# Annual Progress Review

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**May 31, 2024**

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[1] An html version of this report can be found at www.twboughen.github.io/phd/APR/report

# Table of contents

# 1. Introduction

Networks appear in a range of fields of study including but not limited to micro-biology, computer science, sociology, and music. This makes them a valuable source of data to study, this report will be focused on the processes by which these networks form. Often, networks are assumed to have come from the Barabási-Albert model so that a power law can be used to model their degree distribution however a power law, while suitable for the bulk of the data, seems to become less suitable for the most extreme values in the network.

Shown below in Figure 1.1 are some examples of the degree distribution of real networks along with some simulated from two network generative processes given in Section 3.2. The networks in Figure 1.1 are:

- **as-caida**: the undirected network of autonomous systems of the Internet connected with each other from the CAIDA project, collected in 2007. [1]
- **BA simulated**: a network simulated using the Barabási-Albert model with $m = 1$ from Section 3.2
- **Jazz**: the collaboration network between Jazz musicians collected in 2003 [2]
- **Meta**: this is the metabolic network of the roundworm *Caenorhabditis elegans* [3]
- **Pajek-erdos**: This network contains people who have, directly and indirectly, written papers with Paul Erdős as of 2003. [4]
- **Reactome**: This is a network of protein–protein interactions in humans. [5]
- **UA simulated**: a network simulated using the uniform attachment model with $m = 1$ from Section 3.2.

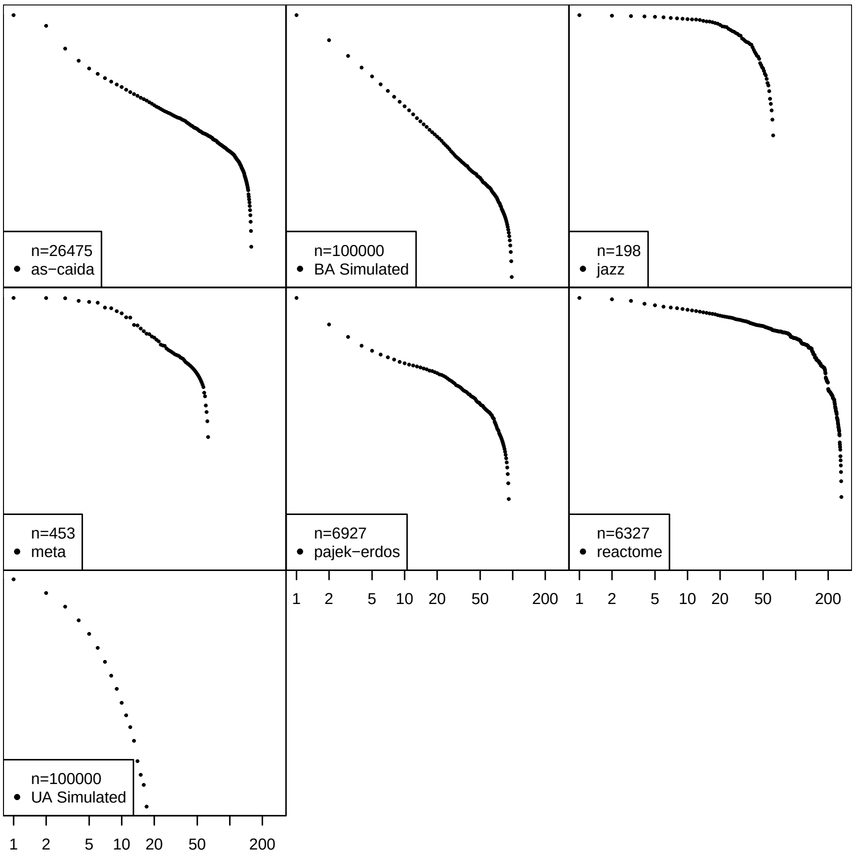

Figure 1.1.: Plots of survival functions of real networks degrees, along with the degree distribution for the UA and the BA models

This report beings with a brief introduction to univariate extreme value theory first for continuous values before moving to considerations for discrete data. Following this, an introduction to networks, network generative processes and some theoretical results for the networks that arise from them.

Since the aim is to gain understanding about the behaviour of the degree distribution of networks at the most extreme values, it seems natural to look to using methods from extreme value theory.

# 2. Univariate Extreme Value Theory

This section begins with a review of the theory and methodology for modelling the extreme values of continuous random variables, before moving to considerations for modelling the extreme values of discrete random variables.

## 2.1. Continuous Extremes

This section is a concise introduction to the two main methods for modelling extreme values: block maxima and peaks over threshold

The first method considers the distribution of block maxima. That is, for a set of independent and identically distributed (iid) random variables $X_1, \dots, X_n$ with common cumulative density function (cdf) $F$, the block maxima approach studies the limiting distribution of $M_n = \max\{X_1, \dots, X_n\}$.

Clearly, as $n \to \infty$, the block maxima $M_n$ converges almost surely to the right endpoint of $F$. However, standardising the block maxima allows for some characterisation of the limiting distribution. The following theorem is a key result in extreme value theory as it derives the general form of all possible limit laws for the standardised block maxima $M_n^*$.

**Theorem 2.1.1** (Fisher–Tippett–Gnedenko Theorem)**.** *With* $X_1, \dots, X_n \overset{\text{iid}}{\sim} F$ *and* $\{a_n\}_{n\geq 0}, \{b_n\}_{n\geq 0}$ *such that:*

$$\lim_{n\to\infty} \Pr\left[\frac{1}{a_n}\{M_n - b_n\} \leq x\right] = G(x),$$

*for some non-degenerate* $G$*.*

*Then* $F$ *is said to be in the (maximum) domain of attraction of* $G$*, denoted* $F \in \mathcal{D}(G)$ *,and* $G$ *is of one of three types:*

- *Gumbel:* $\Lambda(x) = \exp\{-\exp(-x)\}, \quad x \in \mathbb{R}$
- *Fréchet:* $\Phi_\alpha(x) = \exp\{-x^{-\alpha}\}, \quad x \geq 0, \alpha > 0$
- *Negative-Weibull:* $\Psi_\alpha(x) = \exp\{-x^{-a}\}, \quad x < 0, \alpha > 0$

*See [6] for the proof.*

Each of these three types defines a domain of attraction.

