MATH 152 - Unofficial Formula Sheet

Vectors

Basics

Direction Vector	$\vec{ab} = \vec{b} - \vec{a}$
Norm	$\|x\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_a^2}$
Unit Vector	$\hat{u} = \frac{\vec{u}}{\|u\|}$
Perpendicular	$\vec{a}^{\perp} = \det \begin{bmatrix} \hat{i} & \hat{j} \\ a_1 & a_2 \end{bmatrix}$

Dot Product

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$
$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$
$\vec{a} \cdot \vec{b} = \|a\|\|b\| \cos\theta$
$\vec{a} \perp \vec{b}$ if $\vec{a} \cdot \vec{b} = 0$

Cross Product

$$\vec{a} \times \vec{b} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \vec{n}_{\vec{a},\vec{b}}$$

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$
$\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta$
$\vec{a} \parallel \vec{b}$ if $\vec{a} \times \vec{b} = 0$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \det \begin{bmatrix} -\vec{a}- \\ -\vec{b}- \\ -\vec{c}- \end{bmatrix}$$

Projection and Perpendicular

$\text{proj}_{\vec{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{b} = (\vec{a} \cdot \hat{b})\hat{b}$
$\text{perp}_{\vec{b}}(\vec{a}) = \vec{a} - \text{proj}_{\vec{b}}(\vec{a})$

Area and Volume

$A = \|\vec{a} \times \vec{b}\|$

$$A = \left| \det \begin{bmatrix} -\vec{a}- \\ -\vec{b}- \end{bmatrix} \right|$$

$$V = |\vec{a} \cdot (\vec{b} \times \vec{c})| = \left| \det \begin{bmatrix} -\vec{a}- \\ -\vec{b}- \\ -\vec{c}- \end{bmatrix} \right|$$

Lines and Planes

Line Equations

Parametric	$\vec{x} = \vec{a}t + \vec{p}$
Equation Form in $\mathbb{R}^2$	$\vec{x} \cdot \vec{a}^{\perp} = x_1 a_1^{\perp} + x_2 a_2^{\perp} = d$
Equation Form in $\mathbb{R}^3$	$\begin{cases} \vec{x} \cdot \vec{a}^{\perp} = x_1 a_1^{\perp} + x_2 a_2^{\perp} = d_1 \\ \vec{x} \cdot \vec{b}^{\perp} = x_1 b_1^{\perp} + x_2 b_2^{\perp} = d_2 \end{cases}$

Plane Equations

Parametric	$\vec{x} = \vec{a}s + \vec{b}t + \vec{p}$
Equation Form in $\mathbb{R}^3$	$\vec{x} \cdot \vec{n} = d$

Intersection of Objects

Intersection of Planes:

Solve $\begin{bmatrix} n_1 & n_2 & n_3 \\ m_1 & m_2 & m_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

Distance Between Objects

Distance between point $\vec{q}$ and Line $\vec{x} = \vec{p} + \vec{a}t$:
$d = \|\text{perp}_{\vec{a}}(\vec{pq})\| = \|\text{proj}_{\vec{a}^{\perp}}(\vec{pq})\|$
Distance between point $\vec{q}$ and plane $\vec{x} = \vec{p} + \vec{a}s + \vec{b}t$:
$d = \|\text{proj}_{\vec{n}}(\vec{pq})\|$

General Procedure:
set $d = \|\vec{x}_1 - \vec{x}_2\|$ using parametric forms of $\vec{x}$
solve for $t_1, t_2, ...$ using $\vec{\nabla} d = \vec{0}$
plug back in values of $t_1, t_2, ...$ and solve for d

Gaussian Elimination

Step 1:	Set the top left entry to 1 (or as the LCD of the first column)
Step 2:	Use the first row to 'kill off' all other entries in the first column
Step 3:	For column 2, use one row to 'kill off' all the other entries in that column
Step 4:	Repeat process until finished

Linear Independence

$\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent if:

$$\det \begin{pmatrix} -\vec{a}- \\ -\vec{b}- \\ -\vec{c}- \end{pmatrix} = 0$$

or if $\begin{bmatrix} | & | & | \\ \vec{a} & \vec{b} & \vec{c} \\ | & | & | \end{bmatrix}$ has no unique solution.

