.** Since

$$\{x \mid \sup_n f_n(x) > a\} = \bigcup_{n=1}^{\infty} \{x \mid f_n(x) > a\} \quad (2.14)$$

$$\{x \mid \inf_n f_n(x) < a\} = \bigcup_{n=1}^{\infty} \{x \mid f_n(x) < a\} \quad (2.15)$$

Then  $\sup_n f_n(x), \inf_n f_n(x)$  is measurable.

Since  $\sup_{n \geq k} f_n(x), \inf_{n \geq k} f_n(x)$  are measurable, by the previous conclusion, then

$$\limsup_{n \rightarrow \infty} f_n(x) = \inf_k \sup_{n \geq k} \{f_n(x)\} \quad (2.16)$$

$$\liminf_{n \rightarrow \infty} f_n(x) = \sup_k \inf_{n \geq k} \{f_n(x)\} \quad (2.17)$$

are measurable.

□

**Property 4.** If  $\{f_n\}_{n=1}^{\infty}$  is a collection of measurable functions and

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad (2.18)$$

then  $f$  is measurable.

**注.** • 与数列上下极限相同,

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \Leftrightarrow \limsup_{n \rightarrow \infty} f_n(x) = \liminf_{n \rightarrow \infty} f_n(x) = f(x) \quad (2.19)$$

- 此 Property 即说明可测函数列对极限运算封闭. 注意到连续函数列对极限运算并不具备封闭性.(下面给出经典范例)

**例 2.1.1.**

$$\lim_{n \rightarrow \infty} x^n = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases} \quad (2.20)$$

**证明.** Since  $\{f_n\}_{n=1}^{\infty}$  are measurable,  $f(x) = \limsup_{n \rightarrow \infty} f_n(x) = \liminf_{n \rightarrow \infty} f_n(x)$ , then according to Property 3,  $f$  is measurable. □

**Property 5.** If  $f$  and  $g$  are measurable, then

- (i)  $f^k, k \in \mathbb{N}$  are measurable.
- (ii)  $f + g$  and  $fg$  are measurable if both  $f$  and  $g$  are finite-valued.

**证明.**

(i) Since

$$\{f^k > a\} = \{f > a^{\frac{1}{k}}\}, \quad \forall k \text{ is odd} \quad (2.21)$$

$$\{f^k > a\} = \{f > a^{\frac{1}{k}}\} \cup \{f < -a^{\frac{1}{k}}\}, \quad \forall k \text{ is even and } a > 0 \quad (2.22)$$

Therefore,  $f^k, k \in \mathbb{N}$  are measurable.

(ii) Since<sup>1</sup>

$$\{f + g > a\} = \bigcup_{r \in \mathbb{Q}} \{f > a - r\} \cap \{g > r\} \quad (2.23)$$

---

<sup>1</sup>即必  $\exists r \in \mathbb{Q}$ , s. t.  $\{f + g > a\} \supset \{f > a - r\} \cap \{g > r\}$ . (另一侧包含关系  $\subset$  显然易证)

(反证.  $\forall r \in \mathbb{Q}$  上式不成立, 则对于  $r = 0 \in \mathbb{Q}$ ,  $\exists x_0$ , s. t.  $f(x_0) > a$ ,  $g(x_0) > 0$ , 且  $f(x_0) + g(x_0) \leq a$ , 矛盾.)

then  $f + g$  is measurable.

By the previous results in (i) and (ii), since

$$fg = \frac{1}{4}[(f + g)^2 - (f - g)^2] \quad (2.24)$$

Therefore,  $fg$  is also measurable.

□

下面给出数学分析中曾介绍过的几乎处处的定义.

**定义 2.1.2.** A property or statement is said to hold almost everywhere (a.e.) if it is true except on a set of measure zero.

**例 2.1.2.**

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases} \quad (2.25)$$

We say  $f$  is continuous a.e. on  $[0, 1]$  since  $D(f) = \{1\}$  has measure zero.

下面说明几乎处处相等可保持函数可测性.

**命题 2.1.1.** If  $f$  is measurable and  $f = g$  a.e. , then  $g$  is measurable.

**证明.** Since  $f$  is measurable and

$$g = (g - f) + f \quad (2.26)$$

then we shall proof that  $g - f$  is measurable.

Let  $A := \{x \mid g(x) - f(x) \neq 0\}$ , then  $m(A) = 0$ . We get

$$\forall a \geq 0, (g - f)^{-1}((-\infty, a]) = (\mathbb{R}^d \setminus A) \cup N, \text{ where } N \subset A \quad (2.27)$$

Since  $m(A) = 0$ , then  $N$  is measurable and  $m(N) = 0$ . So  $(g - f)^{-1}((-\infty, a])$  is measurable.

Therefore,  $g - f$  is measurable. Then  $g$  is measurable.

□

## 2.2 Measurable functions are nearly simple

本节来介绍一个非常重要的定理. 即可测函数可由简单函数逼近.

**特征函数** 下面先来介绍特征函数的定义.

**定义 2.2.1.** If  $E \subset \mathbb{R}$ , the characteristic / indicator function  $\chi_E / \mathbb{1}_E$  of  $E$  is defined by

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases} \quad (2.28)$$

下面给出可测集与其对应特征函数的关系.

**命题 2.2.1.**  $\chi_E$  is measurable  $\Leftrightarrow E$  is measurable

**证明.** Since

$$\chi_E^{-1}((-\infty, a]) = \begin{cases} \emptyset, & a < 0 \\ E^c, & 0 \leq a < 1 \\ \mathbb{R}^d, & a \geq 1 \end{cases} \quad (2.29)$$

Then  $E$  is measurable  $\Rightarrow \chi_E$  is measurable.

$\chi_E$  is measurable  $\Rightarrow \chi_E^{-1}((-\infty, a]) = E^c$  is measurable.  $\Rightarrow E$  is measurable. □

下面给出特征函数的基本性质.

**命题 2.2.2.** [Property].

(1) If  $A \cap B = \emptyset$ , then

$$\chi_{A \cup B} = \max \{\chi_A, \chi_B\} = \chi_A + \chi_B \quad (2.30)$$

(2)  $\chi_{A \cap B} = \min \{\chi_A, \chi_B\} = \chi_A \cdot \chi_B$ .

*Simple functions* 对特征函数做线性组合, 即可得到简单函数.

定义 2.2.2. A simple function on  $\mathbb{R}^d$  is a finite linear combination

$$f(x) = \sum_{j=1}^n a_j \chi_{E_j}(x) \quad (2.31)$$

where each  $E_j$  is measurable and  $m(E_j) < \infty$ .

