

# Math Notes

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## 1 Complex Analysis

### 1.1 Complex Algebra

Complex numbers arise from the roots of polynomials.

Ex:  $x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm\sqrt{-1}$ . This polynomial has no real roots, however, we can introduce an imaginary number  $i$  such that  $i^2 = -1$ . Then we will have the solution  $x = \pm i$

We can introduce *complex numbers* which are numbers in the form  $z = x + iy$ , where  $x$  is the real part of  $z$ ,  $\Re(z)$ , and  $y$  is the imaginary part of  $z$ ,  $\Im(z)$ . These numbers can also be expressed in vector notation along the complex plane.

### 1.1.1 Complex Arithmetic

Addition, subtraction, and multiplication work the same, just with the addition of the fact  $i^2 = -1$ . For division, we require what is called the conjugate.

The conjugate of a complex number is the same number, just with the sign of the imaginary component flipped.

$$\bar{z} = x - yi$$

where  $\bar{z}$  is the conjugate of  $z$ .

Similarly to vectors, we can also define the modulus (length) of a complex number

$$|z|^2 = x^2 + y^2 = z \cdot \bar{z}$$

Using this, we can define the division of a complex number and also define the real and imaginary components of a complex number.

The general expression for division is:

$$\frac{u}{z} = \frac{s + it}{x + iy} = \frac{(s + it)(x - iy)}{(x + iy)(x - iy)} = \frac{u\bar{z}}{x^2 + y^2} = \frac{u\bar{z}}{|z|^2}$$

The real and imaginary components can be computed as

$$z = x + iy$$

$$\bar{z} = x - iy$$

$$\Re(z) = \frac{z + \bar{z}}{2}$$

$$\Im(z) = \frac{z - \bar{z}}{2i}$$

Ex1: Simplify  $(1 + 2i)(3 + i)(2 - 3i)$

$$\begin{aligned}(1 + 2i)(3 + i)(2 - 3i) &= (3 + i + 6i - 2)(2 - 3i) = (1 + 7i)(2 - 3i) \\ &= 2 - 3i + 14i + 21 \\ &= 23 + 11i\end{aligned}$$

Ex2: Simplify  $\left(\frac{2+i}{1+i}\right)^2$

$$\begin{aligned}\left(\frac{2+i}{1+i}\right)^2 &= \left(\frac{(2+i)(1-i)}{(1+i)(1-i)}\right)^2 = \left(\frac{2+i-2i+1}{2}\right)^2 = \left(\frac{3-i}{2}\right)^2 = \frac{9-6i-1}{4} \\ &= 2 - \frac{3}{2}i\end{aligned}$$

Ex3: Simplify  $(1 + 2i)^5$

$$\begin{aligned}(1 + 2i)^5 &= 1^5 + 5(1)^4(2i) + 10(1)^3(2i)^2 + 10(1)^2(2i)^3 + 5(1)(2i)^4 + (2i)^5 \\ &= 1 + 10i + 10(-4) + 10(-8i) + 5(16) + 32i \\ &= 1 + 10i - 40 - 80i + 80 + 32i \\ &= 41 - 38i\end{aligned}$$

Ex4: Prove

$$\text{if } |z| = 1 \text{ then } \Re\left(\frac{1}{1+z}\right) = \frac{1}{2}$$

*Proof.*

$$\begin{aligned}\Re(w) &= \frac{w + \bar{w}}{2} \\ \overline{\frac{1}{1+z}} &= \frac{1}{1+\bar{z}} \\ \Re\left(\frac{1}{1+z}\right) &= \frac{\frac{1}{1+z} + \frac{1}{1+\bar{z}}}{2} = \frac{1+\bar{z}+1+z}{2(1+z)(1+\bar{z})} = \frac{2+z+\bar{z}}{2(1+z+\bar{z}+|z|^2)} \\ |z| = 1 &\Rightarrow \Re\left(\frac{1}{1+z}\right) = \frac{2+z+\bar{z}}{2(2+z+\bar{z})} = \frac{1}{2}\end{aligned}$$

□

Some properties of  $\bar{z}$ :

- $\overline{\bar{z}} = z$
- $\overline{\bar{z}_1 z_2} = \bar{z}_1 \bar{z}_2$
- $|z_1 z_2| = |z_1| |z_2|$

Some common inequalities come from the triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

*Proof.*

$$\begin{aligned}|z_1 + z_2| &\leq |z_1| + |z_2| \\ |z_1 + z_2| &= \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \\ \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} &\leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} \\ |z_1 + z_2| &\leq |z_1| + |z_2|\end{aligned}$$

□

$$\Rightarrow |z_1 \pm z_2| \geq ||z_1| - |z_2||$$

This leads to a general upper and lower bound that can be derived from these inequalities:

$$||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$$

### 1.1.2 Polar Form of Complex Numbers

Another way to represent complex numbers is through polar form. To use this we must first introduce Euler's identity:

$$e^{i\varphi} = \cos \varphi + i \sin \varphi$$

This then helps us with the equation for the polar form

$$z = a + ib = |z|(\cos \varphi + i \sin \varphi) = |z|e^{i\varphi}$$

For  $z = re^{i\varphi}$  We call  $r$  the magnitude of the complex number and  $\varphi$  is the argument. This can be especially useful for simplifying some complex numbers.

Ex:

$$\left| \frac{(1 + \sqrt{3}i)^{100}}{(\sqrt{3} - i)^{100}} \right| = \frac{|1 + \sqrt{3}i|^{100}}{|\sqrt{3} - i|^{100}} = \frac{2^{100}}{2^{100}} = 1$$

Note that because sinusoidal functions are periodic every  $2\pi$  this means that there infinite ways to express a function in polar coordinates.

$$e^{i2\pi k} = 1, \quad k \in \mathbb{Z} \Rightarrow z = re^{i(\varphi + 2\pi k)}$$

To get around the issue of having infinite possible polar forms for every complex number we define what's called the principal argument to be

$$\text{Arg}(z) = \varphi \in (-\pi, \pi]$$

We define the regular argument to be

$$\arg(z) = \text{Arg}(z) + 2k\pi, \quad k \in \mathbb{Z}$$

Note that  $\text{Arg}(z)$  is singular valued while  $\arg(z)$  is multi-valued.

We can define  $\text{Arg}(z)$  in terms of the real and imaginary parts ( $x$  and  $y$ ) as

$$\text{Arg}(z) = \arctan\left(\frac{y}{x}\right) \pm k\pi$$

What it is specifically depends on what quadrant of the complex plane the point lies in.

- QI:  $\text{Arg}(z) = \arctan\left(\frac{y}{x}\right)$
- QII:  $\text{Arg}(z) = \arctan\left(\frac{y}{x}\right) + \pi$
- QIII:  $\text{Arg}(z) = \arctan\left(\frac{y}{x}\right) - \pi$
- QIV:  $\text{Arg}(z) = \arctan\left(\frac{y}{x}\right)$

Ex:

$$\text{Arg}(-1 - \sqrt{3}i) = \arctan(\sqrt{3}) - \pi = -\frac{2\pi}{3}$$

Ex2:

$$\arg(1 - \sqrt{3}i)$$

$$\text{Arg}(1 - \sqrt{3}i) = -\frac{\pi}{3}$$

$$\arg(1 - \sqrt{3}i) = -\frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z}$$

Ex3:

$$\arg(-1 + 2i)$$

$$\text{Arg}(-1 + 2i) = \arctan(-2) + \pi$$

$$\arg(-1 + 2i) = \pi - \arctan(2) + 2k\pi$$

Ex4: Simplify

$$z = -3 + 3i$$

$$\text{Arg}(z) = \arctan\left(\frac{3}{-3}\right) + \pi = \frac{3\pi}{4}$$

$$|z| = 3\sqrt{2}$$

$$z = 3\sqrt{2}e^{\frac{3\pi}{4}i}$$

Ex5: Simplify

$$z = -3 - 3i$$

$$\text{Arg}(z) = \arctan\left(\frac{-3}{-3}\right) - \pi = -\frac{3\pi}{4}$$

$$|z| = 3\sqrt{2}$$

$$z = 3\sqrt{2}e^{-\frac{3\pi}{4}i}$$

Ex6: Simplify

$$z = \frac{1 - i}{-\sqrt{3} + i} = \frac{u}{v}$$

$$\arg(z) = \arg(u) - \arg(v) = \arctan\left(\frac{-1}{1}\right) - \left(\arctan\left(\frac{1}{-\sqrt{3}}\right) + \pi\right) + 2k\pi$$

$$= -\frac{\pi}{4} - \frac{5\pi}{6} + 2k\pi = -\frac{13\pi}{12} + 2k\pi$$

$$\text{Arg}(z) = \frac{11\pi}{12}$$

$$|z| = \frac{|u|}{|v|} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$z = \frac{e^{\frac{11\pi}{12}i}}{\sqrt{2}}$$

Ex7: Simplify

$$\begin{aligned}
 z &= (\sqrt{3} - i)^2 = w^2 \\
 \arg(w) &= \arctan\left(\frac{-1}{\sqrt{3}}\right) + 2k\pi = -\frac{\pi}{6} \\
 \Rightarrow \arg(z) &= 2\arg(w) = -\frac{\pi}{3} + 4k\pi \\
 \text{Arg}(z) &= -\frac{\pi}{3} \\
 |w| &= 2 \Rightarrow |z| = |w|^2 = 4 \\
 z &= 4e^{-\frac{\pi}{3}i}
 \end{aligned}$$

Ex8: Solve for all values of  $z$

$$\begin{aligned}
 e^z &= -1 - \sqrt{3}i \\
 e^z &= 2e^{-i\frac{2\pi}{3} + 2\pi ik} = e^{\ln 2 - i\frac{2\pi}{3} + 2\pi ik} \\
 z &= \ln 2 - i\frac{2\pi}{3} + 2\pi ik, \quad \forall k \in \mathbb{Z}
 \end{aligned}$$

Properties of  $\text{Arg}(z)$  and  $\arg(z)$

- $\text{Arg}(z_1 z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2)$

*Proof.* Proof by contradiction: assume that  $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$  is true. Take  $z_1 = z_2 = -1$ .

$$\begin{aligned}
 \text{Arg}(z_1) &= \text{Arg}(z_2) = \pi \\
 \Rightarrow \text{Arg}(z_1) + \text{Arg}(z_2) &= 2\pi \\
 \text{Arg}(z_1 z_2) &= \text{Arg}(1) = 0 \\
 \text{Arg}(z_1 z_2) &\neq \text{Arg}(z_1) + \text{Arg}(z_2) \quad \forall z_1, z_2 \neq 0 \in \mathbb{C}
 \end{aligned}$$

□

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\text{Arg}(\bar{z}) \neq -\text{Arg}(z)$

*Proof.* Proof by contradiction: assume that  $\text{Arg}(\bar{z}) = -\text{Arg}(z)$  is true. Take  $z = -1$

$$\begin{aligned}
 \bar{z} &= z = -1 \\
 \text{Arg}(z) &= \pi \\
 \text{Arg}(\bar{z}) &= \pi \\
 -\text{Arg}(z) &= -\pi \\
 \Rightarrow \text{Arg}(\bar{z}) &\neq -\text{Arg}(z) \quad \forall z \in \mathbb{C}
 \end{aligned}$$

□

- $\arg(z) = -\arg(\bar{z})$

*Proof.*

$$z = |z|e^{i\arg(z)} \quad \forall z \in \mathbb{C}$$

$$\bar{z} = |z|e^{-i\arg(z)}$$

$$\Rightarrow \arg(\bar{z}) = -\arg(z)$$

□

### 1.1.3 De Moirre's Formula

Using Euler's identity we can derive a powerful formula called De Moirre's Formula as follows:

$$e^{iN\varphi} = \cos(N\varphi) + i\sin(N\varphi)$$

$$e^{iN\varphi} = (e^{i\varphi})^N = (\cos \varphi + i\sin \varphi)^N$$

$$(\cos \varphi + i\sin \varphi)^N = \cos(N\varphi) + i\sin(N\varphi)$$

Applications of De Moirre's Formula:

Binomial expansion:

$$N = 2 :$$

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$$

$$\sin(2\theta) = 2\cos \theta \sin \theta$$

$$N = 3 :$$

$$(\cos \theta + i\sin \theta)^3 = \cos^3 \theta + 3\cos^2 \theta(i\sin \theta) + 3\cos \theta(i\sin \theta)^2 + (i\sin \theta)^3$$

$$\cos(3\theta) = \cos^3 \theta - 3\cos \theta \sin^2 \theta$$

$$\sin(3\theta) = 3\cos^2 \theta \sin \theta - \sin^3 \theta$$

Ex: Prove

$$\sin(3\theta) = 3\sin \theta - 4\sin^3 \theta$$

*Proof.* Using De Moivre's formula with  $N = 3$

$$(\cos \theta + i\sin \theta)^3 = \cos(3\theta) + i\sin(3\theta)$$

$$(\cos \theta)^3 + 3(\cos \theta)^2(i\sin \theta) + 3\cos \theta(i\sin \theta)^2 + (i\sin \theta)^3 = \cos(3\theta) + i\sin(3\theta)$$

$$\cos^3 \theta - 3\cos \theta \sin^2 \theta + i(3\cos^2 \theta \sin \theta - \sin^3 \theta) = \cos(3\theta) + i\sin(3\theta)$$

$$\Im \{ \cos^3 \theta - 3\cos \theta \sin^2 \theta + i(3\cos^2 \theta \sin \theta - \sin^3 \theta) \} = \Im \{ \cos(3\theta) + i\sin(3\theta) \}$$

$$3\cos^2 \theta \sin \theta - \sin^3 \theta = \sin(3\theta)$$

$$\sin^2 \theta + \cos^2 \theta = 1 \Rightarrow \cos^2 \theta = 1 - \sin^2 \theta$$

$$3(1 - \sin^2 \theta) \sin \theta - \sin^3 \theta = \sin(3\theta)$$

$$3\sin \theta - 3\sin^3 \theta - \sin^3 \theta = \sin(3\theta)$$

$$3\sin \theta - 4\sin^3 \theta = \sin(3\theta)$$

□

Computing trigonometric integrals:

Ex:

$$\begin{aligned}
 & \int_0^{2\pi} \cos^8 \varphi d\varphi \\
 e^{i\varphi} &= \cos \varphi + i \sin \varphi \\
 e^{-i\varphi} &= \cos \varphi - i \sin \varphi \\
 \cos \varphi &= \frac{e^{i\varphi} + e^{-i\varphi}}{2} \\
 \sin \varphi &= \frac{e^{i\varphi} - e^{-i\varphi}}{2i} \\
 \int_0^{2\pi} \left( \frac{e^{i\varphi} + e^{-i\varphi}}{2} \right)^8 d\varphi &= \frac{1}{2^8} \int_0^{2\pi} (e^{i\varphi} + e^{-i\varphi})^8 d\varphi \\
 &= \frac{1}{2^8} \int_0^{2\pi} (e^{i8\varphi} + {}_1C_8 e^{i7\varphi} e^{-i\varphi} + \dots + {}_7C_8 e^{i\varphi} e^{-i7\varphi} + e^{-i8\varphi}) d\varphi \\
 &= \frac{1}{2^8} (0 + \dots + {}_4C_8 2\pi + \dots + 0) = \frac{{}_4C_8}{2^7} \pi
 \end{aligned}$$

Ex2:

$$\begin{aligned}
 \cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\
 \int_0^{2\pi} \cos^6 \theta d\theta &= \int_0^{2\pi} \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right)^6 d\theta \\
 &= \frac{1}{2^6} \int_0^{2\pi} \sum_{k=0}^6 \binom{6}{k} e^{i\theta k} e^{-i\theta(6-k)} d\theta \\
 &= \frac{1}{2^6} \sum_{k=0}^6 \binom{6}{k} \int_0^{2\pi} e^{i\theta(2k-6)} d\theta \\
 \int_0^{2\pi} e^{ik\theta} d\theta &= \left. \frac{e^{ik\theta}}{ik} \right|_0^{2\pi} = \frac{e^{2\pi ik} - 1}{ik} \\
 e^{2\pi ik} &= 1, \quad k \in \mathbb{Z} \Rightarrow \int_0^{2\pi} e^{ik\theta} d\theta = 0, \quad k \neq 0 \in \mathbb{Z} \\
 \Rightarrow \int_0^{2\pi} \cos^6 \theta d\theta &= \frac{1}{2^6} \binom{6}{3} \int_0^{2\pi} d\theta = \frac{(20)(2\pi)}{2^6} = \frac{5\pi}{8}
 \end{aligned}$$

Ex3:

$$\begin{aligned}
 & \int_0^{2\pi} \sin^6(2\theta) d\theta \\
 \sin(2\theta) &= \frac{e^{2i\theta} - e^{-2i\theta}}{2i} \\
 \int_0^{2\pi} \sin^6(2\theta) d\theta &= \int_0^{2\pi} \left( \frac{e^{2i\theta} - e^{-2i\theta}}{2i} \right)^6 d\theta
 \end{aligned}$$



$$\begin{aligned}
&= \frac{1}{(2i)^6} \int_0^{2\pi} \sum_{k=0}^6 \binom{k}{6} (-1)^{6-k} e^{2ik\theta} e^{-2i\theta(6-k)} d\theta \\
&= -\frac{1}{2^6} \sum_{k=0}^6 \binom{k}{6} (-1)^{6-k} \int_0^{2\pi} e^{i\theta(4k-12)} d\theta \\
&\int_0^{2\pi} e^{ik\theta} d\theta = \frac{e^{ik\theta}}{ik} \Big|_0^{2\pi} = \frac{e^{2\pi ik} - 1}{ik} \\
&e^{2\pi ik} = 1, \quad k \in \mathbb{Z} \Rightarrow \int_0^{2\pi} e^{ik\theta} d\theta = 0, \quad k \neq 0 \in \mathbb{Z} \\
&\Rightarrow \int_0^{2\pi} \sin^6(2\theta) d\theta = -\frac{1}{2^6} \binom{3}{6} (-1)^3 \int_0^{2\pi} d\theta \\
&= \frac{5\pi}{8}
\end{aligned}$$

