# *Real Analysis*

## *Measure Theory, Integration, & Hilbert Spaces*[1]

*−TW−*

2024 年 4 月 24 日

[1]参考书籍：

《*Real Analysis − −Measure Theroy, Integration, & Hilbert Spaces*》— *Elias M. Stein*

《*Real Analysis − −Modern Techniques and Their Applications*》— *Gerald B. Folland*

# 序

天道几何，万品流形先自守；

变分无限，孤心测度有同伦。

2024 年 4 月 24 日

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# 目录

# 第一章 *Measure Theory*

## 1.1 *Preliminaries*

**定义 1.1.1.** A (closed) **rectangle** $R$ in $\mathbb{R}^d$ is given by of d one-dimensional closed and bounded intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d] \tag{1.1}$$

where $a_j \leq b_j$ are real numbers , $j = 1, 2, \cdots, d$. In other word , we have

$$R = \{(x_1, \cdots, x_d) \in \mathbb{R}^d \mid a_j \leq x_j \leq b_j, \ \forall j = 1 \sim d\} \tag{1.2}$$

The **volume** of $R$ is

$$|R| = (b_1 - a_1) \cdots (b_d - a_d) \tag{1.3}$$

An **open** rectangle is the product of open intervals , and **the interior of the rectangle $R$** is

$$(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d) \tag{1.4}$$

Also , a **cube** is a rectangle for which $b_1 - a_1 = \cdots = b_d - a_d$.

**定义 1.1.2.** A union of rectangles is said to be **almost disjoint** if the interiors of them are disjoint.

**引理 1.1.1.** If a rectangle is the almost disjoint union of finitely many rectangles , say $R = \bigcup\limits_{k=1}^{N} R_k$, then

$$|R| = \sum_{k=1}^{N} |R_k| \tag{1.5}$$

**注.** 本质上即指的是对于方体的任意的**垂直划分**可转化为"十字形"划分.

**引理 1.1.2.** If $R, R_1, \cdots, R_N$ are rectangles, and $R \subset \bigcup\limits_{k=1}^{U} R_k$, then

$$|R| \le \sum_{k=1}^{N} |R_k| \tag{1.6}$$

**注.** 此即对 *Lemma* 1.1.1 的 slight modification, 即各方体之间不一定再为 almost disjoint.

Now we can give a description of the strcture of open sets in terms of cubes. Begin with the case of $\mathbb{R}$.

**定理 1.1.3.** Every open subset $O$ of $\mathbb{R}$ can be written uniquely as countable union of disjoint open intervals.

**证明.** For each $x \in O$, let $I_x$ be the largest open interval containing $x$ and contained in $O$.

*Step* 1 : Construct $I_x$:

$O$ is open $\Rightarrow x$ is contained in some small open interval contained in $O$.

Let

$$a_x = inf\{a < x \mid (a, x) \subset O\} \tag{1.7}$$

$$b_x = sup\{b > x \mid (x, b) \subset O\} \tag{1.8}$$

Let $I_x = (a_x, b_x)$, then $O = \bigcup\limits_{x \in O} I_x$.

*Step* 2 : Suppose $I_x \cap I_y \ne \varnothing$.

$I_x \cup I_y$ is an open interval s. t. $\begin{cases} x \in I_x \cup I_y \\ I_x \cup I_y \subset O \end{cases}$

Since $I_x$ is maximal, $I_x \cup I_y \subset I_x$. Similarly, $I_x \cup I_y \subset I_y$.

$\Rightarrow I_x = I_y$

$\Rightarrow$ if $I_x \ne I_y$, then $I_x \cap I_y = \varnothing$.

$\Rightarrow Z = \{I_x\}_{x \in O}$ is a disjoint famliy of sets.

*Step* 3 : Since every $I_x$ contains at least a $a_x \in \mathbb{Q}$, construct a map $f$

$$f : Z \longrightarrow \mathbb{Q} \tag{1.9}$$

$$I_x \longmapsto a_x \tag{1.10}$$

$f$ is an injective. $\Rightarrow \{I_x\}_{x \in O}$ is countable. $\Rightarrow O = \bigcup\limits_{j=1}^{\infty} (a_j, b_j)$.

$$\square$$

**定理 1.1.4.** Every open set $O$ of $\mathbb{R}^d$ , $d \geq 1$ , can be written as a countable union of almost disjoint closed cubes.

**证明.** Let

$$Q_k := grid \ of \ 2^{-k}\mathbb{Z}^d, \ k \geq 0 \tag{1.11}$$

$$\underline{A}(O, k) := \{Q \in Q_k \mid Q \subset O\} \tag{1.12}$$

$$\overline{A}(O, k) := \{Q \in Q_k \mid Q \cap O \neq \varnothing\} \tag{1.13}$$

Since $\forall Q \in \underline{A}(O, k), \ \exists q \in Q^\circ, \ \text{s.t.} \ q \in \mathbb{Q}^d,$

According to the Axiom of Choice , $\exists$ the map $f_k : \underline{A}(O, k) \longrightarrow \mathbb{Q}^d$ , which is an injection.

Hence $\underline{A}(O, k)$ is countable.

Let

$$\underline{A}(O) := \bigcup_{k=1}^{\infty} (\underline{A}(O, k) \setminus \underline{A}(O, k-1)) \cup \underline{A}(O, 0) \tag{1.14}$$

Then $\underline{A}(O)$ is also countable. Similarly define $\overline{A}(O)$.

$\forall x \in O$ , let $\delta_x := inf\{|y - x| \mid y \notin O\}$. Since $O$ is open , $\Rightarrow \ \delta_x > 0$.

$$\exists N_x \in \mathbb{N}, \ \text{s.t.} \ 2^{-k}\sqrt{d} \leq \frac{\delta_x}{2} < \delta_x, \forall k \geq N_x \tag{1.15}$$

$$\Rightarrow \forall Q \in \overline{A}(O, N_x), \ \text{s.t.} \ |s - t| \leq 2^{-N_x}\sqrt{d} < \delta_x, \forall s, t \in Q \tag{1.16}$$

$$\Rightarrow Since \ O \subset \overline{A}(O), \ \exists Q_x \in \overline{A}(O, N_x) \subset \overline{A}(O), \ \text{s.t.} \ x \in Q_x \tag{1.17}$$

$$\Rightarrow x \in Q_x \subset O \tag{1.18}$$

$$\Rightarrow x \in Q_x \in \underline{A}(O, N_x) \subset \underline{A}(O) \tag{1.19}$$

$$\Rightarrow O \subset \underline{A}(O) \tag{1.20}$$

Obviously $\underline{A}(O) \subset O$ , so

$$O = \underline{A}(O) = \bigcup_{k=1}^{\infty} (\underline{A}(O, k) \setminus \underline{A}(O, k-1)) \cup \underline{A}(O, 0) \tag{1.21}$$

which is a countable union of almost disjoint closed cubes. $\square$

# 1.2 *The Exterior Measure*

*Definition*  The exterior measure attempts to describe the volume of a set $E$ by approximating it from the outside.

Loosely speaking , the exterior measure $m_*$ assigns to **any subset of** $\mathbb{R}^d$ a first notion of size.

定义 **1.2.1.** If $E$ is a subset of $\mathbb{R}^d$ , the **<u>exterior measure</u>** of $E$ is

$$m_*(E) := \inf \left\{ \sum_{j=1}^{\infty} |Q_j| \ \Big| \ E \subset \bigcup_{j=1}^{\infty} Q_j, \ Q_j \text{ is a closed cube} \right\} \tag{1.22}$$

注.  • **Well definition:** $\forall E \subset \mathbb{R}^d, \ E \subset \bigcup_{n=1}^{\infty} Q_n, \ Q_n = [-n, n]^d \subset \mathbb{R}^d$ , which means $m_*$ can be defined on every subset of $\mathbb{R}^d$.

• It is immediate from the definition that:

For every $\epsilon > 0$ , there exists a covering $E \subset \bigcup_{j=1}^{\infty} Q_j$, s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \tag{1.23}$$

• It is important to note that it would **not suffice** to allow **finite sums** in the definition of $m_*(E)$. If one considered only coverings of $E$ by finite unions of cubes , the quantity is **in general larger** than $m_*(E)$.

(In fact , it is defined as the **outer Jordan content** $J_*(E)$.)

例 **1.2.1.** Consider the set $\mathbb{Q} \cap [0, 1]$.

– For the outer Jordan content , since it's obvious that $J_*(\overline{E}) = J_*(E), \ \forall E \subset \mathbb{R}^d$,

$J_*(\mathbb{Q} \cap [0, 1]) = J_*(\overline{\mathbb{Q} \cap [0, 1]}) = J_*([0, 1]) = 1$

– For the exterior measure , since $\mathbb{Q} \cap [0, 1]$ is countable , let $\mathbb{Q} \cap [0, 1] = \{x_1, x_2, \cdots\}$. Since for all $\epsilon > 0$ ,

$$\mathbb{Q} \cap [0, 1] \subset \bigcup_{j=1}^{\infty} [x_j - \frac{\epsilon}{2^j}, x_j + \frac{\epsilon}{2^j}] \tag{1.24}$$

Hence $m_*(\mathbb{Q} \cap [0, 1]) \leq \epsilon$. For $\epsilon$ is arbitrary , $m_*(\mathbb{Q} \cap [0, 1]) = 0$.

*Examples*    Let's check that whether the exterior measure matches our intuitive idea of volume.

*Example* 1. **The exterior measure of a point is zero.**

证明. It's clear that a point is a cube with $a_j = b_j, \forall j = 1 \sim d$ and which covers itself. $\qquad \square$

*Example* 2. **The exterior measure of a closed cube is equal to its volume.**

证明.

- Let $Q \subset \mathbb{R}^d$ be a closed cube. Since $Q \subset Q$ , $m_*(Q) \le |Q|$ .

- Suppose $Q \subset \bigcup\limits_{j=1}^{\infty} Q_j$ by closed cubes. For fixed $\epsilon > 0$ , $\forall j \in \mathbb{N}$ , choose an open cube $S_j$ ,

$$\text{s. t.} \begin{cases} S_j \supset Q_j \\ |S_j| = (1 + \epsilon)|Q_j| \end{cases} \tag{1.25}$$

Then $Q \subset \bigcup\limits_{j=1}^{\infty} S_j$. Since $Q$ is compact , $\exists S_1, \cdots, S_n \in \{S_j\}_{j=1}^{\infty}$, s. t. $Q \subset \bigcup\limits_{j=1}^{n} S_j$.

Therefore , according to Lemma 1.1.2

$$|Q| \le \sum_{j=1}^{n} |S_j| = (1 + \epsilon) \sum_{j=1}^{n} |Q_j| \le (1 + \epsilon) \sum_{j=1}^{\infty} |Q_j| \tag{1.26}$$

For $\epsilon > 0$ is arbitrary , we get

$$|Q| \le \sum_{j=1}^{\infty} |Q_j| \tag{1.27}$$

$$|Q| \le \inf \sum_{j=1}^{\infty} |Q_j| = m_*(Q) \tag{1.28}$$

$\qquad \square$

*Example* 3. **If $Q$ is an open cube , then $m_*(Q) = |Q|$ .**

证明.

- Since $Q \subset \overline{Q}$ , $m_*(Q) \le |\overline{Q}| = |Q|$.

- We note that for all closed cubes $Q_0$ contained in $Q$ , then $m_*(Q_0) = |Q_0| \le m_*(Q)$.

  For fixed $\epsilon > 0$ which is suffice small , choose a closed cube $Q_0$ contained in $Q$ with a volume $|Q_0| = (1 - \epsilon)|Q|$ , then we have

$$|Q_0| = (1 - \epsilon)|Q| \le m_*(Q) \tag{1.29}$$

For $\epsilon$ is arbitrary , $|Q| \le m_*(Q)$.

$\qquad \square$

*Example* 4. **The exterior measure of a rectangle $R$ is equal to its volume.**

*Example* 5. $m_*(\mathbb{R}^d) = \infty$.

证明. Since any covering of $\mathbb{R}^d$ is also a covering of any cube $Q \subset \mathbb{R}^d$, $m_*(\mathbb{R}^d) \geq m_*(Q)$

$\forall N > 0, \ \exists Q \subset \mathbb{R}^d, \ \text{s.t.} \ |Q| > N$, so $m_*(\mathbb{R}^d) = \infty$. $\qquad\qquad\square$

*Properties*

*Observation* 1. (*Monotonicity*)

If $E_1 \subset E_2$, then $m_*(E_1) \leq m_*(E_2)$.

*Observation* 2. (*Countable sub − additivity*)

If $E \subset \bigcup\limits_{j=1}^{\infty} E_j$, then $m_*(E) \leq \sum\limits_{j=1}^{\infty} m_*(E_j)$.

证明. For a fixed $\epsilon > 0$, for all $E_j$, there exists a covering $\{Q_{j_k}\}_{k=1}^{\infty}$, $E \subset \bigcup\limits_{k=1}^{\infty} Q_{j_k}$, s.t.

$$\sum_{k=1}^{\infty} m_*(Q_{j_k}) \leq m_*(E_j) + \frac{\epsilon}{2^j} \qquad\qquad (1.30)$$

Since $E \subset \bigcup\limits_{j=1}^{\infty} E_j \subset \bigcup\limits_{j=1}^{\infty} \bigcup\limits_{k=1}^{\infty} Q_{j_k}$, $\bigcup\limits_{j=1}^{\infty} \bigcup\limits_{k=1}^{\infty} Q_{j_k}$ covers $E$, then

$$m_*(E) \leq \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} m_*(Q_{j_k}) \leq \sum_{j=1}^{\infty} m_*(E_j) + \epsilon \qquad\qquad (1.31)$$

Since $\epsilon$ is arbitrary, $m_*(E) \leq \sum\limits_{j=1}^{\infty} m_*(E_j)$ $\qquad\qquad\square$

*Observation* 3. If $E \subset \mathbb{R}^d$, then $m_*(E) = \inf\{m_*(O) \mid E \subset O, \ O \ \text{is an open set}\}$.

证明.

- By monotonicity, $m_*(E) \leq m_*(O)$, for all $O$ covers $E$. Then take the infimum.

- For a fixed $\epsilon > 0$ , $\exists$ *covering* $E \subset \bigcup\limits_{j=1}^{\infty} Q_j$, s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \frac{\epsilon}{2} \tag{1.32}$$

For all $Q_j$ , choose an open set $\widetilde{Q_j}$ containing $Q_j$ with a volume $\left|\widetilde{Q_j}\right| \leq \left|Q_j\right| + \frac{\epsilon}{2^{j+1}}$. Let $O = \bigcup\limits_{j=1}^{\infty} \widetilde{Q_j}$ , then by Observation 2 ,

$$m_*(O) \leq \sum_{j=1}^{\infty} m_*(\widetilde{Q_j}) = \sum_{j=1}^{\infty} \left|\widetilde{Q_j}\right| \leq \sum_{j=1}^{\infty} \left|Q_j\right| + \frac{\epsilon}{2} \leq m_*(E) + \epsilon \tag{1.33}$$

Since $\epsilon$ is arbitrary , $m_*(O) \leq m_*(E)$ , so $\inf m_*(O) \leq m_*(E)$.

$\square$

*Observation* 4. If $E = E_1 \cup E_2$ , and $d(E_1, E_2) > 0$ , then

$$m_*(E) = m_*(E_1) + m_*(E_2) \tag{1.34}$$

证明. For a fixed $\epsilon > 0$ , $\exists$ a covering $E \subset \bigcup\limits_{j=1}^{\infty} Q_j$, s. t.

$$\sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \tag{1.35}$$

Subdevide the cubes $Q_j$ and assume that $diam(Q_j) <= \frac{d(E_1, E_2)}{3}$. Then each $Q_j$ can intersect at most one of the two sets $E_1$ or $E_2$. Devide $\{Q_j\}_{j=1}^{\infty}$ into two subsets $\{Q_j\}_{j \in J_1}$, $\{Q_j\}_{j \in J2}$, s. t.

$$E_1 \subset \bigcup_{j \in J_1} Q_j, \ E_2 \subset \bigcup_{j \in J_2} Q_j \tag{1.36}$$

$J_1$ and $J_2$ are both countable. $J_1 \cap J_2 = \varnothing$. Then

$$m_*(E_1) \leq \sum_{j \in J_1} m_*(Q_j), \ m_*(E_2) \leq \sum_{j \in J_2} m_*(Q_j) \tag{1.37}$$

Therefore

$$m_*(E_1) + m_*(E_2) \leq \sum_{j \in J_1} m_*(Q_j) + \sum_{j \in J_2} m_*(Q_j) \leq \sum_{j=1}^{\infty} m_*(Q_j) \leq m_*(E) + \epsilon \tag{1.38}$$

Since $\epsilon$ is arbitrary , $m_*(E_1) + m_*(E_2) \leq m_*(E)$. $\square$

*Observation* 5. If a set $E$ is the countable union of almost disjoint cubes $E = \bigcup\limits_{j=1}^{\infty} Q_j$ , then

$$m_*(E) = \sum_{j=1}^{\infty} |Q_j| \tag{1.39}$$

证明. For a fixed $\epsilon > 0$ , for all $Q_j$ , choose a closed cube $\widetilde{Q_j}$ strictly contained in $Q_j$ with its volume $|\widetilde{Q_j}| \geq |Q_j| - \frac{\epsilon}{2^j}$. Then for every $N \in \mathbb{N}$ , the cubes $\widetilde{Q_1}, \cdots, \widetilde{Q_N}$ are disjoint with a finite distance from one another. By Observation 4 ,

$$m_*(\bigcup_{j=1}^{N} \widetilde{Q_j}) = \sum_{i=1}^{N} |\widetilde{Q_j}| \geq \sum_{j=1}^{N} |Q_j| - \epsilon \tag{1.40}$$

Since $\bigcup\limits_{j=1}^{\infty} \widetilde{Q_j} \subset E$ , we conclude that for every $N$

$$m_*(E) \geq \sum_{j=1}^{N} |Q_j| - \epsilon \tag{1.41}$$

Let $N \to \infty$ , we deduce

$$m_*(E) \geq \sum_{j=1}^{\infty} |Q_j| - \epsilon \tag{1.42}$$

Since $\epsilon$ is arbitrary , $\sum\limits_{j=1}^{\infty} |Q_j| \leq m_*(E)$. $\qquad\qquad\square$

# 1.3 *Measurable sets and the Lebesgue measure*

## 1.3.1 *Measurable sets*

**Definition**

**定义 1.3.1.** A subset $E$ of $\mathbb{R}^d$ is *(Lebesgue) measurable*, if for any $\epsilon > 0$ there exists an open set $O$ with $E \subset O$ and $m_*(O \backslash E) \le \epsilon$.