**注.** 此处定义中并未要求  $\{E_j\}_{j=1}^n$  disjoint. 而事实上这便引出了下面介绍的标准形式.

下面的命题说明了每个简单函数都可写为标准形式 ( $\{E_j\}_{j=1}^n$  disjoint).

命题 2.2.3. Every simple function  $f$  has a standard representaion

$$f = \sum_{k=1}^N a_k \chi_{E_k}, \text{ where } \{E_j\}_{k=1}^N \text{ are disjoint} \quad (2.32)$$

**证明.** Suppose  $f = \sum_{k=1}^N b_k \chi_{E_k}$ ,  $\{E_j\}_{k=1}^N$  may not be disjoint.

Since  $\{E_j\}_{k=1}^N$  is finite, the number of elements of range  $f$  is also finite. Suppose

$$\text{range } f = \{a_1, \dots, a_M\} \quad (2.33)$$

Then let  $F_k = f^{-1}(\{a_k\})$ , then  $\{F_k\}_{k=1}^M$  are disjoint. Therefore, we get the standard representation

$$f = \sum_{k=1}^M a_k \chi_{F_k} \quad (2.34)$$

□

简单函数逼近可测函数 下面给出一个定理, 说明任一可测函数可由简单函数列逼近.

**定理 2.2.1.** Suppose  $f : \mathbb{R}^d \rightarrow [-\infty, \infty]$  is measurable.

Then there exists a sequence  $\{\varphi_n\}$  of simple functions, s. t.

$$0 \leq |\varphi_1| \leq |\varphi_2| \leq \cdots \leq |f| \quad (2.35)$$

$$\lim_{k \rightarrow \infty} \varphi_k(x) = f(x), \text{ for all } x \quad (2.36)$$

and  $\varphi_k \rightarrow f$  uniformly on any set on which  $f$  is bounded.

**证明.** 下面从两方面分类讨论, 即非负函数 & 变号函数,  $f$  有界 & 无界.

(1) 非负函数  $f : \mathbb{R}^d \rightarrow [0, \infty]$ .

1°  $f$  is bounded. Assume  $|f(x)| \leq M$ .

Let<sup>2</sup>

$$E_n^k = f^{-1}\left(\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right), k = 0, \dots, N_n \quad (2.37)$$

$$\varphi_n(x) = \frac{k}{2^n}, \text{ if } x \in E_n^k \quad (2.38)$$

Then

$$\varphi_n(x) = \sum_{k=0}^{N_n} \frac{k}{2^n} \chi_{E_n^k}(x) \quad (2.39)$$

Therefore<sup>3</sup>

$$|\varphi_n(x) - f(x)| \leq \frac{1}{2^n} \rightarrow 0 \text{ (independent of } x) \quad (2.40)$$

$\Rightarrow \varphi_n \rightarrow f$  uniformly.

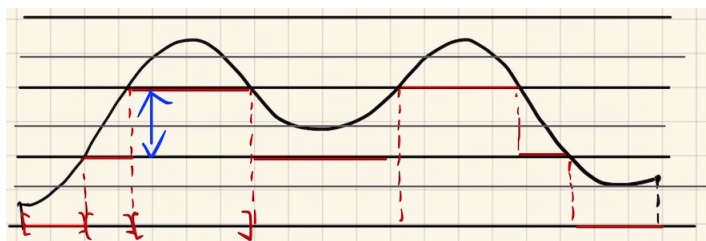


图 2.1: 对  $f$  值域进行分划

<sup>2</sup> $E_n^k$  表示第  $n$  次对值域进行分划后产生的第  $k$  个值域区间, 其中  $\frac{N_n+1}{2^n} \geq M$ .

<sup>3</sup> $|\varphi_n(x) - f(x)|$  小于等于第  $n$  次分划后两个相邻值域区间的步长值, 即  $\frac{1}{2^n}$ .



2°  $f$  is unbounded. (idea: truncation, 将  $f$  截断为一列有界函数列, 并逐点收敛于  $f$ )

Let

$$f_k(x) = \begin{cases} f(x), & \text{if } f(x) \leq k \\ k, & \text{if } f(x) > k \end{cases} \quad (2.41)$$

Then  $f_k(x) \rightarrow f(x), \forall x \in \mathbb{R}^d$ .

Since  $f_k$  is bounded, by the previous result in 1°,

For each  $k, \exists$  a sequence of simple functions  $\{\psi_{kn}\}_{n=1}^{\infty}$ , s. t.

$$\psi_{kn}(x) \rightarrow f_k(x), \forall x \quad (2.42)$$

So we get

$$\begin{array}{ccccccc} \psi_{11} & \psi_{12} & \psi_{13} & \cdots & \rightarrow & f_1 \\ \psi_{21} & \psi_{22} & \psi_{23} & \cdots & \rightarrow & f_2 \\ \psi_{31} & \psi_{32} & \psi_{33} & \cdots & \rightarrow & f_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \downarrow \\ & & & & & f \end{array} \quad (2.43)$$

From the previous results in 1°, we get

$$|\psi_{kn}(x) - f_k(x)| \leq \frac{1}{2^n} \quad (2.44)$$

Let  $n = k$ , then  $|\psi_{kk}(x) - f_k(x)| \leq \frac{1}{2^k}$ . Let  $\varphi_k = \psi_{kk}$ , then

$$|\varphi_k(x) - f(x)| \leq |\varphi_k(x) - f_k(x)| + |f_k(x) - f(x)| \quad (2.45)$$

Since  $f_k(x) \rightarrow f(x)$ , we get  $\varphi_k(x) \rightarrow f(x), \forall x$ , where  $\{\varphi_k = \psi_{kk}\}_{k=1}^{\infty}$  are simple functions.

(2) 变号函数  $f : \mathbb{R}^d \rightarrow [-\infty, \infty]$ .

We denote that

$$f^+(x) := \max\{f(x), 0\} \quad (2.46)$$

$$f^-(x) := \max\{-f(x), 0\} \quad (2.47)$$

By the previous results in (1), there exist sequences of simple functions  $\{\varphi_k\}_{k=1}^{\infty}, \{\psi_k\}_{k=1}^{\infty}$ , s. t.

$$\varphi_k \rightarrow f^+ \text{ and } \psi_k \rightarrow f^- \text{ pointwisely} \quad (2.48)$$

We can observe that  $f = f^+ - f^-$  and  $|f| = f^+ + f^-$ .

Let  $\phi_k(x) = \varphi_k(x) - \psi_k(x)$ , then  $\phi_k$  is a simple function with  $\phi_k \rightarrow f$  pointwisely.

