

# *Real Analysis*

*Measure Theory, Integration, & Hilbert Spaces*<sup>1</sup>

–TW–

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<sup>1</sup>参考书籍：

《*Real Analysis – Measure Theory, Integration, & Hilbert Spaces*》— Elias M. Stein

《*Real Analysis – Modern Techniques and Their Applications*》— Gerald B. Folland

《实变函数论 (第三版)》— 周民强

# 序

天道几何，万品流形先自守；  
变分无限，孤心测度有同伦。

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# 第一章 *Measure Theory*

## 1.1 Preliminaries

定义 1.1.1. A (closed) rectangle  $R$  in  $\mathbb{R}^d$  is given by of  $d$  one-dimensional closed and bounded intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d] \quad (1.1)$$

where  $a_j \leq b_j$  are real numbers,  $j = 1, 2, \cdots, d$ . In other word, we have

$$R = \{(x_1, \cdots, x_d) \in \mathbb{R}^d \mid a_j \leq x_j \leq b_j, \forall j = 1 \sim d\} \quad (1.2)$$

The volume of  $R$  is

$$|R| = (b_1 - a_1) \cdots (b_d - a_d) \quad (1.3)$$

An **open** rectangle is the product of open intervals, and the interior of the rectangle  $R$  is

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d) \quad (1.4)$$

Also, a cube is a rectangle for which  $b_1 - a_1 = \cdots = b_d - a_d$ .

定义 1.1.2. A union of rectangles is said to be almost disjoint if the interiors of them are disjoint.

引理 1.1.1. If a rectangle is the almost disjoint union of finitely many rectangles, say  $R = \bigcup_{k=1}^N R_k$ , then

$$|R| = \sum_{k=1}^N |R_k| \quad (1.5)$$

注. 本质上即指的是对于方体的任意的垂直划分可转化为“十字形”划分.

引理 1.1.2. If  $R, R_1, \dots, R_N$  are rectangles, and  $R \subset \bigcup_{k=1}^N R_k$ , then

$$|R| \leq \sum_{k=1}^N |R_k| \quad (1.6)$$

注. 此即对 Lemma 1.1.1 的 slight modification, 即各方体之间不一定再为 almost disjoint.

Now we can give a description of the strcture of open sets in terms of cubes. Begin with the case of  $\mathbb{R}$ .

定理 1.1.3. Every open subset  $O$  of  $\mathbb{R}$  can be written uniquely as countable union of disjoint open intervals.

证明. For each  $x \in O$ , let  $I_x$  be the largest open interval containing  $x$  and contained in  $O$ .

Step 1 : Construct  $I_x$ :

$O$  is open  $\Rightarrow x$  is contained in some small open interval contained in  $O$ .

Let

$$a_x = \inf\{a < x \mid (a, x) \subset O\} \quad (1.7)$$

$$b_x = \sup\{b > x \mid (x, b) \subset O\} \quad (1.8)$$

Let  $I_x = (a_x, b_x)$ , then  $O = \bigcup_{x \in O} I_x$ .

Step 2 : Suppose  $I_x \cap I_y \neq \emptyset$ .

$$I_x \cup I_y \text{ is an open interval s. t. } \begin{cases} x \in I_x \cup I_y \\ I_x \cup I_y \subset O \end{cases}$$

Since  $I_x$  is maximal,  $I_x \cup I_y \subset I_x$ . Similarly,  $I_x \cup I_y \subset I_y$ .

$$\Rightarrow I_x = I_y$$

$$\Rightarrow \text{if } I_x \neq I_y, \text{ then } I_x \cap I_y = \emptyset.$$

$$\Rightarrow Z = \{I_x\}_{x \in O} \text{ is a disjoint famliy of sets.}$$

Step 3 : Since every  $I_x$  contains at least a  $a_x \in \mathbb{Q}$ , construct a map  $f$

$$f : Z \longrightarrow \mathbb{Q} \quad (1.9)$$

$$I_x \longmapsto a_x \quad (1.10)$$

$$f \text{ is an injective. } \Rightarrow \{I_x\}_{x \in O} \text{ is countable. } \Rightarrow O = \bigcup_{j=1}^{\infty} (a_j, b_j).$$

□

**定理 1.1.4.** Every open set  $O$  of  $\mathbb{R}^d$ ,  $d \geq 1$ , can be written as a countable union of almost disjoint closed cubes.

证明. Let

$$Q_k := \text{grid of } 2^{-k}\mathbb{Z}^d, \quad k \geq 0 \quad (1.11)$$

$$\underline{A}(O, k) := \{Q \in Q_k \mid Q \subset O\} \quad (1.12)$$

$$\overline{A}(O, k) := \{Q \in Q_k \mid Q \cap O \neq \emptyset\} \quad (1.13)$$

Since  $\forall Q \in \underline{A}(O, k)$ ,  $\exists q \in Q^\circ$ , s. t.  $q \in \mathbb{Q}^d$ ,

According to the Axiom of Choice,  $\exists$  the map  $f_k : \underline{A}(O, k) \longrightarrow \mathbb{Q}^d$ , which is an injection.

Hence  $\underline{A}(O, k)$  is countable.

Let

$$\underline{A}(O) := \bigcup_{k=1}^{\infty} (\underline{A}(O, k) \setminus \underline{A}(O, k-1)) \cup \underline{A}(O, 0) \quad (1.14)$$

Then  $\underline{A}(O)$  is also countable. Similarly define  $\overline{A}(O)$ .

$\forall x \in O$ , let  $\delta_x := \inf\{|y - x| \mid y \notin O\}$ . Since  $O$  is open,  $\Rightarrow \delta_x > 0$ .

$$\exists N_x \in \mathbb{N}, \text{ s. t. } 2^{-k} \sqrt{d} \leq \frac{\delta_x}{2} < \delta_x, \forall k \geq N_x \quad (1.15)$$

$$\Rightarrow \forall Q \in \overline{A}(O, N_x), \text{ s. t. } |s - t| \leq 2^{-N_x} \sqrt{d} < \delta_x, \forall s, t \in Q \quad (1.16)$$

$$\Rightarrow \text{Since } O \subset \overline{A}(O), \exists Q_x \in \overline{A}(O, N_x) \subset \overline{A}(O), \text{ s. t. } x \in Q_x \quad (1.17)$$

$$\Rightarrow x \in Q_x \subset O \quad (1.18)$$

$$\Rightarrow x \in Q_x \in \underline{A}(O, N_x) \subset \underline{A}(O) \quad (1.19)$$

$$\Rightarrow O \subset \underline{A}(O) \quad (1.20)$$

Obviously  $\underline{A}(O) \subset O$ , so

$$O = \underline{A}(O) = \bigcup_{k=1}^{\infty} (\underline{A}(O, k) \setminus \underline{A}(O, k-1)) \cup \underline{A}(O, 0) \quad (1.21)$$

which is a countable union of almost disjoint closed cubes. □

## 1.2 The Exterior Measure

*Definition* The exterior measure attempts to describe the volume of a set  $E$  by approximating it from the outside.

Loosely speaking, the exterior measure  $m_*$  assigns to **any subset of  $\mathbb{R}^d$**  a first notion of size.

定义 1.2.1. If  $E$  is a subset of  $\mathbb{R}^d$ , the exterior measure of  $E$  is

$$m_*(E) := \inf \left\{ \sum_{j=1}^{\infty} |Q_j| \mid E \subset \bigcup_{j=1}^{\infty} Q_j, Q_j \text{ is a closed cube} \right\} \quad (1.22)$$

注. • **Well definition:**  $\forall E \subset \mathbb{R}^d$ ,  $E \subset \bigcup_{n=1}^{\infty} Q_n$ ,  $Q_n = [-n, n]^d \subset \mathbb{R}^d$ , which means  $m_*$  can be defined on every subset of  $\mathbb{R}^d$ .

- It is immediate from the definition that:

For every  $\epsilon > 0$ , there exists a covering  $E \subset \bigcup_{j=1}^{\infty} Q_j$ , s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \quad (1.23)$$

- It is important to note that it would **not suffice** to allow **finite sums** in the definition of  $m_*(E)$ . If one considered only coverings of  $E$  by finite unions of cubes, the quantity is **in general larger** than  $m_*(E)$ .

(In fact, it is defined as the **outer Jordan content**  $J_*(E)$ .)

例 1.2.1. Consider the set  $\mathbb{Q} \cap [0, 1]$ .

- For the outer Jordan content, since it's obvious that  $J_*(\overline{E}) = J_*(E)$ ,  $\forall E \subset \mathbb{R}^d$ ,

$$J_*(\mathbb{Q} \cap [0, 1]) = J_*(\overline{\mathbb{Q} \cap [0, 1]}) = J_*([0, 1]) = 1$$

- For the exterior measure, since  $\mathbb{Q} \cap [0, 1]$  is countable, let  $\mathbb{Q} \cap [0, 1] = \{x_1, x_2, \dots\}$ .

Since for all  $\epsilon > 0$ ,

$$\mathbb{Q} \cap [0, 1] \subset \bigcup_{j=1}^{\infty} [x_j - \frac{\epsilon}{2^j}, x_j + \frac{\epsilon}{2^j}] \quad (1.24)$$

Hence  $m_*(\mathbb{Q} \cap [0, 1]) \leq \epsilon$ . For  $\epsilon$  is arbitrary,  $m_*(\mathbb{Q} \cap [0, 1]) = 0$ .



*Examples* Let's check that whether the exterior measure matches our intuitive idea of volume.

**Example 1. The exterior measure of a point is zero.**

证明. It's clear that a point is a cube with  $a_j = b_j, \forall j = 1 \sim d$  and which covers itself.  $\square$

**Example 2. The exterior measure of a closed cube is equal to its volume.**

证明.

- Let  $Q \subset \mathbb{R}^d$  be a closed cube. Since  $Q \subset Q$ ,  $m_*(Q) \leq |Q|$ .
- Suppose  $Q \subset \bigcup_{j=1}^{\infty} Q_j$  by closed cubes. For fixed  $\epsilon > 0$ ,  $\forall j \in \mathbb{N}$ , choose an open cube  $S_j$ ,

$$\text{s. t. } \begin{cases} S_j \supset Q_j \\ |S_j| = (1 + \epsilon) |Q_j| \end{cases} \quad (1.25)$$

Then  $Q \subset \bigcup_{j=1}^{\infty} S_j$ . Since  $Q$  is compact,  $\exists S_1, \dots, S_n \in \{S_j\}_{j=1}^{\infty}$ , s. t.  $Q \subset \bigcup_{j=1}^n S_j$ .

Therefore, according to Lemma 1.1.2

$$|Q| \leq \sum_{j=1}^n |S_j| = (1 + \epsilon) \sum_{j=1}^n |Q_j| \leq (1 + \epsilon) \sum_{j=1}^{\infty} |Q_j| \quad (1.26)$$

For  $\epsilon > 0$  is arbitrary, we get

$$|Q| \leq \sum_{j=1}^{\infty} |Q_j| \quad (1.27)$$

$$|Q| \leq \inf \sum_{j=1}^{\infty} |Q_j| = m_*(Q) \quad (1.28)$$

$\square$

**Example 3. If  $Q$  is an open cube, then  $m_*(Q) = |Q|$ .**

证明.

- Since  $Q \subset \overline{Q}$ ,  $m_*(Q) \leq |\overline{Q}| = |Q|$ .
- We note that for all closed cubes  $Q_0$  contained in  $Q$ , then  $m_*(Q_0) = |Q_0| \leq m_*(Q)$ .

For fixed  $\epsilon > 0$  which is suffice small, choose a closed cube  $Q_0$  contained in  $Q$  with a volume  $|Q_0| = (1 - \epsilon) |Q|$ , then we have

$$|Q_0| = (1 - \epsilon) |Q| \leq m_*(Q) \quad (1.29)$$

For  $\epsilon$  is arbitrary,  $|Q| \leq m_*(Q)$ .

$\square$

**Example 4. The exterior measure of a rectangle  $R$  is equal to its volume.**

**Example 5.**  $m_*(\mathbb{R}^d) = \infty$ .

**证明.** Since any covering of  $\mathbb{R}^d$  is also a covering of any cube  $Q \subset \mathbb{R}^d$ ,  $m_*(\mathbb{R}^d) \geq m_*(Q)$

$\forall N > 0$ ,  $\exists Q \subset \mathbb{R}^d$ , s. t.  $|Q| > N$ , so  $m_*(\mathbb{R}^d) = \infty$ . □

### Properties

*Observation 1. (Monotonicity)*

If  $E_1 \subset E_2$ , then  $m_*(E_1) \leq m_*(E_2)$ .

*Observation 2. (Countable sub – additivity)*

If  $E \subset \bigcup_{j=1}^{\infty} E_j$ , then  $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$ .

**证明.** For a fixed  $\epsilon > 0$ , for all  $E_j$ , there exists a covering  $\{Q_{jk}\}_{k=1}^{\infty}$ ,  $E \subset \bigcup_{k=1}^{\infty} Q_{jk}$ , s. t.

$$\sum_{k=1}^{\infty} m_*(Q_{jk}) \leq m_*(E_j) + \frac{\epsilon}{2^j} \quad (1.30)$$

Since  $E \subset \bigcup_{j=1}^{\infty} E_j \subset \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} Q_{jk}$ ,  $\bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} Q_{jk}$  covers  $E$ , then

$$m_*(E) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} m_*(Q_{jk}) \leq \sum_{j=1}^{\infty} m_*(E_j) + \epsilon \quad (1.31)$$

Since  $\epsilon$  is arbitrary,  $m_*(E) \leq \sum_{j=1}^{\infty} m_*(E_j)$  □

*Observation 3. If  $E \subset \mathbb{R}^d$ , then  $m_*(E) = \inf \{m_*(O) \mid E \subset O, O \text{ is an open set}\}$ .*

**证明.**

- By monotonicity,  $m_*(E) \leq m_*(O)$ , for all  $O$  covers  $E$ . Then take the infimum.

- For a fixed  $\epsilon > 0$ ,  $\exists$  covering  $E \subset \bigcup_{j=1}^{\infty} Q_j$ , s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \frac{\epsilon}{2} \quad (1.32)$$

For all  $Q_j$ , choose an open set  $\tilde{Q}_j$  containing  $Q_j$  with a volume  $|\tilde{Q}_j| \leq |Q_j| + \frac{\epsilon}{2^{j+1}}$ .

Let  $O = \bigcup_{j=1}^{\infty} \tilde{Q}_j$ , then by Observation 2,

$$m_*(O) \leq \sum_{j=1}^{\infty} m_*(\tilde{Q}_j) = \sum_{j=1}^{\infty} |\tilde{Q}_j| \leq \sum_{j=1}^{\infty} |Q_j| + \frac{\epsilon}{2} \leq m_*(E) + \epsilon \quad (1.33)$$

Since  $\epsilon$  is arbitrary,  $m_*(O) \leq m_*(E)$ , so  $\inf m_*(O) \leq m_*(E)$ .

□

Observation 4.

If  $E = E_1 \cup E_2$ , and  $d(E_1, E_2) > 0$ , then

$$m_*(E) = m_*(E_1) + m_*(E_2) \quad (1.34)$$

证明. For a fixed  $\epsilon > 0$ ,  $\exists$  a covering  $E \subset \bigcup_{j=1}^{\infty} Q_j$ , s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \quad (1.35)$$

Subdivide the cubes  $Q_j$  and assume that  $\text{diam}(Q_j) < \frac{d(E_1, E_2)}{3}$ . Then each  $Q_j$  can intersect at most one of the two sets  $E_1$  or  $E_2$ . Devide  $\{Q_j\}_{j=1}^{\infty}$  into two subsets  $\{Q_j\}_{j \in J_1}$ ,  $\{Q_j\}_{j \in J_2}$ , s. t.

$$E_1 \subset \bigcup_{j \in J_1} Q_j, \quad E_2 \subset \bigcup_{j \in J_2} Q_j \quad (1.36)$$

$J_1$  and  $J_2$  are both countable.  $J_1 \cap J_2 = \emptyset$ . Then

$$m_*(E_1) \leq \sum_{j \in J_1} m_*(Q_j), \quad m_*(E_2) \leq \sum_{j \in J_2} m_*(Q_j) \quad (1.37)$$

Therefore

$$m_*(E_1) + m_*(E_2) \leq \sum_{j \in J_1} m_*(Q_j) + \sum_{j \in J_2} m_*(Q_j) \leq \sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \quad (1.38)$$

Since  $\epsilon$  is arbitrary,  $m_*(E_1) + m_*(E_2) \leq m_*(E)$ .

□

*Observation 5.* If a set  $E$  is the countable union of almost disjoint cubes  $E = \bigcup_{j=1}^{\infty} Q_j$ , then

$$m_*(E) = \sum_{j=1}^{\infty} |Q_j| \quad (1.39)$$

**证明.** For a fixed  $\epsilon > 0$ , for all  $Q_j$ , choose a closed cube  $\widetilde{Q}_j$  strictly contained in  $Q_j$  with its volume  $|\widetilde{Q}_j| \geq |Q_j| - \frac{\epsilon}{2^j}$ . Then for every  $N \in \mathbb{N}$ , the cubes  $\widetilde{Q}_1, \dots, \widetilde{Q}_N$  are disjoint with a finite distance from one another. By Observation 4,

$$m_*\left(\bigcup_{j=1}^N \widetilde{Q}_j\right) = \sum_{i=1}^N |\widetilde{Q}_i| \geq \sum_{j=1}^N |Q_j| - \epsilon \quad (1.40)$$

Since  $\bigcup_{j=1}^{\infty} \widetilde{Q}_j \subset E$ , we conclude that for every  $N$

$$m_*(E) \geq \sum_{j=1}^N |Q_j| - \epsilon \quad (1.41)$$

Let  $N \rightarrow \infty$ , we deduce

$$m_*(E) \geq \sum_{j=1}^{\infty} |Q_j| - \epsilon \quad (1.42)$$

Since  $\epsilon$  is arbitrary,  $\sum_{j=1}^{\infty} |Q_j| \leq m_*(E)$ . □

## 1.3 Measurable sets and the Lebesgue measure

### 1.3.1 Measurable sets

#### Definition

**定义 1.3.1.** A subset  $E$  of  $\mathbb{R}^d$  is (Lebesgue) measurable, if for any  $\epsilon > 0$  there exists an open set  $O$  with  $E \subset O$  and  $m_*(O \setminus E) \leq \epsilon$ .

If  $E$  is measurable, we define its (Lebesgue) measurable  $m(E)$  by  $m(E) = m_*(E)$ .

**注.** • 可用映射的观点来理解外测度  $m_*$  与测度  $m$  的关系 (Folland). 即

$$m_* : \mathcal{P}(\mathbb{R}^d) \longrightarrow \overline{\mathbb{R}}_+ = [0, +\infty] \quad (1.43)$$

$$m : \mathcal{M} \longrightarrow \overline{\mathbb{R}}_+ = [0, +\infty] \quad (1.44)$$

$$m = m_* \Big|_{\mathcal{M}} \quad (1.45)$$

其中  $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^d)$  为  $\mathbb{R}^d$  中所有 (Lebesgue) measurable sets 构成的集合.

- 类比于抽象代数中各代数结构的性质, 比如群 (group) 对加法 / 乘法封闭, 我们下面探讨集合族  $\mathcal{M}$  对于可数个集合的运算 (countable unions, countable intersections, complement) 是否封闭. 即通过此引出代数结构  $\sigma$ -algebra.

**Properties** 下面开始探讨 (Lebesgue) measure 的部分性质.

Property 1. Every open set in  $\mathbb{R}^d$  is measurable.

Property 2. If  $m_*(E) = 0$ , then  $E$  is measurable.

**证明.** By Observation 3 in §1.2, for a fixed  $\epsilon > 0$ ,  $\exists E \subset O$  open, s. t.

$$m_*(O) \leq m_*(E) + \epsilon = \epsilon \quad (1.46)$$

Since  $O \setminus E \subset O$ , then  $m_*(O \setminus E) \leq m_*(O) \leq \epsilon$ . □

Property 3. Let  $\{E_j\}_{j=1}^{\infty}$  be a family of measurable sets, then  $\bigcup_{j=1}^{\infty} E_j$  is measurable.

**注.** 即说明集合族  $\mathcal{M}$  对 *countable unions* 封闭.

**证明.** Since  $E_j$  is measurable, for a fixed  $\epsilon > 0$ ,  $\exists E_j \subset O_j$  open, s. t.

$$m_*(O_j \setminus E_j) \leq \frac{\epsilon}{2^j} \quad (1.47)$$

Let  $O = \bigcup_{j=1}^{\infty} O_j \subset \mathbb{R}^d$ , then

$$O \setminus \bigcup_{j=1}^{\infty} E_j = \left( \bigcup_{j=1}^{\infty} O_j \right) \cap \left( \bigcap_{j=1}^{\infty} E_j^c \right) \quad (1.48)$$

$$= \bigcup_{j=1}^{\infty} \left( O_j \cap \left( \bigcap_{k=1}^{\infty} E_k^c \right) \right) \subset \bigcup_{j=1}^{\infty} (O_j \cap E_j^c) = \bigcup_{j=1}^{\infty} (O_j \setminus E_j) \quad (1.49)$$

Therefore

$$m_* \left( O \setminus \bigcup_{j=1}^{\infty} E_j \right) \leq m_* \left( \bigcup_{j=1}^{\infty} (O_j \setminus E_j) \right) \leq \sum_{j=1}^{\infty} m_*(O_j \setminus E_j) \leq \epsilon \quad (1.50)$$

So  $\bigcup_{j=1}^{\infty} E_j$  is measurable. □

Property 4. Closed sets are measurable.

为了证明该性质，先证明如下的分离定理.

**引理 1.3.1.** If  $F$  is closed,  $K$  is compact, and  $K \cap F = \emptyset$ , then  $d(F, K) > 0$ .

**证明.** 反证法. Suppose  $d(F, K) = 0$ , then for any fixed  $n \in \mathbb{N}$ ,  $\exists x_n \in F, y_n \in K$ , s. t.

$$|x_n - y_n| \leq \frac{1}{n} \quad (1.51)$$

Since  $K$  is compact,  $\{y_n\}_{n=1}^{\infty}$  is bounded. Then there exists a subsequence  $\{y_{n_k}\}_{k=1}^{\infty}$ , s. t.

$$y_{n_k} \rightarrow y_0 \in K, \text{ as } k \rightarrow \infty \quad (1.52)$$

Since  $|x_{n_k} - y_{n_k}| \leq \frac{1}{n_k}$ , then

$$|x_{n_k} - y_0| \leq |x_{n_k} - y_{n_k}| + |y_{n_k} - y_0| \rightarrow 0, \text{ as } k \rightarrow \infty \quad (1.53)$$

So  $x_{n_k} \rightarrow y_0 \in F, y_0 \in F \cap K \neq \emptyset$  矛盾. □

下面证明 Property 4.

证明.

- Suppose  $F$  is bounded, then  $F$  is compact.

By Observation 3 in §1.2, for a fixed  $\epsilon > 0$ ,  $\exists F \subset O$  open, s. t.

$$m_*(O) \leq m_*(F) + \epsilon \quad (1.54)$$

Since  $F$  is closed,  $O \setminus F = O \cap F^c$  is open. By Thm1.1.4,  $\exists \{Q_j\}_{j=1}^\infty$ , s. t.

$$O \setminus F = \bigcup_{j=1}^\infty Q_j \quad (1.55)$$

For a fixed  $N \in \mathbb{N}$ , let  $K = \bigcup_{j=1}^N Q_j$ , then  $K$  is compact. By Lemma1.3.1,  $d(K, F) > 0$ .

Since  $K \cup F \subset O$ , by Observation 4 in §1.2,

$$m_*(K) + m_*(F) = m_*(K \cup F) \leq m_*(O) \quad (1.56)$$

So for each fixed  $N \in \mathbb{N}$ ,

$$\sum_{j=1}^N |Q_j| = m_*(K) \leq m_*(O) - m_*(F) \leq \epsilon \quad (1.57)$$

Let  $N \rightarrow \infty$ , we get

$$m_*(O \setminus F) = \sum_{j=1}^\infty |Q_j| \leq \epsilon \quad (1.58)$$

Therefore,  $F$  is measurable.

- For the general situation, since  $\mathbb{R}^d = \bigcup_{j=1}^\infty B_j$ , then

$$F = F \cap \mathbb{R}^d = \bigcup_{j=1}^\infty (F \cap B_j) \quad (1.59)$$

Since  $B_k$  is compact and  $F$  is closed, then  $F \cap B_j$  is compact.

Due to the previous proof,  $F \cap B_j$  is measurable. By Property 3 in §1.3.1,

$$F = \bigcup_{j=1}^\infty (F \cap B_j) \text{ is measurable.} \quad (1.60)$$

□

Property 5. If  $E$  is measurable, then  $E^c$  is measurable.

**注.** 即说明集合族  $\mathcal{M}$  对集合的补运算 *complement* 封闭.

**证明.** Since  $E$  is measurable, then for all fixed  $n \in \mathbb{N}$ ,  $\exists E \subset O_n$  open, s. t.  $m_*(O_n \setminus E) \leq \frac{1}{n}$ .

Let  $S = \bigcup_{j=1}^{\infty} O_j^c \subset E^c$ . Since  $O_j^c$  is closed,  $O_j^c$  is measurable. Then  $S$  is measurable.

Since

$$E^c \setminus S = E^c \cap \left( \bigcap_{j=1}^{\infty} O_j \right) = \bigcap_{j=1}^{\infty} (E^c \cap O_j) \subset E^c \cap O_n = O_n \setminus E, \quad \forall n \in \mathbb{N} \quad (1.61)$$

Then,  $m_*(E^c \setminus S) \leq m_*(O_n \setminus E) \leq \frac{1}{n}$ ,  $\forall n \in \mathbb{N}$ . So  $E^c \setminus S$  is measurable.

Therefore,  $E^c = (E^c \setminus S) \cup S$  is measurable.  $\square$

Property 6. If  $\{E_j\}_{j=1}^{\infty}$  is a family of measurable sets, then  $\bigcap_{j=1}^{\infty} E_j$  is measurable.

**注.** 即说明集合族  $\mathcal{M}$  对 *countable intersections* 封闭.

**证明.** Since

$$\bigcap_{j=1}^{\infty} E_j = \left( \bigcup_{j=1}^{\infty} E_j^c \right)^c \quad (1.62)$$

Then,  $E_j^c$  is measurable and so  $\bigcap_{j=1}^{\infty} E_j$  is measurable.  $\square$

综上, 本节介绍了 (*Lebesgue measurable sets*) 的性质, 并且证明了 *Lebesgue measurable sets* 构成的集合族  $\mathcal{M}$  对 *countable unions*, *countable intersections*, *complement* 运算封闭. 从而  $(\mathcal{M}, \cup, \cap, \text{complement})$  构成代数结构, 即为后续介绍的  *$\sigma$ -algebra*.



### 1.3.2 Lebesgue measure

下面着重来介绍一下 *Lebesgue measure* 的 *properties*.

可数可加性 首先便是可数可加性 *countable additivity*.

定理 1.3.2. If  $E_1, E_2, \dots$  are disjoint measurable sets, then

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} m(E_j) \quad (1.63)$$

证明. Since  $m\left(\bigcup_{j=1}^{\infty} E_j\right) \leq \sum_{j=1}^{\infty} m(E_j)$  always holds, we then proof the reverse inequality.

- Suppose that  $E_j$  is bounded.

Since  $E_j^c$  is measurable, for any fixed  $\epsilon > 0$ , there exists an closed subset  $F_j \subset E_j$ , s. t.

$$m(E_j \setminus F_j) \leq \frac{\epsilon}{2^j} \quad (1.64)$$

Since  $E_j$  is bounded,  $F_j$  is compact.

Let  $K = \bigcup_{j=1}^N F_j$  be a disjoint union of compact sets for some fixed  $N$ , then

$$K \subset \bigcup_{j=1}^{\infty} E_j \quad (1.65)$$

$$m(K) = \sum_{j=1}^N m(F_j) \leq m\left(\bigcup_{j=1}^{\infty} E_j\right) \quad (1.66)$$

Since

$$m(E_j) \leq m(E_j \setminus F_j) + m(F_j) \leq m(F_j) + \frac{\epsilon}{2^j} \quad (1.67)$$

Therefore

$$\sum_{j=1}^N m(E_j) - \epsilon \leq \sum_{j=1}^N m(F_j) \leq m\left(\bigcup_{j=1}^{\infty} E_j\right) \quad (1.68)$$

Let  $N \rightarrow \infty$ , for  $\epsilon$  is arbitrary, we get

$$\sum_{j=1}^{\infty} m(E_j) \leq m\left(\bigcup_{j=1}^{\infty} E_j\right) \quad (1.69)$$

- In the general case, we choose the sequence of cubes  $\{Q_k\}_{k=1}^\infty$ ,  $Q_k = [-k, k]^d \subset \mathbb{R}^d$ .

Let  $S_1 = Q_1$ ,  $S_k = Q_k - Q_{k-1}$ ,  $\forall k \geq 2$ . Then  $\{S_k\}_{k=1}^\infty$  are disjoint and bounded.

Since  $\{S_k\}_{k=1}^\infty$  covers  $\mathbb{R}^d$ ,

$$E_j = \bigcup_{k=1}^{\infty} (E_j \cap S_k) \quad (1.70)$$

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} (E_j \cap S_k) \quad (1.71)$$

Since  $E_j \cap S_k$  is bounded and disjoint, by the previous case,

$$m\left(\bigcup_{j=1}^{\infty} E_j\right) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} m(E_j \cap S_k) = \sum_{j=1}^{\infty} m(E_j) \quad (1.72)$$

□

**单调连续性** 下面我们可以给出单调可测集合列的连续性. *continuity from below/above*

**定理 1.3.3.** Let  $E_1, E_2, \dots$  be measurable sets in  $\mathbb{R}^d$ .

- (i) If  $E_k \nearrow E$ , then  $m(E) = \lim_{n \rightarrow \infty} m(E_n)$ .
- (ii) If  $E_k \searrow E$  and  $m(E_1) < \infty$ , then  $m(E) = \lim_{n \rightarrow \infty} m(E_n)$ .