**Definition 2.1.1** (Domains of Attraction)**.** The three domains of attraction that result from Theorem 2.1.1 have the following equivalent conditions:

For a distribution with cdf $F$ and survival function $\bar{F}$ that has right endpoint $x_F$ given by:

$$x_F = \sup\{x \in \mathbb{R} \cup \{\infty\} : F(x) < 1\}$$

the distribution belongs to each domain of attraction subject to the conditions below:

**If there exists a positive function b**

- Type I/Gumbel/$\mathcal{D}(\Lambda)$:

$$\lim_{x\uparrow x_F} \frac{\bar{F}(x + tb(x))}{\bar{F}(x)} = e^{-t}, \quad \forall t \in \mathbb{R}$$

**If** $x_F = \infty$:

- Type II/Fréchet/$\mathcal{D}(\Phi_\alpha)$:

$$\lim_{x\to\infty} \frac{\bar{F}(tx)}{\bar{F}(x)} = x^{-\alpha}, \quad \forall t > 0 \quad \text{for some } \alpha > 0$$

**If** $x_F < \infty$:

- Type III/Negative-Weibull/$\mathcal{D}(\Psi_\alpha)$:

$$\lim_{h\downarrow 0} \frac{\bar{F}(x_F - xh)}{\bar{F}(x_F - h)} = x^{\alpha}, \quad \alpha > 0$$

The parameter $\alpha$ in Definition 2.1.1 and Theorem 2.1.1 is called the extreme value index.

Here, distributions in the Gumbel domain are referred to as light tailed, distributions in the Negative-Weibull domain are referred to as short tailed, and those in the Fréchet are referred to as heavy tailed. This terminology for heavy tailed distributions is different to some of the literature that defined a heavy tailed distribution as one that decays slower than exponential. However the terminology used here is also widely used.

Throughout this report functions will be referred to as regularly varying or slowly varying, what is meant by this is formally defined below:

**Definition 2.1.2** (Regular Variation)**.** A positive,real valued, measurable function $f$ is said to be regularly varying at infinity with index $\gamma$ if for all $t > 0$:

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = x^{\gamma}.$$

If $\gamma = 0$, then $f$ is instead said to be slowly varying at infinity.

Note that the condition for a distribution to belong to the Fréchet domain of attraction is equivalent to saying that the survival function $\bar{F}$ is regularly varying with index $-\alpha$.

In addition to heavy tailed distributions it is also useful to define what will be referred to as super heavy tailed distributions. This term is often just refers to specific distributions such as the log-Cauchy ,log-Gamma,and log-Weibull distributions but [7] provides a more precise definition below:

**Definition 2.1.3** (Super Heavy Tails)**.** A distribution is with survival function $\bar{F}$ is said to have super heavy tails if:

$$\lim_{x\to\infty} \frac{\bar{F}(tx)}{\bar{F}(x)} = 1, \qquad \forall t > 0$$

That is, a distribution is called super heavy if its survival function is slowly varying.

The three main types of extremal distribution (Gumbel, Fréchet and Negative-Weibull) can be united into one distribution, called the Generalised Extreme Value (GEV) distribution.

**Definition 2.1.4** (Generalised Extreme Value Distribution)**.** Denoted by GEV$(\mu, \sigma, \xi)$ the distribution is characterised by three parameters $\mu \in \mathbb{R}$ the location, $\sigma \in \mathbb{R}^+$ the scale, and the shape $\xi \in \mathbb{R}$. It has support on $\{x \in \mathbb{R} : 1 + \xi(x-\mu)/\sigma > 0\}$ and has cdf given by:

$$G(x) = \begin{cases} \exp\left[-\left\{1 + \frac{\xi(x-\mu)}{\sigma}\right\}_+^{-1/\xi}\right], & \xi \neq 0 \\ \exp\left[-\exp\left\{-\frac{x-\mu}{\sigma}\right\}\right], & \xi = 0. \end{cases}$$

The three types of extremal distribution are obtained from changing the shape parameter $\xi$, which corresponds to $1/\alpha$ in Theorem 2.1.1. This change is generally made so that the largest $\xi$ corresponds to heavier tails of the distribution. Specifically, $\xi < 0$, $\xi = 0$, $\xi > 0$, correspond to the negative Weibull, Gumbel and the Fréchet domains of attraction respectively.

The second method for modelling, is to consider observations above a large threshold, like the limiting distribution of block maxima, the limiting distribution of these extreme values can be characterised by the generalised Pareto (GP) distribution.

**Definition 2.1.5** (Generalised Pareto Distribution)**.** Consider a random variable $X$ with the same cdf $F$ as in Theorem 2.1.1, the Generalised Pareto (GP) distribution can be obtained by using the GEV distribution and conditional probability such that for large enough threshold the GP distribution approximately describes the conditional distribution of threshold exceedances. More precisely, for sufficiently large threshold $u$ and the change of variable to $Y = X - u$:

$$\Pr(Y \leq y | Y > 0) = H(y) = \begin{cases} 1 - \left(1 + \dfrac{\xi y}{\sigma}\right)^{-1/\xi}, & y > 0, \xi \neq 0 \\ 1 - \exp\left(-\dfrac{y}{\sigma}\right), & y > 0, \xi = 0 \end{cases}$$

Since this distribution was obtained using a GEV$(\mu, \sigma^*, \xi)$ the shape parameter $\xi$ is identical in both distributions and the shape parameter $\sigma$ is defined such that $\sigma = \sigma^* + \xi(u - \mu)$.

It is also possible to derive the result without using the GEV, as shown in [REF].

## 2.2. Discrete Extremes

A lot of Section 2.1 is appropriate only for continuous random variables and some of the results may not hold in a discrete setting. In particular, a continuous distribution $F$ being in certain domain of attraction may not necessarily imply that a discretisation of $F$ remains in that domain of attraction.

**Definition 2.2.1** (Discretisation)**.** The discretisation of a distribution with cdf $F$ is given by

$$F^*(n) = F(n) - F(n-1), \quad n \in \mathbb{Z}$$

[8] provides conditions for a discretisation of a continuous distribution to belong to the same domain of attraction. In particular the following theorem which corresponds to Theorem 1 in [8].

**Theorem 2.2.1** (Domain of attraction consistency)**.**

*(a) Every discretisation of distribution in $\mathcal{D}(\Phi_\alpha)$ remains in $\mathcal{D}(\Phi_\alpha)$.*
*(b) The discretisation of a distribution remains in $\mathcal{D}(\Lambda)$ if and only if the original is in $\mathcal{D}(\Lambda) \cap \mathcal{L}$.*

*Where $\mathcal{L}$ is the set of long-tailed distributions that have the property:*

$$\lim_{x \to \infty} \frac{\overline{F}(x+1)}{\overline{F}(x)} = 1$$

In addition [8] introduces a quantity useful for determining the domain of a attraction that a discrete distribution belongs to.

