Rough Calculations

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Deflection Computations:

$$P = mg$$
$$\delta = \frac{\partial U}{\partial P}$$
$$U = \int_0^{L_1+L_2+L_3} \frac{M^2}{2EI} dx$$
$$\delta = \frac{1}{E} \int_0^{L_1+L_2+L_3} \frac{M}{I} \frac{\partial M}{\partial P} dx$$
$$\rho = \frac{m}{V} \Rightarrow m = \rho V = \rho A x$$
$$M(x) = -Px - \frac{1}{2}\rho A_1 g x^2 - \frac{1}{2}\rho A_2 g (x - L_1)^2 \mathbb{1}\{x > L_1\} - \frac{1}{2}\rho A_3 g (x - L_1 - L_2)^2 \mathbb{1}\{x > L_1 + L_2\}$$
$$\frac{\partial M}{\partial P} = -x$$
$$I(x) = I_1 + I_2 \mathbb{1}\{x > L_1\} + I_3 \mathbb{1}\{x > L_1 + L_2\}$$
$$\delta = \frac{1}{E} \left(\int_0^{L_1} \frac{M}{I} \frac{\partial M}{\partial P} dx + \int_{L_1}^{L_1+L_2} \frac{M}{I} \frac{\partial M}{\partial P} dx + \int_{L_1+L_2}^{L_1+L_2+L_3} \frac{M}{I} \frac{\partial M}{\partial P} dx \right)$$
$$\int_0^{L_1} \frac{M}{I} \frac{\partial M}{\partial P} dx = \int_0^{L_1} \frac{mgx^2 + \frac{1}{2}\rho A_1 g x^3}{I_1} dx = \frac{g}{I_1} \left(\frac{mL_1^3}{3} + \frac{\rho A_1 L_1^4}{8} \right)$$
$$\int_{L_1}^{L_1+L_2} \frac{M}{I} \frac{\partial M}{\partial P} dx = \int_{L_1}^{L_1+L_2} \frac{mgx^2 + \frac{1}{2}\rho A_1 g x^3 + \frac{1}{2}\rho A_2 g x (x - L_1)^2}{I_1 + I_2} dx$$
$$= \frac{g}{I_1 + I_2} \left(\frac{m((L_1+L_2)^3 - L_1^3)}{3} + \frac{\rho A_1((L_1+L_2)^4 - L_1^4)}{8} \right)$$
$$+ \frac{g\rho A_2}{I_1 + I_2} \left(\frac{(L_1+L_2)^4 - L_1^4}{8} - \frac{(L_1+L_2)^3 - L_1^3}{3} L_1 + \frac{(L_1+L_2)^2 - L_1^2}{4} L_1^2 \right)$$
$$\int_{L_1+L_2}^{L_1+L_2+L_3} \frac{M}{I} \frac{\partial M}{\partial P} dx = \int_{L_1+L_2}^{L_1+L_2+L_3} \frac{mgx^2 + \frac{1}{2}\rho A_1 g x^3 + \frac{1}{2}\rho A_2 g x (x - L_1)^2 + \frac{1}{2}\rho A_3 g x (x - L_1 - L_2)}{I_1 + I_2 + I_3} dx$$
$$= \frac{g}{I_1 + I_2 + I_3} \left(\frac{m((L_1+L_2+L_3)^3 - (L_1+L_2)^3)}{3} + \frac{\rho A_1((L_1+L_2+L_3)^4 - (L_1+L_2)^4)}{8} \right)$$
$$+ \frac{g\rho A_2}{I_1 + I_2 + I_3} \left(\frac{(L_1+L_2+L_3)^4 - (L_1+L_2)^4}{8} - \frac{(L_1+L_2+L_3)^3 - (L_1+L_2)^3}{3} L_1^2 + \frac{(L_1+L_2+L_3)^2 - (L_1+L_2)^2}{4} L_1^2 \right)$$
$$+ \frac{g\rho A_3}{I_1 + I_2 + I_3} \left(\frac{(L_1+L_2+L_3)^4 - (L_1+L_2)^4}{8} - \frac{(L_1+L_2+L_3)^3 - (L_1+L_2)^3}{3} (L_1+L_2) + \frac{(L_1+L_2+L_3)^2 - (L_1+L_2)^2}{4} (L_1+L_2)^2 \right)$$

And so the deflection is simply

$$\delta = \frac{g}{E} \left(\frac{1}{I_1} \left(\frac{mL_1^3}{3} + \frac{\rho A_1 L_1^4}{8} \right) + \frac{1}{I_1 + I_2} \left(\frac{m((L_1+L_2)^3 - L_1^3)}{3} + \frac{\rho A_1((L_1+L_2)^4 - L_1^4)}{8} \right) \right.$$
$$+ \frac{\rho A_2}{I_1 + I_2} \left(\frac{(L_1+L_2)^4 - L_1^4}{8} - \frac{(L_1+L_2)^3 - L_1^3}{3} L_1 + \frac{(L_1+L_2)^2 - L_1^2}{4} L_1^2 \right)$$
$$+ \frac{1}{I_1 + I_2 + I_3} \left(\frac{m((L_1+L_2+L_3)^3 - (L_1+L_2)^3)}{3} + \frac{\rho A_1((L_1+L_2+L_3)^4 - (L_1+L_2)^4)}{8} \right)$$
$$+ \frac{\rho A_2}{I_1 + I_2 + I_3} \left(\frac{(L_1+L_2+L_3)^4 - (L_1+L_2)^4}{8} - \frac{(L_1+L_2+L_3)^3 - (L_1+L_2)^3}{3} L_1 + \frac{(L_1+L_2+L_3)^2 - (L_1+L_2)^2}{4} L_1^2 \right)$$
$$\left. + \frac{\rho A_3}{I_1 + I_2 + I_3} \left(\frac{(L_1+L_2+L_3)^4 - (L_1+L_2)^4}{8} - \frac{(L_1+L_2+L_3)^3 - (L_1+L_2)^3}{3} (L_1+L_2) + \frac{(L_1+L_2+L_3)^2 - (L_1+L_2)^2}{4} (L_1+L_2)^2 \right) \right)$$