**注.** • 事实上即可写为

$$m(\lim_{n \rightarrow \infty} E_n) = \lim_{n \rightarrow \infty} m(E_n) \quad (1.73)$$

即单调可测集合列可交换极限与测度顺序.

- (ii) 中条件  $m(E_1)$  finite 不可省略, 下面给出一个反例.

**例 1.3.1.** If  $E_n = (n, +\infty)$ , then  $m(E_n) = \infty$  and  $E = \bigcap_{j=1}^{\infty} E_j = \emptyset$ . So

$$m(E) = m(\lim_{n \rightarrow \infty} E_j) = 0, \quad \lim_{n \rightarrow \infty} m(E_j) = \infty \quad (1.74)$$

**证明.**

- (i) Let  $S_1 = E_1$ ,  $S_k = E_k - E_{k-1}$ ,  $\forall k \geq 2$ . Then  $\{S_k\}_{k=1}^\infty$  are disjoint and measurable.

Since  $E = \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} S_k$ , by Thm 1.3.2,

$$m(E) = \sum_{k=1}^{\infty} m(S_k) = \lim_{N \rightarrow \infty} \sum_{k=1}^N m(S_k) = \lim_{N \rightarrow \infty} m\left(\bigcup_{k=1}^N S_k\right) = \lim_{N \rightarrow \infty} m(E_N) \quad (1.75)$$

(ii) Let  $S_1 = E_1$ ,  $S_k = E_k - E_{k+1}$ ,  $\forall k \geq 2$ . Then  $\{S_k\}_{k=1}^\infty$  are disjoint and measurable.

Since  $E_1 = E \cup \left( \bigcup_{k=1}^\infty S_k \right)$ , then

$$m(E_1) = m(E) + \sum_{k=1}^\infty m(S_k) = m(E) + \lim_{N \rightarrow \infty} m\left(\bigcup_{k=1}^N S_k\right) = m(E) + \lim_{N \rightarrow \infty} m(E_1 - E_N) \quad (1.76)$$

For  $E_1 = (E_1 - E_N) \sqcup E_N$  is a disjoint union,

$$m(E_1 - E_N) = m(E_1) - m(E_N) \quad (1.77)$$

Thus

$$m(E_1) = m(E) + \lim_{N \rightarrow \infty} m(E_1 - E_N) = m(E) + m(E_1) - \lim_{N \rightarrow \infty} m(E_N) \quad (1.78)$$

$$m(E) = \lim_{N \rightarrow \infty} m(E_N) \quad (1.79)$$

□

*Geometric insight of measurable sets* 最后我们来给出 (Lebesgue) measurable sets 的几何性质 (与开集、闭集、紧集等之间的关系).

**定理 1.3.4. Lebesgue 测度的正则性.**

Suppose  $E \subset \mathbb{R}^d$  is measurable, then  $\forall \epsilon > 0$  :

- (i)  $\exists$  open  $O \supset E$  with  $m(O \setminus E) \leq \epsilon$ .
- (ii)  $\exists$  closed  $F \subset E$  with  $m(E \setminus F) \leq \epsilon$ .
- (iii) If  $m(E) < \infty$ ,  $\exists$  compact  $K \subset E$  with  $m(E \setminus K) \leq \epsilon$ .
- (iv) If  $m(E) < \infty$ ,  $\exists F = \bigcup_{j=1}^N Q_j$ ,  $\{Q_j\}_{j=1}^\infty$  are closed cubes, s. t.  $m(E \Delta F) \leq \epsilon$ .

**证明.**

- (i) It's just the definition of measurability.
- (ii) Since  $E_j^c$  is measurable,  $\exists$  open  $O_j \supset E_j^c$ , s. t.

$$m(O_j \setminus E_j^c) \leq \epsilon \quad (1.80)$$

Since  $O_j^c \subset E_j$  is closed and  $E_j \setminus O_j^c = O_j \setminus E_j^c$ , let  $F = O_j^c$  closed, then

$$m(E_j \setminus F) = m(O_j \setminus E_j^c) \leq \epsilon \quad (1.81)$$

(iii) By (ii),  $\exists$  closed  $F \subset E$ , s. t.  $m(E \setminus F) \leq \frac{\epsilon}{2}$ .

Let  $B_n$  denote the closed ball centered at the origin of radius  $n$ , then  $B_n$  is compact.

$$F = \bigcup_{j=1}^{\infty} (F \cap B_k) \quad (1.82)$$

Let  $K_n = \bigcup_{k=1}^n (F \cap B_k)$ , then  $K_n$  is compact and  $K_n \nearrow F \Rightarrow E \setminus K_n \nearrow E \setminus F$ .

Since  $m(E \setminus K_1) \leq m(E)$  is finite, by Thm1.3.3(ii)

$$\lim_{n \rightarrow \infty} m(E \setminus K_n) = m(E \setminus F) \quad (1.83)$$

As for  $\epsilon > 0$ ,  $\exists N \in \mathbb{N}$ , s. t. for all  $n \geq N$

$$|m(E \setminus K_n) - m(E \setminus F)| \leq \frac{\epsilon}{2} \quad (1.84)$$

$$m(E \setminus K_n) \leq m(E \setminus F) + \frac{\epsilon}{2} \leq \epsilon \quad (1.85)$$

Therefore,  $m(E \setminus K_N) \leq \epsilon$ , where  $K_N \subset E$  is compact.

(iv)  $\exists$  open  $O \supset E$ , s. t.  $m(O \setminus E) \leq \frac{\epsilon}{2}$ . By Thm1.1.4,  $\exists \{Q_j\}_{j=1}^{\infty}$ , s. t.

$$E \subset O = \bigcup_{j=1}^{\infty} Q_j \quad (1.86)$$

So

$$m(O) = \sum_{j=1}^{\infty} |Q_j| \leq m(O \setminus E) + m(E) \leq \frac{\epsilon}{2} + m(E) \quad (1.87)$$

Since  $m(E)$  is finite,  $\sum_{j=1}^{\infty} |Q_j|$  converges. Then  $\exists N \in \mathbb{N}$ , s. t.

$$\sum_{j=N+1}^{\infty} |Q_j| \leq \frac{\epsilon}{2} \quad (1.88)$$

Let  $F = \bigcup_{j=1}^N Q_j$ . Since  $E \Delta F = (E \setminus F) \sqcup (F \cap E)$ , then

$$m(E \Delta F) = m(E \setminus F) + m(F \setminus E) \quad (1.89)$$

$$\leq m\left(\bigcup_{j=N+1}^{\infty} Q_j\right) + m\left(\bigcup_{j=1}^{\infty} Q_j \setminus E\right) \quad (1.90)$$

$$= \sum_{j=N+1}^{\infty} |Q_j| + \sum_{j=1}^{\infty} |Q_j| - m(E) \quad (1.91)$$

$$\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad (1.92)$$

□

## 1.4 $\sigma$ – algebras and Borel sets

### 1.4.1 $\sigma$ – algebra

首先给出  $\mathbb{R}^d$  中 *algebra* 的定义.

**定义 1.4.1.** Let  $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^d)$ .  $\mathcal{A}$  is called an algebra if

- (1) If  $A_1, \dots, A_n \in \mathcal{A}$ , then  $\bigcup_{j=1}^n A_j \in \mathcal{A}$ .
- (2) If  $A \in \mathcal{A}$ , then  $A^c \in \mathcal{A}$ .

**注.** 容易证明, 若  $\mathcal{A}$  为  $\mathbb{R}^d$  中 *algebra*, 则其对 *finite intersections* 也封闭, 同时  $\emptyset, \mathbb{R}^d \in \mathcal{A}$ .

下面给出  $\mathbb{R}^d$  中  $\sigma$  – *algebra* 的定义.(将 *algebra* 中的 *finite* 条件加强为 *countable*)

**定义 1.4.2.** Let  $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^d)$ .  $\mathcal{M}$  is a  $\sigma$  – algebra if

- (1) If  $A_1, A_2, \dots \in \mathcal{M}$ , then  $\bigcup_{j=1}^{\infty} A_j \in \mathcal{M}$ .
- (2) If  $A \in \mathcal{M}$ , then  $A^c \in \mathcal{M}$ .

**注.** 容易证明  $\mathcal{M}$  对 *countable intersections* 同样封闭,  $\emptyset, \mathbb{R}^d \in \mathcal{M}$ .

**例 1.4.1.** All Lebesgue measurable sets forms a  $\sigma$  – algebra  $\mathcal{M}$ .

类比线性空间、拓扑空间中 (拓扑) 基的概念, 下面给出 **生成  $\sigma$  – algebra** 的概念.

**定义 1.4.3.** Let  $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^d)$ , then the  $\sigma$  – algebra generated by  $\mathcal{A}$  is the smallest  $\sigma$  – algebra containing  $\mathcal{A}$ .

**注.** 即为 the intersection of all  $\sigma$  – *algebras* containing  $\mathcal{A}$ , 这也说明了对于任一给定的集族  $\mathcal{A}$ , 其生成的  $\sigma$  – *algebra* 必存在且唯一.

## 1.4.2 Borel sets

下面给出 *Borel  $\sigma$ -algebra* 及 *Borel sets* 的定义.

**定义 1.4.4.** The Borel  $\sigma$ -algebra is the  $\sigma$ -algebra generated by all open sets in  $\mathbb{R}^d$ , denoted by  $\mathcal{B}_{\mathbb{R}^d}$ .

Elements of this  $\sigma$ -algebra are called Borel sets.

**注.** 事实上, *Borel  $\sigma$ -algebra* 为 Lebesgue countable sets 的一个真子集, 后续会利用 Cantor 集证明.

为了方便研究 *Borel  $\sigma$ -algebra* 的结构, 我们把其中较为复杂 (非平凡) 的元素单独拎出来并称为  $G_\delta, F_\sigma$ .

**定义 1.4.5.** 1. The countable intersections of open sets are called  $G_\delta$  sets.

2. The countable unions of closed sets are called  $F_\sigma$  sets.

下面我们可给出  $\mathcal{B}_{\mathbb{R}^d}$  与 Lebesgue 可测集  $\mathcal{L}$  之间的关系. ( $\mathcal{L}$  只比  $\mathcal{B}_{\mathbb{R}^d}$  多了一些零测集)

**定理 1.4.1.** Lebesgue 测度的正规性.

$E \subset \mathbb{R}^d$  is  $\mathcal{L}$ -measurable

(i) if and only if  $E = G_\delta \setminus N_1$ , for some  $G_\delta$ ,  $m(N_1) = 0$ .

(ii) if and only if  $E = F_\sigma \setminus N_2$ , for some  $F_\sigma$ ,  $m(N_2) = 0$ .

**证明.** Clearly  $E$  is measurable whenever it satisfies either (i) or (ii).

(i) Since  $E$  is measurable,  $\exists$  open sets  $O_n \supset E$ , s. t.

$$m(O_n \setminus E) \leq \frac{1}{n} \quad (1.93)$$

Let  $O = \bigcap_{j=1}^{\infty} O_j$ , then

$$m(O \setminus E) \leq \frac{1}{n}, \quad \forall n \in \mathbb{N} \quad (1.94)$$

Let  $n \rightarrow \infty$ , we get  $m(O \setminus E) = 0$ . Let  $G_\delta = O$ ,  $N_1 = O \setminus E$ . Then  $E = G_\delta \setminus N_1$ .

(ii) Similarly, we can easily proof it by Thm1.3.4(ii).

□

## 1.5 Non – measurable sets

在这一节我们将介绍  $\mathbb{R}$  上一个经典的不可测集 *Vitali set*, 并说明  $\mathbb{R}$  上每个正测度集都有不可测子集.

**Vitali set** Let  $x, y \in [0, 1]$ . Write  $x \sim y \Leftrightarrow x - y \in \mathbb{Q}$ .

$\Rightarrow$  容易验证  $\sim$  为 an equivalence relation.

$\Rightarrow \sim$  partitions  $[0, 1]$ . 记  $[0, 1]$  上等价类为  $\varepsilon_a$ , 则

$$[0, 1] = \bigsqcup_a \varepsilon_a, \{ \varepsilon_a \}_a \text{ are disjoint} \quad (1.95)$$

$\Rightarrow$  By **the Axiom of Choice**, we can choose exactly one element  $x_a$  from each  $\varepsilon_a$ .

$\Rightarrow$  Let  $\mathcal{N} = \{x_a\}_a$ . Then  $\mathcal{N}$  is the Vitali set.

**定理 1.5.1.**  $\mathcal{N}$  is not measurable.

**证明.** Assume that  $\mathcal{N}$  is measurable. Let  $\{r_k\}_{k=1}^\infty$  be an enumeration of  $\mathbb{Q} \cap [-1, 1]$ .

Define

$$\mathcal{N}_k := \mathcal{N} + r_k = \{x_a + r_k\}_a \quad (1.96)$$

Then we shall proof that  $\{\mathcal{N}_k\}_{k=1}^\infty$  are disjoint, and  $[0, 1] \subset \bigcup_{k=1}^\infty \mathcal{N}_k \subset [-1, 2]$ .

- If  $\mathcal{N}_k \cap \mathcal{N}_m \neq \emptyset$ , then  $\exists x_a, x_\beta \in \mathcal{N}, r_k, r_m \in \mathbb{Q} \cap [-1, 1]$ , s. t.

$$x_a + r_k = x_\beta + r_m \quad (1.97)$$

Then  $x_a - x_\beta = r_m - r_k \in \mathbb{Q} \Rightarrow x_a \sim x_\beta \Rightarrow x_a, x_\beta \in \varepsilon_a$  or  $x_a, x_\beta \in \varepsilon_\beta \Rightarrow x_a = x_\beta$  and  $r_k = r_m$ .

Therefore,  $\mathcal{N}_k = \mathcal{N}_m$ .

- Since  $r_k \in [-1, 1]$ ,  $\mathcal{N}_k \in [-1, 2]$ ,  $\forall k$ . Therefore,

$$\bigcup_{k=1}^\infty \mathcal{N}_k \subset [-1, 2] \quad (1.98)$$

- $\forall x \in [0, 1]$ . Since  $\{\varepsilon_a\}_a$  partitions  $[0, 1]$ , there exists  $a_0$ , s. t.

$$x \in \varepsilon_{a_0}, x \sim x_{a_0} \quad (1.99)$$

which means  $x - x_{a_0} \in \mathbb{Q} \cap [-1, 1]$ . Then  $\exists k_0 \in \mathbb{N}$ , s. t.

$$x - x_{a_0} = r_{k_0} \Rightarrow x \in \mathcal{N}_{k_0} \quad (1.100)$$

Therefore,

$$[0, 1] \subset \bigcup_{k=1}^{\infty} \mathcal{N}_k \quad (1.101)$$

Since  $\{\mathcal{N}_k\}_{k=1}^{\infty}$  are disjoint, we get

$$m([0, 1]) \leq \sum_{k=1}^{\infty} m(\mathcal{N}_k) \leq m([-1, 2]) \quad (1.102)$$

Since  $\mathcal{N}_k$  is a translate of  $\mathcal{N}$ , we have  $m(\mathcal{N}) = m(\mathcal{N}_k)$  for each  $k$ . Then

$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}) \leq 3 \Rightarrow \text{Neither } m(\mathcal{N}) = 0 \text{ nor } m(\mathcal{N}) > 0 \text{ is possible.} \quad (1.103)$$

Therefore, it's a contradiction.  $\mathcal{N}$  is non-measurable.  $\square$

**正测度集必有不可测子集** 下面要证明一个结论, 即  $\mathbb{R}$  上任一正测度集必有不可测子集. 这实际上为书<sup>1</sup>Exercises of Chapter 1 的第 32 题 (b).

**命题 1.5.1.** Let  $\mathcal{N}$  denote the non-measurable subset of  $[0, 1]$  constructed in Thm1.5.1.

(a) If  $E$  is a measurable subset of  $\mathcal{N}$ , then  $m(E) = 0$ .

(b) If  $G \subset \mathbb{R}$  with  $m_*(G) > 0$ , then there exists a subset of  $G$  is non-measurable.

**证明.**

(a) Note  $\mathcal{N} = \{x_\alpha\}_{\alpha \in \mathcal{A}}$ , then  $E = \{x_\beta\}_{\beta \in \mathcal{B} \subset \mathcal{A}}$ . Similarly, we can proof

$$\bigcup_{k=1}^{\infty} E_k \subset [-1, 2] \quad (1.104)$$

Since  $\{E_k\}_{k=1}^{\infty}$  are disjoint, and  $E_k$  is a translate of  $E$ , we get

$$\sum_{k=1}^{\infty} m(E) \leq 3 \Rightarrow m(E) = 0 \quad (1.105)$$

(b) Let  $\mathcal{Q} = \{r_k\}_{k=1}^{\infty}$ ,  $\mathcal{N}_k = \mathcal{N} + r_k$ , then

$$\mathbb{R} = \bigsqcup_{k=1}^{\infty} \mathcal{N}_k \quad (1.106)$$

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<sup>1</sup>参考书籍: 《Real Analysis – Measure Theory, Integration, & Hilbert Spaces》— Elias M. Stein



Suppose  $G$  is measurable. Then

$$G = G \cap \mathbb{R} = \bigcup_{k=1}^{\infty} (G \cap \mathcal{N}_k) \quad (1.107)$$

If  $G \cap \mathcal{N}_k$  is measurable, then  $G \cap \mathcal{N}_k \subset \mathcal{N}_k$  is a subset of a non-measurable set  $\mathcal{N}_k$ .

By the previous (a), we get

$$m(G \cap \mathcal{N}_k) = 0 \quad (1.108)$$

Therefore, there exists  $k_0 \in \mathbb{N}$ , s. t.  $G \cap \mathcal{N}_{k_0} \subset G$  is a non-measurable subset of  $G$ .

(otherwise  $m(G) = 0$  contradicts)

□

## 第二章 Measurable Functions

### 2.1 Measurable Functions

定义 下面给出  $\mathbb{R}^d$  上可测函数的定义.(注意值域为扩充实数系  $\bar{\mathbb{R}}$ )

定义 2.1.1. A function defined on a measurable subset  $E \subset \mathbb{R}^d$  is measurable if for all  $a \in \mathbb{R}$ ,

$$f^{-1}([-\infty, a)) = \{x \in E \mid f(x) < a\} \quad (2.1)$$

is measurable.

注. •  $f^{-1}([-\infty, a))$  常简记作  $\{f < a\}$ .

• 下面给出几条等价定义.

(1)  $\{f < a\}$  is measurable.  $\Leftrightarrow \{f \leq a\}$  is measurable.

(2)  $\Leftrightarrow \{f > a\}$  is measurable  $\Leftrightarrow \{f \geq a\}$  is measurable.

(3) If  $f$  is finite-valued, then

$$f \text{ is measurable} \Leftrightarrow \{a < f < b\} \text{ is measurable, } \forall a, b \in \mathbb{R} \quad (2.2)$$

证明.

(1) Since the collection of measurable sets is closed under countable intersections and unions,

$$\{f \leq a\} = \bigcap_{n=1}^{\infty} \{f < a + \frac{1}{n}\} \quad (2.3)$$

$$\{f < a\} = \bigcup_{n=1}^{\infty} \{f \leq a - \frac{1}{n}\} \quad (2.4)$$

Therefore,  $\{f < a\}$  is measurable.  $\Leftrightarrow \{f \leq a\}$  is measurable.

(2) Since the collection of measurable sets is closed under complements, easily proof by (1).

(3) Since  $f$  is finite-valued,

$$\{f < a\} = \bigcup_{n=1}^{\infty} \{-n < f < a\} \quad (2.5)$$

$$\{a < f < b\} = \{f > a\} \cap \{f < b\} \quad (2.6)$$

Therefore, by (2),  $f$  is measurable  $\Leftrightarrow \{a < f < b\}$  is measurable.

□

**Property** 下面给出可测函数的一些性质.

**Property 1.** Let  $-\infty < f(x) < +\infty$  (finite-valued), then

$$f \text{ is measurable} \Leftrightarrow f^{-1}(O) \text{ is measurable } \forall \text{ open set } O \quad (2.7)$$

$$\Leftrightarrow f^{-1}(F) \text{ is measurable } \forall \text{ closed set } F \quad (2.8)$$

**证明.**  $\forall O \subset \mathbb{R}$ , there exists  $\{(a_n, b_n)\}_{n=1}^{\infty}$ , s. t.

$$O = \bigcup_{n=1}^{\infty} (a_n, b_n) \quad (2.9)$$

Then

$$f^{-1}(O) = f^{-1}\left(\bigcup_{n=1}^{\infty} (a_n, b_n)\right) = \bigcup_{n=1}^{\infty} f^{-1}((a_n, b_n)) \quad (2.10)$$

Since  $f$  is finite-valued and measurable, then  $f^{-1}(a_n, b_n)$  is measurable.

Therefore,  $f^{-1}(O)$  is measurable.

□

**Property 2.**  $\{\text{continuous functions}\} \subset \{\text{measurable functions}\}$

(a) If  $f$  is continuous on  $\mathbb{R}^d$ , then  $f$  is measurable.

(b) If  $f$  is measurable, finite-valued and  $\Phi$  is continuous on  $\mathbb{R}$ , then  $\Phi \circ f$  is measurable.

**证明.**

(a) Since  $f$  is continuous,  $\forall O \subset \mathbb{R}$ ,  $f^{-1}(O) \subset \mathbb{R}^d$ . By Property 1,  $f$  is measurable.

(b)  $\forall O \subset \mathbb{R}$ . Since  $\Phi$  is continuous, then  $\Phi^{-1}(O)$  is open.

Since  $f$  is finite-valued and measurable, then  $(\Phi \circ f)^{-1}(O) = f^{-1}(\Phi^{-1}(O))$  is open.

Therefore, by Property 1,  $\Phi \circ f$  is measurable.

□

**Property 3.** Suppose  $\{f_n\}_{n=1}^{\infty}$  is a sequence of measurable functions. Then

$$\sup_n f_n(x), \inf_n f_n(x), \limsup_{n \rightarrow \infty} f_n(x), \liminf_{n \rightarrow \infty} f_n(x) \quad (2.11)$$

are measurable.

**注.** 类比数列的上下极限, 此处

$$\limsup_{n \rightarrow \infty} f_n(x) := \lim_{k \rightarrow \infty} \sup_{n \geq k} \{f_n(x)\} = \inf_k \sup_{n \geq k} \{f_n(x)\} \quad (2.12)$$

$$\liminf_{n \rightarrow \infty} f_n(x) := \lim_{k \rightarrow \infty} \inf_{n \geq k} \{f_n(x)\} = \sup_k \inf_{n \geq k} \{f_n(x)\} \quad (2.13)$$

**证明.** Since

$$\{x \mid \sup_n f_n(x) > a\} = \bigcup_{n=1}^{\infty} \{x \mid f_n(x) > a\} \quad (2.14)$$

$$\{x \mid \inf_n f_n(x) < a\} = \bigcup_{n=1}^{\infty} \{x \mid f_n(x) < a\} \quad (2.15)$$

Then  $\sup_n f_n(x), \inf_n f_n(x)$  is measurable.

Since  $\sup_{n \geq k} f_n(x), \inf_{n \geq k} f_n(x)$  are measurable, by the previous conclusion, then

$$\limsup_{n \rightarrow \infty} f_n(x) = \inf_k \sup_{n \geq k} \{f_n(x)\} \quad (2.16)$$

$$\liminf_{n \rightarrow \infty} f_n(x) = \sup_k \inf_{n \geq k} \{f_n(x)\} \quad (2.17)$$

are measurable.

□

**Property 4.** If  $\{f_n\}_{n=1}^{\infty}$  is a collection of measurable functions and

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \quad (2.18)$$

then  $f$  is measurable.

**注.** • 与数列上下极限相同,

$$\lim_{n \rightarrow \infty} f_n(x) = f(x) \Leftrightarrow \limsup_{n \rightarrow \infty} f_n(x) = \liminf_{n \rightarrow \infty} f_n(x) = f(x) \quad (2.19)$$

- 此 Property 即说明可测函数列对极限运算封闭. 注意到连续函数列对极限运算并不具备封闭性.(下面给出经典范例)

**例 2.1.1.**

$$\lim_{n \rightarrow \infty} x^n = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases} \quad (2.20)$$

**证明.** Since  $\{f_n\}_{n=1}^{\infty}$  are measurable,  $f(x) = \limsup_{n \rightarrow \infty} f_n(x) = \liminf_{n \rightarrow \infty} f_n(x)$ , then according to Property 3,  $f$  is measurable. □

**Property 5.** If  $f$  and  $g$  are measurable, then

- (i)  $f^k, k \in \mathbb{N}$  are measurable.
- (ii)  $f + g$  and  $fg$  are measurable if both  $f$  and  $g$  are finite-valued.

**证明.**

(i) Since

$$\{f^k > a\} = \{f > a^{\frac{1}{k}}\}, \quad \forall k \text{ is odd} \quad (2.21)$$

$$\{f^k > a\} = \{f > a^{\frac{1}{k}}\} \cup \{f < -a^{\frac{1}{k}}\}, \quad \forall k \text{ is even and } a > 0 \quad (2.22)$$

Therefore,  $f^k, k \in \mathbb{N}$  are measurable.

(ii) Since<sup>1</sup>

$$\{f + g > a\} = \bigcup_{r \in \mathbb{Q}} \{f > a - r\} \cap \{g > r\} \quad (2.23)$$

---

<sup>1</sup>即必  $\exists r \in \mathbb{Q}$ , s. t.  $\{f + g > a\} \supset \{f > a - r\} \cap \{g > r\}$ . (另一侧包含关系  $\subset$  显然易证)

(反证.  $\forall r \in \mathbb{Q}$  上式不成立, 则对于  $r = 0 \in \mathbb{Q}$ ,  $\exists x_0$ , s. t.  $f(x_0) > a$ ,  $g(x_0) > 0$ , 且  $f(x_0) + g(x_0) \leq a$ , 矛盾.)

then  $f + g$  is measurable.

By the previous results in (i) and (ii), since

$$fg = \frac{1}{4}[(f + g)^2 - (f - g)^2] \quad (2.24)$$

Therefore,  $fg$  is also measurable.

□

下面给出数学分析中曾介绍过的几乎处处的定义.

**定义 2.1.2.** A property or statement is said to hold almost everywhere (a.e.) if it is true except on a set of measure zero.

**例 2.1.2.**

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases} \quad (2.25)$$

We say  $f$  is continuous a.e. on  $[0, 1]$  since  $D(f) = \{1\}$  has measure zero.

下面说明几乎处处相等可保持函数可测性.

**命题 2.1.1.** If  $f$  is measurable and  $f = g$  a.e. , then  $g$  is measurable.

**证明.** Since  $f$  is measurable and

$$g = (g - f) + f \quad (2.26)$$

then we shall proof that  $g - f$  is measurable.

Let  $A := \{x \mid g(x) - f(x) \neq 0\}$ , then  $m(A) = 0$ . We get

$$\forall a \geq 0, (g - f)^{-1}((-\infty, a]) = (\mathbb{R}^d \setminus A) \cup N, \text{ where } N \subset A \quad (2.27)$$

Since  $m(A) = 0$ , then  $N$  is measurable and  $m(N) = 0$ . So  $(g - f)^{-1}((-\infty, a])$  is measurable.

Therefore,  $g - f$  is measurable. Then  $g$  is measurable.

□

## 2.2 Measurable functions are nearly simple

本节来介绍一个非常重要的定理. 即可测函数可由简单函数逼近.

**特征函数** 下面先来介绍特征函数的定义.

**定义 2.2.1.** If  $E \subset \mathbb{R}$ , the characteristic / indicator function  $\chi_E / \mathbb{1}_E$  of  $E$  is defined by

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases} \quad (2.28)$$

下面给出可测集与其对应特征函数的关系.

**命题 2.2.1.**  $\chi_E$  is measurable  $\Leftrightarrow E$  is measurable

**证明.** Since

$$\chi_E^{-1}((-\infty, a]) = \begin{cases} \emptyset, & a < 0 \\ E^c, & 0 \leq a < 1 \\ \mathbb{R}^d, & a \geq 1 \end{cases} \quad (2.29)$$

Then  $E$  is measurable  $\Rightarrow \chi_E$  is measurable.

$\chi_E$  is measurable  $\Rightarrow \chi_E^{-1}((-\infty, a]) = E^c$  is measurable.  $\Rightarrow E$  is measurable. □

下面给出特征函数的基本性质.

**命题 2.2.2.** [Property].

(1) If  $A \cap B = \emptyset$ , then

$$\chi_{A \cup B} = \max \{\chi_A, \chi_B\} = \chi_A + \chi_B \quad (2.30)$$

(2)  $\chi_{A \cap B} = \min \{\chi_A, \chi_B\} = \chi_A \cdot \chi_B$ .

*Simple functions* 对特征函数做线性组合，即可得到简单函数.

定义 2.2.2. A simple function on  $\mathbb{R}^d$  is a finite linear combination

$$f(x) = \sum_{j=1}^n a_j \chi_{E_j}(x) \quad (2.31)$$

where each  $E_j$  is measurable and  $m(E_j) < \infty$ .

**注.** 此处定义中并未要求  $\{E_j\}_{j=1}^n$  disjoint. 而事实上这便引出了下面介绍的标准形式.

下面的命题说明了每个简单函数都可写为标准形式 ( $\{E_j\}_{j=1}^n$  disjoint).

命题 2.2.3. Every simple function  $f$  has a standard representaion

$$f = \sum_{k=1}^N a_k \chi_{E_k}, \text{ where } \{E_j\}_{k=1}^N \text{ are disjoint} \quad (2.32)$$

**证明.** Suppose  $f = \sum_{k=1}^N b_k \chi_{E_k}$ ,  $\{E_j\}_{k=1}^N$  may not be disjoint.

Since  $\{E_j\}_{k=1}^N$  is finite, the number of elements of range  $f$  is also finite. Suppose

$$\text{range } f = \{a_1, \dots, a_M\} \quad (2.33)$$

Then let  $F_k = f^{-1}(\{a_k\})$ , then  $\{F_k\}_{k=1}^M$  are disjoint. Therefore, we get the standard representation

$$f = \sum_{k=1}^M a_k \chi_{F_k} \quad (2.34)$$

□



简单函数逼近可测函数 下面给出一个定理, 说明任一可测函数可由简单函数列逼近.

