

Real Analysis

*Measure Theory, Integration, & Hilbert Spaces*¹

–TW–

2024 年 5 月 5 日

¹参考书籍：

《*Real Analysis – Measure Theory, Integration, & Hilbert Spaces*》— Elias M. Stein

《*Real Analysis – Modern Techniques and Their Applications*》— Gerald B. Folland

《实变函数论 (第三版)》— 周民强

序

天道几何，万品流形先自守；
变分无限，孤心测度有同伦。

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长夜伴浪破晓梦，梦晓破浪伴夜长

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第一章 *Measure Theory*

1.1 *Preliminaries*

定义 1.1.1. A (closed) **rectangle** R in \mathbb{R}^d is given by of d one-dimensional closed and bounded intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d] \quad (1.1)$$

where $a_j \leq b_j$ are real numbers, $j = 1, 2, \cdots, d$. In other word, we have

$$R = \{(x_1, \cdots, x_d) \in \mathbb{R}^d \mid a_j \leq x_j \leq b_j, \forall j = 1 \sim d\} \quad (1.2)$$

The **volume** of R is

$$|R| = (b_1 - a_1) \cdots (b_d - a_d) \quad (1.3)$$

An **open** rectangle is the product of open intervals, and **the interior of the rectangle** R is

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d) \quad (1.4)$$

Also, a **cube** is a rectangle for which $b_1 - a_1 = \cdots = b_d - a_d$.

定义 1.1.2. A union of rectangles is said to be **almost disjoint** if the interiors of them are disjoint.

引理 1.1.1. If a rectangle is the almost disjoint union of finitely many rectangles, say $R = \bigcup_{k=1}^N R_k$, then

$$|R| = \sum_{k=1}^N |R_k| \quad (1.5)$$

注. 本质上即指的是对于方体的任意的垂直划分可转化为“十字形”划分.

引理 1.1.2. If R, R_1, \dots, R_N are rectangles, and $R \subset \bigcup_{k=1}^N R_k$, then

$$|R| \leq \sum_{k=1}^N |R_k| \quad (1.6)$$

注. 此即对 Lemma 1.1.1 的 slight modification, 即各方体之间不一定再为 almost disjoint.

Now we can give a description of the strcture of open sets in terms of cubes. Begin with the case of \mathbb{R} .

定理 1.1.3. Every open subset O of \mathbb{R} can be written uniquely as countable union of disjoint open intervals.

证明. For each $x \in O$, let I_x be the largest open interval containing x and contained in O .

Step 1 : Construct I_x :

O is open $\Rightarrow x$ is contained in some small open interval contained in O .

Let

$$a_x = \inf\{a < x \mid (a, x) \subset O\} \quad (1.7)$$

$$b_x = \sup\{b > x \mid (x, b) \subset O\} \quad (1.8)$$

Let $I_x = (a_x, b_x)$, then $O = \bigcup_{x \in O} I_x$.

Step 2 : Suppose $I_x \cap I_y \neq \emptyset$.

$$I_x \cup I_y \text{ is an open interval s. t. } \begin{cases} x \in I_x \cup I_y \\ I_x \cup I_y \subset O \end{cases}$$

Since I_x is maximal, $I_x \cup I_y \subset I_x$. Similarly, $I_x \cup I_y \subset I_y$.

$$\Rightarrow I_x = I_y$$

$$\Rightarrow \text{if } I_x \neq I_y, \text{ then } I_x \cap I_y = \emptyset.$$

$$\Rightarrow Z = \{I_x\}_{x \in O} \text{ is a disjoint famliy of sets.}$$

Step 3 : Since every I_x contains at least a $a_x \in \mathbb{Q}$, construct a map f

$$f : Z \longrightarrow \mathbb{Q} \quad (1.9)$$

$$I_x \longmapsto a_x \quad (1.10)$$

$$f \text{ is an injective. } \Rightarrow \{I_x\}_{x \in O} \text{ is countable. } \Rightarrow O = \bigcup_{j=1}^{\infty} (a_j, b_j).$$

□

定理 1.1.4. Every open set O of \mathbb{R}^d , $d \geq 1$, can be written as a countable union of almost disjoint closed cubes.

证明. Let

$$Q_k := \text{grid of } 2^{-k}\mathbb{Z}^d, \quad k \geq 0 \quad (1.11)$$

$$\underline{A}(O, k) := \{Q \in Q_k \mid Q \subset O\} \quad (1.12)$$

$$\overline{A}(O, k) := \{Q \in Q_k \mid Q \cap O \neq \emptyset\} \quad (1.13)$$

Since $\forall Q \in \underline{A}(O, k)$, $\exists q \in Q^\circ$, s. t. $q \in \mathbb{Q}^d$,

According to the Axiom of Choice, \exists the map $f_k : \underline{A}(O, k) \longrightarrow \mathbb{Q}^d$, which is an injection.

Hence $\underline{A}(O, k)$ is countable.

Let

$$\underline{A}(O) := \bigcup_{k=1}^{\infty} (\underline{A}(O, k) \setminus \underline{A}(O, k-1)) \cup \underline{A}(O, 0) \quad (1.14)$$

Then $\underline{A}(O)$ is also countable. Similarly define $\overline{A}(O)$.

$\forall x \in O$, let $\delta_x := \inf\{|y - x| \mid y \notin O\}$. Since O is open, $\Rightarrow \delta_x > 0$.

$$\exists N_x \in \mathbb{N}, \text{ s. t. } 2^{-k} \sqrt{d} \leq \frac{\delta_x}{2} < \delta_x, \forall k \geq N_x \quad (1.15)$$

$$\Rightarrow \forall Q \in \overline{A}(O, N_x), \text{ s. t. } |s - t| \leq 2^{-N_x} \sqrt{d} < \delta_x, \forall s, t \in Q \quad (1.16)$$

$$\Rightarrow \text{Since } O \subset \overline{A}(O), \exists Q_x \in \overline{A}(O, N_x) \subset \overline{A}(O), \text{ s. t. } x \in Q_x \quad (1.17)$$

$$\Rightarrow x \in Q_x \subset O \quad (1.18)$$

$$\Rightarrow x \in Q_x \in \underline{A}(O, N_x) \subset \underline{A}(O) \quad (1.19)$$

$$\Rightarrow O \subset \underline{A}(O) \quad (1.20)$$

Obviously $\underline{A}(O) \subset O$, so

$$O = \underline{A}(O) = \bigcup_{k=1}^{\infty} (\underline{A}(O, k) \setminus \underline{A}(O, k-1)) \cup \underline{A}(O, 0) \quad (1.21)$$

which is a countable union of almost disjoint closed cubes. □

1.2 The Exterior Measure

Definition The exterior measure attempts to describe the volume of a set E by approximating it from the outside.

Loosely speaking, the exterior measure m_* assigns to **any subset of \mathbb{R}^d** a first notion of size.

定义 1.2.1. If E is a subset of \mathbb{R}^d , the exterior measure of E is

$$m_*(E) := \inf \left\{ \sum_{j=1}^{\infty} |Q_j| \mid E \subset \bigcup_{j=1}^{\infty} Q_j, Q_j \text{ is a closed cube} \right\} \quad (1.22)$$

注. • **Well definition:** $\forall E \subset \mathbb{R}^d$, $E \subset \bigcup_{n=1}^{\infty} Q_n$, $Q_n = [-n, n]^d \subset \mathbb{R}^d$, which means m_* can be defined on every subset of \mathbb{R}^d .

- It is immediate from the definition that:

For every $\epsilon > 0$, there exists a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$, s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \quad (1.23)$$

- It is important to note that it would **not suffice** to allow **finite sums** in the definition of $m_*(E)$. If one considered only coverings of E by finite unions of cubes, the quantity is **in general larger** than $m_*(E)$.

(In fact, it is defined as the **outer Jordan content** $J_*(E)$.)

例 1.2.1. Consider the set $\mathbb{Q} \cap [0, 1]$.

- For the outer Jordan content, since it's obvious that $J_*(\overline{E}) = J_*(E)$, $\forall E \subset \mathbb{R}^d$,

$$J_*(\mathbb{Q} \cap [0, 1]) = J_*(\overline{\mathbb{Q} \cap [0, 1]}) = J_*([0, 1]) = 1$$

- For the exterior measure, since $\mathbb{Q} \cap [0, 1]$ is countable, let $\mathbb{Q} \cap [0, 1] = \{x_1, x_2, \dots\}$.

Since for all $\epsilon > 0$,

$$\mathbb{Q} \cap [0, 1] \subset \bigcup_{j=1}^{\infty} [x_j - \frac{\epsilon}{2^j}, x_j + \frac{\epsilon}{2^j}] \quad (1.24)$$

Hence $m_*(\mathbb{Q} \cap [0, 1]) \leq \epsilon$. For ϵ is arbitrary, $m_*(\mathbb{Q} \cap [0, 1]) = 0$.

Examples Let's check that whether the exterior measure matches our intuitive idea of volume.

Example 1. The exterior measure of a point is zero.

证明. It's clear that a point is a cube with $a_j = b_j, \forall j = 1 \sim d$ and which covers itself. \square

Example 2. The exterior measure of a closed cube is equal to its volume.

证明.

- Let $Q \subset \mathbb{R}^d$ be a closed cube. Since $Q \subset Q$, $m_*(Q) \leq |Q|$.
- Suppose $Q \subset \bigcup_{j=1}^{\infty} Q_j$ by closed cubes. For fixed $\epsilon > 0$, $\forall j \in \mathbb{N}$, choose an open cube S_j ,

$$\text{s. t. } \begin{cases} S_j \supset Q_j \\ |S_j| = (1 + \epsilon) |Q_j| \end{cases} \quad (1.25)$$

Then $Q \subset \bigcup_{j=1}^{\infty} S_j$. Since Q is compact, $\exists S_1, \dots, S_n \in \{S_j\}_{j=1}^{\infty}$, s. t. $Q \subset \bigcup_{j=1}^n S_j$.

Therefore, according to Lemma 1.1.2

$$|Q| \leq \sum_{j=1}^n |S_j| = (1 + \epsilon) \sum_{j=1}^n |Q_j| \leq (1 + \epsilon) \sum_{j=1}^{\infty} |Q_j| \quad (1.26)$$

For $\epsilon > 0$ is arbitrary, we get

$$|Q| \leq \sum_{j=1}^{\infty} |Q_j| \quad (1.27)$$

$$|Q| \leq \inf \sum_{j=1}^{\infty} |Q_j| = m_*(Q) \quad (1.28)$$

\square

Example 3. If Q is an open cube, then $m_*(Q) = |Q|$.

证明.

- Since $Q \subset \overline{Q}$, $m_*(Q) \leq |\overline{Q}| = |Q|$.
- We note that for all closed cubes Q_0 contained in Q , then $m_*(Q_0) = |Q_0| \leq m_*(Q)$.

For fixed $\epsilon > 0$ which is suffice small, choose a closed cube Q_0 contained in Q with a volume $|Q_0| = (1 - \epsilon) |Q|$, then we have

$$|Q_0| = (1 - \epsilon) |Q| \leq m_*(Q) \quad (1.29)$$

For ϵ is arbitrary, $|Q| \leq m_*(Q)$.

\square

Example 4. The exterior measure of a rectangle R is equal to its volume.

Example 5. $m_*(\mathbb{R}^d) = \infty$.

证明. Since any covering of \mathbb{R}^d is also a covering of any cube $Q \subset \mathbb{R}^d$, $m_*(\mathbb{R}^d) \geq m_*(Q)$

$\forall N > 0$, $\exists Q \subset \mathbb{R}^d$, s. t. $|Q| > N$, so $m_*(\mathbb{R}^d) = \infty$. □

Properties

Observation 1. (Monotonicity)

If $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$.

Observation 2. (Countable sub – additivity)

If $E \subset \bigcup_{j=1}^{\infty} E_j$, then $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$.

证明. For a fixed $\epsilon > 0$, for all E_j , there exists a covering $\{Q_{jk}\}_{k=1}^{\infty}$, $E \subset \bigcup_{k=1}^{\infty} Q_{jk}$, s. t.

$$\sum_{k=1}^{\infty} m_*(Q_{jk}) \leq m_*(E_j) + \frac{\epsilon}{2^j} \quad (1.30)$$

Since $E \subset \bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} Q_{jk}$, $\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} Q_{jk}$ covers E , then

$$m_*(E) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} m_*(Q_{jk}) \leq \sum_{j=1}^{\infty} m_*(E_j) + \epsilon \quad (1.31)$$

Since ϵ is arbitrary, $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$ □

Observation 3. If $E \subset \mathbb{R}^d$, then $m_(E) = \inf \{m_*(O) \mid E \subset O, O \text{ is an open set}\}$.*

证明.

- By monotonicity, $m_*(E) \leq m_*(O)$, for all O covers E . Then take the infimum.

- For a fixed $\epsilon > 0$, \exists covering $E \subset \bigcup_{j=1}^{\infty} Q_j$, s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \frac{\epsilon}{2} \quad (1.32)$$

For all Q_j , choose an open set \tilde{Q}_j containing Q_j with a volume $|\tilde{Q}_j| \leq |Q_j| + \frac{\epsilon}{2^{j+1}}$.

Let $O = \bigcup_{j=1}^{\infty} \tilde{Q}_j$, then by Observation 2,

$$m_*(O) \leq \sum_{j=1}^{\infty} m_*(\tilde{Q}_j) = \sum_{j=1}^{\infty} |\tilde{Q}_j| \leq \sum_{j=1}^{\infty} |Q_j| + \frac{\epsilon}{2} \leq m_*(E) + \epsilon \quad (1.33)$$

Since ϵ is arbitrary, $m_*(O) \leq m_*(E)$, so $\inf m_*(O) \leq m_*(E)$.

□

Observation 4.

If $E = E_1 \cup E_2$, and $d(E_1, E_2) > 0$, then

$$m_*(E) = m_*(E_1) + m_*(E_2) \quad (1.34)$$

证明. For a fixed $\epsilon > 0$, \exists a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$, s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \quad (1.35)$$

Subdivide the cubes Q_j and assume that $\text{diam}(Q_j) < \frac{d(E_1, E_2)}{3}$. Then each Q_j can intersect at most one of the two sets E_1 or E_2 . Devide $\{Q_j\}_{j=1}^{\infty}$ into two subsets $\{Q_j\}_{j \in J_1}$, $\{Q_j\}_{j \in J_2}$, s. t.

$$E_1 \subset \bigcup_{j \in J_1} Q_j, \quad E_2 \subset \bigcup_{j \in J_2} Q_j \quad (1.36)$$

J_1 and J_2 are both countable. $J_1 \cap J_2 = \emptyset$. Then

$$m_*(E_1) \leq \sum_{j \in J_1} m_*(Q_j), \quad m_*(E_2) \leq \sum_{j \in J_2} m_*(Q_j) \quad (1.37)$$

Therefore

$$m_*(E_1) + m_*(E_2) \leq \sum_{j \in J_1} m_*(Q_j) + \sum_{j \in J_2} m_*(Q_j) \leq \sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \quad (1.38)$$

Since ϵ is arbitrary, $m_*(E_1) + m_*(E_2) \leq m_*(E)$.

□

Observation 5. If a set E is the countable union of almost disjoint cubes $E = \bigcup_{j=1}^{\infty} Q_j$, then

$$m_*(E) = \sum_{j=1}^{\infty} |Q_j| \quad (1.39)$$

证明. For a fixed $\epsilon > 0$, for all Q_j , choose a closed cube \widetilde{Q}_j strictly contained in Q_j with its volume $|\widetilde{Q}_j| \geq |Q_j| - \frac{\epsilon}{2^j}$. Then for every $N \in \mathbb{N}$, the cubes $\widetilde{Q}_1, \dots, \widetilde{Q}_N$ are disjoint with a finite distance from one another. By Observation 4,

$$m_*\left(\bigcup_{j=1}^N \widetilde{Q}_j\right) = \sum_{i=1}^N |\widetilde{Q}_i| \geq \sum_{j=1}^N |Q_j| - \epsilon \quad (1.40)$$

Since $\bigcup_{j=1}^{\infty} \widetilde{Q}_j \subset E$, we conclude that for every N

$$m_*(E) \geq \sum_{j=1}^N |Q_j| - \epsilon \quad (1.41)$$

Let $N \rightarrow \infty$, we deduce

$$m_*(E) \geq \sum_{j=1}^{\infty} |Q_j| - \epsilon \quad (1.42)$$

Since ϵ is arbitrary, $\sum_{j=1}^{\infty} |Q_j| \leq m_*(E)$. □

1.3 Measurable sets and the Lebesgue measure

1.3.1 Measurable sets

Definition

定义 1.3.1. A subset E of \mathbb{R}^d is (Lebesgue) measurable, if for any $\epsilon > 0$ there exists an open set O with $E \subset O$ and $m_*(O \setminus E) \leq \epsilon$.

If E is measurable, we define its (Lebesgue) measurable $m(E)$ by $m(E) = m_*(E)$.

注. • 可用映射的观点来理解外测度 m_* 与测度 m 的关系 (Folland). 即

$$m_* : \mathcal{P}(\mathbb{R}^d) \longrightarrow \overline{\mathbb{R}}_+ = [0, +\infty] \quad (1.43)$$

$$m : \mathcal{M} \longrightarrow \overline{\mathbb{R}}_+ = [0, +\infty] \quad (1.44)$$

$$m = m_* \Big|_{\mathcal{M}} \quad (1.45)$$

其中 $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^d)$ 为 \mathbb{R}^d 中所有 (Lebesgue) measurable sets 构成的集合.

- 类比于抽象代数中各代数结构的性质, 比如群 (group) 对加法 / 乘法封闭, 我们下面探讨集合族 \mathcal{M} 对于可数个集合的运算 (countable unions, countable intersections, complement) 是否封闭. 即通过此引出代数结构 σ -algebra.

Properties 下面开始探讨 (Lebesgue) measure 的部分性质.

Property 1. Every open set in \mathbb{R}^d is measurable.

Property 2. If $m_*(E) = 0$, then E is measurable.

证明. By Observation 3 in §1.2, for a fixed $\epsilon > 0$, $\exists E \subset O$ open, s. t.

$$m_*(O) \leq m_*(E) + \epsilon = \epsilon \quad (1.46)$$

Since $O \setminus E \subset O$, then $m_*(O \setminus E) \leq m_*(O) \leq \epsilon$. □

Property 3. Let $\{E_j\}_{j=1}^{\infty}$ be a family of measurable sets, then $\bigcup_{j=1}^{\infty} E_j$ is measurable.

注. 即说明集合族 \mathcal{M} 对 *countable unions* 封闭.

证明. Since E_j is measurable, for a fixed $\epsilon > 0$, $\exists E_j \subset O_j$ open, s. t.

$$m_*(O_j \setminus E_j) \leq \frac{\epsilon}{2^j} \quad (1.47)$$

Let $O = \bigcup_{j=1}^{\infty} O_j \subset \mathbb{R}^d$, then

$$O \setminus \bigcup_{j=1}^{\infty} E_j = \left(\bigcup_{j=1}^{\infty} O_j \right) \cap \left(\bigcap_{j=1}^{\infty} E_j^c \right) \quad (1.48)$$

$$= \bigcup_{j=1}^{\infty} \left(O_j \cap \left(\bigcap_{k=1}^{\infty} E_k^c \right) \right) \subset \bigcup_{j=1}^{\infty} (O_j \cap E_j^c) = \bigcup_{j=1}^{\infty} (O_j \setminus E_j) \quad (1.49)$$

Therefore

$$m_* \left(O \setminus \bigcup_{j=1}^{\infty} E_j \right) \leq m_* \left(\bigcup_{j=1}^{\infty} (O_j \setminus E_j) \right) \leq \sum_{j=1}^{\infty} m_*(O_j \setminus E_j) \leq \epsilon \quad (1.50)$$

So $\bigcup_{j=1}^{\infty} E_j$ is measurable. □

Property 4. Closed sets are measurable.

为了证明该性质，先证明如下的分离定理.

引理 1.3.1. If F is closed, K is compact, and $K \cap F = \emptyset$, then $d(F, K) > 0$.

证明. 反证法. Suppose $d(F, K) = 0$, then for any fixed $n \in \mathbb{N}$, $\exists x_n \in F, y_n \in K$, s. t.

$$|x_n - y_n| \leq \frac{1}{n} \quad (1.51)$$

Since K is compact, $\{y_n\}_{n=1}^{\infty}$ is bounded. Then there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$, s. t.

$$y_{n_k} \rightarrow y_0 \in K, \text{ as } k \rightarrow \infty \quad (1.52)$$

Since $|x_{n_k} - y_{n_k}| \leq \frac{1}{n_k}$, then

$$|x_{n_k} - y_0| \leq |x_{n_k} - y_{n_k}| + |y_{n_k} - y_0| \rightarrow 0, \text{ as } k \rightarrow \infty \quad (1.53)$$

So $x_{n_k} \rightarrow y_0 \in F, y_0 \in F \cap K \neq \emptyset$ 矛盾. □

下面证明 Property 4.

证明.

- Suppose F is bounded, then F is compact.

By Observation 3 in §1.2, for a fixed $\epsilon > 0$, $\exists F \subset O$ open, s. t.

$$m_*(O) \leq m_*(F) + \epsilon \quad (1.54)$$

Since F is closed, $O \setminus F = O \cap F^c$ is open. By Thm1.1.4, $\exists \{Q_j\}_{j=1}^\infty$, s. t.

$$O \setminus F = \bigcup_{j=1}^\infty Q_j \quad (1.55)$$

For a fixed $N \in \mathbb{N}$, let $K = \bigcup_{j=1}^N Q_j$, then K is compact. By Lemma1.3.1, $d(K, F) > 0$.

Since $K \cup F \subset O$, by Observation 4 in §1.2,

$$m_*(K) + m_*(F) = m_*(K \cup F) \leq m_*(O) \quad (1.56)$$

So for each fixed $N \in \mathbb{N}$,

$$\sum_{j=1}^N |Q_j| = m_*(K) \leq m_*(O) - m_*(F) \leq \epsilon \quad (1.57)$$

Let $N \rightarrow \infty$, we get

$$m_*(O \setminus F) = \sum_{j=1}^\infty |Q_j| \leq \epsilon \quad (1.58)$$

Therefore, F is measurable.

- For the general situation, since $\mathbb{R}^d = \bigcup_{j=1}^\infty B_j$, then

$$F = F \cap \mathbb{R}^d = \bigcup_{j=1}^\infty (F \cap B_j) \quad (1.59)$$

Since B_k is compact and F is closed, then $F \cap B_j$ is compact.

Due to the previous proof, $F \cap B_j$ is measurable. By Property 3 in §1.3.1,

$$F = \bigcup_{j=1}^\infty (F \cap B_j) \text{ is measurable.} \quad (1.60)$$

□

Property 5. If E is measurable, then E^c is measurable.

注. 即说明集合族 \mathcal{M} 对集合的补运算 *complement* 封闭.

证明. Since E is measurable, then for all fixed $n \in \mathbb{N}$, $\exists E \subset O_n$ open, s. t. $m_*(O_n \setminus E) \leq \frac{1}{n}$.

Let $S = \bigcup_{j=1}^{\infty} O_j^c \subset E^c$. Since O_j^c is closed, O_j^c is measurable. Then S is measurable.

Since

$$E^c \setminus S = E^c \cap \left(\bigcap_{j=1}^{\infty} O_j \right) = \bigcap_{j=1}^{\infty} (E^c \cap O_j) \subset E^c \cap O_n = O_n \setminus E, \quad \forall n \in \mathbb{N} \quad (1.61)$$

Then, $m_*(E^c \setminus S) \leq m_*(O_n \setminus E) \leq \frac{1}{n}$, $\forall n \in \mathbb{N}$. So $E^c \setminus S$ is measurable.

Therefore, $E^c = (E^c \setminus S) \cup S$ is measurable. \square

Property 6. If $\{E_j\}_{j=1}^{\infty}$ is a family of measurable sets, then $\bigcap_{j=1}^{\infty} E_j$ is measurable.

注. 即说明集合族 \mathcal{M} 对 *countable intersections* 封闭.

证明. Since

$$\bigcap_{j=1}^{\infty} E_j = \left(\bigcup_{j=1}^{\infty} E_j^c \right)^c \quad (1.62)$$

Then, E_j^c is measurable and so $\bigcap_{j=1}^{\infty} E_j$ is measurable. \square

综上, 本节介绍了 (*Lebesgue measurable sets*) 的性质, 并且证明了 *Lebesgue measurable sets* 构成的集合族 \mathcal{M} 对 *countable unions*, *countable intersections*, *complement* 运算封闭. 从而 $(\mathcal{M}, \cup, \cap, \text{complement})$ 构成代数结构, 即为后续介绍的 *σ -algebra*.

1.3.2 Lebesgue measure

下面着重来介绍一下 *Lebesgue measure* 的 *properties*.

可数可加性 首先便是可数可加性 *countable additivity*.