Ex4: Prove

$$\sum_{k=0}^n \cos(k\theta) = \frac{1}{2} + \frac{\sin\left(\left(n + \frac{1}{2}\right)\theta\right)}{2 \sin\left(\frac{\theta}{2}\right)}$$

*Proof.* De Moivre's formula states

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$$

If we take the conjugate of both sides we get

$$(\cos \theta - i \sin \theta)^n = \cos(n\theta) - i \sin(n\theta)$$

Summing these two equations gives

$$\begin{aligned}
&(\cos \theta + i \sin \theta)^k + (\cos \theta - i \sin \theta)^k = 2 \cos(k\theta) \\
&\cos \theta \pm i \sin \theta = e^{\pm i\theta} \\
&2 \cos(k\theta) = (e^{i\theta})^k + (e^{-i\theta})^k
\end{aligned}$$

We can sum both sides of this to get

$$2 \sum_{k=0}^n \cos(k\theta) = \sum_{k=0}^n (e^{i\theta})^k + \sum_{k=0}^n (e^{-i\theta})^k$$

The formula for the geometric sum is

$$\sum_{k=0}^n z^k = \frac{1 - z^{n+1}}{1 - z}$$

Applying this we get

$$2 \sum_{k=0}^n \cos(k\theta) = \frac{1 - (e^{i\theta})^{n+1}}{1 - e^{i\theta}} + \frac{1 - (e^{-i\theta})^{n+1}}{1 - e^{-i\theta}}$$

$$\begin{aligned}
2 \sum_{k=0}^n \cos(k\theta) &= \frac{(1 - e^{i\theta(n+1)})(1 - e^{-i\theta}) + (1 - e^{-i\theta(n+1)})(1 - e^{i\theta})}{(1 - e^{i\theta})(1 - e^{-i\theta})} \\
2 \sum_{k=0}^n \cos(k\theta) &= \frac{1 - e^{i\theta(n+1)} - e^{i\theta} + e^{i\theta n} + 1 - e^{-i\theta(n+1)} - e^{-i\theta} + e^{-i\theta n}}{1 - e^{i\theta} - e^{-i\theta} + 1} \\
2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{e^{i\theta n} + e^{-i\theta n} - e^{i\theta(n+1)} - e^{-i\theta(n+1)}}{2 - e^{i\theta} - e^{-i\theta}} \\
2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{e^{i\frac{\theta}{2}} \left( e^{-i\theta(n+\frac{1}{2})} - e^{i\theta(n+\frac{1}{2})} \right) + e^{-i\frac{\theta}{2}} \left( e^{i\theta(n+\frac{1}{2})} - e^{-i\theta(n+\frac{1}{2})} \right)}{e^{i\frac{\theta}{2}} \left( e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}} \right) + e^{-i\frac{\theta}{2}} \left( e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}} \right)} \\
2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{\left( e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}} \right) \left( e^{i\theta(n+\frac{1}{2})} - e^{-i\theta(n+\frac{1}{2})} \right)}{- \left( e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}} \right)^2} \\
2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{e^{i\theta(n+\frac{1}{2})} - e^{-i\theta(n+\frac{1}{2})}}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}} \\
2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{\frac{1}{2i} \left( e^{i\theta(n+\frac{1}{2})} - e^{-i\theta(n+\frac{1}{2})} \right)}{\frac{1}{2i} \left( e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}} \right)} \\
\sin(x) &= \frac{e^{ix} - e^{-ix}}{2i} \\
2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{\sin \left( \theta \left( n + \frac{1}{2} \right) \right)}{\sin \left( \frac{\theta}{2} \right)} \\
\sum_{k=0}^n \cos(k\theta) &= \frac{1}{2} + \frac{\sin \left( \left( n + \frac{1}{2} \right) \theta \right)}{2 \sin \left( \frac{\theta}{2} \right)}
\end{aligned}$$

□

### 1.1.4 Geometry in the Complex Plane

Using the notation such that  $z = x + iy$  where  $\Re(z) = x$  and  $\Im(z) = y$  we can define a circle in the complex plane as

$$(x - x_0)^2 + (y - y_0)^2 = r_0^2$$

This is analogous to writing

$$|z - z_0| = r_0$$

The two can be related as follows:

$$\begin{aligned}
|z - z_0| &= r_0 \\
|x + iy - x_0 - iy_0| &= r_0 \\
\sqrt{(x - x_0)^2 + (y - y_0)^2} &= r_0
\end{aligned}$$

$$(x - x_0)^2 + (y - y_0)^2 = r_0^2$$

Ex: describe the circle formed by  $2|z| = |z + 1|$

$$2|z| = |z + 1|$$

$$4|z|^2 = |z + 1|^2$$

$$4x^2 + 4y^2 = (x + 1)^2 + y^2$$

$$4x^2 + 4y^2 = x^2 + 2x + 1 + y^2$$

$$3x^2 + 3y^2 - 2x - 1 = 0$$

$$3x^2 - 2x + \frac{1}{3} + 3y^2 - \frac{4}{3} = 0$$

$$3\left(x - \frac{1}{3}\right)^2 + 3y^2 = \frac{4}{3}$$

$$\left(x - \frac{1}{3}\right)^2 + y^2 = \frac{4}{9}$$

A line in the complex plane can be written as

$$ax + by = c \longleftrightarrow a\frac{z + \bar{z}}{2} + b\frac{z - \bar{z}}{2i} = c$$

Ex: describe the line formed by  $|z - 1 + i| = |z - 2i|$

$$|z - 1 + i| = |z - 2i|$$

$$|z - 1 + i|^2 = |z - 2i|^2$$

$$(x - 1)^2 + (y + 1)^2 = x^2 + (y - 2)^2$$

$$x^2 - 2x + 1 + y^2 + 2y + 1 = x^2 + y^2 - 4y + 4$$

$$-2x + 2 + 2y = -4y + 4$$

$$6y = 2x + 2$$

$$y = \frac{1}{3}(x + 1)$$

We can define an ellipse in the complex plane as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If we have  $a > b$  then we will have a horizontal ellipse and if  $b > a$  then we will have a vertical ellipse.

Assuming that  $a > b$  then we can define the foci points to be at

$$+F = (\sqrt{a^2 - b^2}, 0)$$

$$-F = (-\sqrt{a^2 - b^2}, 0)$$

The equation of an ellipse can also be described by

$$|z - F| + |z + F| = 2a$$

Ex: describe the ellipse formed by  $|z - 1| + |z + 1| = 4$

$$|z - 1| + |z + 1| = 4$$

$$|z - F| + |z + F| = 2a \Rightarrow 2a = 4 \Rightarrow a = 2$$

$$F = \sqrt{a^2 - b^2} = 1 \Rightarrow 1 = 4 - b^2 \Rightarrow b^2 = 3$$

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

Ex2: describe the ellipse formed by  $|z - 1| + |z + 3| = 6$

$$|z - 1| + |z + 3| = 6$$

note that this ellipse is not centered at the origin so we need to shift it

$$|z + 1 - 2| + |z + 1 + 2| = 6$$

$$2a = 6 \Rightarrow a = 3$$

$$F = 2 = \sqrt{a^2 - b^2} \Rightarrow 4 = 9 - b^2 \Rightarrow b^2 = 5$$

$$\frac{(x + 1)^2}{9} + \frac{y^2}{5} = 1$$

### 1.1.5 Roots of a Complex Number

Given  $z_0 = r_0 e^{i\varphi_0}$ , what is  $z_0^{\frac{1}{n}}$ ?

If we let  $w = z_0^{\frac{1}{n}}$  then  $w^n = z_0$

$$w = r e^{i\varphi}, \quad w^n = r^n e^{in\varphi}$$

$$w^n = z_0 \Rightarrow r^n e^{in\varphi} = r_0 e^{i\varphi_0}$$

$$\Rightarrow r^n = r_0 \Rightarrow r = r_0^{\frac{1}{n}}$$

$$e^{in\varphi} = e^{i\varphi_0} \Rightarrow n\varphi = \varphi_0 + 2k\pi$$

$$\varphi = \frac{\varphi_0}{n} + \frac{2k\pi}{n}$$

So all solutions to  $w^n = z_0$  are given by

$$w = r_0^{\frac{1}{n}} e^{i\left(\frac{\varphi_0}{n} + \frac{2k\pi}{n}\right)}, \quad k \in \mathbb{Z}$$

If we normalize  $\varphi_0 = \text{Arg}(z_0)$  then  $k$  will be in the range  $k \in \{0, 1, \dots, n-1\}$ .

Note that the expression

$$w = r_0^{\frac{1}{n}} e^{i\left(\frac{\varphi_0}{n} + \frac{2k\pi}{n}\right)}, \quad k \in \mathbb{Z}$$

is multi-valued. If we want to avoid this, we can take what's called the principal value which is the value when  $k = 0$ :

$$z_0^{\frac{1}{n}} = r_0^{\frac{1}{n}} e^{i\frac{\varphi_0}{n}}$$

Ex: Compute  $(-1)^{\frac{1}{2}}$

$$z_0 = -1 = 1^{\frac{1}{2}} e^{i(\frac{\pi}{2} + \frac{2k\pi}{2})} = e^{i(\frac{\pi}{2} + k\pi)} = \left\{ \dots, e^{-i\frac{3\pi}{2}}, e^{-i\frac{\pi}{2}}, e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{2}}, e^{i\frac{5\pi}{2}}, \dots \right\}$$

note that there are only 2 unique values

$$(-1)^{\frac{1}{2}} = e^{i(\frac{\pi}{2} + k\pi)}, \quad k \in \{0, 1\}$$

The principal value (when  $k = 0$ ) of this equation works out to be  $i$ .

Ex2: Find all solutions to

$$z^7 = i - 1$$

$$z^7 = \sqrt{2} e^{i(\frac{3\pi}{4} + 2\pi k)}$$

$$z = 2^{1/14} e^{i(\frac{3\pi}{28} + \frac{2\pi}{7}k)}, \quad k \in \{0, 1, 2, 3, 4, 5, 6\}$$

Ex3: Find all solutions to

$$z^5 = \frac{2i}{-1 - \sqrt{3}i}$$

$$z^5 = \frac{2e^{i\frac{\pi}{2}}}{2e^{-i\frac{2\pi}{3}}} = e^{i\frac{7\pi}{6}} = e^{-i(\frac{5\pi}{6} + 2\pi k)}$$

$$z = e^{-i(\frac{\pi}{6} + \frac{2\pi}{5}k)}, \quad k \in \{0, 1, 2, 3, 4\}$$

Ex4: Find all solutions to

$$\left( \frac{z}{z+1} \right)^2 = i$$

$$\left( \frac{z}{z+1} \right)^2 = e^{i(\frac{\pi}{2} + 2\pi k)}$$

$$\frac{z}{z+1} = e^{i(\frac{\pi}{4} + \pi k)}, \quad k \in \{0, 1\}$$

$$e^{i(\frac{\pi}{4} + \pi k)} = \left\{ \frac{1}{\sqrt{2}}(1+i), \frac{1}{\sqrt{2}}(-1-i) \right\}$$

$$z = e^{i(\frac{\pi}{4} + k\pi)}(z+1)$$

$$z = \frac{e^{i(\frac{\pi}{4} + k\pi)}}{1 - e^{i(\frac{\pi}{4} + k\pi)}} = \left\{ \frac{1+i}{\sqrt{2}-1-i}, \frac{-1-i}{\sqrt{2}+1+i} \right\}$$

Ex5: Find all solutions to

$$z^2 + 4iz + 1 = 0$$

$$(z^2 + 4iz - 4) + 4 + 1 = 0$$

$$(z+2i)^2 + 5 = 0$$

$$(z+2i)^2 = -5 = 5e^{i(\pi + 2\pi k)}$$

$$z+2i = \sqrt{5} e^{i(\frac{\pi}{2} + \pi k)} = \left\{ \sqrt{5}i, -\sqrt{5}i \right\}$$

$$z = \{(\sqrt{5} - 2)i, -(\sqrt{5} + 2)i\}$$

Ex6: Find all solutions to  $(z + 1)^4 = (1 - i)z^4$

$$(1 - i)z^4 = \sqrt{2}e^{-i\frac{\pi}{4} + 2\pi ki}z^4$$

$$z + 1 = 2^{1/8}e^{-i\frac{\pi}{16} + i\frac{\pi k}{2}}z$$

$$z = \frac{-2^{1/8}}{1 - e^{-i\frac{\pi}{16} + i\frac{\pi k}{2}}}, \quad k \in \{0, 1, 2, 3\}$$

## 1.2 Complex Functions

### 1.2.1 Mapping Properties of Simple Functions

Similar to how functions with real variables map values to a different set of values, complex functions do the same. The main difference is that with complex functions we're mapping a 2 dimensional set of inputs to a 2 dimensional set of outputs.

$$w = f(z) = u + iv$$

We define  $z \in \mathcal{S}$  the image of  $\mathcal{S}$  under  $w$ .

Some common mappings:

- The identity map

$$w = f(z) = z$$

$$\begin{cases} u = x \\ v = y \end{cases}$$

- Translation by  $z_0$

$$w = f(z) = z + z_0$$

$$\begin{cases} u = x + x_0 \\ v = y + y_0 \end{cases}$$

- Stretching ( $a > 1$ ) or contraction ( $a < 1$ )

$$w = f(z) = az = are^{i\varphi}, \quad a \in \mathbb{R}$$

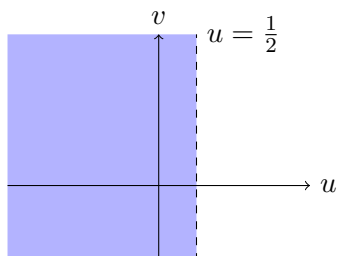
$$\begin{cases} u = ax \\ v = ay \end{cases}$$

- Rotation by  $\varphi_0$

$$w = f(z) = e^{i\varphi_0}z = e^{i(\varphi + \varphi_0)}$$

Using these basic mapping principles we are able to lay the foundation for some more complicated mappings.

Ex: Find the image of  $S = \{|z - 1| \geq 1\}$  under the mapping  $f(z) = \frac{1}{z}$



$$z = \frac{1}{w} = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$$x = \frac{u}{u^2 + v^2}$$

$$y = -\frac{v}{u^2 + v^2}$$

$$|z - 1| \geq 1 \Rightarrow \left| \frac{1}{w} - 1 \right| \geq 1$$

$$\frac{1 - w}{w} \geq 1$$

$$|1 - w| \geq |w|$$

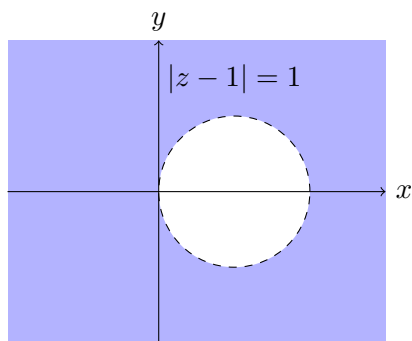
$$|1 - w|^2 \geq |w|^2$$

$$(1 - u)^2 + v^2 \geq u^2 + v^2$$

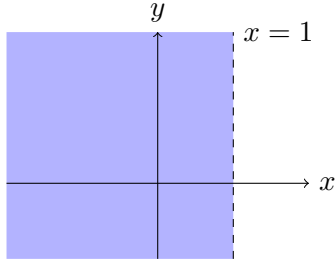
$$-2u + 1 \geq 0$$

$$u \leq \frac{1}{2}$$

$$S' = \left\{ u \leq \frac{1}{2} \right\}$$



Ex2: Find the image of  $S = \{x \leq 1\}$  under the mapping  $f(z) = \frac{1}{z}$

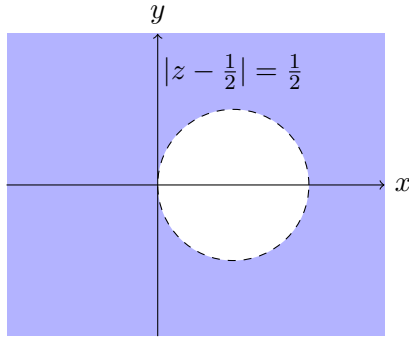


$$z = \frac{1}{w} = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$$x = \frac{u}{u^2 + v^2}$$

$$x \leq 1 \Rightarrow \frac{u}{u^2 + v^2} \leq 1 \Rightarrow u^2 + v^2 \geq u$$

$$(u - \frac{1}{2})^2 + v^2 \geq \frac{1}{4}$$



We see from the previous two examples that circles map to lines and lines map to circles. Let's see why this is the case.

$$a(x^2 + y^2) + bx + cy + d = 0$$

$$a|z|^2 + b\frac{z + \bar{z}}{2} + c\frac{z - \bar{z}}{2i} + d = 0$$

In the case where  $a = 0$ , we have a line. In the case where  $a \neq 0$ , we have a circle.

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$z\bar{z} = |z|^2 = \frac{1}{|w|^2} = \frac{1}{w\bar{w}}$$

$$a\frac{1}{w\bar{w}} + b\frac{z + \bar{z}}{2} + c\frac{z - \bar{z}}{2i} + d = 0$$

$$a\frac{1}{w\bar{w}} + \frac{b}{2}\left(\frac{1}{w} + \frac{1}{\bar{w}}\right) + \frac{c}{2i}\left(\frac{1}{w} - \frac{1}{\bar{w}}\right) + d = 0$$

$$a + \frac{b}{2}(w + \bar{w}) + \frac{c}{2i}(w - \bar{w}) + d(w\bar{w}) = 0$$

If we have a linear transformation of the form  $az + b$  it corresponds to the scaling and translation of the set only. A line will map to a line and a circle will map to a circle.