□

**阶梯函数逼近可测函数** 在证明了可测函数可由简单函数逼近后，我们更进一步，来说明可测函数可由更加简单的**阶梯函数**来逼近。

先给出**阶梯函数**的定义。

**定义 2.2.3.** A **step function** is a finite sum

$$f = \sum_{k=1}^N a_k \chi_{R_k}, \text{ where } R_k \text{ is a rectangle} \quad (2.49)$$

**注.** 阶梯函数 & 简单函数的区别在于，简单函数是作用于有限个**可测集**  $E_k$ ，而阶梯函数是作用于有限个**矩形**  $R_k$ 。

下面的定理说明了 measurable functions are almost step functions.

**定理 2.2.2.** Suppose  $f$  is measurable on  $\mathbb{R}^d$ . Then there exists a sequence of step functions  $\{\psi_k\}_{k=1}^\infty$ , s. t.

$$\lim_{k \rightarrow \infty} \psi_k(x) = f(x), \text{ a.e. } x \quad (2.50)$$

**注.** 首先介绍函数列收敛点集的几种不同的等价表述：

$$\{x \mid \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |f_n(x) - f(x)| < \epsilon\} \quad (2.51)$$

$$\Leftrightarrow \{x \mid \forall n \in \mathbb{N}, \exists N \in \mathbb{N}, \forall k \geq N, |f_k(x) - f(x)| < \frac{1}{n}\} \quad (2.52)$$

$$\Leftrightarrow \bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \{x \mid |f_k(x) - f(x)| < \frac{1}{n}\} \quad (2.53)$$

从而可以得到函数列发散点集 (Negation):

$$\{x \mid \exists n \in \mathbb{N}, \forall N \in \mathbb{N}, \exists k \geq N, |f_k(x) - f(x)| \geq \frac{1}{n}\} \quad (2.54)$$

$$\Leftrightarrow \bigcup_{n=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \{x \mid |f_k(x) - f(x)| \geq \frac{1}{n}\} \quad (2.55)$$

$$\Leftrightarrow \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \{x \mid f_k(x) \neq f(x)\} \quad (2.56)$$

**证明.** (证明思路：先用阶梯函数逼近简单函数，再用简单函数逼近可测函数.)

It suffices to show that  $\chi_E$  can be approxiamted by step functions, for any measurable set  $E$ .

According to Thm1.3.4 (iv)

Let  $f = \chi_E$ , then  $\forall \epsilon > 0, \exists$  cubes  $\bigcup_{j=1}^N Q_j$ , s. t.

$$m(E \Delta \bigcup_{j=1}^N Q_j) \leq \epsilon \quad (2.57)$$

By considering the grid formed by extending the sides of these cubes, there exists almost disjoint rectangles  $\{\tilde{R}_j\}_{j=1}^M$ , s. t.

$$\bigcup_{j=1}^N Q_j = \bigcup_{j=1}^M \tilde{R}_j \quad (2.58)$$

By taking rectangles  $R_j$  contained in  $\tilde{R}_j$ , we can find a collection of disjoint rectangles  $\{R_j\}_{j=1}^M$ , s. t.

$$m(E \Delta \bigsqcup_{j=1}^M R_j) \leq 2\epsilon \quad (2.59)$$

For every  $k \in \mathbb{N}$ , there exists disjoint rectangles  $\{R_j\}_{j=1}^M$ , s. t.

$$m(E \Delta \bigsqcup_{j=1}^M R_j) \leq \frac{1}{2^{k+1}} \quad (2.60)$$

There also exists a step function  $\psi_k$

$$\psi_k(x) := \chi_{\bigcup_{j=1}^M R_j}(x) = \sum_{j=1}^M \chi_{R_j}(x) \quad (2.61)$$

Let

$$E_k := \{x \mid f_k(x) \neq f(x)\} \quad (2.62)$$

Since  $E_k \subset E \Delta \bigsqcup_{j=1}^M R_j$ , then  $m(E_k) \leq \frac{1}{2^k}$ . Let<sup>4</sup>

$$F_j = \bigcup_{j=k+1}^{\infty} E_j, \quad F = \bigcap_{k=1}^{\infty} F_k \quad (2.63)$$

Then  $\psi_k(x) \rightarrow f(x), \forall x \in F^c$ . Since

$$m(F) \leq m(F_k), \quad \forall k \in \mathbb{N} \quad (2.64)$$

$$m(F_k) = m\left(\bigcup_{j=k+1}^{\infty} E_j\right) \leq \sum_{j=k+1}^{\infty} m(E_j) \leq \frac{1}{2^k} \quad (2.65)$$

Therefore,  $m(F) = 0$ .  $\lim_{k \rightarrow \infty} \psi_k(x) = f(x)$ , a.e.  $x$ . □

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<sup>4</sup>根据注中式 (2.56),  $F$  即为函数列  $\{\psi_k\}_{k=1}^{\infty}$  的发散点集, 从而  $\psi_k(x) \rightarrow f(x)$  在  $F^c$  上收敛.

## 第三章 *Integration Theory*

### 3.1 *The Lebesgue integral*

*Lebesgue Integral* 的构造可以分为三步, 分别为构造下列函数的积分:

1. **Simple functions**

2. **Non-negative measurable functions**

$$\int f := \sup \left\{ \int \varphi \mid \varphi \text{ simple}, 0 \leq \varphi \leq f \right\} \quad (3.1)$$

3. **General case**

$$f = f^+ - f^- \quad (3.2)$$

$$\int f := \int f^+ - \int f^- \quad (3.3)$$

#### 3.1.1 *Simple functions*

定义 下面先给出非负简单函数在**标准形式**下的积分定义.

定义 3.1.1. If  $\varphi$  is a non-negative simple function with **standard representation**

$$\varphi(x) = \sum_{k=1}^M a_k \chi_{E_k}(x) \quad (3.4)$$

We define the Lebesgue integral of  $\varphi$  by

$$\int_{\mathbb{R}^d} \varphi(x) dx = \sum_{k=1}^M a_k m(E_k) \quad (3.5)$$

If  $E$  is a measurable subset of  $\mathbb{R}^d$  with finite measure, then

$$\varphi(x) \chi_E(x) = \sum_{k=1}^M a_k \chi_{E_k}(x) \chi_E(x) = \sum_{k=1}^M a_k \chi_{E_k \cap E}(x) \quad (3.6)$$

is also a simple function, and define

$$\int_E \varphi(x) dx = \int_{\mathbb{R}^d} \varphi(x) \chi_E(x) dx \quad (3.7)$$