If $E$ is measurable, we define its *(Lebesgue) measurable* $m(E)$ by $m(E) = m_*(E)$.

**注.**
- 可用映射的观点来理解外测度 $m_*$ 与测度 $m$ 的关系 (*Folland*). 即

$$m_* : \mathcal{P}(\mathbb{R}^d) \longrightarrow \overline{\mathbb{R}}_+ = [0, +\infty] \tag{1.43}$$

$$m : \mathcal{M} \longrightarrow \overline{\mathbb{R}}_+ = [0, +\infty] \tag{1.44}$$

$$m = m_*\Big|_{\mathcal{M}} \tag{1.45}$$

其中 $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^d)$ 为 $\mathbb{R}^d$ 中所有 *(Lebesgue) measurable sets* 构成的集合.

- 类比于抽象代数中各代数结构的性质，比如群 (*group*) 对加法／乘法封闭，我们下面探讨集合族 $\mathcal{M}$ 对于可数个集合的运算 (*countable unions, countable intersections, complement*) 是否封闭. 即通过此引出代数结构 *σ − algebra*.

**Properties** 下面开始探讨 *(Lebesgue) measure* 的部分性质.

Property 1. Every open set in $\mathbb{R}^d$ is measurable.

Property 2. If $m_*(E) = 0$, then $E$ is measurable.

**证明.** By Observation 3 in §1.2, for a fixed $\epsilon > 0$, $\exists E \subset O$ open, s. t.

$$m_*(O) \le m_*(E) + \epsilon = \epsilon \tag{1.46}$$

Since $O \backslash E \subset O$, then $m_*(O \backslash E) \le m_*(O) \le \epsilon$. $\qquad\square$

**Property 3.** Let $\{E_j\}_{j=1}^{\infty}$ be a family of measurable sets, then $\bigcup\limits_{j=1}^{\infty} E_j$ is measurable.

注. 即说明集合族 $\mathcal{M}$ 对 *countable unions* 封闭.

证明. Since $E_j$ is measurable, for a fixed $\epsilon > 0$, $\exists E_j \subset O_j$ open, s.t.

$$m_*(O_j \backslash E_j) \leq \frac{\epsilon}{2^j} \tag{1.47}$$

Let $O = \bigcup\limits_{j=1}^{\infty} O_j \underset{open}{\subset} \mathbb{R}^d$, then

$$O \backslash \bigcup_{j=1}^{\infty} E_j = \left( \bigcup_{j=1}^{\infty} O_j \right) \cap \left( \bigcap_{j=1}^{\infty} E_j^c \right) \tag{1.48}$$

$$= \bigcup_{j=1}^{\infty} \left( O_j \cap \left( \bigcap_{k=1}^{\infty} E_k^c \right) \right) \subset \bigcup_{j=1}^{\infty} \left( O_j \cap E_j^c \right) = \bigcup_{j=1}^{\infty} \left( O_j \backslash E_j \right) \tag{1.49}$$

Therefore

$$m_* \left( O \backslash \bigcup_{j=1}^{\infty} E_j \right) \leq m_* \left( \bigcup_{j=1}^{\infty} \left( O_j \backslash E_j \right) \right) \leq \sum_{j=1}^{\infty} m_* \left( O_j \backslash E_j \right) \leq \epsilon \tag{1.50}$$

So $\bigcup\limits_{j=1}^{\infty} E_j$ is measurable. □

**Property 4.** Closed sets are measurable.

为了证明该性质，先证明如下的分离定理.

引理 **1.3.1.** If $F$ is closed, $K$ is compact, and $K \cap F = \varnothing$, then $d(F, K) > 0$.

证明. 反证法.Suppose $d(F, K) = 0$, then for any fixed $n \in \mathbb{N}$, $\exists x_n \in F, y_n \in K$, s.t.

$$|x_n - y_n| \leq \frac{1}{n} \tag{1.51}$$

Since $K$ is compact, $\{y_n\}_{n=1}^{\infty}$ is bounded. Then there exists a subsequence $\{y_{n_k}\}_{k=1}^{\infty}$, s.t.

$$y_{n_k} \to y_0 \in K, \ as \ k \to \infty \tag{1.52}$$

Since $\left| x_{n_k} - y_{n_k} \right| \leq \frac{1}{n_k}$, then

$$\left| x_{n_k} - y_0 \right| \leq \left| x_{n_k} - y_{n_k} \right| + \left| y_{n_k} - y_0 \right| \to 0, \ as \ k \to \infty \tag{1.53}$$

So $x_{n_k} \to y_0 \in F, y_0 \in F \cap K \neq \varnothing$ 矛盾. □

下面证明 Property 4.

证明.

- Suppose $F$ is bounded, then $F$ is compact.

  By Observation 3 in §1.2, for a fixed $\epsilon > 0$, $\exists F \subset O$ open, s.t.

  $$m_*(O) \leq m_*(F) + \epsilon \tag{1.54}$$

  Since $F$ is closed, $O \backslash F = O \cap F^c$ is open. By Thm1.1.4, $\exists \{Q_j\}_{j=1}^\infty$, s.t.

  $$O \backslash F = \bigcup_{j=1}^\infty Q_j \tag{1.55}$$

  For a fixed $N \in \mathbb{N}$, let $K = \bigcup_{j=1}^N Q_j$, then $K$ is compact. By Lemma1.3.1, $d(K, F) > 0$.
  Since $K \cup F \subset O$, by Observation 4 in §1.2,

  $$m_*(K) + m_*(F) = m_*(K \cup F) \leq m_*(O) \tag{1.56}$$

  So for each fixed $N \in \mathbb{N}$,

  $$\sum_{j=1}^N \left| Q_j \right| = m_*(K) \leq m_*(O) - m_*(F) \leq \epsilon \tag{1.57}$$

  Let $N \to \infty$, we get

  $$m_*(O \backslash F) = \sum_{j=1}^\infty \left| Q_j \right| \leq \epsilon \tag{1.58}$$

  Therefore, $F$ is measurable.

- For the general situation, since $\mathbb{R}^d = \bigcup_{j=1}^\infty B_j$, then

  $$F = F \cap \mathbb{R}^d = \bigcup_{j=1}^\infty \left( F \cap B_j \right) \tag{1.59}$$

  Since $B_k$ is compact and $F$ is closed, then $F \cap B_j$ is compact.

  Due to the previous proof, $F \cap B_j$ is measurable. By Property 3 in §1.3.1,

  $$F = \bigcup_{j=1}^\infty \left( F \cap B_j \right) \text{ is measurable.} \tag{1.60}$$

  $\square$

11

**Property 5.** If $E$ is measurable, then $E^c$ is measurable.

> **注.** 即说明集合族 $M$ 对集合的补运算 *complement* 封闭.

> **证明.** Since $E$ is measurable, then for all fixed $n \in \mathbb{N}$, $\exists E \subset O_n$ open, s. t. $m_*(O_n \backslash E) \le \frac{1}{n}$.
> Let $S = \bigcup\limits_{j=1}^{\infty} O_j^c \subset E^c$. Since $O_j^c$ is closed, $O_j^c$ is measurable. Then $S$ is measurable.
> Since
> $$E^c \backslash S = E^c \cap \left( \bigcap_{j=1}^{\infty} O_j \right) = \bigcap_{j=1}^{\infty} \left( E^c \cap O_j \right) \subset E^c \cap O_n = O_n \backslash E, \ \forall n \in \mathbb{N} \tag{1.61}$$
> Then, $m_*(E^c \backslash S) \le m_*(O_n \backslash E) \le \frac{1}{n}, \ \forall n \in \mathbb{N}$. So $E^c \backslash S$ is measurable.
> Therefore, $E^c = (E^c \backslash S) \cup S$ is measurable. □

**Property 6.** If $\{E_j\}_{j=1}^{\infty}$ is a family of measurable sets, then $\bigcap\limits_{j=1}^{\infty} E_j$ is measurable.

> **注.** 即说明集合族 $M$ 对 *countableintersections* 封闭.

> **证明.** Since
> $$\bigcap_{j=1}^{\infty} E_j = \left( \bigcup_{j=1}^{\infty} E_j^c \right)^c \tag{1.62}$$
> Then, $E_j^c$ is measurable and so $\bigcap\limits_{j=1}^{\infty} E_j$ is measurable. □

综上, 本节介绍了 (*Lebesgue*) *measurable sets* 的性质, 并且证明了 *Lebesgue measurable sets* 构成的集合族 $M$ 对 *countable unions, countable intersections, complement* 运算封闭. 从而 $(M, \cup, \cap, complement)$ 构成代数结构, 即为后续介绍的 *σ − algebra*.

### 1.3.2 *Lebesgue measure*

下面着重来介绍一下 *Lebesgue measure* 的 *properties*.

**可数可加性** 首先便是可数可加性 *countable additivity*.

**定理 1.3.2.** If $E_1, E_2, \cdots$ are disjoint measurable sets, then

$$m(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} m(E_j) \tag{1.63}$$

**证明.** Since $m(\bigcup\limits_{j=1}^{\infty} E_j) \leq \sum\limits_{j=1}^{\infty} m(E_j)$ always holds, we then proof the reverse inequality.

- Suppose that $E_j$ is bounded.

  Since $E_j^c$ is measurable, for any fixed $\epsilon > 0$, there exists an closed subset $F_j \subset E_j$, s.t.

  $$m(E_j \backslash F_j) \leq \frac{\epsilon}{2^j} \tag{1.64}$$

  Since $E_j$ is bounded, $F_j$ is compact.

  Let $K = \bigcup\limits_{j=1}^{N} F_j$ be a disjoint union of compact sets for some fixed $N$, then

  $$K \subset \bigcup_{j=1}^{\infty} E_j \tag{1.65}$$

  $$m(K) = \sum_{j=1}^{N} m(F_j) \leq m(\bigcup_{j=1}^{\infty} E_j) \tag{1.66}$$

  Since

  $$m(E_j) \leq m(E_j \backslash F_j) + m(F_j) \leq m(F_j) + \frac{\epsilon}{2^j} \tag{1.67}$$

  Therefore

  $$\sum_{j=1}^{N} m(E_j) - \epsilon \leq \sum_{j=1}^{N} m(F_j) \leq m(\bigcup_{j=1}^{\infty} E_j) \tag{1.68}$$

  Let $N \to \infty$, for $\epsilon$ is arbitrary, we get

  $$\sum_{j=1}^{\infty} m(E_j) \leq m(\bigcup_{j=1}^{\infty} E_j) \tag{1.69}$$

- In the general case, we choose the sequence of cubes $\{Q_k\}_{k=1}^{\infty}$, $Q_k = [-k, k]^d \subset \mathbb{R}^d$.

  Let $S_1 = Q_1$, $S_k = Q_k - Q_{k-1}$, $\forall k \geq 2$. Then $\{S_k\}_{k=1}^{\infty}$ are disjoint and bounded.

  Since $\{S_k\}_{k=1}^{\infty}$ covers $\mathbb{R}^d$,

$$E_j = \bigcup_{k=1}^{\infty} (E_j \cap S_k) \tag{1.70}$$

$$\bigcup_{j=1}^{\infty} E_j = \bigcup_{j=1}^{\infty} \bigcup_{k=1}^{\infty} (E_j \cap S_k) \tag{1.71}$$

  Since $E_j \cap S_k$ is bounded and disjoint, by the previous case,

$$m(\bigcup_{j=1}^{\infty} E_j) = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} m(E_j \cap S_k) = \sum_{j=1}^{\infty} m(E_j) \tag{1.72}$$

$\square$

**单调连续性** 下面我们可以给出单调可测集合列的连续性.*continuity from below/above*

**定理 1.3.3.** Let $E_1, E_2, \cdots$ be measurable sets in $\mathbb{R}^d$.

(i) If $E_k \nearrow E$, then $m(E) = \lim\limits_{n \to \infty} m(E_n)$.

(ii) If $E_k \searrow E$ and $m(E_1) < \infty$, then $m(E) = \lim\limits_{n \to \infty} m(E_n)$.

**注.** • 事实上即可写为

$$m(\lim_{n \to \infty} E_n) = \lim_{n \to \infty} m(E_n) \tag{1.73}$$

即单调可测集合列**可交换极限与测度顺序**.

- (ii) 中条件 $m(E_1)$ finite 不可省略，下面给出一个反例.

  **例 1.3.1.** If $E_n = (n, +\infty)$, then $m(E_n) = \infty$ and $E = \bigcap_{j=1}^{\infty} E_j = \varnothing$. So

$$m(E) = m(\lim_{n \to \infty} E_j) = 0, \ \lim_{n \to \infty} m(E_j) = \infty \tag{1.74}$$

**证明.**

(i) Let $S_1 = E_1$, $S_k = E_k - E_{k-1}$, $\forall k \geq 2$. Then $\{S_k\}_{k=1}^{\infty}$ are disjoint and measurable.

  Since $E = \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} S_k$, by Thm1.3.2,

$$m(E) = \sum_{k=1}^{\infty} m(S_k) = \lim_{N \to \infty} \sum_{k=1}^{N} m(S_k) = \lim_{N \to \infty} m(\bigcup_{k=1}^{N} S_k) = \lim_{N \to \infty} m(E_N) \tag{1.75}$$

(ii) Let $S_1 = E_1$, $S_k = E_k - E_{k+1}$, $\forall k \geq 2$. Then $\{S_k\}_{k=1}^{\infty}$ are disjoint and measurable.

Since $E_1 = E \cup \left( \bigcup_{k=1}^{\infty} S_k \right)$, then

$$m(E_1) = m(E) + \sum_{k=1}^{\infty} m(S_k) = m(E) + \lim_{N \to \infty} m(\bigcup_{k=1}^{N} S_k) = m(E) + \lim_{N \to \infty} m(E_1 - E_N) \quad (1.76)$$

For $E_1 = (E_1 - E_N) \sqcup E_N$ is a disjoint union,

$$m(E_1 - E_N) = m(E_1) - m(E_N) \quad (1.77)$$

Thus

$$m(E_1) = m(E) + \lim_{N \to \infty} m(E_1 - E_N) = m(E) + m(E_1) - \lim_{N \to \infty} m(E_N) \quad (1.78)$$

$$m(E) = \lim_{N \to \infty} m(E_N) \quad (1.79)$$

$\square$

*Geometric insight of measurable sets* 最后我们来给出 (*Lebesgue*) *measurable sets* 的几何性质 (与开集、闭集、紧集等之间的关系).

**定理 1.3.4.** Suppose $E \subset \mathbb{R}^d$ is measurable, then $\forall \epsilon > 0$ :

(i) $\exists$ open $O \supset E$ with $m(O \backslash E) \leq \epsilon$.

(ii) $\exists$ closed $F \subset E$ with $m(E \backslash F) \leq \epsilon$.

(iii) If $m(E) < \infty$, $\exists$ compact $K \subset E$ with $m(E \backslash K) \leq \epsilon$.

(iv) If $m(E) < \infty$, $\exists F = \bigcup_{j=1}^{N} Q_j$, $\{Q_j\}_{j=1}^{\infty}$ are closed cubes, s. t. $m(E \triangle F) \leq \epsilon$.

**证明.**

(i) It's just the definition of measurability.

(ii) Since $E_j^c$ is measurable, $\exists$ open $O_j \supset E_j^c$, s. t.

$$m(O_j \backslash E_j^c) \leq \epsilon \quad (1.80)$$

Since $O_j^c \subset E_j$ is closed and $E_j \backslash O_j^c = O_j \backslash E_j^c$, let $F = O_j^c$ closed, then

$$m(E_j \backslash F) = m(O_j \backslash E_j^c) \leq \epsilon \quad (1.81)$$

15

(iii) By (ii), $\exists$ closed $F \subset E$, s. t. $m(E\backslash F) \le \frac{\epsilon}{2}$.

Let $B_n$ denote the closed ball centered at the origin of radius n, then $B_n$ is compact.

$$F = \bigcup_{j=1}^{\infty} (F \cap B_k) \tag{1.82}$$

Let $K_n = \bigcup_{k=1}^{n} (F \cap B_k)$, then $K_n$ is compact and $K_n \nearrow F \Rightarrow E\backslash K_n \nearrow E\backslash F$.

Since $m(E\backslash K_1) \le m(E)$ is finite, by Thm1.3.3(ii)

$$\lim_{n\to\infty} m(E\backslash K_n) = m(E\backslash F) \tag{1.83}$$

As for $\epsilon > 0$, $\exists N \in \mathbb{N}$, s. t. for all $n \ge N$

$$|m(E\backslash K_n) - m(E\backslash F)| \le \frac{\epsilon}{2} \tag{1.84}$$

$$m(E\backslash K_n) \le m(E\backslash F) + \frac{\epsilon}{2} \le \epsilon \tag{1.85}$$

Therefore, $m(E\backslash K_N) \le \epsilon$, where $K_N \subset E$ is compact.

(iv) $\exists$ open $O \supset E$, s. t. $m(O\backslash E) \le \frac{\epsilon}{2}$. By Thm1.1.4, $\exists \{Q_j\}_{j=1}^{\infty}$, s. t.

$$E \subset O = \bigcup_{j=1}^{\infty} Q_j \tag{1.86}$$

So

$$m(O) = \sum_{j=1}^{\infty} |Q_j| \le m(O\backslash E) + m(E) \le \frac{\epsilon}{2} + m(E) \tag{1.87}$$

Since $m(E)$ is finite, $\sum_{j=1}^{\infty} |Q_j|$ converges. Then $\exists N \in \mathbb{N}$, s. t.

$$\sum_{j=N+1}^{\infty} |Q_j| \le \frac{\epsilon}{2} \tag{1.88}$$

Let $F = \bigcup_{j=1}^{N} Q_j$. Since $E\triangle F = (E\backslash F) \sqcup (F \cap E)$, then

$$m(E\triangle F) = m(E\backslash F) + m(F\backslash E) \tag{1.89}$$

$$\le m(\bigcup_{j=N+1}^{\infty} Q_j) + m(\bigcup_{j=1}^{\infty} Q_j \backslash E) \tag{1.90}$$

$$= \sum_{j=N+1}^{\infty} |Q_j| + \sum_{j=1}^{\infty} |Q_j| - m(E) \tag{1.91}$$

$$\le \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \tag{1.92}$$

$\square$

# 1.4 σ − algebras and Borel sets

## 1.4.1 σ − algebra

首先给出 $\mathbb{R}^d$ 中 *algebra* 的定义.