We can combine this with the  $w = \frac{1}{z}$  transformation property to get a more general transformation. We call this the *Mobius transformation*:

$$f(z) = \frac{az + b}{cz + d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz + d}$$

where  $a, b, c, d \in \mathbb{C}$

Ex: Find the mapping of  $f(z) = \frac{1}{z+1}$  on the set  $S = \{\Re(z) > 0\}$

$$\begin{aligned} u + iv &= \frac{1}{x + 1 + iy} \Rightarrow x + 1 + iy = \frac{1}{u + iv} \\ x + 1 &= \frac{u}{u^2 + v^2} \\ x > 0 &\Rightarrow x + 1 > 1 \\ \frac{u}{u^2 + v^2} &> 1 \Rightarrow u > u^2 + v^2 \\ u^2 + v^2 - u &+ \frac{1}{4} < \frac{1}{4} \\ \left(u - \frac{1}{2}\right)^2 + v^2 &< \frac{1}{4} \\ S' &= \left\{ w = u + iv \mid \left(u - \frac{1}{2}\right)^2 + v^2 < \left(\frac{1}{2}\right)^2 \right\} \end{aligned}$$

Ex2: Find the mapping of  $f(z) = \frac{z-i}{z+i}$  on  $S = \{|z| < 3\}$

$$\begin{aligned} wz + iw &= z - i \Rightarrow z(w - 1) = -i - iw \Rightarrow z = \frac{i(w + 1)}{1 - w} \\ |z| &= \frac{|w + 1|}{|w - 1|} < 3 \\ |w + 1| &< 3|w - 1| \Rightarrow |w + 1|^2 < 9|w - 1|^2 \\ (u + 1)^2 + v^2 &< 9(u - 1)^2 + 9v^2 \\ u^2 + 2u + 1 + v^2 &< 9u^2 - 18u + 9 + 9v^2 \\ 0 &< 8u^2 - 20u + 8 + 8v^2 \Rightarrow 0 < u^2 - \frac{5}{2}u + 1 + v^2 \\ \frac{9}{16} &< u^2 - \frac{5}{2}u + \frac{25}{16} + v^2 \\ \frac{9}{16} &< \left(u - \frac{5}{4}\right)^2 + v^2 \\ S' &= \left\{ w = u + iv \mid \left(u - \frac{5}{4}\right)^2 + v^2 > \left(\frac{3}{4}\right)^2 \right\} \end{aligned}$$

Another common mapping is the  $f(z) = z^2$  or more generally  $f(z) = z^n$  mapping. For  $w = z^2$ ,

$$w = z^2 = r^2 e^{2i\varphi} \Rightarrow \begin{cases} |w| = |z|^2 \\ \arg(w) = 2 \arg(z) \end{cases}$$

This mapping scales the magnitude but more notably, it doubles the argument. This means that the mapping of a half circle will now be a full circle.

Ex: Find the mapping of  $f(z) = z^2$  on  $S = \{0 \leq \Re(z) \leq 1, \Im(z) = 1\}$

$$w = x^2 + i2xy - y^2$$

$$u = x^2 - y^2 = x^2 - 1 \Rightarrow -1 \leq u \leq 0$$

$$v = 2xy = 2x \Rightarrow 0 \leq v \leq 2$$

$$S' = \{w = u + iv \mid -1 \leq u \leq 0, 0 \leq v \leq 2\}$$

Ex2: Find the mapping of  $f(z) = -2z^5$  on  $S = \{|z| < 1, 0 < \text{Arg}(z) < \frac{\pi}{2}\}$

$$z^5 = -\frac{w}{2} \Rightarrow |z|^5 = \frac{|w|}{2} < 1 \Rightarrow |w| < 2$$

$$5 \arg(z) = \arg(w) \pm \pi$$

$$0 < \arg(w) \pm \pi < \frac{5\pi}{2}$$

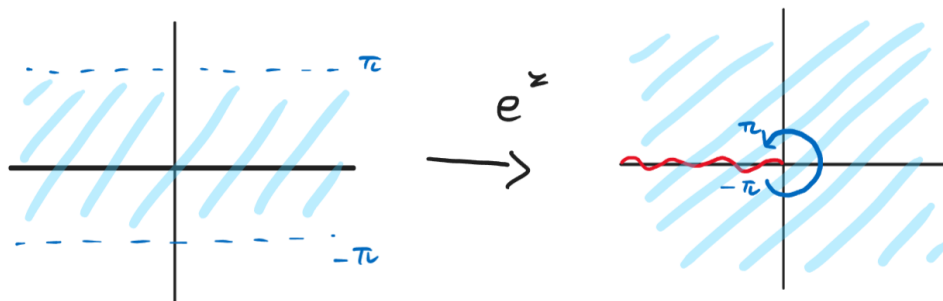
$$-\pi < \arg(w) < \frac{3\pi}{2}$$

$$S' = \{|w| < 2\}$$

Another common mapping is the  $f(z) = e^z$  mapping.

$$w = e^z = e^{x+iy} = e^x e^{iy}$$

$$\begin{cases} |w| = e^x \\ \arg(w) = y \end{cases}$$



This mapping has the property that the magnitude is only dependent on  $x$  and the argument is exactly  $y$ .

Ex: Find the mapping of  $f(z) = e^z$  on  $S = \{\Re(z) = 1\}$

$$w = e^x e^{iy}$$

$$|w| = e, \arg(w) = y$$

$$S' = \{|w| = e\}$$

Ex2: Find the mapping of  $f(z) = e^z$  on  $S = \{0 \leq \Im(z) \leq \frac{\pi}{4}\}$

$$|w| = x$$

$$\arg(w) = y \Rightarrow 0 \leq \arg(w) \leq \frac{\pi}{4}$$

$$S' = \left\{ 0 \leq \text{Arg}(w) \leq \frac{\pi}{4} \right\}$$

Ex3: Find the mapping of  $f(z) = e^{iz}$  on  $S = \left\{ z : -\frac{\pi}{2} \leq \Re(z) \leq \pi, -1 \leq \Im(z) \leq 1 \right\}$   
(Note that multiplying  $z$  by  $i$  rotates it by  $90^\circ$ )

$$w = e^{iz} = e^{ix}e^{-y}$$

$$|w| = e^{-y} \Rightarrow e^{-1} \leq |w| \leq e$$

$$\arg(w) = x \Rightarrow -\frac{\pi}{2} \leq \arg(w) \leq \pi$$

$$S' = \left\{ w \mid e^{-1} \leq |w| \leq e, -\frac{\pi}{2} \leq \arg(w) \leq \pi \right\}$$

Ex4: Prove

$$|e^{-z^3}| \leq 1 \quad \forall \left\{ z \mid -\frac{\pi}{6} \leq \text{Arg}(z) \leq \frac{\pi}{6} \right\}$$

*Proof.* We can express  $-z^3$  as some complex number  $a + ib$  where  $a = \Re(-z^3)$  and  $b = \Im(-z^3)$ .  
Taking the magnitude gives

$$|e^{-z^3}| = |e^{a+ib}| = |e^a e^{ib}| = |e^a| |e^{ib}| = |e^a| = |e^{\Re(-z^3)}|$$

$z$  can be written as

$$z = |z|e^{i\text{Arg}(z)}$$

$$z^3 = |z|^3 e^{i3\text{Arg}(z)} = |z|^3 (\cos(3\text{Arg}(z)) + i\sin(3\text{Arg}(z)))$$

$$-z^3 = -|z|^3 (\cos(3\text{Arg}(z)) + i\sin(3\text{Arg}(z)))$$

$$\Re(-z^3) = -|z|^3 \cos(3\text{Arg}(z))$$

$$\text{Arg}(z) \in \left[ -\frac{\pi}{6}, \frac{\pi}{6} \right]$$

$$\Rightarrow 3\text{Arg}(z) \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$$

$$\cos(3\text{Arg}(z)) \in [0, 1]$$

$$|z|^3 \in \{x \in \mathbb{R} \mid x \geq 0\}$$

$$\Re(-z^3) = -|z|^3 \cos(3\text{Arg}(z)) \in \{x \in \mathbb{R} \mid x \leq 0\}$$

$$e^{\Re(-z^3)} \in [0, 1]$$

$$\Rightarrow |e^{-z^3}| \leq 1$$

□

## 1.2.2 Calculus of Complex Functions

We define the limit of a complex function to be

$$w = f(z) = u + iv$$

$$\lim_{z \rightarrow z_0} f(z) = \lim_{(x,y) \rightarrow (x_0,y_0)} u(x,y) + i \lim_{(x,y) \rightarrow (x_0,y_0)} v(x,y)$$

Note that the notation  $(x,y) \rightarrow (x_0,y_0)$  means that the limit is taken as  $(x,y)$  approaches  $(x_0,y_0)$  along *any* path.

The usual limit arithmetic rules are able to be applied as with real numbers.

In order for  $\lim_{z \rightarrow z_0} f(z)$  to exist, we require that both  $\lim_{(x,y) \rightarrow (x_0,y_0)} u(x,y)$  and  $\lim_{(x,y) \rightarrow (x_0,y_0)} v(x,y)$  exist.

If we define  $z_0 = x_0 + iy_0$  then we can define the derivative of a complex function as

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

If this limit exists then the function is said to be differentiable at  $z_0$ .

Ex:  $f(z) = z$

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{z_0 + \Delta z - z_0}{\Delta z} = 1 \\ &\Rightarrow f'(z_0) = 1 \end{aligned}$$

Ex2:  $f(z) = \bar{z}$

$$\begin{aligned} f'(z) &= \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z} \\ \Delta z &= h_1 + ih_2 \Rightarrow \overline{\Delta z} = h_1 - ih_2 \\ h_2 = 0 : \quad \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z} &= \lim_{h_1 \rightarrow 0} \frac{h_1}{h_1} = 1 \\ h_1 = 0 : \quad \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z} &= \lim_{h_2 \rightarrow 0} \frac{-ih_2}{ih_2} = -1 \\ \lim_{h_1 \rightarrow 0} \frac{h_1}{h_1} &\neq \lim_{h_2 \rightarrow 0} \frac{-ih_2}{ih_2} \quad \therefore \text{the derivative does not exist} \end{aligned}$$

An easy way to determine if a function is differentiable is to use the Cauchy-Riemann equations.

Any path that can be taken to approach  $z_0$  can be written as a linear combination of the paths  $\Delta z = \Delta x$  and  $\Delta z = i\Delta y$  so the derivative must satisfy both of these paths.

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \\ f(z_0) &= u(x_0, y_0) + iv(x_0, y_0) \\ \text{Define } \Delta z &= \Delta x \\ f'(z_0) &= \lim_{\Delta x \rightarrow 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x} \\ f'(z_0) &= \lim_{\Delta x \rightarrow 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \rightarrow 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\ f'(z_0) &= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) \\ \text{Define } \Delta z &= i\Delta y \end{aligned}$$

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta y \rightarrow 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y} \\
f'(z_0) &= \lim_{\Delta y \rightarrow 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \rightarrow 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y} \\
f'(z_0) &= -i \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0) \\
&\Rightarrow \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) = -i \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)
\end{aligned}$$

Splitting the real and imaginary parts we get that the Cauchy-Riemann equations are

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

If the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous then the function is differentiable.

Some functions are not differentiable everywhere, but are differentiable at a point or a set of points.

- If  $f(z)$  is differentiable everywhere in the complex plane then it is said to be **entire**.
- If  $f(z)$  is differentiable in some region  $R$  then it is said to be **analytic** in  $R$ .  
(note that this region cannot be a single point, as the Cauchy-Riemann equations require the partial derivatives to be continuous)

Ex: Show using the Cauchy-Riemann equations that  $f(z) = \bar{z}$  is not differentiable anywhere.

$$\bar{z} = x - iy$$

$$u_x = 1 \neq v_y = -1$$

Ex2: Show that  $f(z) = z^2$  is entire

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$$

$$u(x, y) = x^2 - y^2$$

$$v(x, y) = 2xy$$

$$u_x = 2x = v_y$$

$$u_y = -2y = -v_x$$

Ex3: Show that  $f(z) = \bar{z}$  is differentiable but not analytic at  $z_0 = 0$

$$|z|^2 + 2z = x^2 + 2x + y^2 + i2y$$

$$u_x = 2x + 2 = v_y = 2 \Rightarrow x = 0$$

$$u_y = 2y = -v_x = 0 \Rightarrow y = 0$$

differentiable but not analytic on  $z = \{0\}$

### 1.2.3 Conformal Mappings

Using the Cauchy-Riemann equations,

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

we can get the Laplacian of  $u$  and  $v$ ,

$$\begin{aligned} u_{xx} + u_{yy} &= v_{yx} - v_{xy} = 0 \\ v_{xx} + v_{yy} &= -(u_{yx} + u_{xy}) = 0 \end{aligned}$$

If the Laplacian of  $u$  and  $v$  are both zero then the function is said to be **harmonic**.

If  $\nabla^2 u = 0$  then we can use the Cauchy-Riemann equations to find its harmonic conjugate  $v$ . Ex:

Find the harmonic conjugate of  $u = xy - x + y$

$$\begin{aligned} u_x &= y - 1 = v_y \\ v &= \int (y - 1) dy = \frac{y^2}{2} - y + h(x) \\ u_y &= x + 1 = -v_x = -h'(x) \\ h(x) &= \int (-x - 1) dx = -\frac{x^2}{2} - x + C \\ v &= \frac{y^2}{2} - \frac{x^2}{2} - y - x + C \end{aligned}$$

Ex2: Find the harmonic conjugate of  $u = \ln \sqrt{x^2 + y^2}$

$$\begin{aligned} u_x &= \frac{x}{x^2 + y^2} = v_y \\ v &= \int \frac{x}{x^2 + y^2} dy = \int \frac{1/x}{1 + \frac{y^2}{x^2}} dy = \arctan\left(\frac{y}{x}\right) + h(x) \\ u_y &= \frac{y}{x^2 + y^2} = -v_x = -\frac{y}{1 + \frac{y^2}{x^2}} \left(\frac{-1}{x^2}\right) - h'(x) = \frac{y}{x^2 + y^2} - h'(x) \Rightarrow h'(x) = 0 \\ h(x) &= C \\ v &= \arctan\left(\frac{y}{x}\right) + C \\ v &= \arg(z) + C \end{aligned}$$

Ex3: Find the harmonic conjugate of  $u = \sin x \cosh y$

$$\begin{aligned} u &= \sin x \cosh y \\ u_x &= \cos x \cosh y = v_y \\ v &= \int \cos x \cosh y dy = \cos x \sinh y + h(x) \\ u_y &= \sin x \sinh y = -v_x = -(-\sin x \sinh y) - h'(x) \Rightarrow h'(x) = 0 \\ h(x) &= C \end{aligned}$$

$$v = \cos x \sinh y + C$$

Another property is that

$$|f'(z)|^2 = |\nabla u|^2 = |\nabla v|^2$$

This also implies that

$$\nabla u \cdot \nabla v = 0$$

$$\begin{aligned} f'(z) &= u_x + iv_x = \frac{1}{i}(u_y + iv_y) \\ |f'(z)|^2 &= u_x^2 + v_x^2 = u_y^2 + v_y^2 = |\nabla u|^2 = |\nabla v|^2 \\ \nabla u \cdot \nabla v &= u_x u_y + v_x v_y = u_x v_x + (-v_x)(u_x) = 0 \end{aligned}$$

A conformal mapping is a mapping between two regions that preserves angles. If we have some function  $f(z) = u + iv$  that is analytic then we can create a function of a function as

$$\Phi(u(x, y), v(x, y)) = \phi(x, y)$$

where  $\phi(x, y)$  is a conformal mapping.

A conformal mapping will have the property that

$$\phi_{xx} + \phi_{yy} = |f'(z)|^2(\Phi_{uu} + \Phi_{vv})$$

where  $|f'(z)|^2$  is known as the *conformal factor*.

This can be shown as follows:

$$\begin{aligned} \phi_x &= \Phi_u u_x + \Phi_v v_x \\ \phi_{xx} &= u_{xx} \Phi_u^2 + 2u_x v_x \Phi_u \Phi_v + v_{xx} \Phi_v^2 + \Phi_u u_{xx} + \Phi_v v_{xx} \\ \phi_y &= \Phi_u u_y + \Phi_v v_y \\ \phi_{yy} &= u_{yy} \Phi_u^2 + 2u_y v_y \Phi_u \Phi_v + v_{yy} \Phi_v^2 + \Phi_u u_{yy} + \Phi_v v_{yy} \\ \phi_{xx} + \phi_{yy} &= \Phi_u \nabla^2 u + \Phi_v \nabla^2 v + \Phi_{uu} |\nabla u|^2 + 2\Phi_u \Phi_v \nabla u \cdot \nabla v + \Phi_{vv} |\nabla v|^2 \\ \phi_{xx} + \phi_{yy} &= |f'(z)|^2(\Phi_{uu} + \Phi_{vv}) \end{aligned}$$

Ex:  $f(z) = z^2$  under  $\Phi(u, v) = e^u + v^2$

$$\begin{aligned} f(z) &= x^2 - y^2 + i(2xy) \\ u &= x^2 - y^2 \\ v &= 2xy \\ \phi(x, y) &= e^{x^2 - y^2} + (2xy)^2 \\ \phi_{xx} + \phi_{yy} &= 4|z|^2(e^u + 2) \end{aligned}$$

$f(z)$  is considered a conformal mapping if  $f$  is analytic and  $f'(z) \neq 0$ . As a consequence, if  $\phi_{xx} + \phi_{yy} = 0$  then  $\Phi_{uu} + \Phi_{vv} = 0$ .

