Real Analysis

Measure Theory, Integration, & Hilbert Spaces[1]

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2024 年 4 月 14 日

[1]参考书籍：

《*Real Analysis – –Measure Theroy, Integration, & Hilbert Spaces*》— *Elias M. Stein*

《*Real Analysis – –Modern Techniques and Their Applications*》— *Gerald B. Folland*

序

天道几何，万品流形先自守；

变分无限，孤心测度有同伦。

2024 年 4 月 14 日

长夜伴浪破晓梦，梦晓破浪伴夜长

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第一章　*Measure Theory*

1.1　*Preliminaries*

定义 1.1.1. A (closed) **rectangle** R in $\mathbb{R}^d$ is given by of d one-dimensional closed and bounded intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d] \tag{1.1}$$

where $a_j \leq b_j$ are real numbers , $j = 1, 2, \cdots, d$. In other word , we have

$$R = \{(x_1, \cdots, x_d) \in \mathbb{R}^d \mid a_j \leq x_j \leq b_j,\ \forall j = 1 \sim d\} \tag{1.2}$$

The **volume** of R is

$$|R| = (b_1 - a_1) \cdots (b_d - a_d) \tag{1.3}$$

An **open** rectangle is the product of open intervals , and **the interior of the rectangle** R is

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d) \tag{1.4}$$

Also , a **cube** is a rectangle for which $b_1 - a_1 = \cdots = b_d - a_d$.

定义 1.1.2. A union of rectangles is said to be **almost disjoint** if the interiors of them are disjoint.

引理 1.1.1. If a rectangle is the almost disjoint union of finitely many rectangles , say $R = \bigcup\limits_{k=1}^{N} R_k$, then

$$|R| = \sum_{k=1}^{N} |R_k| \tag{1.5}$$

注. 本质上即指的是对于方体的任意的垂直划分可转化为“十字形”划分.

引理 1.1.2. If $R, R_1, \cdots, R_N$ are rectangles , and $R \subset \bigcup_{k=1}^{U} R_k$, then

$$|R| \leq \sum_{k=1}^{N} |R_k| \tag{1.6}$$

注. 此即对 *Lemma* 1.1.1 的 slight modification , 即各方体之间不一定再为 almost disjoint.

Now we can give a description of the strcture of open sets in terms of cubes. Begin with the case of $\mathbb{R}$.

定理 1.1.3. Every open subset O of $\mathbb{R}$ can be written uniquely as countable union of disjoint open intervals.

证明. For each $x \in O$, let I_x be the largest open interval containing x and contained in O.

Step 1 : Construct I_x:

O is open $\Rightarrow$ x is contained in some small open interval contained in O.

Let

$$a_x = inf\{a < x \mid (a, x) \subset O\} \tag{1.7}$$

$$b_x = sup\{b > x \mid (x, b) \subset O\} \tag{1.8}$$

Let $I_x = (a_x, b_x)$, then $O = \bigcup_{x \in O} I_x$.

Step 2 : Suppose $I_x \cap I_y \neq \varnothing$.

$I_x \cup I_y$ is an open interval s. t. $\begin{cases} x \in I_x \cup I_y \\ I_x \cup I_y \subset O \end{cases}$

Since I_x is maximal , $I_x \cup I_y \subset I_x$. Similarly , $I_x \cup I_y \subset I_y$.

$\Rightarrow \ I_x = I_y$

$\Rightarrow \ if \ I_x \neq I_y, \ then \ I_x \cap I_y = \varnothing.$

$\Rightarrow Z = \{I_x\}_{x \in O}$ is a disjoint famliy of sets.

Step 3 : Since every I_x contains at least a $a_x \in \mathbb{Q}$, construct a map f

$$f : Z \longrightarrow \mathbb{Q} \tag{1.9}$$

$$I_x \longmapsto a_x \tag{1.10}$$

f is an injective. $\Rightarrow$ $\{I_x\}_{x \in O}$ is countable. $\Rightarrow$ $O = \bigcup_{j=1}^{\infty} (a_j, b_j)$.

$\square$

定理 1.1.4. Every open set $\mathcal{O}$ of $\mathbb{R}^d$, $d \geq 1$, can be written as a countable union of almost disjoint closed cubes.

证明. Let

$$\mathcal{Q}_k := grid\ of\ 2^{-k}\mathbb{Z}^d,\ k \geq 0 \tag{1.11}$$

$$\underline{A}(\mathcal{O}, k) := \{Q \in \mathcal{Q}_k \mid Q \subset \mathcal{O}\} \tag{1.12}$$

$$\overline{A}(\mathcal{O}, k) := \{Q \in \mathcal{Q}_k \mid Q \cap \mathcal{O} \neq \varnothing\} \tag{1.13}$$

Since $\forall Q \in \underline{A}(\mathcal{O}, k),\ \exists q \in Q^\circ,\ \text{s.t.}\ q \in \mathbb{Q}^d,$

According to the Axiom of Choice , $\exists$ the map $f_k : \underline{A}(\mathcal{O}, k) \longrightarrow \mathbb{Q}^d$, which is an injection.

Hence $\underline{A}(\mathcal{O}, k)$ is countable.

Let

$$\underline{A}(\mathcal{O}) := \bigcup_{k=1}^{\infty} (\underline{A}(\mathcal{O}, k) \setminus \underline{A}(\mathcal{O}, k-1)) \cup \underline{A}(\mathcal{O}, 0) \tag{1.14}$$

Then $\underline{A}(\mathcal{O})$ is also countable. Similarly define $\overline{A}(\mathcal{O})$.

$\forall x \in \mathcal{O}$, let $\delta_x := inf\{|y - x| \mid y \notin \mathcal{O}\}$. Since $\mathcal{O}$ is open , $\Rightarrow\ \ \delta_x > 0$.

$$\exists N_x \in \mathbb{N},\ \text{s.t.}\ 2^{-k}\sqrt{d} \leq \frac{\delta_x}{2} < \delta_x, \forall k \geq N_x \tag{1.15}$$

$$\Rightarrow \forall Q \in \overline{A}(\mathcal{O}, N_x),\ \text{s.t.}\ |s - t| \leq 2^{-N_x}\sqrt{d} < \delta_x, \forall s, t \in Q \tag{1.16}$$

$$\Rightarrow Since\ \mathcal{O} \subset \overline{A}(\mathcal{O}),\ \exists Q_x \in \overline{A}(\mathcal{O}, N_x) \subset \overline{A}(\mathcal{O}),\ \text{s.t.}\ x \in Q_x \tag{1.17}$$

$$\Rightarrow x \in Q_x \subset \mathcal{O} \tag{1.18}$$

$$\Rightarrow x \in Q_x \in \underline{A}(\mathcal{O}, N_x) \subset \underline{A}(\mathcal{O}) \tag{1.19}$$

$$\Rightarrow \mathcal{O} \subset \underline{A}(\mathcal{O}) \tag{1.20}$$

Obviously $\underline{A}(\mathcal{O}) \subset \mathcal{O}$, so

$$\mathcal{O} = \underline{A}(\mathcal{O}) = \bigcup_{k=1}^{\infty} (\underline{A}(\mathcal{O}, k) \setminus \underline{A}(\mathcal{O}, k-1)) \cup \underline{A}(\mathcal{O}, 0) \tag{1.21}$$

which is a countable union of almost disjoint closed cubes. $\square$

1.2 *The Exterior Measure*

Definition The exterior measure attempts to describe the volume of a set E by approximating it from the outside.

Loosely speaking , the exterior measure m_* assigns to **any subset of** $\mathbb{R}^d$ a first notion of size.

定义 1.2.1. If E is a subset of $\mathbb{R}^d$, the **exterior measure** of E is

$$m_*(E) := \inf\left\{\sum_{j=1}^{\infty}|Q_j| \;\middle|\; E\subset\bigcup_{j=1}^{\infty}Q_j,\ Q_j \textit{ is a closed cube}\right\} \tag{1.22}$$

注. • **Well definition:** $\forall E\subset\mathbb{R}^d,\ E\subset\bigcup_{n=1}^{\infty}Q_n,\ Q_n=[-n,n]^d\subset\mathbb{R}^d$, which means m_* can be defined on every subset of $\mathbb{R}^d$.

- It is immediate from the definition that:

  For every $\epsilon>0$, there exists a covering $E\subset\bigcup_{j=1}^{\infty}Q_j$, s. t.

$$\sum_{j=1}^{\infty}m_*(Q_j)\le m_*(E)+\epsilon \tag{1.23}$$

- It is important to note that it would **not suffice** to allow **finite sums** in the definition of $m_*(E)$. If one considered only coverings of E by finite unions of cubes , the quantity is **in general larger** than $m_*(E)$.

  (In fact , it is defined as the **outer Jordan content** $J_*(E)$.)

例 **1.2.1.** Consider the set $\mathbb{Q}\cap[0,1]$.

- For the outer Jordan content , since it's obvious that $J_*(\overline{E})=J_*(E),\ \forall E\subset\mathbb{R}^d$,

  $J_*(\mathbb{Q}\cap[0,1])=J_*(\overline{\mathbb{Q}\cap[0,1]})=J_*([0,1])=1$

- For the exterior measure , since $\mathbb{Q}\cap[0,1]$ is countable , let $\mathbb{Q}\cap[0,1]=\{x_1,x_2,\cdots\}$.

  Since for all $\epsilon>0$,

$$\mathbb{Q}\cap[0,1]\subset\bigcup_{j=1}^{\infty}[x_j-\frac{\epsilon}{2^j},x_j+\frac{\epsilon}{2^j}] \tag{1.24}$$

  Hence $m_*(\mathbb{Q}\cap[0,1])\le\epsilon$. For ϵ is arbitrary , $m_*(\mathbb{Q}\cap[0,1])=0$.

Examples Let's check that whether the exterior measure matches our intuitive idea of volume.

Example 1. **The exterior measure of a point is zero.**

证明. It's clear that a point is a cube with $a_j = b_j, \forall j = 1 \sim d$ and which covers itself. □

Example 2. **The exterior measure of a closed cube is equal to its volume.**

证明.

- Let $Q \subset \mathbb{R}^d$ be a closed cube. Since $Q \subset Q$, $m_*(Q) \leq |Q|$.
- Suppose $Q \subset \bigcup_{j=1}^{\infty} Q_j$ by closed cubes. For fixed $\epsilon > 0$, $\forall j \in \mathbb{N}$, choose an open cube S_j ,

$$\text{s. t.} \begin{cases} S_j \supset Q_j \\ \left|S_j\right| = (1+\epsilon)\left|Q_j\right| \end{cases} \tag{1.25}$$

Then $Q \subset \bigcup_{j=1}^{\infty} S_j$. Since Q is compact , $\exists S_1, \cdots, S_n \in \{S_j\}_{j=1}^{\infty}$, s. t. $Q \subset \bigcup_{j=1}^{n} S_j$.
Therefore , according to Lemma 1.1.2

$$|Q| \leq \sum_{j=1}^{n} \left|S_j\right| = (1+\epsilon)\sum_{j=1}^{n}\left|Q_j\right| \leq (1+\epsilon)\sum_{j=1}^{\infty}\left|Q_j\right| \tag{1.26}$$

For $\epsilon > 0$ is arbitrary , we get

$$|Q| \leq \sum_{j=1}^{\infty}\left|Q_j\right| \tag{1.27}$$

$$|Q| \leq \inf \sum_{j=1}^{\infty}\left|Q_j\right| = m_*(Q) \tag{1.28}$$

□

Example 3. **If Q is an open cube , then $m_*(Q) = |Q|$.**

证明.

- Since $Q \subset \overline{Q}$, $m_*(Q) \leq \left|\overline{Q}\right| = |Q|$.
- We note that for all closed cubes Q_0 contained in Q , then $m_*(Q_0) = |Q_0| \leq m_*(Q)$.

  For fixed $\epsilon > 0$ which is suffice small , choose a closed cube Q_0 contained in Q with a volume $|Q_0| = (1-\epsilon)|Q|$, then we have

$$|Q_0| = (1-\epsilon)|Q| \leq m_*(Q) \tag{1.29}$$

For ϵ is arbitrary , $|Q| \leq m_*(Q)$.

□

Example 4. **The exterior measure of a rectangle R is equal to its volume.**

Example 5. $m_*(\mathbb{R}^d) = \infty$.

证明. Since any covering of $\mathbb{R}^d$ is also a covering of any cube $Q \subset \mathbb{R}^d$, $m_*(\mathbb{R}^d) \geq m_*(Q)$

$\forall N > 0,\ \exists Q \subset \mathbb{R}^d,$ s. t. $|Q| > N$, so $m_*(\mathbb{R}^d) = \infty$. □

Properties

Observation 1. (*Monotonicity*)

If $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$.

Observation 2. (*Countable sub − additivity*)

If $E \subset \bigcup_{j=1}^{\infty} E_j$, then $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$.

证明. For a fixed $\epsilon > 0$, for all E_j , there exists a covering $\{Q_{j_k}\}_{k=1}^{\infty},\ E \subset \bigcup_{k=1}^{\infty} Q_{j_k},$ s. t.

$$\sum_{k=1}^{\infty} m_*(Q_{j_k}) \leq m_*(E_j) + \frac{\epsilon}{2^j} \tag{1.30}$$

Since $E \subset \bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} Q_{j_k}$, $\bigcup_{j=1}^{\infty}\bigcup_{k=1}^{\infty} Q_{j_k}$ covers E , then

$$m_*(E) \leq \sum_{j=1}^{\infty}\sum_{k=1}^{\infty} m_*(Q_{j_k}) \leq \sum_{j=1}^{\infty} m_*(E_j) + \epsilon \tag{1.31}$$

Since ϵ is arbitrary , $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$ □

Observation 3. If $E \subset \mathbb{R}^d$, then $m_*(E) = \inf\{m_*(O) \mid E \subset O,\ O \textit{ is an open set}\}$.