**定理 2.2.1.** Suppose  $f : \mathbb{R}^d \rightarrow [-\infty, \infty]$  is measurable.

Then there exists a sequence  $\{\varphi_n\}$  of simple functions, s. t.

$$0 \leq |\varphi_1| \leq |\varphi_2| \leq \cdots \leq |f| \quad (2.35)$$

$$\lim_{k \rightarrow \infty} \varphi_k(x) = f(x), \text{ for all } x \quad (2.36)$$

and  $\varphi_k \rightarrow f$  uniformly on any set on which  $f$  is bounded.

**证明.** 下面从两方面分类讨论, 即非负函数 & 变号函数,  $f$  有界 & 无界.

(1) 非负函数  $f : \mathbb{R}^d \rightarrow [0, \infty]$ .

1°  $f$  is bounded. Assume  $|f(x)| \leq M$ .

Let<sup>2</sup>

$$E_n^k = f^{-1}\left(\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right]\right), k = 0, \dots, N_n \quad (2.37)$$

$$\varphi_n(x) = \frac{k}{2^n}, \text{ if } x \in E_n^k \quad (2.38)$$

Then

$$\varphi_n(x) = \sum_{k=0}^{N_n} \frac{k}{2^n} \chi_{E_n^k}(x) \quad (2.39)$$

Therefore<sup>3</sup>

$$|\varphi_n(x) - f(x)| \leq \frac{1}{2^n} \rightarrow 0 \text{ (independent of } x) \quad (2.40)$$

$\Rightarrow \varphi_n \rightarrow f$  uniformly.

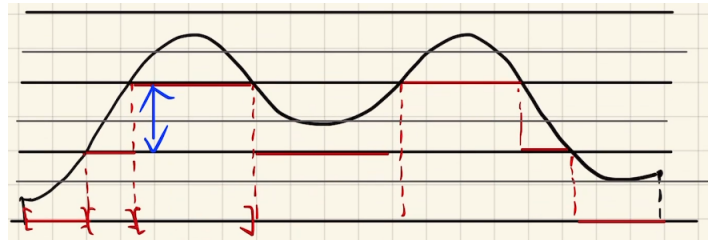


图 2.1: 对  $f$  值域进行分划

<sup>2</sup> $E_n^k$  表示第  $n$  次对值域进行分划后产生的第  $k$  个值域区间, 其中  $\frac{N_n+1}{2^n} \geq M$ .

<sup>3</sup> $|\varphi_n(x) - f(x)|$  小于等于第  $n$  次分划后两个相邻值域区间的步长值, 即  $\frac{1}{2^n}$ .

2°  $f$  is unbounded. (idea: truncation, 将  $f$  截断为一列有界函数列, 并逐点收敛于  $f$ )

Let

$$f_k(x) = \begin{cases} f(x), & \text{if } f(x) \leq k \\ k, & \text{if } f(x) > k \end{cases} \quad (2.41)$$

Then  $f_k(x) \rightarrow f(x), \forall x \in \mathbb{R}^d$ .

Since  $f_k$  is bounded, by the previous result in 1°,

For each  $k, \exists$  a sequence of simple functions  $\{\psi_{kn}\}_{n=1}^{\infty}$ , s. t.

$$\psi_{kn}(x) \rightarrow f_k(x), \forall x \quad (2.42)$$

So we get

$$\begin{array}{ccccccc} \psi_{11} & \psi_{12} & \psi_{13} & \cdots & \rightarrow & f_1 \\ \psi_{21} & \psi_{22} & \psi_{23} & \cdots & \rightarrow & f_2 \\ \psi_{31} & \psi_{32} & \psi_{33} & \cdots & \rightarrow & f_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \downarrow \\ & & & & & f \end{array} \quad (2.43)$$

From the previous results in 1°, we get

$$|\psi_{kn}(x) - f_k(x)| \leq \frac{1}{2^n} \quad (2.44)$$

Let  $n = k$ , then  $|\psi_{kk}(x) - f_k(x)| \leq \frac{1}{2^k}$ . Let  $\varphi_k = \psi_{kk}$ , then

$$|\varphi_k(x) - f(x)| \leq |\varphi_k(x) - f_k(x)| + |f_k(x) - f(x)| \quad (2.45)$$

Since  $f_k(x) \rightarrow f(x)$ , we get  $\varphi_k(x) \rightarrow f(x), \forall x$ , where  $\{\varphi_k = \psi_{kk}\}_{k=1}^{\infty}$  are simple functions.

(2) 变号函数  $f : \mathbb{R}^d \rightarrow [-\infty, \infty]$ .

We denote that

$$f^+(x) := \max\{f(x), 0\} \quad (2.46)$$

$$f^-(x) := \max\{-f(x), 0\} \quad (2.47)$$

By the previous results in (1), there exist sequences of simple functions  $\{\varphi_k\}_{k=1}^{\infty}, \{\psi_k\}_{k=1}^{\infty}$ , s. t.

$$\varphi_k \rightarrow f^+ \text{ and } \psi_k \rightarrow f^- \text{ pointwisely} \quad (2.48)$$

We can observe that  $f = f^+ - f^-$  and  $|f| = f^+ + f^-$ .

Let  $\phi_k(x) = \varphi_k(x) - \psi_k(x)$ , then  $\phi_k$  is a simple function with  $\phi_k \rightarrow f$  pointwisely.

□

**阶梯函数逼近可测函数** 在证明了可测函数可由简单函数逼近后，我们更进一步，来说明可测函数可由更加简单的**阶梯函数**来逼近。

先给出**阶梯函数**的定义。

**定义 2.2.3.** A **step function** is a finite sum

$$f = \sum_{k=1}^N a_k \chi_{R_k}, \text{ where } R_k \text{ is a rectangle} \quad (2.49)$$

**注.** 阶梯函数 & 简单函数的区别在于，简单函数是作用于有限个**可测集**  $E_k$ ，而阶梯函数是作用于有限个**矩形**  $R_k$ 。

下面的定理说明了 measurable functions are almost step functions.

**定理 2.2.2.** Suppose  $f$  is measurable on  $\mathbb{R}^d$ . Then there exists a sequence of step functions  $\{\psi_k\}_{k=1}^\infty$ , s. t.

$$\lim_{k \rightarrow \infty} \psi_k(x) = f(x), \text{ a.e. } x \quad (2.50)$$

**注.** 首先介绍函数列收敛点集的几种不同的等价表述：

$$\{x \mid \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |f_n(x) - f(x)| < \epsilon\} \quad (2.51)$$

$$\Leftrightarrow \{x \mid \forall n \in \mathbb{N}, \exists N \in \mathbb{N}, \forall k \geq N, |f_k(x) - f(x)| < \frac{1}{n}\} \quad (2.52)$$

$$\Leftrightarrow \bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \{x \mid |f_k(x) - f(x)| < \frac{1}{n}\} \quad (2.53)$$

从而可以得到函数列发散点集 (Negation):

$$\{x \mid \exists n \in \mathbb{N}, \forall N \in \mathbb{N}, \exists k \geq N, |f_k(x) - f(x)| \geq \frac{1}{n}\} \quad (2.54)$$

$$\Leftrightarrow \bigcup_{n=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \{x \mid |f_k(x) - f(x)| \geq \frac{1}{n}\} \quad (2.55)$$

$$\Leftrightarrow \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \{x \mid f_k(x) \neq f(x)\} \quad (2.56)$$

**证明.** (证明思路：先用阶梯函数逼近简单函数，再用简单函数逼近可测函数.)

It suffices to show that  $\chi_E$  can be approxiamted by step functions, for any measurable set  $E$ .

According to Thm1.3.4 (iv)

Let  $f = \chi_E$ , then  $\forall \epsilon > 0, \exists$  cubes  $\bigcup_{j=1}^N Q_j$ , s. t.

$$m(E \Delta \bigcup_{j=1}^N Q_j) \leq \epsilon \quad (2.57)$$

By considering the grid formed by extending the sides of these cubes, there exists almost disjoint rectangles  $\{\tilde{R}_j\}_{j=1}^M$ , s. t.

$$\bigcup_{j=1}^N Q_j = \bigcup_{j=1}^M \tilde{R}_j \quad (2.58)$$

By taking rectangles  $R_j$  contained in  $\tilde{R}_j$ , we can find a collection of disjoint rectangles  $\{R_j\}_{j=1}^M$ , s. t.

$$m(E \Delta \bigsqcup_{j=1}^M R_j) \leq 2\epsilon \quad (2.59)$$

For every  $k \in \mathbb{N}$ , there exists disjoint rectangles  $\{R_j\}_{j=1}^M$ , s. t.

$$m(E \Delta \bigsqcup_{j=1}^M R_j) \leq \frac{1}{2^{k+1}} \quad (2.60)$$

There also exists a step function  $\psi_k$

$$\psi_k(x) := \chi_{\bigcup_{j=1}^M R_j}(x) = \sum_{j=1}^M \chi_{R_j}(x) \quad (2.61)$$

Let

$$E_k := \{x \mid f_k(x) \neq f(x)\} \quad (2.62)$$

Since  $E_k \subset E \Delta \bigsqcup_{j=1}^M R_j$ , then  $m(E_k) \leq \frac{1}{2^k}$ . Let<sup>4</sup>

$$F_j = \bigcup_{j=k+1}^{\infty} E_j, \quad F = \bigcap_{k=1}^{\infty} F_k \quad (2.63)$$

Then  $\psi_k(x) \rightarrow f(x), \forall x \in F^c$ . Since

$$m(F) \leq m(F_k), \quad \forall k \in \mathbb{N} \quad (2.64)$$

$$m(F_k) = m\left(\bigcup_{j=k+1}^{\infty} E_j\right) \leq \sum_{j=k+1}^{\infty} m(E_j) \leq \frac{1}{2^k} \quad (2.65)$$

Therefore,  $m(F) = 0$ .  $\lim_{k \rightarrow \infty} \psi_k(x) = f(x)$ , a.e.  $x$ . □

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<sup>4</sup>根据注中式 (2.56),  $F$  即为函数列  $\{\psi_k\}_{k=1}^{\infty}$  的发散点集, 从而  $\psi_k(x) \rightarrow f(x)$  在  $F^c$  上收敛.

## 第三章 *Integration Theory*

### 3.1 *The Lebesgue integral*

*Lebesgue Integral* 的构造可以分为三步, 分别为构造下列函数的积分:

1. **Simple functions**

2. **Non-negative measurable functions**

$$\int f := \sup \left\{ \int \varphi \mid \varphi \text{ simple}, 0 \leq \varphi \leq f \right\} \quad (3.1)$$

3. **General case**

$$f = f^+ - f^- \quad (3.2)$$

$$\int f := \int f^+ - \int f^- \quad (3.3)$$

#### 3.1.1 *Simple functions*

定义 下面先给出非负简单函数在**标准形式**下的积分定义.

定义 3.1.1. If  $\varphi$  is a non-negative simple function with **standard representation**

$$\varphi(x) = \sum_{k=1}^M a_k \chi_{E_k}(x) \quad (3.4)$$

We define the **Lebesgue integral** of  $\varphi$  by

$$\int_{\mathbb{R}^d} \varphi(x) dx = \sum_{k=1}^M a_k m(E_k) \quad (3.5)$$

If  $E$  is a measurable subset of  $\mathbb{R}^d$  with finite measure, then

$$\varphi(x) \chi_E(x) = \sum_{k=1}^M a_k \chi_{E_k}(x) \chi_E(x) = \sum_{k=1}^M a_k \chi_{E_k \cap E}(x) \quad (3.6)$$

is also a simple function, and define

$$\int_E \varphi(x) dx = \int_{\mathbb{R}^d} \varphi(x) \chi_E(x) dx \quad (3.7)$$

**注.** • 此处仅对**标准形式**定义了积分. 事实上, 此处定义的积分与简单函数的表达形式无关 (即**Property 1.**).

- 关于记号, 当测度非常明确时, 大多数情况下可简写, 如

$$\int_E \varphi(x) dx \Rightarrow \int_E \varphi \quad (3.8)$$

$$\int_{\mathbb{R}^d} \varphi(x) dx \Rightarrow \int \varphi \quad (3.9)$$

当为了强调我们选择了何种测度  $\mu$  时, 还可用以下的记号:

$$\int_E \varphi(x) d\mu(x) \quad (3.10)$$

**Property** 下面给出简单函数积分的性质.

**Property 1. Independence of the representation.**

If  $\varphi = \sum_{k=1}^N a_k \chi_{E_k}$  is any representation of  $\varphi$ , then

$$\int \varphi = \sum_{k=1}^N a_k m(E_k) \quad (3.11)$$

在证明这个性质之前, 先来证明一条引理.(书<sup>1</sup>Exercises Of Chapter 2 的第 1 题)

**引理 3.1.1.** Given a collection of sets  $\{F_k\}_{k=1}^n$ , there exists another collection  $\{\widetilde{F}_j\}_{j=1}^N$  with  $N = 2^n - 1$ , so that

$$(i). \quad \bigcup_{k=1}^n F_k = \bigcup_{j=1}^N \widetilde{F}_j \quad (3.12)$$

$$(ii). \quad \{\widetilde{F}_j\}_{j=1}^N \text{ are disjoint} \quad (3.13)$$

$$(iii). \quad F_k = \bigcup_{\widetilde{F}_j \subset F_k} \widetilde{F}_j \quad (3.14)$$

**证明.** Consider the collection

$$\mathcal{F} := \left\{ \bigcup_{k=1}^n G_k - \bigcap_{k=1}^n F_k^c \mid G_k \text{ denotes } F_k \text{ or } F_k^c \right\} \quad (3.15)$$

□

<sup>1</sup>参考书籍: 《Real Analysis – Measure Theory, Integration, & Hilbert Spaces》— Elias M. Stein

下面来证明原命题.

证明. According to Lemma 3.1.1, there exists another decompositon of  $\bigcup_{k=1}^N E_k$ , i.e.

$$\bigcup_{j=1}^M \widetilde{E}_j = \bigcup_{k=1}^N E_k \quad (3.16)$$

where  $\{\widetilde{E}_j\}_{j=1}^M$  are disjoint, and for each  $1 \leq k \leq M$ ,

$$E_k = \bigcup_{\widetilde{E}_j \subset E_k} \widetilde{E}_j \quad (3.17)$$

Let

$$\widetilde{a}_j := \sum_{\widetilde{E}_j \subset E_k} a_k \quad (3.18)$$

Then clearly

$$\varphi = \sum_{j=1}^M \widetilde{a}_j \chi_{\widetilde{E}_j} \quad (3.19)$$

Since  $\{\widetilde{E}_j\}_{j=1}^M$  are disjoint, we get

$$\int \varphi = \sum_{j=1}^M \widetilde{a}_j m(\widetilde{E}_j) = \sum_{j=1}^M \sum_{\widetilde{E}_j \subset E_k} a_k m(\widetilde{E}_j) = \sum_{k=1}^N a_k m(E_k) \quad (3.20)$$

□

## Property 2. Linearity.

If  $\varphi$  and  $\psi$  are non-negative simple, and  $a, b \geq 0$ , then

$$\int (a\varphi + b\psi) = a \int \varphi + b \int \psi \quad (3.21)$$

证明. 下面分为两步来证明.

(a)  $\forall c \geq 0, \int c\varphi = c \int \varphi$ .

Suppose  $\varphi = \sum_{k=1}^M a_k \chi_{E_k}$ , where  $\{E_k\}_{k=1}^M$  are disjoint. Then

$$c\varphi = \sum_{k=1}^M ca_k \chi_{E_k} \quad (3.22)$$

is also a non-negative simple function. Therefore,

$$\int c\varphi = \sum_{k=1}^M ca_k m(E_k) = c \sum_{k=1}^M a_k m(E_k) = c \int \varphi \quad (3.23)$$

$$(b) \int (\varphi + \psi) = \int \varphi + \int \psi.$$

Suppose

$$\varphi = \sum_{k=1}^M a_k \chi_{E_k}, \quad \psi = \sum_{j=1}^N b_j \chi_{F_j} \quad (3.24)$$

where both  $\{E_k\}_{k=1}^M$  and  $\{F_j\}_{j=1}^N$  are disjoint and  $\mathbb{R}^d = \bigcup_{k=1}^M E_k = \bigcup_{j=1}^N F_j$ . Since

$$E_k = E_k \cap \mathbb{R}^d = E_k \cap \bigcap_{j=1}^N F_j = \bigcap_{j=1}^N (E_k \cap F_j) \quad (3.25)$$

Then

$$\varphi = \sum_{k=1}^M a_k \chi_{E_k} = \sum_{k=1}^M a_k \chi_{\bigcap_{j=1}^N (E_k \cap F_j)} = \sum_{k=1}^M \sum_{j=1}^N a_k \chi_{E_k \cap F_j} \quad (3.26)$$

Similarly

$$\psi = \sum_{j=1}^N b_j \chi_{F_j} = \sum_{j=1}^N b_j \chi_{\bigcup_{k=1}^M (E_k \cap F_j)} = \sum_{j=1}^N \sum_{k=1}^M b_j \chi_{E_k \cap F_j} \quad (3.27)$$

Therefore

$$\varphi + \psi = \sum_{j,k} (a_k + b_j) \chi_{E_k \cap F_j} \quad (3.28)$$

$$\int (\varphi + \psi) = \sum_{j,k} (a_k + b_j) m(E_k \cap F_j) \quad (3.29)$$

$$= \sum_{j,k} a_k m(E_k \cap F_j) + \sum_{j,k} b_j m(E_k \cap F_j) \quad (3.30)$$

$$= \int \varphi + \int \psi \quad (3.31)$$

□

### Property 3. Monotonicity.

If  $\varphi \leq \psi$  are non-negative and simple, then

$$\int \varphi \leq \int \psi \quad (3.32)$$

证明. Suppose

$$\varphi = \sum_{k=1}^M a_k \chi_{E_k}, \quad \psi = \sum_{j=1}^N b_j \chi_{F_j} \quad (3.33)$$



where both  $\{E_k\}_{k=1}^M$  and  $\{F_j\}_{j=1}^N$  are disjoint. Similar to the proof in Property 2, we get

$$\psi - \varphi = \sum_{j,k} (b_j - a_k) \chi_{E_k \cap F_j} \quad (3.34)$$

Since  $\varphi(x) \leq \psi(x)$ ,  $\forall x \in \mathbb{R}^d$ , then  $\psi - \varphi$  is non-negative and simple. Therefore,

$$\int (\psi - \varphi) = \sum_{j,k} (b_j - a_k) m(E_k \cap F_j) \geq 0 \Rightarrow \int \varphi \leq \int \psi \quad (3.35)$$

□

**Property 4. Additivity.**

If  $\{E_k\}_{k=1}^\infty$  are disjoint subsets of  $\mathbb{R}^d$  with finite measure, then

$$\int_{\bigcup_{k=1}^\infty E_k} \varphi = \sum_{k=1}^\infty \int_{E_k} \varphi \quad (3.36)$$

**注.** 首先回顾 *abstract measure* 的定义.

**定义 3.1.2.** Let  $X$  be a set and let  $\mathcal{M}$  be a  $\sigma$ -algebra on  $X$ .

A **measure** on  $\mathcal{M}$  is a function  $\mu : \mathcal{M} \rightarrow [0, \infty]$ , s. t.

(i)  $\mu(\emptyset) = 0$ .

(ii) If  $\{E_j\}_{j=1}^\infty \subset \mathcal{M}$  are disjoint, then

$$\mu\left(\bigcup_{j=1}^\infty E_j\right) = \sum_{j=1}^\infty \mu(E_j) \quad (3.37)$$

回到我们积分的性质上来. 下面我们将说明, 对于任一给定的非负简单函数  $\varphi$ , 将  $\varphi$  在任一可测集  $A$  上的积分看作 *Lebesgue  $\sigma$ -algebra*  $\mathcal{L}$  上的映射, 则该映射为定义在  $\mathcal{L}$  上的测度.(从而 Property 4. 作为测度的必要条件自然成立)

**命题 3.1.1.** For any fixed non-negative and simple function  $\varphi$ , the map

$$\mu : \mathcal{L} \rightarrow [0, \infty] \quad (3.38)$$

$$A \mapsto \int_A \varphi \quad (3.39)$$

is a measure on  $\mathcal{L}$ .

证明. Suppose  $\{A_j\}_{j=1}^{\infty} \subset \mathcal{L}$  are disjoint, and

$$\varphi = \sum_{k=1}^M a_k \chi_{E_k}, \text{ where } \{E_k\}_{k=1}^M \text{ are disjoint} \quad (3.40)$$

Let  $A = \bigcup_{j=1}^{\infty} A_j$ , then

$$\int_{\bigcup_{j=1}^{\infty} A_j} \varphi = \int_A \varphi = \int \varphi \chi_A = \int \left( \sum_{k=1}^M a_k \chi_{E_k \cap A} \right) \quad (3.41)$$

$$= \sum_{k=1}^M a_k m(E_k \cap A) \quad (3.42)$$

$$= \sum_{k=1}^M a_k m(E_k \cap \left( \bigcup_{j=1}^{\infty} A_j \right)) \quad (3.43)$$

$$= \sum_{k=1}^M a_k m\left( \bigcap_{j=1}^{\infty} (E_k \cap A_j) \right) \quad (3.44)$$

$$= \sum_{k=1}^M a_k \sum_{j=1}^{\infty} m(E_k \cap A_j) \quad (3.45)$$

$$= \sum_{k=1}^M \sum_{j=1}^{\infty} a_k m(E_k \cap A_j) \quad (3.46)$$

Since positive series always converges in  $[0, \infty]$ , then

$$\int_A \varphi = \sum_{k=1}^M \sum_{j=1}^{\infty} a_k m(E_k \cap A_j) = \sum_{j=1}^{\infty} \sum_{k=1}^M a_k m(E_k \cap A_j) = \sum_{j=1}^{\infty} \int_{A_j} \varphi \quad (3.47)$$

Therefore, the integral on any non-negative simple function is actually a measure on  $\mathcal{L}$ .  $\square$

### 3.1.2 Non – negative measurable functions

为了讨论的方便，先给出非负可测函数的一个记号.

$$\mathcal{M}^+ := \{\text{all non – negative measurable functions}\} \quad (3.48)$$

定义 下面给出非负可测函数的积分的定义.

定义 3.1.3. For  $f \in \mathcal{M}^+$ , we define

$$\int f(x)dx := \sup \left\{ \int \varphi(x)dx \mid 0 \leq \varphi \leq f, \varphi \text{ simple} \right\} \quad (3.49)$$

**注.** 此处对 Non-negative measurable function 积分的定义兼容定义 3.1.1 中对 Non-negative simple function 积分的定义，具体表现为：  $\forall \varphi_0$  non-negative and simple,

$$\sup \left\{ \int \varphi(x)dx \mid 0 \leq \varphi \leq \varphi_0, \varphi \text{ simple} \right\} = \int \varphi_0(x)dx \quad (3.50)$$

性质 下面来验证定义 3.1.3 中定义的积分满足几条基本性质.

**Property 1. Monotonicity.**

Let  $f, g \in \mathcal{M}^+$ . Then

$$\int f \leq \int g \text{ if } f \leq g \quad (3.51)$$

证明. Let

$$A = \{\varphi \text{ simple} \mid 0 \leq \varphi \leq f\} \quad (3.52)$$

$$B = \{\psi \text{ simple} \mid 0 \leq \psi \leq g\} \quad (3.53)$$

Then for all  $\varphi \in A$ ,  $0 \leq \varphi \leq f \leq g \Rightarrow \varphi \in B \Rightarrow A \subset B$ . Since

$$\int f = \sup_{\varphi \in A} \left\{ \int \varphi \right\}, \quad \int g = \sup_{\psi \in B} \left\{ \int \psi \right\} \quad (3.54)$$

Therefore

$$\int f \leq \int g \quad (3.55)$$

□

**Property 2. 齐次性.**

Let  $f \in \mathcal{M}^+$ . If  $c \geq 0$ , then

$$\int cf = c \int f \quad (3.56)$$

证明. Assume  $c > 0$ . Then

$$\int cf = \sup \left\{ \int \varphi \mid 0 \leq \varphi \leq cf, \varphi \text{ simple} \right\} \quad (3.57)$$

$$= \sup \left\{ \int \varphi \mid 0 \leq \frac{\varphi}{c} \leq f, \varphi \text{ simple} \right\} \quad (3.58)$$

$$\stackrel{\psi = \frac{\varphi}{c}}{=} \sup \left\{ \int c\psi \mid 0 \leq \psi \leq f, \psi \text{ simple} \right\} \quad (3.59)$$

$$= c \sup \left\{ \int \psi \mid 0 \leq \psi \leq f, \psi \text{ simple} \right\} \quad (3.60)$$

$$= c \int f \quad (3.61)$$

□

**单调收敛定理** 下面我们正式迈入实分析的“大门”，介绍第一个收敛定理.

**定理 3.1.2. The Monotone Convergence Theorem.**

If  $\{f_n\}_{n=1}^\infty \subset \mathcal{M}^+$ ,  $f_j \leq f_{j+1}$  for all  $j$ , and  $\lim_{n \rightarrow \infty} f_n = f$ , then

$$\int f = \lim_{n \rightarrow \infty} \int f_n \quad (3.62)$$

**注.** • 此即为“单调收敛定理”，这个定理说明了对于单调递增的非负可测函数列，其积分与极限可交换次序. 具体表现为

$$\int f = \int \lim_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \int f_n \quad (3.63)$$

- 该定理还说明了，我们可以给出非负可测函数的另一个更自然的等价定义，即用非负简单函数列的积分逼近非负可测函数的积分.

**定义 3.1.4.** For  $f \in \mathcal{M}^+$ , we can also define

$$\int f := \lim_{n \rightarrow \infty} \int \varphi_n \quad (3.64)$$

where  $\varphi_n \rightarrow f$  and  $0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq f$  by Thm 2.2.1.

并且该定理说明了该积分定义的唯一性及 **well-defined**.

在证明定理前, 先来证明一个引理 (将定理 1.3.3 (i) 拓展到一般的抽象测度上).

**引理 3.1.3.** Let  $X$  be a set,  $\mathcal{M}$  be a  $\sigma$ -algebra on  $X$ ,  $\mu : \mathcal{M} \rightarrow [0, \infty]$  be a measure on  $\mathcal{M}$ .

If  $\{E_n\}_{n=1}^\infty \subset \mathcal{M}$ ,  $E_n \nearrow E$ , then

$$\lim_{n \rightarrow \infty} \mu(E_n) = \mu(E) \quad (3.65)$$

**证明.** 证明过程与 Thm 1.3.3 完全一致 (仅用到了测度的可数可加性).

Let  $S_1 = E_1$ ,  $S_k = E_k - E_{k-1}$ ,  $\forall k \geq 2$ . Then  $\{S_k\}_{n=1}^\infty \subset \mathcal{M}$  are disjoint.

Since  $E = \bigcup_{k=1}^\infty E_k = \bigcup_{k=1}^\infty S_k$ , then

$$\mu(E) = \mu\left(\bigcup_{k=1}^\infty S_k\right) = \sum_{k=1}^\infty \mu(S_k) = \lim_{n \rightarrow \infty} \sum_{k=1}^n \mu(S_k) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_{k=1}^n S_k\right) = \lim_{n \rightarrow \infty} \mu(E_n) \quad (3.66)$$

□

下面证明原定理.

**证明.**

- $\lim_{n \rightarrow \infty} \int f_n \leq \int f$ .

Since  $f_n \leq f$ ,  $\forall n$ , then

$$\int f_n \leq \int f, \quad \forall n \quad (3.67)$$

Since  $\{\int f_n\}_{n=1}^\infty$  always converges in  $[0, \infty]$ , then let  $n \rightarrow \infty$ , we get

$$\lim_{n \rightarrow \infty} \int f_n \leq \int f \quad (3.68)$$

- $\lim_{n \rightarrow \infty} \int f_n \geq \int f$ .

Fix  $0 < a < 1$ , for any  $0 \leq \varphi \leq f$  simple, let

$$E_n = \{x \mid f_n(x) \geq a\varphi(x)\} \quad (3.69)$$

Then since  $\forall x \in E_n$ , we have  $f_{n+1}(x) \geq f_n(x) \geq a\varphi(x) \Rightarrow x \in E_{n+1} \Rightarrow E_n \subset E_{n+1}$ .

Then  $E_n \nearrow$ . Since

$$\int_{\mathbb{R}^d} f_n \geq \int_{E_n} f_n \geq \int_{E_n} a\varphi, \quad \forall n \quad (3.70)$$

Let  $n \rightarrow \infty$ , we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n \geq \lim_{n \rightarrow \infty} \int_{E_n} a\varphi \quad (3.71)$$

Then we have to calculate  $\lim_{n \rightarrow \infty} \int_{E_n} a\varphi$ :

- Since  $a\varphi$  is non-negative and simple, by Prop 3.1.1, the map

$$\mu : \mathcal{L} \longrightarrow [0, \infty] \quad (3.72)$$

$$E \longmapsto \int_E a\varphi \quad (3.73)$$

is a measure on the collection of Lebesgue measurable sets  $\mathcal{L}$ . (将积分视作测度)

Since  $\{E_n\}_{n=1}^\infty \subset \mathcal{L}$  and  $E_n \nearrow$ , by Lemma 3.1.3, we get

$$\lim_{n \rightarrow \infty} \mu(E_n) = \mu\left(\bigcup_{n=1}^\infty E_n\right) \quad (3.74)$$

i.e.

$$\lim_{n \rightarrow \infty} \int_{E_n} a\varphi = \int_{\bigcup_{n=1}^\infty E_n} a\varphi \quad (3.75)$$

For all  $x \in \mathbb{R}^d$ , since  $a\varphi(x) < f(x)$  and  $f_n \rightarrow f$ , there exists  $N_x \in \mathbb{N}$ , s. t.

$$f_n(x) \geq a\varphi(x), \quad \forall n \geq N_x \quad (3.76)$$

which indicates  $x \in E_{N_x}$  for some  $N_x$ . Therefore

$$\bigcup_{n=1}^\infty E_n = \mathbb{R}^d \Rightarrow \lim_{n \rightarrow \infty} \int_{E_n} a\varphi = \int_{\bigcup_{n=1}^\infty E_n} a\varphi = \int_{\mathbb{R}^d} a\varphi \quad (3.77)$$

Therefore, we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n \geq \lim_{n \rightarrow \infty} \int_{E_n} a\varphi = \int_{\mathbb{R}^d} a\varphi \quad (3.78)$$

Let  $a \rightarrow 1$ , then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n \geq \int_{\mathbb{R}^d} \varphi \quad (3.79)$$

Since  $\varphi$  is arbitrary, taking the supremum over  $\varphi$ , we get

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^d} f_n \geq \sup \left\{ \int_{\mathbb{R}^d} \varphi \mid 0 \leq \varphi \leq f, \varphi \text{ simple} \right\} = \int f \quad (3.80)$$

□

函数项级数的可数可加性 接下来我们将给出单调收敛定理在函数项级数上的表达形式，它说明了对于非负可测函数项级数，其积分与求和可交换次序.

