MATH 217 - Formula Sheet

Vectors

Basics

Direction Vector $\vec{ab} = \vec{b} - \vec{a}$

Norm $\|x\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_a^2}$

Unit Vector $\hat{u} = \dfrac{\vec{u}}{\|u\|}$

Perpendicular $\vec{a}^{\perp} = \det \begin{bmatrix} \hat{i} & \hat{j} \\ a_1 & a_2 \end{bmatrix} = \langle -a_2, a_1 \rangle$

Dot Product

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

$\vec{a} \cdot \vec{b} = \|a\|\|b\| \cos\theta$

$\vec{a} \perp \vec{b}$ iff $\vec{a} \cdot \vec{b} = 0$

Cross Product

$$\vec{a} \times \vec{b} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \vec{n}_{\vec{a},\vec{b}}$$

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$\|\vec{a} \times \vec{b}\| = \|\vec{a}\|\|\vec{b}\| \sin\theta$

$\vec{a}$ is parallel to $\vec{b}$ iff $\vec{a} \times \vec{b} = 0$

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \det \begin{bmatrix} -\vec{a}- \\ -\vec{b}- \\ -\vec{c}- \end{bmatrix}$$

Projection and Perpendicular

$\text{proj}_{\vec{b}}(\vec{a}) = \dfrac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2}\vec{b} = (\vec{a} \cdot \hat{b})\hat{b} = \text{comp}_{\vec{b}}(\vec{a})\hat{b}$

$\text{perp}_{\vec{b}}(\vec{a}) = \vec{a} - \text{proj}_{\vec{b}}(\vec{a})$

Area and Volume

$A = \|\vec{a} \times \vec{b}\|$

$A = \left| \det \begin{bmatrix} -\vec{a}- \\ -\vec{b}- \end{bmatrix} \right|$

$V = |\vec{a} \cdot (\vec{b} \times \vec{c})| = \left| \det \begin{bmatrix} -\vec{a}- \\ -\vec{b}- \\ -\vec{c}- \end{bmatrix} \right|$

Lines and Planes

Line Equations

Parametric $\vec{x} = \vec{a}t + \vec{p}$

Equation Form in $\mathbb{R}^2$ $\vec{x} \cdot \vec{a}^{\perp} = x_1 a_1^{\perp} + x_2 a_2^{\perp} = d$

Equation Form in $\mathbb{R}^3$ $\begin{cases} \vec{x} \cdot \vec{a}^{\perp} = x_1 a_1^{\perp} + x_2 a_2^{\perp} = d_1 \\ \vec{x} \cdot \vec{b}^{\perp} = x_1 b_1^{\perp} + x_2 b_2^{\perp} = d_2 \end{cases}$

Plane Equations

Parametric $\vec{x} = \vec{a}s + \vec{b}t + \vec{p}$

Equation Form in $\mathbb{R}^3$ $\vec{x} \cdot \vec{n} = d$

Intersection of Objects

Intersection of Planes:

Solve $\begin{bmatrix} n_1 & n_2 & n_3 \\ m_1 & m_2 & m_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

or do $\vec{n} \times \vec{m}$

Vector Valued functions of one variable

Product Rules

Scalar Times Vector $\frac{d}{dt}(f(t)\vec{r}(t)) = f'\vec{r} + f\vec{r}'$

Dot Product $\frac{d}{dt}(\vec{u}(t) \cdot \vec{v}(t)) = \vec{u}' \cdot \vec{v} + \vec{u} \cdot \vec{v}'$

Cross Product $\frac{d}{dt}(\vec{u}(t) \times \vec{v}(t)) = \vec{u}' \times \vec{v} + \vec{u} \times \vec{v}'$

Arc Length

$$ds = \|\vec{r}'(t)\| dt$$

$$s = \int_{t_0}^{t_f} \|\vec{r}'(t)\| dt = \int_{t_0}^{t_f} \sqrt{(x')^2 + (y')^2 + (z')^2} dt$$