**定义 1.4.1.** Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^d)$. $\mathcal{A}$ is called an <u>algebra</u> if

(1) If $A_1, \cdots, A_n \in \mathcal{A}$, then $\bigcup\limits_{j=1}^{n} A_j \in \mathcal{A}$.

(2) If $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$.

**注.** 容易证明，若 $\mathcal{A}$ 为 $\mathbb{R}^d$ 中 *algebra*，则其对 *finite intersections* 也封闭，同时 $\varnothing, \mathbb{R}^d \in \mathcal{A}$.

下面给出 $\mathbb{R}^d$ 中 *σ − algebra* 的定义.(将 *algebra* 中的 *finite* 条件加强为 *countable*)

**定义 1.4.2.** Let $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^d)$. $\mathcal{M}$ is a <u>σ − algebra</u> if

(1) If $A_1, A_2, \cdots \in \mathcal{M}$, then $\bigcup\limits_{j=1}^{\infty} A_j \in \mathcal{M}$.

(2) If $A \in \mathcal{M}$, then $A^c \in \mathcal{M}$.

**注.** 容易证明 $\mathcal{M}$ 对 *countable intersections* 同样封闭，$\varnothing, \mathbb{R}^d \in \mathcal{M}$.

**例 1.4.1.** All Lebesgue measurable sets forms a *σ − algebra* $\mathcal{M}$.

类比线性空间、拓扑空间中 (拓扑) 基的概念，下面给出**生成** *σ − algebra* 的概念.

**定义 1.4.3.** Let $\mathcal{A} \subset \mathcal{P}(\mathbb{R}^d)$, then the *σ − algebra* **generated by $\mathcal{A}$** is the smallest *σ − algebra* containing $\mathcal{A}$.

**注.** 即为 the intersection of all *σ − algebras* containing $\mathcal{A}$，这也说明了对于任一给定的集族 $\mathcal{A}$，其生成的 *σ − algebra* 必存在且唯一.

### 1.4.2  *Borel sets*

下面给出 *Borel σ − algebra* 及 *Borel sets* 的定义.

**定义 1.4.4.** The *Borel σ − algebra* is the *σ − algebra* generated by all open sets in $\mathbb{R}^d$, denoted by $\mathcal{B}_{\mathbb{R}^d}$.

Elements of this *σ − algebra* are called *Borel sets*.

**注.** 事实上, *Borel σ−algebra* 为 Lebesgue countable sets 的一个真子集, 后续会利用 Cantor 集证明.

为了方便研究 *Borel σ − algebra* 的结构, 我们把其中较为复杂 (非平凡) 的元素单独拎出来并称为 $G_\delta, F_\sigma$.

**定义 1.4.5.**   1. The countable intersections of open sets are called $G_\delta$ sets.

 2. The countable unions of closed sets are called $F_\sigma$ sets.

下面我们可给出 $\mathcal{B}_{\mathbb{R}^d}$ 与 Lebesgue 可测集 $\mathcal{L}$ 之间的关系.($\mathcal{L}$ 只比 $\mathcal{B}_{\mathbb{R}^d}$ 多了一些零测集)

**定理 1.4.1.** $E \subset \mathbb{R}^d$ is $\mathcal{L} − measurable$

 (i)  if and only if $E = G_\delta \backslash N_1$, for some $G_\delta$, $m(N_1) = 0$.

 (ii)  if and only if $E = F_\sigma \backslash N_2$, for some $F_\sigma$, $m(N_2) = 0$.

**证明.** Clearly E is measurable whenever it satisfies either (i) or (ii).

 (i)  Since $E$ is measurable, $\exists$ open sets $O_n \supset E$, s. t.

$$m(O_n \backslash E) \le \frac{1}{n} \tag{1.93}$$

Let $O = \bigcap\limits_{j=1}^{\infty} O_j$, then

$$m(O \backslash E) \le \frac{1}{n}, \ \forall n \in \mathbb{N} \tag{1.94}$$

Let $n \to \infty$, we get $m(O \backslash E) = 0$. Let $G_\delta = O, \ N_1 = O \backslash E$. Then $E = G_\delta \backslash N_1$.

 (ii)  Similarly, we can easily proof it by Thm1.3.4(ii).

$\square$

# 1.5 *Non − measurable sets*

在这一节我们将介绍 $\mathbb{R}$ 上一个经典的不可测集 *Vitali set*，并说明 $\mathbb{R}$ 上每个正测度集都有不可测子集.

**Vitali set**   Let $x, y \in [0, 1]$. Write $x \sim y \Leftrightarrow x − y \in \mathbb{Q}$.

$\Rightarrow$ 容易验证 $\sim$ 为 an equivalence relation.

$\Rightarrow$ $\sim$ partitions $[0 , 1]$. 记 $[0 , 1]$ 上等价类为 $\varepsilon_a$，则

$$[0, 1] = \bigsqcup_a \varepsilon_a, \ \{\varepsilon_a\}_a \ \text{are disjoint} \tag{1.95}$$

$\Rightarrow$ By **the Axiom of Choice**, we can choose exactly one element $x_a$ from each $\varepsilon_a$.

$\Rightarrow$ Let $\mathcal{N} = \{x_a\}_a$. Then $\mathcal{N}$ is the Vitali set.

**定理 1.5.1.** $\mathcal{N}$ is not measurable.

**证明.** Assume that $\mathcal{N}$ is measurable. Let $\{r_k\}_{k=1}^\infty$ be an enumeration of $\mathbb{Q} \cap [−1, 1]$. Define

$$\mathcal{N}_k := N + r_k = \{x_a + r_k\}_a \tag{1.96}$$

Then we shall proof that $\{\mathcal{N}_k\}_{k=1}^\infty$ are disjoint, and $[0, 1] \subset \bigcup_{k=1}^\infty \mathcal{N}_k \subset [−1, 2]$.

- If $\mathcal{N}_k \cap \mathcal{N}_m \neq \varnothing$, then $\exists x_a, x_\beta \in \mathcal{N}, \ r_k, r_m \in \mathbb{Q} \cap [−1, 1]$, s. t.

$$x_a + r_k = x_\beta + r_m \tag{1.97}$$

  Then $x_a − x_\beta = r_m − r_k \in \mathbb{Q} \Rightarrow x_a \sim x_\beta \Rightarrow x_a, x_\beta \in \varepsilon_a$ or $x_a, x_\beta \in \varepsilon_\beta \Rightarrow x_a = x_\beta$ and $r_k = r_m$. Therefore, $\mathcal{N}_k = \mathcal{N}_m$.

- Since $r_k \in [−1, 1]$, $\mathcal{N}_k \in [−1, 2]$, $\forall k$. Therefore,

$$\bigcup_{k=1}^\infty \mathcal{N}_k \subset [−1, 2] \tag{1.98}$$

- $\forall x \in [0, 1]$. Since $\{\varepsilon_a\}_a$ partions $[0, 1]$, there exists $a_0$, s. t.

$$x \in \varepsilon_{a_0}, \ x \sim x_{a_0} \tag{1.99}$$

  which means $x − x_{a_0} \in \mathbb{Q} \cap [−1, 1]$. Then $\exists k_0 \in \mathbb{N}$, s. t.

$$x − x_{a_0} = r_{k_0} \ \Rightarrow \ x \in \mathcal{N}_{k_0} \tag{1.100}$$

Therefore,

$$[0, 1] \subset \bigcup_{k=1}^{\infty} \mathcal{N}_k \tag{1.101}$$

Since $\{\mathcal{N}_k\}_{k=1}^{\infty}$ are disjoint, we get

$$m([0, 1]) \leq \sum_{k=1}^{\infty} m(\mathcal{N}_k) \leq m([-1, 2]) \tag{1.102}$$

Since $\mathcal{N}_k$ is a translate of $\mathcal{N}$, we have $m(\mathcal{N}) = m(\mathcal{N}_k)$ for each k. Then

$$1 \leq \sum_{k=1}^{\infty} m(\mathcal{N}) \leq 3 \quad \Rightarrow \quad \textit{Neither } m(\mathcal{N}) = 0 \textit{ nor } m(\mathcal{N}) > 0 \textit{ is possible.} \tag{1.103}$$

Therefore, it's a contradiction. $\mathcal{N}$ is non-measurable. $\qquad\square$

**正测度集必有不可测子集**　下面要证明一个结论，即 $\mathbb{R}$ 上任一正测度集必有不可测子集. 这实际上为书[1]Exercises of Chapter 1 的第 32 题 (b).

命题 **1.5.1.** Let $\mathcal{N}$ denote the non-measurable subset of $[0, 1]$ constructed in Thm1.5.1.

(a) If $E$ is a measurable subset of $\mathcal{N}$, then $m(E) = 0$.

(b) If $G \subset \mathbb{R}$ with $m_*(G) > 0$, then there exists a subset of $G$ is non-measurable.

证明.

(a) Note $\mathcal{N} = \{x_a\}_{a \in \mathcal{A}}$, then $E = \{x_\beta\}_{\beta \in \mathcal{B} \subset \mathcal{A}}$. Similarly, we can proof

$$\bigcup_{k=1}^{\infty} E_k \subset [-1, 2] \tag{1.104}$$

Since $\{E_k\}_{k=1}^{\infty}$ are disjoint, and $E_k$ is a translate of $E$, we get

$$\sum_{k=1}^{\infty} m(E) \leq 3 \quad \Rightarrow \quad m(E) = 0 \tag{1.105}$$

(b) Let $\mathbb{Q} = \{r_k\}_{k=1}^{\infty}$, $\mathcal{N}_k = \mathcal{N} + r_k$, then

$$\mathbb{R} = \bigsqcup_{k=1}^{\infty} \mathcal{N}_K \tag{1.106}$$

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[1]参考书籍：《*Real Analysis − −Measure Theroy, Integration, & Hilbert Spaces*》— *Elias M. Stein*

Suppose $G$ is measurable. Then

$$G = G \cap \mathbb{R} = \bigsqcup_{k=1}^{\infty} (G \cap \mathcal{N}_k) \tag{1.107}$$

If $G \cap \mathcal{N}_k$ is measurable, then $G \cap \mathcal{N}_k \subset \mathcal{N}_k$ is a subset of a non-measurable set $\mathcal{N}_k$.

By the previous (a), we get

$$m(G \cap \mathcal{N}_k) = 0 \tag{1.108}$$

Therefore, there exists $k_0 \in \mathbb{N}$, s. t. $G \cap \mathcal{N}_{k_0} \subset G$ is a non-measurable subset of $G$.

(otherwise $m(G) = 0$ contradicts)

$\square$

# 第二章　*Measurable Functions*

## 2.1　*Measurable Functions*

**定义**　下面给出 $\mathbb{R}^d$ 上**可测函数**的定义.(注意值域为**扩充实数系** $\overline{\mathbb{R}}$)

**定义 2.1.1.** A function defined on a measurable subset $E \subset \mathbb{R}^d$ is **measurable** if for all $a \in \mathbb{R}$,

$$f^{-1}([-\infty, a)) = \{x \in E \mid f(x) < a\} \tag{2.1}$$

is measurable.

**注.**　• $f^{-1}([-\infty, a))$ 常简记作 $\{f < a\}$.

• 下面给出几条等价定义.

   (1) $\{f < a\}$ is measurable. $\Leftrightarrow \{f \leq a\}$ is measurable.

   (2) $\Leftrightarrow \{f > a\}$ is measurable $\Leftrightarrow \{f \geq a\}$ is measurable.

   (3) If f is finite-valued, then

$$f \text{ is measurable } \Leftrightarrow \{a < f < b\} \text{ is measurable, } \forall a, b \in \mathbb{R} \tag{2.2}$$

**证明.**

   (1) Since the collection of measurable sets is closed under countable intersections and unions,

$$\{f \leq a\} = \bigcap_{n=1}^{\infty} \{f < a + \frac{1}{n}\} \tag{2.3}$$

$$\{f < a\} = \bigcup_{n=1}^{\infty} \{f \leq a - \frac{1}{n}\} \tag{2.4}$$

Therefore, $\{f < a\}$ is measurable. $\Leftrightarrow \{f \leq a\}$ is measurable.

   (2) Since the collection of measurable sets is closed under complements, easily proof by (1).

(3)  Since f is finite-valued,

$$\{f < a\} = \bigcup_{n=1}^{\infty} \{-n < f < a\} \tag{2.5}$$

$$\{a < f < b\} = \{f > a\} \cap \{f < b\} \tag{2.6}$$

Therefore, by (2), $f$ is measurable $\Leftrightarrow \{a < f < b\}$ is measurable.

□

**Property**　下面给出可测函数的一些性质.

**Property 1.** Let $-\infty < f(x) < +\infty$ (finite-valued), then

$$f \ is \ measurable \ \Leftrightarrow \ f^{-1}(O) \ is \ measurable \ \forall open \ set \ O \tag{2.7}$$

$$\Leftrightarrow f^{-1}(F) \ is \ measurable \ \forall closed \ set \ F \tag{2.8}$$

证明. $\forall O \underset{open}{\subset} \mathbb{R}$, there exists $\{(a_n, b_n)\}_{n=1}^{\infty}$, s. t.

$$O = \bigcup_{n=1}^{\infty} (a_n, b_n) \tag{2.9}$$

Then

$$f^{-1}(O) = f^{-1}(\bigcup_{n=1}^{\infty} (a_n, b_n)) = \bigcup_{n=1}^{\infty} f^{-1}((a_n, b_n)) \tag{2.10}$$

Since $f$ is finite-valued and measurable, then $f^{-1}(a_n, b_n)$ is measurable.

Therefore, $f^{-1}(O)$ is measurable.

□

**Property 2.** $\{continuous \ functions\} \subset \{measurable \ functions\}$

(a)  (a)If $f$ is continuous on $\mathbb{R}^d$, then $f$ is measurable.

(b)  If $f$ is measurable, finite-valued and $\Phi$ is continuous on $\mathbb{R}$, then $\Phi \circ f$ is measurable.

证明.

(a)  Since $f$ is continuous, $\forall O \underset{open}{\subset} \mathbb{R}, f^{-1}(O) \underset{open}{\subset} \mathbb{R}^d$. By Property 1, $f$ is measurable.

(b) $\forall O \underset{open}{\subset} \mathbb{R}$. Since $\Phi$ is continuous, then $\Phi^{-1}(O)$ is open.

Since $f$ is finite-valued and measurable, then $(\Phi \circ f)^{-1}(O) = f^{-1}(\Phi^{-1}(O))$ is open.

Therefore, by Property 1, $\Phi \circ f$ is measurable.

$\square$

**Property 3.** Suppose $\{f_n\}_{n=1}^{\infty}$ is a sequence of measurable functions. Then

$$\sup_n f_n(x), \ \inf_n f_n(x), \ \limsup_{n\to\infty} f_n(x), \ \liminf_{n\to\infty} f_n(x) \tag{2.11}$$

are measurable.

注. 类比数列的上下极限，此处

$$\limsup_{n\to\infty} f_n(x) := \lim_{k\to\infty} \sup_{n\geq k}\{f_n(x)\} = \inf_k \sup_{n\geq k}\{f_n(x)\} \tag{2.12}$$

$$\liminf_{n\to\infty} f_n(x) := \lim_{k\to\infty} \inf_{n\geq k}\{f_n(x)\} = \sup_k \inf_{n\geq k}\{f_n(x)\} \tag{2.13}$$

证明. Since

$$\{x \mid \sup_n f_n(x) > a\} = \bigcup_{n=1}^{\infty}\{x \mid f_n(x) > a\} \tag{2.14}$$

$$\{x \mid \inf_n f_n(x) < a\} = \bigcup_{n=1}^{\infty}\{x \mid f_n(x) < a\} \tag{2.15}$$

Then $\sup_n f_n(x), \ \inf_n f_n(x)$ is measurable.

Since $\sup_{n\geq k} f_n(x), \ \inf_{n\geq k} f_n(x)$ are measurable, by the previous conclusion, then

$$\limsup_{n\to\infty} f_n(x) = \inf_k \sup_{n\geq k}\{f_n(x)\} \tag{2.16}$$

$$\liminf_{n\to\infty} f_n(x) = \sup_k \inf_{n\geq k}\{f_n(x)\} \tag{2.17}$$

are measurable.

$\square$

**Property 4.** If $\{f_n\}_{n=1}^{\infty}$ is a collection of measurable functions and

$$\lim_{n\to\infty} f_n(x) = f(x) \tag{2.18}$$

then $f$ is measurable.

注. • 与数列上下极限相同，

$$\lim_{n\to\infty} f_n(x) = f(x) \iff \limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x) = f(x) \tag{2.19}$$

• 此 Property 即说明**可测函数列对极限运算封闭**. 注意到连续函数列对极限运算并不具备封闭性.(下面给出经典范例)

例 **2.1.1.**

$$\lim_{n\to\infty} x^n = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases} \tag{2.20}$$

证明. Since $\{f_n\}_{n=1}^{\infty}$ are measurable, $f(x) = \limsup_{n\to\infty} f_n(x) = \limsup_{n\to\infty} f_n(x)$,
then according to Property 3, $f$ is measurable. □

**Property 5.** If $f$ and $g$ are measurable, then

(i) $f^k$, $k \in \mathbb{N}$ are measurable.

(ii) $f + g$ and $fg$ are measurable if both $f$ and $g$ are finite-valued.

证明.

(i) Since

$$\{f^k > a\} = \{f > a^{\frac{1}{k}}\}, \ \forall k \text{ is odd} \tag{2.21}$$

$$\{f^k > a\} = \{f > a^{\frac{1}{k}}\} \cup \{f < -a^{\frac{1}{k}}\}, \ \forall k \text{ is even and } a > 0 \tag{2.22}$$

Therefore, $f^k$, $k \in \mathbb{N}$ are measurable.