在此之前，先来证明有限项的情况.

(此也可视作非负可测函数积分的**Property 线性性**的一部分.)

**命题 3.1.2. Linearity.**

If  $f, g \in \mathcal{M}^+$ , then

$$\int (f + g) = \int f + \int g \quad (3.81)$$

**证明.** By Thm 2.2.1 and Thm 3.1.2, there exists sequences of non-negative and simple functions  $\{\varphi_n\}_{n=1}^{\infty}$  and  $\{\psi_n\}_{n=1}^{\infty}$ ,  $\varphi_n \rightarrow f$  and  $\psi_n \rightarrow g$ , s. t.

$$\int f = \lim_{n \rightarrow \infty} \int \varphi_n, \quad \int g = \lim_{n \rightarrow \infty} \int \psi_n \quad (3.82)$$

Since  $\varphi_n + \psi_n$  is still non-negative and simple, then

By the Linearity of integral on non-negative and simple functions, (**Property 2.** in §3.1.1)

$$\int (\varphi_n + \psi_n) = \int \varphi_n + \int \psi_n \quad (3.83)$$

Let  $n \rightarrow \infty$ , by Thm 3.1.2, we get (极限与积分交换次序)

$$\int (f + g) = \int f + \int g \quad (3.84)$$

□

根据 Prop 3.1.2, 由归纳法, 容易得到其对任意有限项函数项级数都成立.

下面给出函数项级数上的单调收敛定理.

**定理 3.1.4. Monotone Convergence Theorem (MCT , series version).**

If  $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+$  and  $f = \sum_{n=1}^{\infty} f_n$ , then

$$\int f = \sum_{n=1}^{\infty} \int f_n \quad (3.85)$$

**注.** 该定理说明了对于非负可测函数项级数, 其积分与求和可交换次序.

**证明.** Let  $F_n = \sum_{k=1}^n f_k$ , then  $F_n \nearrow \sum_{k=1}^{\infty} f_k = f$ . By **MCT** (Thm 3.1.2),

$$\lim_{n \rightarrow \infty} \int F_n = \int f \quad (3.86)$$

i.e.

$$\lim_{n \rightarrow \infty} \int \sum_{k=1}^n f_k = \int f \quad (3.87)$$

By the **Linearity** of integral on non-negative functions (Prop 3.1.2),

$$\lim_{n \rightarrow \infty} \int \sum_{k=1}^n f_k = \lim_{n \rightarrow \infty} \sum_{k=1}^n \int f_k = \sum_{k=1}^{\infty} \int f_k = \int f \quad (3.88)$$

□



**积分的唯一性** 在实分析中, 我们并不关心零测集上的各种性质, 进而常常忽略函数在零测集上的情况. 在给出**单调收敛定理**的更一般版本前, 我们先来给出**几乎处处**意义下, 函数积分的唯一性.

下面的命题说明了, 若两个非负可测函数**几乎处处**相等, 则其积分相等.

**命题 3.1.3. Uniqueness.**

If  $f \in \mathcal{M}^+$ , then

$$\int f = 0 \Leftrightarrow f = 0 \text{ a.e.} \quad (3.89)$$

**注.** 根据该命题, 对于任意非负可测函数  $f, g$

$$\int f = \int g \Leftrightarrow \int (f - g) = 0 \Leftrightarrow f - g = 0 \text{ a.e.} \Leftrightarrow f = g \text{ a.e.} \quad (3.90)$$

**证明.**

- 充分性 “ $\Leftarrow$ ” : If  $f = 0$  a.e.

$\forall 0 \leq \varphi \leq f$  simple,  $\varphi = 0$  a.e. . Let  $E = \{x \mid \varphi(x) = 0\}$ , then  $m(E^c) = 0$ .

$$\int \varphi = \int_E \varphi + \int_{E^c} \varphi = 0 + 0 = 0 \quad (3.91)$$

Taking the supremum of  $\varphi$ , we get

$$\int f = \sup \left\{ \int \varphi \mid 0 \leq \varphi \leq f, \varphi \text{ simple} \right\} = 0 \quad (3.92)$$

- 必要性 “ $\Rightarrow$ ” : If  $\int f = 0$ , let

$$E_n := \{x \mid f(x) > \frac{1}{n}\} \quad (3.93)$$

Then

$$\bigcup_{n=1}^{\infty} E_n = \{x \mid f(x) > 0\} = \{f \neq 0\} \quad (3.94)$$

Suppose  $m(\bigcup_{n=1}^{\infty} E_n) > 0$ , then there exists  $N \in \mathbb{N}$ , s. t.  $m(E_N) > 0$ . Then

$$\int f \geq \int_{E_N} f > \frac{1}{N} m(E_N) > 0 \quad (3.95)$$

which is a contradiction to  $\int f = 0$ .

Therefore,  $m(\bigcup_{n=1}^{\infty} E_n) = m(\{f \neq 0\}) = 0, f = 0$  a.e.

□

“几乎处处”版 **MCT** 根据积分的唯一性 (命题 3.1.3), 下面说明在 “几乎处处收敛” 条件下, 单调收敛定理成立 (积分与极限仍可交换次序).

**推论 3.1.5. a.e. MCT.**

If  $\{f_n\}_{n=1}^\infty \subset \mathcal{M}^+, f \in \mathcal{M}^+, f_n \nearrow f$  a.e., then

$$\int f = \lim_{n \rightarrow \infty} \int f_n \quad (3.96)$$

**证明.** Let  $f_n \nearrow f$  on  $E$ , then  $m(E^c) = 0$  and  $f_n - f_n \chi_E = 0$  a.e.

By Prop 3.1.3, we get

$$\int f_n = \int f_n \chi_E \quad (3.97)$$

Since  $f_n \chi_E \nearrow f \chi_E$ , then by **MCT** (Thm 3.1.2, 单调收敛定理)

$$\lim_{n \rightarrow \infty} \int f_n = \lim_{n \rightarrow \infty} \int f_n \chi_E = \int f \chi_E = \int_E f \quad (3.98)$$

Since  $m(E^c) = 0$ , then

$$\int f = \int_E f = \lim_{n \rightarrow \infty} \int f_n \quad (3.99)$$

( $\forall 0 \leq \varphi \leq f$  simple,  $\int \varphi = \int_E \varphi + \int_{E^c} \varphi = \int_E \varphi$ . Taking the supremum of  $\varphi \Rightarrow \int f = \sup \{ \int \varphi \} = \int_E f$ )

□

**Fatou's Lemma** 我们首先来考虑一个问题, 若我们将单调收敛定理 (**MCT**) 中的 “单调” 条件去掉, 结论是否仍然成立 (积分与极限是否仍可交换次序)? 即

Suppose  $f_n \rightarrow f$  a.e., do we have  $\int f_n \rightarrow \int f$ ?

事实上答案为 absolutely no. 下面给出一个反例.

**例 3.1.1.** Consider  $f_n = n \chi_{(0, \frac{1}{n})}$ . Then  $f_n \rightarrow 0$  a.e. on  $[0, 1]$ . However,

$$\int f_n = n \cdot \frac{1}{n} = 1, \quad \forall n \in \mathbb{N} \not\rightarrow 0 \quad (3.100)$$

事实上，将“单调收敛”条件整个去除，我们将得到如下的更一般的 **Fatou's Lemma**.

**定理 3.1.6. Fatou's Lemma.**

If  $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+$ , then

$$\int \liminf_{n \rightarrow \infty} f_n \leq \liminf_{n \rightarrow \infty} \int f_n \quad (3.101)$$

**注.** • 回顾函数列下极限的定义.

$$\liminf_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} f_k \right) \quad (3.102)$$

即对定义域上每一点  $x$ ，取数列  $\{f_n(x)\}_{n=1}^{\infty}$  的下极限，再将所有的  $x$  所对应的下极限拼成一个函数，即定义为函数列  $\{f_n\}_{n=1}^{\infty}$  的下极限.

(上式右侧作用在固定的  $x$  上，即为数列  $\{f_n(x)\}_{n=1}^{\infty}$  下极限的定义.)

• **Fatou's Lemma** 告诉我们，对于任意一列非负可测函数列，其函数列的下极限的积分，要小于每个函数积分后得到的积分数列的下极限.

**证明.** Since

$$\liminf_{n \rightarrow \infty} f_n = \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} f_k \right) \quad (3.103)$$

Let  $g_n = \inf_{k \geq n} f_k$ , then  $g_n \nearrow \lim_{n \rightarrow \infty} g_n$ . By **MCT** (Thm 3.1.2, 单调收敛定理),

$$\int \lim_{n \rightarrow \infty} g_n = \lim_{n \rightarrow \infty} \int g_n \quad (3.104)$$

i.e.

$$\int \liminf_{n \rightarrow \infty} f_k = \lim_{n \rightarrow \infty} \left( \int \inf_{k \geq n} f_k \right) \quad (3.105)$$

For each  $n$ , since  $\inf_{k \geq n} f_k \leq f_j, \forall j \geq n$ , then

$$\int \inf_{k \geq n} f_k \leq \int f_j, \forall j \geq n \quad (3.106)$$

Taking the infimum of  $\{\int f_j\}_{j=n}^{\infty}$ , then

$$\int \inf_{k \geq n} f_k \leq \inf_{j \geq n} \int f_j, \forall n \in \mathbb{N} \quad (3.107)$$

For  $n$  is arbitrary, let  $n \rightarrow \infty$ , we get

$$\lim_{n \rightarrow \infty} \left( \int \inf_{k \geq n} f_k \right) \leq \lim_{n \rightarrow \infty} \left( \inf_{k \geq n} \int f_k \right) = \liminf_{n \rightarrow \infty} \int f_n \quad (3.108)$$

Therefore

$$\int \liminf_{n \rightarrow \infty} f_k = \lim_{n \rightarrow \infty} \left( \int \inf_{k \geq n} f_k \right) \leq \liminf_{n \rightarrow \infty} \int f_n \quad (3.109)$$

□

### 3.1.3 General case

可积函数 跟 Riemann 积分类似，对于 Lebesgue 积分，我们也有可积函数的概念。

下面先让我们回到非负可测函数，定义非负可测函数中可积的概念。

定义 3.1.5. For  $f \in \mathcal{M}^+$ , if

$$\int f < \infty \quad (3.110)$$

Then we say  $f$  is Lebesgue integrable or simply integrable.

下面扩展到一般的可测函数，给出其 **Lebesgue** 积分及可积的定义。

定义 3.1.6. For any  $f$  measurable on  $\mathbb{R}^d$

$$f^+(x) := \max \{f(x), 0\}, \quad f^-(x) := \max \{-f(x), 0\} \quad (3.111)$$

If at least one of  $\int f^+$  and  $\int f^-$  is finite, we define the integral of  $f$

$$\int f := \int f^+ - \int f^- \quad (3.112)$$

We say that  $f$  is (Lebesgue) integrable if  $|f|$  is integrable.

注. • 注意到

$$f = f^+ - f^- \quad (3.113)$$

$$|f| = f^+ + f^- \quad (3.114)$$

• 根据定义，对于任意可测函数  $f$ ，

$$f \text{ integrable} \Leftrightarrow |f| \text{ integrable} \Leftrightarrow \int |f| = \int f^+ + \int f^- < \infty \quad (3.115)$$

$$\Leftrightarrow f^+ \text{ and } f^- \text{ integrable} \quad (3.116)$$

即  $f$  可积  $\Leftrightarrow \int f^+$  和  $\int f^-$  均有界。

**性质** 下面我们将说明, 定义在任一集合  $X$  上的实可积函数构成的空间  $\mathcal{L}^1$  为线性空间, 以及  $f \in \mathcal{L}^1$  时的一些性质.

在此之前, 先给出上述定义的一般的可测函数的积分的基本性质.

**命题 3.1.4.** Suppose  $f, g \in \mathcal{L}$ , then

1. **Linearity:**  $\int (af + bg) = a \int f + b \int g.$

2. **Finite Additivity:**

$$\int_{\bigsqcup_{j=1}^n A_j} f = \sum_{j=1}^n \int_{A_j} f \quad (3.117)$$

where  $\{A_j\}_{j=1}^n$  are disjoint.

3. **Monotonicity:** If  $f \leq g$ , then  $\int f \leq \int g.$

4. **Triangle inequality:**  $|\int f| \leq \int |f|.$

**证明.**

2. : We shall show that  $\int_{\bigsqcup_{j=1}^n A_j} f^+ = \sum_{j=1}^n \int_{A_j} f^+$  and  $\int_{\bigsqcup_{j=1}^n A_j} f^- = \sum_{j=1}^n \int_{A_j} f^-.$

By **Thm 2.2.1**, there exists simple  $\varphi_n \nearrow f^+$ , then by **MCT (Thm 3.1.2, 单调收敛定理)**,

$$\int_{\bigsqcup_{j=1}^n A_j} f^+ = \lim_{n \rightarrow \infty} \int_{\bigsqcup_{j=1}^n A_j} \varphi_n \quad (3.118)$$

Since  $\varphi_n$  are simple, by the **countable additivity** (简单函数的可数可加性), we have

$$\int_{\bigsqcup_{j=1}^n A_j} f^+ = \lim_{n \rightarrow \infty} \int_{\bigsqcup_{j=1}^n A_j} \varphi_n = \lim_{n \rightarrow \infty} \sum_{j=1}^n \int_{A_j} \varphi_n = \sum_{j=1}^n \lim_{n \rightarrow \infty} \int_{A_j} \varphi_n \quad (3.119)$$

$$\stackrel{\text{MCT}}{=} \sum_{j=1}^n \int_{A_j} f^+ \quad (3.120)$$

4. 根据实数域上的三角不等式, we have

$$\left| \int f \right| = \left| \int f^+ - \int f^- \right| \leq \left| \int f^+ \right| + \left| \int f^- \right| = \int f^+ + \int f^- = \int |f| \quad (3.121)$$

□

现在我们便可以来说明，定义在任一集合  $X$  上的实可积函数构成的空间  $\mathcal{L}^1$  为线性空间。

**命题 3.1.5.** The set of integrable real-valued functions on  $X$  is a real vector space.

**证明.**  $\forall f, g \in \mathcal{L}^1$ , if  $a \in \mathbb{R}$ ,

$$\int |f + g| \leq \int (|f| + |g|) = \int |f| + \int |g| < \infty \quad (3.122)$$

$$\int |af| = |a| \int |f| < \infty \quad (3.123)$$

Therefore,  $f + g, af \in \mathcal{L}^1$ .  $\Rightarrow \mathcal{L}^1$  is a real vector space.  $\square$

对于可积函数，我们往往是在整个  $\mathbb{R}^d$  空间上讨论其可积性，类比 **Riemann** 可积函数，合理地猜测其在  $\mathbb{R}^d$  平面上“较远”的地方的积分值应当较小。这就是下面我们要给出的  $\mathcal{L}^1$  可积函数的性质。

**命题 3.1.6.** Suppose  $f \in \mathcal{L}^1(\mathbb{R}^d)$ . Then  $\forall \epsilon > 0$

(i)  $\exists$  a set of finite measure  $B$  such that

$$\int_{B^c} |f| < \epsilon$$

(ii) [**Absolutely Continuity**].

$\exists \delta > 0$  such that

$$\int_E |f| < \epsilon, \forall m(E) < \delta$$

**注.** • (i) 和 (ii) 共同说明了，若  $f \in \mathcal{L}^1(\mathbb{R}^d)$ ，则  $f$  的积分主要集中在一个有限测度区域内，且在很小的区域内  $f$  的积分值趋于零。

• (ii) 本质为测度的绝对连续性 (正测度关于正测度的绝对连续性)。此处令正测度

$$\mu : \mathcal{L} \longrightarrow [0, \infty] \quad (3.124)$$

$$E \longmapsto \mu(E) = \int_E |f| \quad (3.125)$$

则命题 (ii) 可表示为：  $\forall \epsilon > 0, \exists \delta > 0$ , s. t.

$$\mu(E) < \epsilon, \forall m(E) < \delta$$

证明.

(i) : 对定义域做截断.

Suppose  $f \geq 0$ . Let  $B_n = B(0, n)$ ,  $f_n = f\chi_{B_n}$ , then  $f_n \nearrow f$ .

By **MCT (Thm 3.1.2, 单调收敛定理)**,

$$\lim_{n \rightarrow \infty} \int f_n = \int f \quad (3.126)$$

Then  $\forall \epsilon > 0$ ,  $\exists N \in \mathbb{N}$ , s. t.

$$\left| \int f - \int f_N \right| = \int f - \int f_N = \int f(1 - \chi_{B_N}) = \int f\chi_{B_N^c} = \int_{B_N^c} f < \epsilon \quad (3.127)$$

Therefore, let  $B = B_N = B(0, N)$ , the desired result follows.

(ii) : 同样是做截断. 不过此处是对  $f$  的取值做截断.

Let  $B_n = \{x \in \mathbb{R}^d \mid f(x) \leq n\}$ ,  $f_n = f\chi_{B_n}$ . Then  $f_n \nearrow f$ ,  $f_n \leq n$ .

同 (i), By **MCT (Thm 3.1.2, 单调收敛定理)**,

$$\lim_{n \rightarrow \infty} \int f_n = \int f \quad (3.128)$$

$\forall \epsilon > 0$ ,  $\exists N \in \mathbb{N}$ , s. t.

$$\left| \int f - \int f_N \right| = \int (f - f_N) < \frac{\epsilon}{2} \quad (3.129)$$

Pick  $\delta > 0$ , s. t.  $N\delta < \frac{\epsilon}{2}$ . Then for all  $m(E) < \delta$ ,

$$\int_E f = \int_E (f - f_N) + \int_E f_N \leq \int (f - f_N) + N \cdot m(E) \quad (3.130)$$

$$< \frac{\epsilon}{2} + N\delta \quad (3.131)$$

$$< \epsilon \quad (3.132)$$

□

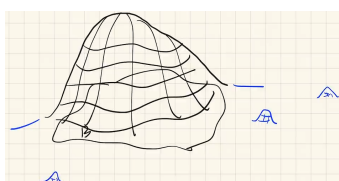


图 3.1: Prop 3.1.6 (i)

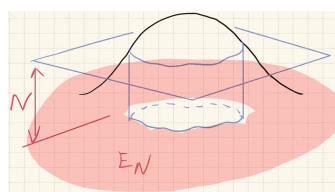


图 3.2: Prop 3.1.6 (ii)



### 3.1.4 The Dominated Convergence Theorem

下面我们来介绍实分析中最最有用的定理——

#### 控制收敛定理 (The Dominated Convergence Theorem).

在 **Riemann** 积分中，对于函数列交换极限与积分的次序的条件太过于奇怪与繁琐，而在 **Lebesgue** 积分中，控制收敛定理则很完美地解决了这一问题。它对于交换极限与积分的次序的条件十分简洁。下面便来介绍这一定理。

#### 定理 3.1.7. The Dominated Convergence Theorem (DCT).

Suppose  $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+$ ,  $f_n \rightarrow f$  a.e.. If  $|f_n| \leq g$ , where  $g \in \mathcal{L}^1(\mathbb{R}^d)$ , then

$$\int |f_n - f| \rightarrow 0, \quad n \rightarrow \infty \quad (3.133)$$

and consequently

$$\int f_n \rightarrow \int f, \quad n \rightarrow \infty \quad (3.134)$$

**证明.** 分别对  $g + f_n$  和  $g - f_n$  利用 **Fatou's Lemma (Thm 3.1.6)** 即可得证。

- Since  $g + f_n \geq 0$ , then by **Fatou's Lemma (Thm 3.1.6)**,

$$\int \liminf_{n \rightarrow \infty} (g + f_n) \leq \liminf_{n \rightarrow \infty} \int (g + f_n) \quad (3.135)$$

Since  $f_n \rightarrow f$ , we have

$$\int g + \int f \leq \int g + \liminf_{n \rightarrow \infty} \int f_n \quad (3.136)$$

$$\int f \leq \liminf_{n \rightarrow \infty} \int f_n \quad (3.137)$$

- Since  $g - f_n \geq 0$ , then by **Fatou's Lemma (Thm 3.1.6)**,

$$\int \liminf_{n \rightarrow \infty} (g - f_n) \leq \liminf_{n \rightarrow \infty} \int (g - f_n) \quad (3.138)$$

$$\int g - \int f \leq \int g + \liminf_{n \rightarrow \infty} (-\int f_n) \quad (3.139)$$

$$= \int g - \limsup_{n \rightarrow \infty} \int f_n \quad (3.140)$$

Then

$$\int f \geq \limsup_{n \rightarrow \infty} \int f_n \quad (3.141)$$

Therefore

$$\limsup_{n \rightarrow \infty} \int f_n \leq \int f \leq \liminf_{n \rightarrow \infty} \int f_n \quad (3.142)$$

which means  $\lim_{n \rightarrow \infty} \int f_n$  exists, and

$$\lim_{n \rightarrow \infty} \int f_n = \int f \quad (3.143)$$

□

### 3.1.5 Complex – Valued Functions

下面我们将实值函数上的 **Lebesgue** 积分推广至复值函数.

先来规定一些记号:

- Let  $f : \mathbb{R}^d \rightarrow \mathbb{C}$ , write  $f(x) = u(x) + iv(x)$ .

下面给出复值函数可测以及可积的定义.

定义 3.1.7. Suppose  $f : \mathbb{R}^d \rightarrow \mathbb{C}, f = u + iv$ , then we say

- $f$  is **measurable** if  $u$  and  $v$  are both measurable.
- $f$  is **Lebesgue integrable** if  $|f|$  is Lebesgue integrable.

**注.** 事实上, 根据此处定义,  $f$  可积  $\Leftrightarrow u$  and  $v$  都可积.

证明.

- $f$  is integrable  $\Rightarrow \int \sqrt{u^2 + v^2} < \infty \Rightarrow \int |u|, \int |v| \leq \int \sqrt{u^2 + v^2} < \infty \Rightarrow u$  and  $v$  可积.
- $u$  and  $v$  可积  $\Rightarrow \int |u|, \int |v| < \infty \Rightarrow \int \sqrt{u^2 + v^2} \leq \int |u| + \int |v| < \infty \Rightarrow f$  可积.

□

下面对命题 3.1.5 的结论进行推广, 即由复值可积函数构成的空间为线性空间.

命题 3.1.7.  $\mathcal{L}^1(\mathbb{R}^d, \mathbb{C})$  is a vector space.

证明. Trivial.

□

## 3.2 $\mathcal{L}^1$ 空间的完备性

引入 在讲 **Riemann** 积分时,我们称 **Riemann** 可积函数构成的空间是不完备的 (not complete). 在提及完备这个概念之前,我们需要先引入衡量“距离”的工具,即范数和度量.

### 3.2.1 范数, 度量

下面给出范数和度量的严格定义.

定义 3.2.1. Let  $X$  be a vector space over  $\mathbb{F}$ , a **norm** is a function:

$$X \longrightarrow \mathbb{R}_{\geq 0} \quad (3.144)$$

$$f \longmapsto \|f\| \quad (3.145)$$

satisfying the following properties:

- (i)  $\|f\| \geq 0, \forall f \in X.$  ( $\|f\| = 0 \Leftrightarrow f = 0 \text{ a.e.}$ )
- (ii)  $\|af\| = |a| \|f\|, \forall a \in \mathbb{F}, f \in X.$
- (iii)  $\|f + g\| \leq \|f\| + \|g\|, \forall f, g \in X.$

**注.** • (i) 中的 “ $\|f\| = 0 \Leftrightarrow f = 0 \text{ a.e.}$ ” 的 “a.e.” 是对于  $X$  取函数空间时的条件, 在实分析的取等条件中基本为默认叙述, 在后续定义中往往省略. 在对  $\mathcal{L}^1$  空间的定义 (定义 3.2.4) 中可以看到其合理性.

- 范数实际上是对  $\mathbb{R}^n$  空间中“与原点之间的距离”这一概念的推广. 将函数视作向量, 则其范数即为到原点的距离, 即模长.
- 若一个线性空间  $X$  上配备了一个范数, 则称其为**赋范向量空间 (赋范线性空间)**.

将函数视作向量，就有其到原点的距离为范数。但若是想要衡量任意两个函数之间的距离，则需要引入下面度量的概念。

定义 3.2.2. A metric on  $X$  is a map

$$d : X \times X \longrightarrow \mathbb{R}_{\geq 0} \quad (3.146)$$

$$(x, y) \longmapsto d(x, y) \quad (3.147)$$

satisfying

$$(i) \quad d(x, y) \geq 0, \forall x, y \in X. \quad (d(x, y) = 0 \Leftrightarrow x = y)$$

$$(ii) \quad d(x, y) = d(y, x), \forall x, y \in X.$$

$$(iii) \quad d(x, y) + d(y, z) \geq d(x, z), \forall x, y, z \in X.$$

**注.** • 若  $X$  为函数空间，则 (i) 中 “ $d(x, y) = 0$ ” 等价条件默认为 “ $x = y$  a.e.” .

• 度量可看作将两个函数 (向量) 的起点均平移至原点后，其两个终点之间的距离。

### 3.2.2 The Space $\mathcal{L}^1(\mathbb{R}^d)$

范数 下面先在所有 **Lebesgue** 可积函数构成的空间上定义范数。

定义 3.2.3. For any integrable function  $f$  on  $\mathbb{R}^d$ , we define the norm of  $f$ ,

$$\|f\| = \int_{\mathbb{R}^d} |f| dx \quad (3.148)$$

**注.** • 由命题 3.1.3 可知，此处  $\|f\| = 0 \Leftrightarrow f = 0$  a.e.

• 容易证明，如此定义的范数满足范数应当满足的三条公理. (定义 3.2.1)

Space  $\mathcal{L}^1(\mathbb{R}^d)$  由于定义 3.2.3 中 “ $\|f\| = 0 \Leftrightarrow f = 0 \text{ a.e.}$ ”，而我们对零测集上的函数性质并不关心，因而引出了如下关于  $\mathcal{L}^1$  空间的定义.

**定义 3.2.4.** 我们在所有 **Lebesgue 可积函数** 构成的空间上定义一个等价关系 “ $\sim$ ”：

$$f \sim g \Leftrightarrow f = g \text{ a.e.}$$

$\mathcal{L}^1(\mathbb{R}^d)$  is the space of equivalence classes of integrable functions.

**注.** 由定义可知， $\mathcal{L}^1(\mathbb{R}^d)$  空间中的元素实际上为函数的等价类 (集合)

$$[f] = \{g \text{ integrable} \mid g \sim f\}$$

而在实际中，我们还是习惯性地当作单独的函数进行运算，这在几乎处处意义下是等价的.

**度量** 下面我们说明，根据定义 3.2.3 中所定义的范数可诱导出  $\mathcal{L}^1(\mathbb{R}^d)$  上的一个度量.

**命题 3.2.1.**

$$d : \mathcal{L}^1(\mathbb{R}^d) \times \mathcal{L}^1(\mathbb{R}^d) \longrightarrow \mathbb{R}_{\geq 0} \quad (3.149)$$

$$(f, g) \longmapsto d(f, g) := \|f - g\| \quad (3.150)$$

defines a metric on  $\mathcal{L}^1(\mathbb{R}^d)$ .

**证明.** 下面即来逐一验证定义 3.2.2 中的三条公理.

- 根据范数的非负性， $\forall f, g \in \mathcal{L}^1(\mathbb{R}^d)$ ， $d(f, g) = \|f - g\| \geq 0$ .

$$d(f, g) = 0 \Leftrightarrow f - g = 0 \text{ a.e.} \Leftrightarrow f = g \text{ in } \mathcal{L}^1(\mathbb{R}^d)$$

- 可交换性.  $\forall f, g \in \mathcal{L}^1(\mathbb{R}^d)$ ,

$$d(f, g) = \|f - g\| = \int_{\mathbb{R}^d} |f - g| = \int_{\mathbb{R}^d} |g - f| = \|g - f\| = d(g, f) \quad (3.151)$$

- 根据范数的三角不等式， $\forall f, g, h \in \mathcal{L}^1(\mathbb{R}^d)$ ,

$$d(f, g) + d(g, h) = \|f - g\| + \|g - h\| \geq \|(f - g) + (g - h)\| = \|f - h\| = d(f, h)$$

□

### 3.2.3 $\mathcal{L}^1$ 空间的完备性

定义 在得到了范数、度量的定义后，我们下面给出完备空间的定义。

定义 3.2.5. A metric space  $X$  is complete if every Cauchy Sequence  $\{x_k\}_{k=1}^{\infty}$  has a limit in  $X$ .