**Definition 2.2.2** (Omega Function)**.** For a distribution $F$ with survival function $\overline{F}$ and some $n \in \mathbb{Z}^+$ let:

$$\Omega(F, n) = \left(\log \frac{\overline{F}(n+1)}{\overline{F}(n+2)}\right)^{-1} - \left(\log \frac{\overline{F}(n)}{\overline{F}(n+1)}\right)^{-1}$$

This quantity plays an important role in Section 4 when determining the domain of attraction to which the degree distribution of a network generative model belongs. In particular a discrete distribution is recoverable to the Fréchet domain of attraction $\mathcal{D}(\Phi_\alpha)$ if:

$$\lim_{n \to \infty} \Omega(F, n) = \alpha^{-1}$$

Applying ideas from Section 2.1 to modelling discrete random variables has been approached from many different directions. What follows is a overview of some of the approaches that have been taken but will see use in this report.

[9] note that using the GP distribution as an approximation in a discrete setting leads to bias in the likelihood function and can lead to it being inadequate for modelling. They propose two other peaks over threshold methods

that rely on parametric families of discrete distributions. The first, what they refer to as the discrete generalised Pareto approximation is based on an extension of the discrete survival function. The second, the generalised Zipf distribution is obtained from an extension of the probability mass function. Both methods are motivated theoretically for modelling of a large class of discrete distributions and are shown in the paper to either match or outperform using the GP to model discrete data directly.

[10] first introduce an extended GP distribution, a continuous distribution that extends the idea of obtaining GP values from a probability integral transform (PIT) of $U(0,1)$ draws and instead considers a PIT of draws from any distribution on $(0,1)$ such as a beta distribution. This distribution is then discretised into their discrete extended GP distribution.

## 2.3. Modelling

The results from Section 2.1 allow the GEV and GP to be fitted to the block maxima and peaks over threshold respectively. An example of where modelling the GEV may be useful are when modelling monthly high temperatures, fitting the GEV to historic data of peak monthly temperatures may allow for future prediction of these temperatures. Fitting the GP may be useful in other scenarios such as modelling the strength of solar flares.

Typically, when fitting the GP, a sufficiently high threshold needs to be specified beforehand. [11] provides some empirical methods for specifying the threshold, one approach is to use a threshold stability plot that uses maximum likelihood to estimate the parameters of the GP for a large range of thresholds. The threshold can be chosen as the point across all of the plots after which the values of the parameters seems stable. One particular issue when fitting the GP to data, is that the likelihoods cannot be compared for different thresholds as changing the threshold changes the amount of data being used.

Another more recent approach shown by [12], uses a spliced threshold mixture to model the threshold exceedances where one distribution is assumed for the bulk of the data and the GP is used for those values above the threshold. To model discrete extremes, the approaches proposed in the literature are use a discretisation of the GP such as the one given below:

**Definition 2.3.1** (Intergral Generalised Pareto Distribution (IGP))**.** Consider a random variable $X$ with cdf $F$, and consider the random variable $Y = \lfloor X \rfloor$. From Definition 2.1.5, $X|X > u \sim GP(\sigma, \xi)$ for some sufficiently large $u \in \mathbb{R}^+$ and it can be obtained that the distribution of $Y|Y > u$ has distribution defined below:

$$\Pr(Y = y > Y > u) = \left(1 + \frac{\xi(y + 1 - \lceil u \rceil)}{\sigma_0 + \xi \lceil u \rceil}\right)_+^{-1/\xi} - \left(1 + \frac{\xi(y - \lceil u \rceil)}{\sigma_0 + \xi \lceil u \rceil}\right)_+^{-1/\xi}$$

For $y = \lceil u \rceil, \lceil u \rceil + 1, \dots$ and $\xi \in \mathbb{R}$ and $u, \sigma_0 \in \mathbb{R}^+$.

A spliced mixture model is given below that uses a general discrete distribution for the bulk of the data and the IGP above some threshold $v$.

**Definition 2.3.2** (IGP Spliced Mixture)**.**

$$f(y) = \begin{cases} (1-\phi)g(x), & y = 1, 2, \dots, v \\ \phi \left[ \left(1 + \frac{\xi(y + 1 - v)}{\sigma_0 + \xi v}\right)_+^{-1/\xi} - \left(1 + \frac{\xi(y - v)}{\sigma_0 + \xi v}\right)_+^{-1/\xi} \right], & y = v + 1, v + 2, \dots \end{cases}$$

where $g$ is the probability mass function of some discrete distribution with support equal to $\{1, 2, \dots, v\}$ and $\xi \in \mathbb{R}$ , $\sigma_0 \in \mathbb{R}^+$ and $v \in \mathbb{Z}^+$.

# 3. Networks

Networks are the data sources that the results from Section 2 will be used to analyse. Networks appear across a wide range of fields when attempting to represent complex systems and the relationships between the components within them.

This section will being with an introduction to the basics of networks and working with them in mathematics and probability, including the concept of degree distribution. Then, a look at a few network generation models and limiting results for the degree distributions of the networks they generate.

## 3.1. Mathematical Definitions

Throughout this section, graphs constructed from vertices and edges will be used as an analogue for these networks, so it is appropriate to begin with some mathematical definitions for exactly what that means.

**Definition 3.1.1** (Graph)**.** A graph $G = (V, E)$ is constructed from a vertex set $V$ and an edge set $E$. The edge set can take on one of two forms depending on if the graph is directed or un-directed. If the graph is directed then $E \subseteq V^2$ i.e the edge set is contained within the set of ordered pairs of vertices, whereas if the graph is **un-directed** then $E \subseteq [V]^2$, where

$$[V]^2 = \{\{u, v\} : u, v \in V\}$$

i.e. the edge set is contained within the set of un-ordered pairs of vertices.

**Definition 3.1.2** (Degree of un-directed graphs)**.** For an un-directed graph a vertex's degree denoted $d(v)$ for $v \in V$ is the number of edges that are connected to vertex $v$:

$$d(v) = |\{e \in E : v \in e\}|$$

**Definition 3.1.3** (Degree of directed graphs)**.** Directed graphs have something analogous, called the in-degree $d_{in}$, out-degree $d_{out}$ and total degree $d_{tot}$. The in-degree of a vertex $v$ is the number edges with endpoint at $v$, whereas the out-degree is the number of edges with start point at $v$ and the total degree is the sum of these i.e.:

$$\begin{aligned} d_{in}(v) &= |\{(w_1, w_2) \in E : w_2 = v\}| \\ d_{out}(v) &= |\{(w_1, w_2) \in E : w_1 = v\}| \\ d_{tot}(v) &= d_{in}(v) + d_{out}(v) \end{aligned}$$

There are many reasons to analyse network like data, one of which is to gain an insight into the mechanics that governed the growth of the network. The next sub-section is focused on presenting several network generative models, that may be able to describe how real networks grow. For now, the focus will be on the degree distributions of these network generative models.