If  $f$  is a conformal mapping then it will preserve angles.

If we have two curves  $C_1$  and  $C_2$  described by the parametrized functions  $z_1(t)$  and  $z_2(t)$  that intersect at a point  $z_0$  then the angle between the two curves is given by  $\theta = \arg(z_2'(t)) - \arg(z_1'(t))$ .

If we then apply a conformal mapping  $w = f(z)$  to the curves then we get  $w_1(t) = f(z_1(t))$  and  $w_2(t) = f(z_2(t))$  and the angle between the two curves is given by  $\theta_w = \arg(w'_2(t)) - \arg(w'_1(t))$ .

$$\begin{aligned}\theta_w &= \arg(f'(z_2)z'_2(t)) - \arg(f'(z_1)z'_1(t)) \\ \theta_w &= \arg(f'(z_2)) - \arg(f'(z_1)) + \arg(z'_2(t)) - \arg(z'_1(t)) \\ \arg(f'(z_1)) &= \arg(f'(z_2)) \\ \Rightarrow \theta_w &= \arg(z'_2(t)) - \arg(z'_1(t)) = \theta\end{aligned}$$

So the angle between the two curves is preserved under a conformal mapping.

Note that  $f'(z) \neq 0$  is a necessary condition in this proof.

If we have a nonconformal mapping then the angle between the two curves will not be preserved. One such case is that of  $f(z) = z^2$  at  $z_0 = 0$ . At this point,  $f'(z_0) = 0$  and the angle between the two curves is doubled.

Conformal mappings also have the property that they map Neumann boundary conditions to Neumann boundary conditions.

If we let  $\hat{n}$  represent the normal vector to the curve  $\phi(x, y)$  then they are related by

$$\frac{\partial \phi}{\partial n} = \nabla \phi \cdot \hat{n}$$

and the mapping is similarly related brcurly

$$\begin{aligned}\frac{\partial \Phi}{\partial n'} &= \nabla \Phi \cdot \hat{n}' \\ \frac{\partial \phi}{\partial n} &= |f'(z)| \frac{\partial \Phi}{\partial n'}\end{aligned}$$

So for Neumann boundary conditions, we will have

$$\frac{\partial \phi}{\partial n} = 0 \Rightarrow \frac{\partial \Phi}{\partial n'} = 0$$

Some examples of conformal mappings come from harmonic functions (having the property that  $\nabla^2 \phi = 0$ ).

Some common harmonic functions are:

- $\phi = C$
- $\phi = ax + by + c$
- $\ln \sqrt{x^2 + y^2}, \mathbb{C} \setminus 0$
- $\phi = \text{Arg}(z), \mathbb{C} \setminus (-\infty]$
- $\phi = x^2 - y^2$



### 1.2.4 Conformal Mapping to Solve Laplace's Equation

Given the useful properties of matching boundary conditions, we can use conformal mappings to help us solve Laplace's equation for a given region.

Ex: Given  $\nabla^2\phi = 0$  for  $1 < x^2 - y^2 < 4$  and  $\phi = 1$  on  $x^2 - y^2 = 1$  and  $\phi = 3$  on  $x^2 - y^2 = 4$ , find  $\phi(x, y)$ .

$$\text{choose } \phi = x^2 - y^2$$

$$\text{choose } \Phi(u, v) = Au + B$$

$$\begin{cases} A(1) + B = 1 \\ A(4) + B = 3 \end{cases} \Rightarrow A = \frac{2}{3}, B = \frac{1}{3}$$

$$\Phi(u, v) = \frac{2}{3}u + \frac{1}{3}$$

$$u = x^2 - y^2$$

$$\phi(x, y) = \frac{2}{3}(x^2 - y^2) + \frac{1}{3}$$

Ex2: Given  $\nabla^2\phi = 0$  within the circular region  $\mathcal{D} = \{1 < x^2 + y^2 < 4\}$  with  $\phi = 1$  on  $x^2 + y^2 = 1$  and  $\phi = -2$  on  $x^2 + y^2 = 4$ , find  $\phi(x, y)$ .

$$\phi(x, y) = A_1 \ln r + A_2$$

$$r = \sqrt{x^2 + y^2}$$

$$r = 1 : \phi = 1 \Rightarrow A_1 \ln(1) + A_2 = 1$$

$$r = 2 : \phi = -2 \Rightarrow A_1 \ln(2) + A_2 = -2$$

$$\Rightarrow A_2 = 1, A_1 = -\frac{3}{\ln 2}$$

$$\phi(x, y) = -\frac{3}{\ln 2} \ln \sqrt{x^2 + y^2} + 1$$

Ex3: Given  $\nabla^2 = 0$  within the strip described by  $\{z : -3 \leq 3\Re(z) - 4\Im(z) \leq 2\}$  with  $\phi = 0$  for  $\{z : -3 = 3\Re(z) - 4\Im(z)\}$  and  $\phi = 4$  for  $\{z : 3\Re(z) - 4\Im(z) = 2\}$  find  $\phi(x, y)$

$$u = 3x - 4y$$

$$u(-3) = 0, u(2) = 4$$

$$\Phi = Au + B$$

$$\Phi(-3) = -3A + B = 0 \Rightarrow B = 3A$$

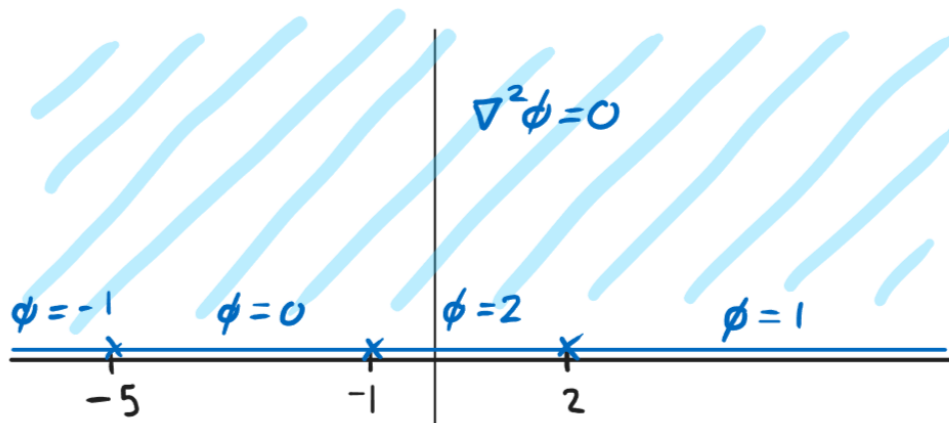
$$\Phi(2) = 2A + 3A = 5A = 4 \Rightarrow A = \frac{4}{5} \Rightarrow B = \frac{12}{5}$$

$$\phi(x, y) = \frac{4}{5}(3x - 4y) + \frac{12}{5}$$

Ex4: Given  $\nabla^2\phi = 0$  for the upper half-plane described by  $\{y > 0 \wedge x \in \mathbb{R}\}$  with the boundary conditions along the x-axis given by

$$\phi(x, 0) = \begin{cases} -1 & x < -5 \\ 0 & -5 < x < -1 \\ 2 & -1 < x < 2 \\ 1 & x > 2 \end{cases}$$

find  $\phi(x, y)$ .



One trick to solve a problem like this is to choose a linear combination of functions of the form  $\text{Arg}(z - z_0)$  with a different point  $z_0$  for every place where the boundary condition changes along the x-axis.

$$\phi = A_1 \text{Arg}(z + 5) + A_2 \text{Arg}(z + 1) + A_3 \text{Arg}(z - 2) + A_4$$

$$\phi(x > 2, 0) = A_4 = 1$$

$$\phi(-1 < x < 2, 0) = \pi A_3 + 1 = 2 \Rightarrow A_3 = \frac{1}{\pi}$$

$$\phi(-5 < x < -1, 0) = \pi A_2 + 1 + 1 = 0 \Rightarrow A_2 = -\frac{2}{\pi}$$

$$\phi(x < -5, 0) = \pi A_1 - 2 + 2 = -1 \Rightarrow A_1 = -\frac{1}{\pi}$$

$$\phi(x, y) = -\frac{1}{\pi} \text{Arg}(z + 5) - \frac{2}{\pi} \text{Arg}(z + 1) + \frac{1}{\pi} \text{Arg}(z - 2) + 1$$

We can also apply these techniques to other types of boundary conditions in some cases.

Ex5: Given  $\nabla^2 \phi = 0$  in the circular region  $\{z : 1 \leq |z| \leq 2\}$  with the boundary conditions  $\phi = 1$  for  $|z| = 1$  and  $\frac{\partial \phi}{\partial r} = 2$  for  $|z| = 2$ , find  $\phi(x, y)$

$$\phi = A \ln r + B$$

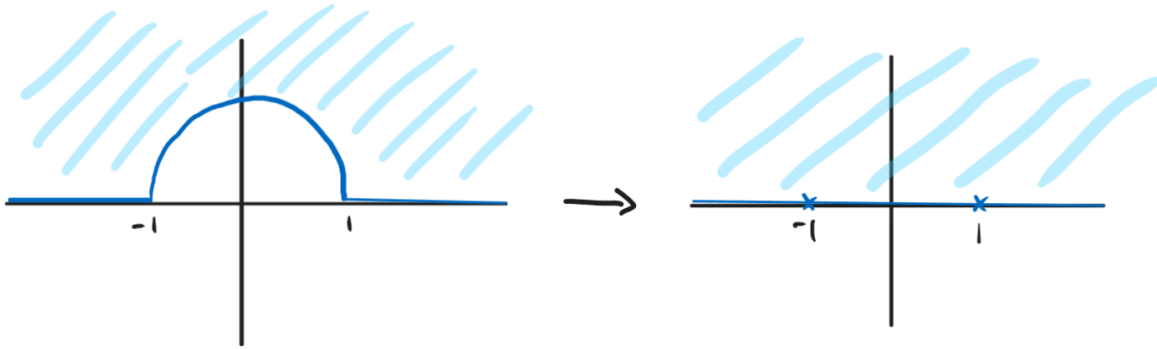
$$\phi(1) = B = 1$$

$$\frac{\partial \phi(2)}{\partial r} = \frac{A}{2} = 2 \Rightarrow A = 4$$

$$\phi = 4 \ln r + 1$$

$$\phi(x, y) = 4 \ln \sqrt{x^2 + y^2} + 1$$

If we have a region that has a semicircle in it we can use the Joukowski mapping to transform it into a region that is easier to work with.



Ex6: Given  $\nabla^2 \phi = 0$  in the upper region of the plane described by  $\{y > 0 \wedge x^2 + y^2 > 9\}$  with the boundary conditions  $\phi = -1$  for  $x < -3$ ,  $\phi = 0$  for  $x^2 + y^2 = 9$ , and  $\phi = 2$  for  $x > 3$ , find  $\phi(x, y)$ .

$$\begin{cases} \phi(x, 0) = -1 & x < -3 \\ \phi(x, y) = 0 & x^2 + y^2 = 9 \\ \phi(x, 0) = 2 & x > 3 \end{cases}$$

$$w = \frac{1}{2} \left( \frac{z}{3} + \frac{3}{z} \right) = u + iv$$

$$\Phi = A_1 \operatorname{Arg}(w + 1) + A_2 \operatorname{Arg}(w - 1) + A_3$$

$$\Phi(u > 1, v) = A_3 = 2$$

$$\Phi(-1 < u < 1, v) = \pi A_2 + 2 = 0 \Rightarrow A_2 = -\frac{2}{\pi}$$

$$\Phi(u < -1, v) = \pi A_1 - 2 + 2 = -1 \Rightarrow A_1 = -\frac{1}{\pi}$$

$$\Phi = -\frac{1}{\pi} \operatorname{Arg}(w + 1) - \frac{2}{\pi} \operatorname{Arg}(w - 1) + 2$$

$$\phi(z) = -\frac{1}{\pi} \operatorname{Arg} \left( \frac{1}{2} \left( \frac{z}{3} + \frac{3}{z} \right) + 1 \right) - \frac{2}{\pi} \operatorname{Arg} \left( \frac{1}{2} \left( \frac{z}{3} + \frac{3}{z} \right) - 1 \right) + 2$$

In the case that we have a semicircle not of radius 1 we can apply a scaling before the Joukowski mapping to get the correct radius.



$$w = \frac{1}{2} \left( \frac{z}{a} + \frac{a}{z} \right)$$

### 1.2.5 Sinusoidal Functions

If we recall Euler's formula  $e^{ix} = \cos x + i \sin x$  we can use with complex numbers to get the following identities for complex sinusoids:

- $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

- $\cos z = \frac{e^{iz} + e^{-iz}}{2}$
- $\sinh z = \frac{e^z - e^{-z}}{2}$
- $\cosh z = \frac{e^z + e^{-z}}{2}$
- $\cos z = \sin\left(\frac{\pi}{2} - z\right)$
- $\sinh z = -i \sin(iz)$
- $\cosh z = \cos(iz) = \sin\left(\frac{\pi}{2} - iz\right)$
- $\frac{d}{dz} \sin z = \cos z$
- $\frac{d}{dz} \cos z = -\sin z$
- $\cos^2 z + \sin^2 z = 1$
- $\frac{d}{dz} \sinh z = \cosh z$
- $\frac{d}{dz} \cosh z = \sinh z$
- $\cosh^2 z - \sinh^2 z = 1$

The most notable difference between the real and complex versions of these functions is that  $|\sin z| \not\leq 1$ . We will see cases of this soon but it is easy to see once we write out the real and imaginary components of  $\sin z$ .

$$\begin{aligned}\sin z &= \sin(x + iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{-y}e^{ix} - e^ye^{-ix}}{2i} \\ \sin z &= \frac{e^{-y}(\cos x + i \sin x) - e^y(\cos x - i \sin x)}{2i} = \frac{e^{-y} - e^y}{2} \cos x + \frac{e^{-y} + e^y}{2i} \sin x \\ \sin z &= \sin x \cosh y + i \cos x \sinh y\end{aligned}$$

Ex: Show that if  $|z| < 1$  then  $|\sin z| < 2$ .

*Proof.*

$$|\sin z| < 2, \quad |z| < 1$$

$$\begin{aligned}\sin z &= \sin x \cosh y + i \cos x \sinh y \\ |\sin z| &= \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y} \\ \cosh^2 y - \sinh^2 y &= 1 \Rightarrow \sinh^2 y = \cosh^2 y - 1 \\ |\sin z| &= \sqrt{\sin^2 x \cosh^2 y + \cos^2 x (\cosh^2 y - 1)} \\ |\sin z| &= \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \cosh^2 y - \cos^2 x} \\ |\sin z| &= \sqrt{\cosh^2 y - \cos^2 x} \leq \cosh y\end{aligned}$$

$$|z| = \sqrt{x^2 + y^2} < 1 \Rightarrow y < 1$$

$$\cosh y < \cosh 1 = \frac{e + e^{-1}}{2} \approx 1.54 < 2$$

$$|\sin z| < 2$$

□

Ex2: Find all solutions to  $\sin z = 4i$

$$\sin(z) = 4i = \sin x \cosh y + i \cos x \sinh y$$

$$\sin x \cosh y = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi$$

$$\text{Case 1: } n = 2k : \cos x = 1$$

$$4 = \cos(2k\pi) \sinh(y) = \sinh y$$

$$\text{Case 2: } n = 2k + 1 : \cos x = -1$$

$$4 = \cos((2k + 1)\pi) \sinh y = -\sinh y$$

$$\sinh y = \frac{e^y - e^{-y}}{2}$$

$$2 \sinh y = e^y - e^{-y}$$

$$e^{2y} - 2 \sinh y e^y - 1 = 0$$

$$e^y = \frac{2 \sinh y \pm \sqrt{4 \sinh^2 y + 4}}{2} = \sinh y \pm \sqrt{\sinh^2 y + 1}$$

$$n = 2k :$$

$$y = \ln(4 \pm \sqrt{17}) = \ln(4 + \sqrt{17})$$

$$n = 2k + 1 :$$

$$y = \ln(-4 \pm \sqrt{17}) = \ln(\sqrt{17} - 4)$$

$$z = \left\{ (x, y) \left| \left( 2k\pi, \ln(4 + \sqrt{17}) \right), \left( (2k + 1)\pi, \ln(\sqrt{17} - 4) \right), k \in \mathbb{Z} \right. \right\}$$

Ex3: Find all solutions to  $\cos z = 0$

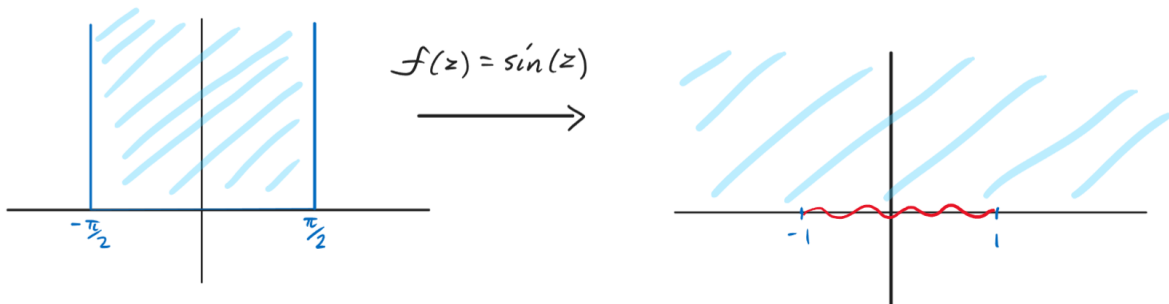
$$\cos(z^4) = 0$$

$$z^4 = \pi n + \frac{\pi}{2} = \left( \pi n + \frac{\pi}{2} \right) e^{2\pi i l}$$

$$z = \left( \pi n + \frac{\pi}{2} \right)^{1/4} e^{i \frac{\pi l}{2}}, \quad l = \{0, 1, 2, 3\}, \quad n \in \mathbb{Z}$$

**Mapping properties of  $\sin z$**

$\sin z$  will map the box  $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, 0 \leq y < \infty$  to the half plane  $v > 0$ .