(ii) Since[1]

$$\{f + g > a\} = \bigcup_{r \in \mathbb{Q}} \{f > a - r\} \cap \{g > r\} \tag{2.23}$$

---

[1]即必 $\exists r \in \mathbb{Q}$, s.t.$\{f + g > a\} \supset \{f > a - r\} \cap \{g > r\}$. (另一侧包含关系 $\subset$ 显然易证)

(反证. $\forall r \in \mathbb{Q}$ 上式不成立，则对于 $r = 0 \in \mathbb{Q}$, $\exists x_0$, s.t.$f(x_0) > a$, $g(x_0) > 0$, 且 $f(x_0) + g(x_0) \le a$, 矛盾.)

then $f + g$ is measurable.

By the previous results in (i) and (ii), since

$$fg = \frac{1}{4}[(f+g)^2 - (f-g)^2] \tag{2.24}$$

Therefore, $fg$ is also measurable.

$\square$

下面给出数学分析中曾介绍过的**几乎处处**的定义.

**定义 2.1.2.** A property or statement is said to hold **almost everywhere (a.e.)** if it is true except on a set of measure zero.

**例 2.1.2.**

$$f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases} \tag{2.25}$$

We say $f$ is continuous a.e. on $[0, 1]$ since $D(f) = \{1\}$ has measure zero.

下面说明几乎处处相等可保持函数可测性.

**命题 2.1.1.** If $f$ is measurble and $f = g$ a.e. , then $g$ is measurable.

**证明.** Since $f$ is measurable and

$$g = (g - f) + f \tag{2.26}$$

then we shall proof that $g - f$ is measurable.

Let $A := \{x \mid g(x) - f(x) \ne 0\}$, then $m(A) = 0$. We get

$$\forall a \ge 0, (g-f)^{-1}((-\infty, a]) = (\mathbb{R}^d \backslash A) \cup N, \ where \ N \subset A \tag{2.27}$$

Since $m(A) = 0$, then $N$ is measurable and $m(N) = 0$. So $(g-f)^{-1}((-\infty, a])$ is measurable.

Therefore, $g - f$ is measurable. Then $g$ is measurable.

$\square$

## 2.2 *Measurable functions are nearly simple*

本节来介绍一个非常重要的定理. 即可测函数可由简单函数逼近.

**特征函数**　下面先来介绍**特征函数**的定义.

**定义 2.2.1.** If $E \subset \mathbb{R}$, the **characteristic / indicator function $\chi_E / \mathbb{1}_E$** of E is defined by

$$\chi_E(x) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{if } x \notin E \end{cases} \tag{2.28}$$

下面给出可测集与其对应特征函数的关系.

**命题 2.2.1.** $\chi_E$ is measurable $\Leftrightarrow$ $E$ is measurable

**证明.** Since

$$\chi_E^{-1}((-\infty, a]) = \begin{cases} \varnothing, & a < 0 \\ E^c, & 0 \le a < 1 \\ \mathbb{R}^d, & a \ge 1 \end{cases} \tag{2.29}$$

Then $E$ is measurable $\Rightarrow \chi_E$ is measurable.

$\chi_E$ is measurable $\Rightarrow \chi_E^{-1}((-\infty, a]) = E^c$ is measurable. $\Rightarrow E$ is measurable. $\qquad \square$

下面给出特征函数的基本性质.

**命题 2.2.2.** [Property].

(1) If $A \cap B = \varnothing$, then

$$\chi_{A \cup B} = \max\{\chi_A, \chi_B\} = \chi_A + \chi_B \tag{2.30}$$

(2) $\chi_{A \cap B} = \min\{\chi_A, \chi_B\} = \chi_A \cdot \chi_B$.

*Simple functions*    对特征函数做线性组合，即可得到**简单函数**.

定义 **2.2.2.** A **simple function** on $\mathbb{R}^d$ is a finite linear combination

$$f(x) = \sum_{j=1}^{n} a_j \chi_{E_j}(x) \tag{2.31}$$

where each $E_j$ is measurable and $m(E_j) < \infty$.

**注.** 此处定义中并未要求 $\{E_j\}_{j=1}^{n}$ disjoint. 而事实上这便引出了下面介绍的**标准形式**.

下面的命题说明了每个简单函数都可写为**标准形式** ($\{E_j\}_{j=1}^{n}$ disjoint).

命题 **2.2.3.** Every simple function $f$ has a **standard representaion**

$$f = \sum_{k=1}^{N} a_k \chi_{E_k}, \; where \; \{E_j\}_{k=1}^{N} \; are \; disjoint \tag{2.32}$$

证明. Suppose $f = \sum_{k=1}^{N} b_k \chi_{E_k}$, $\{E_j\}_{k=1}^{N}$ may not be disjoint.
Since $\{E_j\}_{k=1}^{N}$ is finite, the number of elements of range f is also finite. Suppose

$$range \; f = \{a_1, \cdots, a_M\} \tag{2.33}$$

Then let $F_k = f^{-1}(\{a_k\})$, then $\{F_k\}_{k=1}^{M}$ are disjoint. Therefore, we get the standard representation

$$f = \sum_{k=1}^{M} a_k \chi_{F_k} \tag{2.34}$$

$\square$

**简单函数逼近可测函数**　下面给出一个定理，说明任一可测函数可由简单函数列逼近.

**定理 2.2.1.** Suppose $f : \mathbb{R}^d \longrightarrow [-\infty, \infty]$ is measurable.

Then there exists a sequence $\{\varphi_n\}$ of simple functions, s. t.

$$0 \leq |\varphi_1| \leq |\varphi_2| \leq \cdots \leq |f| \tag{2.35}$$

$$\lim_{k \to \infty} \varphi_k(x) = f(x), \ \textit{for all } x \tag{2.36}$$

and $\varphi_k \to f$ uniformly on any set on which $f$ is bounded.

**证明.** 下面从两方面分类讨论，即非负函数 & 变号函数，$f$ 有界 & 无界.

(1) 非负函数 $f : \mathbb{R}^d \longrightarrow [0, \infty]$.

　　$1°$ $f$ is bounded. Assume $|f(x)| \leq M$.

　　Let[2]

$$E_n^k = f^{-1}((\frac{k}{2^n}, \frac{k+1}{2^n}]), k = 0, \cdots, N_n \tag{2.37}$$

$$\varphi_n(x) = \frac{k}{2^n}, \ \textit{if } x \in E_n^k \tag{2.38}$$

　　Then

$$\varphi_n(x) = \sum_{k=0}^{N_n} \frac{k}{2^n} \chi_{E_n^k}(x) \tag{2.39}$$

　　Therefore[3]

$$|\varphi_n(x) - f(x)| \leq \frac{1}{2^n} \to 0 \ (\textit{independent of } x) \tag{2.40}$$
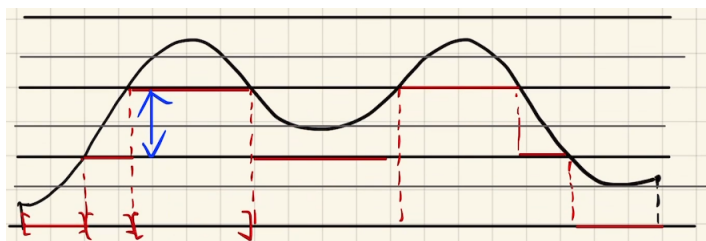
$\Rightarrow \varphi_n \to f$ uniformly.



图 2.1: 对 $f$ 值域进行分划

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[2]$E_n^k$ 表示第 n 次对值域进行分划后产生的第 k 个值域区间，其中 $\frac{N_n+1}{2^n} \geq M$.

[3]$|\varphi_n(x) - f(x)|$ 小于等于第 n 次分划后两个相邻值域区间的步长值，即 $\frac{1}{2^n}$.

$2°$ $f$ is unbounded. (<u>idea</u>: truncation，将 $f$ 截断为一列有界函数列，并逐点收敛于 $f$)

Let

$$f_k(x) = \begin{cases} f(x), & \text{if } f(x) \le k \\ k, & \text{if } f(x) > k \end{cases} \tag{2.41}$$

Then $f_k(x) \to f(x)$, $\forall x \in \mathbb{R}^d$.

Since $f_k$ is bounded, by the previous result in $1°$,

For each $k$, $\exists$ a sequence of simple functions $\{\psi_{kn}\}_{n=1}^\infty$, s. t.

$$\psi_{kn}(x) \to f_k(x), \ \forall x \tag{2.42}$$

So we get

$$\begin{array}{ccccccc} \psi_{11} & \psi_{12} & \psi_{13} & \cdots & \to & f_1 \\ \psi_{21} & \psi_{22} & \psi_{23} & \cdots & \to & f_2 \\ \psi_{31} & \psi_{32} & \psi_{33} & \cdots & \to & f_3 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \downarrow \\ & & & & & f \end{array} \tag{2.43}$$

From the previous results in $1°$, we get

$$|\psi_{kn}(x) - f_k(x)| \le \frac{1}{2^n} \tag{2.44}$$

Let $n = k$, then $|\psi_{kk}(x) - f_k(x)| \le \frac{1}{2^k}$. Let $\varphi_k = \psi_{kk}$, then

$$|\varphi_k(x) - f(x)| \le |\varphi_k(x) - f_k(x)| + |f_k(x) - f(x)| \tag{2.45}$$

Since $f_k(x) \to f(x)$, we get $\varphi_k(x) \to f(x)$, $\forall x$, where $\{\varphi_k = \psi_{kk}\}_{k=1}^\infty$ are simple functions.

(2) 变号函数 $f : \mathbb{R}^d \longrightarrow [-\infty, \infty]$.

We denote that

$$f^+(x) := \max\{f(x), 0\} \tag{2.46}$$

$$f^-(x) := \max\{-f(x), 0\} \tag{2.47}$$

By the previous results in (1), there exist sequences of simple functions $\{\varphi_k\}_{k=1}^\infty$, $\{\psi_k\}_{k=1}^\infty$, s. t.

$$\varphi_k \to f^+ \ and \ \psi_k \to f^- \ pointwisely \tag{2.48}$$

We can observe that $f = f^+ - f^-$ and $|f| = f^+ - f^-$.

Let $\phi_k(x) = \varphi_k(x) - \psi_k(x)$, then $\phi_k$ is a simple function with $\phi_k \to f$ pointwisely.

$\square$

**阶梯函数逼近可测函数**　在证明了可测函数可由简单函数逼近后，我们更进一步，来说明可测函数可由更加简单的**阶梯函数**来逼近.

先给出**阶梯函数**的定义.

**定义 2.2.3.** A **step function** is a finite sum

$$f = \sum_{k=1}^{N} a_k \chi_{R_k}, \text{ where } R_k \text{ is a rectangle} \tag{2.49}$$

**注.** 阶梯函数 & 简单函数的区别在于，简单函数是作用于有限个**可测集** $E_k$，而阶梯函数是作用于有限个**矩形** $R_k$.

下面的定理说明了 measurable functions are almost step functions.

**定理 2.2.2.** Suppose $f$ is measurable on $\mathbb{R}^d$. Then there exists a sequence of step functions $\{\psi_k\}_{k=1}^{\infty}$, s. t.

$$\lim_{k \to \infty} \psi_k(x) = f(x), \ a.e. \ x \tag{2.50}$$

**注.** 首先介绍函数列收敛点集的几种不同的等价表述：

$$\{x \mid \forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, \ |f_n(x) - f(x)| < \epsilon\} \tag{2.51}$$

$$\Leftrightarrow \{x \mid \forall n \in \mathbb{N}, \exists N \in \mathbb{N}, \forall k \geq N, \ |f_k(x) - f(x)| < \frac{1}{n}\} \tag{2.52}$$

$$\Leftrightarrow \bigcap_{n=1}^{\infty} \bigcup_{N=1}^{\infty} \bigcap_{k=N}^{\infty} \{x \mid \ |f_k(x) - f(x)| < \frac{1}{n}\} \tag{2.53}$$

从而可以得到函数列发散点集 (Negation)：

$$\{x \mid \exists n \in \mathbb{N}, \forall N \in \mathbb{N}, \exists k \geq N, \ |f_k(x) - f(x)| \geq \frac{1}{n}\} \tag{2.54}$$

$$\Leftrightarrow \bigcup_{n=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \{x \mid \ |f_k(x) - f(x)| \geq \frac{1}{n}\} \tag{2.55}$$

$$\Leftrightarrow \bigcap_{N=1}^{\infty} \bigcup_{k=N}^{\infty} \{x \mid \ f_k(x) \neq f(x)\} \tag{2.56}$$

**证明.** (证明思路：先用阶梯函数逼近简单函数，再用简单函数逼近可测函数.)

It suffices to show that $\chi_E$ can be approxiamted by step functions, for any measurable set $E$.

According to Thm1.3.4 (iv)

Let $f = \chi_E$, then $\forall \epsilon > 0$, $\exists$ cubes $\bigcup\limits_{j=1}^{N} Q_j$, s. t.

$$m(E \triangle \bigcup_{j=1}^{N} Q_j) \le \epsilon \tag{2.57}$$

By considering the grid formed by extending the sides of these cubes, there exists almost disjoint rectangles $\{\widetilde{R_j}\}_{j=1}^{M}$, s. t.

$$\bigcup_{j=1}^{N} Q_j = \bigcup_{j=1}^{M} \widetilde{R_j} \tag{2.58}$$

By taking ranctangles $R_j$ contained in $\widetilde{R_j}$, we can find a collection of disjoint rectangles $\{R_j\}_{j=1}^{M}$, s. t.

$$m(E \triangle \bigsqcup_{j=1}^{M} R_j) \le 2\epsilon \tag{2.59}$$

For every $k \in \mathbb{N}$, there exists disjoint rectangles $\{R_j\}_{j=1}^{M}$, s. t.

$$m(E \triangle \bigsqcup_{j=1}^{M} R_j) \le \frac{1}{2^{k+1}} \tag{2.60}$$

There also exists a step function $\psi_k$

$$\psi_k(x) := \chi_{\bigcup_{j=1}^{M} R_j}(x) = \sum_{j=1}^{M} \chi_{R_j}(x) \tag{2.61}$$

Let

$$E_k := \{x \mid f_k(x) \ne f(x)\} \tag{2.62}$$

Since $E_k \subset E \triangle \bigsqcup\limits_{j=1}^{M} R_j$, then $m(E_k) \le \frac{1}{2^k}$. Let[4]

$$F_j = \bigcup_{j=k+1}^{\infty} E_j, \quad F = \bigcap_{k=1}^{\infty} F_k \tag{2.63}$$

Then $\psi_k(x) \to f(x)$, $\forall x \in F^c$. Since

$$m(F) \le m(F_k), \quad \forall k \in \mathbb{N} \tag{2.64}$$

$$m(F_k) = m(\bigcup_{j=k+1}^{\infty} E_j) \le \sum_{j=k+1}^{\infty} m(E_j) \le \frac{1}{2^k} \tag{2.65}$$

Therefore, $m(F) = 0$. $\lim\limits_{k \to \infty} \psi_k(x) = f(x)$, $a.e.$ $x$. $\qquad\square$

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[4]根据注中式 (2.56)，$F$ 即为函数列 $\{\psi_k\}_{k=1}^{\infty}$ 的发散点集，从而 $\psi_k(x) \to f(x)$ 在 $F^c$ 上收敛.

# 第三章  *Integration Theory*

## 3.1  *The Lebesgue integral*

*Lebesgue Integral* 的构造可以分为三步，分别为构造下列函数的积分:

1. **Simple functions**

2. **Non-negative measurable functions**

$$\int f := \sup \left\{ \int \varphi \mid \varphi \ simple, \ 0 \le \varphi \le f \right\} \tag{3.1}$$

3. **General case**

$$f = f^{+} - f^{-} \tag{3.2}$$

$$\int f := \int f^{+} - \int f^{-} \tag{3.3}$$

### 3.1.1  *Simple functions*

定义   下面先给出非负简单函数在**标准形式**下的**积分**定义.

定义 **3.1.1.** If $\varphi$ is a non-negative simple function with **standard representation**

$$\varphi(x) = \sum_{k=1}^{M} a_k \chi_{E_k}(x) \tag{3.4}$$

We define the **Lebesgue integral** of $\varphi$ by

$$\int_{\mathbb{R}^d} \varphi(x) dx = \sum_{k=1}^{M} a_k m(E_k) \tag{3.5}$$

If $E$ is a measurable subset of $\mathbb{R}^d$ with finite measure, then

$$\varphi(x)\chi_E(x) = \sum_{k=1}^{M} a_k \chi_{E_k}(x)\chi_E(x) = \sum_{k=1}^{M} a_k \chi_{E_k \cap E}(x) \tag{3.6}$$

is also a simple function, and define

$$\int_E \varphi(x) dx = \int_{\mathbb{R}^d} \varphi(x)\chi_E(x) dx \tag{3.7}$$

**注.** • 此处仅对**标准形式**定义了积分. 事实上，此处定义的积分与简单函数的表达形式无关 (即 **Property 1.**).

• 关于记号，当测度非常明确时，大多数情况下可简写，如

$$\int_E \varphi(x)dx \;\Rightarrow\; \int_E \varphi \tag{3.8}$$

$$\int_{\mathbb{R}^d} \varphi(x)dx \Rightarrow \int \varphi \tag{3.9}$$

当为了强调我们选择了何种测度 $\mu$ 时，还可用以下的记号：

$$\int_E \varphi(x)d\mu(x) \tag{3.10}$$

**Property** 下面给出简单函数积分的性质.

**Property 1. Independence of the representation**.

If $\varphi = \sum_{k=1}^{N} a_k \chi_{E_k}$ is any representation of $\varphi$, then

$$\int \varphi = \sum_{k=1}^{N} a_k m(E_k) \tag{3.11}$$

在证明这个性质之前，先来证明一条引理.(书[1]Exercises Of Chapter 2 的第 1 题)

**引理 3.1.1.** Given a collection of sets $\{F_k\}_{k=1}^{n}$, there exists another collection $\{\widetilde{F}_j\}_{j=1}^{N}$ with $N = 2^n - 1$, so that

$$(i). \qquad \bigcup_{k=1}^{n} F_k = \bigcup_{j=1}^{N} \widetilde{F}_j \tag{3.12}$$

$$(ii). \qquad \{\widetilde{F}_j\}_{j=1}^{N} \ are \ disjoint \tag{3.13}$$

$$(iii). \qquad F_k = \bigcup_{\widetilde{F}_j \subset F_k} \widetilde{F}_j \tag{3.14}$$

**证明.** Consider the collection

$$\mathcal{F} := \{\bigcup_{k=1}^{n} G_k - \bigcap_{k=1}^{n} F_k^c \;\Big|\; G_k \ denotes \ F_k \ or \ F_k^c\} \tag{3.15}$$

$\square$

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[1]参考书籍：《*Real Analysis − −Measure Theroy, Integration, & Hilbert Spaces*》— *Elias M. Stein*

下面来证明原命题.