注. • 完备空间即指空间中的任一柯西列都有收敛到自身的极限。

• 下面给出一个不完备的度量空间的例子。

例 3.2.1. 取一维实数域  $\mathbb{R}$  的子空间  $(0, 1) \subset \mathbb{R}$ ，考虑其上的 Cauchy Sequence  $\{\frac{1}{n}\}_{n=2}^{\infty} \subset (0, 1)$ 。

由于  $\frac{1}{n} \rightarrow 0 \notin (0, 1)$ ，因此度量空间  $(0, 1)$  不完备。

$\mathcal{L}^1$  空间的完备性 下面我们将给出本小节最重要的结论，即  $\mathcal{L}^1$  空间的完备性，这也是其比 Riemann 可积函数所构成的空间的优越性之所在。

定理 3.2.1. (Riesz - Fischer).

$\mathcal{L}^1$  is complete in its metric.

证明. Let  $\{f_n\}_{n=1}^{\infty} \subset \mathcal{L}^1(\mathbb{R}^d)$  be a Cauchy Sequence in  $\mathcal{L}^1$ , then

$$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}, \forall n, m \geq N(\epsilon), \text{ s. t. } \|f_n - f_m\| \leq \epsilon$$

Tacking  $\epsilon = 2^{-k}$ , then  $\exists N(2^{-k}) \geq N(2^{-(k-1)})$ , s. t. for  $n_k = N(2^{-k})$ ,  $n_{k+1} = N(2^{-(k+1)})$ ,

$$\|f_{n_k} - f_{n_{k+1}}\| \leq 2^{-k}$$

下面分为三步进行证明。

• 构建  $f(x)$  并利用  $g(x)$  证明  $f \in \mathcal{L}^1$ ，证明子列  $\{f_{n_j}\}_{j=1}^{\infty}$  收敛到  $f$ 。

Let

$$f = f_{n_1} + \sum_{j=1}^{\infty} (f_{n_{j+1}} - f_{n_j}) \quad (3.152)$$

$$g = |f_{n_1}| + \sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}| \quad (3.153)$$

Then by **MCT (Thm 3.1.2, 控制收敛定理)**

$$\int g = \int |f_{n_1}| + \int \sum_{j=1}^{\infty} |f_{n_{j+1}} - f_{n_j}| = \int |f_{n_1}| + \sum_{j=1}^{\infty} \int |f_{n_{j+1}} - f_{n_j}| \quad (3.154)$$

$$= \int |f_{n_1}| + \sum_{j=1}^{\infty} \|f_{n_{j+1}} - f_{n_j}\| \quad (3.155)$$

$$\leq \int |f_{n_1}| + \sum_{j=1}^{\infty} 2^{-j} < \infty \quad (3.156)$$

Therefore  $g$  is integrable,  $g \in \mathcal{L}^1$ . Since  $|f| \leq g$ , then  $\int |f| < \infty$ .  $f$  is integrable.

Let

$$S_k = f_{n_1} + \sum_{j=1}^k (f_{n_{j+1}} - f_{n_j}) = f_{n_{k+1}}, \quad k = 1, 2, \dots \quad (3.157)$$

$$f \text{ is integrable} \Rightarrow f < \infty \text{ a.e.} \Rightarrow S_k \text{ converges a.e.} \Rightarrow S_k = f_{n_{k+1}} \rightarrow f \text{ a.e.}$$

So we find

$$f_{n_k} \rightarrow f \text{ a.e.}$$

- 将逐点收敛性转化为  $\mathcal{L}^1$  收敛性, 即证  $\|f - f_{n_k}\| \rightarrow 0$ .

We note that

$$|f - f_{n_k}| = \left| \left( f_{n_1} + \sum_{j=1}^{\infty} (f_{n_{j+1}} - f_{n_j}) \right) - \left( f_{n_1} + \sum_{j=1}^{k-1} (f_{n_{j+1}} - f_{n_j}) \right) \right| \quad (3.158)$$

$$= \left| \sum_{j=k}^{\infty} (f_{n_{j+1}} - f_{n_j}) \right| \leq g \quad (3.159)$$

By **DCT (Thm 3.1.7, 控制收敛定理)**, since  $|f - f_{n_k}| \rightarrow 0$  a.e.,  $|f - f_{n_k}| \leq g$ ,  $g$  integrable,

$$\lim_{k \rightarrow \infty} \|f - f_{n_k}\| = \lim_{k \rightarrow \infty} \int |f - f_{n_k}| \stackrel{\text{DCT}}{=} \int \lim_{k \rightarrow \infty} |f - f_{n_k}| = 0 \quad (3.160)$$

Therefore,  $\|f - f_{n_k}\| \rightarrow 0$ . 即  $f_{n_k}$  依  $\mathcal{L}^1$  范数收敛到  $f$ .



- 利用子列  $\{f_{n_k}\}_{k=1}^{\infty}$  作为“桥梁”，证明  $f_n$  依  $\mathcal{L}^1$  范数收敛到  $f$ ，即  $\|f_n - f\| \rightarrow 0$ .  
 $\forall \epsilon > 0$ ，由于  $\{f_n\}_{n=1}^{\infty}$  为  $\mathcal{L}^1$  中 Cauchy Sequence, 因此  $\exists N \in \mathbb{N}$ , s. t.

$$\|f_n - f_m\| < \frac{\epsilon}{2}, \quad \forall n, m > N$$

Since  $\|f_{n_k} - f\| \rightarrow 0$ , then for  $\epsilon > 0$ , pick  $n_k > N$  which s. t.

$$\|f_{n_k} - f\| < \frac{\epsilon}{2}$$

Then

$$\|f_n - f\| \leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\| < \epsilon, \quad \forall n > n_k > N \quad (3.161)$$

Therefore  $\|f_n \rightarrow f\| \rightarrow 0$  with  $f \in \mathcal{L}^1$ .  $\mathcal{L}^1$  is complete in its metric.

□

根据上述定理的证明过程，可以得到下面的推论.

**推论 3.2.2.** If  $\{f_n\}_{n=1}^{\infty}$  converges to  $f$  in  $\mathcal{L}^1$ , then there exists a subsequence  $\{f_{n_k}\}_{k=1}^{\infty}$  such that

$$f_{n_k}(x) \rightarrow f(x) \text{ a.e.}$$

**注.** 即在依  $\mathcal{L}^1$  范数收敛的函数序列中，总存在“几乎处处收敛”意义的子列.

### 3.2.4 $\mathcal{L}^1$ 的稠密子空间

下面说明  $\mathcal{L}^1$  空间中以下的函数集合是稠密的.

**定理 3.2.3.** The following families of functions are dense in  $\mathcal{L}^1(\mathbb{R}^d)$ :

- The simple functions.
- The step functions.
- The continuous functions of compact support.

**证明.** 详情可见视频 [Urysohn 引理与  \$\mathcal{L}^1\$  的稠密子空间](#).

□

### 3.3 Lebesgue 积分的平移不变性

首先给出平移算符及函数平移的符号表达.

**定义 3.3.1.** The translation by a vector  $h$  on  $\mathbb{R}^d$  is denoted by the map  $\tau_h : x \mapsto x - h$ . If  $f$  is a function defined on  $\mathbb{R}^d$ , the translation of  $f$  by  $h \in \mathbb{R}^d$  is the function  $f_h$ , defined by

$$f_h(x) = (f \circ \tau_h)(x) = f(x - h)$$

下面给出 **Lebesgue** 积分的平移不变性.

**定理 3.3.1.** If  $f \in \mathcal{L}^1(\mathbb{R}^d)$ , then  $\forall h \in \mathbb{R}^d, f_h \in \mathcal{L}^1(\mathbb{R}^d)$  and

$$\int_{\mathbb{R}^d} f(x - h) dx = \int_{\mathbb{R}^d} f(x) dx \quad (3.162)$$

**证明.** 下面按 *Lebesgue* 积分的构造过程来证明, 即特征函数  $\Rightarrow$  简单函数  $\Rightarrow$  非负可测.

- **Characteristic Function.**

Suppose  $f = \chi_E$ , where  $E \subset \mathbb{R}^d$  is measurable. Then

$$f_h(x) = f(x - h) = \chi_E(x - h) = \begin{cases} 1, & \text{if } x - h \in E \\ 0, & \text{if } x - h \notin E \end{cases} = \begin{cases} 1, & \text{if } x \in E + h = E_h \\ 0, & \text{if } x \in (E + h)^c = E_h^c \end{cases} \quad (3.163)$$

根据 *Lebesgue* 测度的平移不变性,

$$\int_{\mathbb{R}^d} f_h = m(E_h) = m(E) = \int_{\mathbb{R}^d} f \quad (3.164)$$

- **Simple Function.**

$\forall \varphi = \sum_{k=1}^n a_k \chi_{E_k}$  simple, by the **linearity of integration**,

$$\int_{\mathbb{R}^d} \varphi_h = \sum_{k=1}^n \int_{\mathbb{R}^d} \chi_{(E_k)_h} = \sum_{k=1}^n \int_{\mathbb{R}^d} \chi_{E_k} = \int_{\mathbb{R}^d} \varphi \quad (3.165)$$

- **Non-negative Function.**

$\forall f$  non-negative,  $\exists \{\varphi_n\}_{n=1}^\infty$  simple, s. t.  $\varphi \nearrow f$  and  $\varphi \geq 0$ . Then by **MCT (Thm 3.1.2)**,

$$\int_{\mathbb{R}^d} \varphi_n \rightarrow \int_{\mathbb{R}^d} f \text{ as } n \rightarrow \infty \quad (3.166)$$

Since  $(\varphi_n)_h \nearrow f_h$  and  $\int \varphi_n = \int (\varphi_n)_h$ , then by **MCT (Thm 3.1.2)**,

$$\int_{\mathbb{R}^d} \varphi_n = \int_{\mathbb{R}^d} (\varphi_n)_h \rightarrow \int_{\mathbb{R}^d} f_h \text{ as } n \rightarrow \infty \quad (3.167)$$

Therefore

$$\int_{\mathbb{R}^d} f_h = \int_{\mathbb{R}^d} f \quad (3.168)$$

- **General Case.**

$\forall f \in \mathcal{L}^1(\mathbb{R}^d)$ ,  $f = f^+ - f^-$ , where  $f^+$  and  $f^-$  are non-negative.

Then by the **linearity of integration**,

$$\int_{\mathbb{R}^d} f_h = \int_{\mathbb{R}^d} f_h^+ - \int_{\mathbb{R}^d} f_h^- = \int_{\mathbb{R}^d} f^+ - \int_{\mathbb{R}^d} f^- = \int_{\mathbb{R}^d} f \quad (3.169)$$

□

### 3.4 Lebesgue 可积函数的 $\mathcal{L}^1$ 连续性

引入 Recall 数学分析中连续的等价定义:

$$f \text{ is continuous at } x \Leftrightarrow f(x) - f(x-h) \rightarrow 0 \text{ as } h \rightarrow 0 \quad (3.170)$$

$$\Leftrightarrow |f_h(x) - f(x)| \rightarrow 0 \text{ as } h \rightarrow 0 \quad (3.171)$$

即可大致视作 **Riemann** 可积函数关于 2-范数的连续性.

**Lebesgue** 可积函数的  $\mathcal{L}^1$  连续性 在  $\mathcal{L}^1$  空间中, **Lebesgue** 可积函数也有类似的关于  $\mathcal{L}^1$  范数的连续性. 这就是下面的定理.

**定理 3.4.1.** Suppose  $f \in \mathcal{L}^1(\mathbb{R}^d)$ , then

$$\|f_h - f\|_{\mathcal{L}^1} \rightarrow 0 \text{ as } h \rightarrow 0 \quad (3.172)$$

**证明.** 详见视频[积分的平移不变性与可积函数的  \$\mathcal{L}^1\$  连续性](#).

其中需要用到如下的引理.

**引理 3.4.2.** <sup>2</sup> If  $f \in C_c(\mathbb{R}^d)$ , then  $f$  is uniformly continuous.

**注.**  $f \in C_c(\mathbb{R}^d)$  表示  $f$  为具有紧支集的连续函数.

□

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<sup>2</sup>此为书: 《Real Analysis – Modern Techniques and Their Applications》— Gerald B. Folland

### 3.5 Fubini 定理

为了讨论的方便，下面先给出函数及集合的切片的定义.

**定义 3.5.1.** If  $f$  is a function on  $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ , the slice of  $f$  w.r.t.  $y \in \mathbb{R}^{d_2}$  is the function

$$f^y : \mathbb{R}^{d_1} \longrightarrow \overline{\mathbb{R}} \quad (3.173)$$

$$x \longmapsto f(x, y) \quad (3.174)$$

Similarly, the slice of  $f$  for a fixed  $x \in \mathbb{R}^{d_1}$  is

$$f_x : \mathbb{R}^{d_2} \longrightarrow \overline{\mathbb{R}} \quad (3.175)$$

$$y \longmapsto f(x, y) \quad (3.176)$$

Let  $E \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ , we define its slices by

$$E^y := \{x \in \mathbb{R}^{d_1} \mid (x, y) \in E\}, \quad E_x := \{y \in \mathbb{R}^{d_2} \mid (x, y) \in E\} \quad (3.177)$$

下面给出 **Fubini 定理**.

**定理 3.5.1. Fubini.**

Suppose  $f(x, y)$  is **integrable** on  $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ . Then for a.e.  $y \in \mathbb{R}^{d_2}$ :

- (i) The slice  $f^y$  is integrable on  $\mathbb{R}^{d_1}$ .
- (ii) The function  $y \mapsto \int_{\mathbb{R}^{d_1}} f^y(x) dx$  is integrable on  $\mathbb{R}^{d_2}$ .
- (iii) Moreover,

$$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f^y(x) dx \right) dy = \int_{\mathbb{R}^d} f \quad (3.178)$$

### 3.5.1 Fubini 定理的证明

证明. Let  $\mathcal{F} = \{f \in \mathcal{L}^1(\mathbb{R}^d) \mid f \text{ satisfies (i) } \sim \text{ (iii)}\}$ . It suffices to show that  $\mathcal{L}^1(\mathbb{R}^d) \subset \mathcal{F}$ .

下面仍按照构造 Lebesgue 积分的顺序思路进行证明, 即特征函数  $\Rightarrow$  简单函数  $\Rightarrow$  非负可测.

(其中特征函数部分 (Step 3 ~ 5) 最为复杂繁琐, 后续的证明则是水到渠成)

在此之前, 还要先证明  $\mathcal{F}$  对函数的线性组合及单调函数列的极限封闭.

• **Step 1: Any finite linear combination of functions in  $\mathcal{F}$  also belongs to  $\mathcal{F}$ .**

Suppose  $\{f_k\}_{k=1}^N \subset \mathcal{F}$ . By the condition,  $\forall k, \exists A_k \subset \mathbb{R}^{d_2}, m(A_k) = 0$ , s. t.

$$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in A_k^c.$$

Let  $A = \bigcup_{k=1}^N A_k$ , then  $m(A) = 0$  and

$$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in A^c, \forall k.$$

下面对定理结论逐条验证. By the linearity of integration,  $\forall a_k \in \mathbb{R}$ ,

$$\left(\sum_{k=1}^N a_k f_k\right)^y = \sum_{k=1}^N a_k f_k^y \text{ is integrable on } \mathbb{R}^{d_1} \quad (3.179)$$

$$\int_{\mathbb{R}^{d_1}} \sum_{k=1}^N (a_k f_k)^y(x) dx = \sum_{k=1}^N a_k \int_{\mathbb{R}^{d_1}} f_k^y(x) dx \text{ is integrable on } \mathbb{R}^{d_2} \quad (3.180)$$

$$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} \sum_{k=1}^N (a_k f_k)^y(x) dx \right) dy = \sum_{k=1}^N a_k \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f_k^y(x) dx \right) dy \quad (3.181)$$

$$= \sum_{k=1}^N a_k \int_{\mathbb{R}^d} f_k \quad (3.182)$$

$$= \int_{\mathbb{R}^d} \sum_{k=1}^N a_k f_k \quad (3.183)$$

$$(3.184)$$

Therefore,  $\sum_{k=1}^N a_k f_k \in \mathcal{F}, \forall a_k \in \mathbb{R}$ .

- **Step 2:**  $\mathcal{F}$  对单调函数列的极限封闭, 即  $\forall \{f_k\}_{k=1}^\infty, f_k \nearrow f, f \text{ integrable} \Rightarrow f \in \mathcal{F}$ .

Suppose  $f_k \geq 0$ . By the condition,  $\forall k, \exists A_k \subset \mathbb{R}^d, m(A_k) = 0$ , s. t.

$$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in A_k^c.$$

Let  $A = \bigcup_{k=1}^\infty A_k$ , then  $m(A) = 0$  and

$$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in A^c, \forall k.$$

Since  $f_k^y(x) \nearrow f^y(x)$ , by **MCT (Thm 3.1.2)**

$$\int_{\mathbb{R}^{d_1}} f_k^y(x) dx \rightarrow \int_{\mathbb{R}^{d_1}} f^y(x) dx \text{ as } k \rightarrow \infty \quad (3.185)$$

Let

$$g_k(y) = \int_{\mathbb{R}^{d_1}} f_k^y(x) dx, \quad g(y) = \int_{\mathbb{R}^{d_1}} f^y(x) dx \quad (3.186)$$

Then we have  $g_k(y) \nearrow g(y)$  and  $g_k \geq 0$ . By **MCT (Thm 3.1.2)**

$$\int_{\mathbb{R}^{d_2}} g_k(y) dy \rightarrow \int_{\mathbb{R}^{d_2}} g(y) dy \text{ as } k \rightarrow \infty \quad (3.187)$$

i.e.

$$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f_k^y(x) dx \right) dy \rightarrow \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f^y(x) dx \right) dy \quad (3.188)$$

By the condition (iii), we have

$$\int_{\mathbb{R}^d} f_k \rightarrow \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f^y(x) dx \right) dy \quad (3.189)$$

Since  $f_k \nearrow f, f_k \geq 0$ , by **MCT (Thm 3.1.2)**

$$\int_{\mathbb{R}^d} f_k \rightarrow \int_{\mathbb{R}^d} f \quad (3.190)$$

Therefore

$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f^y(x) dx \right) dy = \int_{\mathbb{R}^{d_2}} g(y) dy \quad (3.191)$$

Since  $f$  is integrable,  $g(y) = \int_{\mathbb{R}^{d_1}} f^y(x) dx$  is integrable on  $\mathbb{R}^{d_2} \Rightarrow \int g < \infty$ .

Then we have

$$g(y) = \int_{\mathbb{R}^{d_1}} f^y(x) dx < \infty \text{ for a.e. } y \in \mathbb{R}^{d_2} \quad (3.192)$$

Therefore  $f^y(x)$  is integrable for a.e.  $y \in \mathbb{R}^{d_2}$ .

Then  $f \in \mathcal{F}$ .

- **Step 3: Any characteristic function of a set  $E$  of type  $G_\delta$  with finite measure belongs to  $\mathcal{F}$ .**

下面对  $E$  进行讨论, 分  $a \sim e$  五种情况来证明:

(a)  $E \subset \mathbb{R}^d$  is a bounded open cube.

Suppose  $E = Q_1 \times Q_2$ , where  $Q_1 \subset \mathbb{R}^{d_1}$  and  $Q_2 \subset \mathbb{R}^{d_2}$  are open cubes.

$\forall y \in \mathbb{R}^{d_2}$ ,  $\chi_E(x, y)$  is measurable in  $x$ , and integrable with

$$g(y) = \int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx = \begin{cases} |Q_1|, & \text{if } y \in Q_2 \\ 0, & \text{if } y \notin Q_2 \end{cases} = |Q_1| \chi_{Q_2}(y) \quad (3.193)$$

Since  $g(y) = |Q_1| \chi_{Q_2}(y)$  is measurable and integrable with

$$\int_{\mathbb{R}^{d_2}} g(y) dy = |Q_1| |Q_2| = |E| = \int_{\mathbb{R}^d} \chi_E \quad (3.194)$$

i.e.

$$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx \right) dy = \int_{\mathbb{R}^d} \chi_E \quad (3.195)$$

Therefore,  $\chi_E \in \mathcal{F}$ .

(b)  $E \subset \mathbb{R}^d$  is a subset of the boundary of some closed cube.

Since  $m(E) = 0$ , we have

$$\int_{\mathbb{R}^d} \chi_E = m(E) = 0 \quad (3.196)$$

After an investigation of various possibilities, we note that (此处细节证明暂且留疑)

$$\forall \text{ a.e. } y \in \mathbb{R}^{d_2}, E^y = \{x \in \mathbb{R}^{d_1} \mid (x, y) \in E\} \text{ has measure 0 in } \mathbb{R}^{d_1}.$$

Then  $\forall \text{ a.e. } y \in \mathbb{R}^{d_2}$ ,  $\chi_E^y(x)$  is integrable on  $\mathbb{R}^{d_1}$  with

$$g(y) = \int_{\mathbb{R}^{d_1}} \chi_E^y(x) dx = \int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx = 0, \quad \forall \text{ a.e. } y \in \mathbb{R}^{d_2} \quad (3.197)$$

So  $g(y)$  is integrable on  $\mathbb{R}^{d_2}$  with

$$\int_{\mathbb{R}^{d_2}} g(y) dy = 0 = \int_{\mathbb{R}^d} \chi_E \quad (3.198)$$

i.e.

$$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx \right) dy = \int_{\mathbb{R}^d} \chi_E \quad (3.199)$$

Therefore,  $\chi_E \in \mathcal{F}$ .



(c)  $E \subset \mathbb{R}^d$  is a finite union of almost disjoint closed cubes.

Suppose  $E = \bigcup_{k=1}^N Q_k$ , where  $\{Q_k^\circ\}_{k=1}^N$  are disjoint.

Let  $A_k = Q_k - Q_k^\circ$  be the boundary of closed cube  $Q_k$ . Then  $\chi_{A_k} \in \mathcal{F}$ . (by **Step 3 (b)**)

$\chi_E$  is a linear combination of  $\chi_{Q_k}$  and  $\chi_{A_k}$ ,  $k = 1 \sim N$ .

Since  $\chi_{Q_k}, \chi_{A_k} \in \mathcal{F}$ ,  $k = 1 \sim N$ , then by **Step 1**,  $\chi_E \in \mathcal{F}$ .

(d)  $E \subset \mathbb{R}^d$  is open and of finite measure.

Since  $E \subset \mathbb{R}^d$  is open, by **Thm 1.1.4**,  $\exists$  almost disjoint closed cubes  $\{Q_k\}_{k=1}^\infty$ , s. t.

$$E = \bigcup_{k=1}^\infty Q_k, \text{ where } \{Q_k^\circ\}_{k=1}^\infty \text{ are disjoint} \quad (3.200)$$

Let

$$f_k = \chi_{\bigcup_{j=1}^k Q_j} \quad (3.201)$$

Then by **Step 3 (c)**,  $f_k \in \mathcal{F}$ , and  $f_k \nearrow f = \chi_E$ ,  $f_k \geq 0$ . By **Step 2**, we have  $f = \chi_E \in \mathcal{F}$ .

(e)  $E \subset \mathbb{R}^d$  is a  $G_\delta$  of finite measure.

By the **definition of  $G_\delta$  (Def 1.4.5)**,

$$E = \bigcap_{k=1}^\infty \widetilde{Q}_k, \text{ where } \widetilde{Q}_k \subset_{\text{open}} \mathbb{R}^d \quad (3.202)$$

Since  $E$  has finite measure,  $\exists \widetilde{O}_0 \subset \mathbb{R}^d$  open, s. t.  $E \subset \widetilde{O}_0$ .

Let

$$O_k = \widetilde{O}_0 \cap \bigcap_{j=1}^k \widetilde{Q}_j \quad (3.203)$$

Then  $O_1 \supset O_2 \supset \dots$  and  $E = \bigcap_{k=1}^\infty O_k$ . Let  $f_k = \chi_{O_k}$ , then  $f_k \in \mathcal{F}$ . (By **Step 3 (d)**)

Since  $f_k \searrow f = \chi_E$ ,  $f_k \in \mathcal{F}$ , then by **Step 2**,  $f = \chi_E \in \mathcal{F}$ .

- **Step 4:** If  $E \subset \mathbb{R}^d$  has measure 0, then  $\chi_E \in \mathcal{F}$ .

By **Thm 1.4.1**,  $\exists$  a set  $G \subset \mathbb{R}^d$  of type  $G_\delta$  with  $E = G \setminus N$ , where  $m(N) = 0$ . Then

$$E \subset G, m(G) = m(E) + m(G \setminus E) = 0.$$

By **Step 3**,  $\chi_G \in \mathcal{F}$ , then

$$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} \chi_G(x, y) dx \right) dy = \int_{\mathbb{R}^d} \chi_G = 0 \quad (3.204)$$

Then

$$\int_{\mathbb{R}^{d_1}} \chi_G(x, y) dx = 0 \text{ for a.e. } y \in \mathbb{R}^{d_2} \quad (3.205)$$

Since

$$\int_{\mathbb{R}^{d_1}} \chi_G(x, y) dx = \int_{\mathbb{R}^{d_1}} \chi_G^y(x) dx = \int_{\mathbb{R}^{d_1}} \chi_{G^y}(x) dx = m(G^y) \quad (3.206)$$

Therefore

$$G^y = \{x \in \mathbb{R}^{d_1} \mid (x, y) \in G\} \text{ has measure 0 for a.e. } y \in \mathbb{R}^{d_2} \quad (3.207)$$

Since  $E^y \subset G^y$ , then  $E^y$  has measure 0 for a.e.  $y \in \mathbb{R}^{d_2}$ .

$$\Rightarrow \int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx = m(E^y) = 0 \text{ for a.e. } y \in \mathbb{R}^{d_2} \quad (3.208)$$

$$\Rightarrow \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx \right) dy = 0 = \int_{\mathbb{R}^d} \chi_E \quad (3.209)$$

$$\Rightarrow \chi_E \in \mathcal{F} \quad (3.210)$$

- **Step 5:** If  $E$  is any measurable subset of  $\mathbb{R}^d$  with finite measure, then  $\chi_E$  belongs to  $\mathcal{F}$ .

By **Thm 1.4.1**,  $\exists$  a finite measure  $G$  of type  $G_\delta$  with  $E \subset G$  and  $m(G - E) = 0$ .

Since  $\chi_E = \chi_G - \chi_{G-E}$ , by **Step 4**,  $\chi_G, \chi_{G-E} \in \mathcal{F}$ , then by **Step 1**,  $\chi_E \in \mathcal{F}$ .

- **Step 6: If  $f$  is integrable, then  $f \in \mathcal{F}$ .**

不妨 Suppose  $f$  non-negative. By **Step 1 and Step 5**,  $\forall \varphi = \sum_{k=1}^N a_k \chi_{E_k}$  simple,  $\varphi \in \mathcal{F}$ .

By **Thm 2.2.1**,  $\exists \{\varphi_k\}_{k=1}^\infty$  simple,  $\varphi_k \nearrow f$ ,  $\varphi_k \geq 0$ . Then by **Step 2**,  $f \in \mathcal{F}$ .

Therefore,

$$\mathcal{L}^1(\mathbb{R}^d) \subset \mathcal{F}$$

□

### 3.5.2 Fubini 定理的应用

**Tonelli 定理** 下面给出一个 **Fubini 定理** 的延伸形式, 就是 **Tonelli 定理**, 常与 **Fubini 定理** 一起使用, 用于判断函数的可积性.

#### 定理 3.5.2. Tonelli.

Suppose  $f(x, y)$  is a **non-negative measurable** function on  $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ . Then for a.e.  $y \in \mathbb{R}^{d_2}$ :

- (i) The slice  $f^y$  is **measurable** on  $\mathbb{R}^{d_1}$ .
- (ii) The function defined by  $y \mapsto \int_{\mathbb{R}^{d_1}} f^y(x) dx$  is **measurable** on  $\mathbb{R}^{d_2}$ .
- (iii) Moreover,

$$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f(x, y) dx \right) dy = \int_{\mathbb{R}^d} f \in [0, \infty] \quad (3.211)$$

**注.** 在尚未知晓  $f$  的可积性时, 可先用 **Tonelli 定理** 计算  $|f|$  的可积性, 从而得到  $f$  的可积性, 再去考虑使用 **Fubini 定理**.

**证明.** Consider the **truncations**

$$f_k(x, y) = \begin{cases} f(x, y), & \text{if } |(x, y)| < k \text{ and } f(x, y) < k \\ 0, & \text{otherwise} \end{cases} \quad (3.212)$$

Since

$$\int_{\mathbb{R}^d} f_k \leq k^{d+1} < \infty \quad (3.213)$$

$f_k$  is integrable for all  $k$ . Then by **Fubini (Thm 3.5.1)**,  $\exists E_k \subset \mathbb{R}^{d_2}$ ,  $m(E_k) = 0$ , s. t.

$$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in E_k^c.$$

$$g_k(y) = \int_{\mathbb{R}^{d_1}} f_k^y(x) dx \text{ is integrable on } \mathbb{R}^{d_2} \text{ a.e.} \quad (3.214)$$

$$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f_k(x, y) dx \right) dy = \int_{\mathbb{R}^d} f_k \quad (3.215)$$

下面开始验证  $f$  满足定理中的各条结论.

- Let  $E = \bigcup_{k=1}^{\infty} E_k$ , then  $m(E) = 0$  and

$$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in E^c, \forall k.$$

$\forall y \in E^c$ , since  $f_k^y(x) \nearrow f^y(x)$ ,  $f_k^y(x)$  integrable on  $\mathbb{R}^{d_1}$ , specifically measurable

Then  $\forall y \in E^c$   $f^y(x)$  is measurable. i.e.  $f^y$  measurable for a.e.  $y \in \mathbb{R}^{d_2}$ .

- Since  $f_k^y(x) \nearrow f^k(x)$ ,  $\forall y \in E^c$ , then by **MCT (Thm 3.1.2)**

$$g_k(y) = \int_{\mathbb{R}^{d_1}} f_k^y(x) dx \nearrow \int_{\mathbb{R}^{d_1}} f^y(x) dx = g(y) \text{ for a.e. } y \in \mathbb{R}^{d_2} \quad (3.216)$$

Since  $g_k(y)$  is integrable on  $\mathbb{R}^{d_2}$  a.e., specifically measurable,

Then  $g(y)$  is measurable on  $\mathbb{R}^{d_2}$  a.e.