证明.

- By monotonicity , $m_*(E) \leq m_*(O)$, for all O covers E. Then take the infimum.

- For a fixed $\epsilon > 0$, $\exists$ *covering* $E \subset \bigcup_{j=1}^{\infty} Q_j$, s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \frac{\epsilon}{2} \tag{1.32}$$

For all Q_j , choose an open set $\widetilde{Q_j}$ containing Q_j with a volume $\left|\widetilde{Q_j}\right| \leq \left|Q_j\right| + \frac{\epsilon}{2^{j+1}}$.
Let $O = \bigcup_{j=1}^{\infty} \widetilde{Q_j}$, then by Observation 2 ,

$$m_*(O) \leq \sum_{j=1}^{\infty} m_*(\widetilde{Q_j}) = \sum_{j=1}^{\infty} \left|\widetilde{Q_j}\right| \leq \sum_{j=1}^{\infty} \left|Q_j\right| + \frac{\epsilon}{2} \leq m_*(E) + \epsilon \tag{1.33}$$

Since ϵ is arbitrary , $m_*(O) \leq m_*(E)$, so $\inf m_*(O) \leq m_*(E)$.

□

Observation 4. If $E = E_1 \cup E_2$, and $d(E_1, E_2) > 0$, then

$$m_*(E) = m_*(E_1) + m_*(E_2) \tag{1.34}$$

证明. For a fixed $\epsilon > 0$, $\exists$ a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$, s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \tag{1.35}$$

Subdevide the cubes Q_j and assume that $diam(Q_j) <= \frac{d(E_1,E_2)}{3}$. Then each Q_j can intersect at most one of the two sets E_1 or E_2. Devide $\{Q_j\}_{j=1}^{\infty}$ into two subsets $\{Q_j\}_{j\in J_1}$, $\{Q_j\}_{j\in J2}$, s. t.

$$E_1 \subset \bigcup_{j\in J_1} Q_j,\ E_2 \subset \bigcup_{j\in J_2} Q_j \tag{1.36}$$

J_1 and J_2 are both countable. $J_1 \cap J_2 = \emptyset$. Then

$$m_*(E_1) \leq \sum_{j\in J_1} m_*(Q_j),\ m_*(E_2) \leq \sum_{j\in J_2} m_*(Q_j) \tag{1.37}$$

Therefore

$$m_*(E_1) + m_*(E_2) \leq \sum_{j\in J_1} m_*(Q_j) + \sum_{j\in J_2} m_*(Q_j) \leq \sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \tag{1.38}$$

Since ϵ is arbitrary , $m_*(E_1) + m_*(E_2) \leq m_*(E)$. □

Observation 5. If a set E is the countable union of almost disjoint cubes $E = \bigcup_{j=1}^{\infty} Q_j$, then

$$m_*(E) = \sum_{j=1}^{\infty} \left|Q_j\right| \tag{1.39}$$

证明. For a fixed $\epsilon > 0$, for all Q_j , choose a closed cube $\widetilde{Q_j}$ strictly contained in Q_j with its volume $\left|\widetilde{Q_j}\right| \geq \left|Q_j\right| - \frac{\epsilon}{2^j}$. Then for every $N \in \mathbb{N}$, the cubes $\widetilde{Q_1}, \cdots, \widetilde{Q_N}$ are disjoint with a finite distance from one another. By Observation 4 ,

$$m_*(\bigcup_{j=1}^{N} \widetilde{Q_j}) = \sum_{i=1}^{N} \left|\widetilde{Q_j}\right| \geq \sum_{j=1}^{N} \left|Q_j\right| - \epsilon \tag{1.40}$$

Since $\bigcup_{j=1}^{\infty} \widetilde{Q_j} \subset E$, we conclude that for every N

$$m_*(E) \geq \sum_{j=1}^{N} \left|Q_j\right| - \epsilon \tag{1.41}$$

Let $N \to \infty$, we deduce

$$m_*(E) \geq \sum_{j=1}^{\infty} \left|Q_j\right| - \epsilon \tag{1.42}$$

Since ϵ is arbitrary , $\sum_{j=1}^{\infty} \left|Q_j\right| \leq m_*(E)$. □

1.3 *Measurable sets and the Lebesgue measure*

1.3.1 *Measurable sets*

Definition

定义 1.3.1. A subset E of $\mathbb{R}^d$ is *(Lebesgue) measurable*, if for any $\epsilon > 0$ there exists an open set O with $E \subset O$ and $m_*(O \backslash E) \leq \epsilon$.

If E is measurable, we define its *(Lebesgue) measurable* $m(E)$ by $m(E) = m_*(E)$.

注.

- 可用映射的观点来理解外测度 m_* 与测度 m 的关系 (*Folland*). 即

$$m_* : \mathcal{P}(\mathbb{R}^d) \longrightarrow \overline{\mathbb{R}}_+ = [0, +\infty] \tag{1.43}$$

$$m : \mathcal{M} \longrightarrow \overline{\mathbb{R}}_+ = [0, +\infty] \tag{1.44}$$

$$m = m_*\Big|_{\mathcal{M}} \tag{1.45}$$

其中 $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^d)$ 为 $\mathbb{R}^d$ 中所有 (*Lebesgue*) *measurable sets* 构成的集合.

- 类比于抽象代数中各代数结构的性质，比如群 (*group*) 对加法 / 乘法封闭，我们下面探讨集合族 $\mathcal{M}$ 对于可数个集合的运算 (*countable unions, countable intersections, complement*) 是否封闭. 即通过此引出代数结构 *σ − algebra*.

Properties 下面开始探讨 (*Lebesgue*) *measure* 的部分性质.

Property 1. Every open set in $\mathbb{R}^d$ is measurable.

Property 2. If $m_*(E) = 0$, then E is measurable.

证明. By Observation 3 in §1.2, for a fixed $\epsilon > 0$, $\exists E \subset O$ open, s. t.

$$m_*(O) \leq m_*(E) + \epsilon = \epsilon \tag{1.46}$$

Since $O \backslash E \subset O$, then $m_*(O \backslash E) \leq m_*(O) \leq \epsilon$. □

Property 3. Let $\{E_j\}_{j=1}^{\infty}$ be a family of measurable sets, then $\bigcup\limits_{j=1}^{\infty} E_j$ is measurable.

注. 即说明集合族 $\mathcal{M}$ 对 *countable unions* 封闭.

证明. Since E_j is measurable, for a fixed $\epsilon > 0$, $\exists E_j \subset O_j$ open, s. t.

$$m_*(O_j \backslash E_j) \le \frac{\epsilon}{2^j} \tag{1.47}$$

Let $O = \bigcup\limits_{j=1}^{\infty} O_j \underset{open}{\subset} \mathbb{R}^d$, then

$$O \backslash \bigcup_{j=1}^{\infty} E_j = \left(\bigcup_{j=1}^{\infty} O_j\right) \cap \left(\bigcap_{j=1}^{\infty} E_j^c\right) \tag{1.48}$$

$$= \bigcup_{j=1}^{\infty}\left(O_j \cap \left(\bigcap_{k=1}^{\infty} E_k^c\right)\right) \subset \bigcup_{j=1}^{\infty}\left(O_j \cap E_j^c\right) = \bigcup_{j=1}^{\infty}\left(O_j \backslash E_j\right) \tag{1.49}$$

Therefore

$$m_*\left(O \backslash \bigcup_{j=1}^{\infty} E_j\right) \le m_*\left(\bigcup_{j=1}^{\infty}\left(O_j \backslash E_j\right)\right) \le \sum_{j=1}^{\infty} m_*\left(O_j \backslash E_j\right) \le \epsilon \tag{1.50}$$

So $\bigcup\limits_{j=1}^{\infty} E_j$ is measurable. □

Property 4. Closed sets are measurable.

为了证明该性质，先证明如下的分离定理.

引理 1.3.1. If F is closed, K is compact, and $K \cap F = \varnothing$, then $d(F, K) > 0$.

证明. 反证法.Suppose $d(F, K) = 0$, then for any fixed $n \in \mathbb{N}$, $\exists x_n \in F, y_n \in K$, s. t.

$$|x_n - y_n| \le \frac{1}{n} \tag{1.51}$$

Since K is compact, $\{y_n\}_{n=1}^{\infty}$ is bounded. Then there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$, s. t.

$$y_{n_k} \to y_0 \in K, \ \ as\ k \to \infty \tag{1.52}$$

Since $\left|x_{n_k} - y_{n_k}\right| \le \frac{1}{n_k}$, then

$$\left|x_{n_k} - y_0\right| \le \left|x_{n_k} - y_{n_k}\right| + \left|y_{n_k} - y_0\right| \to 0, \ \ as\ k \to \infty \tag{1.53}$$

So $x_{n_k} \to y_0 \in F$, $y_0 \in F \cap K \ne \varnothing$ 矛盾. □

下面证明 Property 4.

证明.

- Suppose F is bounded, then F is compact.

  By Observation 3 in §1.2, for a fixed $\epsilon > 0$, $\exists F \subset O$ open, s. t.

$$m_*(O) \leq m_*(F) + \epsilon \tag{1.54}$$

  Since F is closed, $O \backslash F = O \cap F^c$ is open. By Thm1.1.4, $\exists \{Q_j\}_{j=1}^{\infty}$, s. t.

$$O \backslash F = \bigcup_{j=1}^{\infty} Q_j \tag{1.55}$$

  For a fixed $N \in \mathbb{N}$, let $K = \bigcup_{j=1}^{N} Q_j$, then K is compact. By Lemma1.3.1, $d(K, F) > 0$. Since $K \cup F \subset O$, by Observation 4 in §1.2,

$$m_*(K) + m_*(F) = m_*(K \cup F) \leq m_*(O) \tag{1.56}$$

  So for each fixed $N \in \mathbb{N}$,

$$\sum_{j=1}^{N} \left|Q_j\right| = m_*(K) \leq m_*(O) - m_*(F) \leq \epsilon \tag{1.57}$$

  Let $N \to \infty$, we get

$$m_*(O \backslash F) = \sum_{j=1}^{\infty} \left|Q_j\right| \leq \epsilon \tag{1.58}$$

  Therefore, F is measurable.

- For the general situation, since $\mathbb{R}^d = \bigcup_{j=1}^{\infty} B_j$, then

$$F = F \cap \mathbb{R}^d = \bigcup_{j=1}^{\infty} \left(F \cap B_j\right) \tag{1.59}$$

  Since B_k is compact and F is closed, then $F \cap B_j$ is compact.

  Due to the previous proof, $F \cap B_j$ is measurable. By Property 3 in §1.3.1,

$$F = \bigcup_{j=1}^{\infty} \left(F \cap B_j\right) \textit{ is measurable.} \tag{1.60}$$

□

Property 5. If E is measurable, then E^c is measurable.

注. 即说明集合族 $\mathcal{M}$ 对集合的补运算 *complement* 封闭.

证明. Since E is measurable, then for all fixed $n \in \mathbb{N}$, $\exists E \subset O_n$ open, s. t. $m_*(O_n \backslash E) \leq \frac{1}{n}$.
Let $S = \bigcup_{j=1}^{\infty} O_j^c \subset E^c$. Since O_j^c is closed, O_j^c is measurable. Then S is measurable.
Since

$$E^c \backslash S = E^c \cap \left(\bigcap_{j=1}^{\infty} O_j \right) = \bigcap_{j=1}^{\infty} \left(E^c \cap O_j \right) \subset E^c \cap O_n = O_n \backslash E, \ \forall n \in \mathbb{N} \tag{1.61}$$

Then, $m_*(E^c \backslash S) \leq m_*(O_n \backslash E) \leq \frac{1}{n}$, $\forall n \in \mathbb{N}$. So $E^c \backslash S$ is measurable.
Therefore, $E^c = (E^c \backslash S) \cup S$ is measurable. □

Property 6. If $\{E_j\}_{j=1}^{\infty}$ is a family of measurable sets, then $\bigcap_{j=1}^{\infty} E_j$ is measurable.

注. 即说明集合族 $\mathcal{M}$ 对 *countableintersections* 封闭.

证明. Since

$$\bigcap_{j=1}^{\infty} E_j = \left(\bigcup_{j=1}^{\infty} E_j^c \right)^c \tag{1.62}$$

Then, E_j^c is measurable and so $\bigcap_{j=1}^{\infty} E_j$ is measurable. □

综上,本节介绍了 (*Lebesgue*) *measurable sets* 的性质,并且证明了 *Lebesgue measurable sets* 构成的集合族 $\mathcal{M}$ 对 *countable unions*, *countable intersections*, *complement* 运算封闭. 从而 $(\mathcal{M}, \cup, \cap, complement)$ 构成代数结构，即为后续介绍的$\sigma - algebra$.

1.3.2 *Lebesgue measure*

下面着重来介绍一下 *Lebesgue measure* 的 *properties.*

可数可加性 首先便是可数可加性 *countable additivity.*

定理 1.3.2. If $E_1, E_2, \cdots$ are disjoint measurable sets, then

$$m(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} m(E_j) \tag{1.63}$$

证明. Since $m(\bigcup\limits_{j=1}^{\infty} E_j) \leq \sum\limits_{j=1}^{\infty} m(E_j)$ always holds, we then proof the reverse inequality.

- Suppose that E_j is bounded.

  Since E_j^c is measurable, for any fixed $\epsilon > 0$, there exists an closed subset $F_j \subset E_j$, s. t.

$$m(E_j \backslash F_j) \leq \frac{\epsilon}{2^j} \tag{1.64}$$

  Since E_j is bounded, F_j is compact.