定理 1.3.2. If E_1, E_2, \dots are disjoint measurable sets, then

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} m(E_j) \quad (1.63)$$

证明. Since $m\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m(E_j)$ always holds, we then proof the reverse inequality.

- Suppose that E_j is bounded.

Since E_j^c is measurable, for any fixed $\epsilon > 0$, there exists an closed subset $F_j \subset E_j$, s. t.

$$m(E_j \setminus F_j) \leq \frac{\epsilon}{2^j} \quad (1.64)$$

Since E_j is bounded, F_j is compact.

Let $K = \bigcup_{j=1}^N F_j$ be a disjoint union of compact sets for some fixed N , then

$$K \subset \bigcup_{j=1}^{\infty} E_j \quad (1.65)$$

$$m(K) = \sum_{j=1}^N m(F_j) \leq m\left(\bigcup_{j=1}^{\infty} E_j\right) \quad (1.66)$$

Since

$$m(E_j) \leq m(E_j \setminus F_j) + m(F_j) \leq m(F_j) + \frac{\epsilon}{2^j} \quad (1.67)$$

Therefore

$$\sum_{j=1}^N m(E_j) - \epsilon \leq \sum_{j=1}^N m(F_j) \leq m\left(\bigcup_{j=1}^{\infty} E_j\right) \quad (1.68)$$

Let $N \rightarrow \infty$, for ϵ is arbitrary, we get

$$\sum_{j=1}^{\infty} m(E_j) \leq m\left(\bigcup_{j=1}^{\infty} E_j\right) \quad (1.69)$$

- In the general case, we choose the sequence of cubes $\{Q_k\}_{k=1}^\infty$, $Q_k = [-k, k]^d \subset \mathbb{R}^d$.

Let $S_1 = Q_1$, $S_k = Q_k - Q_{k-1}$, $\forall k \geq 2$. Then $\{S_k\}_{k=1}^\infty$ are disjoint and bounded.

Since $\{S_k\}_{k=1}^\infty$ covers \mathbb{R}^d ,

$$E_j = \bigcup_{k=1}^\infty (E_j \cap S_k) \quad (1.70)$$

$$\bigcup_{j=1}^\infty E_j = \bigcup_{j=1}^\infty \bigcup_{k=1}^\infty (E_j \cap S_k) \quad (1.71)$$

Since $E_j \cap S_k$ is bounded and disjoint, by the previous case,

$$m\left(\bigcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty \sum_{k=1}^\infty m(E_j \cap S_k) = \sum_{j=1}^\infty m(E_j) \quad (1.72)$$

□

单调连续性 下面我们可以给出单调可测集合列的连续性. *continuity from below/above*

定理 1.3.3. Let E_1, E_2, \dots be measurable sets in \mathbb{R}^d .

- (i) If $E_k \nearrow E$, then $m(E) = \lim_{n \rightarrow \infty} m(E_n)$.
- (ii) If $E_k \searrow E$ and $m(E_1) < \infty$, then $m(E) = \lim_{n \rightarrow \infty} m(E_n)$.

注. • 事实上即可写为

$$m(\lim_{n \rightarrow \infty} E_n) = \lim_{n \rightarrow \infty} m(E_n) \quad (1.73)$$

即单调可测集合列可交换极限与测度顺序.

- (ii) 中条件 $m(E_1)$ finite 不可省略, 下面给出一个反例.

例 1.3.1. If $E_n = (n, +\infty)$, then $m(E_n) = \infty$ and $E = \bigcap_{j=1}^\infty E_j = \emptyset$. So

$$m(E) = m(\lim_{n \rightarrow \infty} E_j) = 0, \quad \lim_{n \rightarrow \infty} m(E_j) = \infty \quad (1.74)$$

证明.

- (i) Let $S_1 = E_1$, $S_k = E_k - E_{k-1}$, $\forall k \geq 2$. Then $\{S_k\}_{k=1}^\infty$ are disjoint and measurable.

Since $E = \bigcup_{k=1}^\infty E_k = \bigcup_{k=1}^\infty S_k$, by Thm 1.3.2,

$$m(E) = \sum_{k=1}^\infty m(S_k) = \lim_{N \rightarrow \infty} \sum_{k=1}^N m(S_k) = \lim_{N \rightarrow \infty} m\left(\bigcup_{k=1}^N S_k\right) = \lim_{N \rightarrow \infty} m(E_N) \quad (1.75)$$

(ii) Let $S_1 = E_1$, $S_k = E_k - E_{k+1}$, $\forall k \geq 2$. Then $\{S_k\}_{k=1}^\infty$ are disjoint and measurable.

Since $E_1 = E \cup \left(\bigcup_{k=1}^\infty S_k \right)$, then

$$m(E_1) = m(E) + \sum_{k=1}^\infty m(S_k) = m(E) + \lim_{N \rightarrow \infty} m\left(\bigcup_{k=1}^N S_k\right) = m(E) + \lim_{N \rightarrow \infty} m(E_1 - E_N) \quad (1.76)$$

For $E_1 = (E_1 - E_N) \sqcup E_N$ is a disjoint union,

$$m(E_1 - E_N) = m(E_1) - m(E_N) \quad (1.77)$$

Thus

$$m(E_1) = m(E) + \lim_{N \rightarrow \infty} m(E_1 - E_N) = m(E) + m(E_1) - \lim_{N \rightarrow \infty} m(E_N) \quad (1.78)$$

$$m(E) = \lim_{N \rightarrow \infty} m(E_N) \quad (1.79)$$

□

Geometric insight of measurable sets 最后我们来给出 (Lebesgue) measurable sets 的几何性质 (与开集、闭集、紧集等之间的关系).

定理 1.3.4. Lebesgue 测度的正则性.

Suppose $E \subset \mathbb{R}^d$ is measurable, then $\forall \epsilon > 0$:

- (i) \exists open $O \supset E$ with $m(O \setminus E) \leq \epsilon$.
- (ii) \exists closed $F \subset E$ with $m(E \setminus F) \leq \epsilon$.
- (iii) If $m(E) < \infty$, \exists compact $K \subset E$ with $m(E \setminus K) \leq \epsilon$.
- (iv) If $m(E) < \infty$, $\exists F = \bigcup_{j=1}^N Q_j$, $\{Q_j\}_{j=1}^\infty$ are closed cubes, s. t. $m(E \Delta F) \leq \epsilon$.

证明.

(i) It's just the definition of measurability.

(ii) Since E_j^c is measurable, \exists open $O_j \supset E_j^c$, s. t.

$$m(O_j \setminus E_j^c) \leq \epsilon \quad (1.80)$$

Since $O_j^c \subset E_j$ is closed and $E_j \setminus O_j^c = O_j \setminus E_j^c$, let $F = O_j^c$ closed, then

$$m(E_j \setminus F) = m(O_j \setminus E_j^c) \leq \epsilon \quad (1.81)$$

(iii) By (ii), \exists closed $F \subset E$, s. t. $m(E \setminus F) \leq \frac{\epsilon}{2}$.

Let B_n denote the closed ball centered at the origin of radius n , then B_n is compact.

$$F = \bigcup_{j=1}^{\infty} (F \cap B_k) \quad (1.82)$$

Let $K_n = \bigcup_{k=1}^n (F \cap B_k)$, then K_n is compact and $K_n \nearrow F \Rightarrow E \setminus K_n \nearrow E \setminus F$.

Since $m(E \setminus K_1) \leq m(E)$ is finite, by Thm1.3.3(ii)

$$\lim_{n \rightarrow \infty} m(E \setminus K_n) = m(E \setminus F) \quad (1.83)$$

As for $\epsilon > 0$, $\exists N \in \mathbb{N}$, s. t. for all $n \geq N$

$$|m(E \setminus K_n) - m(E \setminus F)| \leq \frac{\epsilon}{2} \quad (1.84)$$

$$m(E \setminus K_n) \leq m(E \setminus F) + \frac{\epsilon}{2} \leq \epsilon \quad (1.85)$$

Therefore, $m(E \setminus K_N) \leq \epsilon$, where $K_N \subset E$ is compact.

(iv) \exists open $O \supset E$, s. t. $m(O \setminus E) \leq \frac{\epsilon}{2}$. By Thm1.1.4, $\exists \{Q_j\}_{j=1}^{\infty}$, s. t.

$$E \subset O = \bigcup_{j=1}^{\infty} Q_j \quad (1.86)$$

So

$$m(O) = \sum_{j=1}^{\infty} |Q_j| \leq m(O \setminus E) + m(E) \leq \frac{\epsilon}{2} + m(E) \quad (1.87)$$

Since $m(E)$ is finite, $\sum_{j=1}^{\infty} |Q_j|$ converges. Then $\exists N \in \mathbb{N}$, s. t.

$$\sum_{j=N+1}^{\infty} |Q_j| \leq \frac{\epsilon}{2} \quad (1.88)$$

Let $F = \bigcup_{j=1}^N Q_j$. Since $E \Delta F = (E \setminus F) \sqcup (F \cap E)$, then

$$m(E \Delta F) = m(E \setminus F) + m(F \setminus E) \quad (1.89)$$

$$\leq m\left(\bigcup_{j=N+1}^{\infty} Q_j\right) + m\left(\bigcup_{j=1}^{\infty} Q_j \setminus E\right) \quad (1.90)$$

$$= \sum_{j=N+1}^{\infty} |Q_j| + \sum_{j=1}^{\infty} |Q_j| - m(E) \quad (1.91)$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (1.92)$$

□

1.4 σ – algebras and Borel sets

1.4.1 σ – algebra

首先给出 \mathbb{R}^d 中 *algebra* 的定义.

定义 1.4.1. Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^d)$. \mathcal{A} is called an algebra if

- (1) If $A_1, \dots, A_n \in \mathcal{A}$, then $\bigcup_{j=1}^n A_j \in \mathcal{A}$.
- (2) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

注. 容易证明, 若 \mathcal{A} 为 \mathbb{R}^d 中 *algebra*, 则其对 *finite intersections* 也封闭, 同时 $\emptyset, \mathbb{R}^d \in \mathcal{A}$.

下面给出 \mathbb{R}^d 中 σ – *algebra* 的定义.(将 *algebra* 中的 *finite* 条件加强为 *countable*)

定义 1.4.2. Let $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^d)$. \mathcal{M} is a σ – algebra if

- (1) If $A_1, A_2, \dots \in \mathcal{M}$, then $\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$.
- (2) If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$.

注. 容易证明 \mathcal{M} 对 *countable intersections* 同样封闭, $\emptyset, \mathbb{R}^d \in \mathcal{M}$.

例 1.4.1. All Lebesgue measurable sets forms a σ – algebra \mathcal{M} .

类比线性空间、拓扑空间中 (拓扑) 基的概念, 下面给出 **生成 σ – algebra** 的概念.

定义 1.4.3. Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^d)$, then the σ – algebra generated by \mathcal{A} is the smallest σ – algebra containing \mathcal{A} .

注. 即为 the intersection of all σ – *algebras* containing \mathcal{A} , 这也说明了对于任一给定的集族 \mathcal{A} , 其生成的 σ – *algebra* 必存在且唯一.

1.4.2 Borel sets

下面给出 *Borel σ -algebra* 及 *Borel sets* 的定义.

定义 1.4.4. The Borel σ -algebra is the σ -algebra generated by all open sets in \mathbb{R}^d , denoted by $\mathcal{B}_{\mathbb{R}^d}$.

Elements of this σ -algebra are called Borel sets.

注. 事实上, *Borel σ -algebra* 为 Lebesgue countable sets 的一个真子集, 后续会利用 Cantor 集证明.

为了方便研究 *Borel σ -algebra* 的结构, 我们把其中较为复杂 (非平凡) 的元素单独拎出来并称为 G_δ, F_σ .

定义 1.4.5. 1. The countable intersections of open sets are called G_δ sets.

2. The countable unions of closed sets are called F_σ sets.

下面我们可给出 $\mathcal{B}_{\mathbb{R}^d}$ 与 Lebesgue 可测集 \mathcal{L} 之间的关系. (\mathcal{L} 只比 $\mathcal{B}_{\mathbb{R}^d}$ 多了一些零测集)

定理 1.4.1. Lebesgue 测度的正规性.

$E \subset \mathbb{R}^d$ is \mathcal{L} -measurable

(i) if and only if $E = G_\delta \setminus N_1$, for some G_δ , $m(N_1) = 0$.

(ii) if and only if $E = F_\sigma \setminus N_2$, for some F_σ , $m(N_2) = 0$.

证明. Clearly E is measurable whenever it satisfies either (i) or (ii).

(i) Since E is measurable, \exists open sets $O_n \supset E$, s. t.

$$m(O_n \setminus E) \leq \frac{1}{n} \quad (1.93)$$

Let $O = \bigcap_{j=1}^{\infty} O_j$, then

$$m(O \setminus E) \leq \frac{1}{n}, \quad \forall n \in \mathbb{N} \quad (1.94)$$

Let $n \rightarrow \infty$, we get $m(O \setminus E) = 0$. Let $G_\delta = O$, $N_1 = O \setminus E$. Then $E = G_\delta \setminus N_1$.

(ii) Similarly, we can easily proof it by Thm1.3.4(ii).

□

1.5 Non – measurable sets

在这一节我们将介绍 \mathbb{R} 上一个经典的不可测集 *Vitali set*, 并说明 \mathbb{R} 上每个正测度集都有不可测子集.

Vitali set Let $x, y \in [0, 1]$. Write $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$.

\Rightarrow 容易验证 \sim 为 an equivalence relation.

$\Rightarrow \sim$ partitions $[0, 1]$. 记 $[0, 1]$ 上等价类为 ε_a , 则

$$[0, 1] = \bigsqcup_a \varepsilon_a, \{ \varepsilon_a \}_a \text{ are disjoint} \quad (1.95)$$

\Rightarrow By **the Axiom of Choice**, we can choose exactly one element x_a from each ε_a .

\Rightarrow Let $\mathcal{N} = \{x_a\}_a$. Then \mathcal{N} is the Vitali set.

定理 1.5.1. \mathcal{N} is not measurable.

证明. Assume that \mathcal{N} is measurable. Let $\{r_k\}_{k=1}^\infty$ be an enumeration of $\mathbb{Q} \cap [-1, 1]$.

Define

$$\mathcal{N}_k := \mathcal{N} + r_k = \{x_a + r_k\}_a \quad (1.96)$$

Then we shall proof that $\{\mathcal{N}_k\}_{k=1}^\infty$ are disjoint, and $[0, 1] \subset \bigcup_{k=1}^\infty \mathcal{N}_k \subset [-1, 2]$.

- If $\mathcal{N}_k \cap \mathcal{N}_m \neq \emptyset$, then $\exists x_a, x_\beta \in \mathcal{N}, r_k, r_m \in \mathbb{Q} \cap [-1, 1]$, s. t.

$$x_a + r_k = x_\beta + r_m \quad (1.97)$$

Then $x_a - x_\beta = r_m - r_k \in \mathbb{Q} \Rightarrow x_a \sim x_\beta \Rightarrow x_a, x_\beta \in \varepsilon_a$ or $x_a, x_\beta \in \varepsilon_\beta \Rightarrow x_a = x_\beta$ and $r_k = r_m$.

Therefore, $\mathcal{N}_k = \mathcal{N}_m$.

- Since $r_k \in [-1, 1], \mathcal{N}_k \in [-1, 2], \forall k$. Therefore,

$$\bigcup_{k=1}^\infty \mathcal{N}_k \subset [-1, 2] \quad (1.98)$$

- $\forall x \in [0, 1]$. Since $\{\varepsilon_a\}_a$ partitions $[0, 1]$, there exists a_0 , s. t.

$$x \in \varepsilon_{a_0}, x \sim x_{a_0} \quad (1.99)$$

which means $x - x_{a_0} \in \mathbb{Q} \cap [-1, 1]$. Then $\exists k_0 \in \mathbb{N}$, s. t.

$$x - x_{a_0} = r_{k_0} \Rightarrow x \in \mathcal{N}_{k_0} \quad (1.100)$$

Therefore,

$$[0, 1] \subset \bigcup_{k=1}^{\infty} \mathcal{N}_k \quad (1.101)$$

Since $\{\mathcal{N}_k\}_{k=1}^{\infty}$ are disjoint, we get

$$m([0, 1]) \leq \sum_{k=1}^{\infty} m(\mathcal{N}_k) \leq m([-1, 2]) \quad (1.102)$$

Since \mathcal{N}_k is a translate of \mathcal{N} , we have $m(\mathcal{N}) = m(\mathcal{N}_k)$ for each k . Then

$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}) \leq 3 \Rightarrow \text{Neither } m(\mathcal{N}) = 0 \text{ nor } m(\mathcal{N}) > 0 \text{ is possible.} \quad (1.103)$$

Therefore, it's a contradiction. \mathcal{N} is non-measurable. \square

正测度集必有不可测子集 下面要证明一个结论, 即 \mathbb{R} 上任一正测度集必有不可测子集. 这实际上为书¹Exercises of Chapter 1 的第 32 题 (b).

命题 1.5.1. Let \mathcal{N} denote the non-measurable subset of $[0, 1]$ constructed in Thm1.5.1.

- (a) If E is a measurable subset of \mathcal{N} , then $m(E) = 0$.
- (b) If $G \subset \mathbb{R}$ with $m_*(G) > 0$, then there exists a subset of G is non-measurable.

证明.

- (a) Note $\mathcal{N} = \{x_\alpha\}_{\alpha \in \mathcal{A}}$, then $E = \{x_\beta\}_{\beta \in \mathcal{B} \subset \mathcal{A}}$. Similarly, we can proof

$$\bigcup_{k=1}^{\infty} E_k \subset [-1, 2] \quad (1.104)$$

Since $\{E_k\}_{k=1}^{\infty}$ are disjoint, and E_k is a translate of E , we get

$$\sum_{k=1}^{\infty} m(E) \leq 3 \Rightarrow m(E) = 0 \quad (1.105)$$

- (b) Let $\mathcal{Q} = \{r_k\}_{k=1}^{\infty}$, $\mathcal{N}_k = \mathcal{N} + r_k$, then

$$\mathbb{R} = \bigcup_{k=1}^{\infty} \mathcal{N}_k \quad (1.106)$$

¹参考书籍: 《Real Analysis – Measure Theory, Integration, & Hilbert Spaces》— Elias M. Stein

Suppose G is measurable. Then

$$G = G \cap \mathbb{R} = \bigcup_{k=1}^{\infty} (G \cap \mathcal{N}_k) \quad (1.107)$$

If $G \cap \mathcal{N}_k$ is measurable, then $G \cap \mathcal{N}_k \subset \mathcal{N}_k$ is a subset of a non-measurable set \mathcal{N}_k .

By the previous (a), we get

$$m(G \cap \mathcal{N}_k) = 0 \quad (1.108)$$

Therefore, there exists $k_0 \in \mathbb{N}$, s. t. $G \cap \mathcal{N}_{k_0} \subset G$ is a non-measurable subset of G .

(otherwise $m(G) = 0$ contradicts)

□

第二章 Measurable Functions

2.1 Measurable Functions

定义 下面给出 \mathbb{R}^d 上可测函数的定义.(注意值域为扩充实数系 $\bar{\mathbb{R}}$)

定义 2.1.1. A function defined on a measurable subset $E \subset \mathbb{R}^d$ is measurable if for all $a \in \mathbb{R}$,

$$f^{-1}([-\infty, a)) = \{x \in E \mid f(x) < a\} \quad (2.1)$$

is measurable.

注. • $f^{-1}([-\infty, a))$ 常简记作 $\{f < a\}$.

• 下面给出几条等价定义.

(1) $\{f < a\}$ is measurable. $\Leftrightarrow \{f \leq a\}$ is measurable.

(2) $\Leftrightarrow \{f > a\}$ is measurable $\Leftrightarrow \{f \geq a\}$ is measurable.

(3) If f is finite-valued, then

$$f \text{ is measurable} \Leftrightarrow \{a < f < b\} \text{ is measurable, } \forall a, b \in \mathbb{R} \quad (2.2)$$

证明.

(1) Since the collection of measurable sets is closed under countable intersections and unions,

$$\{f \leq a\} = \bigcap_{n=1}^{\infty} \{f < a + \frac{1}{n}\} \quad (2.3)$$

$$\{f < a\} = \bigcup_{n=1}^{\infty} \{f \leq a - \frac{1}{n}\} \quad (2.4)$$

Therefore, $\{f < a\}$ is measurable. $\Leftrightarrow \{f \leq a\}$ is measurable.

(2) Since the collection of measurable sets is closed under complements, easily proof by (1).

(3) Since f is finite-valued,

$$\{f < a\} = \bigcup_{n=1}^{\infty} \{-n < f < a\} \quad (2.5)$$

$$\{a < f < b\} = \{f > a\} \cap \{f < b\} \quad (2.6)$$

Therefore, by (2), f is measurable $\Leftrightarrow \{a < f < b\}$ is measurable.

□

Property 下面给出可测函数的一些性质.

Property 1. Let $-\infty < f(x) < +\infty$ (finite-valued), then

$$f \text{ is measurable} \Leftrightarrow f^{-1}(O) \text{ is measurable } \forall \text{ open set } O \quad (2.7)$$

$$\Leftrightarrow f^{-1}(F) \text{ is measurable } \forall \text{ closed set } F \quad (2.8)$$

证明. $\forall O \subset \mathbb{R}$, there exists $\{(a_n, b_n)\}_{n=1}^{\infty}$, s. t.

$$O = \bigcup_{n=1}^{\infty} (a_n, b_n) \quad (2.9)$$

Then

$$f^{-1}(O) = f^{-1}\left(\bigcup_{n=1}^{\infty} (a_n, b_n)\right) = \bigcup_{n=1}^{\infty} f^{-1}((a_n, b_n)) \quad (2.10)$$

Since f is finite-valued and measurable, then $f^{-1}(a_n, b_n)$ is measurable.

Therefore, $f^{-1}(O)$ is measurable.

□

Property 2. $\{\text{continuous functions}\} \subset \{\text{measurable functions}\}$

(a) If f is continuous on \mathbb{R}^d , then f is measurable.

(b) If f is measurable, finite-valued and Φ is continuous on \mathbb{R} , then $\Phi \circ f$ is measurable.

证明.

(a) Since f is continuous, $\forall O \subset \mathbb{R}$, $f^{-1}(O) \subset \mathbb{R}^d$. By Property 1, f is measurable.

(b) $\forall O \subset_{\text{open}} \mathbb{R}$. Since Φ is continuous, then $\Phi^{-1}(O)$ is open.

Since f is finite-valued and measurable, then $(\Phi \circ f)^{-1}(O) = f^{-1}(\Phi^{-1}(O))$ is open.

Therefore, by Property 1, $\Phi \circ f$ is measurable.

□

Property 3. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions. Then

$$\sup_n f_n(x), \inf_n f_n(x), \limsup_{n \rightarrow \infty} f_n(x), \liminf_{n \rightarrow \infty} f_n(x) \quad (2.11)$$

are measurable.

注. 类比数列的上下极限, 此处

$$\limsup_{n \rightarrow \infty} f_n(x) := \lim_{k \rightarrow \infty} \sup_{n \geq k} \{f_n(x)\} = \inf_k \sup_{n \geq k} \{f_n(x)\} \quad (2.12)$$

$$\liminf_{n \rightarrow \infty} f_n(x) := \lim_{k \rightarrow \infty} \inf_{n \geq k} \{f_n(x)\} = \sup_k \inf_{n \geq k} \{f_n(x)\} \quad (2.13)$$

证明. Since

$$\{x \mid \sup_n f_n(x) > a\} = \bigcup_{n=1}^{\infty} \{x \mid f_n(x) > a\} \quad (2.14)$$

$$\{x \mid \inf_n f_n(x) < a\} = \bigcup_{n=1}^{\infty} \{x \mid f_n(x) < a\} \quad (2.15)$$

Then $\sup_n f_n(x), \inf_n f_n(x)$ is measurable.

Since $\sup_{n \geq k} f_n(x), \inf_{n \geq k} f_n(x)$ are measurable, by the previous conclusion, then

$$\limsup_{n \rightarrow \infty} f_n(x) = \inf_k \sup_{n \geq k} \{f_n(x)\} \quad (2.16)$$

$$\liminf_{n \rightarrow \infty} f_n(x) = \sup_k \inf_{n \geq k} \{f_n(x)\} \quad (2.17)$$

are measurable.

□

Property 4. If $\{f_n\}_{n=1}^{\infty}$ is a collection of measurable functions and

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad (2.18)$$

then f is measurable.