Ex: Find the mapping of  $\sin z$  on  $S = \{-\frac{\pi}{2} < x < \frac{\pi}{2}, 0 < y < 1\}$

$$u = \sin x \cosh y$$

$$\sin x \in (-1, 1)$$

$$\cosh y \in (0, \cosh(1))$$

$$u \in (-\cosh(1), \cosh(1))$$

$$v = \cos x \sinh y$$

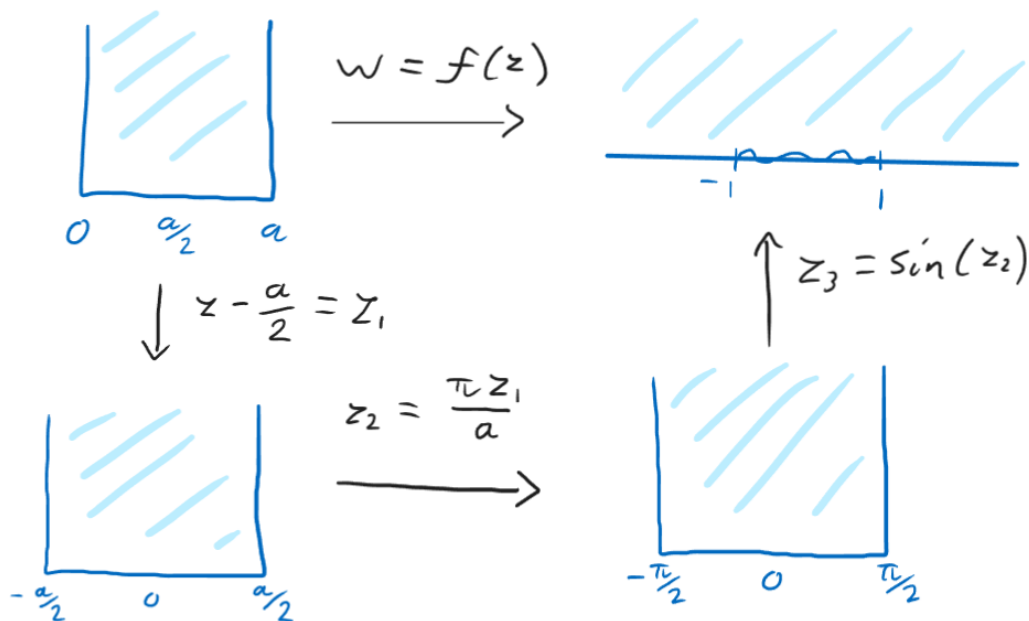
$$\cos x \in (0, 1]$$

$$\sinh y \in (0, \sinh(1))$$

$$v \in (0, \sinh(1))$$

$$S' = \{(u, v) | -\cosh(1) < u < \cosh(1), 0 < v < \sinh(1)\}$$

If the box is offcenter we can also apply a composition of mappings to shift it into the usual form.



Ex2: Find the mapping of  $\sin z$  on  $S = \{-1 < x < 1, y > 0\}$

$$u = \sin x \cosh y$$

$$\sin x \in (-\sin(1), \sin(1))$$

$$\begin{aligned}
&\cosh y \in (1, \infty) \\
&u \in \mathbb{R} \\
&v = \cos x \sinh y \\
&\cos x \in (\cos(1), 1] \\
&\sinh y \in (0, \infty) \\
&v \in (0, \infty) \\
&S' = \{(u, v) | v > 0\}
\end{aligned}$$

Ex3: Solve the Laplace equation  $\nabla^2 u = 0$  in the region  $S = \{0 \leq x \leq 2, 0 \leq y < \infty\}$  with boundary conditions  $\phi = 0$  on  $x = 0$  and  $\phi = 1$  on  $y = 0$  and  $\phi = -2$  on  $x = 2$ .

$$\begin{aligned}
&\text{Map to the half plane using } w = \sin\left(\frac{\pi}{2}(z - 1)\right) \\
&\phi(z) = \Phi(w) = A_1 \text{Arg}(w + 1) + A_2 \text{Arg}(w - 1) + A_3 \\
&u > 1 : \Phi = -2 = A_3 \\
&-1 < u < 1 : \Phi = 1 = A_2\pi - 2 \Rightarrow A_2 = \frac{3}{\pi} \\
&u < -1 : \Phi = 0 = A_1\pi + \frac{3}{\pi}\pi - 2 \Rightarrow A_1 = -\frac{1}{\pi} \\
&\phi(z) = -\frac{1}{\pi} \text{Arg}\left(\sin\left(\frac{\pi}{2}(z - 1)\right) + 1\right) + \frac{3}{\pi} \text{Arg}\left(\sin\left(\frac{\pi}{2}(z - 1)\right) - 1\right) - 2
\end{aligned}$$

### 1.2.6 Logarithmic Functions

$\log z$  is defined as the inverse of  $e^z$ .

$$\begin{aligned}
&e^w = z \Rightarrow w = \log z \\
&w = u + iv \\
&z = re^{i \text{Arg}(z)} \\
&e^u e^{iv} = re^{i \text{Arg}(z)} \\
&e^u = r \Rightarrow u = \ln r \\
&e^{iv} = e^{i \text{Arg}(z)} \Rightarrow v = \text{Arg}(z) + 2k\pi \\
&\log(z) = w = \ln r + i(\text{Arg } z + 2k\pi), \quad k \in \mathbb{Z} \\
&\log(z) = \{w | e^w = z\} = \ln r + i(\text{Arg } z + 2k\pi) = \ln r + i \arg z \\
&\text{Log}(z) = \ln r + i \text{Arg } z
\end{aligned}$$

Note that  $\log(z) = \ln r + i \arg z$  is a multi-valued function and  $\text{Log}(z) = \ln r + i \text{Arg } z$  is the principal value of  $\log z$ .

Ex: Find all values for  $\log 2$  and  $\text{Log } 2$ .

$$\begin{aligned}
&z = 2 = 2e^{i0} \\
&\log 2 = \ln 2 + i(0 + 2k\pi) = \ln 2 + 2k\pi i, \quad k \in \mathbb{Z} \\
&\text{Log } 2 = \ln 2 + i(0) = \ln 2
\end{aligned}$$

Ex2: Find all values for  $\log(-1 - \sqrt{3}i)$  and  $\text{Log}(-1 - \sqrt{3}i)$ .

$$z = -1 - \sqrt{3}i = 2e^{i\frac{2\pi}{3}}$$

$$\log(-1 - \sqrt{3}i) = \ln 2 + i\left(-\frac{2\pi}{3} + 2k\pi\right), \quad k \in \mathbb{Z}$$

$$\text{Log}(-1 - \sqrt{3}i) = \ln 2 + i\left(-\frac{2\pi}{3}\right)$$

Ex3: Find all values for  $\log(e^{1+5i})$  and  $\text{Log}(e^{1+5i})$ .

$$z = e^{1+5i} = e e^{5i} = e e^{i(5+2k\pi)}$$

$$5 + 2k\pi \in (-\pi, \pi] \Rightarrow k = -1$$

$$\text{Arg}(e^{1+5i}) = 5 - 2\pi$$

$$\log(e^{1+5i}) = \ln e + i(5 - 2\pi + 2k\pi) = 1 + i(5 - 2\pi + 2k\pi), \quad k \in \mathbb{Z}$$

$$\text{Log}(e^{1+5i}) = \ln e + i(5 - 2\pi) = 1 + i(5 - 2\pi)$$

Ex4: Find all values for  $\log(-i)$  and  $\text{Log}(-i)$ .

$$-i = e^{-i\frac{\pi}{2}}$$

$$\arg(-i) = -\frac{\pi}{2} + 2\pi k$$

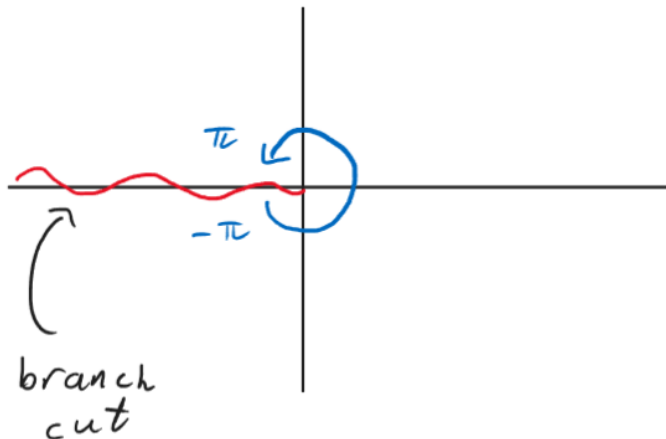
$$\log(-i) = -i\frac{\pi}{2} + i2\pi k$$

$$\text{Log}(-i) = -i\frac{\pi}{2}$$

Properties of  $\log z$  and  $\text{Log } z$ :

- $\log(z_1 z_2) = \log z_1 + \log z_2$   
 $\text{Log}(z_1 z_2) \neq \text{Log } z_1 + \text{Log } z_2$  (same as why  $\text{Arg}(z_1 z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2)$ )
- $e^{\log z} = z$  and  $e^{\text{Log } z} = z$   
but  $\log(e^z) \neq z$  and  $\text{Log}(e^z) \neq z$   
Ex:  $z = 0$ :  $\log(e^0) = \log(1) = \ln 1 + i(0 + 2k\pi) = i2k\pi \neq 0$   
 $\text{Log}(e^{5i}) = i(5 - 2\pi) \neq 5i$   
rather  $z \in \log(e^z)$
- $\log z^n \neq n \log z$   
 $z = 1, n = 2$ :  $\log z^2 = \log 1 = i2k\pi$   
 $2 \log z = 2 \log 1 = 2i(2k')\pi$   
but  $\log z^{\frac{1}{n}} = \frac{1}{n} \log z$
- $\text{Log } z$  is analytic in  $\mathbb{C} \setminus (-\infty, 0]$





$$\text{Log } z = \ln |z| + i \text{Arg } z, \quad -\pi < \text{Arg } z < \pi$$

Ex: Find all solutions to  $e^z = -1 - i$

$$z = \log(-1 - i)$$

$$-1 - i = \sqrt{2}e^{-i\frac{3\pi}{4}}$$

$$z = \frac{1}{2}\ln(2) - i\frac{3\pi}{4} + i2\pi k$$

Ex2: Find the principal value of  $(1 + i)^i = e^{i \text{Log}(1+i)}$

$$1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$$

$$\text{Log}(1 + i) = \frac{1}{2}\ln(2) + i\frac{\pi}{4}$$

$$(1 + i)^i = e^{-\frac{\pi}{4} + i\frac{1}{2}\ln(2)}$$

Ex3: Find the principle value of  $z^i = 1 + i$

$$1 + i = \sqrt{2}e^{i\frac{\pi}{4} + i2\pi k} = e^{\ln(\sqrt{2}) + i\frac{\pi}{4} + i2\pi k}$$

$$z = (1 + i)^{1/i} = (1 + i)^{-i} = e^{\frac{\pi}{4} + 2\pi k} e^{-i \ln(\sqrt{2})}$$

$$z = e^{\frac{\pi}{4}} e^{-i\frac{1}{2}\ln(2)}$$

Ex4: Find the principal value of  $i^i$ .

$$i^i = (e^{i\frac{\pi}{2} + i2\pi k})^i = e^{-\frac{\pi}{2} - 2\pi k} = e^{-\frac{\pi}{2}}$$

Ex5: Find all solutions to  $\text{Log}(z^2 - 1) = \frac{i\pi}{2}$

$$z^2 - 1 = e^{i\frac{\pi}{2}} = i$$

$$z^2 = i + 1 = \sqrt{2}e^{i\frac{\pi}{4} + i2\pi k}$$

$$z = 2^{1/4} e^{i\frac{\pi}{8} + i\pi k}$$

Ex6: Find all solutions to  $e^{2z} + e^z + 1 = 0$

$$e^z = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$e^z = e^{i\pi \pm i\frac{\pi}{3} + i2\pi k}$$

$$z = i\pi \pm i\frac{\pi}{3} + i2\pi k$$

$$z = \pm i\frac{2\pi}{3} + i2\pi k$$

*Derivative of  $\log z$  and  $\text{Log } z$*

$$(\text{Log } z)' = \frac{1}{z}$$

$$\text{Log } z = \ln \sqrt{x^2 + y^2} + i \text{Arg } z = u + iv$$

$$(\text{Log } z)' = u_x + iv_x$$

$$\text{Log } z = \ln |z| + i\varphi, \quad -\pi < \varphi < \pi$$

### 1.2.7 Branch Cuts

There are many different analytic functions for  $\log z$  depending on the branch cut we choose. For example we can choose:

$$\text{Log } z = \ln r + i\varphi, \quad -\pi < \varphi < \pi, \quad \mathbb{C} \setminus (-\infty, 0]$$

$$\log z = \ln r + i\varphi, \quad 0 < \varphi < 2\pi, \quad \mathbb{C} \setminus [0, \infty)$$

$$\log z = \ln r + i\varphi, \quad -\frac{\pi}{2} < \varphi < \frac{5\pi}{2}, \quad \mathbb{C} \setminus i[0, \infty)$$

Ex: Specify a branch cut for  $\log z$  such that it is analytic at  $z = -1, -i, 1$

$$\log z = \ln r + i\varphi, \quad \frac{\pi}{2} < \varphi < \frac{5\pi}{2}$$

The reason we have branch cuts is so that we can define a single-valued analytic function. This is important for functions such as  $\sqrt{z}$  and  $\log z$  as one input can correspond to multiple outputs. For example,  $\sqrt{1} = 1$  and  $\sqrt{1} = -1$ . This is why we need to specify a branch cut for  $\sqrt{z}$  and  $\log z$ . Ex: Construct  $f(z) = z^{1/2}$  such that it is analytic in  $\mathbb{C} \setminus (-\infty, 0]$  and  $f(1) = 1$ .

$$z^{\frac{1}{2}} = e^{\frac{1}{2} \log z} = e^{\frac{1}{2} \ln r + i\frac{1}{2}\varphi} = r^{\frac{1}{2}} e^{i\frac{\varphi}{2}}, \quad -\pi < \varphi < \pi$$

$$f(1) = r^{\frac{1}{2}} e^{i\frac{\varphi}{2}} = 1e^{i0} = 1$$

$$z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{i\frac{\varphi}{2}}, \quad \pi < \varphi < 3\pi$$

$$\rightarrow f(1) = 1^{\frac{1}{2}} e^{i\frac{2\pi}{2}} = -1$$

Ex2: Find all solutions to  $z^{1/2} + 1 - i = 0$  where  $z^{1/2}$  is the principal branch of  $z^{1/2}$ .

$$z^{1/2} = |z|^{1/2} e^{i\frac{\varphi}{2}} = i - 1 = \sqrt{2} e^{i\frac{3\pi}{4} + i2\pi k}$$

$$|z| e^{i\varphi} = 2 e^{i\frac{3\pi}{2} + i4\pi k}$$

$$|z| = 2$$

$$\varphi = \frac{3\pi}{2} + 4\pi k \notin (-\pi, \pi]$$

$\therefore$  no solutions

Ex3: Find all solutions to  $z^{1/2} + 1 - i = 0$

$$z^{1/2} = |z|^{1/2} e^{i\frac{\varphi}{2} + i\pi k_1} i - 1 = \sqrt{2} e^{i\frac{3\pi}{4} + i2\pi k_2}$$

$$|z| e^{i\varphi + i2\pi k_1} = 2 e^{i\frac{3\pi}{2} + i4\pi k_2}$$

$$|z| = 2$$

$$\varphi + 2\pi k_1 = \frac{3\pi}{2} + 4\pi k_2$$

$$\varphi = \frac{3\pi}{2} + 2\pi k_3$$

$$z = 2e^{-i\frac{\pi}{2}} = -2i$$

Ex4: Find where  $\text{Log}(1 + z^2)$  is analytic

$$w = 1 + z^2$$

$$\text{not analytic on } \Re(w) \leq 0 \wedge \Im(w) = 0$$

$$x^2 - y^2 + 1 \leq 0$$

$$2xy = 0 \Rightarrow x = 0 \vee y = 0$$

$$y = 0 : x^2 + 1 \leq 0 \Rightarrow \nexists x \in \mathbb{R} \text{ s.t. } x^2 \leq -1$$

$$x = 0 : y^2 \geq 1 \Rightarrow |y| \geq 1$$

Analytic on the domain

$$\mathbb{C} \setminus \{(x, y) : x = 0, |y| \geq 1\}$$

Ex5: Find where  $\text{Log}\left(\frac{1-z}{1+z}\right)$  is analytic

$$w = \frac{1-z}{1+z}$$

$$\text{not analytic on } \Re(w) \leq 0 \wedge \Im(w) = 0$$

$$w = \frac{(1-z)(1+\bar{z})}{(1+z)(1+\bar{z})} = \frac{1-|z|^2 - z + \bar{z}}{1+|z|^2 + z + \bar{z}} = \frac{1-x^2-y^2-2iy}{1+x^2+y^2+2x}$$

$$y = 0$$

$$\frac{1-x^2}{1+x^2+2x} \leq 0 \Rightarrow 1 \leq x^2 \Rightarrow |x| \geq 1$$

Analytic on

$$\mathbb{C} \setminus \{(x, y) : y = 0, |x| \geq 1\}$$

Ex6: Find where  $f(z) = \text{Log}(1 - z^3)$  is analytic

$$\Re(1 - z^3) \leq 0 \quad \Im(1 - z^3) = 0$$

$$z^3 = r^3 e^{3i\varphi} = r^3 = r^3 (\cos(3\varphi) + i \sin(3\varphi))$$

$$\Im(1 - z^3) = r^3 \sin(3\varphi) = 0 \Rightarrow 3\varphi = n\pi$$

$$\varphi = \frac{n\pi}{3}$$

$$\Re(1 - z^3) = 1 - r^3 \cos(3\varphi) \leq 0$$

$$n = 2k : \cos(3\varphi) = 1 \Rightarrow 1 - r^3 \leq 0 \Rightarrow 1 \leq r$$

$$n = 2k + 1 : \cos(3\varphi) = -1 \Rightarrow 1 + r^3 \leq 0 \Rightarrow 1 \leq -r \Rightarrow -1 \geq r$$

can't have  $r \leq -1$  so can't have  $n = 2k + 1$

$$\Rightarrow \varphi = \frac{2k\pi}{3}, \quad 1 \leq r$$

$$\text{Analytic in } \mathbb{C} \setminus \left\{ |z| \geq 1, \arg(z) = \frac{2k\pi}{3}, k \in \mathbb{Z} \right\}$$