证明. According to Lemma 3.1.1, there exists another decompositon of $\bigcup\limits_{k=1}^{N} E_k$, i.e.

$$\bigcup_{j=1}^{M} \widetilde{E_j} = \bigcup_{k=1}^{N} E_k \tag{3.16}$$

where $\{\widetilde{E_j}\}_{j=1}^{M}$ are disjoint, and for each $1 \le k \le M$,

$$E_k = \bigcup_{\widetilde{E_j} \subset E_k} \widetilde{E_j} \tag{3.17}$$

Let

$$\widetilde{a_j} := \sum_{\widetilde{E_j} \subset E_k} a_k \tag{3.18}$$

Then clearly

$$\varphi = \sum_{j=1}^{M} \widetilde{a_j} \chi_{\widetilde{E_j}} \tag{3.19}$$

Since $\{\widetilde{E_j}\}_{j=1}^{M}$ are disjoint, we get

$$\int \varphi = \sum_{j=1}^{M} \widetilde{a_j} m(\widetilde{E_j}) = \sum_{j=1}^{M} \sum_{\widetilde{E_j} \subset E_k} a_k m(\widetilde{E_j}) = \sum_{k=1}^{N} a_k m(E_k) \tag{3.20}$$

$\square$

**Property 2. Linearity**.

If $\varphi$ and $\psi$ are non-negative simple, and $a, b \ge 0$, then

$$\int (a\varphi + b\psi) = a \int \varphi + b \int \psi \tag{3.21}$$

证明. 下面分为两步来证明.

(a) $\forall c \ge 0, \int c\varphi = c \int \varphi$.

Suppose $\varphi = \sum\limits_{k=1}^{M} a_k \chi_{E_k}$, where $\{E_k\}_{k=1}^{M}$ are disjoint. Then

$$c\varphi = \sum_{k=1}^{M} c a_k \chi_{E_j} \tag{3.22}$$

is also a non-negative simple function. Therefore,

$$\int c\varphi = \sum_{k=1}^{M} c a_k m(E_k) = c \sum_{k=1}^{M} a_k m(E_k) = c \int \varphi \tag{3.23}$$

35

(b) $\int (\varphi + \psi) = \int \varphi + \int \psi$.

Suppose

$$\varphi = \sum_{k=1}^{M} a_k \chi_{E_k}, \ \ \psi = \sum_{j=1}^{N} b_j \chi_{F_j} \tag{3.24}$$

where both $\{E_k\}_{k=1}^{M}$ and $\{F_j\}_{j=1}^{N}$ are disjoint and $\mathbb{R}^d = \bigcup_{k=1}^{M} E_k = \bigcup_{j=1}^{N} F_j$. Since

$$E_k = E_k \cap \mathbb{R}^d = E_k \cap \bigsqcup_{j=1}^{N} F_j = \bigsqcup_{j=1}^{N} (E_k \cap F_j) \tag{3.25}$$

Then

$$\varphi = \sum_{k=1}^{M} a_k \chi_{E_k} = \sum_{k=1}^{M} a_k \chi_{\bigsqcup_{j=1}^{N} (E_k \cap F_j)} = \sum_{k=1}^{M} \sum_{j=1}^{N} a_k \chi_{E_k \cap F_j} \tag{3.26}$$

Similarly

$$\psi = \sum_{j=1}^{N} b_j \chi_{F_j} = \sum_{j=1}^{N} b_k \chi_{\bigsqcup_{k=1}^{M} (E_k \cap F_j)} = \sum_{j=1}^{N} \sum_{k=1}^{M} b_k \chi_{E_k \cap F_j} \tag{3.27}$$

Therefore

$$\varphi + \psi = \sum_{j,k} (a_k + b_j) \chi_{E_k \cap F_j} \tag{3.28}$$

$$\int (\varphi + \psi) = \sum_{j,k} (a_k + b_j) m(E_k \cap F_j) \tag{3.29}$$

$$= \sum_{j,k} a_k m(E_k \cap F_j) + \sum_{j,k} b_j m(E_k \cap F_j) \tag{3.30}$$

$$= \int \varphi + \int \psi \tag{3.31}$$

$\square$

# Property 3. **Monotonicity**.

If $\varphi \leq \psi$ are non-negative and simple, then

$$\int \varphi \leq \int \psi \tag{3.32}$$

证明. Suppose

$$\varphi = \sum_{k=1}^{M} a_k \chi_{E_k}, \ \ \psi = \sum_{j=1}^{N} b_j \chi_{F_j} \tag{3.33}$$

36

where both $\{E_k\}_{k=1}^M$ and $\{F_j\}_{j=1}^N$ are disjoint. Similar to the proof in Property 2, we get

$$\psi - \varphi = \sum_{j,k} (b_j - a_k)\chi_{E_k \cap F_j} \tag{3.34}$$

Since $\varphi(x) \leq \psi(x), \forall x \in \mathbb{R}^d$, then $\psi - \varphi$ is non-negative and simple. Therefore,

$$\int (\psi - \varphi) = \sum_{j,k} (b_j - a_k)m(E_k \cap F_j) \geq 0 \implies \int \varphi \leq \int \psi \tag{3.35}$$

$\square$

## Property 4. Additivity.

If $\{E_k\}_{k=1}^\infty$ are disjoint subsets of $\mathbb{R}^d$ with finite measure, then

$$\int_{\bigcup_{k=1}^\infty E_k} \varphi = \sum_{k=1}^\infty \int_{E_k} \varphi \tag{3.36}$$

注. 首先回顾 *abstract measure* 的定义.

定义 **3.1.2.** Let $X$ be a set and let $\mathcal{M}$ be a $\sigma - algebra$ on $X$.

A **<u>measure</u>** on $\mathcal{M}$ is a function $\mu : \mathcal{M} \longrightarrow [0, \infty]$, s. t.

 (i) $\mu(\varnothing) = 0$.

 (ii) If $\{E_j\}_{j=1}^\infty \subset \mathcal{M}$ are disjoint, then

$$\mu(\bigcup_{j=1}^\infty E_j) = \sum_{j=1}^\infty \mu(E_j) \tag{3.37}$$

回到我们积分的性质上来. 下面我们将说明, 对于任一给定的非负简单函数 $\varphi$, 将 $\varphi$ 在任一可测集 $A$ 上的积分看作 *Lebesgue* $\sigma - algebra$ $\mathcal{L}$ 上的映射, 则该映射为定义在 $\mathcal{L}$ 上的测度.(从而 Property 4. 作为测度的必要条件自然成立)

命题 **3.1.1.** For any fixed non-negative and simple function $\varphi$, the map

$$\mu : \mathcal{L} \longrightarrow [0, \infty] \tag{3.38}$$

$$A \longmapsto \int_A \varphi \tag{3.39}$$

is a measure on $\mathcal{L}$.

证明. Suppose $\{A_j\}_{j=1}^{\infty} \subset \mathcal{L}$ are disjoint, and

$$\varphi = \sum_{k=1}^{M} a_k \chi_{E_k}, \ \ where \ \{E_k\}_{k=1}^{M} \ are \ disjoint \tag{3.40}$$

Let $A = \bigcup_{j=1}^{\infty} A_j$, then

$$\int_{\bigcup_{j=1}^{\infty} A_j} \varphi = \int_A \varphi = \int \varphi \chi_A = \int (\sum_{k=1}^{M} a_k \chi_{E_k \cap A}) \tag{3.41}$$

$$= \sum_{k=1}^{M} a_k m(E_k \cap A) \tag{3.42}$$

$$= \sum_{k=1}^{M} a_k m(E_k \cap (\bigcup_{j=1}^{\infty} A_j)) \tag{3.43}$$

$$= \sum_{k=1}^{M} a_k m(\bigsqcup_{j=1}^{\infty} (E_k \cap A_j)) \tag{3.44}$$

$$= \sum_{k=1}^{M} a_k \sum_{j=1}^{\infty} m(E_k \cap A_j) \tag{3.45}$$

$$= \sum_{k=1}^{M} \sum_{j=1}^{\infty} a_k m(E_k \cap A_j) \tag{3.46}$$

Since positive series always converges in $[0, \infty]$, then

$$\int_A \varphi = \sum_{k=1}^{M} \sum_{j=1}^{\infty} a_k m(E_k \cap A_j) = \sum_{j=1}^{\infty} \sum_{k=1}^{M} a_k m(E_k \cap A_j) = \sum_{j=1}^{\infty} \int_{A_j} \varphi \tag{3.47}$$

Therefore, **the integral on any non-negative simple function is accually a measure on $\mathcal{L}$.** $\square$

### 3.1.2 *Non − negative measurable functions*

为了讨论的方便，先给出**非负可测函数**的一个记号.

$$\mathcal{M}^+ := \{all\ non-negative\ measurable\ functions\} \tag{3.48}$$

**定义** 下面给出非负可测函数的积分的定义.

**定义 3.1.3.** For $f \in \mathcal{M}^+$, we define

$$\int f(x)dx := \sup\left\{\int \varphi(x)dx \mid 0 \le \varphi \le f,\ \varphi\ simple\right\} \tag{3.49}$$

**注.** 此处对 Non-negative measurable function 积分的定义兼容定义 3.1.1 中对 Non-negative simple function 积分的定义，具体表现为：$\forall \varphi_0$ non-negative and simple,

$$\sup\left\{\int \varphi(x)dx \mid 0 \le \varphi \le \varphi_0,\ \varphi\ simple\right\} = \int \varphi_0(x)dx \tag{3.50}$$

**性质** 下面来验证定义 3.1.3 中定义的积分满足几条基本性质.

**Property 1. Monotonicity**.

Let $f, g \in \mathcal{M}^+$. Then

$$\int f \le \int g \quad if\ \ f \le g \tag{3.51}$$

**证明.** Let

$$A = \{\varphi\ simple \mid 0 \le \varphi \le f\} \tag{3.52}$$

$$B = \{\psi\ simple \mid 0 \le \psi \le g\} \tag{3.53}$$

Then for all $\varphi \in A$, $0 \le \varphi \le f \le g \Rightarrow \varphi \in B \Rightarrow A \subset B$. Since

$$\int f = \sup_{\varphi \in A}\left\{\int \varphi\right\},\ \int g = \sup_{\psi \in B}\left\{\int \psi\right\} \tag{3.54}$$

Therefore

$$\int f \le \int g \tag{3.55}$$

$\square$

**Property 2.** 齐次性.

Let $f \in \mathcal{M}^+$. If $c \geq 0$, then

$$\int cf = c \int f \tag{3.56}$$

证明. Assume $c > 0$. Then

$$\int cf = \sup\left\{\int \varphi \mid 0 \leq \varphi \leq cf, \ \varphi \ simple\right\} \tag{3.57}$$

$$= \sup\left\{\int \varphi \mid 0 \leq \frac{\varphi}{c} \leq f, \ \varphi \ simple\right\} \tag{3.58}$$

$$\overset{\psi = \frac{\varphi}{c}}{=} \sup\left\{\int c\psi \mid 0 \leq \psi \leq f, \ \psi \ simple\right\} \tag{3.59}$$

$$= c \sup\left\{\int \psi \mid 0 \leq \psi \leq f, \ \psi \ simple\right\} \tag{3.60}$$

$$= c \int f \tag{3.61}$$

$\square$

**单调收敛定理** 下面我们正式迈入实分析的"大门", 介绍第一个**收敛定理**.

**定理 3.1.2. The Monotone Convergence Theorem**.

If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+, f_j \leq f_{j+1}$ for all $j$, and $\lim_{n\to\infty} f_n = f$, then

$$\int f = \lim_{n\to\infty} \int f_n \tag{3.62}$$

注. • 此即为"**单调收敛定理**", 这个定理说明了对于**单调递增**的**非负可测**函数列, 其**积分与极限可交换次序**. 具体表现为

$$\int f = \int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n \tag{3.63}$$

• 该定理还说明了, 我们可以给出非负可测函数的另一个更自然的等价定义, 即用非负简单函数列的积分逼近非负可测函数的积分.

**定义 3.1.4.** For $f \in \mathcal{M}^+$, we can also define

$$\int f := \lim_{n\to\infty} \int \varphi_n \tag{3.64}$$

where $\varphi_n \to f$ and $0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq f$ by Thm 2.2.1.

并且该定理说明了该积分定义的**唯一性**及 **well-defined**.

在证明定理前，先来证明一个引理 (将定理 1.3.3 (i) 拓展到一般的抽象测度上).

**引理 3.1.3.** Let $X$ be a set, $\mathcal{M}$ be a $\sigma - algebra$ on $X$, $\mu : \mathcal{M} \longrightarrow [0, \infty]$ be a measure on $\mathcal{M}$. If $\{E_n\}_{n=1}^{\infty} \subset \mathcal{M}$, $E_n \nearrow E$, then

$$\lim_{n \to \infty} \mu(E_n) = \mu(E) \tag{3.65}$$

**证明.** 证明过程与 Thm 1.3.3 完全一致 (仅用到了测度的可数可加性).

Let $S_1 = E_1$, $S_k = E_k - E_{k-1}$, $\forall k \geq 2$. Then $\{S_k\}_{n=1}^{\infty} \subset \mathcal{M}$ are disjoint.

Since $E = \bigcup_{k=1}^{\infty} E_k = \bigcup_{k=1}^{\infty} S_k$, then

$$\mu(E) = \mu\left(\bigsqcup_{k=1}^{\infty} S_k\right) = \sum_{k=1}^{\infty} \mu(S_k) = \lim_{n \to \infty} \sum_{k=1}^{n} \mu(S_k) = \lim_{n \to \infty} \mu\left(\bigsqcup_{k=1}^{n} S_k\right) = \lim_{n \to \infty} \mu(E_n) \tag{3.66}$$

$\square$

下面证明原定理.

**证明.**

- $\lim_{n \to \infty} \int f_n \leq \int f$.

  Since $f_n \leq f$, $\forall n$, then

  $$\int f_n \leq \int f, \ \forall n \tag{3.67}$$

  Since $\{\int f_n\}_{n=1}^{\infty}$ always converges in $[0, \infty]$, then let $n \to \infty$, we get

  $$\lim_{n \to \infty} \int f_n \leq \int f \tag{3.68}$$

- $\lim_{n \to \infty} \int f_n \geq \int f$.

  Fix $0 < a < 1$, for any $0 \leq \varphi \leq f$ simple, let

  $$E_n = \{x \mid f_n(x) \geq a\varphi(x)\} \tag{3.69}$$

  Then since $\forall x \in E_n$, we have $f_{n+1}(x) \geq f_n(x) \geq a\varphi(x) \Rightarrow x \in E_{n+1} \Rightarrow E_n \subset E_{n+1}$.

  Then $E_n \nearrow$. Since

  $$\int_{\mathbb{R}^d} f_n \geq \int_{E_n} f_n \geq \int_{E_n} a\varphi, \ \forall n \tag{3.70}$$

  Let $n \to \infty$, we get

  $$\lim_{n \to \infty} \int_{\mathbb{R}^d} f_n \geq \lim_{n \to \infty} \int_{E_n} a\varphi \tag{3.71}$$

41

Then we have to calculate $\lim\limits_{n\to\infty}\int_{E_n} a\varphi$:

– Since $a\varphi$ is non-negative and simple, by Prop 3.1.1, the map

$$\mu : \mathcal{L} \longrightarrow [0, \infty] \tag{3.72}$$

$$E \longmapsto \int_E a\varphi \tag{3.73}$$

is a measure on the collection of Lebesgue measurable sets $\mathcal{L}$. (将积分视作测度)

Since $\{E_n\}_{n=1}^{\infty} \subset \mathcal{L}$ and $E_n \nearrow$, by Lemma 3.1.3, we get

$$\lim_{n\to\infty} \mu(E_n) = \mu(\bigcup_{n=1}^{\infty} E_n) \tag{3.74}$$

i.e.

$$\lim_{n\to\infty} \int_{E_n} a\varphi = \int_{\bigcup_{n=1}^{\infty} E_n} a\varphi \tag{3.75}$$

For all $x \in \mathbb{R}^d$, since $a\varphi(x) < f(x)$ and $f_n \to f$, there exists $N_x \in \mathbb{N}$, s. t.

$$f_n(x) \geq a\varphi(x), \ \forall n \geq N_x \tag{3.76}$$

which indicates $x \in E_{N_x}$ for some $N_x$. Therefore

$$\bigcup_{n=1}^{\infty} E_n = \mathbb{R}^d \ \Rightarrow \ \lim_{n\to\infty} \int_{E_n} a\varphi = \int_{\bigcup_{n=1}^{\infty} E_n} a\varphi = \int_{\mathbb{R}^d} a\varphi \tag{3.77}$$

Therefore, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \geq \lim_{n\to\infty} \int_{E_n} a\varphi = \int_{\mathbb{R}^d} a\varphi \tag{3.78}$$

Let $a \to 1$, then

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \geq \int_{\mathbb{R}^d} \varphi \tag{3.79}$$

Since $\varphi$ is arbitratry, taking the supremum over $\varphi$, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n \geq \sup \{ \int_{\mathbb{R}^d} \varphi \mid 0 \leq \varphi \leq f, \ \varphi \ simple \} = \int f \tag{3.80}$$

□

**函数项级数的可数可加性** 接下来我们将给出**单调收敛定理**在函数项级数上的表达形式，它说明了对于**非负可测函数项级数**，其<u>积分与求和可交换次序</u>.

在此之前，先来证明有限项的情况.

(此也可视作非负可测函数积分的**Property 线性性**的一部分.)