**注.** • 此处仅对**标准形式**定义了积分. 事实上, 此处定义的积分与简单函数的表达式无关 (即**Property 1.**).

- 关于记号, 当测度非常明确时, 大多数情况下可简写, 如

$$\int_E \varphi(x) dx \Rightarrow \int_E \varphi \quad (3.8)$$

$$\int_{\mathbb{R}^d} \varphi(x) dx \Rightarrow \int \varphi \quad (3.9)$$

当为了强调我们选择了何种测度  $\mu$  时, 还可用以下的记号:

$$\int_E \varphi(x) d\mu(x) \quad (3.10)$$

**Property** 下面给出简单函数积分的性质.

**Property 1. Independence of the representation.**

If  $\varphi = \sum_{k=1}^N a_k \chi_{E_k}$  is any representation of  $\varphi$ , then

$$\int \varphi = \sum_{k=1}^N a_k m(E_k) \quad (3.11)$$

在证明这个性质之前, 先来证明一条引理.(书<sup>1</sup>Exercises Of Chapter 2 的第 1 题)

**引理 3.1.1.** Given a collection of sets  $\{F_k\}_{k=1}^n$ , there exists another collection  $\{\widetilde{F}_j\}_{j=1}^N$  with  $N = 2^n - 1$ , so that

$$(i). \quad \bigcup_{k=1}^n F_k = \bigcup_{j=1}^N \widetilde{F}_j \quad (3.12)$$

$$(ii). \quad \{\widetilde{F}_j\}_{j=1}^N \text{ are disjoint} \quad (3.13)$$

$$(iii). \quad F_k = \bigcup_{\widetilde{F}_j \subset F_k} \widetilde{F}_j \quad (3.14)$$

**证明.** Consider the collection

$$\mathcal{F} := \left\{ \bigcup_{k=1}^n G_k - \bigcap_{k=1}^n F_k^c \mid G_k \text{ denotes } F_k \text{ or } F_k^c \right\} \quad (3.15)$$

□

<sup>1</sup>参考书籍: 《Real Analysis – Measure Theory, Integration, & Hilbert Spaces》— Elias M. Stein

下面来证明原命题.

**证明.** According to Lemma 3.1.1, there exists another decompositon of  $\bigcup_{k=1}^N E_k$ , i.e.

$$\bigcup_{j=1}^M \widetilde{E}_j = \bigcup_{k=1}^N E_k \quad (3.16)$$

where  $\{\widetilde{E}_j\}_{j=1}^M$  are disjoint, and for each  $1 \leq k \leq M$ ,

$$E_k = \bigcup_{\widetilde{E}_j \subset E_k} \widetilde{E}_j \quad (3.17)$$

Let

$$\widetilde{a}_j := \sum_{\widetilde{E}_j \subset E_k} a_k \quad (3.18)$$

Then clearly

$$\varphi = \sum_{j=1}^M \widetilde{a}_j \chi_{\widetilde{E}_j} \quad (3.19)$$

Since  $\{\widetilde{E}_j\}_{j=1}^M$  are disjoint, we get

$$\int \varphi = \sum_{j=1}^M \widetilde{a}_j m(\widetilde{E}_j) = \sum_{j=1}^M \sum_{\widetilde{E}_j \subset E_k} a_k m(\widetilde{E}_j) = \sum_{k=1}^N a_k m(E_k) \quad (3.20)$$

□

## Property 2. Linearity.

If  $\varphi$  and  $\psi$  are non-negative simple, and  $a, b \geq 0$ , then

$$\int (a\varphi + b\psi) = a \int \varphi + b \int \psi \quad (3.21)$$

**证明.** 下面分为两步来证明.

(a)  $\forall c \geq 0, \int c\varphi = c \int \varphi$ .

Suppose  $\varphi = \sum_{k=1}^M a_k \chi_{E_k}$ , where  $\{E_k\}_{k=1}^M$  are disjoint. Then

$$c\varphi = \sum_{k=1}^M ca_k \chi_{E_k} \quad (3.22)$$

is also a non-negative simple function. Therefore,

$$\int c\varphi = \sum_{k=1}^M ca_k m(E_k) = c \sum_{k=1}^M a_k m(E_k) = c \int \varphi \quad (3.23)$$

$$(b) \int (\varphi + \psi) = \int \varphi + \int \psi.$$

Suppose

$$\varphi = \sum_{k=1}^M a_k \chi_{E_k}, \quad \psi = \sum_{j=1}^N b_j \chi_{F_j} \quad (3.24)$$

where both  $\{E_k\}_{k=1}^M$  and  $\{F_j\}_{j=1}^N$  are disjoint and  $\mathbb{R}^d = \bigcup_{k=1}^M E_k = \bigcup_{j=1}^N F_j$ . Since

$$E_k = E_k \cap \mathbb{R}^d = E_k \cap \bigcap_{j=1}^N F_j = \bigcap_{j=1}^N (E_k \cap F_j) \quad (3.25)$$

Then

$$\varphi = \sum_{k=1}^M a_k \chi_{E_k} = \sum_{k=1}^M a_k \chi_{\bigcap_{j=1}^N (E_k \cap F_j)} = \sum_{k=1}^M \sum_{j=1}^N a_k \chi_{E_k \cap F_j} \quad (3.26)$$

Similarly

$$\psi = \sum_{j=1}^N b_j \chi_{F_j} = \sum_{j=1}^N b_j \chi_{\bigcup_{k=1}^M (E_k \cap F_j)} = \sum_{j=1}^N \sum_{k=1}^M b_j \chi_{E_k \cap F_j} \quad (3.27)$$

Therefore

$$\varphi + \psi = \sum_{j,k} (a_k + b_j) \chi_{E_k \cap F_j} \quad (3.28)$$

$$\int (\varphi + \psi) = \sum_{j,k} (a_k + b_j) m(E_k \cap F_j) \quad (3.29)$$

$$= \sum_{j,k} a_k m(E_k \cap F_j) + \sum_{j,k} b_j m(E_k \cap F_j) \quad (3.30)$$

$$= \int \varphi + \int \psi \quad (3.31)$$

□

### Property 3. Monotonicity.

If  $\varphi \leq \psi$  are non-negative and simple, then

$$\int \varphi \leq \int \psi \quad (3.32)$$

证明. Suppose

$$\varphi = \sum_{k=1}^M a_k \chi_{E_k}, \quad \psi = \sum_{j=1}^N b_j \chi_{F_j} \quad (3.33)$$

where both  $\{E_k\}_{k=1}^M$  and  $\{F_j\}_{j=1}^N$  are disjoint. Similar to the proof in Property 2, we get

$$\psi - \varphi = \sum_{j,k} (b_j - a_k) \chi_{E_k \cap F_j} \quad (3.34)$$

Since  $\varphi(x) \leq \psi(x)$ ,  $\forall x \in \mathbb{R}^d$ , then  $\psi - \varphi$  is non-negative and simple. Therefore,

$$\int (\psi - \varphi) = \sum_{j,k} (b_j - a_k) m(E_k \cap F_j) \geq 0 \Rightarrow \int \varphi \leq \int \psi \quad (3.35)$$

□

**Property 4. Additivity.**

If  $\{E_k\}_{k=1}^\infty$  are disjoint subsets of  $\mathbb{R}^d$  with finite measure, then

$$\int_{\bigcup_{k=1}^\infty E_k} \varphi = \sum_{k=1}^\infty \int_{E_k} \varphi \quad (3.36)$$

**注.** 首先回顾 *abstract measure* 的定义.

**定义 3.1.2.** Let  $X$  be a set and let  $\mathcal{M}$  be a  $\sigma$ -algebra on  $X$ .

A **measure** on  $\mathcal{M}$  is a function  $\mu : \mathcal{M} \rightarrow [0, \infty]$ , s. t.