注. • 与数列上下极限相同,

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \Leftrightarrow \limsup_{n \rightarrow \infty} f_n(x) = \liminf_{n \rightarrow \infty} f_n(x) = f(x) \quad (2.19)$$

- 此 Property 即说明可测函数列对极限运算封闭. 注意到连续函数列对极限运算并不具备封闭性.(下面给出经典范例)

例 2.1.1.

$$\lim_{n \rightarrow \infty} x^n = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases} \quad (2.20)$$

证明. Since $\{f_n\}_{n=1}^{\infty}$ are measurable, $f(x) = \limsup_{n \rightarrow \infty} f_n(x) = \liminf_{n \rightarrow \infty} f_n(x)$, then according to Property 3, f is measurable. □

Property 5. If f and g are measurable, then

- (i) $f^k, k \in \mathbb{N}$ are measurable.
- (ii) $f + g$ and fg are measurable if both f and g are finite-valued.

证明.

(i) Since

$$\{f^k > a\} = \{f > a^{\frac{1}{k}}\}, \quad \forall k \text{ is odd} \quad (2.21)$$

$$\{f^k > a\} = \{f > a^{\frac{1}{k}}\} \cup \{f < -a^{\frac{1}{k}}\}, \quad \forall k \text{ is even and } a > 0 \quad (2.22)$$

Therefore, $f^k, k \in \mathbb{N}$ are measurable.

(ii) Since¹

$$\{f + g > a\} = \bigcup_{r \in \mathbb{Q}} \{f > a - r\} \cap \{g > r\} \quad (2.23)$$

¹即必 $\exists r \in \mathbb{Q}$, s. t. $\{f + g > a\} \supset \{f > a - r\} \cap \{g > r\}$. (另一侧包含关系 \subset 显然易证)

(反证. $\forall r \in \mathbb{Q}$ 上式不成立, 则对于 $r = 0 \in \mathbb{Q}$, $\exists x_0$, s. t. $f(x_0) > a$, $g(x_0) > 0$, 且 $f(x_0) + g(x_0) \leq a$, 矛盾.)

then $f + g$ is measurable.

By the previous results in (i) and (ii), since

$$fg = \frac{1}{4}[(f + g)^2 - (f - g)^2] \quad (2.24)$$

Therefore, fg is also measurable.

□

下面给出数学分析中曾介绍过的几乎处处的定义.

定义 2.1.2. A property or statement is said to hold almost everywhere (a.e.) if it is true except on a set of measure zero.

例 2.1.2.

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases} \quad (2.25)$$

We say f is continuous a.e. on $[0, 1]$ since $D(f) = \{1\}$ has measure zero.

下面说明几乎处处相等可保持函数可测性.

命题 2.1.1. If f is measurable and $f = g$ a.e. , then g is measurable.

证明. Since f is measurable and

$$g = (g - f) + f \quad (2.26)$$

then we shall proof that $g - f$ is measurable.

Let $A := \{x \mid g(x) - f(x) \neq 0\}$, then $m(A) = 0$. We get

$$\forall a \geq 0, (g - f)^{-1}((-\infty, a]) = (\mathbb{R}^d \setminus A) \cup N, \text{ where } N \subset A \quad (2.27)$$

Since $m(A) = 0$, then N is measurable and $m(N) = 0$. So $(g - f)^{-1}((-\infty, a])$ is measurable.

Therefore, $g - f$ is measurable. Then g is measurable.

□

2.2 Measurable functions are nearly simple

本节来介绍一个非常重要的定理. 即可测函数可由简单函数逼近.

特征函数 下面先来介绍特征函数的定义.

定义 2.2.1. If $E \subset \mathbb{R}$, the characteristic / indicator function $\chi_E / \mathbb{1}_E$ of E is defined by

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases} \quad (2.28)$$

下面给出可测集与其对应特征函数的关系.

命题 2.2.1. χ_E is measurable $\Leftrightarrow E$ is measurable

证明. Since

$$\chi_E^{-1}((-\infty, a]) = \begin{cases} \emptyset, & a < 0 \\ E^c, & 0 \leq a < 1 \\ \mathbb{R}^d, & a \geq 1 \end{cases} \quad (2.29)$$

Then E is measurable $\Rightarrow \chi_E$ is measurable.

χ_E is measurable $\Rightarrow \chi_E^{-1}((-\infty, a]) = E^c$ is measurable. $\Rightarrow E$ is measurable. □

下面给出特征函数的基本性质.

命题 2.2.2. [Property].

(1) If $A \cap B = \emptyset$, then

$$\chi_{A \cup B} = \max \{\chi_A, \chi_B\} = \chi_A + \chi_B \quad (2.30)$$

(2) $\chi_{A \cap B} = \min \{\chi_A, \chi_B\} = \chi_A \cdot \chi_B$.

Simple functions 对特征函数做线性组合，即可得到简单函数.

定义 2.2.2. A simple function on \mathbb{R}^d is a finite linear combination

$$f(x) = \sum_{j=1}^n a_j \chi_{E_j}(x) \quad (2.31)$$

where each E_j is measurable and $m(E_j) < \infty$.

注. 此处定义中并未要求 $\{E_j\}_{j=1}^n$ disjoint. 而事实上这便引出了下面介绍的标准形式.

下面的命题说明了每个简单函数都可写为标准形式 ($\{E_j\}_{j=1}^n$ disjoint).

命题 2.2.3. Every simple function f has a standard representaion

$$f = \sum_{k=1}^N a_k \chi_{E_k}, \text{ where } \{E_j\}_{k=1}^N \text{ are disjoint} \quad (2.32)$$

证明. Suppose $f = \sum_{k=1}^N b_k \chi_{E_k}$, $\{E_j\}_{k=1}^N$ may not be disjoint.

Since $\{E_j\}_{k=1}^N$ is finite, the number of elements of range f is also finite. Suppose

$$\text{range } f = \{a_1, \dots, a_M\} \quad (2.33)$$

Then let $F_k = f^{-1}(\{a_k\})$, then $\{F_k\}_{k=1}^M$ are disjoint. Therefore, we get the standard representation

$$f = \sum_{k=1}^M a_k \chi_{F_k} \quad (2.34)$$

□

简单函数逼近可测函数 下面给出一个定理, 说明任一可测函数可由简单函数列逼近.

定理 2.2.1. Suppose $f : \mathbb{R}^d \rightarrow [-\infty, \infty]$ is measurable.

Then there exists a sequence $\{\varphi_n\}$ of simple functions, s. t.

$$0 \leq |\varphi_1| \leq |\varphi_2| \leq \cdots \leq |f| \quad (2.35)$$

$$\lim_{k \rightarrow \infty} \varphi_k(x) = f(x), \text{ for all } x \quad (2.36)$$

and $\varphi_k \rightarrow f$ uniformly on any set on which f is bounded.

证明. 下面从两方面分类讨论, 即非负函数 & 变号函数, f 有界 & 无界.

(1) 非负函数 $f : \mathbb{R}^d \rightarrow [0, \infty]$.

1° f is bounded. Assume $|f(x)| \leq M$.

Let²

$$E_n^k = f^{-1}\left(\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right), k = 0, \dots, N_n \quad (2.37)$$

$$\varphi_n(x) = \frac{k}{2^n}, \text{ if } x \in E_n^k \quad (2.38)$$

Then

$$\varphi_n(x) = \sum_{k=0}^{N_n} \frac{k}{2^n} \chi_{E_n^k}(x) \quad (2.39)$$

Therefore³

$$|\varphi_n(x) - f(x)| \leq \frac{1}{2^n} \rightarrow 0 \text{ (independent of } x) \quad (2.40)$$

$\Rightarrow \varphi_n \rightarrow f$ uniformly.

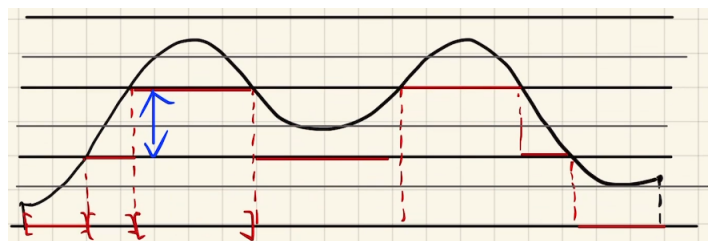


图 2.1: 对 f 值域进行分划

² E_n^k 表示第 n 次对值域进行分划后产生的第 k 个值域区间, 其中 $\frac{N_n+1}{2^n} \geq M$.

³ $|\varphi_n(x) - f(x)|$ 小于等于第 n 次分划后两个相邻值域区间的步长值, 即 $\frac{1}{2^n}$.

2° f is unbounded. (idea: truncation, 将 f 截断为一列有界函数列, 并逐点收敛于 f)

Let

$$f_k(x) = \begin{cases} f(x), & \text{if } f(x) \leq k \\ k, & \text{if } f(x) > k \end{cases} \quad (2.41)$$

Then $f_k(x) \rightarrow f(x), \forall x \in \mathbb{R}^d$.

Since f_k is bounded, by the previous result in 1°,

For each k, \exists a sequence of simple functions $\{\psi_{kn}\}_{n=1}^\infty$, s. t.

$$\psi_{kn}(x) \rightarrow f_k(x), \forall x \quad (2.42)$$

So we get

$$\begin{array}{ccccccc} \psi_{11} & \psi_{12} & \psi_{13} & \cdots & \rightarrow & f_1 \\ \psi_{21} & \psi_{22} & \psi_{23} & \cdots & \rightarrow & f_2 \\ \psi_{31} & \psi_{32} & \psi_{33} & \cdots & \rightarrow & f_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \downarrow \\ & & & & & f \end{array} \quad (2.43)$$

From the previous results in 1°, we get

$$|\psi_{kn}(x) - f_k(x)| \leq \frac{1}{2^n} \quad (2.44)$$

Let $n = k$, then $|\psi_{kk}(x) - f_k(x)| \leq \frac{1}{2^k}$. Let $\varphi_k = \psi_{kk}$, then

$$|\varphi_k(x) - f(x)| \leq |\varphi_k(x) - f_k(x)| + |f_k(x) - f(x)| \quad (2.45)$$

Since $f_k(x) \rightarrow f(x)$, we get $\varphi_k(x) \rightarrow f(x), \forall x$, where $\{\varphi_k = \psi_{kk}\}_{k=1}^\infty$ are simple functions.

(2) 变号函数 $f : \mathbb{R}^d \rightarrow [-\infty, \infty]$.

We denote that

$$f^+(x) := \max\{f(x), 0\} \quad (2.46)$$

$$f^-(x) := \max\{-f(x), 0\} \quad (2.47)$$

By the previous results in (1), there exist sequences of simple functions $\{\varphi_k\}_{k=1}^\infty, \{\psi_k\}_{k=1}^\infty$, s. t.

$$\varphi_k \rightarrow f^+ \text{ and } \psi_k \rightarrow f^- \text{ pointwisely} \quad (2.48)$$

We can observe that $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Let $\phi_k(x) = \varphi_k(x) - \psi_k(x)$, then ϕ_k is a simple function with $\phi_k \rightarrow f$ pointwisely.

□

阶梯函数逼近可测函数 在证明了可测函数可由简单函数逼近后，我们更进一步，来说明可测函数可由更加简单的**阶梯函数**来逼近。

先给出**阶梯函数**的定义。

定义 2.2.3. A **step function** is a finite sum

$$f = \sum_{k=1}^N a_k \chi_{R_k}, \text{ where } R_k \text{ is a rectangle} \quad (2.49)$$

注. 阶梯函数 & 简单函数的区别在于，简单函数是作用于有限个**可测集** E_k ，而阶梯函数是作用于有限个**矩形** R_k 。

下面的定理说明了 measurable functions are almost step functions.

定理 2.2.2. Suppose f is measurable on \mathbb{R}^d . Then there exists a sequence of step functions $\{\psi_k\}_{k=1}^\infty$, s. t.

$$\lim_{k \rightarrow \infty} \psi_k(x) = f(x), \text{ a.e. } x \quad (2.50)$$

注. 首先介绍函数列收敛点集的几种不同的等价表述：

$$\{x \mid \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |f_n(x) - f(x)| < \epsilon\} \quad (2.51)$$

$$\Leftrightarrow \{x \mid \forall n \in \mathbb{N}, \exists N \in \mathbb{N}, \forall k \geq N, |f_k(x) - f(x)| < \frac{1}{n}\} \quad (2.52)$$

$$\Leftrightarrow \bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \{x \mid |f_k(x) - f(x)| < \frac{1}{n}\} \quad (2.53)$$

从而可以得到函数列发散点集 (Negation):

$$\{x \mid \exists n \in \mathbb{N}, \forall N \in \mathbb{N}, \exists k \geq N, |f_k(x) - f(x)| \geq \frac{1}{n}\} \quad (2.54)$$

$$\Leftrightarrow \bigcup_{n=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \{x \mid |f_k(x) - f(x)| \geq \frac{1}{n}\} \quad (2.55)$$

$$\Leftrightarrow \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \{x \mid f_k(x) \neq f(x)\} \quad (2.56)$$

证明. (证明思路：先用阶梯函数逼近简单函数，再用简单函数逼近可测函数.)

It suffices to show that χ_E can be approxiamted by step functions, for any measurable set E .

According to Thm1.3.4 (iv)

Let $f = \chi_E$, then $\forall \epsilon > 0, \exists$ cubes $\bigcup_{j=1}^N Q_j$, s. t.

$$m(E \Delta \bigcup_{j=1}^N Q_j) \leq \epsilon \quad (2.57)$$

By considering the grid formed by extending the sides of these cubes, there exists almost disjoint rectangles $\{\tilde{R}_j\}_{j=1}^M$, s. t.

$$\bigcup_{j=1}^N Q_j = \bigcup_{j=1}^M \tilde{R}_j \quad (2.58)$$

By taking rectangles R_j contained in \tilde{R}_j , we can find a collection of disjoint rectangles $\{R_j\}_{j=1}^M$, s. t.

$$m(E \Delta \bigsqcup_{j=1}^M R_j) \leq 2\epsilon \quad (2.59)$$

For every $k \in \mathbb{N}$, there exists disjoint rectangles $\{R_j\}_{j=1}^M$, s. t.

$$m(E \Delta \bigsqcup_{j=1}^M R_j) \leq \frac{1}{2^{k+1}} \quad (2.60)$$

There also exists a step function ψ_k

$$\psi_k(x) := \chi_{\bigcup_{j=1}^M R_j}(x) = \sum_{j=1}^M \chi_{R_j}(x) \quad (2.61)$$

Let

$$E_k := \{x \mid f_k(x) \neq f(x)\} \quad (2.62)$$

Since $E_k \subset E \Delta \bigsqcup_{j=1}^M R_j$, then $m(E_k) \leq \frac{1}{2^k}$. Let⁴

$$F_j = \bigcup_{j=k+1}^{\infty} E_j, \quad F = \bigcap_{k=1}^{\infty} F_k \quad (2.63)$$

Then $\psi_k(x) \rightarrow f(x), \forall x \in F^c$. Since

$$m(F) \leq m(F_k), \quad \forall k \in \mathbb{N} \quad (2.64)$$

$$m(F_k) = m\left(\bigcup_{j=k+1}^{\infty} E_j\right) \leq \sum_{j=k+1}^{\infty} m(E_j) \leq \frac{1}{2^k} \quad (2.65)$$

Therefore, $m(F) = 0$. $\lim_{k \rightarrow \infty} \psi_k(x) = f(x)$, a.e. x . □

⁴根据注中式 (2.56), F 即为函数列 $\{\psi_k\}_{k=1}^{\infty}$ 的发散点集, 从而 $\psi_k(x) \rightarrow f(x)$ 在 F^c 上收敛.

第三章 *Integration Theory*

3.1 *The Lebesgue integral*

Lebesgue Integral 的构造可以分为三步, 分别为构造下列函数的积分:

1. **Simple functions**

2. **Non-negative measurable functions**

$$\int f := \sup \left\{ \int \varphi \mid \varphi \text{ simple}, 0 \leq \varphi \leq f \right\} \quad (3.1)$$

3. **General case**

$$f = f^+ - f^- \quad (3.2)$$

$$\int f := \int f^+ - \int f^- \quad (3.3)$$

3.1.1 *Simple functions*

定义 下面先给出非负简单函数在**标准形式**下的积分定义.

定义 3.1.1. If φ is a non-negative simple function with **standard representation**

$$\varphi(x) = \sum_{k=1}^M a_k \chi_{E_k}(x) \quad (3.4)$$

We define the Lebesgue integral of φ by

$$\int_{\mathbb{R}^d} \varphi(x) dx = \sum_{k=1}^M a_k m(E_k) \quad (3.5)$$

If E is a measurable subset of \mathbb{R}^d with finite measure, then

$$\varphi(x) \chi_E(x) = \sum_{k=1}^M a_k \chi_{E_k}(x) \chi_E(x) = \sum_{k=1}^M a_k \chi_{E_k \cap E}(x) \quad (3.6)$$

is also a simple function, and define

$$\int_E \varphi(x) dx = \int_{\mathbb{R}^d} \varphi(x) \chi_E(x) dx \quad (3.7)$$

注. • 此处仅对**标准形式**定义了积分. 事实上, 此处定义的积分与简单函数的表达式无关 (即**Property 1.**).

- 关于记号, 当测度非常明确时, 大多数情况下可简写, 如

$$\int_E \varphi(x) dx \Rightarrow \int_E \varphi \quad (3.8)$$

$$\int_{\mathbb{R}^d} \varphi(x) dx \Rightarrow \int \varphi \quad (3.9)$$

当为了强调我们选择了何种测度 μ 时, 还可用以下的记号:

$$\int_E \varphi(x) d\mu(x) \quad (3.10)$$

Property 下面给出简单函数积分的性质.

Property 1. Independence of the representation.

If $\varphi = \sum_{k=1}^N a_k \chi_{E_k}$ is any representation of φ , then

$$\int \varphi = \sum_{k=1}^N a_k m(E_k) \quad (3.11)$$

在证明这个性质之前, 先来证明一条引理.(书¹Exercises Of Chapter 2 的第 1 题)

引理 3.1.1. Given a collection of sets $\{F_k\}_{k=1}^n$, there exists another collection $\{\widetilde{F}_j\}_{j=1}^N$ with $N = 2^n - 1$, so that

$$(i). \quad \bigcup_{k=1}^n F_k = \bigcup_{j=1}^N \widetilde{F}_j \quad (3.12)$$

$$(ii). \quad \{\widetilde{F}_j\}_{j=1}^N \text{ are disjoint} \quad (3.13)$$

$$(iii). \quad F_k = \bigcup_{\widetilde{F}_j \subset F_k} \widetilde{F}_j \quad (3.14)$$

证明. Consider the collection

$$\mathcal{F} := \left\{ \bigcup_{k=1}^n G_k - \bigcap_{k=1}^n F_k^c \mid G_k \text{ denotes } F_k \text{ or } F_k^c \right\} \quad (3.15)$$

□

¹参考书籍: 《Real Analysis – Measure Theory, Integration, & Hilbert Spaces》— Elias M. Stein

下面来证明原命题.

证明. According to Lemma 3.1.1, there exists another decompositon of $\bigcup_{k=1}^N E_k$, i.e.

$$\bigcup_{j=1}^M \widetilde{E}_j = \bigcup_{k=1}^N E_k \quad (3.16)$$

where $\{\widetilde{E}_j\}_{j=1}^M$ are disjoint, and for each $1 \leq k \leq M$,

$$E_k = \bigcup_{\widetilde{E}_j \subset E_k} \widetilde{E}_j \quad (3.17)$$

Let

$$\widetilde{a}_j := \sum_{\widetilde{E}_j \subset E_k} a_k \quad (3.18)$$

Then clearly

$$\varphi = \sum_{j=1}^M \widetilde{a}_j \chi_{\widetilde{E}_j} \quad (3.19)$$

Since $\{\widetilde{E}_j\}_{j=1}^M$ are disjoint, we get

$$\int \varphi = \sum_{j=1}^M \widetilde{a}_j m(\widetilde{E}_j) = \sum_{j=1}^M \sum_{\widetilde{E}_j \subset E_k} a_k m(\widetilde{E}_j) = \sum_{k=1}^N a_k m(E_k) \quad (3.20)$$

□

Property 2. Linearity.

If φ and ψ are non-negative simple, and $a, b \geq 0$, then

$$\int (a\varphi + b\psi) = a \int \varphi + b \int \psi \quad (3.21)$$

证明. 下面分为两步来证明.

(a) $\forall c \geq 0, \int c\varphi = c \int \varphi$.

Suppose $\varphi = \sum_{k=1}^M a_k \chi_{E_k}$, where $\{E_k\}_{k=1}^M$ are disjoint. Then

$$c\varphi = \sum_{k=1}^M ca_k \chi_{E_k} \quad (3.22)$$

is also a non-negative simple function. Therefore,

$$\int c\varphi = \sum_{k=1}^M ca_k m(E_k) = c \sum_{k=1}^M a_k m(E_k) = c \int \varphi \quad (3.23)$$

$$(b) \int (\varphi + \psi) = \int \varphi + \int \psi.$$

Suppose

$$\varphi = \sum_{k=1}^M a_k \chi_{E_k}, \quad \psi = \sum_{j=1}^N b_j \chi_{F_j} \quad (3.24)$$

where both $\{E_k\}_{k=1}^M$ and $\{F_j\}_{j=1}^N$ are disjoint and $\mathbb{R}^d = \bigcup_{k=1}^M E_k = \bigcup_{j=1}^N F_j$. Since

$$E_k = E_k \cap \mathbb{R}^d = E_k \cap \bigsqcup_{j=1}^N F_j = \bigsqcup_{j=1}^N (E_k \cap F_j) \quad (3.25)$$

Then

$$\varphi = \sum_{k=1}^M a_k \chi_{E_k} = \sum_{k=1}^M a_k \chi_{\bigsqcup_{j=1}^N (E_k \cap F_j)} = \sum_{k=1}^M \sum_{j=1}^N a_k \chi_{E_k \cap F_j} \quad (3.26)$$

Similarly

$$\psi = \sum_{j=1}^N b_j \chi_{F_j} = \sum_{j=1}^N b_j \chi_{\bigsqcup_{k=1}^M (E_k \cap F_j)} = \sum_{j=1}^N \sum_{k=1}^M b_k \chi_{E_k \cap F_j} \quad (3.27)$$

Therefore

$$\varphi + \psi = \sum_{j,k} (a_k + b_j) \chi_{E_k \cap F_j} \quad (3.28)$$

$$\int (\varphi + \psi) = \sum_{j,k} (a_k + b_j) m(E_k \cap F_j) \quad (3.29)$$

$$= \sum_{j,k} a_k m(E_k \cap F_j) + \sum_{j,k} b_j m(E_k \cap F_j) \quad (3.30)$$

$$= \int \varphi + \int \psi \quad (3.31)$$

□

Property 3. Monotonicity.

If $\varphi \leq \psi$ are non-negative and simple, then

$$\int \varphi \leq \int \psi \quad (3.32)$$

证明. Suppose

$$\varphi = \sum_{k=1}^M a_k \chi_{E_k}, \quad \psi = \sum_{j=1}^N b_j \chi_{F_j} \quad (3.33)$$

where both $\{E_k\}_{k=1}^M$ and $\{F_j\}_{j=1}^N$ are disjoint. Similar to the proof in Property 2, we get

$$\psi - \varphi = \sum_{j,k} (b_j - a_k) \chi_{E_k \cap F_j} \quad (3.34)$$

Since $\varphi(x) \leq \psi(x)$, $\forall x \in \mathbb{R}^d$, then $\psi - \varphi$ is non-negative and simple. Therefore,

$$\int (\psi - \varphi) = \sum_{j,k} (b_j - a_k) m(E_k \cap F_j) \geq 0 \Rightarrow \int \varphi \leq \int \psi \quad (3.35)$$

□

Property 4. Additivity.

If $\{E_k\}_{k=1}^\infty$ are disjoint subsets of \mathbb{R}^d with finite measure, then

$$\int_{\bigcup_{k=1}^\infty E_k} \varphi = \sum_{k=1}^\infty \int_{E_k} \varphi \quad (3.36)$$

注. 首先回顾 *abstract measure* 的定义.

定义 3.1.2. Let X be a set and let \mathcal{M} be a σ -algebra on X .

A **measure** on \mathcal{M} is a function $\mu : \mathcal{M} \rightarrow [0, \infty]$, s. t.

(i) $\mu(\emptyset) = 0$.

(ii) If $\{E_j\}_{j=1}^\infty \subset \mathcal{M}$ are disjoint, then

$$\mu\left(\bigcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty \mu(E_j) \quad (3.37)$$

回到我们积分的性质上来. 下面我们将说明, 对于任一给定的非负简单函数 φ , 将 φ 在任一可测集 A 上的积分看作 *Lebesgue σ -algebra* \mathcal{L} 上的映射, 则该映射为定义在 \mathcal{L} 上的测度.(从而 Property 4. 作为测度的必要条件自然成立)

命题 3.1.1. For any fixed non-negative and simple function φ , the map

$$\mu : \mathcal{L} \rightarrow [0, \infty] \quad (3.38)$$

$$A \mapsto \int_A \varphi \quad (3.39)$$

is a measure on \mathcal{L} .