Ex7: Find a branch cut for  $f(z) = \sqrt{z(z-1)}$  such that it is analytic for  $\{(x, y) | y = 0, x < 0\}$  such that  $f(2) = \sqrt{2}$

$$\sqrt{z(z-1)} = |z|^{1/2} e^{i\frac{\varphi_1}{2}} |z-1|^{1/2} e^{i\frac{\varphi_2}{2}} = (|z||z-1|)^{1/2} e^{i\frac{\varphi_1+\varphi_2}{2}}$$

$$\varphi_1 \in (-\pi, \pi), \quad \varphi_2 \in (-\pi, \pi)$$

$$\lim_{\varphi_1 \rightarrow \pi} \lim_{\varphi_2 \rightarrow \pi} e^{i\frac{\varphi_1+\varphi_2}{2}} = e^{\pi i}$$

$$\lim_{\varphi_1 \rightarrow -\pi} \lim_{\varphi_2 \rightarrow -\pi} e^{i\frac{\varphi_1+\varphi_2}{2}} = e^{-\pi i} = e^{\pi i}$$

$\therefore$  continuous for  $\{(x, y) | y = 0, x < 0\}$

$$z = 2 : \varphi_1 = 0, \varphi_2 = 0 \Rightarrow \sqrt{z(z-1)} = \sqrt{2}$$

Ex8: Find a branch cut for  $f(z) = \sqrt{z(z-1)}$  such that it is analytic for  $\{(x, y) | y = 0, x < 0\}$  such that  $f(2) = -\sqrt{2}$

$$\sqrt{z(z-1)} = |z|^{1/2} e^{i\frac{\varphi_1}{2}} |z-1|^{1/2} e^{i\frac{\varphi_2}{2}} = (|z||z-1|)^{1/2} e^{i\frac{\varphi_1+\varphi_2}{2}}$$

$$\varphi_1 \in (-\pi, \pi), \quad \varphi_2 \in (-3\pi, -\pi)$$

$$\lim_{\varphi_1 \rightarrow \pi} \lim_{\varphi_2 \rightarrow -\pi} e^{i\frac{\varphi_1+\varphi_2}{2}} = e^{-2\pi i}$$

$$\lim_{\varphi_1 \rightarrow -\pi} \lim_{\varphi_2 \rightarrow -3\pi} e^{i\frac{\varphi_1+\varphi_2}{2}} = e^0 = e^{-2\pi i}$$

$\therefore$  continuous for  $\{(x, y) | y = 0, x < 0\}$

$$z = 2 : \varphi_1 = 0, \varphi_2 = -2\pi \Rightarrow \sqrt{z(z-1)} = \sqrt{2} e^{-i\pi} = -\sqrt{2}$$

Ex9: Find a branch cut for  $f(z) = \log(z^2 + 2z + 2)$  such that it is analytic for  $\{(x, y) | x > -1\}$  such that  $\frac{d}{dz} f(-1) = 0$

$$\log(z^2 + 2z + 2)$$

$$z = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$\log(z^2 + 2z + 2) = \ln(|z - z_1||z - z_2|) + i(\varphi_1 + \varphi_2)$$

$$-\pi < \varphi_1 < \pi, \quad -\pi < \varphi_2 < \pi$$

$$z = -1: \quad \varphi_1 = -\frac{\pi}{2}, \quad \varphi_2 = \frac{\pi}{2}$$

$$\log(z^2 + 2z + 2) \Big|_{z=-1} = 0 + i \left( -\frac{\pi}{2} + \frac{\pi}{2} \right) = 0$$

$$\frac{d}{dz} \log(z^2 + 2z + 2) \Big|_{z=-1} = \frac{2z + 2}{z^2 + 2z + 2} \Big|_{z=-1} = 0$$

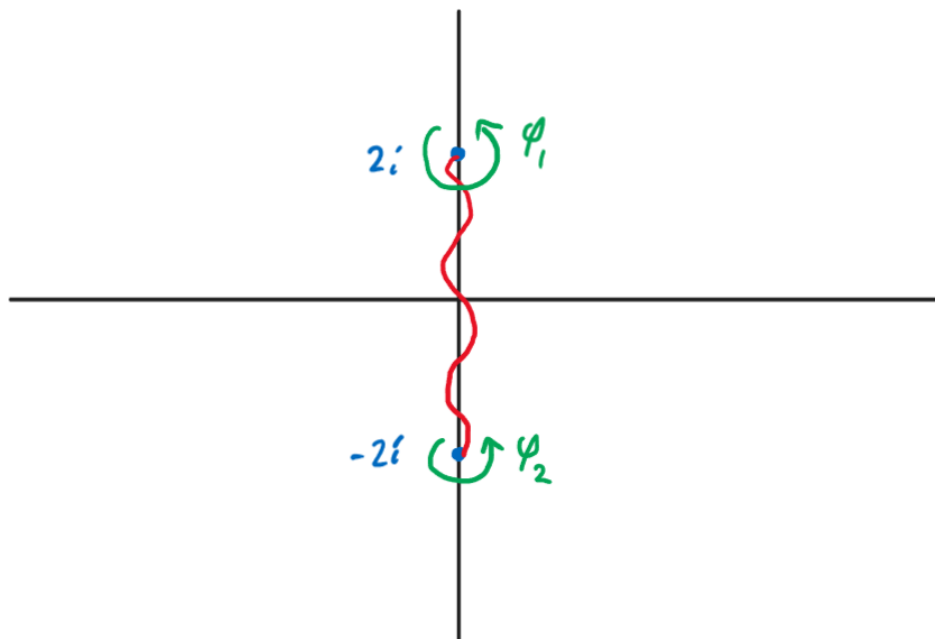
If we have a function such as  $f(z) = \sqrt{z^2 - 1}$  then we will have two branch points at  $z = \pm 1$ . If two of these branch cuts overlap then they have the property of cancelling out. Note that if we have a cube root then we require three overlapping branch cuts to cancel out. For an  $n$ th root we require  $n$  overlapping branch cuts to cancel out.

Ex: Find a branch cut for  $f(z) = \sqrt{4 + z^2}$  such that it is analytic for  $\mathbb{C} \setminus \{x = 0, -2 \leq y \leq 2\}$ .

$$(4 + z^2)^{1/2}, \quad \mathbb{C} \setminus \{x = 0, -2 \leq y \leq 2\}$$

$$4 + z^2 = 0 \Rightarrow z = \pm 2i$$

$$(4 + z^2)^{1/2} = |z + 2i|^{1/2} |z - 2i|^{1/2} e^{i\left(\frac{\varphi_1 + \varphi_2}{2}\right)}$$



Choose  $\frac{\pi}{2} < \varphi_1 < \frac{5\pi}{2}$  and  $\frac{\pi}{2} < \varphi_2 < \frac{5\pi}{2}$

Check continuity at  $y > 2$ :

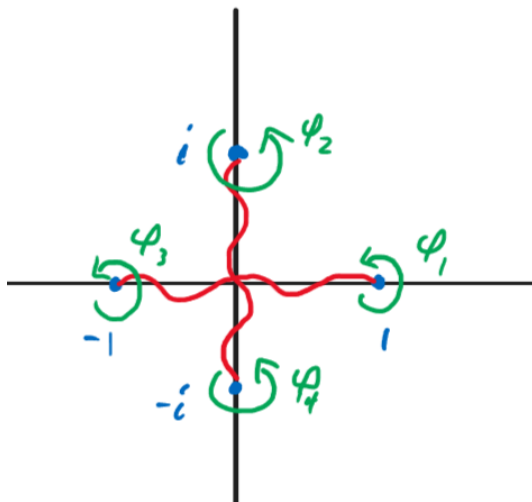
	$\varphi_1$	$\varphi_2$	$e^{i\left(\frac{\varphi_1 + \varphi_2}{2}\right)}$
A	$\frac{5\pi}{2}$	$\frac{5\pi}{2}$	$e^{i\frac{5\pi}{2}} = i$
A'	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$e^{i\frac{\pi}{2}} = i$

Ex2: Find a branch cut for  $f(z) = (z^4 - 1)^{1/2}$  such that it is analytic for  $|z| > 1$ .

$$(z^4 - 1)^{1/2}, \{ |z| > 1 \}$$

$$z^4 - 1 = 0 \Rightarrow z = \pm 1, \pm i$$

$$(z^4 - 1)^{1/2} = |z + 1|^{1/2} |z - 1|^{1/2} |z + i|^{1/2} |z - i|^{1/2} e^{i\left(\frac{\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4}{2}\right)}$$



Choose  $-\pi < \varphi_1 < \pi$ ,  $\frac{\pi}{2} < \varphi_2 < \frac{5\pi}{2}$ ,  $-\pi < \varphi_3 < \pi$ , and  $\frac{\pi}{2} < \varphi_4 < \frac{5\pi}{2}$

Check continuity at  $y > 1$ :

	$\varphi_2$	$\varphi_4$	$e^{i\left(\frac{\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4}{2}\right)}$
A	$\frac{5\pi}{2}$	$\frac{5\pi}{2}$	$e^{i\frac{5\pi}{2}} e^{i\left(\frac{\varphi_1 + \varphi_3}{2}\right)} = ie^{i\left(\frac{\varphi_1 + \varphi_3}{2}\right)}$
A'	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$e^{i\frac{\pi}{2}} e^{i\left(\frac{\varphi_1 + \varphi_3}{2}\right)} = ie^{i\left(\frac{\varphi_1 + \varphi_3}{2}\right)}$

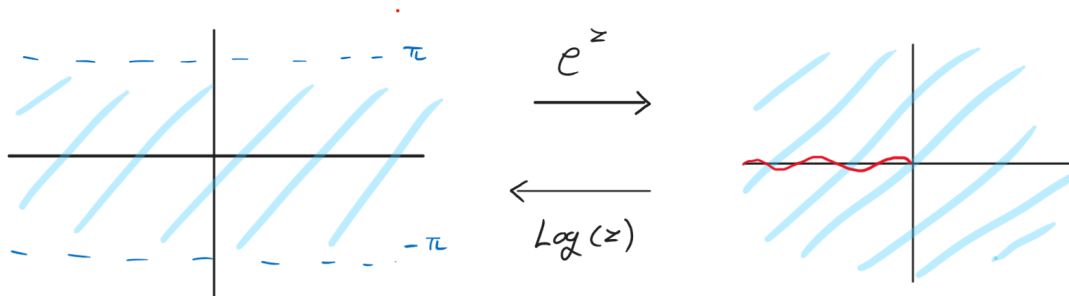
Check continuity at  $x < 1$ :

	$\varphi_1$	$\varphi_3$	$e^{i\left(\frac{\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4}{2}\right)}$
B	$\pi$	$\pi$	$e^{i\pi} e^{i\left(\frac{\varphi_2 + \varphi_4}{2}\right)} = -e^{i\left(\frac{\varphi_2 + \varphi_4}{2}\right)}$
B'	$-\pi$	$-\pi$	$e^{-i\pi} = -e^{i\left(\frac{\varphi_2 + \varphi_4}{2}\right)}$

### 1.2.8 Inverse Functions

To get a one-to-one function we often need to restrict the domain of the function.

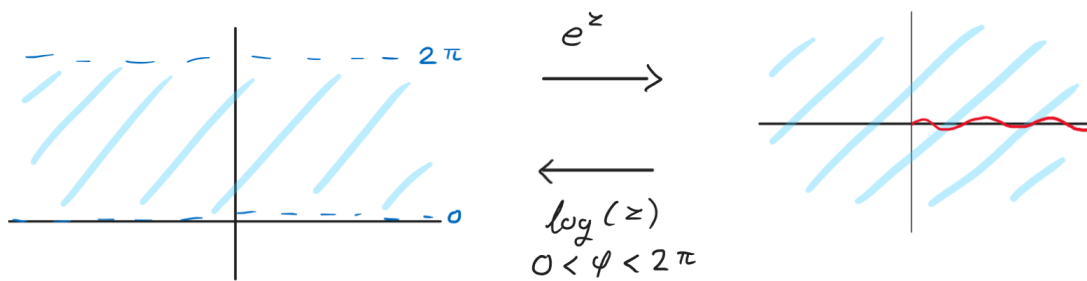
Inverse of  $e^z$  is  $z = \text{Log}(w)$  or a branch cut of  $\log(w)$



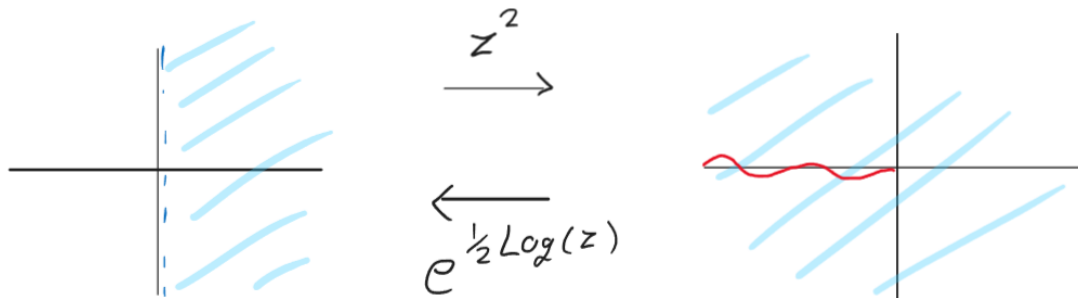
$$e^z : \{-\pi < y < \pi\} \rightarrow \mathbb{C} \setminus (-\infty, 0]$$

$$\text{Log}(z) : \mathbb{C} \setminus (-\infty, 0] \rightarrow \{-\pi < v < \pi\}$$

The inverse of  $e^z$  over  $\{0 < y < 2\pi\}$  is  $\log z = \ln r + i\varphi$ ,  $0 < \varphi < 2\pi$ .

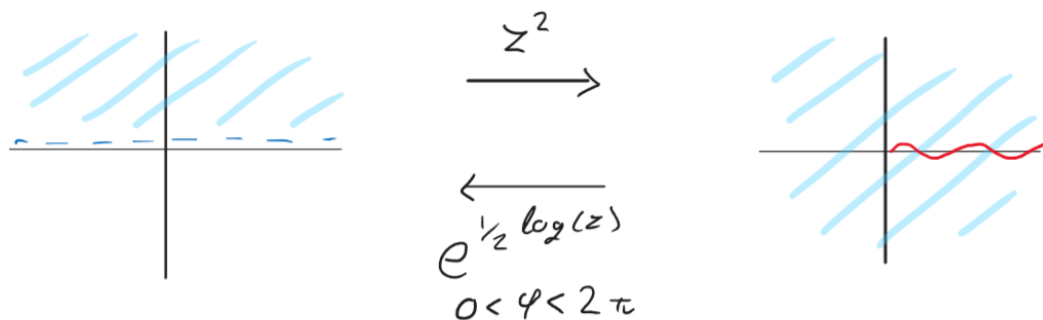


The inverse function for  $z^2$  over  $\{x > 0\}$  is  $z^{\frac{1}{2}} = e^{\frac{1}{2} \text{Log}(z)}$



$$z^{\frac{1}{2}} : \{x > 0\} \rightarrow \mathbb{C} \setminus (-\infty, 0]$$

The inverse function for  $z^2$  over  $\{y > 0\}$  is  $z^{\frac{1}{2}} = e^{\frac{1}{2} \log(z)}$ ,  $0 < \varphi < 2\pi$



$$z^2 : \{y > 0\} \rightarrow \mathbb{C} \setminus [0, \infty)$$

Ex: Find the inverse function of  $e^z$  on the domain  $\{\frac{\pi}{4} < y < \frac{9\pi}{4}\}$

$$w = e^x e^{iy}$$

$$\log(w) = \ln(e^x) + iy = x + iy = z$$

$$\log(w)$$

$$f^{-1}(z) = \log(z), \left\{ \frac{\pi}{4} < \arg(z) < \frac{9\pi}{4} \right\}$$

Ex2: Find the inverse function of  $z^4$  on the domain  $\{x > 0, y > 0\}$

$$z = w^{1/4} = |w|^{1/4} e^{i \frac{\arg(w)}{4}}$$

$$0 < \arg(z) < \frac{\pi}{2} \Rightarrow 0 < \frac{\arg(w)}{4} < \frac{\pi}{2} \Rightarrow 0 < \arg(w) < 2\pi$$

$$f^{-1}(z) = z^{1/4}, \{0 < \arg(z) < 2\pi\}$$

The inverse of  $\sin z$  is a little bit less obvious. Let us find all inverse functions for  $\sin z$ .

$$\sin w = z$$

$$\frac{e^{iw} - e^{-iw}}{2i} = z$$

$$e^{iw} - e^{-iw} = 2iz$$

$$e^{2iw} - 2ize^{iw} - 1 = 0$$

$$e^{iw} = \frac{2iz \pm \sqrt{(-2iz)^2 + 4}}{2}$$

$$e^{iw} = iz \pm \sqrt{1 - z^2}$$

$$e^{iw} = iz + i(z^2 - 1)^{\frac{1}{2}}$$

$$iw = \log \left( iz + i(z^2 - 1)^{\frac{1}{2}} \right)$$

$$w = -i \log \left( iz + i(z^2 - 1)^{\frac{1}{2}} \right)$$



The general solutions to  $\sin w = z$  are given by the above expression. This expression contains two multivalued parts,  $\log$  and  $(z^2 - 1)^{\frac{1}{2}}$ .