- Since  $g_k(y) \nearrow g(y)$ ,  $\forall$  a.e.  $y \in \mathbb{R}^{d_2}$ , then by **MCT (Thm 3.1.2)**

$$\int_{\mathbb{R}^{d_2}} g_k(y) dy \rightarrow \int_{\mathbb{R}^{d_2}} g(y) dy \quad (3.217)$$

i.e.

$$\int_{\mathbb{R}^d} f_k = \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f_k(x, y) dx \right) dy \rightarrow \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f(x, y) dx \right) dy \quad (3.218)$$

Since  $f_k \nearrow f$ , by **MCT (Thm 3.1.2)**

$$\int_{\mathbb{R}^d} f_k \rightarrow \int_{\mathbb{R}^d} f \quad (3.219)$$

Therefore

$$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f(x, y) dx \right) dy = \int_{\mathbb{R}^d} f \in [0, \infty] \quad (3.220)$$

□

乘积测度 下面给出乘积测度在 **Lebesgue** 测度下的一些表现性质. 具体证明可见书<sup>3</sup>P82~85, 基本都是 Trivial 的.

**推论 3.5.3.** If  $E$  is a measurable set in  $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ , then for a.e.  $y \in \mathbb{R}^{d_2}$ , the slice

$$E^y = \{x \in \mathbb{R}^{d_1} \mid (x, y) \in E\} \text{ is a measurable subset of } \mathbb{R}^{d_1}.$$

Moreover,  $m(E^y)$  is a measurable function of  $y$  and

$$m(E) = \int_{\mathbb{R}^{d_2}} m(E^y) dy \quad (3.221)$$

**注.** • 该命题为 **Tonelli 定理 (Thm 3.5.2)** 的推论, 考虑  $f = \chi_E$  即可轻松得证.

• 该推论说明了对于任一可测集  $E$ , 其切片  $E^y$  都是几乎处处可测的.

有了推论 3.5.3, 我们自然会去思考一般情况下其逆命题是否成立, 即

$$E_y \text{ measurable for a.e. } y \in \mathbb{R}^{d_2} \Rightarrow E \subset \mathbb{R}^d \text{ measurable?}$$

然而答案是否定的. 下面给出一个反例.

**例 3.5.1.** Let  $\mathcal{N}$  denote a non-measurable subset  $\mathbb{R}$  (正测度集必有不可测子集, **Prop 1.5.1**).

Then define

$$E = [0, 1] \times \mathcal{N} \subset \mathbb{R} \times \mathbb{R}$$

We see that

$$E^y = \begin{cases} [0, 1], & \text{if } y \in \mathcal{N} \\ \emptyset, & \text{if } y \notin \mathcal{N} \end{cases}$$

Thus  $E^y$  is measurable for every  $y \in \mathbb{R}$ . However, if  $E$  is measurable, then by **Cor 3.5.3**,

$$E_x = \{y \in \mathbb{R} \mid (x, y) \in E\} = \begin{cases} \mathcal{N}, & \text{if } x \in [0, 1] \\ \emptyset, & \text{if } x \notin [0, 1] \end{cases}$$

which is a contradiction for  $\mathcal{N}$  is non-measurable.

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<sup>3</sup> 《Real Analysis – Measure Theory, Integration, & Hilbert Spaces》— Elias M. Stein

下面对推论 3.5.3 进行一定程度的推广，得到如下命题.

**命题 3.5.1.** If  $E = E_1 \times E_2$  is a measurable subset of  $\mathbb{R}^d$ , and  $m_*(E_2) > 0$ , then  $E_1$  is measurable.

而我们接下来将说明，若两个集合均可测，则他们的 **Descartes 积也是可测集**. 而这事实上就是抽象测度中**乘积测度**的定义. 在此之前，先来说明一个证明时需要用到的引理.

**引理 3.5.4.** If  $E_1 \subset \mathbb{R}^{d_1}$  and  $E_2 \subset \mathbb{R}^{d_2}$ , then

$$m_*(E_1 \times E_2) \leq m_*(E_1)m_*(E_2)$$

下面便给出乘积测度在 **Lebesgue 测度**下的定义.

**命题 3.5.2.** Suppose  $E_1$  and  $E_2$  are measurable subsets of  $\mathbb{R}^{d_1}$  and  $\mathbb{R}^{d_2}$ , respectively. Then  $E = E_1 \times E_2$  is a measurable subset of  $\mathbb{R}^d$ . Moreover,

$$m(E) = m(E_1)m(E_2)$$

**几何联系** 在 **Riemann** 积分中我们都熟知积分  $\int f$  即代表  $f$  下方所围成区域的“体积”. 而下面我们将说明, 在 **Lebesgue** 积分中, 积分与几何直观之间的联系. (Stein P85~ 86)

在此之前先给出一个命题, 此为命题 **3.5.2**的推论.

**推论 3.5.5.** Suppose  $f$  is a measurable function on  $\mathbb{R}^{d_1}$ . Then the function  $\tilde{f}$  defined by

$$\tilde{f}(x, y) = f(x)$$

is measurable on  $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ .

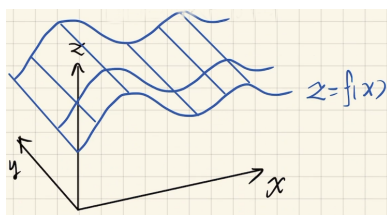


图 3.3: Prop 3.5.5

下面给出 **Lebesgue** 积分与几何直观之间的联系.

**推论 3.5.6.** Suppose  $f(x)$  is a non-negative function on  $\mathbb{R}^d$ , and let

$$\mathcal{A} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} \mid 0 \leq y \leq f(x)\}$$

Then:

(i)  $f$  is measurable on  $\mathbb{R}^d$  iff  $\mathcal{A}$  is measurable on  $\mathbb{R}^{d+1}$ .

1. If the conditions in (i) hold, then

$$\int_{\mathbb{R}^d} f = m(\mathcal{A}) \tag{3.222}$$

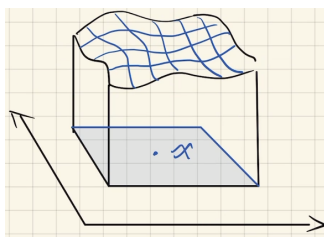


图 3.4: Prop 3.5.6



### 3.6 Lebesgue 积分与 Riemann 积分的联系

下面我们将说明, **Lebesgue** 积分可视为 **Riemann** 积分的延拓, 它很好地囊括了 **Riemann** 积分的定义.

在此之前, 先来给出 **MCT (Thm 3.1.2, 单调收敛定理)** 在单调递减函数列上的表述.

**定理 3.6.1. Monotone Convergence Theorem (decreasing).**

Let  $\{f_n\}_{n=1}^\infty$  be non-negative,  $f_n \searrow f$ ,  $\int f_1 < \infty$ , then

$$\lim_{n \rightarrow \infty} \int f_n = \int f \quad (3.223)$$

**证明.** Let  $g_n = f_1 - f_n$ ,  $n \in \mathbb{N}$ . Then  $g_n \geq 0$  and  $g_n \nearrow g = f_1 - f$ . By **MCT (Thm 3.1.2)**

$$\int g_n \rightarrow \int g \text{ as } n \rightarrow \infty \quad (3.224)$$

i.e.

$$\int (f_1 - f_n) \rightarrow \int (f_1 - f) \text{ as } n \rightarrow \infty \quad (3.225)$$

Therefore

$$\int f_n \rightarrow \int f \quad (3.226)$$

□

下面说明 **Riemann** 可积函数的积分即为其 **Lebesgue** 积分.

**定理 3.6.2.** Suppose  $f$  is Riemann integrable, then

$$\int_{[a,b]}^{\mathcal{R}} f(x) dx = \int_{[a,b]}^{\mathcal{L}} f(x) dx \quad (3.227)$$

**证明.** 详细证明可见书<sup>4</sup>§4.4 或视频 [Lebesgue 积分与 Riemann 积分的联系](#). □

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<sup>4</sup> 《实变函数论 (第三版)》——周民强

### 3.7 Lebesgue 积分的伸缩变换

下面我们给出 **Lebesgue** 积分的伸缩变换公式. 这实质上为一般的抽象测度的变量替换公式在 **Lebesgue** 测度下的特例, 而此处我们的证明方法为 **Lebesgue** 测度下的方法, 依赖于  $\mathbb{R}^d$  中的几何直观, 较为枯燥繁琐, 不具有一般性. 在后续学习抽象测度时会给出一般性的方法论.

**命题 3.7.1. Lebesgue 积分的伸缩变换公式.**

- $m(\delta E) = |\delta| m(E), \delta \in \mathbb{R}, E \subset \mathbb{R}.$
- $\int f(x)dx = |\delta| \int f(\delta x)dx, \delta \in \mathbb{R}, f \in \mathcal{L}^1(\mathbb{R}).$
- $\int f(x)dx = \delta_1 \cdots \delta_d \int f(\delta x)dx, \delta \in \mathbb{R}^d, \delta_j > 0, f \in \mathcal{L}^1(\mathbb{R}^d).$
- $m(\delta E) = \delta_1 \cdots \delta_d m(E), \delta_j > 0, E \subset \mathbb{R}^d.$

**证明.** 可见视频 [积分的伸缩变换](#) 或参考书<sup>5</sup>P73~74.

□

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<sup>5</sup> 《Real Analysis – Measure Theory, Integration, & Hilbert Spaces》— Elias M. Stein

## 3.8 Littlewood 三原则

**Motivation** 尽管我们建立起了围绕 Lebesgue 测度为中心的新的理论体系, 但我们仍应当重视其与数学分析中概念的联系. 而 Littlewood 便总结归纳出了这样三条 principles:

- (i) Every (measurable) set is **nearly** a finite union of intervals.
- (ii) Every function (of class  $\mathcal{L}^n$ ) is **nearly** continuous.
- (iii) Every convergent sequence is **nearly** uniformly convergent.

不难发现其叙述显得并不太严谨, 其中的 **nearly** 一词需要我们给予严格的数学定义.

Littlewood 三原则告诉了我们可测函数与连续函数之间的联系, 包括收敛函数列与一致收敛的关系. 其中第一条原则即为定理 1.3.4 (iv).

下面我们从后往前依次给出第三、二条原则, 即 Egorov 定理与 Lusin 定理. 这在抽象测度中仍然起着重要作用.

### 3.8.1 Egorov 定理

关于 Littlewood 三原则中的 (iii), 实际上在数学分析中已不陌生. 下面给出一个经典例子.

**例 3.8.1.** Consider the sequence  $f_n(x) = x^n$ ,  $x \in [0, 1]$ . Then  $f_n$  converges on  $[0, 1]$  to  $f$ :

$$f(x) = \begin{cases} 0, & \text{if } x \in [0, 1) \\ 1, & \text{if } x = 1 \end{cases}$$

So  $f_n \rightarrow f$  but not uniformly on  $[0, 1]$ .

However, if we consider the closed interval  $[0, 1 - \epsilon]$  or any closed interval  $[a, b]$  except 1, then

$$f_n \Rightarrow f \text{ uniformly on } [0, 1 - \epsilon] \text{ or } [a, b].$$

which implies “convergent sequence is nearly uniformly convergent”.

下面给出 **Egorov** 定理的表述.

**定理 3.8.1. Egorov (Almost Uniform Convergence, 近一致收敛).**

Suppose  $\{f_k\}_{k=1}^{\infty}$  is a sequence of measurable functions on a measurable set  $A$  with  $m(A) < \infty$ , and  $f_k(x) \rightarrow f(x)$  on  $A$  a.e. Given  $\epsilon > 0$ , we can find a set  $E \subset A$  s. t.

$$m(E) < \epsilon \text{ and } f_k \Rightarrow f \text{ uniformly on } E^c.$$

**注.** • 此处若将 **Lebesgue** 测度  $m$  换为一般的抽象测度  $\mu$ , 即可得到抽象测度下的 **Egorov** 定理. (可见书<sup>6</sup>P62 Thm 2.33)

• 在证明定理前, 先回顾一下函数列收敛点集 & 发散点集的表述.

– 收敛点集.

$$x \in \text{收敛点集} \Leftrightarrow \forall \epsilon > 0, \exists N, \forall n \geq N, |f_n(x) - f(x)| < \epsilon \quad (3.228)$$

$$\stackrel{\text{离散}}{\Leftrightarrow} \forall k \in \mathbb{N}, \exists N, \forall n \geq N, |f_n(x) - f(x)| < \frac{1}{k} \quad (3.229)$$

$$\Rightarrow C(f) = \bigcap_{k=1}^{\infty} \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \left\{ x \mid |f_n(x) - f(x)| < \frac{1}{k} \right\} \quad (3.230)$$

– 发散点集.

$$x \in \text{发散点集} \Leftrightarrow \exists \epsilon > 0, \forall N, \exists n \geq N, |f_n(x) - f(x)| \geq \epsilon \quad (3.231)$$

$$\stackrel{\text{离散}}{\Leftrightarrow} \exists k \in \mathbb{N}, \forall N, \exists n \geq N, |f_n(x) - f(x)| \geq \frac{1}{k} \quad (3.232)$$

$$\Rightarrow D(f) = \bigcup_{k=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \left\{ x \mid |f_n(x) - f(x)| \geq \frac{1}{k} \right\} \quad (3.233)$$

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<sup>6</sup> 《Real Analysis – Modern Techniques and Their Applications》— Gerald B. Folland

证明. Let

$$E_{\textcolor{red}{m}}(k) = \bigcup_{n=\textcolor{red}{m}}^{\infty} \left\{ x \mid |f_n(x) - f(x)| \geq \frac{1}{k} \right\} \quad (3.234)$$

Then  $E_m(k) \searrow$  in  $m$ .

Since  $f_k(x) \rightarrow f(x)$  on  $A$  a.e., then  $m(D(f)) = 0$ . Since

$$\bigcap_{m=1}^{\infty} E_m(k) \subset D(f) = \bigcup_{k=1}^{\infty} \left( \bigcap_{m=1}^{\infty} E_m(k) \right) \quad (3.235)$$

Then  $m\left(\bigcap_{m=1}^{\infty} E_m(k)\right) = 0$ . Then by **Thm 1.3.3**,

$$m\left(\bigcap_{m=1}^{\infty} E_m(k)\right) = m\left(\lim_{N \rightarrow \infty} \bigcap_{m=1}^N E_m(k)\right) = \lim_{N \rightarrow \infty} m\left(\bigcap_{m=1}^N E_m(k)\right) = 0 \quad (3.236)$$

即得到数列  $\left\{ \bigcap_{m=1}^N E_m(k) \right\}_{N=1}^{\infty}$  极限为 0. Then for any fixed  $\epsilon > 0$ ,  $\exists N_k \in \mathbb{N}$ , s. t.

$$m\left(\bigcap_{m=1}^{N_k} E_m(k)\right) = m(E_{N_k}(k)) < \frac{\epsilon}{2^k} \quad (3.237)$$

Let  $E = \bigcup_{k=1}^{\infty} E_{N_k}(k)$ , then  $m(E) < \epsilon$  and

$$E^c = \bigcap_{k=1}^{\infty} E_{N_k}^c(k) = \bigcap_{k=1}^{\infty} \bigcap_{n=N_k}^{\infty} \left\{ x \mid |f_n(x) - f(x)| < \frac{1}{k} \right\} \quad (3.238)$$

Then we get for a fixed  $k_0 \in \mathbb{N}$ ,  $\exists N_{k_0}$ ,  $\forall n \geq N_{k_0}$ , s. t.

$$|f_n(x) - f(x)| < \frac{1}{k_0} \text{ for all } x \in E^c.$$

Therefore,  $f_n \Rightarrow f$  uniformly on  $E^c$  with  $m(E) < \epsilon$ . □

### 3.8.2 *Lusin* 定理

下面给出 **Littlewood** 三原则中的第 (ii) 点, 可测函数 **nearly** 连续, 即 **Lusin** 定理.

**定理 3.8.2. Lusin.**

Suppose  $f : E \rightarrow \mathbb{R}$  is measurable and finite-valued on  $E$  with  $m(E) < \infty$ . Then for every  $\epsilon > 0$ , there exists a compact set  $F \subset E$ , s. t.

$$m(F^c) < \epsilon \text{ and } f|_F \text{ is continuous.}$$

证明. 可见书<sup>7</sup>**P34 Thm 4.5** 或视频 [可测函数与连续函数的联系](#).

□

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<sup>7</sup> 《Real Analysis – Measure Theory, Integration, & Hilbert Spaces》— Elias M. Stein

## 第四章 *Differentiation and Integration*

**Motivation** 在 **Riemann** 积分的框架下, 我们知道积分和微分可以视作一对互逆的运算. 而在这一章, 我们将在全新的 **Lebesgue** 测度的框架下重新审视积分和微分之间的关系.

下面先来描述一下想要解决的问题.

- Let  $f \in \mathcal{L}^1(\mathbb{R}^d)$ . 对于变上限积分  $F(x) = \int_a^x f(y)dy$ , 我们知道根据 **Riemann** 积分下的微积分基本定理, 对  $F$  求导就会回到被积函数  $f$  本身. 那么我们会好奇:
  - 在 **Lebesgue** 积分的框架下, 这个结论是否还成立?
  - 如果成立的话, 又对哪些  $x$  成立呢?

此时回顾求导的定义, 即对于差商 (此处改写为更具一般性的符号  $I = (x, x+h)$ )

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h} \int_x^{x+h} f(y)dy = \frac{1}{|I|} \int_I f(y)dy \quad (4.1)$$

对差商中的增量  $h \rightarrow 0$ , 即得到导数的定义. 那么我们的问题就转化为了

$$\lim_{\substack{|I| \rightarrow 0 \\ I \ni x}} \frac{1}{|I|} \int_I f(y)dy = f(x) \text{ holds for which } x? \quad (4.2)$$

更一般地, 将上述问题从一维实直线  $\mathbb{R}$  推广至  $\mathbb{R}^d$  空间上, 将区间  $I$  用开球  $B$  替换, 得

$$\lim_{\substack{m(B) \rightarrow 0 \\ B \ni x}} \frac{1}{m(B)} \int_B f(y)dy = f(x) \text{ holds for which } x? \quad (4.3)$$

**注.** – 此处看似是随着开球  $B$  的测度减小,  $x \in B$  在跟着  $B$  “跑”, 但实际上则相反:

对于每个固定的  $x$ , 让包含着  $x$  的球  $B \ni x$  不断减小其测度, 最后取极限

而这也正是此处极限条件写为 “ $B \ni x$ ” 而非 “ $x \in B$ ” 的原因, 逻辑更清晰.

- 事实上该结论对于几乎处处的  $x$  都成立 (若  $f$  **Lebesgue** 可积), 这就是后面要讲的 **Lebesgue** 微分定理.

## 4.1 Hardy – Littlewood 极大函数 (非球心)

定义 下面我们给出 **Hardy-Littlewood** 极大函数的定义.

定义 4.1.1. If  $f \in \mathcal{L}^1(\mathbb{R}^d)$ , we define its **maximal function**  $Mf$  by

$$Mf(x) = \sup_{B \ni x} \frac{1}{m(B)} \int_B |f(y)| dy \quad (4.4)$$

**注.** 我们目前并不知道球面测度的具体数值与计算方法, 但事实上我们也并不需要知道其具体数值, 具体表现在:

设  $\mathbb{R}^d$  中单位球  $B(0, 1)$  的测度为  $m(B(0, 1)) = v_d$ .  $\forall B(x, r) \subset \mathbb{R}^d$ , 根据 **Lebesgue** 测度的平移不变性和伸缩变换公式 (**Prop 3.7.1**)

$$B(0, r) = rB(0, 1) \Rightarrow m(B(x, r)) = m(B(0, r)) = r^d m(B(0, 1)) = r^d v_d$$

性质 下面来说明 **Hardy-Littlewood** 极大函数的三条性质.

命题 4.1.1. Suppose  $f \in \mathcal{L}^1(\mathbb{R}^d)$ . Then:

- (i)  $Mf$  is measurable.
- (ii)  $Mf(x) < \infty$  for a.e.  $x$ .
- (iii) **weak-type inequality.**

$Mf$  satisfies

$$m(\{x \in \mathbb{R}^d \mid Mf(x) > a\}) \leq \frac{A}{a} \|f\|_{\mathcal{L}^1}, \quad \forall a > 0 \quad (4.5)$$

where  $A = 3^d$ .

证明.

- (i) Let  $E_a = \{x \in \mathbb{R}^d \mid Mf(x) > a\}$ . 下面证明  $E_a$  open.

$\forall x \in E_a$ , by the **definition of  $Mf$**  (**Def 4.1.1**),  $\exists B_x \ni x$ , s. t.

$$\frac{1}{m(B_x)} \int_{B_x} |f| > a \quad (4.6)$$

Then  $\forall y \in B_x$ ,  $B_x$  is also an open ball containing  $y$ , so we have  $y \in E_a$ . i.e.  $B_x \subset E_a$ .

Therefore,  $E_a$  is open, specifically measurable for all  $a$ . Then  $Mf$  is measurable.



(ii) 下面说明 (iii)  $\Rightarrow$  (ii):

Let  $E_a = \{x \in \mathbb{R}^d \mid Mf(x) > a\}$ . Then  $E_n \searrow E = \{x \in \mathbb{R}^d \mid Mf(x) = \infty\}$ .

Since  $f \in \mathcal{L}^1(\mathbb{R}^d)$ ,  $\|f\|_{\mathcal{L}^1}$  is finite. Then by (iii),  $m(E_1) < \infty$ .

Then by **Thm 1.3.3**,

$$m(E) = \lim_{n \rightarrow \infty} m(E_n) \leq \lim_{n \rightarrow \infty} \frac{A}{n} \|f\|_{\mathcal{L}^1} = 0 \quad (4.7)$$

Therefore  $m(E) = 0$ . i.e.  $Mf(x) < \infty$  for a.e.  $x$ .

(iii) 在证明 (iii) 之前, 先来介绍 **Vitali 覆盖引理**.

**引理 4.1.1. Vitali Covering Lemma (Elementary Version).**

Suppose  $\mathcal{B} = \{B_1, B_2, \dots, B_N\}$ ,  $B_i \subset \mathbb{R}^d$  are open balls, then there is a disjoint subcollection  $B_{i_1}, \dots, B_{i_k}$  that satisfies

$$m\left(\bigcup_{l=1}^N B_l\right) \leq 3^d \sum_{j=1}^k m(B_{i_j}) \quad (4.8)$$

**注.** 这是 **Vitali 覆盖引理**的初等版本 (有限版本), 更一般的版本是对一列球结论成立.

**证明.** 详见视频(非球心)Hardy-Littlewood 极大函数 23:10 (类似贪心算法的迭代步骤)  $\square$

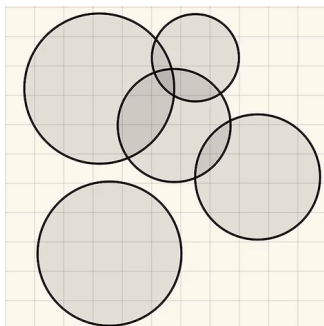


图 4.1: Lemma 4.1.1

下面继续来证明 (iii):

Fix  $a > 0$ ,  $\forall x \in B_a$ ,  $\exists$  open ball  $B_x$ , s. t.

$$\frac{1}{m(B_x)} \int_{B_x} |f| > a \quad (4.9)$$

So we have  $E_a \subset \bigcup_{x \in E_a} B_x$ .

Since  $E_a$  is measurable (by (i)), then by **Thm 1.3.4 (Lebesgue 测度的内正则性)**,

$\forall \epsilon > 0$ ,  $\exists$  compact  $K_\epsilon \subset E_a$ , s. t.

$$m(E_a \setminus K_\epsilon) \leq \epsilon$$

i.e.

$$m(E_a) - m(K_\epsilon) \leq \epsilon$$

Since  $K_\epsilon$  is compact,  $K_\epsilon \subset \bigcup_{x \in K_\epsilon} B_x$ , there exists a subcollection  $B_{x_1}, \dots, B_{x_N}$ , s. t.

$$K_\epsilon \subset \bigcup_{l=1}^N B_{x_l}$$

Then by **Vitali Covering Lemma (Lemma 4.1.1)**, there exists a subcollection  $B_{x_{i_1}}, \dots, B_{x_{i_k}}$ , s. t.

$$m\left(\bigcup_{l=1}^N B_{x_l}\right) \leq 3^d \sum_{j=1}^k m(B_{x_{i_j}})$$

Therefore

$$m(K_\epsilon) \leq m\left(\bigcup_{l=1}^N B_{x_l}\right) \leq 3^d \sum_{j=1}^k m(B_{x_{i_j}}) \quad (4.10)$$

$$= \frac{3^d}{a} \sum_{j=1}^k a \cdot m(B_{x_{i_j}}) \quad (4.11)$$

$$\leq \frac{3^d}{a} \int_{\bigcup_{j=1}^k B_{x_{i_j}}} |f| \quad (4.12)$$

$$\leq \frac{3^d}{a} \int_{\mathbb{R}^d} |f| \quad (4.13)$$

$$= \frac{3^d}{a} \|f\|_{\mathcal{L}^1} \quad (4.14)$$

Then

$$m(E_a) \leq m(K_\epsilon) + \epsilon \leq \frac{A}{a} \|f\|_{\mathcal{L}^1} + \epsilon \quad (4.15)$$

where  $A = 3^d$ ,  $\epsilon > 0$ .

Since  $\epsilon$  is arbitrary, let  $\epsilon \rightarrow 0$ , we have

$$m(E_a) \leq \frac{A}{a} \|f\|_{\mathcal{L}^1}, \quad A = 3^d, \quad \forall a > 0 \quad (4.16)$$

□

## 4.2 Lebesgue 微分定理 (非球心)

在这一节我们将利用 **Hardy-Littlewood** 极大函数来证明 **Lebesgue** 微分定理.

### 4.2.1 Chebyshev's Inequality

在此之前, 我们先来证明一个非常有用的不等式, 即切比雪夫不等式.

**定理 4.2.1. Chebyshev's Inequality.**

If  $g \in \mathcal{L}^1(\mathbb{R}^d)$ , then

$$m(\{x \in \mathbb{R}^d \mid |g(x)| > a\}) \leq \frac{1}{a} \|g\|_{\mathcal{L}^1}, \quad \forall a > 0 \quad (4.17)$$

**证明.** Let  $E_a = \{x \in \mathbb{R}^d \mid |g(x)| > a\}$ . Then

$$\|g\|_{\mathcal{L}^1} = \int_{\mathbb{R}^d} |g| \geq \int_{E_a} |g| \geq \int_{E_a} a = a \cdot m(E_a) \quad (4.18)$$

□

### 4.2.2 The Lebesgue Differentiation Theorem

下面我们就来给出 **Lebesgue** 微分定理.

**定理 4.2.2.** If  $f \in \mathcal{L}^1(\mathbb{R}^d)$ , then

$$\lim_{\substack{m(B) \rightarrow 0 \\ B \ni x}} \frac{1}{m(B)} \int_B f(y) dy = f(x) \text{ for a.e. } x \quad (4.19)$$

**注.** • **Lebesgue** 微分定理说明了对于几乎处处的  $x$ , 当包含  $x$  的球体  $B$  的测度趋于 0 时,  $f$  在球体  $B$  上积分的平均值就会收敛到  $f(x)$ .

- 定理左侧实际上是关于集合  $B$  的函数的一个极限过程, 用  $\epsilon - \delta$  语言叙述如下:

$\forall \epsilon > 0, \exists \delta > 0$ , s. t. for all  $B \ni x$  and  $m(B) < \delta$ , we have

$$\left| \frac{1}{m(B)} \int_B f(y) dy - f(x) \right| \leq \epsilon \quad (4.20)$$

- 要证明该定理, 首先需要说明等式左侧极限的存在性, 但这并不好说明. 为了跳过说明其存在性的问题, 我们需要引入类似“上极限”的函数, 即:

If suffices to show

$$\lim_{\delta \rightarrow 0} \sup_{\substack{m(B) < \delta \\ B \ni x}} \left| \frac{1}{m(B)} \int_B f(y) dy - f(x) \right| = 0 \text{ for a.e. } x \quad (4.21)$$

由于极限内的函数随着  $\delta$  递减而单调递减, 又存在下界 0, 因此在  $\delta = 0$  处必存在右极限. 这样就跳过了原极限是否存在的问题.

- 事实上此处极限“怪异”的本质原因在于开球  $B$  的选取的任意性, 若将其定义为以  $x$  为球心,  $r$  为半径的球, 则可直接令  $r \rightarrow 0$  变为正常的函数极限, 即

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) dy = f(x) \quad (4.22)$$

在下一节我们会从 **Hardy-Littlewood** 极大函数开始, 以此方法重新说明 **Lebesgue** 微分定理.

证明. Let

$$E_a = \left\{ x \in \mathbb{R}^d \mid \lim_{\delta \rightarrow 0} \sup_{\substack{m(B) < \delta \\ B \ni x}} \left| \frac{1}{m(B)} \int_B f(y) dy - f(x) \right| > 2a \right\} \quad (4.23)$$

Then we **WTS (want to show)**:

$$m(E_a) = 0, \forall a \geq 0$$

Fix  $a \geq 0$ . By **Thm 3.2.3**,  $C_c(\mathbb{R}^d)$  is dense in  $\mathcal{L}^1(\mathbb{R}^d)$  (有紧支集连续函数), then  $\forall \epsilon > 0, \exists g \in C_c(\mathbb{R}^d)$ , s. t.

$$\|f - g\|_{\mathcal{L}^1} < \epsilon$$

Since  $g$  is uniformly continuous, then  $\exists \delta > 0$ , s. t.

$$\left| \frac{1}{m(B)} \int_B g(y) dy - g(x) \right| \leq \frac{1}{m(B)} \int_B |g(y) - g(x)| dy < \frac{1}{m(B)} \int_B \epsilon dy = \epsilon \quad (4.24)$$

for all  $B \ni x$  and  $m(B) < \delta$ .