**命题 3.1.2. Linearity**.

If $f, g \in \mathcal{M}^+$, then

$$\int (f + g) = \int f + \int g \tag{3.81}$$

**证明.** By Thm 2.2.1 and Thm 3.1.2, there exists sequences of non-negative and simple functions $\{\varphi_n\}_{n=1}^{\infty}$ and $\{\psi_n\}_{n=1}^{\infty}$, $\varphi_n \to f$ and $\psi_n \to g$, s. t.

$$\int f = \lim_{n \to \infty} \int \varphi_n, \ \int g = \lim_{n \to \infty} \int \psi_n \tag{3.82}$$

Since $\varphi_n + \psi_n$ is still non-negative and simple, then

By the Linearity of integral on non-negative and simple functions, (**Property 2.** in §3.1.1)

$$\int (\varphi_n + \psi_n) = \int \varphi_n + \int \psi_n \tag{3.83}$$

Let $n \to \infty$, by Thm 3.1.2, we get (**极限与积分交换次序**)

$$\int (f + g) = \int f + \int g \tag{3.84}$$

$\square$

根据 Prop 3.1.2，由归纳法，容易得到其对任意**有限项**函数项级数都成立.

下面给出函数项级数上的**单调收敛定理**.

**定理 3.1.4. Monotone Convergence Theorem (MCT , series version)**.

If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+$ and $f = \sum\limits_{n=1}^{\infty} f_n$, then

$$\int f = \sum_{n=1}^{\infty} \int f_n \qquad (3.85)$$

**注.** 该定理说明了对于**非负可测函数项级数**，其<u>积分与求和可交换次序</u>.

**证明.** Let $F_n = \sum\limits_{k=1}^{n} f_k$, then $F_n \nearrow \sum\limits_{k=1}^{\infty} f_k = f$. By **MCT** (Thm 3.1.2),

$$\lim_{n \to \infty} \int F_n = \int f \qquad (3.86)$$

i.e.

$$\lim_{n \to \infty} \int \sum_{k=1}^{n} f_k = \int f \qquad (3.87)$$

By the **Linearity** of integral on non-negative functions (Prop 3.1.2),

$$\lim_{n \to \infty} \int \sum_{k=1}^{n} f_k = \lim_{n \to \infty} \sum_{k=1}^{n} \int f_k = \sum_{k=1}^{\infty} \int f_k = \int f \qquad (3.88)$$

$\square$

**积分的唯一性** 在实分析中，我们并不关心零测集上的各种性质，进而常常忽略函数在零测集上的情况. 在给出**单调收敛定理**的更一般版本前，我们先来给出**几乎处处**意义下，函数**积分的唯一性**.

下面的命题说明了，若两个非负可测函数**几乎处处**相等，则其积分相等.

**命题 3.1.3. Uniqueness**.

If $f \in \mathcal{M}^+$, then

$$\int f = 0 \iff f = 0 \ a.e. \tag{3.89}$$

**注.** 根据该命题，对于任意非负可测函数 $f, g$

$$\int f = \int g \iff \int (f - g) = 0 \iff f - g = 0 \ a.e. \iff f = g \ a.e. \tag{3.90}$$

**证明.**

- 充分性 "$\Leftarrow$": If $f = 0$ a.e.

  $\forall 0 \le \varphi \le f$ simple, $\varphi = 0$ a.e. . Let $E = \{x \mid \varphi(x) = 0\}$, then $m(E^c) = 0$.

$$\int \varphi = \int_E \varphi + \int_{E^c} \varphi = 0 + 0 = 0 \tag{3.91}$$

  Taking the supremum of $\varphi$, we get

$$\int f = \sup \left\{ \int \varphi \mid 0 \le \varphi \le f, \ \varphi \ simple \right\} = 0 \tag{3.92}$$

- 必要性 "$\Rightarrow$": If $\int f = 0$, let

$$E_n := \left\{ x \mid f(x) > \frac{1}{n} \right\} \tag{3.93}$$

  Then

$$\bigcup_{n=1}^{\infty} E_n = \{x \mid f(x) > 0\} = \{f \ne 0\} \tag{3.94}$$

  Suppose $m(\bigcup_{n=1}^{\infty} E_n) > 0$, then there exists $N \in \mathbb{N}$, s.t. $m(E_N) > 0$. Then

$$\int f \ge \int_{E_N} f > \frac{1}{N} m(E_N) > 0 \tag{3.95}$$

  which is a contradiction to $\int f = 0$.

  Therefore, $m(\bigcup_{n=1}^{\infty} E_n) = m(\{f \ne 0\}) = 0, f = 0$ a.e.

$\square$

**"几乎处处"版 MCT** 根据积分的唯一性 (命题 3.1.3)，下面说明在**"几乎处处收敛"**条件下，**单调收敛定理**成立 (积分与极限仍可交换次序).

**推论 3.1.5. a.e. MCT**.

If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+, f \in \mathcal{M}^+, f_n \nearrow f$ a.e. , then

$$\int f = \lim_{n \to \infty} \int f_n \tag{3.96}$$

**证明.** Let $f_n \nearrow f$ on $E$, then $m(E^c) = 0$ and $f_n - f_n \chi_E = 0$ a.e.
By Prop 3.1.3, we get

$$\int f_n = \int f_n \chi_E \tag{3.97}$$

Since $f_n \chi_E \nearrow f \chi_E$, then by **MCT** (Thm 3.1.2, 单调收敛定理)

$$\lim_{n \to \infty} \int f_n = \lim_{n \to \infty} \int f_n \chi_E = \int f \chi_E = \int_E f \tag{3.98}$$

Since $m(E^c) = 0$, then

$$\int f = \int_E f = \lim_{n \to \infty} \int f_n \tag{3.99}$$

($\forall 0 \leq \varphi \leq f$ simple, $\int \varphi = \int_E \varphi + \int_{E_c} \varphi = \int_E \varphi$. Taking the supremum of $\varphi \Rightarrow \int f = \sup\{\int \varphi\} = \int_E f$)

$\square$

**Fatou's Lemma** 我们首先来考虑一个问题，若我们将**单调收敛定理 (MCT)** 中的**"单调"**条件去掉，结论是否仍然成立 (积分与极限是否仍可交换次序)？即

$$\text{Suppose } f_n \to f \text{ a.e. , do we have } \int f_n \to \int f?$$

事实上答案为 absolutely no. 下面给出一个反例.

**例 3.1.1.** Consider $f_n = n\chi_{(0, \frac{1}{n})}$. Then $f_n \to 0$ a.e. on $[0, 1]$. However,

$$\int f_n = n \cdot \frac{1}{n} = 1, \ \forall n \in \mathbb{N} \nrightarrow 0 \tag{3.100}$$

事实上，将**"单调收敛"**条件整个去除，我们将得到如下的更一般的 **Fatou's Lemma**.

**定理 3.1.6. Fatou's Lemma**.

If $\{f_n\}_{n=1}^{\infty} \subset \mathcal{M}^+$, then

$$\int \liminf_{n \to \infty} f_n \leq \liminf_{n \to \infty} \int f_n \tag{3.101}$$

**注.** • 回顾函数列下极限的定义.

$$\liminf_{n \to \infty} f_n = \lim_{n \to \infty} \left( \inf_{k \geq n} f_k \right) \tag{3.102}$$

即对定义域上每一点 $x$，取数列 $\{f_n(x)\}_{n=1}^{\infty}$ 的下极限，再将所有的 $x$ 所对应的下极限拼成一个函数，即定义为函数列 $\{f_n\}_{n=1}^{\infty}$ 的下极限.

(上式右侧作用在固定的 $x$ 上，即为数列 $\{f_n(x)\}_{n=1}^{\infty}$ 下极限的定义.)

• **Fatou's Lemma** 告诉我们，对于任意一列非负可测函数列，其**函数列的下极限的积分**，要小于每个函数积分后得到的**积分数列的下极限**.

**证明.** Since

$$\liminf_{n \to \infty} f_n = \lim_{n \to \infty} \left( \inf_{k \geq n} f_k \right) \tag{3.103}$$

Let $g_n = \inf_{k \geq n} f_k$, then $g_n \nearrow \lim_{n \to \infty} g_n$. By **MCT** (Thm 3.1.2, 单调收敛定理),

$$\int \lim_{n \to \infty} g_n = \lim_{n \to \infty} \int g_n \tag{3.104}$$

i.e.

$$\int \liminf_{n \to \infty} f_k = \lim_{n \to \infty} \left( \int \inf_{k \geq n} f_k \right) \tag{3.105}$$

For each $n$, since $\inf_{k \geq n} f_k \leq f_j, \forall j \geq n$, then

$$\int \inf_{k \geq n} f_k \leq \int f_j, \ \forall j \geq n \tag{3.106}$$

Taking the infimum of $\{\int f_j\}_{j=n}^{\infty}$, then

$$\int \inf_{k \geq n} f_k \leq \inf_{j \geq n} \int f_j, \ \forall n \in \mathbb{N} \tag{3.107}$$

For $n$ is arbitrary, let $n \to \infty$, we get

$$\lim_{n\to\infty} \left( \int \inf_{k \geq n} f_k \right) \leq \lim_{n\to\infty} \left( \inf_{k \geq n} \int f_k \right) = \liminf_{n\to\infty} \int f_n \tag{3.108}$$

Therefore

$$\int \liminf_{n\to\infty} f_k = \lim_{n\to\infty} \left( \int \inf_{k \geq n} f_k \right) \leq \liminf_{n\to\infty} \int f_n \tag{3.109}$$

$\square$

### 3.1.3 *General case*

**可积函数** 跟 Riemann 积分类似，对于 Lebesgue 积分，我们也有**可积函数**的概念.

下面先让我们回到**非负可测**函数，定义非负可测函数中**可积**的概念.

**定义 3.1.5.** For $f \in \mathcal{M}^+$, if

$$\int f < \infty \tag{3.110}$$

Then we say $f$ is **Lebesgue integrable** or simply **integrable**.

下面扩展到一般的可测函数，给出其 **Lebesgue 积分**及**可积**的定义.

**定义 3.1.6.** For any $f$ measurable on $\mathbb{R}^d$

$$f^+(x) := \max\{f(x), 0\} \,,\; f^-(x) := \max\{-f(x), 0\} \tag{3.111}$$

If at least one of $\int f^+$ and $\int f^-$ is finite, we define the **integral of $f$**

$$\int f := \int f^+ - \int f^- \tag{3.112}$$

We say that $f$ is **(Lebesgue) integrable** if $|f|$ is integrable.

**注.** • 注意到

$$f = f^+ - f^- \tag{3.113}$$

$$|f| = f^+ + f^- \tag{3.114}$$

• 根据定义，对于任意可测函数 $f$，

$$f \text{ integrable} \;\Leftrightarrow\; |f| \text{ integrable} \;\Leftrightarrow\; \int |f| = \int f^+ + \int f^- < \infty \tag{3.115}$$

$$\Leftrightarrow\; f^+ \text{ and } f^- \text{ integrable} \tag{3.116}$$

即 $f$ 可积 $\Leftrightarrow \int f^+$ 和 $\int f^-$ 均有界.

**性质** 下面我们将说明，定义在任一集合 $X$ 上的**实可积函数**构成的空间 $\mathcal{L}^1$ 为**线性空间**，以及 $f \in \mathcal{L}^1$ 时的一些性质.

在此之前，先给出上述定义的一般的可测函数的积分的**基本性质**.

**命题 3.1.4.** Suppose $f, g \in \mathcal{L}$, then

1. **Linearity**: $\int (af + bg) = a \int f + b \int g$.

2. **Finite Additivity**:
$$\int_{\bigsqcup_{j=1}^{n} A_j} f = \sum_{j=1}^{n} \int_{A_j} f \tag{3.117}$$
where $\{A_j\}_{j=1}^{n}$ are disjoint.

3. **Monotonicity**: If $f \le g$, then $\int f \le \int g$.

4. **Triangle inequality**: $\left| \int f \right| \le \int |f|$.

**证明.**

2. : We shall show that $\int_{\bigsqcup_{j=1}^{n} A_j} f^+ = \sum_{j=1}^{n} \int_{A_j} f^+$ and $\int_{\bigsqcup_{j=1}^{n} A_j} f^- = \sum_{j=1}^{n} \int_{A_j} f^-$.

By **Thm 2.2.1**, there exists simple $\varphi_n \nearrow f^+$, then by **MCT (Thm 3.1.2, 单调收敛定理)**,
$$\int_{\bigsqcup_{j=1}^{n} A_j} f^+ = \lim_{n \to \infty} \int_{\bigsqcup_{j=1}^{n} A_j} \varphi_n \tag{3.118}$$

Since $\varphi_n$ are simple, by the **countable additivity** (简单函数的可数可加性), we have
$$\int_{\bigsqcup_{j=1}^{n} A_j} f^+ = \lim_{n \to \infty} \int_{\bigsqcup_{j=1}^{n} A_j} \varphi_n = \lim_{n \to \infty} \sum_{j=1}^{n} \int_{A_j} \varphi_n = \sum_{j=1}^{n} \lim_{n \to \infty} \int_{A_j} \varphi_n \tag{3.119}$$
$$\stackrel{\mathbf{MCT}}{=} \sum_{j=1}^{n} \int_{A_j} f^+ \tag{3.120}$$

4. 根据实数域上的三角不等式，we have
$$\left| \int f \right| = \left| \int f^+ - \int f^- \right| \le \left| \int f^+ \right| + \left| \int f^- \right| = \int f^+ + \int f^- = \int |f| \tag{3.121}$$

$\square$

现在我们便可以来说明，定义在任一集合 $X$ 上的**实可积函数**构成的空间 $\mathcal{L}^1$ 为**线性空间**.

**命题 3.1.5.** The set of integrable real-valued functions on $X$ is a real vector space.

**证明.** $\forall f, g \in \mathcal{L}^1$, if $a \in \mathbb{R}$,

$$\int |f + g| \leq \int (|f| + |g|) = \int |f| + \int |g| < \infty \tag{3.122}$$

$$\int |af| = |a| \int |f| < \infty \tag{3.123}$$

Therefore, $f + g, af \in \mathcal{L}^1$. $\Rightarrow$ $\mathcal{L}^1$ is a real vector space. $\qquad\square$

对于可积函数，我们往往是在整个 $\mathbb{R}^d$ 空间上讨论其可积性，类比 **Riemann** 可积函数，合理地猜测其在 $\mathbb{R}^d$ 平面上 **"较远"** 的地方的积分值应当较小. 这就是下面我们要给出的 $\mathcal{L}^1$ 可积函数的性质.

**命题 3.1.6.** Suppose $f \in \mathcal{L}^1(\mathbb{R}^d)$. Then $\forall \epsilon > 0$

(i) $\exists$ a set of finite measure $B$ such that

$$\int_{B^c} |f| < \epsilon$$

(ii) [**Absolutely Continuity**].

$\exists \delta > 0$ such that

$$\int_E |f| < \epsilon, \ \forall m(E) < \delta$$

**注.** • (i) 和 (ii) 共同说明了，若 $f \in \mathcal{L}^1(\mathbb{R}^d)$，则 $f$ 的积分主要集中在一个**有限测度**区域内，且在很小的区域内 $f$ 的积分值趋于零.

• (ii) 本质为测度的**绝对连续性** (正测度关于正测度的绝对连续性). 此处令正测度

$$\mu : \mathcal{L} \longrightarrow [0, \infty] \tag{3.124}$$

$$E \longmapsto \mu(E) = \int_E |f| \tag{3.125}$$

则命题 (ii) 可表示为：$\forall \epsilon > 0, \exists \delta > 0, \text{s.t.}$

$$\mu(E) < \epsilon, \ \forall m(E) < \delta$$

51

证明.

(i) : 对定义域做截断.

Suppose $f \geq 0$. Let $B_n = B(0, n)$, $f_n = f\chi_{B_n}$, then $f_n \nearrow f$.

By **MCT (Thm 3.1.2, 单调收敛定理)**,

$$\lim_{n \to \infty} \int f_n = \int f \tag{3.126}$$

Then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, s. t.

$$\left| \int f - \int f_N \right| = \int f - \int f_N = \int f(1 - \chi_{B_N}) = \int f\chi_{B_N^c} = \int_{B_N^c} f < \epsilon \tag{3.127}$$

Therefore, let $B = B_N = B(0, N)$, the desired result follows.

(ii) : 同样是做截断. 不过此处是对 $f$ 的取值做截断.

Let $B_n = \{x \in \mathbb{R}^d \mid f(x) \leq n\}$, $f_n = f\chi_{B_n}$. Then $f_n \nearrow f$, $f_n \leq n$.

同 (i), By **MCT (Thm 3.1.2, 单调收敛定理)**,

$$\lim_{n \to \infty} \int f_n = \int f \tag{3.128}$$

$\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, s. t.

$$\left| \int f - \int f_N \right| = \int (f - f_N) < \frac{\epsilon}{2} \tag{3.129}$$

Pick $\delta > 0$, s. t. $N\delta < \frac{\epsilon}{2}$. Then for all $m(E) < \delta$,

$$\int_E f = \int_E (f - f_N) + \int_E f_N \leq \int (f - f_N) + N \cdot m(E) \tag{3.130}$$

$$< \frac{\epsilon}{2} + N\delta \tag{3.131}$$
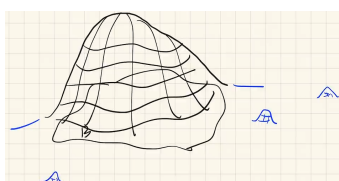
$$< \epsilon \tag{3.132}$$
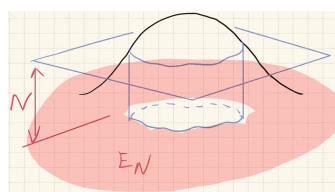
$\square$



图 3.1: Prop 3.1.6 (i)



图 3.2: Prop 3.1.6 (ii)

### 3.1.4 *The Dominated Convergence Theorem*

下面我们来介绍实分析中最最有用的定理——

#### 控制收敛定理 (The Dominated Convergence Theorem).

在 **Riemann** 积分中，对于函数列**交换极限与积分的次序**的条件太过于奇怪与繁琐，而在 **Lebesgue** 积分中，**控制收敛定理**则很完美地解决了这一问题. 它对于**交换极限与积分的次序**的条件十分简洁. 下面便来介绍这一定理.