Divergence and Curl

Divergence and Curl

Divergence $\text{div}(\vec{F}) = \vec{\nabla} \cdot \vec{F}$

Curl $\text{curl}(\vec{F}) = \vec{\nabla} \times \vec{F}$

$\text{curl}(\vec{F}) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

$\vec{F}$ is a potential if $\vec{F}$ is simply connected and $\text{curl}(\vec{F}) = \vec{0}$

Partial Derivatives

Basics

Gradient $\vec{\nabla} f = \left\langle \dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z} \right\rangle$

Directional Derivative $(D_{\hat{u}} f)(x_0, y_0) = (\vec{\nabla} f)(x_0, y_0) \cdot \hat{u}$

Chain Rule $\dfrac{dz}{dt} = \dfrac{\partial z}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial z}{\partial y} \cdot \dfrac{dy}{dt}$

Chain Rule for Functions of Several Variables

$$\begin{bmatrix} \frac{\partial z}{\partial s} \\ \frac{\partial z}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} \end{bmatrix}$$

Radial vectors $\vec{\nabla} r^n = n r^{n-1} \vec{r}$

Divergence product rule $\vec{\nabla} \cdot (f\vec{F}) = \vec{\nabla} f \cdot \vec{F} + (f)(\vec{\nabla} \cdot \vec{F})$

Tangent Planes

Plane tangent to the graph $z = f(x, y)$:

$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

Plane tangent to level surface $f(x, y, z) = c$ at $P = (x_0, y_0, z_0)$:

$f_x(P)(x - x_0) + f_y(P)(y - y_0) + f_z(P)(z - z_0) = 0$

Linear Approximation

Of 1 Variable $\Delta f \approx f'(x_0)\Delta x$

Of 2 Variables $\Delta f \approx f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y$

Classification of Critical Points and Optimization

$$D(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - f_{xy}^2$$

If $D(x_0, y_0) < 0$, (x_0, y_0) is a saddle

If $D(x_0, y_0) > 0$ and $f_{xx} < 0$, (x_0, y_0) is a local max

If $D(x_0, y_0) > 0$ and $f_{xx} > 0$, (x_0, y_0) is a local min

If $D(x_0, y_0) = 0$ then (x_0, y_0) is not an ordinary critical point.

Lagrange Multipliers

$$\begin{cases} (\vec{\nabla} f)(x_0, y_0, z_0) = \lambda(\vec{\nabla} g)(x_0, y_0, z_0) \\ g(x, y, z_0) = 0 \end{cases}$$

Multiple Integrals

Change of Coordinates

Polar	$dA = r dr d\theta$
	$r^2 = x^2 + y^2$
	$x = r\cos\theta,\ y = r\sin\theta$
Cylindrical	$dV = r dz dr d\theta$
	$r^2 = x^2 + y^2$
	$x = r\cos\theta,\ y = r\sin\theta,\ z = z$
Spherical	$dV = \rho^2 \sin\phi d\rho d\phi d\theta$
	$\rho^2 = x^2 + y^2 + z^2$
	$x = \rho\cos\theta\sin\phi$
	$y = \rho\sin\theta\sin\phi$
	$z = \rho\cos\phi$
General	$dA = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du dv$

Applications

Area	$A = \iint_R dA$
Average value	$\overline{f(x,y)} = \frac{1}{\text{Area}(R)} \iint_R f(x,y) dA$
Average height	$\overline{y} = \frac{1}{\text{Area}(R)} \iint_R y dA$
Mass of region	$m = \iint_R \rho(x,y) dA$
Center of mass	$\overline{x} = \frac{1}{\text{Mass}(R)} \iint_R x\rho(x,y) dA$
Moment of inertia	$I_a = \iint_R D(x,y)^2 \rho(x,y) dA$
Surface Area	$S = \iint_R \sqrt{1 + f_x^2 + f_y^2} dA$