  Let $K = \bigcup\limits_{j=1}^{N} F_j$ be a disjoint union of compact sets for some fixed N, then

$$K \subset \bigcup_{j=1}^{\infty} E_j \tag{1.65}$$

$$m(K) = \sum_{j=1}^{N} m(F_j) \leq m(\bigcup_{j=1}^{\infty} E_j) \tag{1.66}$$

  Since

$$m(E_j) \leq m(E_j \backslash F_j) + m(F_j) \leq m(F_j) + \frac{\epsilon}{2^j} \tag{1.67}$$

  Therefore

$$\sum_{j=1}^{N} m(E_j) - \epsilon \leq \sum_{j=1}^{N} m(F_j) \leq m(\bigcup_{j=1}^{\infty} E_j) \tag{1.68}$$

  Let $N \to \infty$, for ϵ is arbitrary, we get

$$\sum_{j=1}^{\infty} m(E_j) \leq m(\bigcup_{j=1}^{\infty} E_j) \tag{1.69}$$

- In the general case, we choose the sequence of cubes $\{Q_k\}_{k=1}^{\infty}$, $Q_k = [-k, k]^d \subset \mathbb{R}^d$.

  Let $S_1 = Q_1$, $S_k = Q_k - Q_{k-1}$, $\forall k \geq 2$. Then $\{S_k\}_{k=1}^{\infty}$ are disjoint and bounded.

  Since $\{S_k\}_{k=1}^{\infty}$ covers $\mathbb{R}^d$,

$$E_j = \bigcup_{k=1}^{\infty} (E_j \cap S_k) \tag{1.70}$$

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} (E_j \cap S_k) \tag{1.71}$$

  Since $E_j \cap S_k$ is bounded and disjoint, by the previous case,

$$m(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} m(E_j \cap S_k) = \sum_{j=1}^{\infty} m(E_j) \tag{1.72}$$

□

单调连续性 下面我们可以给出单调可测集合列的连续性.*continuity from below/above*

定理 1.3.3. Let $E_1, E_2, \cdots$ be measurable sets in $\mathbb{R}^d$.

(i) If $E_k \nearrow E$, then $m(E) = \lim\limits_{n \to \infty} m(E_n)$.

(ii) If $E_k \searrow E$ and $m(E_1) < \infty$, then $m(E) = \lim\limits_{n \to \infty} m(E_n)$.

注. • 事实上即可写为

$$m(\lim_{n \to \infty} E_n) = \lim_{n \to \infty} m(E_n) \tag{1.73}$$

即单调可测集合列**可交换极限与测度顺序**.

- (ii) 中条件 $m(E_1)$ finite 不可省略，下面给出一个反例.

  例 1.3.1. If $E_n = (n, +\infty)$, then $m(E_n) = \infty$ and $E = \bigcap\limits_{j=1}^{\infty} E_j = \varnothing$. So

$$m(E) = m(\lim_{n \to \infty} E_j) = 0, \quad \lim_{n \to \infty} m(E_j) = \infty \tag{1.74}$$

证明.

(i) Let $S_1 = E_1$, $S_k = E_k - E_{k-1}$, $\forall k \geq 2$. Then $\{S_k\}_{k=1}^{\infty}$ are disjoint and measurable.

Since $E = \bigcup\limits_{k=1}^{\infty} E_k = \bigcup\limits_{k=1}^{\infty} S_k$, by Thm1.3.2,

$$m(E) = \sum_{k=1}^{\infty} m(S_k) = \lim_{N \to \infty} \sum_{k=1}^{N} m(S_k) = \lim_{N \to \infty} m(\bigcup_{k=1}^{N} S_k) = \lim_{N \to \infty} m(E_N) \tag{1.75}$$

(ii) Let $S_1 = E_1$, $S_k = E_k - E_{k+1}$, $\forall k \geq 2$. Then $\{S_k\}_{k=1}^{\infty}$ are disjoint and measurable.
Since $E_1 = E \cup \left(\bigcup_{k=1}^{\infty} S_k\right)$, then

$$m(E_1) = m(E) + \sum_{k=1}^{\infty} m(S_k) = m(E) + \lim_{N\to\infty} m(\bigcup_{k=1}^{N} S_k) = m(E) + \lim_{N\to\infty} m(E_1 - E_N) \tag{1.76}$$

For $E_1 = (E_1 - E_N) \sqcup E_N$ is a disjoint union,

$$m(E_1 - E_N) = m(E_1) - m(E_N) \tag{1.77}$$

Thus

$$m(E_1) = m(E) + \lim_{N\to\infty} m(E_1 - E_N) = m(E) + m(E_1) - \lim_{N\to\infty} m(E_N) \tag{1.78}$$

$$m(E) = \lim_{N\to\infty} m(E_N) \tag{1.79}$$

□

Geometric insight of measurable sets 最后我们来给出 (*Lebesgue*) *measurable sets* 的几何性质 (与开集、闭集、紧集等之间的关系).

定理 1.3.4. Suppose $E \subset \mathbb{R}^d$ is measurable, then $\forall \epsilon > 0$:

(i) $\exists$ open $O \supset E$ with $m(O \backslash E) \leq \epsilon$.

(ii) $\exists$ closed $F \subset E$ with $m(E \backslash F) \leq \epsilon$.

(iii) If $m(E) < \infty$, $\exists$ compact $K \subset E$ with $m(E \backslash K) \leq \epsilon$.

(iv) If $m(E) < \infty$, $\exists F = \bigcup_{j=1}^{N} Q_j$, $\{Q_j\}_{j=1}^{\infty}$ are closed cubes, s. t. $m(E \triangle F) \leq \epsilon$.

证明.

(i) It's just the definition of measurability.

(ii) Since E_j^c is measurable, $\exists$ open $O_j \supset E_j^c$, s. t.

$$m(O_j \backslash E_j^c) \leq \epsilon \tag{1.80}$$

Since $O_j^c \subset E_j$ is closed and $E_j \backslash O_j^c = O_j \backslash E_j^c$, let $F = O_j^c$ closed, then

$$m(E_j \backslash F) = m(O_j \backslash E_j^c) \leq \epsilon \tag{1.81}$$

(iii) By (ii), $\exists$ closed $F \subset E$, s. t. $m(E\backslash F) \leq \frac{\epsilon}{2}$.

Let B_n denote the closed ball centered at the origin of radius n, then B_n is compact.

$$F = \bigcup_{j=1}^{\infty} (F \cap B_k) \tag{1.82}$$

Let $K_n = \bigcup_{k=1}^{n} (F \cap B_k)$, then K_n is compact and $K_n \nearrow F \Rightarrow E\backslash K_n \nearrow E\backslash F$.
Since $m(E\backslash K_1) \leq m(E)$ is finite, by Thm1.3.3(ii)

$$\lim_{n\to\infty} m(E\backslash K_n) = m(E\backslash F) \tag{1.83}$$

As for $\epsilon > 0$, $\exists N \in \mathbb{N}$, s. t. for all $n \geq N$

$$|m(E\backslash K_n) - m(E\backslash F)| \leq \frac{\epsilon}{2} \tag{1.84}$$

$$m(E\backslash K_n) \leq m(E\backslash F) + \frac{\epsilon}{2} \leq \epsilon \tag{1.85}$$

Therefore, $m(E\backslash K_N) \leq \epsilon$, where $K_N \subset E$ is compact.

(iv) $\exists$ open $O \supset E$, s. t. $m(O\backslash E) \leq \frac{\epsilon}{2}$. By Thm1.1.4, $\exists \{Q_j\}_{j=1}^{\infty}$, s. t.

$$E \subset O = \bigcup_{j=1}^{\infty} Q_j \tag{1.86}$$

So

$$m(O) = \sum_{j=1}^{\infty} \left|Q_j\right| \leq m(O\backslash E) + m(E) \leq \frac{\epsilon}{2} + m(E) \tag{1.87}$$

Since $m(E)$ is finite, $\sum_{j=1}^{\infty} \left|Q_j\right|$ converges. Then $\exists N \in \mathbb{N}$, s. t.

$$\sum_{j=N+1}^{\infty} \left|Q_j\right| \leq \frac{\epsilon}{2} \tag{1.88}$$

Let $F = \bigcup_{j=1}^{N} Q_j$. Since $E \triangle F = (E\backslash F) \sqcup (F \cap E)$, then

$$\begin{aligned}
m(E \triangle F) &= m(E\backslash F) + m(F\backslash E) && (1.89)\\
&\leq m(\bigcup_{j=N+1}^{\infty} Q_j) + m(\bigcup_{j=1}^{\infty} Q_j\backslash E) && (1.90)\\
&= \sum_{j=N+1}^{\infty} \left|Q_j\right| + \sum_{j=1}^{\infty} \left|Q_j\right| - m(E) && (1.91)\\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon && (1.92)
\end{aligned}$$

□

1.4 $\sigma-$ *algebras and Borel sets*

1.4.1 $\sigma-$ *algebra*

首先给出 $\mathbb{R}^d$ 中 *algebra* 的定义.

定义 1.4.1. Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^d)$. $\mathcal{A}$ is called an *algebra* if

(1) If $A_1, \cdots, A_n \in \mathcal{A}$, then $\bigcup_{j=1}^{n} A_j \in \mathcal{A}$.

(2) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

注. 容易证明，若 $\mathcal{A}$ 为 $\mathbb{R}^d$ 中 *algebra*，则其对 *finite intersections* 也封闭，同时 $\varnothing, \mathbb{R}^d \in \mathcal{A}$.

下面给出 $\mathbb{R}^d$ 中 $\sigma-$ *algebra* 的定义.(将 *algebra* 中的 *finite* 条件加强为 *countable*)

定义 1.4.2. Let $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^d)$. $\mathcal{M}$ is a $\sigma-$ *algebra* if

(1) If $A_1, A_2, \cdots \in \mathcal{M}$, then $\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$.

(2) If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$.

注. 容易证明 $\mathcal{M}$ 对 *countable intersections* 同样封闭，$\varnothing, \mathbb{R}^d \in \mathcal{M}$.

例 1.4.1. All Lebesgue measurable sets forms a $\sigma-$ *algebra* $\mathcal{M}$.

类比线性空间、拓扑空间中 (拓扑) 基的概念，下面给出**生成** $\sigma-$ *algebra* 的概念.

定义 1.4.3. Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^d)$, then the $\sigma-$ *algebra* **generated by** $\mathcal{A}$ is the smallest $\sigma-$ *algebra* containing $\mathcal{A}$.

注. 即为 the intersection of all $\sigma-$ *algebras* containing $\mathcal{A}$，这也说明了对于任一给定的集族 $\mathcal{A}$，其生成的 $\sigma-$ *algebra* 必存在且唯一.

1.4.2 *Borel sets*

下面给出 *Borel* $\sigma - algebra$ 及 *Borel sets* 的定义.

定义 1.4.4. The *Borel* $\sigma - algebra$ is the $\sigma - algebra$ generated by all open sets in $\mathbb{R}^d$, denoted by $\mathcal{B}_{\mathbb{R}^d}$.

Elements of this $\sigma - algebra$ are called *Borel sets*.

注. 事实上，*Borel* $\sigma - algebra$ 为 Lebesgue countable sets 的一个真子集，后续会利用 Cantor 集证明.

为了方便研究 *Borel* $\sigma - algebra$ 的结构，我们把其中较为复杂 (非平凡) 的元素单独拎出来并称为 G_δ, F_σ.

定义 1.4.5.

1. The countable intersections of open sets are called G_δ sets.
2. The countable unions of closed sets are called F_σ sets.

下面我们可给出 $\mathcal{B}_{\mathbb{R}^d}$ 与 Lebesgue 可测集 $\mathcal{L}$ 之间的关系.($\mathcal{L}$ 只比 $\mathcal{B}_{\mathbb{R}^d}$ 多了一些零测集)

定理 1.4.1. $E \subset \mathbb{R}^d$ is $\mathcal{L} - measurable$

(i) if and only if $E = G_\delta \backslash N_1$, for some G_δ, $m(N_1) = 0$.

(ii) if and only if $E = F_\sigma \backslash N_2$, for some F_σ, $m(N_2) = 0$.

证明. Clearly E is measurable whenever it satisfies either (i) or (ii).

(i) Since E is measurable, $\exists$ open sets $O_n \supset E$, s. t.

$$m(O_n \backslash E) \leq \frac{1}{n} \tag{1.93}$$

Let $O = \bigcap_{j=1}^{\infty} O_j$, then

$$m(O \backslash E) \leq \frac{1}{n}, \ \forall n \in \mathbb{N} \tag{1.94}$$

Let $n \to \infty$, we get $m(O \backslash E) = 0$. Let $G_\delta = O, \ N_1 = O \backslash E$. Then $E = G_\delta \backslash N_1$.

(ii) Similarly, we can easily proof it by Thm1.3.4(ii).

□

1.5 Non − measurable sets

在这一节我们将介绍 $\mathbb{R}$ 上一个经典的不可测集 *Vitali set*，并说明 $\mathbb{R}$ 上每个正测度集都有不可测子集.

Vitali set Let $x, y \in [0,1]$. Write $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$.

$\Rightarrow$ 容易验证 $\sim$ 为 an equivalence relation.

$\Rightarrow \sim$ partions [0 , 1]. 记 [0 , 1] 上等价类为 ε_α，则

$$[0,1] = \bigsqcup_\alpha \varepsilon_\alpha, \ \{\varepsilon_\alpha\}_\alpha \ \textit{are disjoint} \tag{1.95}$$

$\Rightarrow$ By **the Axiom of Choice**, we can choose exactly one element x_α from each ε_α.

$\Rightarrow$ Let $\mathcal{N} = \{x_\alpha\}_\alpha$. Then $\mathcal{N}$ is the Vitali set.

定理 1.5.1. $\mathcal{N}$ is not measurable.