(i)  $\mu(\emptyset) = 0$ .

(ii) If  $\{E_j\}_{j=1}^\infty \subset \mathcal{M}$  are disjoint, then

$$\mu\left(\bigcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty \mu(E_j) \quad (3.37)$$

回到我们积分的性质上来. 下面我们将说明, 对于任一给定的非负简单函数  $\varphi$ , 将  $\varphi$  在任一可测集  $A$  上的积分看作 *Lebesgue  $\sigma$ -algebra*  $\mathcal{L}$  上的映射, 则该映射为定义在  $\mathcal{L}$  上的测度.(从而 Property 4. 作为测度的必要条件自然成立)

**命题 3.1.1.** For any fixed non-negative and simple function  $\varphi$ , the map

$$\mu : \mathcal{L} \rightarrow [0, \infty] \quad (3.38)$$

$$A \mapsto \int_A \varphi \quad (3.39)$$

is a measure on  $\mathcal{L}$ .



证明. Suppose  $\{A_j\}_{j=1}^{\infty} \subset \mathcal{L}$  are disjoint, and

$$\varphi = \sum_{k=1}^M a_k \chi_{E_k}, \text{ where } \{E_k\}_{k=1}^M \text{ are disjoint} \quad (3.40)$$

Let  $A = \bigcup_{j=1}^{\infty} A_j$ , then

$$\int_{\bigcup_{j=1}^{\infty} A_j} \varphi = \int_A \varphi = \int \varphi \chi_A = \int \left( \sum_{k=1}^M a_k \chi_{E_k \cap A} \right) \quad (3.41)$$

$$= \sum_{k=1}^M a_k m(E_k \cap A) \quad (3.42)$$

$$= \sum_{k=1}^M a_k m(E_k \cap \left( \bigcup_{j=1}^{\infty} A_j \right)) \quad (3.43)$$

$$= \sum_{k=1}^M a_k m\left( \bigcap_{j=1}^{\infty} (E_k \cap A_j) \right) \quad (3.44)$$

$$= \sum_{k=1}^M a_k \sum_{j=1}^{\infty} m(E_k \cap A_j) \quad (3.45)$$

$$= \sum_{k=1}^M \sum_{j=1}^{\infty} a_k m(E_k \cap A_j) \quad (3.46)$$

Since positive series always converges in  $[0, \infty]$ , then

$$\int_A \varphi = \sum_{k=1}^M \sum_{j=1}^{\infty} a_k m(E_k \cap A_j) = \sum_{j=1}^{\infty} \sum_{k=1}^M a_k m(E_k \cap A_j) = \sum_{j=1}^{\infty} \int_{A_j} \varphi \quad (3.47)$$

Therefore, the integral on any non-negative simple function is actually a measure on  $\mathcal{L}$ .  $\square$

### 3.1.2 Non – negative measurable functions

为了讨论的方便，先给出非负可测函数的一个记号.

$$\mathcal{M}^+ := \{\text{all non – negative measurable functions}\} \quad (3.48)$$

定义 下面给出非负可测函数的积分的定义.

定义 3.1.3. For  $f \in \mathcal{M}^+$ , we define

$$\int f(x)dx := \sup \left\{ \int \varphi(x)dx \mid 0 \leq \varphi \leq f, \varphi \text{ simple} \right\} \quad (3.49)$$

**注.** 此处对 Non-negative measurable function 积分的定义兼容定义 3.1.1 中对 Non-negative simple function 积分的定义，具体表现为：  $\forall \varphi_0$  non-negative and simple,

$$\sup \left\{ \int \varphi(x)dx \mid 0 \leq \varphi \leq \varphi_0, \varphi \text{ simple} \right\} = \int \varphi_0(x)dx \quad (3.50)$$

性质 下面来验证定义 3.1.3 中定义的积分满足几条基本性质.

**Property 1. Monotonicity.**

Let  $f, g \in \mathcal{M}^+$ . Then

$$\int f \leq \int g \text{ if } f \leq g \quad (3.51)$$

证明. Let

$$A = \{\varphi \text{ simple} \mid 0 \leq \varphi \leq f\} \quad (3.52)$$

$$B = \{\psi \text{ simple} \mid 0 \leq \psi \leq g\} \quad (3.53)$$

Then for all  $\varphi \in A$ ,  $0 \leq \varphi \leq f \leq g \Rightarrow \varphi \in B \Rightarrow A \subset B$ . Since

$$\int f = \sup_{\varphi \in A} \left\{ \int \varphi \right\}, \quad \int g = \sup_{\psi \in B} \left\{ \int \psi \right\} \quad (3.54)$$

Therefore

$$\int f \leq \int g \quad (3.55)$$

□

**Property 2. 齐次性.**

Let  $f \in \mathcal{M}^+$ . If  $c \geq 0$ , then

$$\int cf = c \int f \quad (3.56)$$

证明. Assume  $c > 0$ . Then

$$\int cf = \sup \left\{ \int \varphi \mid 0 \leq \varphi \leq cf, \varphi \text{ simple} \right\} \quad (3.57)$$

$$= \sup \left\{ \int \varphi \mid 0 \leq \frac{\varphi}{c} \leq f, \varphi \text{ simple} \right\} \quad (3.58)$$

$$\stackrel{\psi = \frac{\varphi}{c}}{=} \sup \left\{ \int c\psi \mid 0 \leq \psi \leq f, \psi \text{ simple} \right\} \quad (3.59)$$

$$= c \sup \left\{ \int \psi \mid 0 \leq \psi \leq f, \psi \text{ simple} \right\} \quad (3.60)$$

$$= c \int f \quad (3.61)$$

□

**单调收敛定理** 下面我们正式迈入实分析的“大门”，介绍第一个收敛定理.

**定理 3.1.2. The Monotone Convergence Theorem.**

If  $\{f_n\}_{n=1}^\infty \subset \mathcal{M}^+$ ,  $f_j \leq f_{j+1}$  for all  $j$ , and  $\lim_{n \rightarrow \infty} f_n = f$ , then

$$\int f = \lim_{n \rightarrow \infty} \int f_n \quad (3.62)$$

**注.** • 此即为“单调收敛定理”，这个定理说明了对于单调递增的非负可测函数列，其积分与极限可交换次序. 具体表现为

$$\int f = \int \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \int f_n \quad (3.63)$$

- 该定理还说明了，我们可以给出非负可测函数的另一个更自然的等价定义，即用非负简单函数列的积分逼近非负可测函数的积分.