证明. Suppose $\{A_j\}_{j=1}^{\infty} \subset \mathcal{L}$ are disjoint, and

$$\varphi = \sum_{k=1}^M a_k \chi_{E_k}, \text{ where } \{E_k\}_{k=1}^M \text{ are disjoint} \quad (3.40)$$

Let $A = \bigcup_{j=1}^{\infty} A_j$, then

$$\int_{\bigcup_{j=1}^{\infty} A_j} \varphi = \int_A \varphi = \int \varphi \chi_A = \int \left(\sum_{k=1}^M a_k \chi_{E_k \cap A} \right) \quad (3.41)$$

$$= \sum_{k=1}^M a_k m(E_k \cap A) \quad (3.42)$$

$$= \sum_{k=1}^M a_k m(E_k \cap \left(\bigcup_{j=1}^{\infty} A_j \right)) \quad (3.43)$$

$$= \sum_{k=1}^M a_k m\left(\bigsqcup_{j=1}^{\infty} (E_k \cap A_j)\right) \quad (3.44)$$

$$= \sum_{k=1}^M a_k \sum_{j=1}^{\infty} m(E_k \cap A_j) \quad (3.45)$$

$$= \sum_{k=1}^M \sum_{j=1}^{\infty} a_k m(E_k \cap A_j) \quad (3.46)$$

Since positive series always converges in $[0, \infty]$, then

$$\int_A \varphi = \sum_{k=1}^M \sum_{j=1}^{\infty} a_k m(E_k \cap A_j) = \sum_{j=1}^{\infty} \sum_{k=1}^M a_k m(E_k \cap A_j) = \sum_{j=1}^{\infty} \int_{A_j} \varphi \quad (3.47)$$

Therefore, the integral on any non-negative simple function is actually a measure on \mathcal{L} . \square

3.1.2 Non – negative measurable functions

为了讨论的方便，先给出非负可测函数的一个记号.

$$\mathcal{M}^+ := \{\text{all non – negative measurable functions}\} \quad (3.48)$$

定义 下面给出非负可测函数的积分的定义.

定义 3.1.3. For $f \in \mathcal{M}^+$, we define

$$\int f(x)dx := \sup \left\{ \int \varphi(x)dx \mid 0 \leq \varphi \leq f, \varphi \text{ simple} \right\} \quad (3.49)$$

注. 此处对 Non-negative measurable function 积分的定义兼容定义 3.1.1 中对 Non-negative simple function 积分的定义，具体表现为： $\forall \varphi_0$ non-negative and simple,

$$\sup \left\{ \int \varphi(x)dx \mid 0 \leq \varphi \leq \varphi_0, \varphi \text{ simple} \right\} = \int \varphi_0(x)dx \quad (3.50)$$

性质 下面来验证定义 3.1.3 中定义的积分满足几条基本性质.

Property 1. Monotonicity.

Let $f, g \in \mathcal{M}^+$. Then

$$\int f \leq \int g \text{ if } f \leq g \quad (3.51)$$

证明. Let

$$A = \{\varphi \text{ simple} \mid 0 \leq \varphi \leq f\} \quad (3.52)$$

$$B = \{\psi \text{ simple} \mid 0 \leq \psi \leq g\} \quad (3.53)$$

Then for all $\varphi \in A$, $0 \leq \varphi \leq f \leq g \Rightarrow \varphi \in B \Rightarrow A \subset B$. Since

$$\int f = \sup_{\varphi \in A} \left\{ \int \varphi \right\}, \quad \int g = \sup_{\psi \in B} \left\{ \int \psi \right\} \quad (3.54)$$

Therefore

$$\int f \leq \int g \quad (3.55)$$

□

Property 2. 齐次性.

Let $f \in \mathcal{M}^+$. If $c \geq 0$, then

$$\int cf = c \int f \quad (3.56)$$

证明. Assume $c > 0$. Then

$$\int cf = \sup \left\{ \int \varphi \mid 0 \leq \varphi \leq cf, \varphi \text{ simple} \right\} \quad (3.57)$$

$$= \sup \left\{ \int \varphi \mid 0 \leq \frac{\varphi}{c} \leq f, \varphi \text{ simple} \right\} \quad (3.58)$$

$$\stackrel{\psi = \frac{\varphi}{c}}{=} \sup \left\{ \int c\psi \mid 0 \leq \psi \leq f, \psi \text{ simple} \right\} \quad (3.59)$$

$$= c \sup \left\{ \int \psi \mid 0 \leq \psi \leq f, \psi \text{ simple} \right\} \quad (3.60)$$

$$= c \int f \quad (3.61)$$

□

单调收敛定理 下面我们正式迈入实分析的“大门”，介绍第一个收敛定理.

定理 3.1.2. The Monotone Convergence Theorem.

If $\{f_n\}_{n=1}^\infty \subset \mathcal{M}^+$, $f_j \leq f_{j+1}$ for all j , and $\lim_{n \rightarrow \infty} f_n = f$, then

$$\int f = \lim_{n \rightarrow \infty} \int f_n \quad (3.62)$$

注. • 此即为“单调收敛定理”，这个定理说明了对于单调递增的非负可测函数列，其积分与极限可交换次序. 具体表现为

$$\int f = \int \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \int f_n \quad (3.63)$$

- 该定理还说明了，我们可以给出非负可测函数的另一个更自然的等价定义，即用非负简单函数列的积分逼近非负可测函数的积分.

定义 3.1.4. For $f \in \mathcal{M}^+$, we can also define

$$\int f := \lim_{n \rightarrow \infty} \int \varphi_n \quad (3.64)$$

where $\varphi_n \rightarrow f$ and $0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq f$ by Thm 2.2.1.

并且该定理说明了该积分定义的唯一性及 **well-defined**.

在证明定理前, 先来证明一个引理 (将定理 1.3.3 (i) 拓展到一般的抽象测度上).

引理 3.1.3. Let X be a set, \mathcal{M} be a σ -algebra on X , $\mu : \mathcal{M} \rightarrow [0, \infty]$ be a measure on \mathcal{M} .

If $\{E_n\}_{n=1}^\infty \subset \mathcal{M}$, $E_n \nearrow E$, then

$$\lim_{n \rightarrow \infty} \mu(E_n) = \mu(E) \quad (3.65)$$

证明. 证明过程与 Thm 1.3.3 完全一致 (仅用到了测度的可数可加性).

Let $S_1 = E_1$, $S_k = E_k - E_{k-1}$, $\forall k \geq 2$. Then $\{S_k\}_{n=1}^\infty \subset \mathcal{M}$ are disjoint.

Since $E = \bigcup_{k=1}^\infty E_k = \bigcup_{k=1}^\infty S_k$, then

$$\mu(E) = \mu\left(\bigcup_{k=1}^\infty S_k\right) = \sum_{k=1}^\infty \mu(S_k) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(S_k) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{k=1}^n S_k\right) = \lim_{n \rightarrow \infty} \mu(E_n) \quad (3.66)$$

□

下面证明原定理.

证明.

- $\lim_{n \rightarrow \infty} \int f_n \leq \int f$.

Since $f_n \leq f$, $\forall n$, then

$$\int f_n \leq \int f, \quad \forall n \quad (3.67)$$

Since $\{\int f_n\}_{n=1}^\infty$ always converges in $[0, \infty]$, then let $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \int f_n \leq \int f \quad (3.68)$$

- $\lim_{n \rightarrow \infty} \int f_n \geq \int f$.

Fix $0 < a < 1$, for any $0 \leq \varphi \leq f$ simple, let

$$E_n = \{x \mid f_n(x) \geq a\varphi(x)\} \quad (3.69)$$

Then since $\forall x \in E_n$, we have $f_{n+1}(x) \geq f_n(x) \geq a\varphi(x) \Rightarrow x \in E_{n+1} \Rightarrow E_n \subset E_{n+1}$.

Then $E_n \nearrow$. Since

$$\int_{\mathbb{R}^d} f_n \geq \int_{E_n} f_n \geq \int_{E_n} a\varphi, \quad \forall n \quad (3.70)$$

Let $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n \geq \lim_{n \rightarrow \infty} \int_{E_n} a\varphi \quad (3.71)$$

Then we have to calculate $\lim_{n \rightarrow \infty} \int_{E_n} a\varphi$:

– Since $a\varphi$ is non-negative and simple, by Prop 3.1.1, the map

$$\mu : \mathcal{L} \longrightarrow [0, \infty] \quad (3.72)$$

$$E \longmapsto \int_E a\varphi \quad (3.73)$$

is a measure on the collection of Lebesgue measurable sets \mathcal{L} . (将积分视作测度)

Since $\{E_n\}_{n=1}^\infty \subset \mathcal{L}$ and $E_n \nearrow$, by Lemma 3.1.3, we get

$$\lim_{n \rightarrow \infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^\infty E_n\right) \quad (3.74)$$

i.e.

$$\lim_{n \rightarrow \infty} \int_{E_n} a\varphi = \int_{\bigcup_{n=1}^\infty E_n} a\varphi \quad (3.75)$$

For all $x \in \mathbb{R}^d$, since $a\varphi(x) < f(x)$ and $f_n \rightarrow f$, there exists $N_x \in \mathbb{N}$, s. t.

$$f_n(x) \geq a\varphi(x), \quad \forall n \geq N_x \quad (3.76)$$

which indicates $x \in E_{N_x}$ for some N_x . Therefore

$$\bigcup_{n=1}^\infty E_n = \mathbb{R}^d \Rightarrow \lim_{n \rightarrow \infty} \int_{E_n} a\varphi = \int_{\bigcup_{n=1}^\infty E_n} a\varphi = \int_{\mathbb{R}^d} a\varphi \quad (3.77)$$

Therefore, we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n \geq \lim_{n \rightarrow \infty} \int_{E_n} a\varphi = \int_{\mathbb{R}^d} a\varphi \quad (3.78)$$

Let $a \rightarrow 1$, then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n \geq \int_{\mathbb{R}^d} \varphi \quad (3.79)$$

Since φ is arbitrary, taking the supremum over φ , we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n \geq \sup \left\{ \int_{\mathbb{R}^d} \varphi \mid 0 \leq \varphi \leq f, \varphi \text{ simple} \right\} = \int f \quad (3.80)$$

□

函数项级数的可数可加性 接下来我们将给出单调收敛定理在函数项级数上的表达形式，它说明了对于非负可测函数项级数，其积分与求和可交换次序.

在此之前，先来证明有限项的情况.

(此也可视作非负可测函数积分的**Property 线性性**的一部分.)

命题 3.1.2. Linearity.

If $f, g \in \mathcal{M}^+$, then

$$\int (f + g) = \int f + \int g \quad (3.81)$$

证明. By Thm 2.2.1 and Thm 3.1.2, there exists sequences of non-negative and simple functions $\{\varphi_n\}_{n=1}^{\infty}$ and $\{\psi_n\}_{n=1}^{\infty}$, $\varphi_n \rightarrow f$ and $\psi_n \rightarrow g$, s. t.

$$\int f = \lim_{n \rightarrow \infty} \int \varphi_n, \quad \int g = \lim_{n \rightarrow \infty} \int \psi_n \quad (3.82)$$

Since $\varphi_n + \psi_n$ is still non-negative and simple, then

By the Linearity of integral on non-negative and simple functions, (**Property 2.** in §3.1.1)

$$\int (\varphi_n + \psi_n) = \int \varphi_n + \int \psi_n \quad (3.83)$$

Let $n \rightarrow \infty$, by Thm 3.1.2, we get (极限与积分交换次序)

$$\int (f + g) = \int f + \int g \quad (3.84)$$

□

根据 Prop 3.1.2, 由归纳法, 容易得到其对任意有限项函数项级数都成立.

下面给出函数项级数上的单调收敛定理.

定理 3.1.4. Monotone Convergence Theorem (MCT , series version).

If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+$ and $f = \sum_{n=1}^{\infty} f_n$, then

$$\int f = \sum_{n=1}^{\infty} \int f_n \quad (3.85)$$

注. 该定理说明了对于非负可测函数项级数, 其积分与求和可交换次序.

证明. Let $F_n = \sum_{k=1}^n f_k$, then $F_n \nearrow \sum_{k=1}^{\infty} f_k = f$. By **MCT** (Thm 3.1.2),

$$\lim_{n \rightarrow \infty} \int F_n = \int f \quad (3.86)$$

i.e.

$$\lim_{n \rightarrow \infty} \int \sum_{k=1}^n f_k = \int f \quad (3.87)$$

By the **Linearity** of integral on non-negative functions (Prop 3.1.2),

$$\lim_{n \rightarrow \infty} \int \sum_{k=1}^n f_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int f_k = \sum_{k=1}^{\infty} \int f_k = \int f \quad (3.88)$$

□

积分的唯一性 在实分析中, 我们并不关心零测集上的各种性质, 进而常常忽略函数在零测集上的情况. 在给出**单调收敛定理**的更一般版本前, 我们先来给出**几乎处处**意义下, 函数积分的唯一性.

下面的命题说明了, 若两个非负可测函数**几乎处处**相等, 则其积分相等.

命题 3.1.3. Uniqueness.

If $f \in \mathcal{M}^+$, then

$$\int f = 0 \Leftrightarrow f = 0 \text{ a.e.} \quad (3.89)$$

注. 根据该命题, 对于任意非负可测函数 f, g

$$\int f = \int g \Leftrightarrow \int (f - g) = 0 \Leftrightarrow f - g = 0 \text{ a.e.} \Leftrightarrow f = g \text{ a.e.} \quad (3.90)$$

证明.

- 充分性 “ \Leftarrow ” : If $f = 0$ a.e.

$\forall 0 \leq \varphi \leq f$ simple, $\varphi = 0$ a.e. . Let $E = \{x \mid \varphi(x) = 0\}$, then $m(E^c) = 0$.

$$\int \varphi = \int_E \varphi + \int_{E^c} \varphi = 0 + 0 = 0 \quad (3.91)$$

Taking the supremum of φ , we get

$$\int f = \sup \left\{ \int \varphi \mid 0 \leq \varphi \leq f, \varphi \text{ simple} \right\} = 0 \quad (3.92)$$

- 必要性 “ \Rightarrow ” : If $\int f = 0$, let

$$E_n := \{x \mid f(x) > \frac{1}{n}\} \quad (3.93)$$

Then

$$\bigcup_{n=1}^{\infty} E_n = \{x \mid f(x) > 0\} = \{f \neq 0\} \quad (3.94)$$

Suppose $m(\bigcup_{n=1}^{\infty} E_n) > 0$, then there exists $N \in \mathbb{N}$, s. t. $m(E_N) > 0$. Then

$$\int f \geq \int_{E_N} f > \frac{1}{N} m(E_N) > 0 \quad (3.95)$$

which is a contradiction to $\int f = 0$.

Therefore, $m(\bigcup_{n=1}^{\infty} E_n) = m(\{f \neq 0\}) = 0, f = 0$ a.e.

□

“几乎处处”版 **MCT** 根据积分的唯一性 (命题 3.1.3), 下面说明在 “几乎处处收敛” 条件下, 单调收敛定理成立 (积分与极限仍可交换次序).

推论 3.1.5. a.e. MCT.

If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+, f \in \mathcal{M}^+, f_n \nearrow f$ a.e., then

$$\int f = \lim_{n \rightarrow \infty} \int f_n \quad (3.96)$$

证明. Let $f_n \nearrow f$ on E , then $m(E^c) = 0$ and $f_n - f_n \chi_E = 0$ a.e.

By Prop 3.1.3, we get

$$\int f_n = \int f_n \chi_E \quad (3.97)$$

Since $f_n \chi_E \nearrow f \chi_E$, then by **MCT** (Thm 3.1.2, 单调收敛定理)

$$\lim_{n \rightarrow \infty} \int f_n = \lim_{n \rightarrow \infty} \int f_n \chi_E = \int f \chi_E = \int_E f \quad (3.98)$$

Since $m(E^c) = 0$, then

$$\int f = \int_E f = \lim_{n \rightarrow \infty} \int f_n \quad (3.99)$$

($\forall 0 \leq \varphi \leq f$ simple, $\int \varphi = \int_E \varphi + \int_{E^c} \varphi = \int_E \varphi$. Taking the supremum of $\varphi \Rightarrow \int f = \sup \{ \int \varphi \} = \int_E f$)

□

Fatou's Lemma 我们首先来考虑一个问题, 若我们将单调收敛定理 (**MCT**) 中的 “单调” 条件去掉, 结论是否仍然成立 (积分与极限是否仍可交换次序)? 即

Suppose $f_n \rightarrow f$ a.e., do we have $\int f_n \rightarrow \int f$?

事实上答案为 absolutely no. 下面给出一个反例.

例 3.1.1. Consider $f_n = n \chi_{(0, \frac{1}{n})}$. Then $f_n \rightarrow 0$ a.e. on $[0, 1]$. However,

$$\int f_n = n \cdot \frac{1}{n} = 1, \quad \forall n \in \mathbb{N} \not\rightarrow 0 \quad (3.100)$$

事实上，将“单调收敛”条件整个去除，我们将得到如下的更一般的 **Fatou's Lemma**.

定理 3.1.6. Fatou's Lemma.

If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+$, then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n \quad (3.101)$$

注. • 回顾函数列下极限的定义.

$$\liminf_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \left(\inf_{k \geq n} f_k \right) \quad (3.102)$$

即对定义域上每一点 x ，取数列 $\{f_n(x)\}_{n=1}^{\infty}$ 的下极限，再将所有的 x 所对应的下极限拼成一个函数，即定义为函数列 $\{f_n\}_{n=1}^{\infty}$ 的下极限.

(上式右侧作用在固定的 x 上，即为数列 $\{f_n(x)\}_{n=1}^{\infty}$ 下极限的定义.)

- **Fatou's Lemma** 告诉我们，对于任意一列非负可测函数列，其函数列的下极限的积分，要小于每个函数积分后得到的积分数列的下极限.

证明. Since

$$\liminf_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \left(\inf_{k \geq n} f_k \right) \quad (3.103)$$

Let $g_n = \inf_{k \geq n} f_k$, then $g_n \nearrow \lim_{n \rightarrow \infty} g_n$. By **MCT** (Thm 3.1.2, 单调收敛定理),

$$\int \lim_{n \rightarrow \infty} g_n = \lim_{n \rightarrow \infty} \int g_n \quad (3.104)$$

i.e.

$$\int \liminf_{n \rightarrow \infty} f_k = \lim_{n \rightarrow \infty} \left(\int \inf_{k \geq n} f_k \right) \quad (3.105)$$

For each n , since $\inf_{k \geq n} f_k \leq f_j, \forall j \geq n$, then

$$\int \inf_{k \geq n} f_k \leq \int f_j, \forall j \geq n \quad (3.106)$$

Taking the infimum of $\{\int f_j\}_{j=n}^{\infty}$, then

$$\int \inf_{k \geq n} f_k \leq \inf_{j \geq n} \int f_j, \forall n \in \mathbb{N} \quad (3.107)$$

For n is arbitrary, let $n \rightarrow \infty$, we get

$$\lim_{n \rightarrow \infty} \left(\int \inf_{k \geq n} f_k \right) \leq \lim_{n \rightarrow \infty} \left(\inf_{k \geq n} \int f_k \right) = \liminf_{n \rightarrow \infty} \int f_n \quad (3.108)$$

Therefore

$$\int \liminf_{n \rightarrow \infty} f_k = \lim_{n \rightarrow \infty} \left(\int \inf_{k \geq n} f_k \right) \leq \liminf_{n \rightarrow \infty} \int f_n \quad (3.109)$$

□

3.1.3 General case

可积函数 跟 Riemann 积分类似，对于 Lebesgue 积分，我们也有可积函数的概念。

下面先让我们回到非负可测函数，定义非负可测函数中可积的概念。

定义 3.1.5. For $f \in \mathcal{M}^+$, if

$$\int f < \infty \quad (3.110)$$

Then we say f is Lebesgue integrable or simply integrable.

下面扩展到一般的可测函数，给出其 **Lebesgue** 积分及可积的定义。

定义 3.1.6. For any f measurable on \mathbb{R}^d

$$f^+(x) := \max \{f(x), 0\}, \quad f^-(x) := \max \{-f(x), 0\} \quad (3.111)$$

If at least one of $\int f^+$ and $\int f^-$ is finite, we define the integral of f

$$\int f := \int f^+ - \int f^- \quad (3.112)$$

We say that f is (Lebesgue) integrable if $|f|$ is integrable.

注. • 注意到

$$f = f^+ - f^- \quad (3.113)$$

$$|f| = f^+ + f^- \quad (3.114)$$

• 根据定义，对于任意可测函数 f ，

$$f \text{ integrable} \Leftrightarrow |f| \text{ integrable} \Leftrightarrow \int |f| = \int f^+ + \int f^- < \infty \quad (3.115)$$

$$\Leftrightarrow f^+ \text{ and } f^- \text{ integrable} \quad (3.116)$$

即 f 可积 $\Leftrightarrow \int f^+$ 和 $\int f^-$ 均有界。

性质 下面我们将说明, 定义在任一集合 X 上的实可积函数构成的空间 \mathcal{L}^1 为线性空间, 以及 $f \in \mathcal{L}^1$ 时的一些性质.

在此之前, 先给出上述定义的一般的可测函数的积分的基本性质.

命题 3.1.4. Suppose $f, g \in \mathcal{L}$, then

1. **Linearity:** $\int (af + bg) = a \int f + b \int g.$

2. **Finite Additivity:**

$$\int_{\bigsqcup_{j=1}^n A_j} f = \sum_{j=1}^n \int_{A_j} f \quad (3.117)$$

where $\{A_j\}_{j=1}^n$ are disjoint.

3. **Monotonicity:** If $f \leq g$, then $\int f \leq \int g.$

4. **Triangle inequality:** $|\int f| \leq \int |f|.$

证明.

2. : We shall show that $\int_{\bigsqcup_{j=1}^n A_j} f^+ = \sum_{j=1}^n \int_{A_j} f^+$ and $\int_{\bigsqcup_{j=1}^n A_j} f^- = \sum_{j=1}^n \int_{A_j} f^-.$

By **Thm 2.2.1**, there exists simple $\varphi_n \nearrow f^+$, then by **MCT (Thm 3.1.2, 单调收敛定理)**,

$$\int_{\bigsqcup_{j=1}^n A_j} f^+ = \lim_{n \rightarrow \infty} \int_{\bigsqcup_{j=1}^n A_j} \varphi_n \quad (3.118)$$

Since φ_n are simple, by the **countable additivity** (简单函数的可数可加性), we have

$$\int_{\bigsqcup_{j=1}^n A_j} f^+ = \lim_{n \rightarrow \infty} \int_{\bigsqcup_{j=1}^n A_j} \varphi_n = \lim_{n \rightarrow \infty} \sum_{j=1}^n \int_{A_j} \varphi_n = \sum_{j=1}^n \lim_{n \rightarrow \infty} \int_{A_j} \varphi_n \quad (3.119)$$

$$\stackrel{\text{MCT}}{=} \sum_{j=1}^n \int_{A_j} f^+ \quad (3.120)$$

4. 根据实数域上的三角不等式, we have

$$\left| \int f \right| = \left| \int f^+ - \int f^- \right| \leq \left| \int f^+ \right| + \left| \int f^- \right| = \int f^+ + \int f^- = \int |f| \quad (3.121)$$

□

现在我们便可以来说明，定义在任一集合 X 上的实可积函数构成的空间 \mathcal{L}^1 为线性空间。

命题 3.1.5. The set of integrable real-valued functions on X is a real vector space.

证明. $\forall f, g \in \mathcal{L}^1$, if $a \in \mathbb{R}$,

$$\int |f + g| \leq \int (|f| + |g|) = \int |f| + \int |g| < \infty \quad (3.122)$$

$$\int |af| = |a| \int |f| < \infty \quad (3.123)$$

Therefore, $f + g, af \in \mathcal{L}^1$. $\Rightarrow \mathcal{L}^1$ is a real vector space. \square

对于可积函数，我们往往是在整个 \mathbb{R}^d 空间上讨论其可积性，类比 **Riemann** 可积函数，合理地猜测其在 \mathbb{R}^d 平面上“较远”的地方的积分值应当较小。这就是下面我们要给出的 \mathcal{L}^1 可积函数的性质。

命题 3.1.6. Suppose $f \in \mathcal{L}^1(\mathbb{R}^d)$. Then $\forall \epsilon > 0$

(i) \exists a set of finite measure B such that

$$\int_{B^c} |f| < \epsilon$$

(ii) [**Absolutely Continuity**].

$\exists \delta > 0$ such that

$$\int_E |f| < \epsilon, \forall m(E) < \delta$$

注. • (i) 和 (ii) 共同说明了，若 $f \in \mathcal{L}^1(\mathbb{R}^d)$ ，则 f 的积分主要集中在一个有限测度区域内，且在很小的区域内 f 的积分值趋于零。

• (ii) 本质为测度的绝对连续性 (正测度关于正测度的绝对连续性)。此处令正测度

$$\mu : \mathcal{L} \longrightarrow [0, \infty] \quad (3.124)$$

$$E \longmapsto \mu(E) = \int_E |f| \quad (3.125)$$

则命题 (ii) 可表示为： $\forall \epsilon > 0, \exists \delta > 0$, s. t.

$$\mu(E) < \epsilon, \forall m(E) < \delta$$

证明.