证明. Assume that $\mathcal{N}$ is measurable. Let $\{r_k\}_{k=1}^\infty$ be an enumeration of $\mathbb{Q} \cap [-1,1]$. Define

$$\mathcal{N}_k := N + r_k = \{x_\alpha + r_k\}_\alpha \tag{1.96}$$

Then we shall proof that $\{\mathcal{N}_k\}_{k=1}^\infty$ are disjoint, and $[0,1] \subset \bigcup_{k=1}^\infty \mathcal{N}_k \subset [-1,2]$.

- If $\mathcal{N}_k \cap \mathcal{N}_m \neq \varnothing$, then $\exists x_\alpha, x_\beta \in \mathcal{N},\ r_k, r_m \in \mathbb{Q} \cap [-1,1]$, s. t.

$$x_\alpha + r_k = x_\beta + r_m \tag{1.97}$$

  Then $x_\alpha - x_\beta = r_m - r_k \in \mathbb{Q} \Rightarrow x_\alpha \sim x_\beta \Rightarrow x_\alpha, x_\beta \in \varepsilon_\alpha$ or $x_\alpha, x_\beta \in \varepsilon_\beta \Rightarrow x_\alpha = x_\beta$ and $r_k = r_m$. Therefore, $\mathcal{N}_k = \mathcal{N}_m$.

- Since $r_k \in [-1,1]$, $\mathcal{N}_k \in [-1,2]$, $\forall k$. Therefore,

$$\bigcup_{k=1}^\infty \mathcal{N}_k \subset [-1,2] \tag{1.98}$$

- $\forall x \in [0,1]$. Since $\{\varepsilon_\alpha\}_\alpha$ partions $[0,1]$, there exists α_0, s. t.

$$x \in \varepsilon_{\alpha_0},\ x \sim x_{\alpha_0} \tag{1.99}$$

  which means $x - x_{\alpha_0} \in \mathbb{Q} \cap [-1,1]$. Then $\exists k_0 \in \mathbb{N}$, s. t.

$$x - x_{\alpha_0} = r_{k_0} \ \Rightarrow \ x \in \mathcal{N}_{k_0} \tag{1.100}$$

Therefore,

$$[0,1] \subset \bigcup_{k=1}^{\infty} \mathcal{N}_k \tag{1.101}$$

Since $\{\mathcal{N}_k\}_{k=1}^{\infty}$ are disjoint, we get

$$m([0,1]) \leq \sum_{k=1}^{\infty} m(\mathcal{N}_k) \leq m([-1,2]) \tag{1.102}$$

Since $\mathcal{N}_k$ is a translate of $\mathcal{N}$, we have $m(\mathcal{N}) = m(\mathcal{N}_k)$ for each k. Then

$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}) \leq 3 \ \Rightarrow \ \textit{Neither } m(\mathcal{N}) = 0 \textit{ nor } m(\mathcal{N}) > 0 \textit{ is possible.} \tag{1.103}$$

Therefore, it's a contradiction. $\mathcal{N}$ is non-measurable. □

正测度集必有不可测子集 下面要证明一个结论，即 $\mathbb{R}$ 上任一正测度集必有不可测子集. 这实际上为书[1]Exercises of Chapter 1 的第 32 题 (b).

命题 1.5.1. Let $\mathcal{N}$ denote the non-measurable subset of $[0,1]$ constructed in Thm1.5.1.

(a) If E is a measurable subset of $\mathcal{N}$, then $m(E) = 0$.

(b) If $G \subset \mathbb{R}$ with $m_*(G) > 0$, then there exists a subset of G is non-measurable.

证明.

(a) Note $\mathcal{N} = \{x_\alpha\}_{\alpha \in \mathcal{A}}$, then $E = \{x_\beta\}_{\beta \in \mathcal{B} \subset \mathcal{A}}$. Similarly, we can proof

$$\bigcup_{k=1}^{\infty} E_k \subset [-1,2] \tag{1.104}$$

Since $\{E_k\}_{k=1}^{\infty}$ are disjoint, and E_k is a translate of E, we get

$$\sum_{k=1}^{\infty} m(E) \leq 3 \ \Rightarrow \ m(E) = 0 \tag{1.105}$$

(b) Let $\mathbb{Q} = \{r_k\}_{k=1}^{\infty}$, $\mathcal{N}_k = \mathcal{N} + r_k$, then

$$\mathbb{R} = \bigsqcup_{k=1}^{\infty} \mathcal{N}_K \tag{1.106}$$

[1]参考书籍：《*Real Analysis – –Measure Theroy, Integration, & Hilbert Spaces*》— *Elias M. Stein*

Suppose G is measurable. Then

$$G = G \cap \mathbb{R} = \bigsqcup_{k=1}^{\infty} (G \cap \mathcal{N}_k) \tag{1.107}$$

If $G \cap \mathcal{N}_k$ is measurable, then $G \cap \mathcal{N}_k \subset \mathcal{N}_k$ is a subset of a non-measurable set $\mathcal{N}_k$. By the previous (a), we get

$$m(G \cap \mathcal{N}_k) = 0 \tag{1.108}$$

Therefore, there exists $k_0 \in \mathbb{N}$, s. t. $G \cap \mathcal{N}_{k_0} \subset G$ is a non-measurable subset of G. (otherwise $m(G) = 0$ contradicts)

□

第二章 *Measurable Functions*

2.1 *Measurable Functions*

定义 下面给出 $\mathbb{R}^d$ 上**可测函数**的定义.(注意值域为**扩充实数系** $\overline{\mathbb{R}}$)

定义 2.1.1. A function defined on a measurable subset $E \subset \mathbb{R}^d$ is **measurable** if for all $a \in \mathbb{R}$,

$$f^{-1}([-\infty, a)) = \{x \in E \mid f(x) < a\} \tag{2.1}$$

is measurable.

注. • $f^{-1}([-\infty, a))$ 常简记作 $\{f < a\}$.

• 下面给出几条等价定义.

(1) $\{f < a\}$ is measurable. $\Leftrightarrow$ $\{f \le a\}$ is measurable.

(2) $\Leftrightarrow$ $\{f > a\}$ is measurable $\Leftrightarrow$ $\{f \ge a\}$ is measurable.

(3) If f is finite-valued, then

$$f \textit{ is measurable } \Leftrightarrow \{a < f < b\} \textit{ is measurable}, \ \forall a, b \in \mathbb{R} \tag{2.2}$$

证明.

(1) Since the collection of measurable sets is closed under countable intersections and unions,

$$\{f \le a\} = \bigcap_{n=1}^{\infty} \{f < a + \frac{1}{n}\} \tag{2.3}$$

$$\{f < a\} = \bigcup_{n=1}^{\infty} \{f \le a - \frac{1}{n}\} \tag{2.4}$$

Therefore, $\{f < a\}$ is measurable. $\Leftrightarrow$ $\{f \le a\}$ is measurable.

(2) Since the collection of measurable sets is closed under complements, easily proof by (1).

(3) Since f is finite-valued,

$$\{f < a\} = \bigcup_{n=1}^{\infty} \{-n < f < a\} \tag{2.5}$$

$$\{a < f < b\} = \{f > a\} \cap \{f < b\} \tag{2.6}$$

Therefore, by (2), f is measurable $\Leftrightarrow$ $\{a < f < b\}$ is measurable.

□

Property 下面给出可测函数的一些性质.

Property 1. Let $-\infty < f(x) < +\infty$ (finite-valued), then

$$f \textit{ is measurable } \Leftrightarrow f^{-1}(O) \textit{ is measurable } \forall \textit{open set } O \tag{2.7}$$

$$\Leftrightarrow f^{-1}(F) \textit{ is measurable } \forall \textit{closed set } F \tag{2.8}$$

证明. $\forall O \underset{open}{\subset} \mathbb{R}$, there exists $\{(a_n, b_n)\}_{n=1}^{\infty}$, s. t.

$$O = \bigcup_{n=1}^{\infty} (a_n, b_n) \tag{2.9}$$

Then

$$f^{-1}(O) = f^{-1}(\bigcup_{n=1}^{\infty} (a_n, b_n)) = \bigcup_{n=1}^{\infty} f^{-1}((a_n, b_n)) \tag{2.10}$$

Since f is finite-valued and measurable, then $f^{-1}(a_n, b_n)$ is measurable.

Therefore, $f^{-1}(O)$ is measurable. □

Property 2. $\{\textit{continuous functions}\} \subset \{\textit{measurable functions}\}$

(a) (a)If f is continuous on $\mathbb{R}^d$, then f is measurable.

(b) If f is measurable, finite-valued and Φ is continuous on $\mathbb{R}$, then $\Phi \circ f$ is measurable.

证明.

(a) Since f is continuous, $\forall O \underset{open}{\subset} \mathbb{R}$, $f^{-1}(O) \underset{open}{\subset} \mathbb{R}^d$. By Property 1, f is measurable.

(b) $\forall O \underset{open}{\subset} \mathbb{R}$. Since Φ is continuous, then $\Phi^{-1}(O)$ is open.

Since f is finite-valued and measurable, then $(\Phi \circ f)^{-1}(O) = f^{-1}(\Phi^{-1}(O))$ is open.

Therefore, by Property 1, $\Phi \circ f$ is measurable.

$\square$

Property 3. Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions. Then

$$\sup_n f_n(x),\ \inf_n f_n(x),\ \limsup_{n\to\infty} f_n(x),\ \liminf_{n\to\infty} f_n(x) \tag{2.11}$$

are measurable.

注. 类比数列的上下极限，此处

$$\limsup_{n\to\infty} f_n(x) := \lim_{k\to\infty} \sup_{n\geq k} \{f_n(x)\} = \inf_k \sup_{n\geq k} \{f_n(x)\} \tag{2.12}$$

$$\liminf_{n\to\infty} f_n(x) := \lim_{k\to\infty} \inf_{n\geq k} \{f_n(x)\} = \sup_k \inf_{n\geq k} \{f_n(x)\} \tag{2.13}$$

证明. Since

$$\{x \mid \sup_n f_n(x) > a\} = \bigcup_{n=1}^{\infty} \{x \mid f_n(x) > a\} \tag{2.14}$$

$$\{x \mid \inf_n f_n(x) < a\} = \bigcup_{n=1}^{\infty} \{x \mid f_n(x) < a\} \tag{2.15}$$

Then $\sup\limits_n f_n(x)$, $\inf\limits_n f_n(x)$ is measurable.

Since $\sup\limits_{n\geq k} f_n(x)$, $\inf\limits_{n\geq k} f_n(x)$ are measurable, by the previous conclusion, then

$$\limsup_{n\to\infty} f_n(x) = \inf_k \sup_{n\geq k} \{f_n(x)\} \tag{2.16}$$

$$\liminf_{n\to\infty} f_n(x) = \sup_k \inf_{n\geq k} \{f_n(x)\} \tag{2.17}$$

are measurable. $\square$

Property 4. If $\{f_n\}_{n=1}^{\infty}$ is a collection of measurable functions and

$$\lim_{n\to\infty} f_n(x) = f(x) \tag{2.18}$$

then f is measurable.

注. • 与数列上下极限相同,

$$\lim_{n\to\infty} f_n(x) = f(x) \ \Leftrightarrow \ \limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x) = f(x) \tag{2.19}$$

• 此 Property 即说明可测函数列对极限运算封闭. 注意到连续函数列对极限运算并不具备封闭性.(下面给出经典范例)

例 2.1.1.

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases} \tag{2.20}$$

证明. Since $\{f_n\}_{n=1}^{\infty}$ are measurable, $f(x) = \limsup\limits_{n\to\infty} f_n(x) = \limsup\limits_{n\to\infty} f_n(x)$, then according to Property 3, f is measurable. □

Property 5. If f and g are measurable, then

(i) f^k, $k \in \mathbb{N}$ are measurable.

(ii) $f + g$ and fg are measurable if both f and g are finite-valued.

证明.

(i) Since

$$\{f^k > a\} = \{f > a^{\frac{1}{k}}\}, \ \forall k \text{ is odd} \tag{2.21}$$

$$\{f^k > a\} = \{f > a^{\frac{1}{k}}\} \cup \{f < -a^{\frac{1}{k}}\}, \ \forall k \text{ is even and } a > 0 \tag{2.22}$$

Therefore, f^k, $k \in \mathbb{N}$ are measurable.

(ii) Since[1]

$$\{f + g > a\} = \bigcup_{r\in\mathbb{Q}} \{f > a - r\} \cap \{g > r\} \tag{2.23}$$

---

[1]即必 $\exists r \in \mathbb{Q}$, s. t.$\{f + g > a\} \supset \{f > a - r\} \cap \{g > r\}$. (另一侧包含关系 $\subset$ 显然易证)
(反证. $\forall r \in \mathbb{Q}$ 上式不成立，则对于 $r = 0 \in \mathbb{Q}$, $\exists x_0$, s. t. $f(x_0) > a$, $g(x_0) > 0$, 且 $f(x_0) + g(x_0) \le a$, 矛盾.)

then $f+g$ is measurable.

By the previous results in (i) and (ii), since

$$fg = \frac{1}{4}[(f+g)^2 - (f-g)^2] \tag{2.24}$$

Therefore, fg is also measurable.

□

下面给出数学分析中曾介绍过的**几乎处处**的定义.

定义 2.1.2. A property or statement is said to hold **almost everywhere (a.e.)** if it is true except on a set of measure zero.

例 2.1.2.

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases} \tag{2.25}$$

We say f is continuous a.e. on $[0,1]$ since $D(f) = \{1\}$ has measure zero.

下面说明几乎处处相等可保持函数可测性.

命题 2.1.1. If f is measurble and $f = g$ a.e. , then g is measurable.

证明. Since f is measurable and

$$g = (g-f) + f \tag{2.26}$$

then we shall proof that $g-f$ is measurable.