## 3.2. Network Generative Models

It is useful to be able to model the way a network may have grown using simple rules as the subsequent model can then be used to simulate how the network may grow in future and provide insights into the underlying mechanics of the system the network represents. These models are also sometimes called mechanistic models in the literature. Also, although they are referred to as network generative models, graphs are still being used in the rules that govern how the generative model works. The focus here is on preferential attachment models, but it should be noted that network generative models are not limited to this class of models. Some other well known models include the Erdős-Réyni model[REF] and the small-world model[REF].

This section begins by detailing a fairly simple generative model and its limiting results for the degree distribution, followed by two special cases of the first model and their results.

## General Preferential Attachment (GPA)

Under this model, at each time step one vertex is added to the network and brings an edge with it that connects the existing vertices with a probability proportional to some function of the vertices degrees.

**Definition 3.2.1** (General Preferential Attachment Model)**.** Starting with a graph $G_1 = (V_1, E_1) = (\{1, \dots, m_0\}, \emptyset)$. At each following time step $t > 1$ the graph $G_t = (V_t, E_t)$ is generated by the following rules:

1. **Growth:** Add a new vertex to the vertex set i.e.

$$V_t = V_{t-1} \cup \{t\}$$

2. **Preferential Attachment:** Add $m \leq m_0$ edges connecting the new vertex those already in the graph $\{1, \dots, t-1\}$ selected at random with weights proportional to a function of their degree i.e.:

$$E_t = E_{t-1} \cup \{\tilde{e}_1, \dots, \tilde{e}_m\}$$

where $\tilde{e}_j = \{t, \tilde{v}\}$ and $\tilde{v} = i$ with weights

$$\frac{g(d(i))}{\sum_{w \in V_{t-1}} g(d(i))}, \qquad i \in V_{t-1}$$

for some function $g : \mathbb{Z} \mapsto \mathbb{R}^+ \setminus \{0\}$, which will be referred to as the preferential attachment function

There are some asymptotic results that have been derived for the case when $m = 1$, making the process generate a random tree.

### Limiting Degree Distribution

In [13] the limiting degree distribution was calculated in terms of the preferential attachment function and does not have a general explicit form. It is defined as follows, let $\lambda^*$ be the solution, if it exists, to:

$$1 = \sum_{n=1}^{\infty} \prod_{i=1}^{n-1} \frac{g(i)}{g(i) + \lambda}$$

then the limiting degree distribution of a network resulting from the GPA model has probability mass function (pmf):

$$f(k) = \frac{\lambda^*}{g(k) + \lambda^*} \prod_{i=0}^{k-1} \frac{g(i)}{g(i) + \lambda^*}$$

## Barabási-Albert (BA)

The GPA model has several special cases, when $g$ is the identity function i.e $g(k) = k$, it becomes the BA model [14].

**Definition 3.2.2** (Barabási-Albert Model)**.** Starting with a graph $G_1 = (V_1, E_1)$ where $V_1 = \{1, \dots, m_0\}$ and $E_1 = \{\{v\} : v \in V_1\}$ i.e a graph with $m_0$ vertices with one self-loop each. At each time step $t > 1$ the graph $G_t = (V_1, E_1)$ is generated by the following rules:

1. **Growth:** Add a new vertex to the vertex set i.e.

$$V_t = V_{t-1} \cup \{t\}$$

2. **Preferential Attachment:** Add $m \leq m_0$ edges between the new vertex and those already in the graph with probability proportional to each vertices degree i.e.

$$E_t = E_{t-1} \cup \{\tilde{e}_1, \dots, \tilde{e}_m\}$$

where each new edge $\tilde{e}_i = \{t, \tilde{v}_i\}(i = 1, \dots, m)$ has $\tilde{v}_i$ sampled independently without replacement from $V_{t-1}$ with probability:

$$\frac{d(\tilde{v}_i)}{\sum_{u \in V_{t-1}} d(u)}$$

### Limiting Degree Distribution

In [15] it was shown that for large values of $t$, the limiting degree distribution of a network produces by this model is:

$$f(k) = \frac{2m(m+1)}{k(k+1)(k+2)}, \qquad k \geq m$$

[16] states that since this probability mass function is regularly varying with exponent 2, then so is its cumulative mass function and it is in the Fréchet domain of attraction $\mathcal{D}(\Phi_2)$.

## Uniform Attachment (UA)

The final special case presented here is obtained from setting the preferential attachment function $g$ to be some constant value.

**Definition 3.2.3** (Uniform Attachment Model)**.** Start with a graph $G_1 = (V_1, E_1) = (\{1, \dots, m_0\}, \emptyset)$, at each time step $t > 1$ the graph is denoted by $G_t = (V_t, E_t)$ and generated by repeating the following two steps:

1. **Growth:** Add a new vertex to the vertex set i.e.

$$V_t = V_{t-1} \cup \{t\}$$

2. **Uniform Attachment:** Add $m \leq m_0$ edges between the new vertex and those already in the graph with probability proportional to each vertices degree i.e.

$$E_t = E_{t-1} \cup \{\tilde{e}_1, \dots, \tilde{e}_m\}$$

where each new edge $\tilde{e}_i = \{t, \tilde{v}_i\}(i = 1, \dots, m)$ has $\tilde{v}_i$ sampled independently without replacement from $V_{t-1}$ with probability:

$$\frac{1}{\sum_{u \in V_{t-1}} 1} = \frac{1}{|V_{t-1}|}$$

### Limiting Degree Distribution

As showing in [14] the expected degree distribution of this model for large values of $t$ is approximately:

$$f(k) = \frac{e}{m} \exp\left(-\frac{k}{m}\right), \qquad k \geq m$$

Although this was not shown rigorously and treats the degree of a vertex as a continuous random variable, this is an shifted exponential distribution with left endpoint $m$ and rate parameter $1/m$ and as such it belongs to the Gumbel domain of attraction.

If $m = 1$, it is possible to get a more precise result from the result regarding the limiting degree distribution of the GPA. By setting the preferential attachment function $g(k) = \lambda^*$, the can be shown that the limiting degree distribution is:

$$f(k) = \left(\frac{1}{2}\right)^k, \qquad k = 1, 2, \dots$$

This is a geometric distribution and is recoverable to the Gumbel domain of attraction as stated in [8].