(i) : 对定义域做截断.

Suppose $f \geq 0$. Let $B_n = B(0, n)$, $f_n = f\chi_{B_n}$, then $f_n \nearrow f$.

By **MCT (Thm 3.1.2, 单调收敛定理)**,

$$\lim_{n \rightarrow \infty} \int f_n = \int f \quad (3.126)$$

Then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, s. t.

$$\left| \int f - \int f_N \right| = \int f - \int f_N = \int f(1 - \chi_{B_N}) = \int f\chi_{B_N^c} = \int_{B_N^c} f < \epsilon \quad (3.127)$$

Therefore, let $B = B_N = B(0, N)$, the desired result follows.

(ii) : 同样是做截断. 不过此处是对 f 的取值做截断.

Let $B_n = \{x \in \mathbb{R}^d \mid f(x) \leq n\}$, $f_n = f\chi_{B_n}$. Then $f_n \nearrow f$, $f_n \leq n$.

同 (i), By **MCT (Thm 3.1.2, 单调收敛定理)**,

$$\lim_{n \rightarrow \infty} \int f_n = \int f \quad (3.128)$$

$\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, s. t.

$$\left| \int f - \int f_N \right| = \int (f - f_N) < \frac{\epsilon}{2} \quad (3.129)$$

Pick $\delta > 0$, s. t. $N\delta < \frac{\epsilon}{2}$. Then for all $m(E) < \delta$,

$$\int_E f = \int_E (f - f_N) + \int_E f_N \leq \int (f - f_N) + N \cdot m(E) \quad (3.130)$$

$$< \frac{\epsilon}{2} + N\delta \quad (3.131)$$

$$< \epsilon \quad (3.132)$$

□

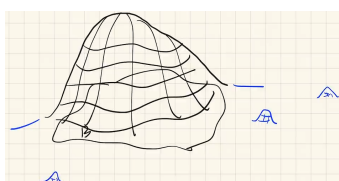


图 3.1: Prop 3.1.6 (i)

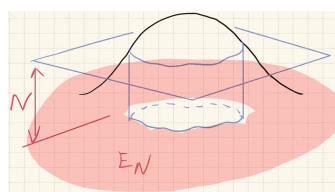


图 3.2: Prop 3.1.6 (ii)

3.1.4 The Dominated Convergence Theorem

下面我们来介绍实分析中最最有用的定理——

控制收敛定理 (The Dominated Convergence Theorem).

在 **Riemann** 积分中，对于函数列交换极限与积分的次序的条件太过于奇怪与繁琐，而在 **Lebesgue** 积分中，控制收敛定理则很完美地解决了这一问题。它对于交换极限与积分的次序的条件十分简洁。下面便来介绍这一定理。

定理 3.1.7. The Dominated Convergence Theorem (DCT).

Suppose $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+$, $f_n \rightarrow f$ a.e.. If $|f_n| \leq g$, where $g \in \mathcal{L}^1(\mathbb{R}^d)$, then

$$\int |f_n - f| \rightarrow 0, \quad n \rightarrow \infty \quad (3.133)$$

and consequently

$$\int f_n \rightarrow \int f, \quad n \rightarrow \infty \quad (3.134)$$

证明. 分别对 $g + f_n$ 和 $g - f_n$ 利用 **Fatou's Lemma (Thm 3.1.6)** 即可得证。

- Since $g + f_n \geq 0$, then by **Fatou's Lemma (Thm 3.1.6)**,

$$\int \liminf_{n \rightarrow \infty} (g + f_n) \leq \liminf_{n \rightarrow \infty} \int (g + f_n) \quad (3.135)$$

Since $f_n \rightarrow f$, we have

$$\int g + \int f \leq \int g + \liminf_{n \rightarrow \infty} \int f_n \quad (3.136)$$

$$\int f \leq \liminf_{n \rightarrow \infty} \int f_n \quad (3.137)$$

- Since $g - f_n \geq 0$, then by **Fatou's Lemma (Thm 3.1.6)**,

$$\int \liminf_{n \rightarrow \infty} (g - f_n) \leq \liminf_{n \rightarrow \infty} \int (g - f_n) \quad (3.138)$$

$$\int g - \int f \leq \int g + \liminf_{n \rightarrow \infty} (-\int f_n) \quad (3.139)$$

$$= \int g - \limsup_{n \rightarrow \infty} \int f_n \quad (3.140)$$

Then

$$\int f \geq \limsup_{n \rightarrow \infty} \int f_n \quad (3.141)$$

Therefore

$$\limsup_{n \rightarrow \infty} \int f_n \leq \int f \leq \liminf_{n \rightarrow \infty} \int f_n \quad (3.142)$$

which means $\lim_{n \rightarrow \infty} \int f_n$ exists, and

$$\lim_{n \rightarrow \infty} \int f_n = \int f \quad (3.143)$$

□

3.1.5 Complex – Valued Functions

下面我们将实值函数上的 **Lebesgue** 积分推广至复值函数.

先来规定一些记号:

- Let $f : \mathbb{R}^d \rightarrow \mathbb{C}$, write $f(x) = u(x) + iv(x)$.

下面给出复值函数可测以及可积的定义.

定义 3.1.7. Suppose $f : \mathbb{R}^d \rightarrow \mathbb{C}$, $f = u + iv$, then we say

- f is **measurable** if u and v are both measurable.
- f is **Lebesgue integrable** if $|f|$ is Lebesgue integrable.

注. 事实上, 根据此处定义, f 可积 $\Leftrightarrow u$ and v 都可积.

证明.

- f is integrable $\Rightarrow \int \sqrt{u^2 + v^2} < \infty \Rightarrow \int |u|, \int |v| \leq \int \sqrt{u^2 + v^2} < \infty \Rightarrow u$ and v 可积.
- u and v 可积 $\Rightarrow \int |u|, \int |v| < \infty \Rightarrow \int \sqrt{u^2 + v^2} \leq \int |u| + \int |v| < \infty \Rightarrow f$ 可积.

□

下面对命题 3.1.5 的结论进行推广, 即由复值可积函数构成的空间为线性空间.

命题 3.1.7. $\mathcal{L}^1(\mathbb{R}^d, \mathbb{C})$ is a vector space.

证明. Trivial.

□

3.2 \mathcal{L}^1 空间的完备性

引入 在讲 **Riemann** 积分时,我们称 **Riemann** 可积函数构成的空间是不完备的 (not complete). 在提及完备这个概念之前,我们需要先引入衡量“距离”的工具,即范数和度量.

3.2.1 范数, 度量

下面给出范数和度量的严格定义.

定义 3.2.1. Let X be a vector space over \mathbb{F} , a **norm** is a function:

$$X \longrightarrow \mathbb{R}_{\geq 0} \quad (3.144)$$

$$f \longmapsto \|f\| \quad (3.145)$$

satisfying the following properties:

- (i) $\|f\| \geq 0, \forall f \in X.$ ($\|f\| = 0 \Leftrightarrow f = 0 \text{ a.e.}$)
- (ii) $\|af\| = |a| \|f\|, \forall a \in \mathbb{F}, f \in X.$
- (iii) $\|f + g\| \leq \|f\| + \|g\|, \forall f, g \in X.$

注. • (i) 中的 “ $\|f\| = 0 \Leftrightarrow f = 0 \text{ a.e.}$ ” 的 “a.e.” 是针对 X 取函数空间时的条件, 在实分析的取等条件中基本为默认叙述, 在后续定义中往往省略. 在对 \mathcal{L}^1 空间的定义 (定义 3.2.4) 中可以看到其合理性.

- 范数实际上是对 \mathbb{R}^n 空间中“与原点之间的距离”这一概念的推广. 将函数视作向量, 则其范数即为到原点的距离, 即模长.
- 若一个线性空间 X 上配备了一个范数, 则称其为**赋范向量空间 (赋范线性空间)**.

将函数视作向量，就有其到原点的距离为范数。但若是想要衡量任意两个函数之间的距离，则需要引入下面度量的概念。

定义 3.2.2. A metric on X is a map

$$d : X \times X \longrightarrow \mathbb{R}_{\geq 0} \quad (3.146)$$

$$(x, y) \longmapsto d(x, y) \quad (3.147)$$

satisfying

$$(i) \quad d(x, y) \geq 0, \forall x, y \in X. \quad (d(x, y) = 0 \Leftrightarrow x = y)$$

$$(ii) \quad d(x, y) = d(y, x), \forall x, y \in X.$$

$$(iii) \quad d(x, y) + d(y, z) \geq d(x, z), \forall x, y, z \in X.$$

注. • 若 X 为函数空间，则 (i) 中 “ $d(x, y) = 0$ ” 等价条件默认为 “ $x = y$ a.e.” .

• 度量可看作将两个函数 (向量) 的起点均平移至原点后，其两个终点之间的距离。

3.2.2 The Space $\mathcal{L}^1(\mathbb{R}^d)$

范数 下面先在所有 **Lebesgue** 可积函数构成的空间上定义范数。

定义 3.2.3. For any integrable function f on \mathbb{R}^d , we define the norm of f ,

$$\|f\| = \int_{\mathbb{R}^d} |f| dx \quad (3.148)$$

注. • 由命题 3.1.3 可知，此处 $\|f\| = 0 \Leftrightarrow f = 0$ a.e.

• 容易证明，如此定义的范数满足范数应当满足的三条公理. (定义 3.2.1)

Space $\mathcal{L}^1(\mathbb{R}^d)$ 由于定义 3.2.3 中 “ $\|f\| = 0 \Leftrightarrow f = 0 \text{ a.e.}$ ”，而我们对零测集上的函数性质并不关心，因而引出了如下关于 \mathcal{L}^1 空间的定义.

定义 3.2.4. 我们在所有 Lebesgue 可积函数构成的空间上定义一个等价关系 “ \sim ”：

$$f \sim g \Leftrightarrow f = g \text{ a.e.}$$

$\mathcal{L}^1(\mathbb{R}^d)$ is the space of equivalence classes of integrable functions.

注. 由定义可知， $\mathcal{L}^1(\mathbb{R}^d)$ 空间中的元素实际上为函数的等价类 (集合)

$$[f] = \{g \text{ integrable} \mid g \sim f\}$$

而实际中，我们还是习惯性地当作单独的函数进行运算，这在几乎处处意义下是等价的.

度量 下面我们说明，根据定义 3.2.3 中所定义的范数可诱导出 $\mathcal{L}^1(\mathbb{R}^d)$ 上的一个度量.

命题 3.2.1.

$$d : \mathcal{L}^1(\mathbb{R}^d) \times \mathcal{L}^1(\mathbb{R}^d) \longrightarrow \mathbb{R}_{\geq 0} \quad (3.149)$$

$$(f, g) \longmapsto d(f, g) := \|f - g\| \quad (3.150)$$

defines a metric on $\mathcal{L}^1(\mathbb{R}^d)$.

证明. 下面即来逐一验证定义 3.2.2 中的三条公理.

- 根据范数的非负性， $\forall f, g \in \mathcal{L}^1(\mathbb{R}^d)$ ， $d(f, g) = \|f - g\| \geq 0$.

$$d(f, g) = 0 \Leftrightarrow f - g = 0 \text{ a.e.} \Leftrightarrow f = g \text{ in } \mathcal{L}^1(\mathbb{R}^d)$$

- 可交换性. $\forall f, g \in \mathcal{L}^1(\mathbb{R}^d)$,

$$d(f, g) = \|f - g\| = \int_{\mathbb{R}^d} |f - g| = \int_{\mathbb{R}^d} |g - f| = \|g - f\| = d(g, f) \quad (3.151)$$

- 根据范数的三角不等式， $\forall f, g, h \in \mathcal{L}^1(\mathbb{R}^d)$,

$$d(f, g) + d(g, h) = \|f - g\| + \|g - h\| \geq \|(f - g) + (g - h)\| = \|f - h\| = d(f, h)$$

□

3.2.3 \mathcal{L}^1 空间的完备性

定义 在得到了范数、度量的定义后，我们下面给出完备空间的定义。

定义 3.2.5. A metric space X is complete if every Cauchy Sequence $\{x_k\}_{k=1}^{\infty}$ has a limit in X .

注. • 完备空间即指空间中的任一柯西列都有收敛到自身的极限。

• 下面给出一个不完备的度量空间的例子。

例 3.2.1. 取一维实数域 \mathbb{R} 的子空间 $(0, 1) \subset \mathbb{R}$ ，考虑其上的 Cauchy Sequence $\{\frac{1}{n}\}_{n=2}^{\infty} \subset (0, 1)$ 。

由于 $\frac{1}{n} \rightarrow 0 \notin (0, 1)$ ，因此度量空间 $(0, 1)$ 不完备。

\mathcal{L}^1 空间的完备性 下面我们将给出本小节最重要的结论，即 \mathcal{L}^1 空间的完备性，这也是其比 Riemann 可积函数所构成的空间的优越性之所在。

定理 3.2.1. (Riesz - Fischer).

\mathcal{L}^1 is complete in its metric.

证明. Let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{L}^1(\mathbb{R}^d)$ be a Cauchy Sequence in \mathcal{L}^1 , then

$$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}, \forall n, m \geq N(\epsilon), \text{ s. t. } \|f_n - f_m\| \leq \epsilon$$

Tacking $\epsilon = 2^{-k}$, then $\exists N(2^{-k}) \geq N(2^{-(k-1)})$, s. t. for $n_k = N(2^{-k})$, $n_{k+1} = N(2^{-(k+1)})$,

$$\|f_{n_k} - f_{n_{k+1}}\| \leq 2^{-k}$$

下面分为三步进行证明。

• 构建 $f(x)$ 并利用 $g(x)$ 证明 $f \in \mathcal{L}^1$ ，证明子列 $\{f_{n_j}\}_{j=1}^{\infty}$ 收敛到 f 。

Let

$$f = f_{n_1} + \sum_{j=1}^{\infty} (f_{n_{j+1}} - f_{n_j}) \quad (3.152)$$

$$g = |f_{n_1}| + \sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}| \quad (3.153)$$

Then by **MCT (Thm 3.1.2, 控制收敛定理)**

$$\int g = \int |f_{n_1}| + \int \sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}| = \int |f_{n_1}| + \sum_{j=1}^{\infty} \int |f_{n_{j+1}} - f_{n_j}| \quad (3.154)$$

$$= \int |f_{n_1}| + \sum_{j=1}^{\infty} \|f_{n_{j+1}} - f_{n_j}\| \quad (3.155)$$

$$\leq \int |f_{n_1}| + \sum_{j=1}^{\infty} 2^{-j} < \infty \quad (3.156)$$

Therefore g is integrable, $g \in \mathcal{L}^1$. Since $|f| \leq g$, then $\int |f| < \infty$. f is integrable.

Let

$$S_k = f_{n_1} + \sum_{j=1}^k (f_{n_{j+1}} - f_{n_j}) = f_{n_{k+1}}, \quad k = 1, 2, \dots \quad (3.157)$$

$$f \text{ is integrable} \Rightarrow f < \infty \text{ a.e.} \Rightarrow S_k \text{ converges a.e.} \Rightarrow S_k = f_{n_{k+1}} \rightarrow f \text{ a.e.}$$

So we find

$$f_{n_k} \rightarrow f \text{ a.e.}$$

- 将逐点收敛性转化为 \mathcal{L}^1 收敛性, 即证 $\|f - f_{n_k}\| \rightarrow 0$.

We note that

$$|f - f_{n_k}| = \left| \left(f_{n_1} + \sum_{j=1}^{\infty} (f_{n_{j+1}} - f_{n_j}) \right) - \left(f_{n_1} + \sum_{j=1}^{k-1} (f_{n_{j+1}} - f_{n_j}) \right) \right| \quad (3.158)$$

$$= \left| \sum_{j=k}^{\infty} (f_{n_{j+1}} - f_{n_j}) \right| \leq g \quad (3.159)$$

By **DCT (Thm 3.1.7, 控制收敛定理)**, since $|f - f_{n_k}| \rightarrow 0$ a.e., $|f - f_{n_k}| \leq g$, g integrable,

$$\lim_{k \rightarrow \infty} \|f - f_{n_k}\| = \lim_{k \rightarrow \infty} \int |f - f_{n_k}| \stackrel{\text{DCT}}{=} \int \lim_{k \rightarrow \infty} |f - f_{n_k}| = 0 \quad (3.160)$$

Therefore, $\|f - f_{n_k}\| \rightarrow 0$. 即 f_{n_k} 依 \mathcal{L}^1 范数收敛到 f .

- 利用子列 $\{f_{n_k}\}_{k=1}^{\infty}$ 作为“桥梁”，证明 f_n 依 \mathcal{L}^1 范数收敛到 f ，即 $\|f_n - f\| \rightarrow 0$.
 $\forall \epsilon > 0$ ，由于 $\{f_n\}_{n=1}^{\infty}$ 为 \mathcal{L}^1 中 Cauchy Sequence, 因此 $\exists N \in \mathbb{N}$, s. t.

$$\|f_n - f_m\| < \frac{\epsilon}{2}, \quad \forall n, m > N$$

Since $\|f_{n_k} - f\| \rightarrow 0$, then for $\epsilon > 0$, pick $n_k > N$ which s. t.

$$\|f_{n_k} - f\| < \frac{\epsilon}{2}$$

Then

$$\|f_n - f\| \leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\| < \epsilon, \quad \forall n > n_k > N \quad (3.161)$$

Therefore $\|f_n \rightarrow f\| \rightarrow 0$ with $f \in \mathcal{L}^1$. \mathcal{L}^1 is complete in its metric.

□

根据上述定理的证明过程，可以得到下面的推论.

推论 3.2.2. If $\{f_n\}_{n=1}^{\infty}$ converges to f in \mathcal{L}^1 , then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that

$$f_{n_k}(x) \rightarrow f(x) \text{ a.e.}$$

注. 即在依 \mathcal{L}^1 范数收敛的函数序列中，总存在“几乎处处收敛”意义的子列.

3.2.4 \mathcal{L}^1 的稠密子空间

下面说明 \mathcal{L}^1 空间中以下的函数集合是稠密的.

定理 3.2.3. The following families of functions are dense in $\mathcal{L}^1(\mathbb{R}^d)$:

- The simple functions.
- The step functions.
- The continuous functions of compact support.

证明. 详情可见视频 [Urysohn 引理与 \$\mathcal{L}^1\$ 的稠密子空间](#).

□

3.3 Lebesgue 积分的平移不变性

首先给出平移算符及函数平移的符号表达.

定义 3.3.1. The translation by a vector h on \mathbb{R}^d is denoted by the map $\tau_h : x \mapsto x - h$. If f is a function defined on \mathbb{R}^d , the translation of f by $h \in \mathbb{R}^d$ is the function f_h , defined by

$$f_h(x) = (f \circ \tau_h)(x) = f(x - h)$$

下面给出 **Lebesgue** 积分的平移不变性.

定理 3.3.1. If $f \in \mathcal{L}^1(\mathbb{R}^d)$, then $\forall h \in \mathbb{R}^d, f_h \in \mathcal{L}^1(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} f(x - h) dx = \int_{\mathbb{R}^d} f(x) dx \quad (3.162)$$

证明. 下面按 *Lebesgue* 积分的构造过程来证明, 即特征函数 \Rightarrow 简单函数 \Rightarrow 非负可测.

- **Characteristic Function.**

Suppose $f = \chi_E$, where $E \subset \mathbb{R}^d$ is measurable. Then

$$f_h(x) = f(x - h) = \chi_E(x - h) = \begin{cases} 1, & \text{if } x - h \in E \\ 0, & \text{if } x - h \notin E \end{cases} = \begin{cases} 1, & \text{if } x \in E + h = E_h \\ 0, & \text{if } x \in (E + h)^c = E_h^c \end{cases} \quad (3.163)$$

根据 *Lebesgue* 测度的平移不变性,

$$\int_{\mathbb{R}^d} f_h = m(E_h) = m(E) = \int_{\mathbb{R}^d} f \quad (3.164)$$

- **Simple Function.**

$\forall \varphi = \sum_{k=1}^n a_k \chi_{E_k}$ simple, by the **linearity of integration**,

$$\int_{\mathbb{R}^d} \varphi_h = \sum_{k=1}^n \int_{\mathbb{R}^d} \chi_{(E_k)_h} = \sum_{k=1}^n \int_{\mathbb{R}^d} \chi_{E_k} = \int_{\mathbb{R}^d} \varphi \quad (3.165)$$

- **Non-negative Function.**

$\forall f$ non-negative, $\exists \{\varphi_n\}_{n=1}^{\infty}$ simple, s. t. $\varphi \nearrow f$ and $\varphi \geq 0$. Then by **MCT (Thm 3.1.2)**,

$$\int_{\mathbb{R}^d} \varphi_n \rightarrow \int_{\mathbb{R}^d} f \text{ as } n \rightarrow \infty \quad (3.166)$$

Since $(\varphi_n)_h \nearrow f_h$ and $\int \varphi_n = \int (\varphi_n)_h$, then by **MCT (Thm 3.1.2)**,

$$\int_{\mathbb{R}^d} \varphi_n = \int_{\mathbb{R}^d} (\varphi_n)_h \rightarrow \int_{\mathbb{R}^d} f_h \text{ as } n \rightarrow \infty \quad (3.167)$$

Therefore

$$\int_{\mathbb{R}^d} f_h = \int_{\mathbb{R}^d} f \quad (3.168)$$

- **General Case.**

$\forall f \in \mathcal{L}^1(\mathbb{R}^d)$, $f = f^+ - f^-$, where f^+ and f^- are non-negative.

Then by the **linearity of integration**,

$$\int_{\mathbb{R}^d} f_h = \int_{\mathbb{R}^d} f_h^+ - \int_{\mathbb{R}^d} f_h^- = \int_{\mathbb{R}^d} f^+ - \int_{\mathbb{R}^d} f^- = \int_{\mathbb{R}^d} f \quad (3.169)$$

□

3.4 Lebesgue 可积函数的 \mathcal{L}^1 连续性

引入 Recall 数学分析中连续的等价定义:

$$f \text{ is continuous at } x \Leftrightarrow f(x) - f(x-h) \rightarrow 0 \text{ as } h \rightarrow 0 \quad (3.170)$$

$$\Leftrightarrow |f_h(x) - f(x)| \rightarrow 0 \text{ as } h \rightarrow 0 \quad (3.171)$$

即可大致视作 **Riemann** 可积函数关于 2-范数的连续性.

Lebesgue 可积函数的 \mathcal{L}^1 连续性 在 \mathcal{L}^1 空间中, **Lebesgue** 可积函数也有类似的关于 \mathcal{L}^1 范数的连续性. 这就是下面的定理.

定理 3.4.1. Suppose $f \in \mathcal{L}^1(\mathbb{R}^d)$, then

$$\|f_h - f\|_{\mathcal{L}^1} \rightarrow 0 \text{ as } h \rightarrow 0 \quad (3.172)$$

证明. 详见视频[积分的平移不变性与可积函数的 \$\mathcal{L}^1\$ 连续性](#).

其中需要用到如下的引理.

引理 3.4.2. ² If $f \in C_c(\mathbb{R}^d)$, then f is uniformly continuous.

注. $f \in C_c(\mathbb{R}^d)$ 表示 f 为具有紧支集的连续函数.

□

²此为书: 《Real Analysis – Modern Techniques and Their Applications》— Gerald B. Folland

3.5 Fubini 定理

为了讨论的方便，下面先给出函数及集合的切片的定义.

定义 3.5.1. If f is a function on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, the slice of f w.r.t. $y \in \mathbb{R}^{d_2}$ is the function

$$f^y : \mathbb{R}^{d_1} \longrightarrow \overline{\mathbb{R}} \quad (3.173)$$

$$x \longmapsto f(x, y) \quad (3.174)$$

Similarly, the slice of f for a fixed $x \in \mathbb{R}^{d_1}$ is

$$f_x : \mathbb{R}^{d_2} \longrightarrow \overline{\mathbb{R}} \quad (3.175)$$

$$y \longmapsto f(x, y) \quad (3.176)$$

Let $E \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, we define its slices by

$$E^y := \{x \in \mathbb{R}^{d_1} \mid (x, y) \in E\}, \quad E_x := \{y \in \mathbb{R}^{d_2} \mid (x, y) \in E\} \quad (3.177)$$

下面给出 **Fubini 定理**.

定理 3.5.1. Fubini.