Line and Surface Integrals

Line and Work Integrals

$$\int_C f ds = \int_C f(\vec{r}(t)) \|\vec{r}'(t)\| dt$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \hat{T} ds = \int_C \vec{F}(t) \cdot \vec{r}'(t) dt$$

Surface and Flux Integrals

$$\iint_S f dS = \pm \iint_S f(\vec{r}(u,v)) \|\vec{r}_u \times \vec{r}_v\| du dv$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \vec{F} \cdot \hat{n} dS = \pm \iint_S \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) du dv$$

Integral Theorems and Differential Forms

Integral Theorems

$$\iint_R (Q_x - P_y) dA = \oint_{\partial R} \vec{F} \cdot d\vec{r}$$

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

$$\iiint_E (\vec{\nabla} \cdot \vec{F}) dV = \oiint_{\partial E} \vec{F} \cdot d\vec{S}$$

Differential Forms

$$\Omega^k(U) \cdot \Omega^l(U) \to \Omega^{k+l}(U)$$

$$d:\ \Omega^k(U) \to \Omega^{k+1}(U)$$

$$\text{for } \alpha \in \Omega^k(U),\ \beta \in \Omega^l(U),$$

$$\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$$

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

$$\int_{\partial M} \alpha = \int_M d\alpha$$

k-form	function/vector field
$\Omega^0(U)$	f
$\Omega^1(U)$	$F_1 dx + F_2 dy + F_3 dz$
$\Omega^2(U)$	$F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$
$\Omega^3(U)$	$f dx \wedge dy \wedge dz$

Trigonometry

Basic Trig Identities

Pythagorean	$\sin^2\theta + \cos^2\theta = 1$
	$\sec^2\theta - \tan^2\theta = 1$
	$\csc^2\theta - \cot^2\theta = 1$
Double Angle	$\sin(2\theta) = 2\sin\theta\cos\theta$
	$\cos(2\theta) = \cos^2\theta - \sin^2\theta$
	$\cos(2\theta) = 2\cos^2\theta - 1$
	$\cos(2\theta) = 1 - 2\sin^2\theta$
Half Angle	$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$
	$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$
Integrals	$\int_0^{2\pi} \sin^2\theta d\theta = \int_0^{2\pi} \cos^2\theta d\theta = \pi$

Integration Summary

Integrals of a function $\int_C f ds$

- Use a parameterization

$$\int_C f ds = \int_a^b f(\vec{r}(t)) \|d\vec{r}'(t)\| dt$$

Integrals of a vector field (work integrals): $\int_C \vec{F} \cdot d\vec{r}$

- If $\vec{F}$ is conservative, use FTL ($\vec{\nabla} f = \vec{F}$)

$$\int_C \vec{F} \cdot d\vec{r} = f(P_2) - f(P_1)$$

(if $\vec{F}$ is not conservative, we can try $\vec{F} = \vec{F}_1 + \vec{F}_2$)

- If C is closed, use Green's/Stoke's Theorem

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

(if C is not closed, we can try $\partial S = C + C'$)

- Compute using a parameterization

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

Integral of a function over a surface

- Use a parameterization

$$\iint_S f dS = \iint_S f(\vec{r}(u,v)) \|\vec{r}_u \times \vec{r}_v\| du dv$$

Integral of a vector field over a surface: $\iint_S \vec{F} \cdot d\vec{S}$

- If $\vec{F} = \vec{\nabla} \times \vec{G}$, apply Stoke's Theorem

$$\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{r}$$

- If S is closed, apply Divergence Theorem

$$\oiint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint (\vec{\nabla} \cdot \vec{F}) dV$$

(If S is not closed, we can try $\partial E = S + S'$)

- Compute using a parameterization

$$\iint_S \vec{F} \cdot d\vec{S} = \pm \iint_S F(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) du dv$$

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