# 4. Methods

The aim of this section is to investigate the degree distribution of real networks and compare them to the results obtained for the generative models in Section 3.2. First, a look at what the degree distributions of real networks look like.

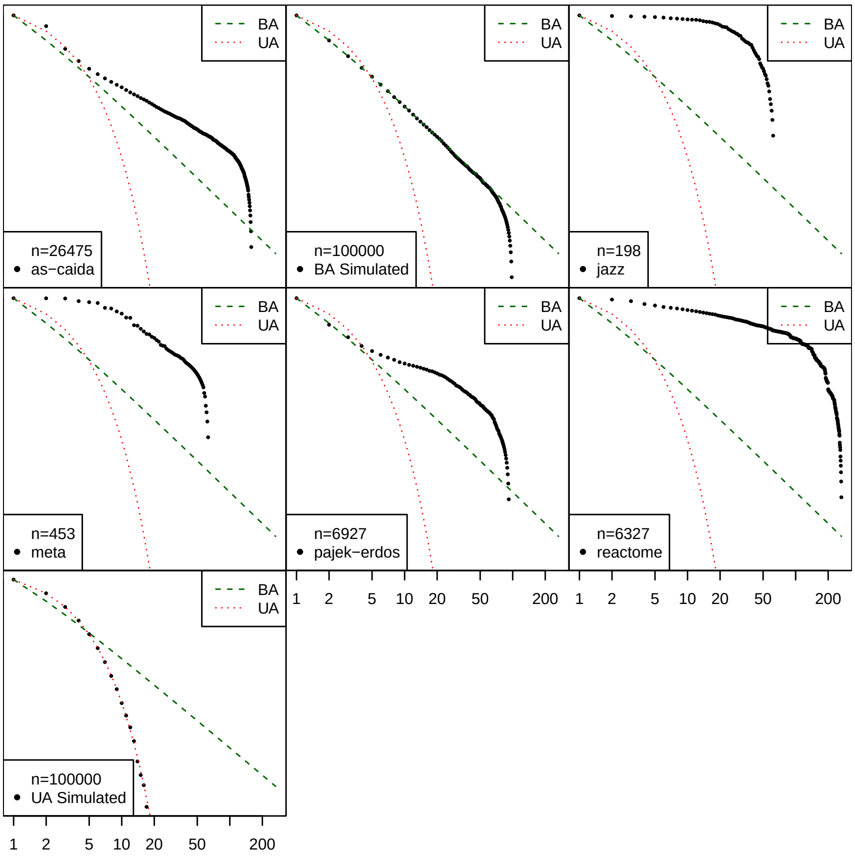

Figure 4.1.: Plots of survival functions of real networks degrees, along with the degree distribution for the UA and the BA models

Figure 4.1 shows the survival function of the degrees of various real networks as well as “BA Simulated” and “UA Simulated” which were generated using the corresponding schemes in Section 3.2. Additionally, the theoretical limiting degree distribution of both the UA model and the BA model (for m=1) are included on the plots. Visually it seems that neither of these models are adequate for modelling the growth of the real networks shown here.

To further investigate this, Section 4.1 considers fitting a model to these data that will provide insight into what would be needed from a network generative model such that it flexible enough to capture the variation of shapes of degree distribution in real networks.

## 4.1. Modelling degree distributions

As mentioned in Section 2.3, the method used here to model the extreme values of the data will be a spliced threshold mixture. Specifically, it will be a spliced threshold mixture of a power law and a discretisation of the generalised Pareto distribution similar to what is defined in [17].

**Definition 4.1.1** (Power-Law IGP Distribution)**.**

$$f(y) = \begin{cases} (1-\phi)\dfrac{y^{-(\alpha+1)}}{\sum_{k=1}^{v}} k^{\alpha+1}, & y = 1,2,\dots,v \\ \phi\left[\left(1+\dfrac{\xi(y+1-v)}{\sigma_0+\xi v}\right)_+^{-1/\xi} - \left(1+\dfrac{\xi(y-v)}{\sigma_0+\xi v}\right)_+^{-1/\xi}\right], & y = v+1, v+2, \dots \end{cases}$$

where $\alpha \in \mathbb{R}^+$ is the power law index and $\xi \in \mathbb{R}$ , $\sigma_0 \in \mathbb{R}^+$ and $v \in \mathbb{Z}^+$.

## 4.2. Fitting model to the data

The values of the parameters in the model for each data set were estimated under the Bayesian framework using a Metropolis within Gibbs sampler. Below are plots showing the same data as in Figure 4.1 but with the mean and 95% credible intervals of the survival function of the model for each data-set.

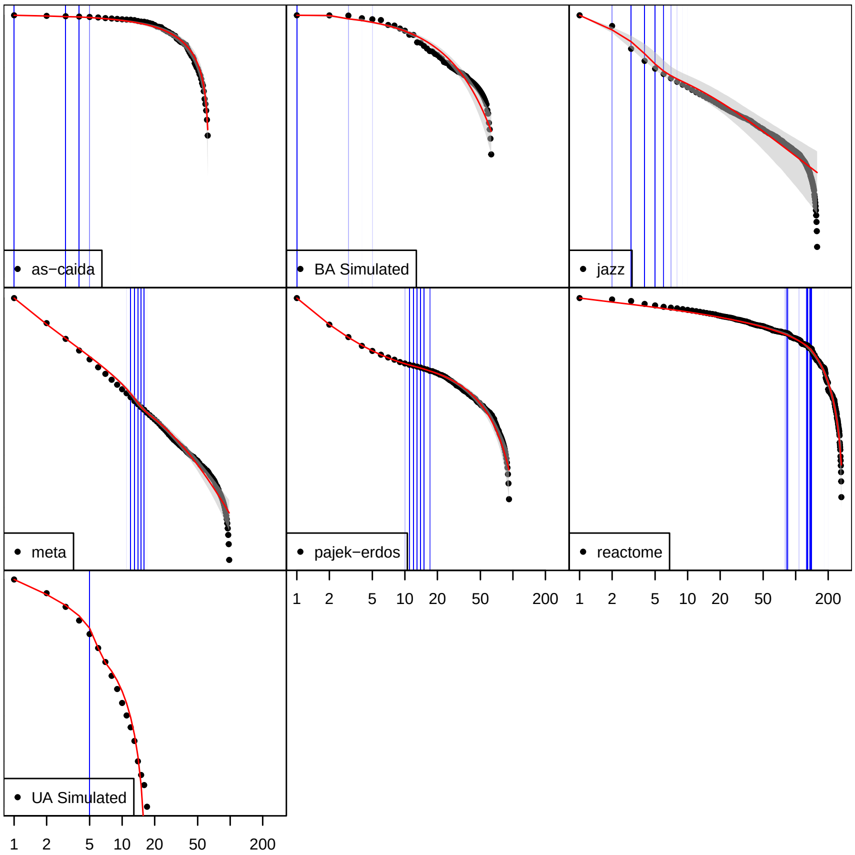

Figure 4.2.: Survival functions from Figure 4.1, with mean and 95% CI for the fitted model