**定义 3.1.4.** For  $f \in \mathcal{M}^+$ , we can also define

$$\int f := \lim_{n \rightarrow \infty} \int \varphi_n \quad (3.64)$$

where  $\varphi_n \rightarrow f$  and  $0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq f$  by Thm 2.2.1.

并且该定理说明了该积分定义的唯一性及 **well-defined**.

在证明定理前, 先来证明一个引理 (将定理 1.3.3 (i) 拓展到一般的抽象测度上).

**引理 3.1.3.** Let  $X$  be a set,  $\mathcal{M}$  be a  $\sigma$ -algebra on  $X$ ,  $\mu : \mathcal{M} \rightarrow [0, \infty]$  be a measure on  $\mathcal{M}$ .

If  $\{E_n\}_{n=1}^\infty \subset \mathcal{M}$ ,  $E_n \nearrow E$ , then

$$\lim_{n \rightarrow \infty} \mu(E_n) = \mu(E) \quad (3.65)$$

**证明.** 证明过程与 Thm 1.3.3 完全一致 (仅用到了测度的可数可加性).

Let  $S_1 = E_1$ ,  $S_k = E_k - E_{k-1}$ ,  $\forall k \geq 2$ . Then  $\{S_k\}_{n=1}^\infty \subset \mathcal{M}$  are disjoint.

Since  $E = \bigcup_{k=1}^\infty E_k = \bigcup_{k=1}^\infty S_k$ , then

$$\mu(E) = \mu\left(\bigcup_{k=1}^\infty S_k\right) = \sum_{k=1}^\infty \mu(S_k) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(S_k) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{k=1}^n S_k\right) = \lim_{n \rightarrow \infty} \mu(E_n) \quad (3.66)$$

□

下面证明原定理.

**证明.**

- $\lim_{n \rightarrow \infty} \int f_n \leq \int f$ .

Since  $f_n \leq f$ ,  $\forall n$ , then

$$\int f_n \leq \int f, \quad \forall n \quad (3.67)$$

Since  $\{\int f_n\}_{n=1}^\infty$  always converges in  $[0, \infty]$ , then let  $n \rightarrow \infty$ , we get

$$\lim_{n \rightarrow \infty} \int f_n \leq \int f \quad (3.68)$$

- $\lim_{n \rightarrow \infty} \int f_n \geq \int f$ .

Fix  $0 < a < 1$ , for any  $0 \leq \varphi \leq f$  simple, let

$$E_n = \{x \mid f_n(x) \geq a\varphi(x)\} \quad (3.69)$$

Then since  $\forall x \in E_n$ , we have  $f_{n+1}(x) \geq f_n(x) \geq a\varphi(x) \Rightarrow x \in E_{n+1} \Rightarrow E_n \subset E_{n+1}$ .

Then  $E_n \nearrow$ . Since

$$\int_{\mathbb{R}^d} f_n \geq \int_{E_n} f_n \geq \int_{E_n} a\varphi, \quad \forall n \quad (3.70)$$

Let  $n \rightarrow \infty$ , we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n \geq \lim_{n \rightarrow \infty} \int_{E_n} a\varphi \quad (3.71)$$

Then we have to calculate  $\lim_{n \rightarrow \infty} \int_{E_n} a\varphi$ :

– Since  $a\varphi$  is non-negative and simple, by Prop 3.1.1, the map

$$\mu : \mathcal{L} \longrightarrow [0, \infty] \quad (3.72)$$

$$E \longmapsto \int_E a\varphi \quad (3.73)$$

is a measure on the collection of Lebesgue measurable sets  $\mathcal{L}$ . (将积分视作测度)

Since  $\{E_n\}_{n=1}^\infty \subset \mathcal{L}$  and  $E_n \nearrow$ , by Lemma 3.1.3, we get

$$\lim_{n \rightarrow \infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^\infty E_n\right) \quad (3.74)$$

i.e.

$$\lim_{n \rightarrow \infty} \int_{E_n} a\varphi = \int_{\bigcup_{n=1}^\infty E_n} a\varphi \quad (3.75)$$

For all  $x \in \mathbb{R}^d$ , since  $a\varphi(x) < f(x)$  and  $f_n \rightarrow f$ , there exists  $N_x \in \mathbb{N}$ , s. t.

$$f_n(x) \geq a\varphi(x), \quad \forall n \geq N_x \quad (3.76)$$

which indicates  $x \in E_{N_x}$  for some  $N_x$ . Therefore

$$\bigcup_{n=1}^\infty E_n = \mathbb{R}^d \Rightarrow \lim_{n \rightarrow \infty} \int_{E_n} a\varphi = \int_{\bigcup_{n=1}^\infty E_n} a\varphi = \int_{\mathbb{R}^d} a\varphi \quad (3.77)$$

Therefore, we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n \geq \lim_{n \rightarrow \infty} \int_{E_n} a\varphi = \int_{\mathbb{R}^d} a\varphi \quad (3.78)$$

Let  $a \rightarrow 1$ , then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n \geq \int_{\mathbb{R}^d} \varphi \quad (3.79)$$

Since  $\varphi$  is arbitrary, taking the supremum over  $\varphi$ , we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n \geq \sup \left\{ \int_{\mathbb{R}^d} \varphi \mid 0 \leq \varphi \leq f, \varphi \text{ simple} \right\} = \int f \quad (3.80)$$

□

函数项级数的可数可加性 接下来我们将给出单调收敛定理在函数项级数上的表达形式，它说明了对于非负可测函数项级数，其积分与求和可交换次序.

在此之前，先来证明有限项的情况.

(此也可视作非负可测函数积分的**Property 线性性**的一部分.)

**命题 3.1.2. Linearity.**

If  $f, g \in \mathcal{M}^+$ , then

$$\int (f + g) = \int f + \int g \quad (3.81)$$

**证明.** By Thm 2.2.1 and Thm 3.1.2, there exists sequences of non-negative and simple functions  $\{\varphi_n\}_{n=1}^{\infty}$  and  $\{\psi_n\}_{n=1}^{\infty}$ ,  $\varphi_n \rightarrow f$  and  $\psi_n \rightarrow g$ , s. t.

$$\int f = \lim_{n \rightarrow \infty} \int \varphi_n, \quad \int g = \lim_{n \rightarrow \infty} \int \psi_n \quad (3.82)$$

Since  $\varphi_n + \psi_n$  is still non-negative and simple, then

By the Linearity of integral on non-negative and simple functions, (**Property 2.** in §3.1.1)

$$\int (\varphi_n + \psi_n) = \int \varphi_n + \int \psi_n \quad (3.83)$$

Let  $n \rightarrow \infty$ , by Thm 3.1.2, we get (极限与积分交换次序)

$$\int (f + g) = \int f + \int g \quad (3.84)$$

□

根据 Prop 3.1.2, 由归纳法, 容易得到其对任意有限项函数项级数都成立.