Linear Transformations

Rules

$T(\vec{x}) = A\vec{x}$
$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$
$T(s\vec{x}) = sT(\vec{x})$
$(S \circ T)\vec{x} = BA\vec{x}$
$A = [T(\vec{e}_1)|T(\vec{e}_2)| \cdots |T(\vec{e}_n)]$

Rotations

$$\text{Rot}_{\theta} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$\text{Rot}_{\theta}^2(\vec{x}) = \text{Rot}_{2\theta}(\vec{x})$

Projections

$$\text{Proj}_{\hat{a}} = \begin{bmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix}$$

$$\text{Proj}_{\theta} = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$$

$\text{Proj}^2(\vec{x}) = \text{Proj}(\vec{x})$

Reflections

$\text{Ref} = 2\text{Proj} - I$

$$\text{Ref}_{\theta} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$$

$$\text{Ref}_{\hat{a}} = \begin{bmatrix} 2a_1^2 - 1 & 2a_1 a_2 \\ 2a_1 a_2 & 2a_2^2 - 1 \end{bmatrix}$$

$\text{Ref}^2(\vec{x}) = \vec{x}$

Transpose

Identities

$(A^T)^T = A$
$(A+B)^T = A^T + B^T$
$(kA)^T = kA^T$
$(AB)^T = B^T A^T$

Decomposition of a Matrix

jth column $= A\vec{e}_j$
ith row $= \vec{e}_i^T A$
Entry $a_{ij} = \vec{e}_i^T A \vec{e}_j$

Dot Product

$\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b}$

Determinants

2x2 and 3x3 Determinants

$\det(A) = |A|$

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb$$

$$\det \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

General

$\det(A) = \det(A^T)$
$\det(A^{-1}) = \frac{1}{\det(A)}$
$\det(A^x) = \det(A)^x$
$\det(AB) = \det(A)\det(B)$
$\det(kA) = k^n \det(A)$ where n is the matrix size
$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n$

Simplifying Determinants

Swap Rows	$\det(B) = -\det(A)$
Multiply Row by k:	$\det(B) = k\det(A)$
Add Factor of a Row	$\det(B) = \det(A)$

The determinant of a triangular matrix is the product of the diagonals

Inverse

A is invertible iff $\det(A) \neq 0$
If A is invertible then:
$A^{-1}A = AA^{-1} = I$
$(A^{-1})^{-1} = A$
$(kA)^{-1} = \frac{1}{k}A^{-1}$
$(AB)^{-1} = B^{-1}A^{-1}$
$(A^T)^{-1} = (A^{-1})^T$

Complex Numbers

Definitions

Imaginary Number	$i^2 = -1$
Complex Number	$z = a + ib$
Conjugate	$\overline{z} = a - ib$
Real Part	$\Re(z) = \frac{z+\overline{z}}{2}$
Imaginary Part	$\Im(z) = \frac{z-\overline{z}}{2}$
Norm	$\lvert z\rvert = \sqrt{a^2+b^2}$

Operations and Identities

$\overline{zu} = \overline{z} \cdot \overline{u}$
$\overline{(z \pm u)} = \overline{z} \pm \overline{u}$
$z \cdot \overline{z} = |z|^2$
$|zu| = |z||u|$
$\frac{u}{z} = \frac{u\overline{z}}{|z|^2}$

Polar Form

$e^{i\theta} = \cos\theta + i\sin\theta =$
$z = |z|e^{i\theta}$
$\arg(z) = \theta = \arctan(\frac{b}{a})$

Eigen-Analysis

General Formulas (to be updated by Friday)

$\det(A - I\lambda) = 0$
$A\vec{v} = \lambda\vec{v}$

Solving Method

Step 1: Use $\det(A - I\lambda) = 0$ to solve for λ
Step 2: Use $[A - I\lambda_i][\vec{v}_i] = \vec{0}$ to solve for each eigenvector $\vec{v}_i$
Step 3: To find $A^n\vec{x}$, set $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots$
Step 4: Solve $\vec{v}_1\ \vec{v}_2\ \cdots][\vec{c}] = \vec{x}$ to solve for coefficients
Step 5: Set $A^n\vec{x} = c_1\lambda_1^n\vec{v}_2 + c_2\lambda_2^n\vec{v}_2 + \cdots$
(Step 6:) Each conjugate pair can be expressed as $2\Re(c_i\lambda_i^n\vec{v}_i)$

Differential Equations

Solving Method

Step 1: Set $\frac{d\vec{x}}{dt} = A\vec{x}$
Step 2: Find eigenvalues and eigenvectors of A
Step 3: Write $\vec{x}(t) = c_1e^{\lambda_1 t}\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2 + \cdots$ as the general form of the solution.
Step 4: If imaginary, express conjugate pairs as $c_1\vec{P}(t) + c_2\vec{Q}(t)$
Where $\vec{P}(t) = \Re(e^{\alpha t}\vec{v}_1(\cos(\beta t) + i\sin(\beta t))$
and $\vec{Q}(t) = \Im(e^{\alpha t}\vec{v}_1(\cos(\beta t) + i\sin(\beta t))$
Step 5: Use initial conditions to solve for coefficients

LCR Circuits

Step 1: Write voltage and current equations
Step 2: Express i and E in terms of I and V
Step 3: Write differential equations of form $\frac{dV}{dt} = \pm\frac{i}{C}$ and $\frac{dI}{dt} = \pm\frac{E}{L}$
Step 4: Express system of differential equations in matrix form
Step 5: Solve differential equation for various $I(t)$ and $V(t)$

Compiled by Tyler Wilson 2021