As show by Figure 4.2 the model seems to fit the data quite well, below are some plot summarising each of the parameters for each of the models:

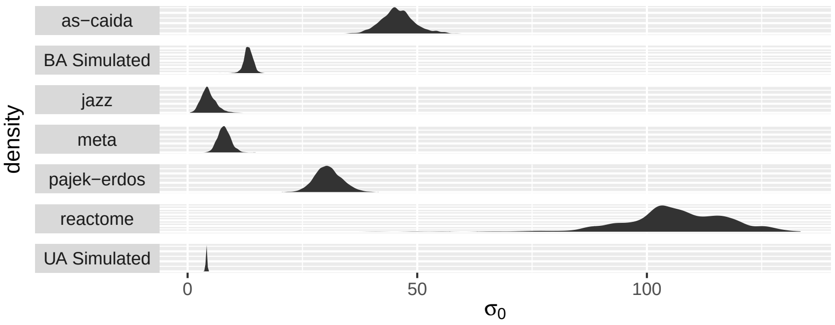

Figure 4.3.: Posterior of scale ($\sigma$)

Figure 4.4 shows that for 'arenas-meta' and 'UAsim' the posterior of the threshold is extremely concentrated, so much so that only one threshold is used. In the case of 'arenas-meta', this value is 1 meaning that the power law index $\alpha$ (Figure 4.5) is free to be any value that is permitted by the prior, which is anything on the positive real line, explaining the very diffuse posterior. The threshold for 'UAsim' is so low due to the magnitude of the values in the data and is much more concentrated because of the sample size.

The variety of values that $\alpha$ and $\xi$ take across all of the data sets shown in Figure 4.5 and Figure 4.6, makes it clear that none of these models could have been the result of either the BA model or the UA model when $m = 1$. Changing $m$ may indeed change the degree distribution, but it would also change the left endpoint of the degree distributions as each vertex would join the graph with $m$ edges leaving no vertices with degree less that $m$.

## 4.3. General Preferential Attachment Analyses

So far it has been shown that neither the BA model nor the UA model can adequately capture the range of type of degree distributions of real networks. So, a natural place to start when attempting to expand the range of possible degree distributions is the more general model, the GPA. This section, will use results from [8] and Section 2 to investigate the possible types of degree distribution that may arise from different preferential attachment functions in the GPA model.

### The Preferential Attachment Function

From here on the preferential functions that will be used for the GPA model will be of the form:

$$g(k) = k^{\gamma}, \qquad \gamma > 0.$$

This allows for investigating the cases where the preferential attachment function is sub-linear ($\gamma < 1$) and when it is super-linear ($\gamma > 1$).

As discussed in Section 2.2, the limiting value of $\Omega(F, n)$ can give a lot of information about the behaviour of a discrete distribution at extreme values. Below is a plot showing the value of this quantity as $n$ increases for various different values of $\gamma$.

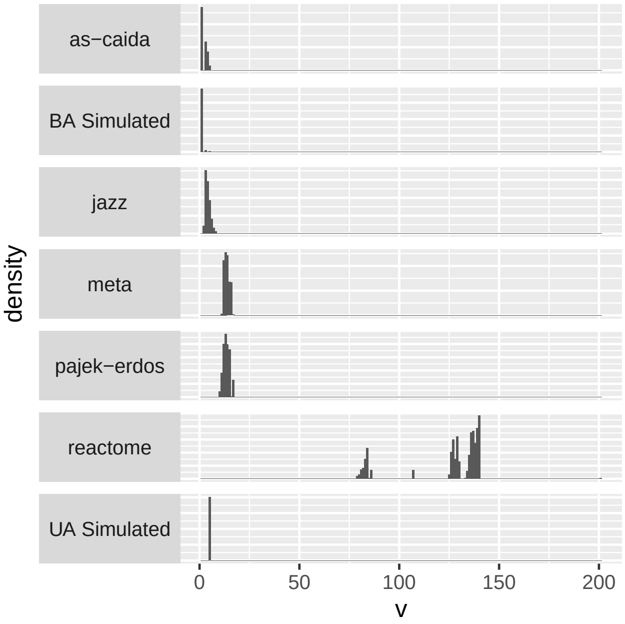

Figure 4.4.: Posterior of threshold ($v$)

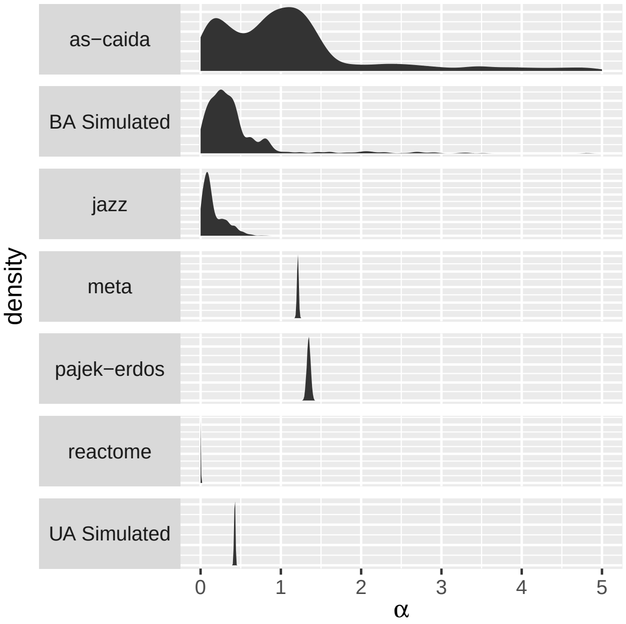

Figure 4.5.: Posterior of power law index ($\alpha$)