下面给出函数项级数上的单调收敛定理.

**定理 3.1.4. Monotone Convergence Theorem (MCT , series version).**

If  $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+$  and  $f = \sum_{n=1}^{\infty} f_n$ , then

$$\int f = \sum_{n=1}^{\infty} \int f_n \quad (3.85)$$

**注.** 该定理说明了对于非负可测函数项级数, 其积分与求和可交换次序.

**证明.** Let  $F_n = \sum_{k=1}^n f_k$ , then  $F_n \nearrow \sum_{k=1}^{\infty} f_k = f$ . By **MCT** (Thm 3.1.2),

$$\lim_{n \rightarrow \infty} \int F_n = \int f \quad (3.86)$$

i.e.

$$\lim_{n \rightarrow \infty} \int \sum_{k=1}^n f_k = \int f \quad (3.87)$$

By the **Linearity** of integral on non-negative functions (Prop 3.1.2),

$$\lim_{n \rightarrow \infty} \int \sum_{k=1}^n f_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int f_k = \sum_{k=1}^{\infty} \int f_k = \int f \quad (3.88)$$

□

**积分的唯一性** 在实分析中, 我们并不关心零测集上的各种性质, 进而常常忽略函数在零测集上的情况. 在给出**单调收敛定理**的更一般版本前, 我们先来给出**几乎处处**意义下, 函数积分的唯一性.

下面的命题说明了, 若两个非负可测函数**几乎处处**相等, 则其积分相等.

**命题 3.1.3. Uniqueness.**

If  $f \in \mathcal{M}^+$ , then

$$\int f = 0 \Leftrightarrow f = 0 \text{ a.e.} \quad (3.89)$$

**注.** 根据该命题, 对于任意非负可测函数  $f, g$

$$\int f = \int g \Leftrightarrow \int (f - g) = 0 \Leftrightarrow f - g = 0 \text{ a.e.} \Leftrightarrow f = g \text{ a.e.} \quad (3.90)$$

**证明.**

- 充分性 “ $\Leftarrow$ ” : If  $f = 0$  a.e.

$\forall 0 \leq \varphi \leq f$  simple,  $\varphi = 0$  a.e. . Let  $E = \{x \mid \varphi(x) = 0\}$ , then  $m(E^c) = 0$ .

$$\int \varphi = \int_E \varphi + \int_{E^c} \varphi = 0 + 0 = 0 \quad (3.91)$$

Taking the supremum of  $\varphi$ , we get

$$\int f = \sup \left\{ \int \varphi \mid 0 \leq \varphi \leq f, \varphi \text{ simple} \right\} = 0 \quad (3.92)$$

- 必要性 “ $\Rightarrow$ ” : If  $\int f = 0$ , let

$$E_n := \{x \mid f(x) > \frac{1}{n}\} \quad (3.93)$$

Then

$$\bigcup_{n=1}^{\infty} E_n = \{x \mid f(x) > 0\} = \{f \neq 0\} \quad (3.94)$$

Suppose  $m(\bigcup_{n=1}^{\infty} E_n) > 0$ , then there exists  $N \in \mathbb{N}$ , s. t.  $m(E_N) > 0$ . Then

$$\int f \geq \int_{E_N} f > \frac{1}{N} m(E_N) > 0 \quad (3.95)$$

which is a contradiction to  $\int f = 0$ .

Therefore,  $m(\bigcup_{n=1}^{\infty} E_n) = m(\{f \neq 0\}) = 0, f = 0$  a.e.

□



“几乎处处”版 **MCT** 根据积分的唯一性 (命题 3.1.3), 下面说明在 “几乎处处收敛” 条件下, 单调收敛定理成立 (积分与极限仍可交换次序).

**推论 3.1.5. a.e. MCT.**

If  $\{f_n\}_{n=1}^\infty \subset \mathcal{M}^+, f \in \mathcal{M}^+, f_n \nearrow f$  a.e., then

$$\int f = \lim_{n \rightarrow \infty} \int f_n \quad (3.96)$$

**证明.** Let  $f_n \nearrow f$  on  $E$ , then  $m(E^c) = 0$  and  $f_n - f_n \chi_E = 0$  a.e.

By Prop 3.1.3, we get

$$\int f_n = \int f_n \chi_E \quad (3.97)$$

Since  $f_n \chi_E \nearrow f \chi_E$ , then by **MCT** (Thm 3.1.2, 单调收敛定理)

$$\lim_{n \rightarrow \infty} \int f_n = \lim_{n \rightarrow \infty} \int f_n \chi_E = \int f \chi_E = \int_E f \quad (3.98)$$

Since  $m(E^c) = 0$ , then

$$\int f = \int_E f = \lim_{n \rightarrow \infty} \int f_n \quad (3.99)$$

( $\forall 0 \leq \varphi \leq f$  simple,  $\int \varphi = \int_E \varphi + \int_{E^c} \varphi = \int_E \varphi$ . Taking the supremum of  $\varphi \Rightarrow \int f = \sup \{ \int \varphi \} = \int_E f$ )

□

**Fatou's Lemma** 我们首先来考虑一个问题, 若我们将单调收敛定理 (**MCT**) 中的 “单调” 条件去掉, 结论是否仍然成立 (积分与极限是否仍可交换次序)? 即

Suppose  $f_n \rightarrow f$  a.e., do we have  $\int f_n \rightarrow \int f$ ?

事实上答案为 absolutely no. 下面给出一个反例.

**例 3.1.1.** Consider  $f_n = n \chi_{(0, \frac{1}{n})}$ . Then  $f_n \rightarrow 0$  a.e. on  $[0, 1]$ . However,

$$\int f_n = n \cdot \frac{1}{n} = 1, \quad \forall n \in \mathbb{N} \not\rightarrow 0 \quad (3.100)$$

事实上，将“单调收敛”条件整个去除，我们将得到如下的更一般的 **Fatou's Lemma**.

**定理 3.1.6. Fatou's Lemma.**

If  $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+$ , then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n \quad (3.101)$$

**注.** • 回顾函数列下极限的定义.

$$\liminf_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} f_k \right) \quad (3.102)$$

即对定义域上每一点  $x$ ，取数列  $\{f_n(x)\}_{n=1}^{\infty}$  的下极限，再将所有的  $x$  所对应的下极限拼成一个函数，即定义为函数列  $\{f_n\}_{n=1}^{\infty}$  的下极限.