Let $A := \{x \mid g(x) - f(x) \neq 0\}$, then $m(A) = 0$. We get

$$\forall a \ge 0, (g-f)^{-1}((-\infty, a]) = (\mathbb{R}^d \backslash A) \cup N, \ \ where\ N \subset A \tag{2.27}$$

Since $m(A) = 0$, then N is measurable and $m(N) = 0$. So $(g-f)^{-1}((-\infty, a])$ is measurable. Therefore, $g-f$ is measurable. Then g is measurable. □

2.2 *Measurable functions are nearly simple*

本节来介绍一个非常重要的定理. 即可测函数可由简单函数逼近.

特征函数 下面先来介绍**特征函数**的定义.

定义 2.2.1. If $E \subset \mathbb{R}$, the **characteristic / indicator function** $\chi_E / \mathbb{1}_E$ of E is defined by

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases} \tag{2.28}$$

下面给出可测集与其对应特征函数的关系.

命题 2.2.1. χ_E is measurable $\Leftrightarrow$ E is measurable

证明. Since

$$\chi_E^{-1}((-\infty, a]) = \begin{cases} \varnothing, & a < 0 \\ E^c, & 0 \le a < 1 \\ \mathbb{R}^d, & a \ge 1 \end{cases} \tag{2.29}$$

Then E is measurable $\Rightarrow$ χ_E is measurable.

χ_E is measurable $\Rightarrow$ $\chi_E^{-1}((-\infty, a]) = E^c$ is measurable. $\Rightarrow$ E is measurable. □

下面给出特征函数的基本性质.

命题 2.2.2. [Property].

(1) If $A \cap B = \varnothing$, then

$$\chi_{A\cup B} = \max\{\chi_A, \chi_B\} = \chi_A + \chi_B \tag{2.30}$$

(2) $\chi_{A\cap B} = \min\{\chi_A, \chi_B\} = \chi_A \cdot \chi_B$.

Simple functions 对特征函数做线性组合，即可得到**简单函数**.

定义 2.2.2. A **simple function** on $\mathbb{R}^d$ is a finite linear combination

$$f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x) \tag{2.31}$$

where each E_j is measurable and $m(E_j) < \infty$.

注. 此处定义中并未要求 $\{E_j\}_{j=1}^n$ disjoint. 而事实上这便引出了下面介绍的**标准形式**.

下面的命题说明了每个简单函数都可写为**标准形式** ($\{E_j\}_{j=1}^n$ disjoint).

命题 2.2.3. Every simple function f has a **standard representaion**

$$f = \sum_{k=1}^{N} a_k \chi_{E_k}, \ where\ \{E_j\}_{k=1}^{N}\ are\ disjoint \tag{2.32}$$

证明. Suppose $f = \sum_{k=1}^{N} b_k \chi_{E_k}$, $\{E_j\}_{k=1}^N$ may not be disjoint.
Since $\{E_j\}_{k=1}^N$ is finite, the number of elements of range f is also finite. Suppose

$$range\ f = \{a_1, \cdots, a_M\} \tag{2.33}$$

Then let $F_k = f^{-1}(\{a_k\})$, then $\{F_k\}_{k=1}^M$ are disjoint. Therefore, we get the standard representation

$$f = \sum_{k=1}^{M} a_k \chi_{F_k} \tag{2.34}$$

□

简单函数逼近可测函数 下面给出一个定理，说明任一可测函数可由简单函数列逼近.

定理 2.2.1. Suppose $f: \mathbb{R}^d \longrightarrow [-\infty, \infty]$ is measurable.
Then there exists a sequence $\{\varphi_n\}$ of simple functions, s. t.

$$0 \le |\varphi_1| \le |\varphi_2| \le \cdots \le |f| \tag{2.35}$$

$$\lim_{k\to\infty} \varphi_k(x) = f(x),\ for\ all\ x \tag{2.36}$$

and $\varphi_k \to f$ uniformly on any set on which f is bounded.

证明. 下面从两方面分类讨论，即非负函数 & 变号函数，f 有界 & 无界.

(1) 非负函数 $f: \mathbb{R}^d \longrightarrow [0, \infty]$.

1° f is bounded. Assume $|f(x)| \le M$.

Let[2]

$$E_n^k = f^{-1}\left(\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right), k = 0, \cdots, N_n \tag{2.37}$$

$$\varphi_n(x) = \frac{k}{2^n},\ if\ x \in E_n^k \tag{2.38}$$

Then

$$\varphi_n(x) = \sum_{k=0}^{N_n} \frac{k}{2^n} \chi_{E_n^k}(x) \tag{2.39}$$

Therefore[3]

$$|\varphi_n(x) - f(x)| \le \frac{1}{2^n} \to 0\ (independent\ of\ x) \tag{2.40}$$

$\Rightarrow \varphi_n \to f$ uniformly.

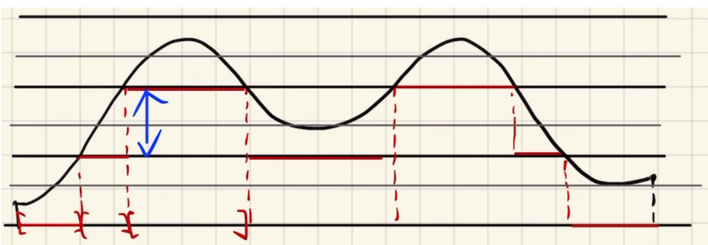

图 2.1: 对 f 值域进行分划

[2] E_n^k 表示第 n 次对值域进行分划后产生的第 k 个值域区间，其中 $\frac{N_n+1}{2^n} \ge M$.

[3] $|\varphi_n(x) - f(x)|$ 小于等于第 n 次分划后两个相邻值域区间的步长值，即 $\frac{1}{2^n}$.

2° f is unbounded. (idea: truncation，将 f 截断为一列有界函数列，并逐点收敛于 f)

Let

$$f_k(x) = \begin{cases} f(x), & \text{if } f(x) \le k \\ k, & \text{if } f(x) > k \end{cases} \tag{2.41}$$

Then $f_k(x) \to f(x)$, $\forall x \in \mathbb{R}^d$.

Since f_k is bounded, by the previous result in 1°,

For each k, $\exists$ a sequence of simple functions $\{\psi_{kn}\}_{n=1}^{\infty}$, s. t.

$$\psi_{kn}(x) \to f_k(x), \ \forall x \tag{2.42}$$

So we get

$$\begin{matrix} \psi_{11} & \psi_{12} & \psi_{13} & \cdots & \to & f_1 \\ \psi_{21} & \psi_{22} & \psi_{23} & \cdots & \to & f_2 \\ \psi_{31} & \psi_{32} & \psi_{33} & \cdots & \to & f_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \downarrow \\ & & & & & f \end{matrix} \tag{2.43}$$

From the previous results in 1°, we get

$$|\psi_{kn}(x) - f_k(x)| \le \frac{1}{2^n} \tag{2.44}$$

Let $n = k$, then $|\psi_{kk}(x) - f_k(x)| \le \frac{1}{2^k}$. Let $\varphi_k = \psi_{kk}$, then

$$|\varphi_k(x) - f(x)| \le |\varphi_k(x) - f_k(x)| + |f_k(x) - f(x)| \tag{2.45}$$

Since $f_k(x) \to f(x)$, we get $\varphi_k(x) \to f(x)$, $\forall x$, where $\{\varphi_k = \psi_{kk}\}_{k=1}^{\infty}$ are simple functions.

(2) 变号函数 $f : \mathbb{R}^d \longrightarrow [-\infty, \infty]$.

We denote that

$$f^+(x) := \max\{f(x), 0\} \tag{2.46}$$

$$f^-(x) := \max\{-f(x), 0\} \tag{2.47}$$

By the previous results in (1), there exist sequences of simple functions $\{\varphi_k\}_{k=1}^{\infty}, \{\psi_k\}_{k=1}^{\infty}$, s. t.

$$\varphi_k \to f^+ \text{ and } \psi_k \to f^- \text{ pointwisely} \tag{2.48}$$

We can observe that $f = f^+ - f^-$ and $|f| = f^+ - f^-$.

Let $\phi_k(x) = \varphi_k(x) - \psi_k(x)$, then ϕ_k is a simple function with $\phi_k \to f$ pointwisely.

□

阶梯函数逼近可测函数 在证明了可测函数可由简单函数逼近后，我们更进一步，来说明可测函数可由更加简单的**阶梯函数**来逼近.

先给出**阶梯函数**的定义.

定义 2.2.3. A **step function** is a finite sum

$$f=\sum_{k=1}^{N} a_k \chi_{R_k},\ \textit{where } R_k \textit{ is a rectangle} \tag{2.49}$$

注. 阶梯函数 & 简单函数的区别在于，简单函数是作用于有限个**可测集** E_k，而阶梯函数是作用于有限个**矩形** R_k.

下面的定理说明了 measurable functions are almost step functions.

定理 2.2.2. Suppose f is measurable on $\mathbb{R}^d$. Then there exists a sequence of step functions $\{\psi_k\}_{k=1}^{\infty}$, s. t.

$$\lim_{k\to\infty}\psi_k(x)=f(x),\ a.e.\ x \tag{2.50}$$

注. 首先介绍函数列收敛点集的几种不同的等价表述：

$$\{x \mid \forall \epsilon>0, \exists N\in\mathbb{N}, \forall n\geq N,\ |f_n(x)-f(x)|<\epsilon\} \tag{2.51}$$

$$\Leftrightarrow \{x \mid \forall n\in\mathbb{N}, \exists N\in\mathbb{N}, \forall k\geq N,\ |f_k(x)-f(x)|<\frac{1}{n}\} \tag{2.52}$$

$$\Leftrightarrow \bigcap_{n=1}^{\infty}\bigcup_{N=1}^{\infty}\bigcap_{k=N}^{\infty}\{x \mid\ |f_k(x)-f(x)|<\frac{1}{n}\} \tag{2.53}$$

从而可以得到函数列发散点集 (Negation)：

$$\{x \mid \exists n\in\mathbb{N}, \forall N\in\mathbb{N}, \exists k\geq N,\ |f_k(x)-f(x)|\geq\frac{1}{n}\} \tag{2.54}$$

$$\Leftrightarrow \bigcup_{n=1}^{\infty}\bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty}\{x \mid\ |f_k(x)-f(x)|\geq\frac{1}{n}\} \tag{2.55}$$

$$\Leftrightarrow \bigcap_{N=1}^{\infty}\bigcup_{k=N}^{\infty}\{x \mid\ f_k(x)\neq f(x)\} \tag{2.56}$$

证明. (证明思路：先用阶梯函数逼近简单函数，再用简单函数逼近可测函数.)

It suffices to show that χ_E can be approxiamted by step functions, for any measurable set E.

According to Thm1.3.4 (iv)

Let $f = \chi_E$, then $\forall \epsilon > 0$, $\exists$ cubes $\bigcup_{j=1}^{N} Q_j$, s. t.

$$m(E \triangle \bigcup_{j=1}^{N} Q_j) \leq \epsilon \tag{2.57}$$

By considering the grid formed by extending the sides of these cubes, there exists almost disjoint rectangles $\{\widetilde{R}_j\}_{j=1}^{M}$, s. t.

$$\bigcup_{j=1}^{N} Q_j = \bigcup_{j=1}^{M} \widetilde{R}_j \tag{2.58}$$

By taking ranctangles R_j contained in $\widetilde{R}_j$, we can find a collection of disjoint rectangles $\{R_j\}_{j=1}^{M}$, s. t.

$$m(E \triangle \bigsqcup_{j=1}^{M} R_j) \leq 2\epsilon \tag{2.59}$$

For every $k \in \mathbb{N}$, there exists disjoint rectangles $\{R_j\}_{j=1}^{M}$, s. t.

$$m(E \triangle \bigsqcup_{j=1}^{M} R_j) \leq \frac{1}{2^{k+1}} \tag{2.60}$$

There also exists a step function ψ_k

$$\psi_k(x) := \chi_{\bigcup_{j=1}^{M} R_j}(x) = \sum_{j=1}^{M} \chi_{R_j}(x) \tag{2.61}$$

Let

$$E_k := \{x \mid f_k(x) \neq f(x)\} \tag{2.62}$$

Since $E_k \subset E \triangle \bigsqcup_{j=1}^{M} R_j$, then $m(E_k) \leq \frac{1}{2^k}$. Let[4]

$$F_j = \bigcup_{j=k+1}^{\infty} E_j, \;\; F = \bigcap_{k=1}^{\infty} F_k \tag{2.63}$$

Then $\psi_k(x) \to f(x)$, $\forall x \in F^c$. Since

$$m(F) \leq m(F_k), \;\; \forall k \in \mathbb{N} \tag{2.64}$$

$$m(F_k) = m(\bigcup_{j=k+1}^{\infty} E_j) \leq \sum_{j=k+1}^{\infty} m(E_j) \leq \frac{1}{2^k} \tag{2.65}$$

Therefore, $m(F) = 0$. $\lim_{k\to\infty} \psi_k(x) = f(x)$, $a.e.\ x$. □

[4]根据注中式 (2.56)，F 即为函数列 $\{\psi_k\}_{k=1}^{\infty}$ 的发散点集，从而 $\psi_k(x) \to f(x)$ 在 F^c 上收敛.

第三章 *Integration Theory*

3.1 *The Lebesgue integral*

Lebesgue Integral 的构造可以分为三步，分别为构造下列函数的积分:

1. **Simple functions**
2. **Non-negative measurable functions**

$$\int f := \sup\{\int \varphi \mid \varphi \textit{ simple}, \ 0 \le \varphi \le f\} \tag{3.1}$$

3. **General case**

$$f = f^+ - f^- \tag{3.2}$$

$$\int f := \int f^+ - \int f^- \tag{3.3}$$

3.1.1 *Simple functions*

定义 下面先给出非负简单函数在**标准形式**下的**积分**定义.