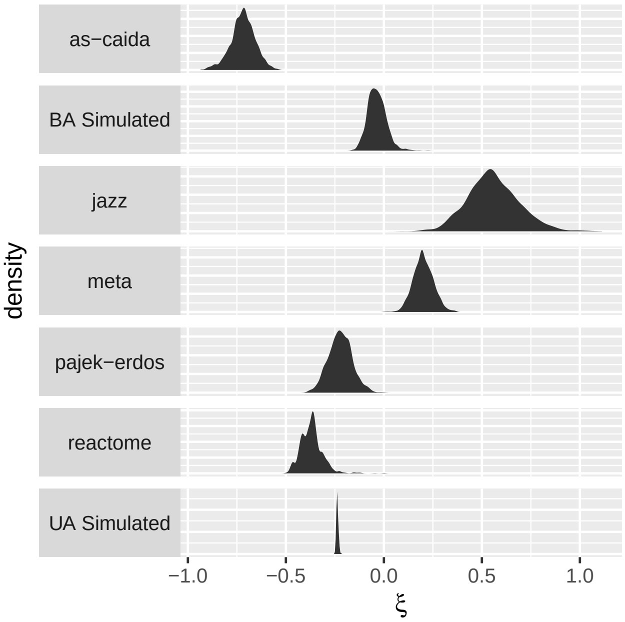

Figure 4.6.: Posterior of shape ($\xi$)

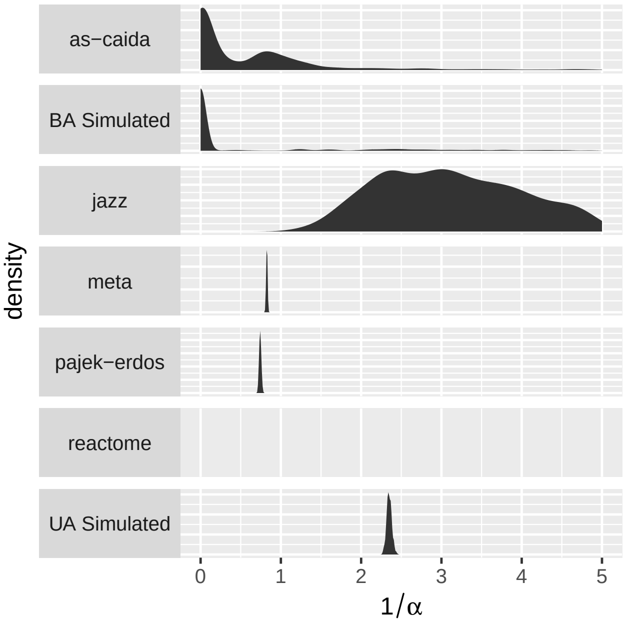

Figure 4.7.: Posterior of inverse of power law index ($1/\alpha$)

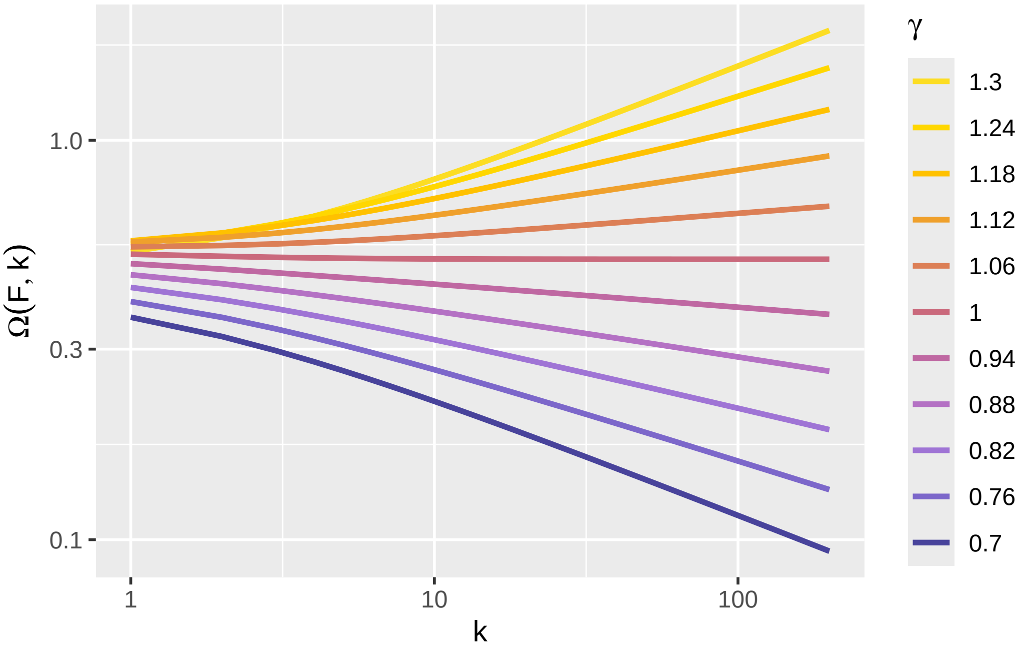

Figure 4.8.: Plot of $\Omega(F, n)$ for various $\gamma \in (0.7, 1.3)$

Figure 4.8 shows that for $\gamma < 1$ $\Omega(F, n)$ seems to approach 0 as $n$ increases, whereas for $\gamma = 1$ $\Omega(F, n)$ seems to converge to finite non-zero limit which is to be expected as this corresponds to the BA model which has limiting degree distribution in the Fréchet domain of attraction. However, for $\gamma > 1$ the value of $\Omega(F, n)$ appears to diverge and does not approach a finite limit.

[8] does not provide any results in particular for the case of $\Omega(F, n)$ diverging but if the definition of slow variation and thus super-heavy tails is viewed as regular variation in the limit as $\alpha$ goes to infinity then the following can be obtained.

**Conjecture 4.3.1.** *For a distribution $F$ with survival function $\overline{F}$ and some $n \in \mathbb{Z}^+$, if:*

$$\lim_{n\to\infty} \Omega(F, n) = \lim_{\alpha\downarrow 0} \alpha^{-1} = \infty$$

*then $F$ has super heavy tails.*

This is further supported by Figure 4.9 below, which shows the value of the quantity from Definition 2.1.3 for increasing values of $n$ and values of $\gamma$ in the range $(1, 2)$. The plot shows the quantity approaching 1 for all values of $\gamma$ as $n$ increases, suggesting that the limiting degree distribution of the GPA model with $g(k) = k^\gamma, \gamma > 1$ has super heavy tails.