下面对  $m(E_a)$  进行估计.  $\forall x \in E_a$ ,

$$\left| \frac{1}{m(B)} \int_B f(y) dy - f(x) \right| \leq \left| \frac{1}{m(B)} \int_B (f(y) - g(y)) dy \right| + \left| \frac{1}{m(B)} \int_B g(y) dy - g(x) \right| + |g(x) - f(x)| \quad (4.25)$$

对上述不等式中的开球  $B \ni x$  取上确界  $\sup$ , 得

$$\sup_{B \ni x} \left| \frac{1}{m(B)} \int_B f(y) dy - f(x) \right| \leq \sup_{B \ni x} \left| \frac{1}{m(B)} \int_B (f(y) - g(y)) dy \right| + \sup_{B \ni x} \left| \frac{1}{m(B)} \int_B g(y) dy - g(x) \right| + |g(x) - f(x)| \quad (4.26)$$

再令  $m(B) \rightarrow 0$ , 由于根据式 (4.24),

$$\lim_{\delta \rightarrow 0} \sup_{\substack{m(B) < \delta \\ B \ni x}} \left| \frac{1}{m(B)} \int_B g(y) dy - g(x) \right| = 0 \quad (4.27)$$

因此

$$\lim_{\delta \rightarrow 0} \sup_{\substack{m(B) < \delta \\ B \ni x}} \left| \frac{1}{m(B)} \int_B f(y) dy - f(x) \right| \leq \lim_{\delta \rightarrow 0} \sup_{\substack{m(B) < \delta \\ B \ni x}} \left| \frac{1}{m(B)} \int_B (f(y) - g(y)) dy \right| + 0 + |g(x) - f(x)| \quad (4.28)$$

下面对红色部分进行估计. 根据对  $\delta$  的单调性可知,

$$\lim_{\delta \rightarrow 0} \sup_{\substack{m(B) < \delta \\ B \ni x}} \left| \frac{1}{m(B)} \int_B (f(y) - g(y)) dy \right| \leq \sup_{B \ni x} \frac{1}{m(B)} \int_B |f(y) - g(y)| dy = M(f - g)(x) \quad (4.29)$$

又因为对于  $\forall x \in E_a$ ,

$$\lim_{\delta \rightarrow 0} \sup_{\substack{m(B) < \delta \\ B \ni x}} \left| \frac{1}{m(B)} \int_B f(y) dy - f(x) \right| > 2a \quad (4.30)$$

所以

$$M(f - g)(x) + |g(x) - f(x)| > 2a \quad (4.31)$$

$$\Rightarrow M(f - g)(x) > a \text{ or } |g(x) - f(x)| > a \quad (4.32)$$

$$\Rightarrow E_a \subset \{x \in \mathbb{R}^d \mid M(f - g)(x) > a\} \cup \{x \in \mathbb{R}^d \mid |g - f| > a\} \quad (4.33)$$

下面分别来估计 the purple one 和 the orange one 的测度.

- 由于  $|f - g| \in \mathcal{L}^1 \mathbb{R}^d$ , 因此根据 **Chebyshev's Inequality (Thm 4.2.1)**,

$$m(\{x \in \mathbb{R}^d \mid |g - f| > a\}) \leq \frac{1}{a} \|f - g\|_{\mathcal{L}^1} \quad (4.34)$$

- 根据 **Hardy-Littlewood 极大函数的 weak-type inequality (Prop 4.1.1 (iii))**,

$$m(\{x \in \mathbb{R}^d \mid M(f - g)(x) > a\}) \leq \frac{A}{a} \|f - g\|_{\mathcal{L}^1} \quad (4.35)$$

从而根据  $\|f - g\|_{\mathcal{L}^1} < \epsilon$ ,

$$m(E_a) \leq m(\{x \in \mathbb{R}^d \mid M(f - g)(x) > a\}) + m(\{x \in \mathbb{R}^d \mid |g - f| > a\}) \quad (4.36)$$

$$\leq \frac{A+1}{a} \|f - g\|_{\mathcal{L}^1} \quad (4.37)$$

$$< \frac{A+1}{a} \epsilon, \quad \forall a \geq 0 \quad (4.38)$$

Since  $\epsilon > 0$  is arbitrary, let  $\epsilon \rightarrow 0$ , we get

$$m(E_a) = 0, \quad \forall a \geq 0 \quad (4.39)$$

□

## 4.3 Hardy – Littlewood 极大函数 & Lebesgue 微分定理 (球心)

### 4.3.1 Hardy – Littlewood 极大函数

本节的重点是给出 **Hardy-Littlewood 极大函数 (centered)** 的定义并证明其连续性.

**Preliminaries** 在此之前, 先来给出一些记号与命题.

We define 开球 & 球面

$$B(x, r) := \{y \in \mathbb{R}^d \mid |y - x| < r\}$$

$$S(x, r) := \{y \in \mathbb{R}^d \mid |y - x| = r\}$$

下面说明球面为零测集.

**命题 4.3.1.**  $m(S(x, r)) = 0, \forall x \in \mathbb{R}^d, r \geq 0.$

**证明.** 根据 **Lebesgue 测度的平移不变性和伸缩变换公式 (Prop 3.7.1)**, it suffices to show

$$m(S(0, 1)) = 0$$

反证法. Suppose  $m(S(0, 1)) > 0$ . By **Prop 3.7.1**,

$$m(rS(0, 1)) = r^d m(S(0, 1)) \geq m(S(0, 1)), \forall r \geq 1.$$

Consider the compact set  $\{x \in \mathbb{R}^d \mid 1 \leq |x| \leq 2\}$ . We have

$$\bigcup_{k=1}^{\infty} S(0, 1 + \frac{1}{k}) = \bigcup_{k=1}^{\infty} (1 + \frac{1}{k})S(0, 1) \subset \{x \in \mathbb{R}^d \mid 1 \leq |x| \leq 2\} \quad (4.40)$$

However

$$m\left(\bigcup_{k=1}^{\infty} S(0, 1 + \frac{1}{k})\right) = \sum_{k=1}^{\infty} m\left(S(0, 1 + \frac{1}{k})\right) \geq \sum_{k=1}^{\infty} m(S(0, 1)) = \infty \quad (4.41)$$

which is a contradiction for  $\{1 \leq |x| \leq 2\}$  is compact.  $\square$

下面我们给出当球心收敛时，开球的特征函数的收敛性.

**命题 4.3.2.**  $\forall (x_j, r_j) \rightarrow (x, r)$  on  $\mathbb{R}^d \times \mathbb{R}$ ,

$$\chi_{B(x_j, r_j)}(y) \rightarrow \chi_{B(x, r)}(y) \text{ on } \mathbb{R}^d \setminus S(x, r)$$

**注.** 该命题在开球  $B(x_j, r_j)$  的边界，即  $S(x, r)$  上不一定成立. 下面给出一个反例.

**例 4.3.1.** In  $\mathbb{R}$ , take  $x_j = 0, r_j = 1 + \frac{1}{j+1}$ . Then  $(x_j, r_j) \rightarrow (0, 1)$ .

$$\chi_{B(x_j, r_j)} = \chi_{(-1-\frac{1}{j+1}, 1+\frac{1}{j+1})} \rightarrow \chi_{[-1, 1]}$$

and  $\chi_{[-1, 1]}(x) \neq \chi_{(-1, 1)}(x)$ , for  $x = -1$  or  $1$ .

**证明.** 下面分别对  $|y| < r$  与  $|y - x| > r$  两种情况进行讨论.

(i)  $|y - x| < r$ . WTS

$$|x_j - y| < r_j, \forall j > N \text{ for some } N$$

Since  $|x_j - y| < |x_j - x| + |x - y|$ , it suffices to show

$$|x_j - x| + |x - y| < r_j$$

Suppose  $|x - y| = r - \epsilon, \epsilon > 0$ . Then

$$\Leftrightarrow |x_j - x| + r - \epsilon < r_j$$

$$\Leftrightarrow |x_j - x| + r - r_j < \epsilon$$

It suffices to show

$$|x_j - x| + |r_j - r| < \epsilon$$

Since  $(x_j, r_j) \rightarrow (x, r)$ , then  $\exists N \in \mathbb{N}$ , s. t.

$$|x_j - x| < \frac{\epsilon}{3}, |r_j - r| < \frac{\epsilon}{3}, \forall j > N$$

Then  $|x_j - y| < r_j, \forall j > N$ .

(ii)  $|y - x| > r$ . 同理 Suppose  $|x - y| = r + \epsilon, \epsilon > 0$ . WTS  $|x_j - y| > r_j, \forall j > N$  for some  $N$ .

It suffices to show

$$|x_j - y| > |x - y| - |x_j - x| = r + \epsilon - |x_j - x| > r_j$$

$$\Leftrightarrow |x_j - x| + r_j - r < \epsilon$$

□



**Average Value of  $f$  on  $B(x, r)$**  在说明  $f$  的连续性之前, 先来说明去掉  $\sup$  的函数的连续性.

定义 4.3.1. We define the average value of  $f$  on  $B(x, r)$

$$A_rf(x) = \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) dy \quad (4.42)$$

下面给出局部可积的概念.

定义 4.3.2. A measurable function  $f : \mathbb{R}^d \rightarrow \mathbb{C}$  is called locally integrable if

$$\int_K |f| < \infty \text{ for every bounded measurable set } K \subset \mathbb{R}^d \quad (4.43)$$

We write  $f \in \mathcal{L}_{loc}^1(\mathbb{R}^d)$ .

例 4.3.2.  $f(x) = e^x$  is locally integrable but not integrable on  $\mathbb{R}$ .

下面给出  $A_rf(x)$  的连续性.

引理 4.3.1. If  $f \in \mathcal{L}_{loc}^1(\mathbb{R}^d)$ , then

$A_rf(x)$  is jointly continuous in  $r$  and  $x$  ( $r > 0, x \in \mathbb{R}^d$ ).

证明. 下面分为两步进行证明.

- $\int_{B(x, r)} f(y) dy$  is continuous.

$$\int_{B(0, r)} f(y) dy = \int_{\mathbb{R}^d} f(y) \chi_{B(x, r)}(y) dy \quad (4.44)$$

Fix  $(x, r) \in \mathbb{R}^d \times \mathbb{R}$ ,  $\forall (x_j, r_j) \rightarrow (x, r)$ , by **Prop 4.3.2**,

$$\chi_{B(x_j, r_j)}(y) \rightarrow \chi_{B(x, r)}(y) \text{ on } \mathbb{R}^d \setminus S(x, r)$$

Since by **Prop 4.3.1**,  $m(S(x, r)) = 0$ , we have

$$\begin{aligned} \chi_{B(x_j, r_j)}(y) &\rightarrow \chi_{B(x, r)}(y) \text{ for a.e. } y \\ \Rightarrow f(y) \chi_{B(x_j, r_j)}(y) &\rightarrow f(y) \chi_{B(x, r)}(y) \text{ for a.e. } y \end{aligned}$$

Since  $(x_j, r_j) \rightarrow (x, r)$ ,  $\exists N$ , s. t.  $B(x_j, r_j) \subset B(x, 100r)$ ,  $\forall j > N$ . Then

$$|f(y)\chi_{B(x_j, r_j)}(y)| \leq |f(y)\chi_{B(x, 100r)}(y)| \in \mathcal{L}^1 \text{ for a.e. } y$$

Therefore, by **DCT (Thm 3.1.7, 控制收敛定理)**

$$\int_{\mathbb{R}^d} f(y)\chi_{B(x_j, r_j)}(y)dy \rightarrow \int_{\mathbb{R}^d} f(y)\chi_{B(x, r)}(y)dy \quad (4.45)$$

i.e.

$$\int_{B(x_j, r_j)} f(y)dy \rightarrow \int_{B(x, r)} f(y)dy, \quad \forall (x_j, r_j) \rightarrow (x, r) \quad (4.46)$$

Then by **Heine 归结原理**,  $\int_{B(x, r)} f(y)dy$  is continuous.

- $A_r f(x)$  is continuous. Since  $m(B(x, r)) = r^d m(B(0, 1)) = r^d v_d$ , then

$$A_r f(x) = v_d^{-1} r^{-d} \int_{B(x, r)} f(y)dy \text{ is continuous} \quad (4.47)$$

□

**Hardy-Littlewood 极大函数** 下面先给出 **Hardy-Littlewood 极大函数 (球心)** 的定义.

定义 4.3.3. If  $f \in \mathcal{L}_{loc}^1$ , we define its Hardy-Littlewood maximal function  $Hf$  by

$$Hf(x) = \sup_{r>0} A_r |f|(x) = \sup_{r>0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y)| dy \quad (4.48)$$

下面说明 **Hardy-Littlewood 极大函数** 的连续性.

推论 4.3.2.  $Hf$  is continuous.

证明.  $\forall (a, \infty) \subset \mathbb{R}$ , by **Lemma 4.3.1**,  $A_r |f|$  is continuous, then

$$(Hf)^{-1}((a, \infty)) = \bigcup_{r>0} (A_r |f|)^{-1}((a, \infty)) \text{ is open, } \forall a \in \mathbb{R} \quad (4.49)$$

Therefore  $Hf$  is continuous. □

此版本的 **Hardy-Littlewood 极大函数** 同样有 **weak-type inequality**.

**命题 4.3.3. weak-type inequality.**

If  $f \in \mathcal{L}^1$ , then

$$m(\{x \in \mathbb{R}^d \mid Hf(x) > a\}) \leq \frac{A}{a} \|f\|_{\mathcal{L}^1}, \quad \forall a > 0 \quad (4.50)$$

where  $A = 3^d$ .

证明. 与命题 4.1.1 (iii) 证明类似. □

### 4.3.2 Lebesgue 微分定理

函数的上极限 首先来回顾一下函数的上极限的定义.

定义 4.3.4.  $\forall$  函数  $f: E \subset \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $x_0$  为  $E$  的聚点, 定义  $f$  在  $x_0$  的上极限 为

$$\limsup_{x \rightarrow x_0} f(x) = \lim_{\delta \rightarrow 0^+} \sup_{0 < |x - x_0| < \delta} f(x) \quad (4.51)$$

同理可定义函数  $f$  在  $x_0$  点的下极限.

**注.** 不难证明, 该定义与通常数学分析书<sup>1</sup>上的定义等价, 即

$$\limsup_{x \rightarrow x_0} f(x) = \sup \{l \in \mathbb{R} \mid \exists \{x_n\}_{n=1}^{\infty} \subset E, x_n \rightarrow x_0, \text{ s. t. } f(x_n) \rightarrow l\} \quad (4.52)$$

下面利用函数的上极限给出函数极限的等价定义.

命题 4.3.4.  $\forall$  函数  $f: E \subset \mathbb{R}^d \rightarrow \mathbb{R}$ ,  $x_0$  为  $E$  的聚点, 则

$$\lim_{x \rightarrow x_0} f(x) = c \Leftrightarrow \limsup_{x \rightarrow x_0} |f(x) - c| = 0$$

**证明.** 此处证明采用数学分析中的定义更方便. 根据 **Heine** 归结原理,

$$\lim_{x \rightarrow x_0} f(x) = c \Leftrightarrow \forall \{x_j\}_{j=1}^{\infty}, x_j \rightarrow x_0, \text{ s. t. } f(x_j) \rightarrow c \quad (4.53)$$

$$\Leftrightarrow E = \{l \in \mathbb{R} \mid \exists \{x_n\}_{n=1}^{\infty} \subset E, x_n \rightarrow x_0, \text{ s. t. } |f(x_n) - c| \rightarrow l\} = \{0\} \quad (4.54)$$

$$\Leftrightarrow \limsup_{x \rightarrow x_0} |f(x) - c| = \sup E = 0 \quad (4.55)$$

□

<sup>1</sup>此处参考书籍:《数学分析教程(上册)(第一版)》——常庚哲、史济怀编 § 2.11 定义 2.22

**Lebesgue 微分定理** 下面给出 **Lebesgue 微分定理 (球心)**.

**定理 4.3.3. Lebesgue Differentiation Theorem.**

If  $f \in \mathcal{L}_{loc}^1$ , then

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) dy = f(x) \text{ for a.e. } x \in \mathbb{R}^d \quad (4.56)$$

**证明.** Since  $f \in \mathcal{L}_{loc}^1$ , then for all  $x \in \mathbb{R}^d$ ,  $\exists N$ , s. t.  $|x| < N$  and

$$A_r f(x) = \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) dy \text{ depends only on } f(y) \text{ for } |y| \leq N + 1. \quad (4.57)$$

(with  $r \leq 1$  and  $|x| < N$ )

Then we can replace  $f$  with  $f\chi_{B(0, N+1)} \in \mathcal{L}^1$ . 于是我们不妨设  $f \in \mathcal{L}^1$ . 根据 **Prop 4.3.4**, 要证

$$\lim_{r \rightarrow 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) dy = \lim_{r \rightarrow 0} A_r f(x) = f(x) \text{ for a.e. } x \in \mathbb{R}^d \quad (4.58)$$

即证

$$\limsup_{r \rightarrow 0} |A_r f(x) - f(x)| = 0 \text{ for a.e. } x \in \mathbb{R}^d \quad (4.59)$$

下面来估计  $\limsup_{r \rightarrow 0} |A_r f(x) - f(x)|$ .

Fix  $\epsilon > 0$ , 根据 **Thm 3.2.3**,  $\exists g \in C_c(\mathbb{R}^d)$ , s. t.

$$\|f - g\|_{\mathcal{L}^1} < \epsilon$$

Since  $A_r g(x)$  is continuous (by **Lemma 4.3.1**), we have

$$|A_r g(x) - g(x)| = |A_r g(x) - A_0 g(x)| < \epsilon \text{ for all small } r.$$

Then

$$\limsup_{r \rightarrow 0} |A_r f(x) - f(x)| = \limsup_{r \rightarrow 0} |A_r(f - g)(x) + (A_r g - g)(x) + (g - f)(x)| \quad (4.60)$$

$$\leq \lim_{\delta \rightarrow 0^+} \sup_{0 < r < \delta} |A_r(f - g)(x)| + 0 + |g(x) - f(x)| \quad (4.61)$$

Since

$$\sup_{0 < r < \delta} |A_r(f - g)(x)| \leq \sup_{r > 0} |A_r(f - g)(x)| = \sup_{r > 0} \left| \frac{1}{m(B(x, r))} \int_{B(x, r)} f(y) - g(y) dy \right| \quad (4.62)$$

$$\leq \sup_{r > 0} \frac{1}{m(B(x, r))} \int_{B(x, r)} |f(y) - g(y)| dy \quad (4.63)$$

$$= H(f - g)(x), \quad \forall \delta > 0 \quad (4.64)$$

Let  $\delta \rightarrow 0^+$ , we have

$$\limsup_{r \rightarrow 0} |A_r f(x) - f(x)| \leq \lim_{\delta \rightarrow 0^+} \sup_{0 < r < \delta} |A_r(f - g)(x)| + |g(x) - f(x)| \quad (4.65)$$

$$\leq H(f - g)(x) + |g(x) - f(x)| \quad (4.66)$$

后续证明与 **Thm 4.2.2** 一致, 即定义  $E_a$ , 并分别对  $H(f - g)(x)$  与  $|g - f|$  运用 **weak-type inequality (Prop 4.3.3)** 与 **Chebyshev's Inequality (Thm 4.2.1)**, 即可证明  $m(E_a) = 0, \forall a \geq 0$ . 从而得证.  $\square$

## 4.4 有界变差函数

引入 在 **Riemann** 积分中, 我们知道对于一阶连续可微函数, 我们有微积分基本定理

$$F(b) - F(a) = \int_a^b F'(x)dx \quad (4.67)$$

而对于 **Lebesgue** 积分, 我们也想要得到该命题成立的条件, 且最好为**充要条件**. 可以举例证明, 仅仅  $F$  连续并不能保证  $F$  可导 (可见视频 [a continuous but nowhere differentiable function](#)). 同时仅仅要求  $F$  导数存在也可能出现  $F'$  不可积的情况, 如下反例.

**例 4.4.1.** (书<sup>2</sup> P147 Ex 12).

Consider the function  $F(x) = x^2 \sin \frac{1}{x^2}$ ,  $x \neq 0$ , with  $F(0) = 0$ . Show that  $F'(x)$  exists for every  $x$ , but  $F'$  is not integrable on  $[-1, 1]$ .

**证明.** 详细证明可见视频 [微积分基本定理: 问题引入](#).

□

为了解决上述问题, 我们在这一节将引入一种新类型的函数, 叫做**有界变差函数**.

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<sup>2</sup> 《Real Analysis – Measure Theory, Integration, & Hilbert Spaces》— Elias M. Stein

### 4.4.1 有界变差函数的概念

引入 为了便于理解，我们将有界变差函数与平面上的曲线相联系. 首先来回顾数学分析中有关平面上的曲线的相关概念.

定义 4.4.1. Let  $\gamma$  be a parametrized curve in the plane given by  $z(t) = (x(t), y(t))$ , where  $a \leq t \leq b$ . Here  $x(t)$  and  $y(t)$  are continuous real-valued functions on  $[a, b]$ .

The curve is rectifiable if  $\exists M < \infty$ , s. t. for any partition  $a = t_0 < t_1 < \cdots < t_N = b$  of  $[a, b]$ ,

$$\sum_{j=1}^N |z(t_j) - z(t_{j-1})| \leq M \quad (4.68)$$

The Length  $L(\gamma)$  of the curve is defined as

$$L(\gamma) = \sup_{\text{all partitions}} \sum_{j=1}^N |z(t_j) - z(t_{j-1})| \quad (4.69)$$

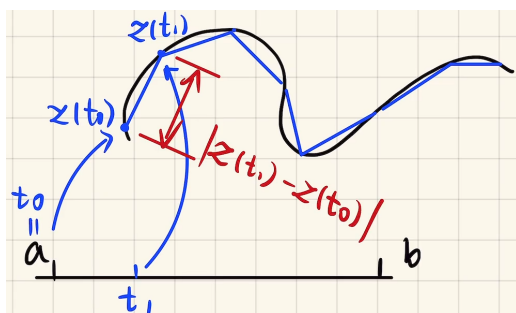


图 4.2: rectifiable curve

**注.** 为了表述的方便，后续我们常将  $z$  的值域视作复平面，即  $z(t) = x(t) + iy(t)$ .

在定义了曲线可求长的概念后，我们自然要问，在什么情况下曲线可求长？即

What condition on  $x(t)$  and  $y(t)$  guarantees rectifiability of  $\gamma$ ?

为了解决这个问题，下面我们给出有界变差函数的定义，并会在后续给出这个问题的充要条件，即

$\gamma$  rectifiable  $\Leftrightarrow x(t), y(t)$  均为有界变差函数.



定义 下面给出有界变差函数的定义.

定义 4.4.2. Suppose  $F : [a, b] \rightarrow \mathbb{C}$ , and  $\mathcal{P}$  is a partition  $a = t_0 < t_1 < \cdots < t_N = b$ . The variation of  $F$  on this partition  $\mathcal{P}$  is defined by

$$V_{\mathcal{P}}(F) = \sum_{j=1}^N |F(t_j) - F(t_{j-1})| \quad (4.70)$$

$F$  is said to be of bounded variation (BV) if  $V_{\mathcal{P}}(F)$  is bounded over all partitions.

**注.** 有界变差函数不要求函数连续, 而在考虑平面曲线时默认函数连续.

下面就能够来回答“引入”中提到的问题, 即平面曲线可求长的充要条件.

定理 4.4.1. A curve parametrized by  $F(t) = x(t) + iy(t)$ ,  $a \leq t \leq b$ , is rectifiable  $\Leftrightarrow$  both  $x(t)$  and  $y(t)$  are of BV.

证明.  $\forall$  partition  $\mathcal{P}$ :  $a = t_0 < t_1 < \cdots < t_N = b$ , we have

$$V_{\mathcal{P}}(F) = \sum_{j=1}^N |F(t_j) - F(t_{j-1})| \quad (4.71)$$

Since  $|a + bi| \leq |a| + |b| \leq 2|a + bi|$ ,

$$|F(t_j) - F(t_{j-1})| = |x(t_j) - x(t_{j-1}) + i(y(t_j) - y(t_{j-1}))| \quad (4.72)$$

Then

- $\Leftarrow$ :  $|F(t_j) - F(t_{j-1})| \leq |x(t_j) - x(t_{j-1})| + |y(t_j) - y(t_{j-1})|$ , then  $\exists M < \infty$ , s. t.

$$V_{\mathcal{P}}(F) = \sum_{j=1}^N |F(t_j) - F(t_{j-1})| \quad (4.73)$$

$$= \sum_{j=1}^N |x(t_j) - x(t_{j-1})| + \sum_{j=1}^N |y(t_j) - y(t_{j-1})| \quad (4.74)$$

$$\leq 2M, \quad \forall \text{partition } \mathcal{P} \quad (4.75)$$

Therefore, the curve  $F(t) = x(t) + iy(t)$  is rectifiable.

- $\Rightarrow$ :  $|x(t_j) - x(t_{j-1})| + |y(t_j) - y(t_{j-1})| \leq 2|F(t_j) - F(t_{j-1})|$ , then  $\exists M < \infty$ , s. t.

$$\sum_{j=1}^N |x(t_j) - x(t_{j-1})| + \sum_{j=1}^N |y(t_j) - y(t_{j-1})| \leq 2 \sum_{j=1}^N |F(t_j) - F(t_{j-1})| = 2V_{\mathcal{P}}(F) \leq 2M \quad (4.76)$$

Therefore, both  $x(t)$  and  $y(t)$  are of BV.

□

例子 下面来给出一些有界变差函数的例子.

例 4.4.2. •  $x, x^2$  is of BV on  $[a, b]$ ,  $\forall [a, b] \subset \mathbb{R}$ .

证明.  $\forall$  partition  $\mathcal{P}$ :  $a = x_0 < x_1 < \cdots < x_N = b$ , since  $x$  is strictly increasing, then

$$V_{\mathcal{P}}(x) = \sum_{j=1}^N |x_j - x_{j-1}| = \sum_{j=1}^N x_j - x_{j-1} = b - a < \infty \quad (4.77)$$

Also for  $x^2$ ,

$$V_{\mathcal{P}}(x^2) = \sum_{j=1}^N |x_j^2 - x_{j-1}^2| = \sum_{j=1}^N |x_j + x_{j-1}| |x_j - x_{j-1}| \leq 2b \sum_{j=1}^N |x_j - x_{j-1}| = 2b(b - a) < \infty \quad (4.78)$$

Therefore, both  $x$  and  $x^2$  are of BV on  $[a, b]$ ,  $\forall [a, b] \subset \mathbb{R}$ . □

- If  $F$  is real-valued, monotonic, and bounded, then  $F$  is of BV.

证明.  $\forall$  partition  $\mathcal{P}$ :  $a = t_0 < t_1 < \cdots < t_N = b$ .

Since  $F$  is bounded,  $\exists M < \infty$ , s. t.  $|F| \leq M$ . 不妨设  $F$  单调递增,

$$V_{\mathcal{P}}(F) = \sum_{j=1}^N |F(t_j) - F(t_{j-1})| = F(b) - F(a) \leq 2M, \quad \forall \text{partition } \mathcal{P} \quad (4.79)$$

So  $F$  is of BV. □

• (书<sup>3</sup> P147 Ex 11).

If  $a, b > 0$ , let

$$f(x) = \begin{cases} x^a \sin(x^{-b}), & 0 \leq x \leq 1 \\ 0, & x = 0 \end{cases} \quad (4.80)$$

Then

$$f \text{ is of BV in } [0, 1] \Leftrightarrow a > b$$

证明.

– 先来考虑简单情形, 即  $f(x) = \sin \frac{1}{x}$ .

根据直觉, 随着  $x \rightarrow 0^+$ ,  $\sin \frac{1}{x}$  的震荡越剧烈, 当分划足够密时, 其变差中应当会出现各项为 1 的无穷级数, 从而发散. 下面取一个特殊分划进行证明.

对于  $\forall$  奇数  $k \in \mathbb{N}$ , 取

$$\frac{1}{x_k} = 2k\pi + \frac{\pi}{2}, \frac{1}{x_{k+1}} = 2k\pi + \pi$$

于是

$$V_{\mathcal{P}}(f) = \sum_{j=1}^N \left| \sin \frac{1}{x_j} - \sin \frac{1}{x_{j-1}} \right| = \sum_{j=1}^N 1 = N, \text{ which is related to } \mathcal{P} \quad (4.81)$$

故  $\forall M < \infty$ , 当分划  $\mathcal{P}$  足够密时,  $V_{\mathcal{P}}(f) > M$ . 故  $f$  非 BV.