**定理 3.1.7. The Dominated Convergence Theorem (DCT).**
Suppose $\{f_n\}_{n=1}^\infty \subset \mathcal{M}^+, f_n \to f$ a.e.. If $|f_n| \le g$, where $g \in \mathcal{L}^1(\mathbb{R}^d)$, then

$$\int |f_n - f| \to 0, \ n \to \infty \tag{3.133}$$

and consequently

$$\int f_n \to \int f, \ n \to \infty \tag{3.134}$$

**证明.** 分别对 $g + f_n$ 和 $g - f_n$ 利用 **Fatou's Lemma (Thm 3.1.6)** 即可得证.

- Since $g + f_n \ge 0$, then by **Fatou's Lemma (Thm 3.1.6)**,

$$\int \liminf_{n\to\infty} (g + f_n) \le \liminf_{n\to\infty} \int (g + f_n) \tag{3.135}$$

Since $f_n \to f$, we have

$$\int g + \int f \le \int g + \liminf_{n\to\infty} \int f_n \tag{3.136}$$

$$\int f \le \liminf_{n\to\infty} \int f_n \tag{3.137}$$

- Since $g - f_n \ge 0$, then by **Fatou's Lemma (Thm 3.1.6)**,

$$\int \liminf_{n\to\infty} (g - f_n) \le \liminf_{n\to\infty} \int (g - f_n) \tag{3.138}$$

$$\int g - \int f \le \int g + \liminf_{n\to\infty} \left(- \int f_n\right) \tag{3.139}$$

$$= \int g - \limsup_{n\to\infty} \int f_n \tag{3.140}$$

Then

$$\int f \ge \limsup_{n\to\infty} \int f_n \tag{3.141}$$

Therefore

$$\limsup_{n\to\infty} \int f_n \leq \int f \leq \liminf_{n\to\infty} \int f_n \tag{3.142}$$

which means $\lim_{n\to\infty} \int f_n$ exists, and

$$\lim_{n\to\infty} \int f_n = \int f \tag{3.143}$$

$\square$

### 3.1.5 *Complex – Valued Functions*

下面我们将**实值函数**上的 **Lebesgue** 积分推广至**复值函数**.

先来规定一些记号：

- Let $f : \mathbb{R}^d \to \mathbb{C}$, write $f(x) = u(x) + iv(x)$.

下面给出复值函数**可测**以及**可积**的定义.

**定义 3.1.7.** Suppose $f : \mathbb{R}^d \to \mathbb{C}, f = u + iv$, then we say

- $f$ is **measurable** if $u$ and $v$ are both measurable.

- $f$ is **Lebesgue integrable** if $|f|$ is Lebesgue integrable.

**注.** 事实上，根据此处定义，$f$ 可积 $\Leftrightarrow$ $u$ and $v$ 都可积.

**证明.**

- $f$ is integrable $\Rightarrow$ $\int \sqrt{u^2 + v^2} < \infty$ $\Rightarrow$ $\int |u|, \int |v| \le \int \sqrt{u^2 + v^2} < \infty$ $\Rightarrow$ $u$ and $v$ 可积.

- $u$ and $v$ 可积 $\Rightarrow$ $\int |u|, \int |v| < \infty$ $\Rightarrow$ $\int \sqrt{u^2 + v^2} \le \int |u| + \int |v| < \infty$ $\Rightarrow$ $f$ 可积.

$\square$

下面对**命题 3.1.5** 的结论进行推广，即由**复值可积函数**构成的空间为**线性空间**.

**命题 3.1.7.** $\mathcal{L}^1(\mathbb{R}^d, \mathbb{C})$ is a vector space.

**证明.** Trivial.

$\square$

# 3.2   $\mathcal{L}^1$ 空间的完备性

**引入**    在讲 **Riemann** 积分时，我们称 **Riemann** 可积函数构成的空间是**不完备的 (not complete)**. 在提及**完备**这个概念之前，我们需要先引入衡量"距离"的工具，即**范数**和**度量**.

## 3.2.1   范数，度量

下面给出**范数**和**度量**的严格定义.

**定义 3.2.1.** Let $X$ be a vector space over $\mathbb{F}$, a **<u>norm</u>** is a function:

$$X \longrightarrow \mathbb{R}_{\geq 0} \tag{3.144}$$

$$f \longmapsto \|f\| \tag{3.145}$$

satisfying the following properties:

(i)  $\|f\| \geq 0, \forall f \in X.$        ($\|f\| = 0 \iff f = 0 \ a.e.$)

(ii)  $\|af\| = |a| \|f\|, \forall a \in \mathbb{F}, f \in X.$

(iii)  $\|f + g\| \leq \|f\| + \|g\|, \forall f, g \in X.$


> **注.**      • (i) 中的"$\|f\| = 0 \iff f = 0 \ a.e.$"的"**a.e.**"是对于 $X$ 取函数空间时的条件，在实分析的取等条件中基本为默认叙述，在后续定义中往往省略. 在对 $\mathcal{L}^1$ 空间的定义 (**定义 3.2.4**) 中可以看到其合理性.
>
> • **范数**实际上是对 $\mathbb{R}^n$ 空间中**"与原点之间的距离"**这一概念的推广. 将函数视作向量，则其范数即为到原点的距离，即**模长**.
>
> • 若一个**线性空间** $X$ 上配备了一个**范数**，则称其为<span style="color:blue">赋范向量空间 (赋范线性空间)</span>.

将函数视作向量，就有其**到原点的距离**为范数. 但若是想要衡量**任意两个函数之间的距离**，则需要引入下面**度量**的概念.

**定义 3.2.2.** A **<u>metric</u>** on $X$ is a map

$$d : X \times X \longrightarrow \mathbb{R}_{\geq 0} \qquad (3.146)$$

$$(x, y) \longmapsto d(x, y) \qquad (3.147)$$

satisfying

(i) $d(x, y) \geq 0, \forall x, y \in X.$ $\qquad$ $(d(x, y) = 0 \iff x = y)$

(ii) $d(x, y) = d(y, x), \forall x, y \in X.$

(iii) $d(x, y) + d(y, z) \geq d(x, z), \forall x, y, z \in X.$

**注.** • 若 $X$ 为函数空间，则 (i) 中 "$d(x, y) = 0$" 等价条件默认为 "$x = y$ $a.e.$".

• **度量**可看作将两个函数 (向量) 的起点均平移至原点后，其两个终点之间的**距离**.

## 3.2.2 $\quad$ *The Space $\mathcal{L}^1(\mathbb{R}^d)$*

**范数** $\quad$ 下面先在所有 **Lebesgue** 可积函数构成的空间上定义**范数**.

**定义 3.2.3.** For any integrable function $f$ on $\mathbb{R}^d$, we define the **<u>norm</u>** of $f$,

$$\|f\| = \int_{\mathbb{R}^d} |f| \, dx \qquad (3.148)$$

**注.** • 由**命题 3.1.3**可知，此处 $\|f\| = 0 \iff f = 0$ $a.e.$

• 容易证明，如此定义的**范数**满足范数应当满足的三条公理. (**定义 3.2.1**)

**Space** $\mathcal{L}^1(\mathbb{R}^d)$   由于**定义 3.2.3**中"$\|f\| = 0 \iff f = 0 \ a.e.$",而我们对零测集上的函数性质并不关心,因而引出了如下关于 $\mathcal{L}^1$ 空间的定义.

**定义 3.2.4.** 我们在所有 **Lebesgue 可积函数**构成的空间上定义一个等价关系"~":

$$f \sim g \iff f = g \ a.e.$$

$\mathcal{L}^1(\mathbb{R}^d)$ is **the space of equivalences classes** of integrable functions.

**注.** 由定义可知,$\mathcal{L}^1(\mathbb{R}^d)$ 空间中的元素实际上为**函数的等价类 (集合)**

$$[f] = \{g \ integrable \mid g \sim f\}$$

而在实际中,我们还是习惯性地当作单独的函数进行运算,这在**几乎处处**的意义下时等价的.

**度量**   下面我们说明,根据**定义 3.2.3** 中所定义的**范数**可诱导出 $\mathcal{L}^1(\mathbb{R}^d)$ 上的一个**度量**.

**命题 3.2.1.**

$$d : \mathcal{L}^1(\mathbb{R}^d) \times \mathcal{L}^1(\mathbb{R}^d) \longrightarrow \mathbb{R}_{\geq 0} \tag{3.149}$$

$$(f, g) \longmapsto d(f, g) := \|f - g\| \tag{3.150}$$

defines a **metric** on $\mathcal{L}^1(\mathbb{R}^d)$.

**证明.** 下面即来逐一验证**定义 3.2.2** 中的三条公理.

- 根据范数的非负性,$\forall f, g \in \mathcal{L}^1(\mathbb{R}^d)$,$d(f, g) = \|f - g\| \geq 0$.

$$d(f, g) = 0 \iff f - g = 0 \ a.e. \iff f = g \ in \ \mathcal{L}^1(\mathbb{R}^d)$$

- 可交换性. $\forall f, g \in \mathcal{L}^1(\mathbb{R}^d)$,

$$d(f, g) = \|f - g\| = \int_{\mathbb{R}^d} |f - g| = \int_{\mathbb{R}^d} |g - f| = \|g - f\| = d(g, f) \tag{3.151}$$

- 根据范数的三角不等式,$\forall f, g, h \in \mathcal{L}^1(\mathbb{R}^d)$,

$$d(f, g) + d(g, h) = \|f - g\| + \|g - h\| \geq \|(f - g) + (g - h)\| = \|f - h\| = d(f, h)$$

$\square$

### 3.2.3 $\mathcal{L}^1$ 空间的完备性

**定义** 在得到了**范数**、**度量**的定义后，我们下面给出**完备空间**的定义.

> **定义 3.2.5.** A **metric space** $X$ is **<u>complete</u>** if every Cauchy Sequence $\{x_k\}_{k=1}^\infty$ has a limit in $X$.

> **注.** • **完备空间**即指空间中的任一柯西列都有**收敛到自身**的极限.

• 下面给出一个**不完备的度量空间**的例子.

> **例 3.2.1.** 取一维实数域 $\mathbb{R}$ 的子空间 $(0, 1) \subset \mathbb{R}$，考虑其上的 Cauchy Sequence $\{\frac{1}{n}\}_{n=2}^\infty \subset (0, 1)$.
>
> 由于 $\frac{1}{n} \to 0 \notin (0, 1)$，因此度量空间 $(0, 1)$ 不完备.

**$\mathcal{L}^1$ 空间的完备性** 下面我们将给出本小节最重要的结论，即 $\mathcal{L}^1$ 空间的**完备性**，这也是其比 **Riemann 可积函数**所构成的空间的优越性之所在.

> **定理 3.2.1. (Riesz - Fischer).**
>
> $$\mathcal{L}^1 \text{ is complete in its metric.}$$

**证明.** Let $\{f_n\}_{n=1}^\infty \subset \mathcal{L}^1(\mathbb{R}^d)$ be a Cauchy Sequence in $\mathcal{L}^1$, then

$$\forall \epsilon > 0, \exists N(\epsilon) \in \mathbb{N}, \forall n, m \geq N(\epsilon), \text{s.t.} \|f_n - f_m\| \leq \epsilon$$

Tacking $\epsilon = 2^{-k}$, then $\exists N(2^{-k}) \geq N^{2^{-(k-1)}}$, s.t. for $n_k = N(2^{-k}), n_{k+1} = N(2^{-(k+1)})$,

$$\|f_{n_k} - f_{n_{k+1}}\| \leq 2^{-k}$$

下面分为三步进行证明.

• 构建 $f(x)$ 并利用 $g(x)$ 证明 $f \in \mathcal{L}^1$，证明子列 $\{f_{n_j}\}_{j=1}^\infty$ 收敛到 $f$.

Let

$$f = f_{n_1} + \sum_{j=1}^\infty (f_{n_{j+1}} - f_{n_j}) \tag{3.152}$$

$$g = |f_{n_1}| + \sum_{j=1}^\infty |f_{n_{j+1}} - f_{n_j}| \tag{3.153}$$

Then by **MCT (Thm 3.1.2, 控制收敛定理)**

$$\int g = \int \left|f_{n_1}\right| + \int \sum_{j=1}^{\infty} \left|f_{n_{j+1}} - f_{n_j}\right| = \int \left|f_{n_1}\right| + \sum_{j=1}^{\infty} \int \left|f_{n_{j+1}} - f_{n_j}\right| \tag{3.154}$$

$$= \int \left|f_{n_1}\right| + \sum_{j=1}^{\infty} \|f_{n_{j+1}} - f_{n_j}\| \tag{3.155}$$

$$\leq \int \left|f_{n_1}\right| + \sum_{j=1}^{\infty} 2^{-j} < \infty \tag{3.156}$$

Therefore $g$ is integrable, $g \in \mathcal{L}^1$. Since $|f| \leq g$, then $\int |f| < \infty$. $f$ is integrable.

Let

$$S_k = f_{n_1} + \sum_{j=1}^{k} (f_{n_{j+1}} - f_{n_j}) = f_{n_{k+1}}, \ \ k = 1, 2, \cdots \tag{3.157}$$

$$f \text{ is integrable} \ \Rightarrow \ f < \infty \ a.e. \ \Rightarrow \ S_k \text{ converges a.e.} \ \Rightarrow \ S_k = f_{n_{k+1}} \to f \ a.e.$$

So we find

$$f_{n_k} \to f \ a.e.$$

- 将逐点收敛性转化为 $\mathcal{L}^1$ 收敛性，即证 $\|f - f_{n_k}\| \to 0$.

  We note that

$$\left|f - f_{n_k}\right| = \left\| \left(f_{n_1} + \sum_{j=1}^{\infty} (f_{n_{j+1}} - f_{n_j})\right) - \left(f_{n_1} + \sum_{j=1}^{k-1} (f_{n_{j+1}} - f_{n_j})\right) \right\| \tag{3.158}$$

$$= \left| \sum_{j=k}^{\infty} (f_{n_{j+1}} - f_{n_j}) \right| \leq g \tag{3.159}$$

By **DCT (Thm 3.1.7, 控制收敛定理)**, since $\left|f - f_{n_k}\right| \to 0 \ a.e.$, $\left|f - f_{n_k}\right| \leq g$, $g$ integrable,

$$\lim_{k \to \infty} \|f - f_{n_k}\| = \lim_{k \to \infty} \int \left|f - f_{n_k}\right| \overset{\text{DCT}}{=} \int \lim_{k \to \infty} \left|f - f_{n_k}\right| = 0 \tag{3.160}$$

Therefore, $\|f - f_{n_k}\| \to 0$. 即 $f_{n_k}$ 依 $\mathcal{L}^1$ 范数收敛到 $f$.

- 利用子列 $\{f_{n_k}\}_{k=1}^{\infty}$ 作为"桥梁"，证明 $f_n$ 依 $\mathcal{L}^1$ 范数收敛到 $f$，即 $\|f_n - f\| \to 0$.

  $\forall \epsilon > 0$，由于 $\{f_n\}_{n=1}^{\infty}$ 为 $\mathcal{L}^1$ 中 Cauchy Sequence，因此 $\exists N \in \mathbb{N}$, s. t.

$$\|f_n - f_m\| < \tfrac{\epsilon}{2}, \ \forall n, m > N$$

  Since $\|f_{n_k} - f\| \to 0$, then for $\epsilon > 0$, pick $n_k > N$ which s. t.

$$\|f_{n_k} - f\| < \tfrac{\epsilon}{2}$$

  Then

$$\|f_n - f\| \le \|f_n - f_{n_k}\| + \|f_{n_k} - f\| < \epsilon, \ \forall n > n_k > N \tag{3.161}$$

  Therefore $\|f_n \to f\| \to 0$ with $f \in \mathcal{L}^1$. $\mathcal{L}^1$ is complete in its metric.

  $\square$

根据上述定理的证明过程，可以得到下面的推论.

**推论 3.2.2.** If $\{f_n\}_{n=1}^{\infty}$ converges to $f$ in $\mathcal{L}^1$, then there exists a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that

$$f_{n_k}(x) \to f(x) \ a.e.$$

**注.** 即在**依 $\mathcal{L}^1$ 范数收敛**的函数序列中，总存在**"几乎处处收敛"**意义的子列.

## 3.2.4  $\mathcal{L}^1$ 的稠密子空间

下面说明 $\mathcal{L}^1$ 空间中以下的函数集合是**稠密的**.

**定理 3.2.3.** The following families of functions are dense in $\mathcal{L}^1(\mathbb{R}^d)$:

 (i)  The simple functions.

 (ii)  The step functions.

(iii)  The continuous functions of compact support.

**证明.** 详情可见视频Urysohn 引理与 $\mathcal{L}^1$ 的稠密子空间. $\square$

# 3.3 *Lebesgue* 积分的平移不变性

首先给出**平移算符**及**函数平移**的符号表达.

**定义 3.3.1.** The **translation** by a vector $h$ on $\mathbb{R}^d$ is denoted by the map $\tau_h : x \mapsto x - h$. If $f$ is a function defined on $\mathbb{R}^d$, the **translation** of $f$ by $h \in \mathbb{R}^d$ is the function $f_h$, defined by

$$f_h(x) = (f \circ \tau_h)(x) = f(x - h)$$

下面给出 **Lebesgue 积分的平移不变性**.

**定理 3.3.1.** If $f \in \mathcal{L}^1(\mathbb{R}^d)$, then $\forall h \in \mathbb{R}^d, f_h \in \mathcal{L}^1(\mathbb{R}^d)$ and

$$\int_{\mathbb{R}^d} f(x - h)dx = \int_{\mathbb{R}^d} f(x)dx \tag{3.162}$$

**证明.** 下面按 *Lebesgue* 积分的构造过程来证明，即**特征函数** $\Rightarrow$ **简单函数** $\Rightarrow$ **非负可测**.

- **Characteristic Function**.

  Suppose $f = \chi_E$, where $E \subset \mathbb{R}^d$ is measurable. Then

$$f_h(x) = f(x - h) = \chi_E(x - h) = \begin{cases} 1, & \text{if } x - h \in E \\ 0, & \text{if } x - h \notin E \end{cases} = \begin{cases} 1, & \text{if } x \in E + h = E_h \\ 0, & \text{if } x \in (E + h)^c = E_h^c \end{cases} \tag{3.163}$$

  根据 *Lebesgue* 测度的**平移不变性**，

$$\int_{\mathbb{R}^d} f_h = m(E_h) = m(E) = \int_{\mathbb{R}^d} f \tag{3.164}$$

- **Simple Function**.

  $\forall \varphi = \sum\limits_{k=1}^{n} a_k \chi_{E_k}$ simple, by the **linearity of integration**,

$$\int_{\mathbb{R}^d} \varphi_h = \sum_{k=1}^{n} \int_{\mathbb{R}^d} \chi_{(E_k)_h} = \sum_{k=1}^{n} \int_{\mathbb{R}^d} \chi_{E_k} = \int_{\mathbb{R}^d} \varphi \tag{3.165}$$

- **Non-negative Function**.