定义 3.1.1. If φ is a non-negative simple function with **standard representation**

$$\varphi(x) = \sum_{k=1}^{M} a_k \chi_{E_k}(x) \tag{3.4}$$

We define the **Lebesgue integral** of φ by

$$\int_{\mathbb{R}^d} \varphi(x)dx = \sum_{k=1}^{M} a_k m(E_k) \tag{3.5}$$

If E is a measurable subset of $\mathbb{R}^d$ with finite measure, then

$$\varphi(x)\chi_E(x) = \sum_{k=1}^{M} a_k \chi_{E_k}(x)\chi_E(x) = \sum_{k=1}^{M} a_k \chi_{E_k \cap E}(x) \tag{3.6}$$

is also a simple function, and define

$$\int_E \varphi(x)dx = \int_{\mathbb{R}^d} \varphi(x)\chi_E(x)dx \tag{3.7}$$

注. • 此处仅对**标准形式**定义了积分. 事实上，此处定义的积分与简单函数的表达形式无关 (即**Property 1.**).

• 关于记号，当测度非常明确时，大多数情况下可简写，如

$$\int_E \varphi(x)dx \Rightarrow \int_E \varphi \tag{3.8}$$

$$\int_{\mathbb{R}^d} \varphi(x)dx \Rightarrow \int \varphi \tag{3.9}$$

当为了强调我们选择了何种测度 μ 时，还可用以下的记号：

$$\int_E \varphi(x)d\mu(x) \tag{3.10}$$

Property 下面给出简单函数积分的性质.

Property 1. **Independence of the representation.**

If $\varphi = \sum\limits_{k=1}^{N} a_k \chi_{E_k}$ is any representation of φ, then

$$\int \varphi = \sum_{k=1}^{N} a_k m(E_k) \tag{3.11}$$

在证明这个性质之前，先来证明一条引理.(书[1]Exercises Of Chapter 2 的第 1 题)

引理 3.1.1. Given a collection of sets $\{F_k\}_{k=1}^n$, there exists another collection $\{\widetilde{F_j}\}_{j=1}^N$ with $N = 2^n - 1$, so that

$$(i). \quad \bigcup_{k=1}^{n} F_k = \bigcup_{j=1}^{N} \widetilde{F_j} \tag{3.12}$$

$$(ii). \quad \{\widetilde{F_j}\}_{j=1}^{N} \ are\ disjoint \tag{3.13}$$

$$(iii). \quad F_k = \bigcup_{\widetilde{F_j} \subset F_k} \widetilde{F_j} \tag{3.14}$$

证明. Consider the collection

$$\mathcal{F} := \{\bigcup_{k=1}^{n} G_k - \bigcap_{k=1}^{n} F_k^c \,\Big|\, G_k\ denotes\ F_k\ or\ F_k^c\} \tag{3.15}$$

□

[1]参考书籍：《*Real Analysis – –Measure Theroy, Integration, & Hilbert Spaces*》— *Elias M. Stein*

下面来证明原命题.

证明. According to Lemma 3.1.1, there exists another decompositon of $\bigcup_{k=1}^{N} E_k$, i.e.

$$\bigcup_{j=1}^{M} \widetilde{E}_j = \bigcup_{k=1}^{N} E_k \tag{3.16}$$

where $\{\widetilde{E}_j\}_{j=1}^{M}$ are disjoint, and for each $1 \leq k \leq M$,

$$E_k = \bigcup_{\widetilde{E}_j \subset E_k} \widetilde{E}_j \tag{3.17}$$

Let

$$\widetilde{a}_j := \sum_{\widetilde{E}_j \subset E_k} a_k \tag{3.18}$$

Then clearly

$$\varphi = \sum_{j=1}^{M} \widetilde{a}_j \chi_{\widetilde{E}_j} \tag{3.19}$$

Since $\{\widetilde{E}_j\}_{j=1}^{M}$ are disjoint, we get

$$\int \varphi = \sum_{j=1}^{M} \widetilde{a}_j m(\widetilde{E}_j) = \sum_{j=1}^{M} \sum_{\widetilde{E}_j \subset E_k} a_k m(\widetilde{E}_j) = \sum_{k=1}^{N} a_k m(E_k) \tag{3.20}$$

□

Property 2. **Linearity**.

If φ and ψ are non-negative simple, and $a, b \geq 0$, then

$$\int (a\varphi + b\psi) = a \int \varphi + b \int \psi \tag{3.21}$$

证明. 下面分为两步来证明.

(a) $\forall c \geq 0, \int c\varphi = c \int \varphi$.

Suppose $\varphi = \sum_{k=1}^{M} a_k \chi_{E_k}$, where $\{E_k\}_{k=1}^{M}$ are disjoint. Then

$$c\varphi = \sum_{k=1}^{M} c a_k \chi_{E_j} \tag{3.22}$$

is also a non-negative simple function. Therefore,

$$\int c\varphi = \sum_{k=1}^{M} c a_k m(E_k) = c \sum_{k=1}^{M} a_k m(E_k) = c \int \varphi \tag{3.23}$$

(b) $\int(\varphi+\psi)=\int\varphi+\int\psi$.

Suppose

$$\varphi=\sum_{k=1}^{M}a_k\chi_{E_k},\quad \psi=\sum_{j=1}^{N}b_j\chi_{F_j} \tag{3.24}$$

where both $\{E_k\}_{k=1}^M$ and $\{F_j\}_{j=1}^N$ are disjoint and $\mathbb{R}^d=\bigcup_{k=1}^{M}E_k=\bigcup_{j=1}^{N}F_j$. Since

$$E_k=E_k\cap\mathbb{R}^d=E_k\cap\bigsqcup_{j=1}^{N}F_j=\bigsqcup_{j=1}^{N}(E_k\cap F_j) \tag{3.25}$$

Then

$$\varphi=\sum_{k=1}^{M}a_k\chi_{E_k}=\sum_{k=1}^{M}a_k\chi_{\bigsqcup_{j=1}^{N}(E_k\cap F_j)}=\sum_{k=1}^{M}\sum_{j=1}^{N}a_k\chi_{E_k\cap F_j} \tag{3.26}$$

Similarly

$$\psi=\sum_{j=1}^{N}b_j\chi_{F_j}=\sum_{j=1}^{N}b_k\chi_{\bigsqcup_{k=1}^{M}(E_k\cap F_j)}=\sum_{j=1}^{N}\sum_{k=1}^{M}b_k\chi_{E_k\cap F_j} \tag{3.27}$$

Therefore

$$\varphi+\psi=\sum_{j,k}(a_k+b_j)\chi_{E_k\cap F_j} \tag{3.28}$$

$$\int(\varphi+\psi)=\sum_{j,k}(a_k+b_j)m(E_k\cap F_j) \tag{3.29}$$

$$=\sum_{j,k}a_km(E_k\cap F_j)+\sum_{j,k}b_jm(E_k\cap F_j) \tag{3.30}$$

$$=\int\varphi+\int\psi \tag{3.31}$$

□

Property 3. Monotonicity.

If $\varphi\le\psi$ are non-negative and simple, then

$$\int\varphi\le\int\psi \tag{3.32}$$

证明. Suppose

$$\varphi=\sum_{k=1}^{M}a_k\chi_{E_k},\quad \psi=\sum_{j=1}^{N}b_j\chi_{F_j} \tag{3.33}$$

where both $\{E_k\}_{k=1}^{M}$ and $\{F_j\}_{j=1}^{N}$ are disjoint. Similar to the proof in Property 2, we get

$$\psi - \varphi = \sum_{j,k} (b_j - a_k)\chi_{E_k \cap F_j} \tag{3.34}$$

Since $\varphi(x) \leq \psi(x)$, $\forall x \in \mathbb{R}^d$, then $\psi - \varphi$ is non-negative and simple. Therefore,

$$\int (\psi - \varphi) = \sum_{j,k} (b_j - a_k) m(E_k \cap F_j) \geq 0 \;\Rightarrow\; \int \varphi \leq \int \psi \tag{3.35}$$

□

Property 4. **Additivity**.

If $\{E_k\}_{k=1}^{\infty}$ are disjoint subsets of $\mathbb{R}^d$ with finite measure, then

$$\int_{\bigcup_{k=1}^{\infty} E_k} \varphi = \sum_{k=1}^{\infty} \int_{E_k} \varphi \tag{3.36}$$

注. 首先回顾 *abstract measure* 的定义.

定义 3.1.2. Let X be a set and let $\mathcal{M}$ be a $\sigma - algebra$ on X.

A **measure** on $\mathcal{M}$ is a function $\mu : \mathcal{M} \longrightarrow [0, \infty]$, s. t.

(i) $\mu(\varnothing) = 0$.

(ii) If $\{E_j\}_{j=1}^{\infty} \subset \mathcal{M}$ are disjoint, then

$$\mu(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \mu(E_j) \tag{3.37}$$

回到我们积分的性质上来. 下面我们将说明，对于任一给定的非负简单函数 φ，将 φ 在任一可测集 A 上的积分看作 *Lebesgue* $\sigma - algebra$ $\mathcal{L}$ 上的映射，则该映射为定义在 $\mathcal{L}$ 上的测度.(从而 Property 4. 作为测度的必要条件自然成立)

命题 3.1.1. For any fixed non-negative and simple function φ, the map

$$\mu : \mathcal{L} \longrightarrow [0, \infty] \tag{3.38}$$

$$A \longmapsto \int_A \varphi \tag{3.39}$$

is a measure on $\mathcal{L}$.

证明. Suppose $\{A_j\}_{j=1}^{\infty} \subset \mathcal{L}$ are disjoint, and

$$\varphi = \sum_{k=1}^{M} a_k \chi_{E_k}, \;\; where\ \{E_k\}_{k=1}^{M}\ are\ disjoint \tag{3.40}$$

Let $A = \bigcup\limits_{j=1}^{\infty} A_j$, then

$$\int_{\bigcup_{j=1}^{\infty} A_j} \varphi = \int_A \varphi = \int \varphi \chi_A = \int (\sum_{k=1}^{M} a_k \chi_{E_k \cap A}) \tag{3.41}$$

$$= \sum_{k=1}^{M} a_k m(E_k \cap A) \tag{3.42}$$

$$= \sum_{k=1}^{M} a_k m(E_k \cap (\bigcup_{j=1}^{\infty} A_j)) \tag{3.43}$$

$$= \sum_{k=1}^{M} a_k m(\bigsqcup_{j=1}^{\infty} (E_k \cap A_j)) \tag{3.44}$$

$$= \sum_{k=1}^{M} a_k \sum_{j=1}^{\infty} m(E_k \cap A_j) \tag{3.45}$$

$$= \sum_{k=1}^{M} \sum_{j=1}^{\infty} a_k m(E_k \cap A_j) \tag{3.46}$$

Since positive series always converges in $[0, \infty]$, then

$$\int_A \varphi = \sum_{k=1}^{M} \sum_{j=1}^{\infty} a_k m(E_k \cap A_j) = \sum_{j=1}^{\infty} \sum_{k=1}^{M} a_k m(E_k \cap A_j) = \sum_{j=1}^{\infty} \int_{A_j} \varphi \tag{3.47}$$

Therefore, **the integral on any non-negative simple function is accually a measure on $\mathcal{L}$**. □

3.1.2 $Non-negative\ measurable\ functions$

为了讨论的方便，先给出**非负可测函数**的一个记号.

$$\mathcal{M}^{+} := \{all\ non-negative\ measurable\ functions\} \tag{3.48}$$

定义 下面给出非负可测函数的积分的定义.

定义 3.1.3. For $f \in \mathcal{M}^{+}$, we define

$$\int f(x)dx := \sup\{\int \varphi(x)dx \mid 0 \le \varphi \le f,\ \varphi\ simple\} \tag{3.49}$$

注. 此处对 Non-negative measurable function 积分的定义兼容定义 3.1.1 中对 Non-negative simple function 积分的定义，具体表现为：$\forall \varphi_0$ non-negative and simple,

$$\sup\{\int \varphi(x)dx \mid 0 \le \varphi \le \varphi_0,\ \varphi\ simple\} = \int \varphi_0(x)dx \tag{3.50}$$

性质 下面来验证定义 3.1.3 中定义的积分满足几条基本性质.

Property 1. **Monotonicity**.

Let $f, g \in \mathcal{M}^{+}$. Then

$$\int f \le \int g \ \ if \ \ f \le g \tag{3.51}$$

证明. Let

$$A = \{\varphi\ simple \mid 0 \le \varphi \le f\} \tag{3.52}$$

$$B = \{\psi\ simple \mid 0 \le \psi \le g\} \tag{3.53}$$

Then for all $\varphi \in A$, $0 \le \varphi \le f \le g \Rightarrow \varphi \in B \Rightarrow A \subset B$. Since

$$\int f = \sup_{\varphi \in A}\{\int \varphi\},\ \int g = \sup_{\psi \in B}\{\int \psi\} \tag{3.54}$$

Therefore

$$\int f \le \int g \tag{3.55}$$

□

Property 2. 齐次性.

Let $f \in \mathcal{M}^+$. If $c \geq 0$, then

$$\int cf = c \int f \tag{3.56}$$

证明. Assume $c > 0$. Then

$$\int cf = \sup\{\int \varphi \mid 0 \leq \varphi \leq cf,\ \varphi \text{ simple}\} \tag{3.57}$$

$$= \sup\{\int \varphi \mid 0 \leq \frac{\varphi}{c} \leq f,\ \varphi \text{ simple}\} \tag{3.58}$$

$$\overset{\psi = \frac{\varphi}{c}}{=} \sup\{\int c\psi \mid 0 \leq \psi \leq f,\ \psi \text{ simple}\} \tag{3.59}$$

$$= c\sup\{\int \psi \mid 0 \leq \psi \leq f,\ \psi \text{ simple}\} \tag{3.60}$$

$$= c\int f \tag{3.61}$$

□

单调收敛定理 下面我们正式迈入实分析的**"大门"**，介绍第一个**收敛定理**.