Suppose $f(x, y)$ is **integrable** on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for a.e. $y \in \mathbb{R}^{d_2}$:

- (i) The slice f^y is integrable on \mathbb{R}^{d_1} .
- (ii) The function $y \mapsto \int_{\mathbb{R}^{d_1}} f^y(x) dx$ is integrable on \mathbb{R}^{d_2} .
- (iii) Moreover,

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f^y(x) dx \right) dy = \int_{\mathbb{R}^d} f \quad (3.178)$$

3.5.1 Fubini 定理的证明

证明. Let $\mathcal{F} = \{f \in \mathcal{L}^1(\mathbb{R}^d) \mid f \text{ satisfies (i) } \sim \text{ (iii)}\}$. It suffices to show that $\mathcal{L}^1(\mathbb{R}^d) \subset \mathcal{F}$.

下面仍按照构造 Lebesgue 积分的顺序思路进行证明, 即特征函数 \Rightarrow 简单函数 \Rightarrow 非负可测.

(其中特征函数部分 (Step 3 ~ 5) 最为复杂繁琐, 后续的证明则是水到渠成)

在此之前, 还要先证明 \mathcal{F} 对函数的线性组合及单调函数列的极限封闭.

• **Step 1: Any finite linear combination of functions in \mathcal{F} also belongs to \mathcal{F} .**

Suppose $\{f_k\}_{k=1}^N \subset \mathcal{F}$. By the condition, $\forall k, \exists A_k \subset \mathbb{R}^{d_2}, m(A_k) = 0$, s. t.

$$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in A_k^c.$$

Let $A = \bigcup_{k=1}^N A_k$, then $m(A) = 0$ and

$$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in A^c, \forall k.$$

下面对定理结论逐条验证. By the linearity of integration, $\forall a_k \in \mathbb{R}$,

$$\left(\sum_{k=1}^N a_k f_k\right)^y = \sum_{k=1}^N a_k f_k^y \text{ is integrable on } \mathbb{R}^{d_1} \quad (3.179)$$

$$\int_{\mathbb{R}^{d_1}} \sum_{k=1}^N (a_k f_k)^y(x) dx = \sum_{k=1}^N a_k \int_{\mathbb{R}^{d_1}} f_k^y(x) dx \text{ is integrable on } \mathbb{R}^{d_2} \quad (3.180)$$

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \sum_{k=1}^N (a_k f_k)^y(x) dx \right) dy = \sum_{k=1}^N a_k \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f_k^y(x) dx \right) dy \quad (3.181)$$

$$= \sum_{k=1}^N a_k \int_{\mathbb{R}^d} f_k \quad (3.182)$$

$$= \int_{\mathbb{R}^d} \sum_{k=1}^N a_k f_k \quad (3.183)$$

$$(3.184)$$

Therefore, $\sum_{k=1}^N a_k f_k \in \mathcal{F}, \forall a_k \in \mathbb{R}$.

- **Step 2:** \mathcal{F} 对单调函数列的极限封闭, 即 $\forall \{f_k\}_{k=1}^\infty, f_k \nearrow f, f \text{ integrable} \Rightarrow f \in \mathcal{F}$.

Suppose $f_k \geq 0$. By the condition, $\forall k, \exists A_k \subset \mathbb{R}^d, m(A_k) = 0$, s. t.

$$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in A_k^c.$$

Let $A = \bigcup_{k=1}^\infty A_k$, then $m(A) = 0$ and

$$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in A^c, \forall k.$$

Since $f_k^y(x) \nearrow f^y(x)$, by **MCT (Thm 3.1.2)**

$$\int_{\mathbb{R}^{d_1}} f_k^y(x) dx \rightarrow \int_{\mathbb{R}^{d_1}} f^y(x) dx \text{ as } k \rightarrow \infty \quad (3.185)$$

Let

$$g_k(y) = \int_{\mathbb{R}^{d_1}} f_k^y(x) dx, \quad g(y) = \int_{\mathbb{R}^{d_1}} f^y(x) dx \quad (3.186)$$

Then we have $g_k(y) \nearrow g(y)$ and $g_k \geq 0$. By **MCT (Thm 3.1.2)**

$$\int_{\mathbb{R}^{d_2}} g_k(y) dy \rightarrow \int_{\mathbb{R}^{d_2}} g(y) dy \text{ as } k \rightarrow \infty \quad (3.187)$$

i.e.

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f_k^y(x) dx \right) dy \rightarrow \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f^y(x) dx \right) dy \quad (3.188)$$

By the condition (iii), we have

$$\int_{\mathbb{R}^d} f_k \rightarrow \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f^y(x) dx \right) dy \quad (3.189)$$

Since $f_k \nearrow f, f_k \geq 0$, by **MCT (Thm 3.1.2)**

$$\int_{\mathbb{R}^d} f_k \rightarrow \int_{\mathbb{R}^d} f \quad (3.190)$$

Therefore

$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f^y(x) dx \right) dy = \int_{\mathbb{R}^{d_2}} g(y) dy \quad (3.191)$$

Since f is integrable, $g(y) = \int_{\mathbb{R}^{d_1}} f^y(x) dx$ is integrable on $\mathbb{R}^{d_2} \Rightarrow \int g < \infty$.

Then we have

$$g(y) = \int_{\mathbb{R}^{d_1}} f^y(x) dx < \infty \text{ for a.e. } y \in \mathbb{R}^{d_2} \quad (3.192)$$

Therefore $f^y(x)$ is integrable for a.e. $y \in \mathbb{R}^{d_2}$.

Then $f \in \mathcal{F}$.

- **Step 3: Any characteristic function of a set E of type G_δ with finite measure belongs to \mathcal{F} .**

下面对 E 进行讨论, 分 $a \sim e$ 五种情况来证明:

(a) $E \subset \mathbb{R}^d$ is a bounded open cube.

Suppose $E = Q_1 \times Q_2$, where $Q_1 \subset \mathbb{R}^{d_1}$ and $Q_2 \subset \mathbb{R}^{d_2}$ are open cubes.

$\forall y \in \mathbb{R}^{d_2}$, $\chi_E(x, y)$ is measurable in x , and integrable with

$$g(y) = \int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx = \begin{cases} |Q_1|, & \text{if } y \in Q_2 \\ 0, & \text{if } y \notin Q_2 \end{cases} = |Q_1| \chi_{Q_2}(y) \quad (3.193)$$

Since $g(y) = |Q_1| \chi_{Q_2}(y)$ is measurable and integrable with

$$\int_{\mathbb{R}^{d_2}} g(y) dy = |Q_1| |Q_2| = |E| = \int_{\mathbb{R}^d} \chi_E \quad (3.194)$$

i.e.

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx \right) dy = \int_{\mathbb{R}^d} \chi_E \quad (3.195)$$

Therefore, $\chi_E \in \mathcal{F}$.

(b) $E \subset \mathbb{R}^d$ is a subset of the boundary of some closed cube.

Since $m(E) = 0$, we have

$$\int_{\mathbb{R}^d} \chi_E = m(E) = 0 \quad (3.196)$$

After an investigation of various possibilities, we note that (此处细节证明暂且留疑)

$$\forall \text{ a.e. } y \in \mathbb{R}^{d_2}, E^y = \{x \in \mathbb{R}^{d_1} \mid (x, y) \in E\} \text{ has measure 0 in } \mathbb{R}^{d_1}.$$

Then $\forall \text{ a.e. } y \in \mathbb{R}^{d_2}$, $\chi_E^y(x)$ is integrable on \mathbb{R}^{d_1} with

$$g(y) = \int_{\mathbb{R}^{d_1}} \chi_E^y(x) dx = \int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx = 0, \quad \forall \text{ a.e. } y \in \mathbb{R}^{d_2} \quad (3.197)$$

So $g(y)$ is integrable on \mathbb{R}^{d_2} with

$$\int_{\mathbb{R}^{d_2}} g(y) dy = 0 = \int_{\mathbb{R}^d} \chi_E \quad (3.198)$$

i.e.

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx \right) dy = \int_{\mathbb{R}^d} \chi_E \quad (3.199)$$

Therefore, $\chi_E \in \mathcal{F}$.

(c) $E \subset \mathbb{R}^d$ is a finite union of almost disjoint closed cubes.

Suppose $E = \bigcup_{k=1}^N Q_k$, where $\{Q_k^\circ\}_{k=1}^N$ are disjoint.

Let $A_k = Q_k - Q_k^\circ$ be the boundary of closed cube Q_k . Then $\chi_{A_k} \in \mathcal{F}$. (by **Step 3 (b)**)

χ_E is a linear combination of χ_{Q_k} and χ_{A_k} , $k = 1 \sim N$.

Since $\chi_{Q_k}, \chi_{A_k} \in \mathcal{F}$, $k = 1 \sim N$, then by **Step 1**, $\chi_E \in \mathcal{F}$.

(d) $E \subset \mathbb{R}^d$ is open and of finite measure.

Since $E \subset \mathbb{R}^d$ is open, by **Thm 1.1.4**, \exists almost disjoint closed cubes $\{Q_k\}_{k=1}^\infty$, s. t.

$$E = \bigcup_{k=1}^\infty Q_k, \text{ where } \{Q_k^\circ\}_{k=1}^\infty \text{ are disjoint} \quad (3.200)$$

Let

$$f_k = \chi_{\bigcup_{j=1}^k Q_j} \quad (3.201)$$

Then by **Step 3 (c)**, $f_k \in \mathcal{F}$, and $f_k \nearrow f = \chi_E$, $f_k \geq 0$. By **Step 2**, we have $f = \chi_E \in \mathcal{F}$.

(e) $E \subset \mathbb{R}^d$ is a G_δ of finite measure.

By the **definition of G_δ (Def 1.4.5)**,

$$E = \bigcap_{k=1}^\infty \widetilde{Q}_k, \text{ where } \widetilde{Q}_k \subset \mathbb{R}^d \text{ open} \quad (3.202)$$

Since E has finite measure, $\exists \widetilde{O}_0 \subset \mathbb{R}^d$ open, s. t. $E \subset \widetilde{O}_0$.

Let

$$O_k = \widetilde{O}_0 \cap \bigcap_{j=1}^k \widetilde{Q}_j \quad (3.203)$$

Then $O_1 \supset O_2 \supset \dots$ and $E = \bigcap_{k=1}^\infty O_k$. Let $f_k = \chi_{O_k}$, then $f_k \in \mathcal{F}$. (By **Step 3 (d)**)

Since $f_k \searrow f = \chi_E$, $f_k \in \mathcal{F}$, then by **Step 2**, $f = \chi_E \in \mathcal{F}$.

- **Step 4:** If $E \subset \mathbb{R}^d$ has measure 0, then $\chi_E \in \mathcal{F}$.

By **Thm 1.4.1**, \exists a set $G \subset \mathbb{R}^d$ of type G_δ with $E = G \setminus N$, where $m(N) = 0$. Then

$$E \subset G, m(G) = m(E) + m(G \setminus E) = 0.$$

By **Step 3**, $\chi_G \in \mathcal{F}$, then

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \chi_G(x, y) dx \right) dy = \int_{\mathbb{R}^d} \chi_G = 0 \quad (3.204)$$

Then

$$\int_{\mathbb{R}^{d_1}} \chi_G(x, y) dx = 0 \text{ for a.e. } y \in \mathbb{R}^{d_2} \quad (3.205)$$

Since

$$\int_{\mathbb{R}^{d_1}} \chi_G(x, y) dx = \int_{\mathbb{R}^{d_1}} \chi_G^y(x) dx = \int_{\mathbb{R}^{d_1}} \chi_{G^y}(x) dx = m(G^y) \quad (3.206)$$

Therefore

$$G^y = \{x \in \mathbb{R}^{d_1} \mid (x, y) \in G\} \text{ has measure 0 for a.e. } y \in \mathbb{R}^{d_2} \quad (3.207)$$

Since $E^y \subset G^y$, then E^y has measure 0 for a.e. $y \in \mathbb{R}^{d_2}$.

$$\Rightarrow \int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx = m(E^y) = 0 \text{ for a.e. } y \in \mathbb{R}^{d_2} \quad (3.208)$$

$$\Rightarrow \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx \right) dy = 0 = \int_{\mathbb{R}^d} \chi_E \quad (3.209)$$

$$\Rightarrow \chi_E \in \mathcal{F} \quad (3.210)$$

- **Step 5:** If E is any measurable subset of \mathbb{R}^d with finite measure, then χ_E belongs to \mathcal{F} .

By **Thm 1.4.1**, \exists a finite measure G of type G_δ with $E \subset G$ and $m(G - E) = 0$.

Since $\chi_E = \chi_G - \chi_{G-E}$, by **Step 4**, $\chi_G, \chi_{G-E} \in \mathcal{F}$, then by **Step 1**, $\chi_E \in \mathcal{F}$.

- **Step 6: If f is integrable, then $f \in \mathcal{F}$.**

不妨 Suppose f non-negative. By **Step 1 and Step 5**, $\forall \varphi = \sum_{k=1}^N a_k \chi_{E_k}$ simple, $\varphi \in \mathcal{F}$.

By **Thm 2.2.1**, $\exists \{\varphi_k\}_{k=1}^\infty$ simple, $\varphi_k \nearrow f$, $\varphi_k \geq 0$. Then by **Step 2**, $f \in \mathcal{F}$.

Therefore,

$$\mathcal{L}^1(\mathbb{R}^d) \subset \mathcal{F}$$

□

3.5.2 Fubini 定理的应用

Tonelli 定理 下面给出一个 **Fubini 定理** 的延伸形式, 就是 **Tonelli 定理**, 常与 **Fubini 定理** 一起使用, 用于判断函数的可积性.

定理 3.5.2. Tonelli.

Suppose $f(x, y)$ is a **non-negative measurable** function on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for a.e. $y \in \mathbb{R}^{d_2}$:

- (i) The slice f^y is **measurable** on \mathbb{R}^{d_1} .
- (ii) The function defined by $y \mapsto \int_{\mathbb{R}^{d_1}} f^y(x) dx$ is **measurable** on \mathbb{R}^{d_2} .
- (iii) Moreover,

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x, y) dx \right) dy = \int_{\mathbb{R}^d} f \in [0, \infty] \quad (3.211)$$

注. 在尚未知晓 f 的可积性时, 可先用 **Tonelli 定理** 计算 $|f|$ 的可积性, 从而得到 f 的可积性, 再去考虑使用 **Fubini 定理**.

证明. Consider the **truncations**

$$f_k(x, y) = \begin{cases} f(x, y), & \text{if } |(x, y)| < k \text{ and } f(x, y) < k \\ 0, & \text{otherwise} \end{cases} \quad (3.212)$$

Since

$$\int_{\mathbb{R}^d} f_k \leq k^{d+1} < \infty \quad (3.213)$$

f_k is integrable for all k . Then by **Fubini (Thm 3.5.1)**, $\exists E_k \subset \mathbb{R}^{d_2}$, $m(E_k) = 0$, s. t.

$$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in E_k^c.$$

$$g_k(y) = \int_{\mathbb{R}^{d_1}} f_k^y(x) dx \text{ is integrable on } \mathbb{R}^{d_2} \text{ a.e.} \quad (3.214)$$

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f_k(x, y) dx \right) dy = \int_{\mathbb{R}^d} f_k \quad (3.215)$$

下面开始验证 f 满足定理中的各条结论.

- Let $E = \bigcup_{k=1}^{\infty} E_k$, then $m(E) = 0$ and

$$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in E^c, \forall k.$$

$\forall y \in E^c$, since $f_k^y(x) \nearrow f^y(x)$, $f_k^y(x)$ integrable on \mathbb{R}^{d_1} , specifically measurable

Then $\forall y \in E^c$ $f^y(x)$ is measurable. i.e. f^y measurable for a.e. $y \in \mathbb{R}^{d_2}$.

- Since $f_k^y(x) \nearrow f^k(x)$, $\forall y \in E^c$, then by **MCT (Thm 3.1.2)**

$$g_k(y) = \int_{\mathbb{R}^{d_1}} f_k^y(x) dx \nearrow \int_{\mathbb{R}^{d_1}} f^y(x) dx = g(y) \text{ for a.e. } y \in \mathbb{R}^{d_2} \quad (3.216)$$

Since $g_k(y)$ is integrable on \mathbb{R}^{d_2} a.e., specifically measurable,

Then $g(y)$ is measurable on \mathbb{R}^{d_2} a.e.

- Since $g_k(y) \nearrow g(y)$, \forall a.e. $y \in \mathbb{R}^{d_2}$, then by **MCT (Thm 3.1.2)**

$$\int_{\mathbb{R}^{d_2}} g_k(y) dy \rightarrow \int_{\mathbb{R}^{d_2}} g(y) dy \quad (3.217)$$

i.e.

$$\int_{\mathbb{R}^d} f_k = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f_k(x, y) dx \right) dy \rightarrow \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x, y) dx \right) dy \quad (3.218)$$

Since $f_k \nearrow f$, by **MCT (Thm 3.1.2)**

$$\int_{\mathbb{R}^d} f_k \rightarrow \int_{\mathbb{R}^d} f \quad (3.219)$$

Therefore

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x, y) dx \right) dy = \int_{\mathbb{R}^d} f \in [0, \infty] \quad (3.220)$$

□

乘积测度 下面给出乘积测度在 **Lebesgue** 测度下的一些表现性质. 具体证明可见书³P82~85, 基本都是 Trivial 的.

推论 3.5.3. If E is a measurable set in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, then for a.e. $y \in \mathbb{R}^{d_2}$, the slice

$$E^y = \{x \in \mathbb{R}^{d_1} \mid (x, y) \in E\} \text{ is a measurable subset of } \mathbb{R}^{d_1}.$$

Moreover, $m(E^y)$ is a measurable function of y and

$$m(E) = \int_{\mathbb{R}^{d_2}} m(E^y) dy \quad (3.221)$$

注. • 该命题为 **Tonelli 定理 (Thm 3.5.2)** 的推论, 考虑 $f = \chi_E$ 即可轻松得证.

• 该推论说明了对于任一可测集 E , 其切片 E^y 都是几乎处处可测的.

有了推论 3.5.3, 我们自然会去思考一般情况下其逆命题是否成立, 即

$$E_y \text{ measurable for a.e. } y \in \mathbb{R}^{d_2} \Rightarrow E \subset \mathbb{R}^d \text{ measurable?}$$

然而答案是否定的. 下面给出一个反例.

例 3.5.1. Let \mathcal{N} denote a non-measurable subset \mathbb{R} (正测度集必有不可测子集, **Prop 1.5.1**).

Then define

$$E = [0, 1] \times \mathcal{N} \subset \mathbb{R} \times \mathbb{R}$$

We see that

$$E^y = \begin{cases} [0, 1], & \text{if } y \in \mathcal{N} \\ \emptyset, & \text{if } y \notin \mathcal{N} \end{cases}$$

Thus E^y is measurable for every $y \in \mathbb{R}$. However, if E is measurable, then by **Cor 3.5.3**,

$$E_x = \{y \in \mathbb{R} \mid (x, y) \in E\} = \begin{cases} \mathcal{N}, & \text{if } x \in [0, 1] \\ \emptyset, & \text{if } x \notin [0, 1] \end{cases}$$

which is a contradiction for \mathcal{N} is non-measurable.

³ 《Real Analysis – Measure Theory, Integration, & Hilbert Spaces》— Elias M. Stein

下面对推论 3.5.3 进行一定程度的推广，得到如下命题.

命题 3.5.1. If $E = E_1 \times E_2$ is a measurable subset of \mathbb{R}^d , and $m_*(E_2) > 0$, then E_1 is measurable.

而我们接下来将说明，若两个集合均可测，则他们的 **Descartes 积也是可测集**. 而这事实上就是抽象测度中**乘积测度**的定义. 在此之前，先来说明一个证明时需要用到的引理.

引理 3.5.4. If $E_1 \subset \mathbb{R}^{d_1}$ and $E_2 \subset \mathbb{R}^{d_2}$, then

$$m_*(E_1 \times E_2) \leq m_*(E_1)m_*(E_2)$$

下面便给出乘积测度在 **Lebesgue 测度**下的定义.

命题 3.5.2. Suppose E_1 and E_2 are measurable subsets of \mathbb{R}^{d_1} and \mathbb{R}^{d_2} , respectively. Then $E = E_1 \times E_2$ is a measurable subset of \mathbb{R}^d . Moreover,

$$m(E) = m(E_1)m(E_2)$$

几何联系 在 **Riemann** 积分中我们都熟知积分 $\int f$ 即代表 f 下方所围成区域的“体积”. 而下面我们将说明, 在 **Lebesgue** 积分中, 积分与几何直观之间的联系. (Stein P85~ 86)

在此之前先给出一个命题, 此为命题 **3.5.2**的推论.

推论 3.5.5. Suppose f is a measurable function on \mathbb{R}^{d_1} . Then the function \tilde{f} defined by

$$\tilde{f}(x, y) = f(x)$$

is measurable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$.

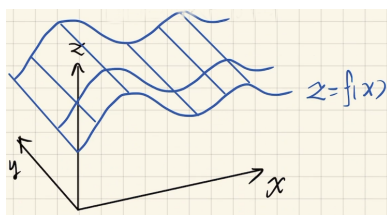


图 3.3: Prop 3.5.5

下面给出 **Lebesgue** 积分与几何直观之间的联系.

推论 3.5.6. Suppose $f(x)$ is a non-negative function on \mathbb{R}^d , and let

$$\mathcal{A} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} \mid 0 \leq y \leq f(x)\}$$

Then:

(i) f is measurable on \mathbb{R}^d iff \mathcal{A} is measurable on \mathbb{R}^{d+1} .

1. If the conditions in (i) hold, then

$$\int_{\mathbb{R}^d} f = m(\mathcal{A}) \tag{3.222}$$

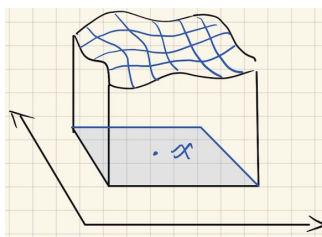


图 3.4: Prop 3.5.6

3.6 Lebesgue 积分与 Riemann 积分的联系

下面我们将说明, **Lebesgue** 积分可视为 **Riemann** 积分的延拓, 它很好地囊括了 **Riemann** 积分的定义.

在此之前, 先来给出 **MCT (Thm 3.1.2, 单调收敛定理)** 在单调递减函数列上的表述.

定理 3.6.1. Monotone Convergence Theorem (decreasing).

Let $\{f_n\}_{n=1}^\infty$ be non-negative, $f_n \searrow f$, $\int f_1 < \infty$, then

$$\lim_{n \rightarrow \infty} \int f_n = \int f \quad (3.223)$$

证明. Let $g_n = f_1 - f_n$, $n \in \mathbb{N}$. Then $g_n \geq 0$ and $g_n \nearrow g = f_1 - f$. By **MCT (Thm 3.1.2)**

$$\int g_n \rightarrow \int g \text{ as } n \rightarrow \infty \quad (3.224)$$

i.e.

$$\int (f_1 - f_n) \rightarrow \int (f_1 - f) \text{ as } n \rightarrow \infty \quad (3.225)$$

Therefore

$$\int f_n \rightarrow \int f \quad (3.226)$$

□

下面说明 **Riemann** 可积函数的积分即为其 **Lebesgue** 积分.

定理 3.6.2. Suppose f is Riemann integrable, then

$$\int_{[a,b]}^{\mathcal{R}} f(x) dx = \int_{[a,b]}^{\mathcal{L}} f(x) dx \quad (3.227)$$

证明. 详细证明可见书⁴§4.4 或视频 [Lebesgue 积分与 Riemann 积分的联系](#). □

⁴ 《实变函数论 (第三版)》——周民强

3.7 Lebesgue 积分的伸缩变换

下面我们给出 **Lebesgue** 积分的伸缩变换公式. 这实质上为一般的抽象测度的变量替换公式在 **Lebesgue** 测度下的特例, 而此处我们的证明方法为 **Lebesgue** 测度下的方法, 依赖于 \mathbb{R}^d 中的几何直观, 较为枯燥繁琐, 不具有一般性. 在后续学习抽象测度时会给出一般性的方法论.

命题 3.7.1. Lebesgue 积分的伸缩变换公式.

- $m(\delta E) = |\delta| m(E), \delta \in \mathbb{R}, E \subset \mathbb{R}.$
- $\int f(x)dx = |\delta| \int f(\delta x)dx, \delta \in \mathbb{R}, f \in \mathcal{L}^1(\mathbb{R}).$
- $\int f(x)dx = \delta_1 \cdots \delta_d \int f(\delta x)dx, \delta \in \mathbb{R}^d, \delta_j > 0, f \in \mathcal{L}^1(\mathbb{R}^d).$
- $m(\delta E) = \delta_1 \cdots \delta_d m(E), \delta_j > 0, E \subset \mathbb{R}^d.$

证明. 可见视频 [积分的伸缩变换](#) 或参考书⁵P73~74.