Ex: Find all solutions to  $\sin w = -i$

$$\sin(w) = -i$$

$$\sin x \cosh y + i \cos x \sinh y = -i$$

$$\sin x \cosh y = 0 \Rightarrow x = n\pi$$

$$\cos x \sinh y = -1$$

$$n = 2k :$$

$$\sinh y = -1$$

$$n = 2k + 1 :$$

$$-\sinh y = -1 \Rightarrow \sinh y = 1$$

$$\sinh y = \frac{e^y - e^{-y}}{2}$$

$$e^{2y} - 2 \sinh y e^y - 1 = 0$$

$$e^y = \frac{2 \sinh y \pm \sqrt{4 \sinh^2 y + 4}}{2} = \sinh y \pm \sqrt{\sinh^2 y + 1}$$

$$y = \ln \left( \sinh y \pm \sqrt{\sinh^2 y + 1} \right)$$

$$n = 2k :$$

$$y = \ln \left( -1 \pm \sqrt{2} \right) = \ln \left( \sqrt{2} - 1 \right)$$

$$n = 2k + 1 :$$

$$y = \ln \left( 1 \pm \sqrt{2} \right) = \ln \left( 1 + \sqrt{2} \right)$$

$$w = \left\{ 2k_1\pi + i \ln \left( \sqrt{2} - 1 \right), (2k_2 + 1)\pi + i \ln \left( 1 + \sqrt{2} \right), k_1, k_2 \in \mathbb{Z} \right\}$$

Ex2: Find all solutions to  $\cos w = -i$

$$\cos(w) = -i$$

$$\cos(w) = \cos x \cosh y - i \sin x \sinh y$$

$$\cos x \cosh y = 0 \Rightarrow x = \frac{\pi}{2} + n\pi$$

$$n = 2k :$$

$$-\sinh y = -1 \Rightarrow \sinh y = 1$$

$$n = 2k + 1 :$$

$$\sinh y = -1$$

$$\sinh y = \frac{e^y - e^{-y}}{2}$$

$$e^{2y} - 2 \sinh y e^y - 1 = 0$$

$$e^y = \frac{2 \sinh y \pm \sqrt{4 \sinh^2 y + 4}}{2} = \sinh y \pm \sqrt{\sinh^2 y + 1}$$

$$y = \ln \left( \sinh y \pm \sqrt{\sinh^2 y + 1} \right)$$

$$n = 2k :$$

$$y = \ln \left( 1 \pm \sqrt{2} \right) = \ln \left( 1 + \sqrt{2} \right)$$

$$n = 2k + 1 :$$

$$y = \ln \left( -1 \pm \sqrt{2} \right) = \ln \left( \sqrt{2} - 1 \right)$$

$$z = \left\{ \frac{\pi}{2} + 2k_1\pi + i \ln \left( 1 + \sqrt{2} \right), \frac{\pi}{2} + (2k_2 + 1)\pi + i \ln \left( \sqrt{2} - 1 \right), k_1, k_2 \in \mathbb{Z} \right\}$$

The principal value of the inverse function of  $\sin z$  is called  $f(z) = \arcsin z$  and is defined such that  $f(0) = 0$ .

The principal value of the inverse function of  $\cos z$  is called  $f(z) = \arccos z$  and is defined such that  $f(0) = \frac{\pi}{2}$ .

These functions are defined as

$$\arcsin z = -i \operatorname{Log} \left( iz + i(z^2 - 1)^{\frac{1}{2}} \right)$$

$$\text{where } (z^2 - 1)^{\frac{1}{2}} = |z - 1|^{\frac{1}{2}} |z + 1|^{\frac{1}{2}} e^{i \frac{\arg(z-1) + \arg(z+1)}{2}}$$

$$\text{such that } 0 < \arg(z - 1) < 2\pi \text{ and } \pi < \arg(z + 1) < 3\pi$$

$$\arccos z = -i \operatorname{Log} \left( z + (z^2 - 1)^{\frac{1}{2}} \right)$$

$$\text{where } (z^2 - 1)^{\frac{1}{2}} = |z - 1|^{\frac{1}{2}} |z + 1|^{\frac{1}{2}} e^{i \frac{\arg(z-1) + \arg(z+1)}{2}}$$

Ex: Find all solutions to  $w = \arcsin(-i)$

$$w = \arcsin(-i)$$

$$\arcsin(z) = -i \operatorname{Log}(iz + i(z^2 - 1)^{1/2})$$

$$(z^2 - 1)^{1/2} = |z + 1|^{1/2} |z - 1|^{1/2} e^{i \left( \frac{\varphi_1 + \varphi_2}{2} \right)}, \varphi_1 \in (0, 2\pi), \varphi_2 \in (\pi, 3\pi)$$

$$(z^2 - 1)^{1/2} \Big|_{z=-i} \Rightarrow \varphi_1 = \frac{5\pi}{4}, \varphi_2 = \frac{3\pi}{4} + \pi$$

$$(z^2 - 1)^{1/2} \Big|_{z=-i} = \sqrt{2} e^{i \frac{3\pi}{2}} = -\sqrt{2}i$$

$$\arcsin(-i) = -i \operatorname{Log}(1 + i(-\sqrt{2}i)) = -i \operatorname{Log}(1 + \sqrt{2})$$

$$\arcsin(-i) = -i \ln \left( 1 + \sqrt{2} \right)$$

Ex2: Find all solutions to  $w = \arccos(-i)$

$$w = \arccos(-i)$$

$$w = -i \operatorname{Log}(z + (z^2 - 1)^{1/2})$$

$$(z^2 - 1)^{1/2} = |z + 1|^{1/2} |z - 1|^{1/2} e^{i \left( \frac{\varphi_1 + \varphi_2}{2} \right)}, \varphi_1 \in (0, 2\pi), \varphi_2 \in (-\pi, \pi)$$

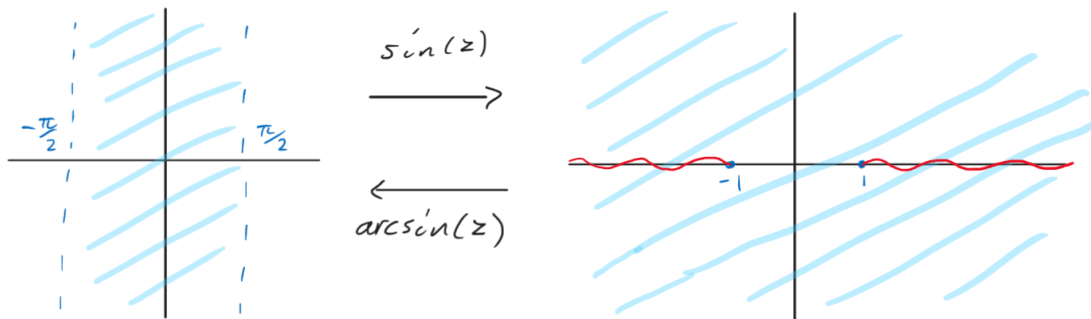
$$(z^2 - 1)^{1/2} \Big|_{z=-i} \Rightarrow \varphi_1 = \frac{5\pi}{4}, \varphi_2 = -\frac{\pi}{4}$$

$$(z^2 - 1)^{1/2} \Big|_{z=-i} = \sqrt{2}e^{i\frac{\pi}{2}} = \sqrt{2}i$$

$$w = -i \operatorname{Log}(-i + \sqrt{2}i) = -i \operatorname{Log}((\sqrt{2} - 1)e^{i\frac{\pi}{2}}) = -i \left( \ln(\sqrt{2} - 1) + i\frac{\pi}{2} \right)$$

$$w = \frac{\pi}{2} - i \ln(\sqrt{2} - 1)$$

Mapping properties of  $\sin z$  and  $\arcsin z$ :



We can also apply this mapping to solve Laplace's equation.

Ex:

$$\Delta \phi = 0, \{x > 0, y > 0\}$$

$$\text{BCs : } \begin{cases} \phi = 1 & x = 0, y > 0 \\ \phi_y = 0 & 0 < x < 1, y = 0 \\ \phi = 2 & x > 1, y = 0 \end{cases}$$

$$w = \arcsin(z)$$

$$\Phi = Au + B$$

$$\Phi(u = 0) = 1 \Rightarrow B = 1$$

$$\Phi\left(u = \frac{\pi}{2}\right) = 2 = A\frac{\pi}{2} + 1 \Rightarrow A = \frac{2}{\pi}$$

$$\Phi = \frac{2}{\pi}u + 1$$

$$u + iv = \arcsin(x + iy)$$

$$\sin(u + iv) = x + iy$$

$$\begin{cases} x = \sin u \cosh v \\ y = \cos u \sinh v \end{cases} \Rightarrow \begin{cases} \cosh v = \frac{x}{\sin u} \\ \sinh v = \frac{y}{\cos u} \end{cases} \Rightarrow \cosh^2 v - \sinh^2 v = \frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1$$

$$t = \sin^2 u$$

$$\sin^2 u + \cos^2 u = 1 \Rightarrow \cos^2 u = 1 - t$$

$$\frac{x^2}{t} - \frac{y^2}{1-t} = 1$$

$$t(1-t) = (1-t)x^2 - ty^2$$

$$\begin{aligned}
t - t^2 &= x^2 - tx^2 - ty^2 \\
t^2 + t(-x^2 - y^2 - 1) + x^2 \\
t &= \frac{x^2 + y^2 + 1 \pm \sqrt{(x^2 + y^2 + 1)^2 - 4x^2}}{2} \\
0 \leq \sin^2 u \leq 1 &\Rightarrow 0 \leq t \leq 1 \Rightarrow t = \frac{x^2 + y^2 + 1 - \sqrt{(x^2 + y^2 + 1)^2 - 4x^2}}{2} \\
\sin^2 u &= \frac{x^2 + y^2 + 1 - \sqrt{(x^2 + y^2 + 1)^2 - 4x^2}}{2} \\
0 < u < \frac{\pi}{2} &\Rightarrow 0 < \sin u < 1 \\
\sin u &= \sqrt{\frac{x^2 + y^2 + 1 - \sqrt{(x^2 + y^2 + 1)^2 - 4x^2}}{2}} \\
u &= \arcsin \left( \sqrt{\frac{x^2 + y^2 + 1 - \sqrt{(x^2 + y^2 + 1)^2 - 4x^2}}{2}} \right) \\
\phi &= \frac{2}{\pi} \arcsin \left( \sqrt{\frac{x^2 + y^2 + 1 - \sqrt{(x^2 + y^2 + 1)^2 - 4x^2}}{2}} \right) + 1
\end{aligned}$$

We can also define inverse functions for  $\sinh z$  and  $\cosh z$  using the same method as for  $\sin z$  and  $\cos z$ .

Ex: Find the inverse of  $\sinh(z)$  such that  $f^{-1}(0) = \ln(1)$ .

$$\begin{aligned}
w &= \sinh(z) = \frac{e^z - e^{-z}}{2} \\
2w &= e^z - e^{-z} \\
e^{2z} - 2we^z - 1 & \\
e^z &= \frac{2w \pm \sqrt{4w^2 + 4}}{2} = w \pm \sqrt{w^2 + 1} \\
|e^z| > 0 &\Rightarrow |w \pm \sqrt{w^2 + 1}| \\
\log(e^z) &= \ln(e^x) + i(y + 2\pi k) = \log \left( w \pm (w^2 + 1)^{1/2} \right) \\
z &= \log \left( w + (w^2 + 1)^{1/2} \right) \\
(w^2 + 1)^{1/2} &= |w + i|^{1/2} |w - i|^{1/2} e^{i \left( \frac{\varphi_1 + \varphi_2}{2} \right)}, \quad \varphi_1 \in \left( \frac{\pi}{2}, \frac{5\pi}{2} \right), \quad \varphi_2 \in \left( \frac{3\pi}{2}, \frac{7\pi}{2} \right) \\
(w^2 + 1)^{1/2} \Big|_{w=0} &\Rightarrow \varphi_1 = \frac{3\pi}{2}, \quad \varphi_2 = \frac{5\pi}{2} \\
(w^2 + 1)^{1/2} \Big|_{w=0} &= e^{i2\pi} = 1 \\
0 &= \log(1) = \ln(1) + i2\pi k \\
f^{-1}(z) &= \text{Log} \left( z + |z + i|^{1/2} |z - i|^{1/2} e^{i \left( \frac{\varphi_1 + \varphi_2}{2} \right)} \right), \quad \varphi_1 \in \left( \frac{\pi}{2}, \frac{5\pi}{2} \right), \quad \varphi_2 \in \left( \frac{3\pi}{2}, \frac{7\pi}{2} \right)
\end{aligned}$$

## 1.3 Integration of Complex Functions

### 1.3.1 Path Integrals

Given that complex functions exist in a two-dimensional space, it is natural to consider integration over a path in the complex plane.

We can come up with a parameterization to express this path as a function of a real variable  $t$ :

$$w(t) = x(t) + iy(t)$$

$$\int_a^b w(t)dt = \int_a^b x(t)dt + i \int_a^b y(t)dt$$

Complex integrals will have the same properties as real integrals, such as linearity. One new property for complex integrals is:

$$\left| \int_a^b w(t)dt \right| \leq \left| \int_a^b |w(t)|dt \right|$$

*Proof.*

$$\begin{aligned} \int_a^b w(t)dt &= \rho e^{i\varphi} \\ \rho &= \left| \int_a^b w(t)dt \right| \\ \rho &= e^{-i\varphi} \int_a^b w(t)dt = \int_a^b w(t)e^{-i\varphi}dt \\ \rho &= \left| \int_a^b w(t)dt \right| = \int_a^b e^{-it} w(t)dt = \Re \left( \int_a^b e^{-i\varphi} w(t)dt \right) \\ \rho &= \int_a^b \Re(e^{-i\varphi} w(t)) dt \leq \int_a^b |e^{-i\varphi} w(t)| dt = \int_a^b |w(t)|dt \end{aligned}$$

□

A path or contour  $C$  is a piecewise smooth curve in the complex plane.

$$C = z(t) = x(t) + iy(t), \quad a \leq t \leq b$$

A path also has a direction, which is the direction of increasing  $t$ .

$$\begin{aligned} \int_C f(z)dz &= \int_a^b f(z(t))z'(t)dt \\ dz &= z'(t)dt = (x'(t) + iy'(t))dt \end{aligned}$$

Ex:  $f(x, y) = x - 2xyi$

$$\begin{aligned} \int_C (x - 2xyi)dz, \quad C &= \{y = x^2, 0 \leq x \leq 1\} \cup \{y = 1, -1 \leq x \leq 1\} \\ f_1 &= x - 2x^3i \end{aligned}$$

$$z = x + iy = x + ix^2 \Rightarrow dz = (1 + 2xi)dx$$

$$\int_0^1 (x - 2x^3i)(1 + 2xi)dx = \int_0^1 (x + 2x^2i - 2x^3i + 4x^4)dx = \frac{1}{2} + \frac{2}{3}i - \frac{2}{4}i + \frac{4}{5}$$

$$f_2 = x - 2xi$$

$$z = x + iy = x + i \Rightarrow dz = dx$$

$$\int_{-1}^1 (x - 2xi)dx = \left[ \frac{x^2}{2} - x^2i \right]_{-1}^1 = 0$$

$$\int_C (x - 2xyi)dz = \frac{13}{10} + \frac{1}{6}i$$

A closed path (has the same start and end points) is called a contour. It is the same as a path integral but we will see later that contours have special properties with complex functions.