– 对于一般情况, 下面先讨论一种特殊分划  $\mathcal{P}$ . 即 (**Monotonic Partition**, 单调划分)

$$\left[ \frac{1}{x_{4k+1}^b}, \frac{1}{x_{4k+2}^b} \right] = \left[ 2k\pi + \frac{\pi}{2}, 2k\pi + \pi \right], \left[ \frac{1}{x_{4k+3}^b}, \frac{1}{x_{4k+4}^b} \right] = \left[ 2k\pi + \frac{3\pi}{2}, 2k\pi + 2\pi \right] \quad (4.82)$$

则以上述第一种的分划  $[x_{4k+1}, x_{4k+2}]$  举例,

$$\left| x_{k+1}^a \sin \frac{1}{x_{k+1}^b} - x_k^a \sin \frac{1}{x_k^b} \right| = \left| x_{k+1}^a \sin \frac{1}{x_{k+1}^b} \right| = \frac{1}{(2k\pi + \frac{\pi}{2})^{\frac{a}{b}}} = O\left(\frac{1}{k^{\frac{a}{b}}}\right) \quad (4.83)$$

同理对于  $[x_{4k+2}, x_{4k+3}]$ ,  $[x_{4k+3}, x_{4k+4}]$ ,  $[x_{4k+4}, x_{4k+5}]$ , 均可得到

$$\left| x_{k+1}^a \sin \frac{1}{x_{k+1}^b} - x_k^a \sin \frac{1}{x_k^b} \right| = O\left(\frac{1}{k^{\frac{a}{b}}}\right) \quad (4.84)$$

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<sup>3</sup> 《Real Analysis – Measure Theory, Integration, & Hilbert Spaces》— Elias M. Stein

于是

$$\sum_{k=0}^N \left| x_{k+1}^a \sin \frac{1}{x_{k+1}^b} - x_k^a \sin \frac{1}{x_k^b} \right| = O\left(\sum_{k=1}^N \frac{1}{k^{\frac{a}{b}}}\right) \quad (4.85)$$

根据 **p**-级数  $\sum_n \frac{1}{n^p}$  的收敛性 (收敛  $\Leftrightarrow p > 1$ ) 可得,

$$\sum_{k=0}^N \left| x_{k+1}^a \sin \frac{1}{x_{k+1}^b} - x_k^a \sin \frac{1}{x_k^b} \right| < \infty \quad (4.86)$$

$$\Leftrightarrow \frac{a}{b} > 1 \Leftrightarrow a > b \quad (4.87)$$

由于对于任一分划  $\mathcal{P}$ , 有:

“加密分割, 变差不减”

因此对于  $\forall$  分割, 我们可以在其中加入如上节点, 其变差不减, 但因为上述节点中,  $\sin \frac{1}{x^b}$  在每个区间  $[x_k, x_{k+1}]$  均单调, 所以在各区间  $[x_k, x_{k+1}]$  中的变差可直接去除绝对值, 并得到

$$V_{\mathcal{P}}(f) = O\left(\sum_{k=1}^N \frac{1}{k^{\frac{a}{b}}}\right) \quad (4.88)$$

于是

$$f \text{ is of BV} \Leftrightarrow \frac{a}{b} > 1 \Leftrightarrow a > b$$

□

## 4.4.2 有界变差函数的刻画

介绍 本小节将给出有界变差函数的一个刻画, 即

任一有界变差函数可差分为两个有界递增函数之差.

同时还将研究函数的全变差的性质. 而这一切都是为了后续研究微积分基本定理做准备.

定义 回顾函数的变差的概念 (Def 4.4.2). 在此基础上, 我们下面给出全变差的定义.

定义 4.4.3. Suppose  $f : [a, b] \rightarrow \mathbb{C}$ . The **total variation** of  $f$  on  $[a, x]$  ( $a \leq x \leq b$ ) is defined by

$$V_f([a, x]) = \sup_{\text{all partitions}} \sum_{j=1}^n |f(t_j) - f(t_{j-1})| \quad (4.89)$$

In particular, if  $f$  is real-valued, i.e.  $f : [a, b] \rightarrow \mathbb{R}$ . Then the **positive variation** of  $f$  on  $[a, x]$  is

$$P_f([a, x]) = \sup_{\substack{\text{all partitions} \\ (+)}} \sum f(t_j) - f(t_{j-1}) \quad (4.90)$$

Also the **negative variation** of  $f$  on  $[a, x]$  is

$$N_f([a, x]) = \sup_{\substack{\text{all partitions} \\ (-)}} \sum -[f(t_j) - f(t_{j-1})] \quad (4.91)$$

**注.** • 全变差对任一复值函数均可定义, 而正变差和负变差则只对实值函数有定义. 在后续的讨论中基本默认  $f$  为实值函数.

- 下面对定义中的符号  $(+)$  和  $(-)$  进行说明, 即

$$(+) := \{j \mid f(t_j) \geq f(t_{j-1})\} \quad (4.92)$$

$$(-) := \{j \mid f(t_j) \leq f(t_{j-1})\} \quad (4.93)$$

- 常常将全变差  $V_f([a, b])$  简记为  $V_a^b(f)$ .

**有界变差函数的刻画** 在刻画有界变差函数之前, 先来给出一个引理. 它说明了对于实值有界变差函数  $f$ , 其全变差与正、负变差之间的关系, 以及  $f$  与正、负变差的关系.

**引理 4.4.2.** Suppose  $f$  is real-valued and of BV on  $[a, b]$ . Then for all  $x \in [a, b]$ , we have

$$f(x) - f(a) = P_f([a, x]) - N_f([a, x])$$

and

$$V_f([a, x]) = P_f([a, x]) + N_f([a, x])$$

**证明.**

- $f(x) - f(a) = P_f([a, x]) - N_f([a, x])$ :

$\forall \epsilon > 0, \exists$  a partition  $\mathcal{P}: a = t_0 < t_1 < \cdots < t_N = b$ , s. t.

$$\left| P_f - \sum_{(+)} f(t_j) - f(t_{j-1}) \right| \leq \epsilon \text{ and } \left| N_f - \sum_{(-)} -[f(t_j) - f(t_{j-1})] \right| \leq \epsilon \quad (4.94)$$

Then

$$-\epsilon + \sum_{(+)} f(t_j) - f(t_{j-1}) \leq P_f \leq \sum_{(+)} f(t_j) - f(t_{j-1}) + \epsilon \quad (4.95)$$

$$-\epsilon + \sum_{(-)} -[f(t_j) - f(t_{j-1})] \leq N_f \leq \sum_{(-)} -[f(t_j) - f(t_{j-1})] + \epsilon \quad (4.96)$$

Since

$$f(x) - f(a) = \left( \sum_{(+)} f(t_j) - f(t_{j-1}) \right) - \left( \sum_{(-)} -[f(t_j) - f(t_{j-1})] \right) \quad (4.97)$$

Then

$$P_f - N_f \in [f(x) - f(a) - 2\epsilon, f(x) - f(a) + 2\epsilon] \quad (4.98)$$

$$\Rightarrow |(P_f - N_f) - (f(x) - f(a))| \leq 2\epsilon, \forall \epsilon > 0 \quad (4.99)$$

Since  $\epsilon$  is arbitrary, letting  $\epsilon \rightarrow 0$ , we get  $f(x) - f(a) = P_f - N_f$ .

- $V_f([a, x]) = P_f([a, x]) + N_f([a, x])$ :

$\forall$  partition  $\mathcal{P}$ :  $a = t_0 < t_1 < \cdots < t_N = b$ , s. t.

$$V_{\mathcal{P}}(f) = \sum_{j=1}^N |f(t_j) - f(t_{j-1})| = \left( \sum_{(+)} f(t_j) - f(t_{j-1}) \right) + \left( \sum_{(-)} -[f(t_j) - f(t_{j-1})] \right) \quad (4.100)$$

- $V_f([a, x]) \leq P_f([a, x]) + N_f([a, x])$ :

分别对右侧两项取上确界 through all partitions, we have

$$\sum_{j=1}^N |f(t_j) - f(t_{j-1})| \leq P_f([a, x]) + N_f([a, x]) \quad (4.101)$$

再对左侧取上确界 through all partitions, then

$$V_f([a, x]) \leq P_f([a, x]) + N_f([a, x]) \quad (4.102)$$

- $P_f([a, x]) + N_f([a, x]) \leq V_f([a, x])$ :

Similarly, 先对左侧取上确界, 再对右侧分别取上确界, 得到

$$P_f([a, x]) + N_f([a, x]) \leq V_f([a, x]) \quad (4.103)$$

综上,  $V_f([a, x]) = P_f([a, x]) + N_f([a, x])$ .

□

下面我们说明，任一有界变差函数可差分为两个有界递增函数之差.

**定理 4.4.3.** A real-valued function  $f : [a, b] \rightarrow \mathbb{R}$  on  $[a, b]$  is of BV

$\Leftrightarrow f$  is the difference of two increasing bounded functions.

证明.

- $\Leftarrow$ : Suppose  $f = f_1 - f_2$ , where  $f_j$  is increasing and bounded on  $[a, b]$ ,  $j = 1, 2$ .

Then by **Example 4.4.2**,  $f_j$  is of BV on  $[a, b]$ ,  $j = 1, 2$ .

$\forall$  partition  $\mathcal{P}$ :  $a = t_0 < t_1 < \cdots < t_N = b$ , since

$$|f(t_j) - f(t_{j-1})| \leq |f_1(t_j) - f_1(t_{j-1})| + |f_2(t_j) - f_2(t_{j-1})|, \quad \forall j = 1 \sim N \quad (4.104)$$

Then since both  $f_1$  and  $f_2$  are of BV,  $\exists M < \infty$ , s. t.

$$\sum_{j=1}^N |f(t_j) - f(t_{j-1})| \leq \sum_{j=1}^N |f_1(t_j) - f_1(t_{j-1})| + \sum_{j=1}^N |f_2(t_j) - f_2(t_{j-1})| \leq 2M \quad (4.105)$$

Therefore,  $f$  is of BV on  $[a, b]$ .

- $\Rightarrow$ : By **Lemma 4.4.2**,  $f(x) - f(a) = P_f([a, x]) - N_f([a, x])$ ,  $\forall x \in [a, b]$ .

It's trivial to show that  $P_f([a, x])$  is increasing in  $x$ .

Similarly, we get  $N_f([a, x])$  is increasing in  $x$ . Therefore

$$f(x) = (P_f([a, x]) + f(a)) - N_f([a, x])$$

where  $P_f([a, x]) + f(a)$  and  $N_f([a, x])$  are increasing and bounded.

□



### 4.4.3 有界变差函数的全变差的性质

在本小节的最后，我们来讨论一下实值有界变差函数的全变差的性质.

**命题 4.4.1.** Let  $f \in BV([a, b])$  and be real-valued. Then

- (i)  $\forall c \in (a, b), V_f([a, b]) = V_f([a, c]) + V_f([c, b]).$
- (ii)  $V_f([a, x])$  and  $U(x) = V_f([a, x]) - f(x)$  are both increasing in  $x$  on  $[a, b].$
- (iii)  $V_f([a, x])$  is continuous at  $x_0 \Leftrightarrow f$  is continuous at  $x_0.$

**证明.**

- (i) 不难证明,  $\forall c \in (a, b),$

$$V_f([a, b]) = \sup \left[ \sum_{j=1}^k |f(t_j) - f(t_{j-1})| + \sum_{j=k+1}^N |f(t_j) - f(t_{j-1})| \right] \quad (4.106)$$

The sup is taken over all partitions with  $a = t_0 < t_1 < \cdots < t_k = c < \cdots < t_N = b.$

Since

$$\sum_{j=0}^N |f(t_j) - f(t_{j-1})| = \sum_{j=1}^k |f(t_j) - f(t_{j-1})| + \sum_{j=k+1}^N |f(t_j) - f(t_{j-1})| \quad (4.107)$$

Then 先对左侧取上确界 over all partitions on  $[a, b],$  we have

$$V_f([a, b]) \geq \sum_{j=1}^k |f(t_j) - f(t_{j-1})| + \sum_{j=k+1}^N |f(t_j) - f(t_{j-1})| \quad (4.108)$$

再对左侧两项分别取上确界 over all partitions on  $[a, c]$  and  $[c, b],$  we have

$$V_f([a, b]) \geq V_f([a, c]) + V_f([c, b]) \quad (4.109)$$

Similarly, 改变两侧取上确界次序, 可得

$$V_f([a, b]) \leq V_f([a, c]) + V_f([c, b]) \quad (4.110)$$

Therefore,  $V_f([a, b]) = V_f([a, c]) + V_f([c, b]), \forall c \in (a, b).$

(ii) Since  $P_f([a, x])$  and  $N_f([a, x])$  are both increasing, then by **Lemma 4.4.2**,

$V_f([a, x]) = P_f([a, x]) + N_f([a, x])$  is increasing in  $x$  on  $[a, b]$ .

$\forall x \geq y$ , we have

$$V(x) - V(y) = V_f([a, x]) - V_f([a, y]) \quad (4.111)$$

$$= V_y^x(f) \geq |f(x) - f(y)| \geq f(x) - f(y) \quad (4.112)$$

Therefore  $V(x) - f(x) \geq V(y) - f(y)$ ,  $\forall x \geq y$ . i.e.

$$U(x) \geq U(y), \forall x \geq y$$

(iii) •  $\Rightarrow$ : 根据 (ii) 的证明过程 (式 (4.111)), we get

$$|V(x) - V(y)| \geq V(x) - V(y) \geq |f(x) - f(y)|, \forall x, y \in [a, b]$$

Suppose  $V(x)$  is continuous at  $x_0$ , then  $\forall \epsilon > 0, \exists \delta > 0$ , s. t.

$$|f(x) - f(x_0)| \leq |V(x) - V(x_0)| \leq \epsilon, \forall |x - x_0| < \delta$$

Then  $f(x)$  is continuous at  $x_0$ .

•  $\Leftarrow$ : Suppose  $f(x)$  is continuous at  $x_0$ . Fix  $\epsilon > 0$ , then  $\exists \delta_0 > 0$ , s. t.

$$|f(x_0 + h) - f(x)| < \frac{\epsilon}{2}, \forall |h| < \delta_0$$

It suffices to show

$$|V(x_0 + h) - V(x)| < \epsilon, \forall \text{ small } h$$

考虑全变差  $V_a^{x_0}(f)$  and  $V_{x_0}^b(f)$ , for fixed  $\epsilon > 0, \exists$  partitions

$$a = t_0 < t_1 < \cdots < t_n = x_0 \quad (4.113)$$

$$x_0 = s_0 < s_1 < \cdots < s_m = b \quad (4.114)$$

s. t.

$$\left| V_f([a, x_0]) - \sum_{j=1}^n |f(t_j) - f(t_{j-1})| \right| < \frac{\epsilon}{2} \quad (4.115)$$

$$\left| V_f([x_0, b]) - \sum_{j=1}^m |f(s_j) - f(s_{j-1})| \right| < \frac{\epsilon}{2} \quad (4.116)$$

Now update  $h$  with  $|h| < h_0 = \min \{ \delta_0, x_0 - t_{n-1}, s_1 - x_0 \}$ .

下面先对  $h > 0$  的情况讨论.

$$V(x_0 + h) - V(x_0) = V_{x_0}^b(f) - V_{x_0+h}^b(f) \quad (4.117)$$

$$\leq \sum_{j=1}^m |f(t_j) - f(t_{j-1})| + \frac{\epsilon}{2} - V_{x_0+h}^b(f) \quad (4.118)$$

$$= |f(s_0) - f(s_1)| + \sum_{j=2}^m |f(t_j) - f(t_{j-1})| + \frac{\epsilon}{2} - V_{x_0+h}^b(f) \quad (4.119)$$

$$\leq |f(x_0 + h) - f(x_0)| + |f(s_1) - f(x_0 + h)| + \sum_{j=2}^m |f(t_j) - f(t_{j-1})| + \frac{\epsilon}{2} - V_{x_0+h}^b(f) \quad (4.120)$$

Since  $x_0 + h < s_1 < \dots < s_m = b$  is a partition of  $[x_0 + h, b]$ , then

$$|f(s_1) - f(x_0 + h)| + \sum_{j=2}^m |f(t_j) - f(t_{j-1})| \leq V_{x_0+h}^b(f) \quad (4.121)$$

Therefore

$$V(x_0 + h) - V(x_0) \leq |f(x_0 + h) - f(x_0)| + \frac{\epsilon}{2} \quad (4.122)$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \quad \forall |h| < h_0 \quad (4.123)$$

Similarly, 对于  $h < 0$  的情况, 我们可同样估计  $V(x_0) - V(x_0 + h) = V_a^{x_0}(f) - V_a^{x_0+h}(f)$ , 从而得出结论.

综上,  $V(x)$  is continuous at  $x_0$ .

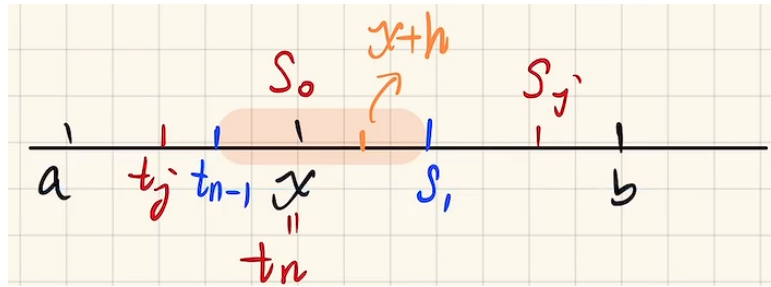


图 4.3: Partitions on  $[a, x_0]$  and  $[x_0, b]$

□

## 4.5 升阳引理 & Dini 导数

在后面一节我们会证明有界变差函数几乎处处可微. 但在此之前, 需要给出本节的 **Rising Sun Lemma** 和 **Dini** 导数的相关内容作为铺垫.

### 4.5.1 Rising Sun Lemma

这里直接给出升阳引理的内容.

**引理 4.5.1. Rising Sun Lemma.**

Suppose  $G$  is real-valued and continuous on  $\mathbb{R}$ . Let

$$E = \{x \in \mathbb{R} \mid G(x+h) > G(x) \text{ for some } h = h_x > 0\}$$

If  $E$  is nonempty, then it must be open, and hence can be written as

$$E = \bigcup_{k=1}^{\infty} (a_k, b_k) \quad (4.124)$$

If  $(a_k, b_k)$  is finite, then

$$G(a_k) = G(b_k)$$

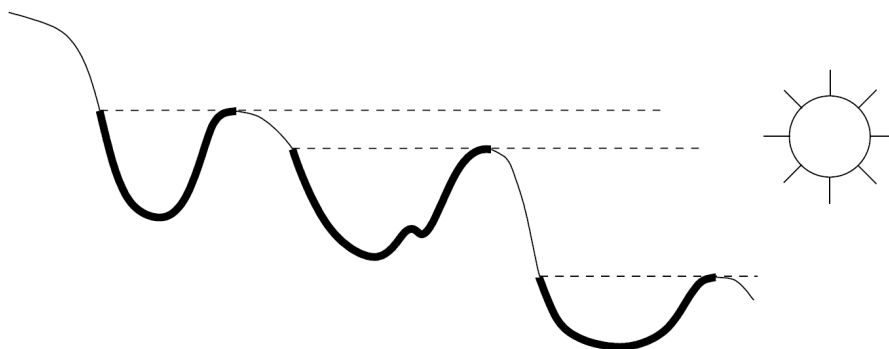


图 4.4: Rising Sun Lemma

证明. Suppose  $E$  is nonempty. 下面分三个部分证明.

- $E$  is open.

From the definition of  $E$ , we get

$$E = \{x \in \mathbb{R} \mid G(x+h) > G(x) \text{ for some } h > 0\} \quad (4.125)$$

$$= \bigcup_{h>0} \{x \in \mathbb{R} \mid G(x+h) - G(x) > 0\} \quad (4.126)$$

Let  $\widetilde{G}_h(x) = G(x+h) - G(x)$ , then  $G_h$  is continuous. 根据连续函数开集的原象为开集,

$$\{x \in \mathbb{R} \mid \widetilde{G}_h(x) > 0\} \text{ is open, } \forall h > 0$$

Then

$$E = \bigcup_{h>0} \{x \in \mathbb{R} \mid \widetilde{G}_h(x) > 0\} \text{ is open} \quad (4.127)$$

- $E = \bigcup_{k=1}^{\infty} (a_k, b_k)$ .

By **Thm 1.1.3** (开集构成定理), it's trivial.

- If  $(a_k, b_k)$  is finite, then  $G(a_k) = G(b_k)$ .

–  $G(b_k) \leq G(a_k)$ :

Since  $a_k \notin E$ , then

$$G(a_k + h) \leq G(a_k), \forall h > 0$$

Since  $b_k > a_k$ , then  $G(b_k) \leq G(a_k)$ .

–  $G(b_k) = G(a_k)$ : (排除  $G(b_k)$  严格小于  $G(a_k)$  的情况)

反证法. Suppose  $G(b_k) < G(a_k)$ .

Since  $G$  is continuous,  $[a_k, b_k] \subset \mathbb{R}$  is connected,

then by **Intermediate Value Thm** (介值定理),

$$A = \left\{ x \in [a_k, b_k] \mid G(x) = \frac{G(a_k) + G(b_k)}{2} \right\} \neq \emptyset \quad (4.128)$$

Since  $b_k$  为  $A$  的上界, then by 确界原理,  $\exists c = \sup A$ .

下面证明  $c \in A$ , 即  $c = \sup A = \max A$ .

\* 反证法. Suppose  $c \notin A$ , i.e.

$$G(c) \neq \frac{G(a_k) + G(b_k)}{2}$$

Since  $c = \sup A$ , then  $\exists \{x_j\}_{j=1}^{\infty} \subset A$ , s. t.

$$x_j \rightarrow c \text{ and } G(x_j) = \frac{G(a_k) + G(b_k)}{2}, \forall j$$

Since  $G$  is continuous, then for  $\epsilon = \left| G(c) - \left( \frac{G(a_k) + G(b_k)}{2} \right) \right| / 2 > 0$ ,  $\exists \delta > 0$ , s. t.

$$G(x) \neq \frac{G(a_k) + G(b_k)}{2}, \forall x \in (c - \delta, c + \delta) \cap [a_k, b_k]$$

which is a contradiction for  $x_j \rightarrow c$  and  $G(x_j) = \frac{G(a_k) + G(b_k)}{2}$ .

综上,  $c = \max A$ . 下面根据  $E$  的性质给出矛盾.

Since  $c \in E$ ,  $\exists h_c > 0$ , s. t.

$$G(c + h_c) > G(c) = \frac{G(a_k) + G(b_k)}{2}$$

Then  $c + h_c < b_k$  (otherwise  $b_k \in E$  for  $G(c + h_c) > G(b_k)$  with  $c + h_c > b_k$ ).

Then by **Intermediate Value Thm** (介值定理),  $\exists c + h_c < d < b_k$ , s. t.

$$G(d) = \frac{G(a_k) + G(b_k)}{2} \text{ with } d > c + h_c > c$$

which is a contradiction with  $c = \max A$ .

Therefore,  $G(a_k) = G(b_k)$ .

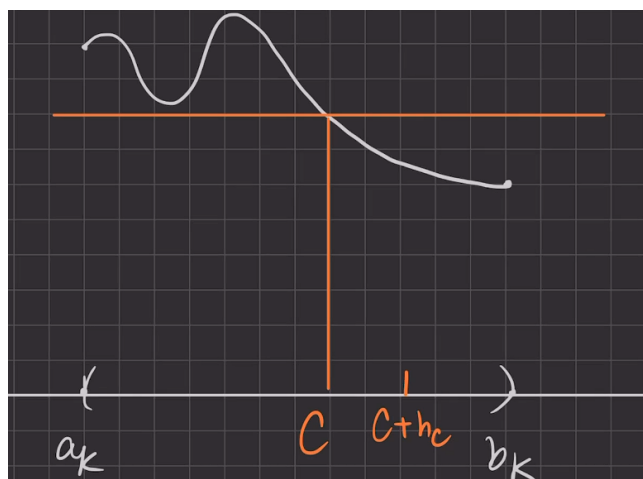


图 4.5: Case  $G(a_k) > G(b_k)$

□

下面给出升阳引理定义在闭区间上的推论.

**推论 4.5.2.** Suppose  $G : [a, b] \rightarrow \mathbb{R}$  is continuous and let

$$E = \{x \in (a, b) \mid G(x + h) > G(x) \text{ for some } h = h_x > 0\}$$

then  $E$  is either empty or open. In the latter case,

$$E = \bigcup_{k=1}^{\infty} (a_k, b_k) \text{ and } G(b_k) = G(a_k) \quad (4.129)$$

except possibly when  $a_k = a$ , in which case we only have

$$G(a_k) \leq G(b_k)$$

**证明.** 与 **Rising Sun Lemma (Lemma 4.5.1)** 证明过程一致.

Suppose  $a_k = a$ . 下用反证法. Suppose  $G(a_k) = G(a) > G(b_k)$ . 同理,  $\exists$  biggest  $c \in (a, b_k)$ , s. t.

$$G(c) = \frac{G(a) + G(b_k)}{2}$$

But  $c \in E \Rightarrow \exists d \in (c, b_k)$ , s. t.  $G(d) = G(c)$ , a contradiction for  $c$  is the biggest. □

## 4.5.2 Dini 导数

引入 探讨函数的可微性，即探讨如下极限的存在性：

$$\lim_{h \rightarrow 0} \Delta_h(f)(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \quad (4.130)$$

而我们又知道，上述极限存在  $\Leftrightarrow$  下述极限存在，i.e.

$$\Leftrightarrow \lim_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h} \quad (4.131)$$

$$\Leftrightarrow \limsup_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h} = \liminf_{h \rightarrow 0^+} \frac{f(x+h) - f(x)}{h} \quad (4.132)$$

$$= \limsup_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h} = \liminf_{h \rightarrow 0^-} \frac{f(x+h) - f(x)}{h} \quad (4.133)$$

i.e.

$$\limsup_{h \rightarrow 0^+} \Delta_h(f)(x) = \liminf_{h \rightarrow 0^+} \Delta_h(f)(x) \quad (4.134)$$

$$= \limsup_{h \rightarrow 0^-} \Delta_h(f)(x) = \liminf_{h \rightarrow 0^-} \Delta_h(f)(x) \quad (4.135)$$

而上述四个极限即为下面要定义的 **Dini 导数**。

定义 下面给出 **Dini 导数** 的定义 / 记号。

定义 4.5.1. Suppose  $F$  is real-valued. The four **Dini numbers** at  $x$  is defined by

$$D^+(F)(x) = \limsup_{h \rightarrow 0^+} \Delta_h(F)(x) \quad (4.136)$$

$$D_+(F)(x) = \liminf_{h \rightarrow 0^+} \Delta_h(F)(x) \quad (4.137)$$

$$D^-(F)(x) = \limsup_{h \rightarrow 0^-} \Delta_h(F)(x) \quad (4.138)$$

$$D_-(F)(x) = \liminf_{h \rightarrow 0^-} \Delta_h(F)(x) \quad (4.139)$$



性质 下面说明, 连续函数的 **Dini** 导数可测.

引理 4.5.3. <sup>4</sup> Suppose  $F$  is continuous on  $[a, b]$ , then

$$D^+(F)(x) = \limsup_{h \rightarrow 0^+} \Delta_h(F)(x) \quad (4.140)$$

is measurable.

证明. (证明的重点是将各种连续型的极限转化为离散型, 再利用可测函数列对极限的封闭性得证)

根据函数上极限的定义 (Def 4.3.4),

$$D^+(F)(x) = \limsup_{h \rightarrow 0^+} \Delta_h(F)(x) = \lim_{\delta \rightarrow 0^+} \sup_{0 < h < \delta} \frac{F(x+h) - F(x)}{h} \quad (4.141)$$

下面先将最外层的  $\delta \rightarrow 0^+$  过程离散化. Let

$$G(\delta) = \sup_{0 < h < \delta} \frac{F(x+h) - F(x)}{h} \quad (4.142)$$

Then  $G \nearrow$  as  $\delta \nearrow$ . 由于单调函数必存在极限 (包含  $\pm\infty$ ), 因此根据 **Heine** 归结原理,

$$\limsup_{h \rightarrow 0^+} \Delta_h(F)(x) = \lim_{n \rightarrow \infty} \sup_{0 < h < \frac{1}{n}} \frac{F(x+h) - F(x)}{h} \quad (4.143)$$

Since 可测函数列对极限封闭, it suffices to show for each  $n \in \mathbb{N}$ ,

$$\sup_{0 < h < \frac{1}{n}} \Delta_h(F)(x) \text{ is measurable} \quad (4.144)$$

为了证明上述函数可测, 我们的思路还是一样, 即用可测函数列逼近. 但上述  $\sup$  过程  $0 < h < \frac{1}{n}$  仍为连续型过程, 下面通过说明该过程与  $h \in (0, \frac{1}{n}) \cap \mathbb{Q}$  的等价性, 来将  $\sup$  过程离散化, 从而方便利用可测函数列来逼近.

**Claim:** Fix  $x \in [a, b]$ . Let  $f(h) = \Delta_h(F)(x)$ , then

$$\sup_{0 < h < \frac{1}{n}} f(h) = \sup_{h \in (0, \frac{1}{n}) \cap \mathbb{Q}} f(h) \quad (4.145)$$

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<sup>4</sup> 《Real Analysis – Measure Theory, Integration, & Hilbert Spaces》— Elias M. Stein P147 Ex 14.

- 证明. Let  $A = \sup_{0 < h < \frac{1}{n}} f(h)$ ,  $B = \sup_{h \in (0, \frac{1}{n}) \cap \mathbb{Q}} f(h)$ .

显然  $A \geq B$ . 下证  $A \leq B$ .

Since  $A = \sup_{0 < h < \frac{1}{n}} f(h)$ ,  $\exists \{x_j\}_{j=1}^{\infty} \subset (0, \frac{1}{n})$  s. t.

$$f(x_j) \rightarrow A$$

由于对于  $\forall j \in \mathbb{N}$ ,  $x_j$  可由有理数列  $\{y_{kj}\}_{k=1}^{\infty} \subset (0, \frac{1}{n}) \cap \mathbb{Q}$  逼近, i.e.

$$y_{kj} \rightarrow x_j \text{ as } k \rightarrow \infty$$

Since  $f(h) = \Delta_h(F)(x)$  is continuous in  $h$ , then

$$f(y_{kj}) \rightarrow f(x_j), \forall j \in \mathbb{N} \text{ as } k \rightarrow \infty$$

Now take  $\{y_{kk}\}_{k=1}^{\infty} \subset (0, \frac{1}{n}) \cap \mathbb{Q}$ , we have

$$f(y_{kk}) \rightarrow \lim_{k \rightarrow \infty} f(x_k) = A \text{ as } k \rightarrow \infty$$

Therefore,  $A \leq B$ . 综上  $A = B$ . □

因为  $\sup_{h \in (0, \frac{1}{n}) \cap \mathbb{Q}} \Delta_h(F)(x)$  为一列可测函数  $\Delta_h(F)(x)$ ,  $h \in (0, \frac{1}{n}) \cap \mathbb{Q}$  的上确界, 而可测函数列对上确界封闭, 所以  $\sup_{h \in (0, \frac{1}{n}) \cap \mathbb{Q}} \Delta_h(F)(x)$  measurable.

Therefore,

$$\sup_{0 < h < \frac{1}{n}} \Delta_h(F)(x) = \sup_{h \in (0, \frac{1}{n}) \cap \mathbb{Q}} \Delta_h(F)(x) \text{ measurable, } \forall n \in \mathbb{N}$$

从而得证. □