定理 3.1.2. The Monotone Convergence Theorem.

If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+, f_j \leq f_{j+1}$ for all j, and $\lim_{n\to\infty} f_n = f$, then

$$\int f = \lim_{n\to\infty} \int f_n \tag{3.62}$$

注.

- 此即为**"单调收敛定理"**，这个定理说明了对于**单调递增**的**非负可测**函数列，其积分与极限可交换次序. 具体表现为

$$\int f = \int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n \tag{3.63}$$

- 该定理还说明了，我们可以给出非负可测函数的另一个更自然的等价定义，即用非负简单函数列的积分逼近非负可测函数的积分.

定义 3.1.4. For $f \in \mathcal{M}^+$, we can also define

$$\int f := \lim_{n\to\infty} \int \varphi_n \tag{3.64}$$

where $\varphi_n \to f$ and $0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq f$ by Thm 2.2.1.

并且该定理说明了该积分定义的**唯一性**及 **well-defined**.

在证明定理前，先来证明一个引理 (将定理 1.3.3 (i) 拓展到一般的抽象测度上).

引理 3.1.3. Let X be a set, $\mathcal{M}$ be a $\sigma-algebra$ on X, $\mu:\mathcal{M}\longrightarrow[0,\infty]$ be a measure on $\mathcal{M}$. If $\{E_n\}_{n=1}^{\infty}\subset\mathcal{M}$, $E_n\nearrow E$, then

$$\lim_{n\to\infty}\mu(E_n)=\mu(E) \tag{3.65}$$

证明. 证明过程与 Thm 1.3.3 完全一致 (仅用到了测度的可数可加性).

Let $S_1=E_1$, $S_k=E_k-E_{k-1}$, $\forall k\geq 2$. Then $\{S_k\}_{k=1}^{\infty}\subset\mathcal{M}$ are disjoint.

Since $E=\bigcup\limits_{k=1}^{\infty}E_k=\bigcup\limits_{k=1}^{\infty}S_k$, then

$$\mu(E)=\mu(\bigsqcup_{k=1}^{\infty}S_k)=\sum_{k=1}^{\infty}\mu(S_k)=\lim_{n\to\infty}\sum_{k=1}^{n}\mu(S_k)=\lim_{n\to\infty}\mu(\bigsqcup_{k=1}^{n}S_k)=\lim_{n\to\infty}\mu(E_n) \tag{3.66}$$

□

下面证明原定理.

证明.

- $\lim\limits_{n\to\infty}\int f_n\leq\int f$.

  Since $f_n\leq f$, $\forall n$, then

$$\int f_n\leq\int f,\ \forall n \tag{3.67}$$

  Since $\{\int f_n\}_{n=1}^{\infty}$ always converges in $[0,\infty]$, then let $n\to\infty$, we get

$$\lim_{n\to\infty}\int f_n\leq\int f \tag{3.68}$$

- $\lim\limits_{n\to\infty}\int f_n\geq\int f$.

  Fix $0<a<1$, for any $0\leq\varphi\leq f$ simple, let

$$E_n=\{x\mid f_n(x)\geq a\varphi(x)\} \tag{3.69}$$

  Then since $\forall x\in E_n$, we have $f_{n+1}(x)\geq f_n(x)\geq a\varphi(x)\Rightarrow x\in E_{n+1}\Rightarrow E_n\subset E_{n+1}$.

  Then $E_n\nearrow$. Since

$$\int_{\mathbb{R}^d}f_n\geq\int_{E_n}f_n\geq\int_{E_n}a\varphi,\ \forall n \tag{3.70}$$

  Let $n\to\infty$, we get

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}f_n\geq\lim_{n\to\infty}\int_{E_n}a\varphi \tag{3.71}$$

Then we have to calculate $\lim_{n\to\infty}\int_{E_n} a\varphi$:

- Since $a\varphi$ is non-negative and simple, by Prop 3.1.1, the map

$$\mu : \mathcal{L} \longrightarrow [0, \infty] \tag{3.72}$$

$$E \longmapsto \int_E a\varphi \tag{3.73}$$

is a measure on the collection of Lebesgue measurable sets $\mathcal{L}$. (将积分视作测度)

Since $\{E_n\}_{n=1}^{\infty} \subset \mathcal{L}$ and $E_n \nearrow$, by Lemma 3.1.3, we get

$$\lim_{n\to\infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n) \tag{3.74}$$

i.e.

$$\lim_{n\to\infty} \int_{E_n} a\varphi = \int_{\bigcup_{n=1}^{\infty} E_n} a\varphi \tag{3.75}$$

For all $x \in \mathbb{R}^d$, since $a\varphi(x) < f(x)$ and $f_n \to f$, there exists $N_x \in \mathbb{N}$, s. t.

$$f_n(x) \geq a\varphi(x), \ \forall n \geq N_x \tag{3.76}$$

which indicates $x \in E_{N_x}$ for some N_x. Therefore

$$\bigcup_{n=1}^{\infty} E_n = \mathbb{R}^d \ \Rightarrow \ \lim_{n\to\infty} \int_{E_n} a\varphi = \int_{\bigcup_{n=1}^{\infty} E_n} a\varphi = \int_{\mathbb{R}^d} a\varphi \tag{3.77}$$

Therefore, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \geq \lim_{n\to\infty} \int_{E_n} a\varphi = \int_{\mathbb{R}^d} a\varphi \tag{3.78}$$

Let $a \to 1$, then

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \geq \int_{\mathbb{R}^d} \varphi \tag{3.79}$$

Since φ is arbitratry, taking the supremum over φ, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \geq \sup\{\int_{\mathbb{R}^d} \varphi \mid 0 \leq \varphi \leq f, \ \varphi \ simple\} = \int f \tag{3.80}$$

□

函数项级数的可数可加性　接下来我们将给出**单调收敛定理**在**函数项级数**上的表达形式，它说明了对于**非负可测函数项级数**，其**积分与求和可交换次序**.

在此之前，先来证明有限项的情况.

(此也可视作非负可测函数积分的**Property 线性性**的一部分.)

命题 3.1.2. Linearity.

If $f, g \in \mathcal{M}^+$, then

$$\int (f+g) = \int f + \int g \tag{3.81}$$

证明. By Thm 2.2.1 and Thm 3.1.2, there exists sequences of non-negative and simple functions $\{\varphi_n\}_{n=1}^{\infty}$ and $\{\psi_n\}_{n=1}^{\infty}$, $\varphi_n \to f$ and $\psi_n \to g$, s. t.

$$\int f = \lim_{n\to\infty} \int \varphi_n, \quad \int g = \lim_{n\to\infty} \int \psi_n \tag{3.82}$$

Since $\varphi_n + \psi_n$ is still non-negative and simple, then

By the Linearity of integral on non-negative and simple functions, (**Property 2.** in §3.1.1)

$$\int (\varphi_n + \psi_n) = \int \varphi_n + \int \psi_n \tag{3.83}$$

Let $n \to \infty$, by Thm 3.1.2, we get (**极限与积分交换次序**)

$$\int (f+g) = \int f + \int g \tag{3.84}$$

□

根据 Prop 3.1.2，由归纳法，容易得到其对任意**有限项**函数项级数都成立.

下面给出函数项级数上的**单调收敛定理**.

定理 3.1.4. Monotone Convergence Theorem (MCT , series version).
If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+$ and $f = \sum_{n=1}^{\infty} f_n$, then

$$\int f = \sum_{n=1}^{\infty} \int f_n \tag{3.85}$$

注. 该定理说明了对于**非负可测函数项级数**，其**积分与求和可交换次序**.

证明. Let $F_n = \sum_{k=1}^{n} f_k$, then $F_n \nearrow \sum_{k=1}^{\infty} f_k = f$. By **MCT** (Thm 3.1.2),

$$\lim_{n\to\infty} \int F_n = \int f \tag{3.86}$$

i.e.

$$\lim_{n\to\infty} \int \sum_{k=1}^{n} f_k = \int f \tag{3.87}$$

By the **Linearity** of integral on non-negative functions (Prop 3.1.2),

$$\lim_{n\to\infty} \int \sum_{k=1}^{n} f_k = \lim_{n\to\infty} \sum_{k=1}^{n} \int f_k = \sum_{k=1}^{\infty} \int f_k = \int f \tag{3.88}$$

□

积分的唯一性 在实分析中，我们并不关心零测集上的各种性质，进而常常忽略函数在零测集上的情况. 在给出**单调收敛定理**的更一般版本前，我们先来给出**几乎处处**意义下，函数**积分的唯一性**.

下面的命题说明了，若两个非负可测函数**几乎处处**相等，则其积分相等.

命题 3.1.3. Uniqueness.

If $f \in \mathcal{M}^+$, then

$$\int f = 0 \ \Leftrightarrow \ f = 0 \ a.e. \tag{3.89}$$

注. 根据该命题，对于任意非负可测函数 f, g

$$\int f = \int g \ \Leftrightarrow \ \int (f - g) = 0 \ \Leftrightarrow \ f - g = 0 \ a.e. \ \Leftrightarrow \ f = g \ a.e. \tag{3.90}$$

证明.

- 充分性 “$\Leftarrow$” : If $f = 0$ a.e.

  $\forall 0 \le \varphi \le f$ simple, $\varphi = 0$ a.e. . Let $E = \{x \mid \varphi(x) = 0\}$, then $m(E^c) = 0$.

$$\int \varphi = \int_E \varphi + \int_{E^c} \varphi = 0 + 0 = 0 \tag{3.91}$$

  Taking the supremum of φ, we get

$$\int f = \sup\{\int \varphi \mid 0 \le \varphi \le f, \ \varphi \ simple\} = 0 \tag{3.92}$$

- 必要性 “$\Rightarrow$” : If $\int f = 0$, let

$$E_n := \{x \mid f(x) > \frac{1}{n}\} \tag{3.93}$$

  Then

$$\bigcup_{n=1}^{\infty} E_n = \{x \mid f(x) > 0\} = \{f \neq 0\} \tag{3.94}$$

  Suppose $m(\bigcup_{n=1}^{\infty} E_n) > 0$, then there exists $N \in \mathbb{N}$, s. t. $m(E_N) > 0$. Then

$$\int f \ge \int_{E_N} f > \frac{1}{N} m(E_N) > 0 \tag{3.95}$$

  which is a contradiction to $\int f = 0$.

  Therefore, $m(\bigcup_{n=1}^{\infty} E_n) = m(\{f \neq 0\}) = 0, f = 0$ a.e.

$\square$

“几乎处处”版 MCT 根据积分的唯一性 (命题 3.1.3)，下面说明在**“几乎处处收敛”**条件下，**单调收敛定理**成立 (积分与极限仍可交换次序).

推论 3.1.5. a.e. MCT.
If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+, f \in \mathcal{M}^+, f_n \nearrow f$ a.e. , then

$$\int f = \lim_{n\to\infty} \int f_n \tag{3.96}$$

证明. Let $f_n \nearrow f$ on E, then $m(E^c) = 0$ and $f_n - f_n\chi_E = 0$ a.e.
By Prop 3.1.3, we get

$$\int f_n = \int f_n\chi_E \tag{3.97}$$

Since $f_n\chi_E \nearrow f\chi_E$, then by **MCT** (Thm 3.1.2, 单调收敛定理)

$$\lim_{n\to\infty} \int f_n = \lim_{n\to\infty} \int f_n\chi_E = \int f\chi_E = \int_E f \tag{3.98}$$

Since $m(E^c) = 0$, then

$$\int f = \int_E f = \lim_{n\to\infty} \int f_n \tag{3.99}$$

($\forall 0 \le \varphi \le f$ simple, $\int \varphi = \int_E \varphi + \int_{E_c} \varphi = \int_E \varphi$. Taking the supremum of $\varphi \Rightarrow \int f = \sup\{\int \varphi\} = \int_E f$)

□

Fatou’s Lemma 我们首先来考虑一个问题，若我们将**单调收敛定理 (MCT)** 中的**“单调”**条件去掉，结论是否仍然成立 (积分与极限是否仍可交换次序)？即

$$\text{Suppose } f_n \to f \text{ a.e. , do we have } \int f_n \to \int f?$$

事实上答案为 absolutely no. 下面给出一个反例.

例 3.1.1. Consider $f_n = n\chi_{(0,\frac{1}{n})}$. Then $f_n \to 0$ a.e. on $[0, 1]$. However,

$$\int f_n = n \cdot \frac{1}{n} = 1, \ \forall n \in \mathbb{N} \not\to 0 \tag{3.100}$$

事实上，将**“单调收敛”**条件整个去除，我们将得到如下的更一般的 **Fatou's Lemma**.

定理 3.1.6. Fatou's Lemma.
If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+$, then

$$\int \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int f_n \tag{3.101}$$

注. • 回顾函数列下极限的定义.

$$\liminf_{n\to\infty} f_n = \lim_{n\to\infty} \left(\inf_{k\ge n} f_k\right) \tag{3.102}$$

即对定义域上每一点 x，取数列 $\{f_n(x)\}_{n=1}^{\infty}$ 的下极限，再将所有的 x 所对应的下极限拼成一个函数，即定义为函数列 $\{f_n\}_{n=1}^{\infty}$ 的下极限.

(上式右侧作用在固定的 x 上，即为数列 $\{f_n(x)\}_{n=1}^{\infty}$ 下极限的定义.)

• **Fatou's Lemma** 告诉我们，对于任意一列非负可测函数列，其**函数列的下极限的积分**，要小于每个函数积分后得到的**积分数列的下极限**.