  $\forall f$ non-negative, $\exists \{\varphi_n\}_{n=1}^{\infty}$ simple, s. t. $\varphi \nearrow f$ and $\varphi \geq 0$. Then by **MCT (Thm 3.1.2)**,

  $$\int_{\mathbb{R}^d} \varphi_n \to \int_{\mathbb{R}^d} f \ as \ n \to \infty \tag{3.166}$$

  Since $(\varphi_n)_h \nearrow f_h$ and $\int \varphi_n = \int (\varphi_n)_h$, then by **MCT (Thm 3.1.2)**,

  $$\int_{\mathbb{R}^d} \varphi_n = \int_{\mathbb{R}^d} (\varphi_n)_h \to \int_{\mathbb{R}^d} f_h \ as \ n \to \infty \tag{3.167}$$

  Therefore

  $$\int_{\mathbb{R}^d} f_h = \int_{\mathbb{R}^d} f \tag{3.168}$$

- **General Case**.

  $\forall f \in \mathcal{L}^1(\mathbb{R}^d), f = f^+ - f^-$, where $f^+$ and $f^-$ are non-negative.

  Then by the **linearity of integration**,

  $$\int_{\mathbb{R}^d} f_h = \int_{\mathbb{R}^d} f_h^+ - \int_{\mathbb{R}^d} f_h^- = \int_{\mathbb{R}^d} f^+ - \int_{\mathbb{R}^d} f^- = \int_{\mathbb{R}_d} f \tag{3.169}$$

  $\square$

# 3.4　*Lebesgue* 可积函数的 $\mathcal{L}^1$ 连续性

**引入**　Recall 数学分析中连续的**等价定义**:

$$f \text{ is continous at } x \iff f(x) - f(x-h) \to 0 \text{ as } h \to 0 \tag{3.170}$$

$$\iff |f_h(x) - f(x)| \to 0 \text{ as } h \to 0 \tag{3.171}$$

即可大致视作 **Riemann** 可积函数关于 **2-范数**的连续性.

**Lebesgue 可积函数的 $\mathcal{L}^1$ 连续性**　在 $\mathcal{L}^1$ 空间中，**Lebesgue** 可积函数也有类似的关于 $\mathcal{L}^1$ 范数的连续性. 这就是下面的定理.

**定理 3.4.1.** Suppose $f \in \mathcal{L}^1(\mathbb{R}^d)$, then

$$\|f_h - f\|_{\mathcal{L}^1} \to 0 \text{ as } h \to 0 \tag{3.172}$$

**证明.** 详见视频<span style="color:magenta">积分的平移不变性与可积函数的 $\mathcal{L}^1$ 连续性</span>.
其中需要用到如下的引理.

**引理 3.4.2.** [2] If $f \in C_c(\mathbb{R}^d)$, then $f$ is uniformly continuous.

<span style="color:magenta">**注.**</span> $f \in C_c(\mathbb{R}^d)$ 表示 $f$ 为具有紧支集的连续函数.

$\square$

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[2]此为书：《*Real Analysis −−Modern Techniques and Their Applications*》— *Gerald B. Folland*
**P238 Lemma 8.4**

# 3.5  *Fubini* 定理

为了讨论的方便，下面先给出函数及集合的**切片**的定义.

**定义 3.5.1.** If $f$ is a function on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, the <u>**slice**</u> of $f$ w.r.t. $y \in \mathbb{R}^{d_2}$ is the function

$$f^y : \mathbb{R}^{d_1} \longrightarrow \overline{\mathbb{R}} \tag{3.173}$$

$$x \longmapsto f(x, y) \tag{3.174}$$

Similarly, the <u>**slice**</u> of $f$ for a fixed $x \in \mathbb{R}^{d_1}$ is

$$f_x : \mathbb{R}^{d_2} \longrightarrow \overline{\mathbb{R}} \tag{3.175}$$

$$y \longmapsto f(x, y) \tag{3.176}$$

Let $E \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, we define its <u>**slices**</u> by

$$E^y := \{x \in \mathbb{R}^{d_1} \mid (x, y) \in E\}, \ E_x := \{y \in \mathbb{R}^{d_2} \mid (x, y) \in E\} \tag{3.177}$$

下面给出 **Fubini** 定理.

**定理 3.5.1. Fubini**.

Suppose $f(x, y)$ is **integrable** on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for a.e. $y \in \mathbb{R}^{d_2}$:

(i) The slice $f^y$ is integrable on $\mathbb{R}^{d_1}$.

(ii) The function $y \mapsto \int_{\mathbb{R}^{d_1}} f^y(x) dx$ is integrable on $\mathbb{R}^{d_2}$.

(iii) Moreover,

$$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f^y(x) dx \right) dy = \int_{\mathbb{R}^d} f \tag{3.178}$$

### 3.5.1 *Fubini* 定理的证明

**证明.** Let $\mathcal{F} = \{f \in \mathcal{L}^1(\mathbb{R}^d) \mid f \text{ satisfies } (i) \sim (iii)\}$. It suffices to show that $\mathcal{L}^1(\mathbb{R}^d) \subset \mathcal{F}$.
下面仍按照构造 *Lebesgue* 积分的顺序思路进行证明，即**特征函数** $\Rightarrow$ **简单函数** $\Rightarrow$ **非负可测**.

(其中**特征函数**部分 (**Step 3 ~ 5**) 最为复杂繁琐，后续的证明则是水到渠成)

在此之前，还要先证明 $\mathcal{F}$ 对**函数的线性组合**及**单调函数列的极限**封闭.

- **Step 1: Any finite linear combination of functions in $\mathcal{F}$ also belongs to $\mathcal{F}$.**
  Suppose $\{f_k\}_{k=1}^N \subset \mathcal{F}$. By the condition, $\forall k$, $\exists A_k \subset \mathbb{R}^{d_2}$, $m(A_k) = 0$, s. t.

  $$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in A_k^c.$$

  Let $A = \bigcup_{k=1}^N A_k$, then $m(A) = 0$ and

  $$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in A^c, \forall k.$$

  下面对定理结论逐条验证. By the linearity of integration, $\forall a_k \in \mathbb{R}$,

  $$(\sum_{k=1}^N a_k f_k)^y = \sum_{k=1}^N a_k f_k^y \text{ is integrable on } \mathbb{R}^{d_1} \tag{3.179}$$

  $$\int_{\mathbb{R}^{d_1}} \sum_{k=1}^N (a_k f_k)^y(x) dx = \sum_{k=1}^N a_k \int_{\mathbb{R}^{d_1}} f_k^y(x) dx \text{ is integrable on } \mathbb{R}^{d_2} \tag{3.180}$$

  $$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} \sum_{k=1}^N (a_k f_k)^y(x) dx \right) dy = \sum_{k=1}^N a_k \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f_k^y(x) dx \right) dy \tag{3.181}$$

  $$= \sum_{k=1}^N a_k \int_{\mathbb{R}^d} f_k \tag{3.182}$$

  $$= \int_{\mathbb{R}^d} \sum_{k=1}^N a_k f_k \tag{3.183}$$

  $$\tag{3.184}$$

  Therefore, $\sum_{k=1}^N a_k f_k \in \mathcal{F}$, $\forall a_k \in \mathbb{R}$.

- **Step 2:** $\mathcal{F}$ 对单调函数列的极限封闭，即 $\forall \{f_k\}_{k=1}^{\infty}$, $f_k \nearrow f$, $f$ **integrable** $\Rightarrow f \in \mathcal{F}$.

Suppose $f_k \geq 0$. By the condition, $\forall k$, $\exists A_k \subset \mathbb{R}^d$, $m(A_k) = 0$, s. t.

$$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in A_k^c.$$

Let $A = \bigcup_{k=1}^{\infty} A_k$, then $m(A) = 0$ and

$$f_k^y(x) \text{ is integrable on } \mathbb{R}^{d_1}, \forall y \in A^c, \forall k.$$

Since $f_k^y(x) \nearrow f^y(x)$, by **MCT (Thm 3.1.2)**

$$\int_{\mathbb{R}^{d_1}} f_k^y(x) dx \to \int_{\mathbb{R}^{d_1}} f^y(x) dx \text{ as } k \to \infty \tag{3.185}$$

Let

$$g_k(y) = \int_{\mathbb{R}^{d_1}} f_k^y(x) dx, \quad g(y) = \int_{\mathbb{R}^{d_1}} f^y(x) dx \tag{3.186}$$

Then we have $g_k(y) \nearrow g(y)$ and $g_k \geq 0$. By **MCT (Thm 3.1.2)**

$$\int_{\mathbb{R}^{d_2}} g_k(y) dy \to \int_{\mathbb{R}^{d_2}} g(y) dy \text{ as } k \to \infty \tag{3.187}$$

i.e.

$$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f_k^y(x) dx \right) dy \to \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f^y(x) dx \right) dy \tag{3.188}$$

By the condition (iii), we have

$$\int_{\mathbb{R}^d} f_k \to \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f^y(x) dx \right) dy \tag{3.189}$$

Since $f_k \nearrow f, f_k \geq 0$, by **MCT (Thm 3.1.2)**

$$\int_{\mathbb{R}^d} f_k \to \int_{\mathbb{R}^d} f \tag{3.190}$$

Therefore

$$\int_{\mathbb{R}^d} f = \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} f^y(x) dx \right) dy = \int_{\mathbb{R}^{d_2}} g(y) dy \tag{3.191}$$

Since $f$ is integrable, $g(y) = \int f^y(x) dx$ is integrable on $\mathbb{R}^{d_2}$ $\Rightarrow \int g < \infty$.

Then we have

$$g(y) = \int_{\mathbb{R}^{d_1}} f^y(x) dx < \infty \text{ for a.e. } y \in \mathbb{R}^{d_2} \tag{3.192}$$

Therefore $f^y(x)$ is integrable for a.e. $y \in \mathbb{R}^{d_2}$.

Then $f \in \mathcal{F}$.

- **Step 3: Any characteristic function of a set $E$ of type $G_\delta$ with finite measure belongs to $\mathcal{F}$.**

  下面对 $E$ 进行讨论，分 $a \sim e$ 五种情况来证明：

  **(a)** $E \subset \mathbb{R}^d$ **is a bounded open cube**.

  Suppose $E = Q_1 \times Q_2$, where $Q_1 \subset \mathbb{R}^{d_1}$ and $Q_2 \subset \mathbb{R}^{d_2}$ are open cubes.

  $\forall y \in \mathbb{R}^{d_2}$, $\chi_E(x, y)$ is measurable in $x$, and integrable with

  $$g(y) = \int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx = \begin{cases} |Q_1|, & \text{if } y \in Q_2 \\ 0, & \text{if } y \notin Q_2 \end{cases} = |Q_1| \chi_{Q_2}(y) \tag{3.193}$$

  Since $g(y) = |Q_1| \chi_{Q_2}(y)$ is measurable and integrable with

  $$\int_{\mathbb{R}^{d_2}} g(y) dy = |Q_1||Q_2| = |E| = \int_{\mathbb{R}^d} \chi_E \tag{3.194}$$

  i.e.

  $$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx \right) dy = \int_{\mathbb{R}^d} \chi_E \tag{3.195}$$

  Therefore, $\chi_E \in \mathcal{F}$.

  **(b)** $E \subset \mathbb{R}^d$ **is a subset of the boundary of some closed cube**.

  Since $m(E) = 0$, we have

  $$\int_{\mathbb{R}^d} \chi_E = m(E) = 0 \tag{3.196}$$

  After an investigation of various possibilities, we note that (此处细节证明暂且留疑)

  $$\forall \text{ a.e. } y \in \mathbb{R}^{d_2}, E^y = \{x \in \mathbb{R}^{d_1} \mid (x, y) \in E\} \text{ has measure } 0 \text{ in } \mathbb{R}^{d_1}.$$

  Then $\forall$ a.e. $y \in \mathbb{R}^{d_2}$, $\chi_E^y(x)$ is integrable on $\mathbb{R}^{d_1}$ with

  $$g(y) = \int_{\mathbb{R}^{d_1}} \chi_E^y(x) dx = \int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx = 0, \ \forall \text{ a.e. } y \in \mathbb{R}^{d_2} \tag{3.197}$$

  So $g(y)$ is integrable on $\mathbb{R}^{d_2}$ with

  $$\int_{\mathbb{R}^{d_2}} g(y) dy = 0 = \int_{\mathbb{R}^d} \chi_E \tag{3.198}$$

  i.e.

  $$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx \right) dy = \int_{\mathbb{R}^d} \chi_E \tag{3.199}$$

  Therefore, $\chi_E \in \mathcal{F}$.

**(c)** $E \subset \mathbb{R}^d$ **is a finite union of almost disjoint closed cubes**.

Suppose $E = \bigcup_{k=1}^{N} Q_k$, where $\{Q_k^\circ\}_{k=1}^{N}$ are disjoint.

Let $A_k = Q_k - Q_k^\circ$ be the boundary of closed cube $Q_k$. Then $\chi_{A_k} \in \mathcal{F}$. (by **Step 3 (b)**)

$\chi_E$ is a linear combination of $\chi_{Q_k}$ and $\chi_{A_k}$, $k = 1 \sim N$.

Since $\chi_{Q_k}, \chi_{A_k} \in \mathcal{F}$, $k = 1 \sim N$, then by **Step 1**, $\chi_E \in \mathcal{F}$.

**(d)** $E \subset \mathbb{R}^d$ **is open and of finite measure**.

Since $E \subset \mathbb{R}^d$ is open, by **Thm 1.1.4**, $\exists$ alomst disjoint closed cubes $\{Q_k\}_{k=1}^{\infty}$, s. t.

$$E = \bigcup_{k=1}^{\infty} Q_k, \ where \ \{Q_k^\circ\}_{k=1}^{\infty} \ are \ disjoint \tag{3.200}$$

Let

$$f_k = \chi_{\bigcup_{j=1}^{k} Q_j} \tag{3.201}$$

Then by **Step 3 (c)**, $f_k \in \mathcal{F}$, and $f_k \nearrow f = \chi_E, f_k \geq 0$. By **Step 2**, we have $f = \chi_E \in \mathcal{F}$.

**(e)** $E \subset \mathbb{R}^d$ **is a $G_\delta$ of finite measure**.

By the **definition of $G_\delta$ (Def 1.4.5)**,

$$E = \bigcap_{k=1}^{\infty} \widetilde{Q_k}, \ where \ \widetilde{Q_k} \underset{open}{\subset} \mathbb{R}^d \tag{3.202}$$

Since $E$ has finite measure, $\exists \widetilde{O_0} \subset \mathbb{R}^d$ open, s. t. $E \subset \widetilde{O_0}$.

Let

$$O_k = O_0 \cap \bigcap_{j=1}^{k} \widetilde{O_j} \tag{3.203}$$

Then $O_1 \supset O_2 \supset \cdots$ and $E = \bigcap_{k=1}^{\infty} O_k$. Let $f_k = \chi_{O_k}$, then $f_k \in \mathcal{F}$. (By **Step 3 (d)**)

Since $f_k \searrow f = \chi_E, f_k \in \mathcal{F}$, then by **Step 2**, $f = \chi_E \in \mathcal{F}$.

- **Step 4: If $E \subset \mathbb{R}^d$ has measure 0, then $\chi_E \in \mathcal{F}$.**

  By **Thm 1.4.1**, $\exists$ a set $G \subset \mathbb{R}^d$ of type $G_\delta$ with $E = G \backslash N$, where $m(N) = 0$. Then

  $$E \subset G, \ m(G) = m(E) + m(G \backslash E) = 0.$$

  By **Step 3**, $\chi_G \in \mathcal{F}$, then

  $$\int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} \chi_G(x, y) dx \right) dy = \int_{\mathbb{R}^d} \chi_G = 0 \tag{3.204}$$

  Then

  $$\int_{\mathbb{R}^{d_1}} \chi_G(x, y) dx = 0 \ for \ a.e. \ y \in \mathbb{R}^{d_2} \tag{3.205}$$

  Since

  $$\int_{\mathbb{R}^{d_1}} \chi_G(x, y) dx = \int_{\mathbb{R}^{d_1}} \chi_G^y(x) dx = \int_{\mathbb{R}^{d_1}} \chi_{G^y}(x) dx = m(G^y) \tag{3.206}$$

  Therefore

  $$G^y = \{x \in \mathbb{R}^{d_1} \mid (x, y) \in G\} \ has \ measure \ 0 \ for \ a.e. \ y \in \mathbb{R}^{d_2} \tag{3.207}$$

  Since $E^y \subset G^y$, then $E^y$ has measure 0 for a.e. $y \in \mathbb{R}^{d_2}$.

  $$\Rightarrow \int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx = m(E^y) = 0 \ for \ a.e. \ y \in \mathbb{R}^{d_2} \tag{3.208}$$

  $$\Rightarrow \int_{\mathbb{R}^{d_2}} \left( \int_{\mathbb{R}^{d_1}} \chi_E(x, y) dx \right) dy = 0 = \int_{\mathbb{R}^d} \chi_E \tag{3.209}$$

  $$\Rightarrow \chi_E \in \mathcal{F} \tag{3.210}$$

- **Step 5: If $E$ is any measurable subset of $\mathbb{R}^d$ with finite measure, then $\chi_E$ belongs to $\mathcal{F}$.**

  By **Thm 1.4.1**, $\exists$ a finite measure $G$ of type $G_\delta$ with $E \subset G$ and $m(G - E) = 0$.

  Since $\chi_E = \chi_G - \chi_{G-E}$, by **Step 4**, $\chi_G, \chi_{G-E} \in \mathcal{F}$, then by **Step 1**, $\chi_E \in \mathcal{F}$.

- **Step 6: If $f$ is integrable, then $f \in \mathcal{F}$.**

  不妨 Suppose $f$ non-negative. By **Step 1 and Step 5**, $\forall \varphi = \sum\limits_{k=1}^{N} a_k \chi_{E_k}$ simple, $\varphi \in \mathcal{F}$.

  By **Thm 2.2.1**, $\exists \{\varphi_k\}_{k=1}^{\infty}$ simple, $\varphi_k \nearrow f$, $\varphi_k \geq 0$. Then by **Step 2**, $f \in \mathcal{F}$.

Therefore,

$$\mathcal{L}^1(\mathbb{R}^d) \subset \mathcal{F}$$

$\square$

### 3.5.2 *Fubini* 定理的应用