(上式右侧作用在固定的  $x$  上，即为数列  $\{f_n(x)\}_{n=1}^{\infty}$  下极限的定义.)

- **Fatou's Lemma** 告诉我们，对于任意一列非负可测函数列，其函数列的下极限的积分，要小于每个函数积分后得到的积分数列的下极限.

**证明.** Since

$$\liminf_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} f_k \right) \quad (3.103)$$

Let  $g_n = \inf_{k \geq n} f_k$ , then  $g_n \nearrow \lim_{n \rightarrow \infty} g_n$ . By **MCT** (Thm 3.1.2, 单调收敛定理),

$$\int \lim_{n \rightarrow \infty} g_n = \lim_{n \rightarrow \infty} \int g_n \quad (3.104)$$

i.e.

$$\int \liminf_{n \rightarrow \infty} f_k = \lim_{n \rightarrow \infty} \left( \int \inf_{k \geq n} f_k \right) \quad (3.105)$$

For each  $n$ , since  $\inf_{k \geq n} f_k \leq f_j, \forall j \geq n$ , then

$$\int \inf_{k \geq n} f_k \leq \int f_j, \forall j \geq n \quad (3.106)$$

Taking the infimum of  $\{\int f_j\}_{j=n}^{\infty}$ , then

$$\int \inf_{k \geq n} f_k \leq \inf_{j \geq n} \int f_j, \forall n \in \mathbb{N} \quad (3.107)$$

For  $n$  is arbitrary, let  $n \rightarrow \infty$ , we get

$$\lim_{n \rightarrow \infty} \left( \int \inf_{k \geq n} f_k \right) \leq \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} \int f_k \right) = \liminf_{n \rightarrow \infty} \int f_n \quad (3.108)$$

Therefore

$$\int \liminf_{n \rightarrow \infty} f_k = \lim_{n \rightarrow \infty} \left( \int \inf_{k \geq n} f_k \right) \leq \liminf_{n \rightarrow \infty} \int f_n \quad (3.109)$$

□

### 3.1.3 General case

可积函数 跟 Riemann 积分类似，对于 Lebesgue 积分，我们也有可积函数的概念。

下面先让我们回到非负可测函数，定义非负可测函数中可积的概念。

定义 3.1.5. For  $f \in \mathcal{M}^+$ , if

$$\int f < \infty \quad (3.110)$$

Then we say  $f$  is Lebesgue integrable or simply integrable.

下面扩展到一般的可测函数，给出其 **Lebesgue** 积分及可积的定义。

定义 3.1.6. For any  $f$  measurable on  $\mathbb{R}^d$

$$f^+(x) := \max \{f(x), 0\}, \quad f^-(x) := \max \{-f(x), 0\} \quad (3.111)$$

If at least one of  $\int f^+$  and  $\int f^-$  is finite, we define the integral of  $f$

$$\int f := \int f^+ - \int f^- \quad (3.112)$$

We say that  $f$  is (Lebesgue) integrable if  $|f|$  is integrable.

注. • 注意到

$$f = f^+ - f^- \quad (3.113)$$

$$|f| = f^+ + f^- \quad (3.114)$$

• 根据定义，对于任意可测函数  $f$ ，

$$f \text{ integrable} \Leftrightarrow |f| \text{ integrable} \Leftrightarrow \int |f| = \int f^+ + \int f^- < \infty \quad (3.115)$$

$$\Leftrightarrow f^+ \text{ and } f^- \text{ integrable} \quad (3.116)$$

即  $f$  可积  $\Leftrightarrow \int f^+$  和  $\int f^-$  均有界。

性质 下面我们将说明, 定义在任一集合  $X$  上的实可积函数构成的空间  $\mathcal{L}^1$  为线性空间.

在此之前, 先给出上述定义的一般的可测函数的积分的基本性质.

命题 3.1.4. Suppose  $f, g \in \mathcal{L}$ , then

1. **Linearity:**  $\int (af + bg) = a \int f + b \int g.$

2. **Finite Additivity:**

$$\int_{\bigsqcup_{j=1}^n A_j} f = \sum_{j=1}^n \int_{A_j} f \quad (3.117)$$

where  $\{A_j\}_{j=1}^n$  are disjoint.

3. **Monotonicity:** If  $f \leq g$ , then  $\int f \leq \int g.$

4. **Triangle inequality:**  $|\int f| \leq \int |f|.$

证明.

2. : We shall show that  $\int_{\bigsqcup_{j=1}^n A_j} f^+ = \sum_{j=1}^n \int_{A_j} f^+$  and  $\int_{\bigsqcup_{j=1}^n A_j} f^- = \sum_{j=1}^n \int_{A_j} f^-.$

By **Thm 2.2.1**, there exists simple  $\varphi_n \nearrow f^+$ , then by **MCT (Thm 3.1.2, 单调收敛定理)**,

$$\int_{\bigsqcup_{j=1}^n A_j} f^+ = \lim_{n \rightarrow \infty} \int_{\bigsqcup_{j=1}^n A_j} \varphi_n \quad (3.118)$$

Since  $\varphi_n$  are simple, by the **countable additivity** (简单函数的可数可加性), we have

$$\int_{\bigsqcup_{j=1}^n A_j} f^+ = \lim_{n \rightarrow \infty} \int_{\bigsqcup_{j=1}^n A_j} \varphi_n = \lim_{n \rightarrow \infty} \sum_{j=1}^n \int_{A_j} \varphi_n = \sum_{j=1}^n \lim_{n \rightarrow \infty} \int_{A_j} \varphi_n \quad (3.119)$$

$$\stackrel{\text{MCT}}{=} \sum_{j=1}^n \int_{A_j} f^+ \quad (3.120)$$

4. 根据实数域上的三角不等式, we have

$$\left| \int f \right| = \left| \int f^+ - \int f^- \right| \leq \left| \int f^+ \right| + \left| \int f^- \right| = \int f^+ + \int f^- = \int |f| \quad (3.121)$$

□

现在我们便可以来说明，定义在任一集合  $X$  上的实可积函数构成的空间  $\mathcal{L}^1$  为线性空间。

**命题 3.1.5.** The set of integrable real-valued functions on  $X$  is a real vector space.

**证明.**  $\forall f, g \in \mathcal{L}^1$ , if  $a \in \mathbb{R}$ ,

$$\int |f + g| \leq \int (|f| + |g|) = \int |f| + \int |g| < \infty \quad (3.122)$$

$$\int |af| = |a| \int |f| < \infty \quad (3.123)$$

Therefore,  $f + g, af \in \mathcal{L}^1$ .  $\Rightarrow \mathcal{L}^1$  is a real vector space. □