证明. Since

$$\liminf_{n\to\infty} f_n = \lim_{n\to\infty} (\inf_{k\ge n} f_k) \tag{3.103}$$

Let $g_n = \inf_{k\ge n} f_k$, then $g_n \nearrow \lim_{n\to\infty} g_n$. By **MCT** (Thm 3.1.2, 单调收敛定理),

$$\int \lim_{n\to\infty} g_n = \lim_{n\to\infty} \int g_n \tag{3.104}$$

i.e.

$$\int \liminf_{n\to\infty} f_k = \lim_{n\to\infty} (\int \inf_{k\ge n} f_k) \tag{3.105}$$

For each n, since $\inf_{k\ge n} f_k \le f_j$, $\forall j \ge n$, then

$$\int \inf_{k\ge n} f_k \le \int f_j, \ \forall j \ge n \tag{3.106}$$

Taking the infimum of $\{\int f_j\}_{j=n}^{\infty}$, then

$$\int \inf_{k\ge n} f_k \le \inf_{j\ge n} \int f_j, \ \forall n \in \mathbb{N} \tag{3.107}$$

For n is arbitrary, let $n \to \infty$, we get

$$\lim_{n\to\infty} \left(\int \inf_{k\geq n} f_k\right) \leq \lim_{n\to\infty} \left(\inf_{k\geq n} \int f_k\right) = \liminf_{n\to\infty} \int f_n \tag{3.108}$$

Therefore

$$\int \liminf_{n\to\infty} f_k = \lim_{n\to\infty} \left(\int \inf_{k\geq n} f_k\right) \leq \liminf_{n\to\infty} \int f_n \tag{3.109}$$

□

3.1.3 *General case*

可积函数 跟 Riemann 积分类似，对于 Lebesgue 积分，我们也有**可积函数**的概念.

下面先让我们回到**非负可测**函数，定义非负可测函数中**可积**的概念.

定义 3.1.5. For $f \in \mathcal{M}^+$, if

$$\int f < \infty \tag{3.110}$$

Then we say f is **Lebesgue integrable** or simply **integrable**.

下面扩展到一般的可测函数，给出其 **Lebesgue 积分**及**可积**的定义.

定义 3.1.6. For any f measurable on $\mathbb{R}^d$

$$f^+(x) := \max\{f(x), 0\}\ ,\ f^-(x) := \max\{-f(x), 0\} \tag{3.111}$$

If at least one of $\int f^+$ and $\int f^-$ is finite, we define the **integral of f**

$$\int f := \int f^+ - \int f^- \tag{3.112}$$

We say that f is **(Lebesgue) integrable** if $|f|$ is integrable.

注.

- 注意到

$$f = f^+ - f^- \tag{3.113}$$
$$|f| = f^+ + f^- \tag{3.114}$$

- 根据定义，对于任意可测函数 f，

$$f \textit{ integrable} \Leftrightarrow |f| \textit{ integrable} \Leftrightarrow \int |f| = \int f^+ + \int f^- < \infty \tag{3.115}$$
$$\Leftrightarrow f^+ \textit{ and } f^- \textit{ integrable} \tag{3.116}$$

即 f 可积 $\Leftrightarrow \int f^+$ 和 $\int f^-$ 均有界.

性质 下面我们将说明，定义在任一集合 X 上的**实可积函数**构成的空间 $\mathcal{L}^1$ 为**线性空间**，以及 $f \in \mathcal{L}^1$ 时的一些性质.

在此之前，先给出上述定义的一般的可测函数的积分的**基本性质**.

命题 3.1.4. Suppose $f, g \in \mathcal{L}$, then

1. **Linearity**: $\int (af + bg) = a \int f + b \int g$.

2. **Finite Additivity**:
$$\int_{\bigsqcup_{j=1}^n A_j} f = \sum_{j=1}^n \int_{A_j} f \tag{3.117}$$
where $\{A_j\}_{j=1}^n$ are disjoint.

3. **Monotonicity**: If $f \le g$, then $\int f \le \int g$.

4. **Triangle inequality**: $\left|\int f\right| \le \int |f|$.

证明.

2. : We shall show that $\int_{\bigsqcup_{j=1}^n A_j} f^+ = \sum_{j=1}^n \int_{A_j} f^+$ and $\int_{\bigsqcup_{j=1}^n A_j} f^- = \sum_{j=1}^n \int_{A_j} f^-$.
By **Thm 2.2.1**, there exists simple $\varphi_n \nearrow f^+$, then by **MCT (Thm 3.1.2, 单调收敛定理)**,
$$\int_{\bigsqcup_{j=1}^n A_j} f^+ = \lim_{n\to\infty} \int_{\bigsqcup_{j=1}^n A_j} \varphi_n \tag{3.118}$$
Since φ_n are simple, by the **countable additivity (简单函数的可数可加性)**, we have
$$\int_{\bigsqcup_{j=1}^n A_j} f^+ = \lim_{n\to\infty} \int_{\bigsqcup_{j=1}^n A_j} \varphi_n = \lim_{n\to\infty} \sum_{j=1}^n \int_{A_j} \varphi_n = \sum_{j=1}^n \lim_{n\to\infty} \int_{A_j} \varphi_n \tag{3.119}$$
$$\overset{\textbf{MCT}}{=} \sum_{j=1}^n \int_{A_j} f^+ \tag{3.120}$$

4. 根据实数域上的三角不等式，we have
$$\left|\int f\right| = \left|\int f^+ - \int f^-\right| \le \left|\int f^+\right| + \left|\int f^-\right| = \int f^+ + \int f^- = \int |f| \tag{3.121}$$

□

现在我们便可以来说明，定义在任一集合 X 上的**实可积函数**构成的空间 $\mathcal{L}^1$ 为**线性空间**.

命题 3.1.5. The set of integrable real-valued functions on X is a real vector space.

证明. $\forall f, g \in \mathcal{L}^1$, if $a \in \mathbb{R}$,

$$\int |f+g| \leq \int (|f|+|g|) = \int |f| + \int |g| < \infty \tag{3.122}$$

$$\int |af| = |a| \int |f| < \infty \tag{3.123}$$

Therefore, $f+g, af \in \mathcal{L}^1$. $\Rightarrow$ $\mathcal{L}^1$ is a real vector space. □

对于可积函数，我们往往是在整个 $\mathbb{R}^d$ 空间上讨论其可积性，类比 **Riemann** 可积函数，合理地猜测其在 $\mathbb{R}^d$ 平面上 **"较远"** 的地方的积分值应当较小. 这就是下面我们要给出的 $\mathcal{L}^1$ 可积函数的性质.

命题 3.1.6. Suppose $f \in \mathcal{L}^1(\mathbb{R}^d)$. Then $\forall \epsilon > 0$

(i) $\exists$ a set of finite measure B such that

$$\int_{B^c} |f| < \epsilon$$

(ii) [**Absolutely Continuity**].

$\exists \delta > 0$ such that

$$\int_E |f| < \epsilon, \ \forall m(E) < \delta$$

注. • (i) 和 (ii) 共同说明了，若 $f \in \mathcal{L}^1(\mathbb{R}^d)$，则 f 的积分主要集中在一个**有限测度**区域内，且在很小的区域内 f 的积分值趋于零.

• (ii) 本质为测度的**绝对连续性** (**正测度关于正测度的绝对连续性**). 此处令正测度

$$\mu : \mathcal{L} \longrightarrow [0, \infty] \tag{3.124}$$

$$E \longmapsto \mu(E) = \int_E |f| \tag{3.125}$$

则命题 (ii) 可表示为：$\forall \epsilon > 0, \exists \delta > 0$, s. t.

$$\mu(E) < \epsilon, \ \forall m(E) < \delta$$

证明.

(i) : 对定义域做截断.

Suppose $f \geq 0$. Let $B_n = B(0, n), f_n = f\chi_{B_n}$, then $f_n \nearrow f$.

By **MCT (Thm 3.1.2, 单调收敛定理)**,

$$\lim_{n\to\infty} \int f_n = \int f \tag{3.126}$$

Then $\forall \epsilon > 0, \exists N \in \mathbb{N}$, s. t.

$$\left| \int f - \int f_N \right| = \int f - \int f_N = \int f(1 - \chi_{B_N}) = \int f\chi_{B_N^c} = \int_{B_N^c} f < \epsilon \tag{3.127}$$

Therefore, let $B = B_N = B(0, N)$, the desired result follows.

(ii) : 同样是做截断. 不过此处是对 f 的取值做截断.

Let $B_n = \{x \in \mathbb{R}^d \mid f(x) \leq n\}, f_n = f\chi_{B_n}$. Then $f_n \nearrow f, f_n \leq n$.

同 (i), By **MCT (Thm 3.1.2, 单调收敛定理)**,

$$\lim_{n\to\infty} \int f_n = \int f \tag{3.128}$$

$\forall \epsilon > 0, \exists N \in \mathbb{N}$, s. t.

$$\left| \int f - \int f_N \right| = \int (f - f_N) < \frac{\epsilon}{2} \tag{3.129}$$

Pick $\delta > 0$, s. t. $N\delta < \frac{\epsilon}{2}$. Then for all $m(E) < \delta$,

$$\int_E f = \int_E (f - f_N) + \int_E f_N \leq \int (f - f_N) + N \cdot m(E) \tag{3.130}$$

$$< \frac{\epsilon}{2} + N\delta \tag{3.131}$$

$$< \epsilon \tag{3.132}$$

□

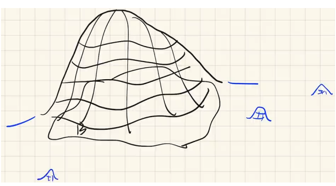

图 3.1: Prop 3.1.6 (i)

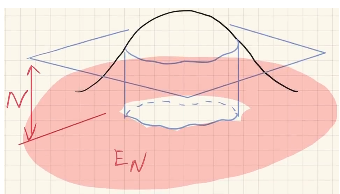

图 3.2: Prop 3.1.6 (ii)

3.1.4 *The Dominated Convergence Theorem*

下面我们来介绍实分析中最最有用的定理——

<u>**控制收敛定理 (The Dominated Convergence Theorem)**</u>.

在 **Riemann 积分**中，对于函数列**交换极限与积分的次序**的条件太过于奇怪与繁琐，而在 **Lebesgue 积分**中，**控制收敛定理**则很完美地解决了这一问题. 它对于**交换极限与积分的次序**的条件十分简洁. 下面便来介绍这一定理.

定理 3.1.7. The Dominated Convergence Theorem (DCT).
Suppose $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+, f_n \to f$ a.e.. If $|f_n| \le g$, where $g \in \mathcal{L}^1(\mathbb{R}^d)$, then

$$\int |f_n - f| \to 0,\ n \to \infty \tag{3.133}$$

and consequently

$$\int f_n \to \int f,\ n \to \infty \tag{3.134}$$

证明. 分别对 $g + f_n$ 和 $g - f_n$ 利用 **Fatou's Lemma (Thm 3.1.6)** 即可得证.

- Since $g + f_n \ge 0$, then by **Fatou's Lemma (Thm 3.1.6)**,

$$\int \liminf_{n\to\infty} (g + f_n) \le \liminf_{n\to\infty} \int (g + f_n) \tag{3.135}$$

  Since $f_n \to f$, we have

$$\int g + \int f \le \int g + \liminf_{n\to\infty} \int f_n \tag{3.136}$$

$$\int f \le \liminf_{n\to\infty} \int f_n \tag{3.137}$$

- Since $g - f_n \ge 0$, then by **Fatou's Lemma (Thm 3.1.6)**,

$$\int \liminf_{n\to\infty} (g - f_n) \le \liminf_{n\to\infty} \int (g - f_n) \tag{3.138}$$

$$\int g - \int f \le \int g + \liminf_{n\to\infty} (-\int f_n) \tag{3.139}$$

$$= \int g - \limsup_{n\to\infty} \int f_n \tag{3.140}$$

  Then

$$\int f \ge \limsup_{n\to\infty} \int f_n \tag{3.141}$$

Therefore

$$\limsup_{n\to\infty} \int f_n \leq \int f \leq \liminf_{n\to\infty} \int f_n \tag{3.142}$$

which means $\lim_{n\to\infty} \int f_n$ exists, and

$$\lim_{n\to\infty} \int f_n = \int f \tag{3.143}$$

□

3.1.5 *Complex – Valued Functions*

下面我们将**实值函数**上的 **Lebesgue 积分**推广至**复值函数**.

先来规定一些记号：

- Let $f : \mathbb{R}^d \to \mathbb{C}$, write $f(x) = u(x) + iv(x)$.

下面给出复值函数**可测**以及**可积**的定义.

定义 3.1.7. Suppose $f : \mathbb{R}^d \to \mathbb{C}, f = u + iv$, then we say

- f is **measurable** if u and v are both measurable.
- f is **Lebesgue integrable** if $|f|$ is Lebesgue integrable.

注. 事实上，根据此处定义，f 可积 $\Leftrightarrow$ u and v 都可积.

证明.

- f is integrable $\Rightarrow \int \sqrt{u^2 + v^2} < \infty \Rightarrow \int |u|, \int |v| \le \int \sqrt{u^2 + v^2} < \infty \Rightarrow$ u and v 可积.
- u and v 可积 $\Rightarrow \int |u|, \int |v| < \infty \Rightarrow \int \sqrt{u^2 + v^2} \le \int |u| + \int |v| < \infty \Rightarrow$ f 可积.

□

下面对**命题 3.1.5** 的结论进行推广，即由**复值可积函数**构成的空间为**线性空间**.

命题 3.1.7. $\mathcal{L}^1(\mathbb{R}^d, \mathbb{C})$ is a vector space.

证明. Trivial. □