Math 257 Summary Sheet

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Ordinary Differential Equations

Linear ODEs: $y' + p(x)y = g(x)$ will have a solution of the form

$$\frac{d}{dx}(yr) = rg,\ \ r = e^{\int p(x)dx}$$

Constant coefficients: $ay''+by'+cy = 0$. You can write the characteristic equation as $ar^2+br+c = 0$ and the general solution will be

$$y(x) = \begin{cases} Ae^{r_1 x} + Be^{r_2 x} & r_1 \neq r_2 \in \mathbb{R} \\ Ae^{rx} + Bxe^{rx} & r_1 = r_2 \\ e^{\lambda x}\left(A\sin(\mu x) + B\cos(\mu x)\right) & r = \lambda \pm i\mu \end{cases}$$

Cauchy-Euler: $ax^2y''+bxy'+cy = 0$. You can write the characteristic equation as $ar(r-1) + br + c = 0$ and the general solution will be

$$y(x) = \begin{cases} Ax^{r_1} + Bx^{r_2} & r_1 \neq r_2 \in \mathbb{R} \\ Ax^r + Bx^r \ln|x| & r_1 = r_2 \\ x^{\lambda}\left(A\sin(\mu\ln|x|) + B\cos(\mu\ln|x|)\right) & r = \lambda \pm i\mu \end{cases}$$

Nonhomogeneous equations: You can write the solution as $y(x) = y_c+y_p$ and can use undetermined coefficients to find y_p

$f(x)$	guess
$e^{\alpha x}$	$ae^{\alpha x}$
$\sin(\omega x)$	$a\cos(\omega x) + b\sin(\omega x)$
$\cos(\omega x)$	$a\cos(\omega x) + b\sin(\omega x)$
t^n	$a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$

If there is any overlap with the complementary solution then you multiply your guess by x

Series Solutions

For writing series solutions of $y'' + p(x)y' + q(x)y = 0$ about $x = x_0$
If x_0 is an ordinary point,

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

If the limit as $x \to x_0$ of any of $p(x),\ q(x),\ p'(x),\ q'(x)$ does not exist, x_0 is a singular point.
Then, if $\lim_{x\to x_0}(x - x_0)p(x) = \alpha_0$ and $\lim_{x\to x_0}(x - x_0)^2 q(x) = \beta_0$ both exist, x_0 is a regular singular point.
The indicial equation is given by $r(r-1) + \alpha_0 r + \beta_0 = 0$ and the general solution will depend on

the type of solution of r.
If $r_1 - r_2 \notin \mathbb{Z}$ (most common case for us) then,

$$y_1 = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}, \ y_2 = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_2}$$

If $r_1 = r_2$ then,

$$y_1 = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r}, \ y_2 = y_1(x) \ln(x - x_0) + \sum_{n=1}^{\infty} b_n (x - x_0)^{n+r}$$

If $|r_1 - r_2| \in \mathbb{N}$ then,

$$y_1 = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+r_1}, \ y_2 = a y_1(x) \ln(x - x_0) + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+r_2}$$

Homogeneous Heat Equation

General solutions for different boundary conditions:

- $u(0,t) = u(L,t) = 0$ (Dirichlet)

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$
$$X_n = \sin\left(\frac{n\pi x}{L}\right)$$
$$T_n = e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t}$$
$$n \geq 1$$

- $u_x(0,t) = u_x(L,t) = 0$ (Neumann)

$$\lambda_n = 0, \ \left(\frac{n\pi}{L}\right)^2$$
$$X_n = 1, \ \cos\left(\frac{n\pi x}{L}\right)$$
$$T_n = 1, \ e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t}$$
$$n \geq 1$$

- $u(0,t) = u(L,t)$ and $u_x(0,t) = u_x(L,t)$ (Periodic)

$$\lambda_n = 0, \ \left(\frac{n\pi}{L}\right)^2$$
$$X_n = 1, \ \sin\left(\frac{n\pi x}{L}\right), \ \cos\left(\frac{n\pi x}{L}\right)$$
$$T_n = 1, \ e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t}$$
$$n \geq 1$$

- $u(0,t) = u_x(L,t) = 0$ (Mixed type 1)

$$\begin{aligned}
\lambda_n &= \left(\frac{2n-1}{2L}\pi\right)^2 \\
X_n &= \sin\left(\frac{2n-1}{2L}\pi x\right) \\
T_n &= e^{-\alpha^2\left(\frac{2n-1}{2L}\pi\right)^2 t} \\
n &\geq 1
\end{aligned}$$

- $u_x(0,t) = u(L,t)$ (Mixed type 2)

$$\begin{aligned}
\lambda_n &= \left(\frac{2n-1}{2L}\pi\right)^2 \\
X_n &= \cos\left(\frac{2n-1}{2L}\pi x\right) \\
T_n &= e^{-\alpha^2\left(\frac{2n-1}{2L}\pi\right)^2 t} \\
n &\geq 1
\end{aligned}$$

The solution in each case will be of the form $u(x,t) = \sum\limits_{n=n_0}^{\infty} X_n(x)T_n(t)$ where the sum starts at either $n_0 = 0$ or $n_0 = 1$ depending on the boundary conditions.

Fourier Series

The Fourier series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

We can use the Fourier series to create a function identical to our IC, $u(x,0)$ and get the coefficients in our PDE.
Another way to do this is by exploiting orthogonality in which case, the following integrals will be of use.

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & m \neq n \\ L & m = n \end{cases}$$

$$\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases} 0 & m \neq n \\ L & m = n \neq 0 \\ 2L & m = n = 0 \end{cases}$$

$$\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0$$
$$\cos(n\pi) = (-1)^n$$
$$\sin(n\pi) = 0$$

Finite Difference Approximations

Want to solve to get u_i^{k+1} in terms of u^k terms so we can solve for the next time step.
Formulas:

Forward: $f'(x_0) = \dfrac{f(x_0+\Delta x) - f(x_0)}{\Delta x} + \mathcal{O}(\Delta x)$

Backward: $f'(x_0) = \dfrac{f(x_0) - f(x_0-\Delta x)}{\Delta x} + \mathcal{O}(\Delta x)$

Centre: $f'(x_0) = \dfrac{f(x_0+\Delta x) - f(x_0-\Delta x)}{2\Delta x} + \mathcal{O}(\Delta x^2)$

2nd Order: $f''(x_0) = \dfrac{f(x_0+\Delta x) - 2f(x_0) + f(x_0-\Delta x)}{\Delta x^2} + \mathcal{O}(\Delta x^2)$

More formulas can be derived using Taylor series as a starting point.
Index notation: we write $x = i\Delta x$ and $t = k\Delta t$ and so $u_i^k = u(i\Delta x, k\Delta t)$ where i is the step in x and k is the time step.
Method: We use the above formulas to write expressions for u_t and u_{xx}, plug them into our PDE, and solve for u_i^{k+1}. The expression with $\mathcal{O}(\Delta x^2, \Delta t)$ is given by

$$u_i^{k+1} = \alpha^2 \frac{\Delta t}{\Delta x^2}\left(u_{i+1}^k - 2u_i^k + u_{i-1}^k\right) + u_i^k$$

From here we use our IC to get points for u_i^0 and use the BCs to get information about the points at the edges. For example,

$$u(0,t) = 0 \Rightarrow u_0^k = 0 \ \forall k$$
$$u_x(0,t) = 0 \Rightarrow \frac{u_1^k - u_{-1}^k}{2\Delta x} = 0 \Rightarrow u_{-1}^k = u_1^k$$