□

⁵ 《Real Analysis – Measure Theory, Integration, & Hilbert Spaces》— Elias M. Stein

3.8 Littlewood 三原则

Motivation 尽管我们建立起了围绕 Lebesgue 测度为中心的新的理论体系, 但我们仍应当重视其与数学分析中概念的联系. 而 Littlewood 便总结归纳出了这样三条 principles:

- (i) Every (measurable) set is **nearly** a finite union of intervals.
- (ii) Every function (of class \mathcal{L}^n) is **nearly** continuous.
- (iii) Every convergent sequence is **nearly** uniformly convergent.

不难发现其叙述显得并不太严谨, 其中的 **nearly** 一词需要我们给予严格的数学定义.

Littlewood 三原则告诉了我们可测函数与连续函数之间的联系, 包括收敛函数列与一致收敛的关系. 其中第一条原则即为定理 1.3.4 (iv).

下面我们从后往前依次给出第三、二条原则, 即 Egorov 定理与 Lusin 定理. 这在抽象测度中仍然起着重要作用.

3.8.1 Egorov 定理

关于 Littlewood 三原则中的 (iii), 实际上在数学分析中已不陌生. 下面给出一个经典例子.

例 3.8.1. Consider the sequence $f_n(x) = x^n$, $x \in [0, 1]$. Then f_n converges on $[0, 1]$ to f :

$$f(x) = \begin{cases} 0, & \text{if } x \in [0, 1) \\ 1, & \text{if } x = 1 \end{cases}$$

So $f_n \rightarrow f$ but not uniformly on $[0, 1]$.

However, if we consider the closed interval $[0, 1 - \epsilon]$ or any closed interval $[a, b]$ except 1, then

$$f_n \Rightarrow f \text{ uniformly on } [0, 1 - \epsilon] \text{ or } [a, b].$$

which implies “convergent sequence is nearly uniformly convergent”.

下面给出 **Egorov 定理** 的表述.

定理 3.8.1. Egorov (Almost Uniform Convergence, 近一致收敛).

Suppose $\{f_k\}_{k=1}^{\infty}$ is a sequence of measurable functions on a measurable set A with $m(A) < \infty$, and $f_k(x) \rightarrow f(x)$ on A a.e. Given $\epsilon > 0$, we can find a set $E \subset A$ s. t.

$$m(E) < \epsilon \text{ and } f_k \Rightarrow f \text{ uniformly on } E^c.$$

注. • 此处若将 **Lebesgue 测度 m** 换为一般的抽象测度 μ , 即可得到抽象测度下的 **Egorov 定理**. (可见书⁶P62 Thm 2.33)

• 在证明定理前, 先回顾一下函数列收敛点集 & 发散点集的表述.

– 收敛点集.

$$x \in \text{收敛点集} \Leftrightarrow \forall \epsilon > 0, \exists N, \forall n \geq N, |f_n(x) - f(x)| < \epsilon \quad (3.228)$$

$$\stackrel{\text{离散}}{\Leftrightarrow} \forall k \in \mathbb{N}, \exists N, \forall n \geq N, |f_n(x) - f(x)| < \frac{1}{k} \quad (3.229)$$

$$\Rightarrow C(f) = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ x \mid |f_n(x) - f(x)| < \frac{1}{k} \right\} \quad (3.230)$$

– 发散点集.

$$x \in \text{发散点集} \Leftrightarrow \exists \epsilon > 0, \forall N, \exists n \geq N, |f_n(x) - f(x)| \geq \epsilon \quad (3.231)$$

$$\stackrel{\text{离散}}{\Leftrightarrow} \exists k \in \mathbb{N}, \forall N, \exists n \geq N, |f_n(x) - f(x)| \geq \frac{1}{k} \quad (3.232)$$

$$\Rightarrow D(f) = \bigcup_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \left\{ x \mid |f_n(x) - f(x)| \geq \frac{1}{k} \right\} \quad (3.233)$$

⁶ 《Real Analysis – Modern Techniques and Their Applications》— Gerald B. Folland

证明. Let

$$E_{\textcolor{red}{m}}(k) = \bigcup_{n=\textcolor{red}{m}}^{\infty} \left\{ x \mid |f_n(x) - f(x)| \geq \frac{1}{k} \right\} \quad (3.234)$$

Then $E_m(k) \searrow$ in m .

Since $f_k(x) \rightarrow f(x)$ on A a.e., then $m(D(f)) = 0$. Since

$$\bigcap_{m=1}^{\infty} E_m(k) \subset D(f) = \bigcup_{k=1}^{\infty} \left(\bigcap_{m=1}^{\infty} E_m(k) \right) \quad (3.235)$$

Then $m\left(\bigcap_{m=1}^{\infty} E_m(k)\right) = 0$. Then by **Thm 1.3.3**,

$$m\left(\bigcap_{m=1}^{\infty} E_m(k)\right) = m\left(\lim_{N \rightarrow \infty} \bigcap_{m=1}^N E_m(k)\right) = \lim_{N \rightarrow \infty} m\left(\bigcap_{m=1}^N E_m(k)\right) = 0 \quad (3.236)$$

即得到数列 $\left\{ \bigcap_{m=1}^N E_m(k) \right\}_{N=1}^{\infty}$ 极限为 0. Then for any fixed $\epsilon > 0$, $\exists N_k \in \mathbb{N}$, s. t.

$$m\left(\bigcap_{m=1}^{N_k} E_m(k)\right) = m(E_{N_k}(k)) < \frac{\epsilon}{2^k} \quad (3.237)$$

Let $E = \bigcup_{k=1}^{\infty} E_{N_k}(k)$, then $m(E) < \epsilon$ and

$$E^c = \bigcap_{k=1}^{\infty} E_{N_k}^c(k) = \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} \left\{ x \mid |f_n(x) - f(x)| < \frac{1}{k} \right\} \quad (3.238)$$

Then we get for a fixed $k_0 \in \mathbb{N}$, $\exists N_{k_0}$, $\forall n \geq N_{k_0}$, s. t.

$$|f_n(x) - f(x)| < \frac{1}{k_0} \text{ for all } x \in E^c.$$

Therefore, $f_n \Rightarrow f$ uniformly on E^c with $m(E) < \epsilon$. □

3.8.2 *Lusin* 定理

下面给出 **Littlewood** 三原则中的第 (ii) 点, 可测函数 **nearly** 连续, 即 **Lusin** 定理.

定理 3.8.2. *Lusin*.

Suppose $f : E \rightarrow \mathbb{R}$ is measurable and finite-valued on E with $m(E) < \infty$. Then for every $\epsilon > 0$, there exists a compact set $F \subset E$, s. t.

$$m(F^c) < \epsilon \text{ and } f|_F \text{ is continuous.}$$

证明. 可见书⁷**P34 Thm 4.5** 或视频 [可测函数与连续函数的联系](#).

□

⁷ 《*Real Analysis – Measure Theory, Integration, & Hilbert Spaces*》— *Elias M. Stein*

第四章 *Differentiation and Integration*

Motivation 在 **Riemann** 积分的框架下, 我们知道积分和微分可以视作一对互逆的运算. 而在这一章, 我们将在全新的 **Lebesgue** 测度的框架下重新审视积分和微分之间的关系.

下面先来描述一下想要解决的问题.

- Let $f \in \mathcal{L}^1(\mathbb{R}^d)$. 对于变上限积分 $F(x) = \int_a^x f(y)dy$, 我们知道根据 **Riemann** 积分下的微积分基本定理, 对 F 求导就会回到被积函数 f 本身. 那么我们会好奇:
 - 在 **Lebesgue** 积分的框架下, 这个结论是否还成立?
 - 如果成立的话, 又对哪些 x 成立呢?

此时回顾求导的定义, 即对于差商 (此处改写为更具一般性的符号 $I = (x, x+h)$)

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(y)dy = \frac{1}{|I|} \int_I f(y)dy \quad (4.1)$$

对差商中的增量 $h \rightarrow 0$, 即得到导数的定义. 那么我们的问题就转化为了

$$\lim_{\substack{|I| \rightarrow 0 \\ I \ni x}} \frac{1}{|I|} \int_I f(y)dy = f(x) \text{ holds for which } x? \quad (4.2)$$

更一般地, 将上述问题从一维实直线 \mathbb{R} 推广至 \mathbb{R}^d 空间上, 将区间 I 用开球 B 替换, 得

$$\lim_{\substack{m(B) \rightarrow 0 \\ B \ni x}} \frac{1}{m(B)} \int_B f(y)dy = f(x) \text{ holds for which } x? \quad (4.3)$$

注. – 此处看似是随着开球 B 的测度减小, $x \in B$ 在跟着 B “跑”, 但实际上则相反:

对于每个固定的 x , 让包含着 x 的球 $B \ni x$ 不断减小其测度, 最后取极限

而这也正是此处极限条件写为 “ $B \ni x$ ” 而非 “ $x \in B$ ” 的原因, 逻辑更清晰.

- 事实上该结论对于几乎处处的 x 都成立 (若 f **Lebesgue** 可积), 这就是后面要讲的 **Lebesgue** 微分定理.

4.1 Hardy – Littlewood 极大函数 (非球心)

定义 下面我们给出 **Hardy-Littlewood** 极大函数的定义.

定义 4.1.1. If $f \in \mathcal{L}^1(\mathbb{R}^d)$, we define its **maximal function** Mf by

$$Mf(x) = \sup_{B \ni x} \frac{1}{m(B)} \int_B |f(y)| dy \quad (4.4)$$

注. 我们目前并不知道球面测度的具体数值与计算方法, 但事实上我们也并不需要知道其具体数值, 具体表现在:

设 \mathbb{R}^d 中单位球 $B(0, 1)$ 的测度为 $m(B(0, 1)) = v_d$. $\forall B(x, r) \subset \mathbb{R}^d$, 根据 **Lebesgue** 测度的平移不变性和伸缩变换公式 (**Prop 3.7.1**)

$$B(0, r) = rB(0, 1) \Rightarrow m(B(x, r)) = m(B(0, r)) = r^d m(B(0, 1)) = r^d v_d$$

性质 下面来说明 **Hardy-Littlewood** 极大函数的三条性质.

命题 4.1.1. Suppose $f \in \mathcal{L}^1(\mathbb{R}^d)$. Then:

- (i) Mf is measurable.
- (ii) $Mf(x) < \infty$ for a.e. x .
- (iii) **weak-type inequality.**

Mf satisfies

$$m(\{x \in \mathbb{R}^d \mid Mf(x) > a\}) \leq \frac{A}{a} \|f\|_{\mathcal{L}^1}, \quad \forall a > 0 \quad (4.5)$$

where $A = 3^d$.

证明.

- (i) Let $E_a = \{x \in \mathbb{R}^d \mid Mf(x) > a\}$. 下面证明 E_a open.

$\forall x \in E_a$, by the **definition of Mf** (**Def 4.1.1**), $\exists B_x \ni x$, s. t.

$$\frac{1}{m(B_x)} \int_{B_x} |f| > a \quad (4.6)$$

Then $\forall y \in B_x$, B_x is also an open ball containing y , so we have $y \in E_a$. i.e. $B_x \subset E_a$.

Therefore, E_a is open, specifically measurable for all a . Then Mf is measurable.

(ii) 下面说明 (iii) \Rightarrow (ii):

Let $E_a = \{x \in \mathbb{R}^d \mid Mf(x) > a\}$. Then $E_n \searrow E = \{x \in \mathbb{R}^d \mid Mf(x) = \infty\}$.

Since $f \in \mathcal{L}^1(\mathbb{R}^d)$, $\|f\|_{\mathcal{L}^1}$ is finite. Then by (iii), $m(E_1) < \infty$.

Then by **Thm 1.3.3**,

$$m(E) = \lim_{n \rightarrow \infty} m(E_n) \leq \lim_{n \rightarrow \infty} \frac{A}{n} \|f\|_{\mathcal{L}^1} = 0 \quad (4.7)$$

Therefore $m(E) = 0$. i.e. $Mf(x) < \infty$ for a.e. x .

(iii) 在证明 (iii) 之前, 先来介绍 **Vitali 覆盖引理**.

引理 4.1.1. Vitali Covering Lemma (Elementary Version).

Suppose $\mathcal{B} = \{B_1, B_2, \dots, B_N\}$, $B_i \subset \mathbb{R}^d$ are open balls, then there is a disjoint subcollection B_{i_1}, \dots, B_{i_k} that satisfies

$$m\left(\bigcup_{l=1}^N B_l\right) \leq 3^d \sum_{j=1}^k m(B_{i_j}) \quad (4.8)$$

注. 这是 **Vitali 覆盖引理**的初等版本 (有限版本), 更一般的版本是对一列球结论成立.

证明. 详见视频(非球心)Hardy-Littlewood 极大函数 23:10 (类似贪心算法的迭代步骤) \square

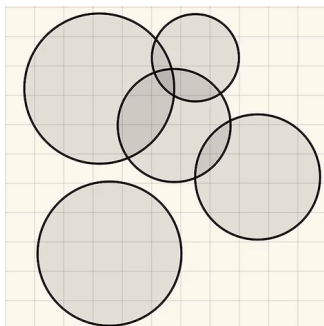


图 4.1: Lemma 4.1.1

下面继续来证明 (iii):

Fix $a > 0$, $\forall x \in B_a$, \exists open ball B_x , s. t.

$$\frac{1}{m(B_x)} \int_{B_x} |f| > a \quad (4.9)$$

So we have $E_a \subset \bigcup_{x \in E_a} B_x$.

Since E_a is measurable (by (i)), then by **Thm 1.3.4 (Lebesgue 测度的内正则性)**,

$\forall \epsilon > 0$, \exists compact $K_\epsilon \subset E_a$, s. t.

$$m(E_a \setminus K_\epsilon) \leq \epsilon$$

i.e.

$$m(E_a) - m(K_\epsilon) \leq \epsilon$$

Since K_ϵ is compact, $K_\epsilon \subset \bigcup_{x \in K_\epsilon} B_x$, there exists a subcollection B_{x_1}, \dots, B_{x_N} , s. t.

$$K_\epsilon \subset \bigcup_{l=1}^N B_{x_l}$$

Then by **Vitali Covering Lemma (Lemma 4.1.1)**, there exists a subcollection $B_{x_{i_1}}, \dots, B_{x_{i_k}}$, s. t.

$$m\left(\bigcup_{l=1}^N B_{x_l}\right) \leq 3^d \sum_{j=1}^k m(B_{x_{i_j}})$$

Therefore

$$m(K_\epsilon) \leq m\left(\bigcup_{l=1}^N B_{x_l}\right) \leq 3^d \sum_{j=1}^k m(B_{x_{i_j}}) \quad (4.10)$$

$$= \frac{3^d}{a} \sum_{j=1}^k a \cdot m(B_{x_{i_j}}) \quad (4.11)$$

$$\leq \frac{3^d}{a} \int_{\bigcup_{j=1}^k B_{x_{i_j}}} |f| \quad (4.12)$$

$$\leq \frac{3^d}{a} \int_{\mathbb{R}^d} |f| \quad (4.13)$$

$$= \frac{3^d}{a} \|f\|_{\mathcal{L}^1} \quad (4.14)$$

Then

$$m(E_a) \leq m(K_\epsilon) + \epsilon \leq \frac{A}{a} \|f\|_{\mathcal{L}^1} + \epsilon \quad (4.15)$$

where $A = 3^d$, $\epsilon > 0$.

Since ϵ is arbitrary, let $\epsilon \rightarrow 0$, we have

$$m(E_a) \leq \frac{A}{a} \|f\|_{\mathcal{L}^1}, \quad A = 3^d, \quad \forall a > 0 \quad (4.16)$$

□

4.2 Lebesgue 微分定理 (非球心)

在这一节我们将利用 **Hardy-Littlewood** 极大函数来证明 **Lebesgue** 微分定理.

4.2.1 Chebyshev's Inequality

在此之前, 我们先来证明一个非常有用的不等式, 即切比雪夫不等式.

定理 4.2.1. Chebyshev's Inequality.

If $g \in \mathcal{L}^1(\mathbb{R}^d)$, then

$$m(\{x \in \mathbb{R}^d \mid |g(x)| > a\}) \leq \frac{1}{a} \|g\|_{\mathcal{L}^1}, \quad \forall a > 0 \quad (4.17)$$

证明. Let $E_a = \{x \in \mathbb{R}^d \mid |g(x)| > a\}$. Then

$$\|g\|_{\mathcal{L}^1} = \int_{\mathbb{R}^d} |g| \geq \int_{E_a} |g| \geq \int_{E_a} a = a \cdot m(E_a) \quad (4.18)$$

□

4.2.2 The Lebesgue Differentiation Theorem

下面我们就来给出 **Lebesgue** 微分定理.

定理 4.2.2. If $f \in \mathcal{L}^1(\mathbb{R}^d)$, then

$$\lim_{\substack{m(B) \rightarrow 0 \\ B \ni x}} \frac{1}{m(B)} \int_B f(y) dy = f(x) \text{ for a.e. } x \quad (4.19)$$

注. • **Lebesgue** 微分定理说明了对于几乎处处的 x , 当包含 x 的球体 B 的测度趋于 0 时, f 在球体 B 上积分的平均值就会收敛到 $f(x)$.

- 定理左侧实际上是关于集合 B 的函数的一个极限过程, 用 $\epsilon - \delta$ 语言叙述如下:

$\forall \epsilon > 0, \exists \delta > 0$, s. t. for all $B \ni x$ and $m(B) < \delta$, we have

$$\left| \frac{1}{m(B)} \int_B f(y) dy - f(x) \right| \leq \epsilon \quad (4.20)$$

- 要证明该定理, 首先需要说明等式左侧极限的存在性, 但这并不好说明. 为了跳过说明其存在性的问题, 我们需要引入类似“上极限”的函数, 即:

If suffices to show

$$\lim_{\delta \rightarrow 0} \sup_{\substack{m(B) < \delta \\ B \ni x}} \left| \frac{1}{m(B)} \int_B f(y) dy - f(x) \right| = 0 \text{ for a.e. } x \quad (4.21)$$

由于极限内的函数随着 δ 递减而单调递减, 又存在下界 0, 因此在 $\delta = 0$ 处必存在右极限. 这样就跳过了原极限是否存在的问题.

- 事实上此处极限“怪异”的本质原因在于开球 B 的选取的任意性, 若将其定义为以 x 为球心, r 为半径的球, 则可直接令 $r \rightarrow 0$ 变为正常的函数极限, 即

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) dy = f(x) \quad (4.22)$$

在下一节我们会从 **Hardy-Littlewood** 极大函数开始, 以此方法重新说明 **Lebesgue** 微分定理.

证明. Let

$$E_a = \left\{ x \in \mathbb{R}^d \mid \lim_{\delta \rightarrow 0} \sup_{\substack{m(B) < \delta \\ B \ni x}} \left| \frac{1}{m(B)} \int_B f(y) dy - f(x) \right| > 2a \right\} \quad (4.23)$$

Then we **WTS (want to show)**:

$$m(E_a) = 0, \forall a \geq 0$$

Fix $a \geq 0$. By **Thm 3.2.3**, $C_c(\mathbb{R}^d)$ is dense in $\mathcal{L}^1(\mathbb{R}^d)$ (有紧支集连续函数), then $\forall \epsilon > 0, \exists g \in C_c(\mathbb{R}^d)$, s. t.

$$\|f - g\|_{\mathcal{L}^1} < \epsilon$$

Since g is uniformly continuous, then $\exists \delta > 0$, s. t.

$$\left| \frac{1}{m(B)} \int_B g(y) dy - g(x) \right| \leq \frac{1}{m(B)} \int_B |g(y) - g(x)| dy < \frac{1}{m(B)} \int_B \epsilon dy = \epsilon \quad (4.24)$$

for all $B \ni x$ and $m(B) < \delta$.

下面对 $m(E_a)$ 进行估计. $\forall x \in E_a$,

$$\left| \frac{1}{m(B)} \int_B f(y) dy - f(x) \right| \leq \left| \frac{1}{m(B)} \int_B (f(y) - g(y)) dy \right| + \left| \frac{1}{m(B)} \int_B g(y) dy - g(x) \right| + |g(x) - f(x)| \quad (4.25)$$

对上述不等式中的开球 $B \ni x$ 取上确界 \sup , 得

$$\sup_{B \ni x} \left| \frac{1}{m(B)} \int_B f(y) dy - f(x) \right| \leq \sup_{B \ni x} \left| \frac{1}{m(B)} \int_B (f(y) - g(y)) dy \right| + \sup_{B \ni x} \left| \frac{1}{m(B)} \int_B g(y) dy - g(x) \right| + |g(x) - f(x)| \quad (4.26)$$

再令 $m(B) \rightarrow 0$, 由于根据式 (4.24),

$$\lim_{\delta \rightarrow 0} \sup_{\substack{m(B) < \delta \\ B \ni x}} \left| \frac{1}{m(B)} \int_B g(y) dy - g(x) \right| = 0 \quad (4.27)$$

因此

$$\lim_{\delta \rightarrow 0} \sup_{\substack{m(B) < \delta \\ B \ni x}} \left| \frac{1}{m(B)} \int_B f(y) dy - f(x) \right| \leq \lim_{\delta \rightarrow 0} \sup_{\substack{m(B) < \delta \\ B \ni x}} \left| \frac{1}{m(B)} \int_B (f(y) - g(y)) dy \right| + 0 + |g(x) - f(x)| \quad (4.28)$$

下面对红色部分进行估计. 根据对 δ 的单调性可知,

$$\lim_{\delta \rightarrow 0} \sup_{\substack{m(B) < \delta \\ B \ni x}} \left| \frac{1}{m(B)} \int_B (f(y) - g(y)) dy \right| \leq \sup_{B \ni x} \frac{1}{m(B)} \int_B |f(y) - g(y)| dy = M(f - g)(x) \quad (4.29)$$

又因为对于 $\forall x \in E_a$,

$$\lim_{\delta \rightarrow 0} \sup_{\substack{m(B) < \delta \\ B \ni x}} \left| \frac{1}{m(B)} \int_B f(y) dy - f(x) \right| > 2a \quad (4.30)$$

所以

$$M(f - g)(x) + |g(x) - f(x)| > 2a \quad (4.31)$$

$$\Rightarrow M(f - g)(x) > a \text{ or } |g(x) - f(x)| > a \quad (4.32)$$

$$\Rightarrow E_a \subset \{x \in \mathbb{R}^d \mid M(f - g)(x) > a\} \cup \{x \in \mathbb{R}^d \mid |g - f| > a\} \quad (4.33)$$

下面分别来估计 the purple one 和 the orange one 的测度.

- 由于 $|f - g| \in \mathcal{L}^1 \mathbb{R}^d$, 因此根据 **Chebyshev's Inequality (Thm 4.2.1)**,

$$m(\{x \in \mathbb{R}^d \mid |g - f| > a\}) \leq \frac{1}{a} \|f - g\|_{\mathcal{L}^1} \quad (4.34)$$

- 根据 **Hardy-Littlewood 极大函数的 weak-type inequality (Prop 4.1.1 (iii))**,

$$m(\{x \in \mathbb{R}^d \mid M(f - g)(x) > a\}) \leq \frac{A}{a} \|f - g\|_{\mathcal{L}^1} \quad (4.35)$$

从而根据 $\|f - g\|_{\mathcal{L}^1} < \epsilon$,

$$m(E_a) \leq m(\{x \in \mathbb{R}^d \mid M(f - g)(x) > a\}) + m(\{x \in \mathbb{R}^d \mid |g - f| > a\}) \quad (4.36)$$

$$\leq \frac{A+1}{a} \|f - g\|_{\mathcal{L}^1} \quad (4.37)$$

$$< \frac{A+1}{a} \epsilon, \quad \forall a \geq 0 \quad (4.38)$$

Since $\epsilon > 0$ is arbitrary, let $\epsilon \rightarrow 0$, we get

$$m(E_a) = 0, \quad \forall a \geq 0 \quad (4.39)$$

□

4.3 Hardy – Littlewood 极大函数 & Lebesgue 微分定理 (球心)

4.3.1 Hardy – Littlewood 极大函数

本节的重点是给出 **Hardy-Littlewood 极大函数 (centered)** 的定义并证明其连续性.

Preliminaries 在此之前, 先来给出一些记号与命题.

We define 开球 & 球面

$$B(x, r) := \{y \in \mathbb{R}^d \mid |y - x| < r\}$$

$$S(x, r) := \{y \in \mathbb{R}^d \mid |y - x| = r\}$$

下面说明球面为零测集.

命题 4.3.1. $m(S(x, r)) = 0, \forall x \in \mathbb{R}^d, r \geq 0.$

证明. 根据 **Lebesgue 测度的平移不变性和伸缩变换公式 (Prop 3.7.1)**, it suffices to show

$$m(S(0, 1)) = 0$$

反证法. Suppose $m(S(0, 1)) > 0$. By **Prop 3.7.1**,

$$m(rS(0, 1)) = r^d m(S(0, 1)) \geq m(S(0, 1)), \forall r \geq 1.$$

Consider the compact set $\{x \in \mathbb{R}^d \mid 1 \leq |x| \leq 2\}$. We have

$$\bigcup_{k=1}^{\infty} S(0, 1 + \frac{1}{k}) = \bigcup_{k=1}^{\infty} (1 + \frac{1}{k})S(0, 1) \subset \{x \in \mathbb{R}^d \mid 1 \leq |x| \leq 2\} \quad (4.40)$$

However

$$m\left(\bigcup_{k=1}^{\infty} S(0, 1 + \frac{1}{k})\right) = \sum_{k=1}^{\infty} m\left(S(0, 1 + \frac{1}{k})\right) \geq \sum_{k=1}^{\infty} m(S(0, 1)) = \infty \quad (4.41)$$

which is a contradiction for $\{1 \leq |x| \leq 2\}$ is compact. \square

下面我们给出当球心收敛时，开球的特征函数的收敛性.