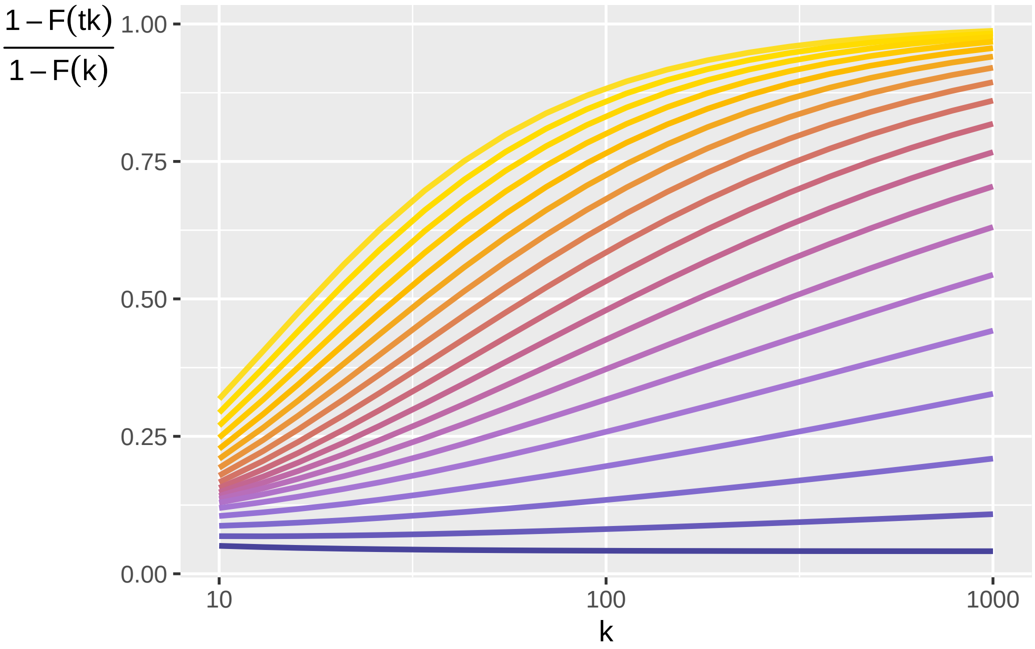

Figure 4.9.: Plot testing slow variation for $\gamma \in (1, 2)$

Figure 4.9 is for $t = 5$, and the condition for super heavy tails the quantity to approach one for all $t > 0$, other plots were created for other values of $t$ and changing $t$ just changes the rate of convergence to one.

## 4.4. Potential Novelty

The results from this subsection suggest that for super-linear preferential attachment functions the GPA model has limiting degree distribution with super heavy tails. This, along with results for the linear case in Section 3.2 and sub-linear cases in [15] lead to the following conjecture.

**Conjecture 4.4.1.** *The GPA model is only capable of producing three different types of degree distribution:*

1. *Gumbel: sub-linear preferential attachment function i.e.* $g(k) = k^{\gamma}, \gamma \in (0, 1)$
2. *Fréchet* $\mathcal{D}(\Phi_2)$*: linear preferential attachment function i.e.* $g(k) = k$
3. *Super heavy tails: super-linear preferential attachment function i.e.*$g(k) = k^{\gamma}, \gamma > 1$

This means that under the framework presented here, even the GPA model is no where near close to being able to capture the range of types of degree distribution found in real networks.

# 5. Discussion and Next Steps

The generative models considered so far are very simple, which is good since the goal is to find as simple a model as possible. However, it is clear now and perhaps unsurprising that the models shown here are not capable of modelling realistic network growth. This section is dedicated to discussing next steps for this project and address some questions left open.

The models from Section 3.2 are very limited when it comes to the actual growth of the network. Whenever a new vertex joins the network it brings a fixed number of edges with it that remain permanently, this is the only way that edges are added and thus the degrees changed. Below are some modifications that could be made to address this issue:

- Allow removal of edges throughout the networks growth.
- Allow edges to be made between already existing vertices in the network.
- Bring a random number of edges when a vertex is added.

It is possible to include all of these into a model with the modification that at each time step you do one of three different steps (Growth, Connection, Removal) with certain probabilities where the number of edges added at each growth step is a discrete random variable. Something similar could be done for the connection and removal steps.

Additionally, all of the models assume a constant preferential attachment function both over time and across vertices. To address this the preferential attachment function could be allowed to change over time and perhaps differ between vertices. This would allow a vertex to 'age' in a sense, and could also allow for 'categories' of vertices that share the same preferential attachment function which differs from those in other 'categories'.

# A. Updated Project Plan

Progress so far is in line with the original project plan, so a lot of this plan remains the same.

## A.1. Objectives and more specific goals

### Continue investigating possible modifications to Barabási-Albert model (months 9-24)

So far, it has become clear that modifying the preferential attachment function alone is not enough to make the BA model more suitable for real networks. The next step is to investigate other possible modifications, for example adding the ability to remove edge along with the possibility of adding edges throughout the networks growth but not necessarily when a new vertex is added, [18] and [19] are possible starting points.

In addition to adding the possibility of edge removal, modifying the preferential attachment function over time could also be possible.

#### Monitor changes in model parameters of various real networks (months 18-24)

Fitting the model for degree distributions to real networks and monitoring the changes in parameters over time will be an invaluable resource when it comes time to develop modifications to the generative model. This will be an ongoing process during some other steps but will require a set of functions and/or an R package to be developed that can efficiently estimate the parameters.

#### Use these changes to inform the possbile modifications (months 24-30)

As modifications are made to the generative model, the changes in model parameters over time can also be recorded for the modified generative models to make sure that they align with real networks.

#### Investigate properties of the new model (months 30-36)

Once a set of modifications has been developed, the theoretical properties of networks that grow under the new model will be important to study.

## A.2. Outcomes

#### Validate current model for degree distribution is suitable (~month 12)

The current model seems to perform fairly well when fitting to real networks, but a more robust method for testing how well the model fits could be used to make sure that it is suitable. There are perhaps some indications that a third component should be added to the spliced mixture model.

#### Create set of functions to fit the model (~month 18)

Once a suitable model for the degree distribution has been decided on, an efficient and fairly uninvolved way to fit the model needs to be developed. This is to aid with monitoring changes in parameters over time, as it will need to be fitted to many sets of data over many time frames.

### Develop modifiations of the Barabási-Albert model (~month 30)

A new model will be developed based on modifications of the Barabási-Albert model that can more accurately describe how real networks have grown.

# B. Training

## B.1. School Training

I attended two weeks of APTS courses, one in Warwick which included two modules on Statistical Computing and Statistical Inference, and another in Nottingham which included modules on Applied Stochastic Processes and Statistical Modelling.

## B.2. Conferences

I attended one conference this year, the SAgE PGR conference where I presented a poster summarising the work I had done up until that point.

## B.3. Training still required

There is still some training I still need to do:

- c++ training.
- Training on Rocket HPC.

## Funding and Stipend

The funding for this project expires on **17th March 2027**.

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