命题 4.3.2. $\forall (x_j, r_j) \rightarrow (x, r)$ on $\mathbb{R}^d \times \mathbb{R}$,

$$\chi_{B(x_j, r_j)}(y) \rightarrow \chi_{B(x, r)}(y) \text{ on } \mathbb{R}^d \setminus S(x, r)$$

注. 该命题在开球 $B(x_j, r_j)$ 的边界，即 $S(x, r)$ 上不一定成立. 下面给出一个反例.

例 4.3.1. In \mathbb{R} , take $x_j = 0, r_j = 1 + \frac{1}{j+1}$. Then $(x_j, r_j) \rightarrow (0, 1)$.

$$\chi_{B(x_j, r_j)} = \chi_{(-1-\frac{1}{j+1}, 1+\frac{1}{j+1})} \rightarrow \chi_{[-1, 1]}$$

and $\chi_{[-1, 1]}(x) \neq \chi_{(-1, 1)}(x)$, for $x = -1$ or 1 .

证明. 下面分别对 $|y| < r$ 与 $|y - x| > r$ 两种情况进行讨论.

(i) $|y - x| < r$. WTS

$$|x_j - y| < r_j, \forall j > N \text{ for some } N$$

Since $|x_j - y| < |x_j - x| + |x - y|$, it suffices to show

$$|x_j - x| + |x - y| < r_j$$

Suppose $|x - y| = r - \epsilon, \epsilon > 0$. Then

$$\Leftrightarrow |x_j - x| + r - \epsilon < r_j$$

$$\Leftrightarrow |x_j - x| + r - r_j < \epsilon$$

It suffices to show

$$|x_j - x| + |r_j - r| < \epsilon$$

Since $(x_j, r_j) \rightarrow (x, r)$, then $\exists N \in \mathbb{N}$, s. t.

$$|x_j - x| < \frac{\epsilon}{3}, |r_j - r| < \frac{\epsilon}{3}, \forall j > N$$

Then $|x_j - y| < r_j, \forall j > N$.

(ii) $|y - x| > r$. 同理 Suppose $|x - y| = r + \epsilon, \epsilon > 0$. WTS $|x_j - y| > r_j, \forall j > N$ for some N .

It suffices to show

$$|x_j - y| > |x - y| - |x_j - x| = r + \epsilon - |x_j - x| > r_j$$

$$\Leftrightarrow |x_j - x| + r_j - r < \epsilon$$

□

Average Value of f on $B(x, r)$ 在说明 f 的连续性之前, 先来说明去掉 \sup 的函数的连续性.

定义 4.3.1. We define the average value of f on $B(x, r)$

$$A_rf(x) = \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) dy \quad (4.42)$$

下面给出局部可积的概念.

定义 4.3.2. A measurable function $f : \mathbb{R}^d \rightarrow \mathbb{C}$ is called locally integrable if

$$\int_K |f| < \infty \text{ for every bounded measurable set } K \subset \mathbb{R}^d \quad (4.43)$$

We write $f \in \mathcal{L}_{loc}^1(\mathbb{R}^d)$.

例 4.3.2. $f(x) = e^x$ is locally integrable but not integrable on \mathbb{R} .

下面给出 $A_rf(x)$ 的连续性.

引理 4.3.1. If $f \in \mathcal{L}_{loc}^1(\mathbb{R}^d)$, then

$A_rf(x)$ is jointly continuous in r and x ($r > 0, x \in \mathbb{R}^d$).

证明. 下面分为两步进行证明.

- $\int_{B(x, r)} f(y) dy$ is continuous.

$$\int_{B(0, r)} f(y) dy = \int_{\mathbb{R}^d} f(y) \chi_{B(x, r)}(y) dy \quad (4.44)$$

Fix $(x, r) \in \mathbb{R}^d \times \mathbb{R}$, $\forall (x_j, r_j) \rightarrow (x, r)$, by **Prop 4.3.2**,

$$\chi_{B(x_j, r_j)}(y) \rightarrow \chi_{B(x, r)}(y) \text{ on } \mathbb{R}^d \setminus S(x, r)$$

Since by **Prop 4.3.1**, $m(S(x, r)) = 0$, we have

$$\begin{aligned} \chi_{B(x_j, r_j)}(y) &\rightarrow \chi_{B(x, r)}(y) \text{ for a.e. } y \\ \Rightarrow f(y) \chi_{B(x_j, r_j)}(y) &\rightarrow f(y) \chi_{B(x, r)}(y) \text{ for a.e. } y \end{aligned}$$

Since $(x_j, r_j) \rightarrow (x, r)$, $\exists N$, s. t. $B(x_j, r_j) \subset B(x, 100r)$, $\forall j > N$. Then

$$|f(y)\chi_{B(x_j, r_j)}(y)| \leq |f(y)\chi_{B(x, 100r)}(y)| \in \mathcal{L}^1 \text{ for a.e. } y$$

Therefore, by **DCT (Thm 3.1.7, 控制收敛定理)**

$$\int_{\mathbb{R}^d} f(y)\chi_{B(x_j, r_j)}(y)dy \rightarrow \int_{\mathbb{R}^d} f(y)\chi_{B(x, r)}(y)dy \quad (4.45)$$

i.e.

$$\int_{B(x_j, r_j)} f(y)dy \rightarrow \int_{B(x, r)} f(y)dy, \quad \forall (x_j, r_j) \rightarrow (x, r) \quad (4.46)$$

Then by **Heine 归结原理**, $\int_{B(x, r)} f(y)dy$ is continuous.

- $A_r f(x)$ is continuous. Since $m(B(x, r)) = r^d m(B(0, 1)) = r^d v_d$, then

$$A_r f(x) = v_d^{-1} r^{-d} \int_{B(x, r)} f(y)dy \text{ is continuous} \quad (4.47)$$

□

Hardy-Littlewood 极大函数 下面先给出 **Hardy-Littlewood 极大函数 (球心)** 的定义.

定义 4.3.3. If $f \in \mathcal{L}_{loc}^1$, we define its Hardy-Littlewood maximal function Hf by

$$Hf(x) = \sup_{r>0} A_r |f|(x) = \sup_{r>0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y)| dy \quad (4.48)$$

下面说明 **Hardy-Littlewood 极大函数** 的连续性.

推论 4.3.2. Hf is continuous.

证明. $\forall (a, \infty) \subset \mathbb{R}$, by **Lemma 4.3.1**, $A_r |f|$ is continuous, then

$$(Hf)^{-1}((a, \infty)) = \bigcup_{r>0} (A_r |f|)^{-1}((a, \infty)) \text{ is open, } \forall a \in \mathbb{R} \quad (4.49)$$

Therefore Hf is continuous. □

此版本的 **Hardy-Littlewood 极大函数** 同样有 **weak-type inequality**.

命题 4.3.3. **weak-type inequality**.

If $f \in \mathcal{L}^1$, then

$$m(\{x \in \mathbb{R}^d \mid Hf(x) > a\}) \leq \frac{A}{a} \|f\|_{\mathcal{L}^1}, \quad \forall a > 0 \quad (4.50)$$

where $A = 3^d$.

证明. 与命题 4.1.1 (iii) 证明类似. □

4.3.2 Lebesgue 微分定理

函数的上极限 首先来回顾一下函数的上极限的定义.

定义 4.3.4. \forall 函数 $f: E \subset \mathbb{R}^d \rightarrow \mathbb{R}$, x_0 为 E 的聚点, 定义 f 在 x_0 的上极限 为

$$\limsup_{x \rightarrow x_0} f(x) = \lim_{\delta \rightarrow 0^+} \sup_{0 < |x - x_0| < \delta} f(x) \quad (4.51)$$

同理可定义函数 f 在 x_0 点的下极限.

注. 不难证明, 该定义与通常数学分析书¹上的定义等价, 即

$$\limsup_{x \rightarrow x_0} f(x) = \sup \{l \in \mathbb{R} \mid \exists \{x_n\}_{n=1}^{\infty} \subset E, x_n \rightarrow x_0, \text{ s. t. } f(x_n) \rightarrow l\} \quad (4.52)$$

下面利用函数的上极限给出函数极限的等价定义.

命题 4.3.4. \forall 函数 $f: E \subset \mathbb{R}^d \rightarrow \mathbb{R}$, x_0 为 E 的聚点, 则

$$\lim_{x \rightarrow x_0} f(x) = c \Leftrightarrow \limsup_{x \rightarrow x_0} |f(x) - c| = 0$$

证明. 此处证明采用数学分析中的定义更方便. 根据 **Heine** 归结原理,

$$\lim_{x \rightarrow x_0} f(x) = c \Leftrightarrow \forall \{x_j\}_{j=1}^{\infty}, x_j \rightarrow x_0, \text{ s. t. } f(x_j) \rightarrow c \quad (4.53)$$

$$\Leftrightarrow E = \{l \in \mathbb{R} \mid \exists \{x_n\}_{n=1}^{\infty} \subset E, x_n \rightarrow x_0, \text{ s. t. } |f(x_n) - c| \rightarrow l\} = \{0\} \quad (4.54)$$

$$\Leftrightarrow \limsup_{x \rightarrow x_0} |f(x) - c| = \sup E = 0 \quad (4.55)$$

□

¹此处参考书籍:《数学分析教程(上册)(第一版)》——常庚哲、史济怀编 § 2.11 定义 2.22

Lebesgue 微分定理 下面给出 **Lebesgue 微分定理 (球心)**.

定理 4.3.3. Lebesgue Differentiation Theorem.

If $f \in \mathcal{L}_{loc}^1$, then

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) dy = f(x) \text{ for a.e. } x \in \mathbb{R}^d \quad (4.56)$$

证明. Since $f \in \mathcal{L}_{loc}^1$, then for all $x \in \mathbb{R}^d$, $\exists N$, s. t. $|x| < N$ and

$$A_r f(x) = \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) dy \text{ depends only on } f(y) \text{ for } |y| \leq N + 1. \quad (4.57)$$

(with $r \leq 1$ and $|x| < N$)

Then we can replace f with $f\chi_{B(0, N+1)} \in \mathcal{L}^1$. 于是我们不妨设 $f \in \mathcal{L}^1$. 根据 **Prop 4.3.4**, 要证

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) dy = \lim_{r \rightarrow 0} A_r f(x) = f(x) \text{ for a.e. } x \in \mathbb{R}^d \quad (4.58)$$

即证

$$\limsup_{r \rightarrow 0} |A_r f(x) - f(x)| = 0 \text{ for a.e. } x \in \mathbb{R}^d \quad (4.59)$$

下面来估计 $\limsup_{r \rightarrow 0} |A_r f(x) - f(x)|$.

Fix $\epsilon > 0$, 根据 **Thm 3.2.3**, $\exists g \in C_c(\mathbb{R}^d)$, s. t.

$$\|f - g\|_{\mathcal{L}^1} < \epsilon$$

Since $A_r g(x)$ is continuous (by **Lemma 4.3.1**), we have

$$|A_r g(x) - g(x)| = |A_r g(x) - A_0 g(x)| < \epsilon \text{ for all small } r.$$

Then

$$\limsup_{r \rightarrow 0} |A_r f(x) - f(x)| = \limsup_{r \rightarrow 0} |A_r(f - g)(x) + (A_r g - g)(x) + (g - f)(x)| \quad (4.60)$$

$$\leq \lim_{\delta \rightarrow 0^+} \sup_{0 < r < \delta} |A_r(f - g)(x)| + 0 + |g(x) - f(x)| \quad (4.61)$$

Since

$$\sup_{0 < r < \delta} |A_r(f - g)(x)| \leq \sup_{r > 0} |A_r(f - g)(x)| = \sup_{r > 0} \left| \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) - g(y) dy \right| \quad (4.62)$$

$$\leq \sup_{r > 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - g(y)| dy \quad (4.63)$$

$$= H(f - g)(x), \quad \forall \delta > 0 \quad (4.64)$$

Let $\delta \rightarrow 0^+$, we have

$$\limsup_{r \rightarrow 0} |A_r f(x) - f(x)| \leq \lim_{\delta \rightarrow 0^+} \sup_{0 < r < \delta} |A_r(f - g)(x)| + |g(x) - f(x)| \quad (4.65)$$

$$\leq H(f - g)(x) + |g(x) - f(x)| \quad (4.66)$$

后续证明与 **Thm 4.2.2** 一致, 即定义 E_a , 并分别对 $H(f - g)(x)$ 与 $|g - f|$ 运用 **weak-type inequality (Prop 4.3.3)** 与 **Chebyshev's Inequality (Thm 4.2.1)**, 即可证明 $m(E_a) = 0, \forall a \geq 0$. 从而得证. \square

4.4 有界变差函数

引入 在 **Riemann** 积分中, 我们知道对于一阶连续可微函数, 我们有微积分基本定理

$$F(b) - F(a) = \int_a^b F'(x)dx \quad (4.67)$$

而对于 **Lebesgue** 积分, 我们也想要得到该命题成立的条件, 且最好为**充要条件**. 可以举例证明, 仅仅 F 连续并不能保证 F 可导 (可见视频 [a continuous but nowhere differentiable function](#)). 同时仅仅要求 F 导数存在也可能出现 F' 不可积的情况, 如下反例.

例 4.4.1. (书² P147 Ex 12).

Consider the function $F(x) = x^2 \sin \frac{1}{x^2}$, $x \neq 0$, with $F(0) = 0$. Show that $F'(x)$ exists for every x , but F' is not integrable on $[-1, 1]$.

证明. 详细证明可见视频 [微积分基本定理: 问题引入](#).

□

为了解决上述问题, 我们在这一节将引入一种新类型的函数, 叫做**有界变差函数**.

² 《Real Analysis – Measure Theory, Integration, & Hilbert Spaces》— Elias M. Stein

4.4.1 有界变差函数的概念

引入 为了便于理解，我们将有界变差函数与平面上的曲线相联系. 首先来回顾数学分析中有关平面上的曲线的相关概念.

定义 4.4.1. Let γ be a parametrized curve in the plane given by $z(t) = (x(t), y(t))$, where $a \leq t \leq b$. Here $x(t)$ and $y(t)$ are continuous real-valued functions on $[a, b]$.

The curve is rectifiable if $\exists M < \infty$, s. t. for any partition $a = t_0 < t_1 < \cdots < t_N = b$ of $[a, b]$,

$$\sum_{j=1}^N |z(t_j) - z(t_{j-1})| \leq M \quad (4.68)$$

The Length $L(\gamma)$ of the curve is defined as

$$L(\gamma) = \sup_{\text{all partitions}} \sum_{j=1}^N |z(t_j) - z(t_{j-1})| \quad (4.69)$$

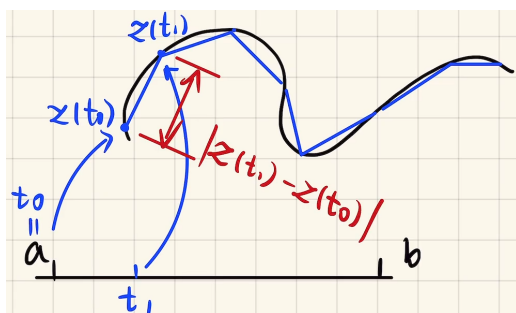


图 4.2: rectifiable curve

注. 为了表述的方便，后续我们常将 z 的值域视作复平面，即 $z(t) = x(t) + iy(t)$.

在定义了曲线可求长的概念后，我们自然要问，在什么情况下曲线可求长？即

What condition on $x(t)$ and $y(t)$ guarantees rectifiability of γ ?

为了解决这个问题，下面我们给出有界变差函数的定义，并会在后续给出这个问题的充要条件，即

γ rectifiable $\Leftrightarrow x(t), y(t)$ 均为有界变差函数.

定义 下面给出有界变差函数的定义.

定义 4.4.2. Suppose $F : [a, b] \rightarrow \mathbb{C}$, and \mathcal{P} is a partition $a = t_0 < t_1 < \cdots < t_N = b$. The variation of F on this partition \mathcal{P} is defined by

$$V_{\mathcal{P}}(F) = \sum_{j=1}^N |F(t_j) - F(t_{j-1})| \quad (4.70)$$

F is said to be of bounded variation (BV) if $V_{\mathcal{P}}(F)$ is bounded over all partitions.

下面就能够来回答“引入”中提到的问题，即平面曲线可求长的充要条件.

定理 4.4.1. A curve parametrized by $F(t) = x(t) + iy(t)$, $a \leq t \leq b$, is rectifiable \Leftrightarrow both $x(t)$ and $y(t)$ are of BV.

证明. \forall partition \mathcal{P} : $a = t_0 < t_1 < \cdots < t_N = b$, we have

$$V_{\mathcal{P}}(F) = \sum_{j=1}^N |F(t_j) - F(t_{j-1})| \quad (4.71)$$

Since $|a + bi| \leq |a| + |b| \leq 2|a + bi|$,

$$|F(t_j) - F(t_{j-1})| = |x(t_j) - x(t_{j-1}) + i(y(t_j) - y(t_{j-1}))| \quad (4.72)$$

Then

- \Leftarrow : $|F(t_j) - F(t_{j-1})| \leq |x(t_j) - x(t_{j-1})| + |y(t_j) - y(t_{j-1})|$, then $\exists M < \infty$, s. t.

$$V_{\mathcal{P}}(F) = \sum_{j=1}^N |F(t_j) - F(t_{j-1})| \quad (4.73)$$

$$= \sum_{j=1}^N |x(t_j) - x(t_{j-1})| + \sum_{j=1}^N |y(t_j) - y(t_{j-1})| \quad (4.74)$$

$$\leq 2M, \quad \forall \text{partition } \mathcal{P} \quad (4.75)$$

Therefore, the curve $F(t) = x(t) + iy(t)$ is rectifiable.

- \Rightarrow : $|x(t_j) - x(t_{j-1})| + |y(t_j) - y(t_{j-1})| \leq 2|F(t_j) - F(t_{j-1})|$, then $\exists M < \infty$, s. t.

$$\sum_{j=1}^N |x(t_j) - x(t_{j-1})| + \sum_{j=1}^N |y(t_j) - y(t_{j-1})| \leq 2 \sum_{j=1}^N |F(t_j) - F(t_{j-1})| = 2V_{\mathcal{P}}(F) \leq 2M \quad (4.76)$$

Therefore, both $x(t)$ and $y(t)$ are of BV.

□

例子 下面来给出一些有界变差函数的例子.

例 4.4.2. • x, x^2 is of BV on $[a, b]$, $\forall [a, b] \subset \mathbb{R}$.

证明. \forall partition \mathcal{P} : $a = x_0 < x_1 < \cdots < x_N = b$, since x is strictly increasing, then

$$V_{\mathcal{P}}(x) = \sum_{j=1}^N |x_j - x_{j-1}| = \sum_{j=1}^N x_j - x_{j-1} = b - a < \infty \quad (4.77)$$

Also for x^2 ,

$$V_{\mathcal{P}}(x^2) = \sum_{j=1}^N |x_j^2 - x_{j-1}^2| = \sum_{j=1}^N |x_j + x_{j-1}| |x_j - x_{j-1}| \leq 2b \sum_{j=1}^N |x_j - x_{j-1}| = 2b(b - a) < \infty \quad (4.78)$$

Therefore, both x and x^2 are of BV on $[a, b]$, $\forall [a, b] \subset \mathbb{R}$. □

- If F is real-valued, monotonic, and bounded, then F is of BV.

证明. \forall partition \mathcal{P} : $a = t_0 < t_1 < \cdots < t_N = b$.

Since F is bounded, $\exists M < \infty$, s. t. $|F| \leq M$. 不妨设 F 单调递增,

$$V_{\mathcal{P}}(F) = \sum_{j=1}^N |F(t_j) - F(t_{j-1})| = F(b) - F(a) \leq 2M, \quad \forall \text{partition } \mathcal{P} \quad (4.79)$$

So F is of BV. □

• (书³ P147 Ex 11).

If $a, b > 0$, let

$$f(x) = \begin{cases} x^a \sin(x^{-b}), & 0 \leq x \leq 1 \\ 0, & x = 0 \end{cases} \quad (4.80)$$

Then

$$f \text{ is of BV in } [0, 1] \Leftrightarrow a > b$$

证明.

– 先来考虑简单情形, 即 $f(x) = \sin \frac{1}{x}$.

根据直觉, 随着 $x \rightarrow 0^+$, $\sin \frac{1}{x}$ 的震荡越剧烈, 当分划足够密时, 其变差中应当会出现各项为 1 的无穷级数, 从而发散. 下面取一个特殊分划进行证明.

对于 \forall 奇数 $k \in \mathbb{N}$, 取

$$\frac{1}{x_k} = 2k\pi + \frac{\pi}{2}, \frac{1}{x_{k+1}} = 2k\pi + \pi$$

于是

$$V_{\mathcal{P}}(f) = \sum_{j=1}^N \left| \sin \frac{1}{x_j} - \sin \frac{1}{x_{j-1}} \right| = \sum_{j=1}^N 1 = N, \text{ which is related to } \mathcal{P} \quad (4.81)$$

故 $\forall M < \infty$, 当分划 \mathcal{P} 足够密时, $V_{\mathcal{P}}(f) > M$. 故 f 非 BV.

– 对于一般情况, 下面先讨论一种特殊分划 \mathcal{P} . 即 (**Monotonic Partition**, 单调划分)

$$\left[\frac{1}{x_{4k+1}^b}, \frac{1}{x_{4k+2}^b} \right] = \left[2k\pi + \frac{\pi}{2}, 2k\pi + \pi \right], \left[\frac{1}{x_{4k+3}^b}, \frac{1}{x_{4k+4}^b} \right] = \left[2k\pi + \frac{3\pi}{2}, 2k\pi + 2\pi \right] \quad (4.82)$$

则以上述第一种的分划 $[x_{4k+1}, x_{4k+2}]$ 举例,

$$\left| x_{k+1}^a \sin \frac{1}{x_{k+1}^b} - x_k^a \sin \frac{1}{x_k^b} \right| = \left| x_{k+1}^a \sin \frac{1}{x_{k+1}^b} \right| = \frac{1}{(2k\pi + \frac{\pi}{2})^{\frac{a}{b}}} = O\left(\frac{1}{k^{\frac{a}{b}}}\right) \quad (4.83)$$

同理对于 $[x_{4k+2}, x_{4k+3}]$, $[x_{4k+3}, x_{4k+4}]$, $[x_{4k+4}, x_{4k+5}]$, 均可得到

$$\left| x_{k+1}^a \sin \frac{1}{x_{k+1}^b} - x_k^a \sin \frac{1}{x_k^b} \right| = O\left(\frac{1}{k^{\frac{a}{b}}}\right) \quad (4.84)$$

³ 《Real Analysis – Measure Theory, Integration, & Hilbert Spaces》— Elias M. Stein

于是

$$\sum_{k=0}^N \left| x_{k+1}^a \sin \frac{1}{x_{k+1}^b} - x_k^a \sin \frac{1}{x_k^b} \right| = O\left(\sum_{k=1}^N \frac{1}{k^{\frac{a}{b}}}\right) \quad (4.85)$$

根据 **p**-级数 $\sum_n \frac{1}{n^p}$ 的收敛性 (收敛 $\Leftrightarrow p > 1$) 可得,

$$\sum_{k=0}^N \left| x_{k+1}^a \sin \frac{1}{x_{k+1}^b} - x_k^a \sin \frac{1}{x_k^b} \right| < \infty \quad (4.86)$$

$$\Leftrightarrow \frac{a}{b} > 1 \Leftrightarrow a > b \quad (4.87)$$

由于对于任一分划 \mathcal{P} , 有:

“加密分割, 变差不减”

因此对于 \forall 分割, 我们可以在其中加入如上节点, 其变差不减, 但因为上述节点中, $\sin \frac{1}{x^b}$ 在每个区间 $[x_k, x_{k+1}]$ 均单调, 所以在各区间 $[x_k, x_{k+1}]$ 中的变差可放缩后去除绝对值, 并得到

$$V_{\mathcal{P}}(f) = O\left(\sum_{k=1}^N \frac{1}{k^{\frac{a}{b}}}\right) \quad (4.88)$$

于是

$$f \text{ is of BV} \Leftrightarrow \frac{a}{b} > 1 \Leftrightarrow a > b$$

□

4.4.2 有界变差函数的刻画