Ex2:  $f(z) = \bar{z}^2$  on the contour  $C$  that is the box with vertices  $0, 1, 1 + i, i, 0$  in the counterclockwise direction.

$$\bar{z}^2 = (x - iy)^2 = x^2 - 2xyi - y^2$$

$$y = 0 : f_1 = x^2$$

$$z = x + iy = x \Rightarrow dz = dx$$

$$\int_{C_1} f_1 dz = \int_0^1 x^2 dx = \frac{1}{3}$$

$$x = 1 : f_2 = 1 - 2yi - y^2$$

$$z = x + iy = 1 + iy \Rightarrow dz = idy$$

$$\int_{C_2} f_2 dz = \int_0^1 (1 - 2yi - y^2)idy = \left[ iy + y^2 - \frac{y^3}{3}i \right]_0^1 = i + 1 - \frac{i}{3} = \frac{2}{3}i + 1$$

$$y = 1 : f_3 = x^2 - 2xi - 1$$

$$z = x + iy = x + i \Rightarrow dz = dx$$

$$\int_{C_3} f_3 dz = - \int_0^1 (x^2 - 2xi - 1)dx = -\frac{1}{3} + i + 1 = i + \frac{2}{3}$$

$$x = 0 : f_4 = -y^2$$

$$z = x + iy = iy \Rightarrow dz = idy$$

$$\int_{C_4} f_4 dz = - \int_0^1 (-y^2)idy = \frac{i}{3}$$

$$\int_C \bar{z}^2 dz = 2 + 2i$$

Ex3:  $f(z) = \text{Log}(z)$

$$C = \{|z| = 1, \Re(z) \geq 0\}$$

$$z = e^{it}, -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$

$$dz = ie^{it}dt$$

$$\begin{aligned}
\int_C \text{Log}(z) dz &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \text{Log}(e^{it}) (ie^{it}) dt = - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} te^{it} dt \\
&= - \frac{te^{it}}{i} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{it}}{i} dt = -e^{it} \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -2i
\end{aligned}$$

### 1.3.2 Fundamental Theorem of Calculus

A neat property of complex integrals is that they are path independent. This means that the value of the integral is the same for any path between the start and end points. This is not true for real integrals.

Ex: Given the function  $f(z) = z^2 + 1$  we will compute three different integrals between the points  $z_1 = -i$  and  $z_2 = 1$  and show that they are all equal.

Path 1:

$$\begin{aligned}
C &= \{y = x - 1\} \\
z^2 + 1 &= (x + iy)^2 + 1 = x^2 + 2xyi - y^2 + 1 \\
\int_C (x^2 - y^2 + 1 + 2xyi) dz \\
z = x + iy = x + i(x - 1) &\Rightarrow dz = (1 + i)dx \\
\int_0^1 (x^2 - (x - 1)^2 + 1 + 2x(x - 1)i)(1 + i) dx \\
&= \int_0^1 (2x + 2x^2i - 2xi)(1 + i) dx = \left[ x^2 + \frac{2x^3}{3}i - x^2i \right]_0^1 (1 + i) \\
&= \left( 1 - \frac{i}{3} \right) (1 + i) = \frac{4}{3} + \frac{2}{3}i
\end{aligned}$$

Path 2:

$$\begin{aligned}
C &= \{x = 0, -1 \leq y \leq 0\} \cup \{y = 0, 0 \leq x \leq 1\} \\
C_1 : \int_{-1}^0 (-y^2 + 1)idy &= \left[ y - \frac{y^3}{3} \right]_{-1}^0 i = \frac{2}{3}i \\
C_2 : \int_0^1 (x^2 + 1)dx &= \left[ \frac{x^3}{3} + x \right]_0^1 = \frac{4}{3} \\
\int_C (z^2 + 1)dz &= \frac{4}{3} + \frac{2}{3}i
\end{aligned}$$

Path 3:

$$\begin{aligned}
z &= e^{it}, \quad -\frac{\pi}{2} \leq t \leq 0 \\
z^2 + 1 &= e^{2it} + 1 \\
dz &= ie^{it} dt \\
\int_{-\frac{\pi}{2}}^0 (e^{2it} + 1) ie^{it} dt &= i \int_{-\frac{\pi}{2}}^0 (e^{3it} + e^{it}) dt = i \left[ \frac{e^{3it}}{3i} + \frac{e^{it}}{i} \right]_{-\frac{\pi}{2}}^0 = \frac{4}{3} - \frac{e^{-i\frac{3\pi}{2}}}{3} - e^{-i\frac{\pi}{2}}
\end{aligned}$$

$$= \frac{4}{3} + \frac{2}{3}i$$

This finding leads us to the Fundamental Theorem of Calculus for complex integrals.

$$\int_C f(z)dz = F(z_2) - F(z_1)$$

where  $F'(z) = f(z)$ .

When an antiderivative of  $f(z)$  exists it becomes far easier to compute the integral.

Ex:

$$\int_C \sin z dz, \quad C : e^{it}, \quad -\frac{\pi}{2} \leq t \leq \frac{5\pi}{4}$$

$$\int_C \sin z dz = -\cos z_1 + \cos z_0$$

$$z_0 = e^{-i\frac{\pi}{2}} = -i$$

$$z_1 = e^{i\frac{5\pi}{4}} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$

$$\cos(z) = \cos(x) \cosh(y) - i \sin(x) \sinh(y)$$

$$\cos(z_0) = \cosh(1)$$

$$\cos(z_1) = \cos\left(\frac{1}{\sqrt{2}}\right) \cosh\left(\frac{1}{\sqrt{2}}\right) - i \sin\left(\frac{1}{\sqrt{2}}\right) \sinh\left(\frac{1}{\sqrt{2}}\right)$$

$$\int_C \sin z dz = \cos(-i) - \cos\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = \cosh(1) - \cos\left(\frac{1}{\sqrt{2}}\right) \cosh\left(\frac{1}{\sqrt{2}}\right) + i \sin\left(\frac{1}{\sqrt{2}}\right) \sinh\left(\frac{1}{\sqrt{2}}\right)$$

Ex2:

$$\int_C \frac{dz}{z}, \quad C : \frac{x^2}{4} + y^2 = 1, \quad y \geq 0$$

$$\int_C \frac{dz}{z} = \log^*(z) = \ln|z| + i\varphi \Big|_{z_0}^{z_1}, \quad \varphi \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$$

$$y = 0 : \frac{x^2}{4} = 1 \Rightarrow x = \pm 2$$

$$z_0 = 2, \quad z_1 = -2$$

$$\int_C \frac{dz}{z} = \ln 2 + \pi i - \ln 2 = \pi i$$

Ex3:

$$\int_C \frac{dz}{z^2}, \quad C : \frac{x^2}{4} + y^2 = 1, \quad y \geq 0$$

$$\int_C \frac{dz}{z^2} = -\frac{1}{z_1} + \frac{1}{z_0}$$

$$z_0 = 2, \quad z_1 = -2$$

$$\int_C \frac{dz}{z^2} = \frac{1}{2} + \frac{1}{2} = 1$$



Ex4:

$$\begin{aligned} \int_C \frac{dz}{z}, \quad C : z = e^{it}, \quad 0 \leq t \leq \frac{3\pi}{2} \\ \int_C \frac{dz}{z} = \ln|z| + i\varphi \Big|_{z_0}^{z_1}, \quad \varphi \in \left(-\frac{\pi}{4}, \frac{7\pi}{4}\right) \\ z_0 = e^0 = 1 \\ z_1 = e^{i\frac{3\pi}{2}} = -i \\ \int_C \frac{dz}{z} = \left(\ln(1) + \frac{3\pi}{2}i\right) - (\ln(1) + 0) = \frac{3\pi}{2}i \end{aligned}$$

Ex5:

$$\begin{aligned} \int_C z^{1/3} dz, \quad C : r = 2 \cos\left(\frac{\theta}{2}\right), \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ \int_C z^{1/3} dz = \frac{3}{4} z^{4/3} \Big|_{z_0}^{z_1} = \frac{3}{4} |z|^{4/3} e^{i\varphi} \Big|_{z_0}^{z_1}, \quad \varphi \in (-\pi, \pi) \\ r\left(\pm\frac{\pi}{2}\right) = 2 \cos\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} = \sqrt{2} \\ z_0 = -\sqrt{2}i, \quad z_1 = \sqrt{2}i \\ \int_C z^{1/3} dz = \frac{3}{4} (\sqrt{2})^{4/3} \left(e^{i\frac{2\pi}{3}} - e^{-i\frac{2\pi}{3}}\right) \\ \int_C z^{1/3} dz = \frac{3}{2^{1/3}} \frac{i}{2i} \left(e^{i\frac{2\pi}{3}} - e^{-i\frac{2\pi}{3}}\right) = \frac{3i}{2^{1/3}} \sin\left(\frac{2\pi}{3}\right) \end{aligned}$$

Ex6:

$$\begin{aligned} \int_C \operatorname{Log}(z) dz, \quad C : r = 2 \cos\left(\frac{\theta}{2}\right), \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \\ z_0 = -\sqrt{2}i, \quad z_1 = \sqrt{2}i \\ \int_C \operatorname{Log} z dz = [z \operatorname{Log} z - z]_{-\sqrt{2}i}^{\sqrt{2}i} = \sqrt{2}i \left(\ln \sqrt{2} + i\frac{\pi}{2} - 1\right) + \sqrt{2}i \left(\ln \sqrt{2} - i\frac{\pi}{2} - 1\right) \\ = \left(2\sqrt{2} \ln \sqrt{2} - 2\sqrt{2}\right) i \end{aligned}$$

Another property of complex integrals we will make use of often is the inequality

$$\left| \int_C f(z) dz \right| \leq \max_{z \in C} |f(z)| \cdot \operatorname{Length}(C)$$

*Proof.*

$$\begin{aligned} \left| \int_C f(z) dz \right| &= \left| \int_a^b f(z(t)) z'(t) dt \right| \\ &\leq \left| \int_a^b |f(z(t))| |z'(t)| dt \right| \end{aligned}$$

$$\begin{aligned}
&\leq \max_{a \leq t \leq b} |f(z(t))| \cdot \int_a^b \sqrt{x^2 + y^2} dt \\
&\leq \max_{z \in C} |f(z)| \cdot \text{Length}(C)
\end{aligned}$$

□

Ex: Prove the inequality

$$\left| \int_C \frac{dz}{z^2 + i} \right| \leq \frac{3\pi}{4}$$

$C$  is the circle  $|z| = 3$  traversed once

*Proof.*

$$\left| \int_C f(z) dz \right| \leq \max_{z \in C} |f(z)| \text{Length}(C)$$

$$\text{Length}(C) = 6\pi$$

$$\left| \int_C \frac{dz}{z^2 + i} \right| \leq 6\pi \max_{z \in C} \left| \frac{1}{z^2 + i} \right|$$

$$\max_{z \in C} \left| \frac{1}{z^2 + i} \right| = \frac{1}{\min_{z \in C} |z^2 + i|}$$

$$|z^2 + i| \geq ||z^2| - |i|| = |9 - 1| = 8$$

$$\min_{z \in C} |z^2 + i| = 8$$

$$\max_{z \in C} \left| \frac{1}{z^2 + i} \right| = \frac{1}{8}$$

$$\left| \int_C \frac{dz}{z^2 + i} \right| \leq \frac{6\pi}{8} = \frac{3\pi}{4}$$

□

Ex2: Prove the inequality

$$\left| \int_C \text{Log}(z) dz \right| \leq \frac{\pi^2}{2}$$

where  $C$  is described by  $e^{it}$ ,  $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

*Proof.* Note that the curve is a semicircle of radius 1 where  $x \geq 0$ .

$$\left| \int_C f(z) dz \right| \leq \max_{z \in C} |f(z)| \text{Length}(C)$$

$$\text{Length}(C) = \pi$$

$$f(z) = \text{Log } z = \ln |z| + i\varphi, \quad \varphi \in (-\pi, \pi)$$

$$\max_{z \in C} |f(z)| = \max_{z \in C} |\text{Log } z| = \max_{z \in C} |\ln |z| + i\varphi|$$

$$z = e^{it}$$

$$|z| = 1$$

$$\varphi = t$$

$$\max_{z \in C} |f(z)| = \max_{-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}} |it| = \max_{-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}} |t| = \frac{\pi}{2}$$

$$\left| \int_C \text{Log}(z) dz \right| \leq \frac{\pi}{2} \cdot \pi = \frac{\pi^2}{2}$$

□

Ex3: Prove the inequality

$$\left| \int_C \frac{e^{3z}}{e^z + 1} dz \right| \leq \frac{2\pi e^{3R}}{e^R - 1}$$

where  $C$  is the vertical line segment from  $z = R$  to  $z = R + 2\pi i$  such that  $R \in \mathbb{R}$  and  $R > 0$ .

*Proof.*

$$\left| \int_C f(z) dz \right| \leq \max_{z \in C} |f(z)| \text{Length}(C)$$

$$\text{Length}(C) = 2\pi$$

$$f(z) = \frac{e^{3z}}{e^z + 1}$$

Let us describe  $C$  by

$$C : z = R + it, \quad 0 \leq t \leq 2\pi$$

$$\max_{z \in C} |f(z)| = \max_{z \in C} \left| \frac{e^{3z}}{e^z + 1} \right| = \max_{0 \leq t \leq 2\pi} \left| \frac{e^{3R} e^{3it}}{e^R e^{it} + 1} \right| = \max_{0 \leq t \leq 2\pi} \frac{e^{3R}}{|e^R e^{it} + 1|}$$

$$\max_{0 \leq t \leq 2\pi} \frac{1}{|e^R e^{it} + 1|} = \frac{1}{\min_{0 \leq t \leq 2\pi} |e^R e^{it} + 1|}$$

$$\min_{0 \leq t \leq 2\pi} |e^R e^{it} + 1| = \min_{0 \leq t \leq 2\pi} |e^R (\cos(t) + i \sin(t)) + 1| = \min_{0 \leq t \leq 2\pi} \sqrt{(e^R \cos(t) + 1)^2 + e^{2R} \sin^2(t)}$$

$$= \min_{0 \leq t \leq 2\pi} \sqrt{e^{2R} \cos^2 t + e^{2R} \sin^2 t + 2e^R \cos t + 1} = \min_{0 \leq t \leq 2\pi} \sqrt{e^{2R} + 2e^R \cos t + 1}$$

$$\min_{0 \leq t \leq 2\pi} \cos t = -1$$

$$\min_{0 \leq t \leq 2\pi} |e^R e^{it} + 1| = \sqrt{e^{2R} - 2e^R + 1} = \sqrt{(e^R - 1)^2} = |e^R - 1|$$

$$R > 0 \Rightarrow e^R > 1 \Rightarrow |e^R - 1| = e^R - 1$$

$$\max_{z \in C} \left| \frac{e^{3z}}{e^z + 1} \right| = \frac{e^{3R}}{e^R - 1}$$

$$\left| \int_C \frac{e^{3z}}{e^z + 1} dz \right| \leq \frac{2\pi e^{3R}}{e^R - 1}$$

□

### 1.3.3 Loop Integrals

A loop,  $C$ , is a path such that  $z_2 = z_1$  (the start and end points are the same).

A simple closed loop is a loop with no other intersections. For example, a figure 8 is not a simple closed loop.

Using the FTC (Fundamental Theorem of Calculus) we can see that if  $f(z)$  is continuous and has an antiderivative  $F(z)$  in  $\mathcal{D}$  then

$$\int_C f(z)dz = 0$$

for any loops in  $\mathcal{D}$ .

Ex:

$$f(z) = \frac{\cos z}{z^2 + 6z + 10}$$
$$z = \frac{-6 \pm \sqrt{36 - 40}}{2} = -3 \pm i$$

$f(z)$  is analytic on  $|z| \leq 2$

$$\therefore \int_{|z|=2} f(z)dz = 0$$

Ex2:

$$f(z) = \text{Log}(2z + 5)$$
$$w = 2z + 5$$

not analytic where  $\Re(w) < 0$ ,  $\Im(w) = 0$

$$w = 2x + 5 + iy$$
$$y = 0$$
$$2x + 5 < 0 \Rightarrow x < -\frac{5}{2}$$

$f(z)$  is analytic on  $|z| \leq 2$

$$\therefore \int_{|z|=2} f(z)dz = 0$$

Ex3:

$$f(z) = \arcsin\left(\frac{z}{3}\right)$$
$$w = \frac{z}{3}$$

not analytic where  $|\Re(w)| > 1$ ,  $\Im(w) = 0$

$$w = \frac{x}{3} + i\frac{y}{3}$$
$$y = 0$$

$$\frac{|x|}{3} > 1 \Rightarrow |x| > 3$$

$f(z)$  is analytic on  $|z| \leq 2$

$$\therefore \int_{|z|=2} f(z) dz = 0$$

Ex4:

$$f(z) = \tan\left(\frac{z}{2}\right) = \frac{\sin\left(\frac{z}{2}\right)}{\cos\left(\frac{z}{2}\right)}$$

$$\cos\left(\frac{z}{2}\right) = 0$$

$$\frac{z}{2} = \frac{\pi}{2} + n\pi, \quad n \in \mathbb{Z}$$

$$z = \pi + 2\pi n = \{\dots, -\pi, \pi, \dots\}$$

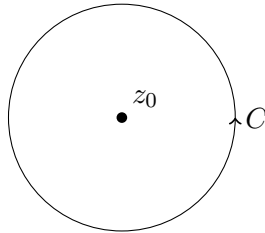
$f(z)$  is analytic on  $|z| \leq 2$

$$\therefore \int_{|z|=2} f(z) dz = 0$$

Let us look at the slightly more general case of

$$\int_C (z - z_0)^n dz$$

where  $C$  is a simple closed loop and  $z_0$  is a point inside  $C$ .



If  $n \geq 0$ ,

$$\begin{aligned} \left( \frac{1}{n+1} (z - z_0)^{n+1} \right)' &= (z - z_0)^n \\ \Rightarrow \int_C (z - z_0)^n dz &= 0 \end{aligned}$$

If  $n < 0$  and  $n \neq -1$ ,

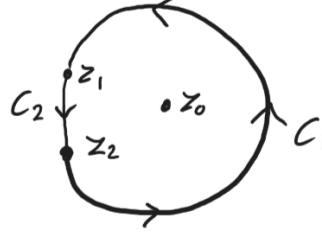
$$\begin{aligned} \left( \frac{1}{n+1} (z - z_0)^{n+1} \right)' &= (z - z_0)^n \\ \Rightarrow \int_C (z - z_0)^n dz &= 0 \end{aligned}$$

If  $n = -1$ ,

$$\int_C \frac{dz}{z - z_0}$$

Let us assume that  $z_0 = 0$  by shifting everything over by  $z_0$ .

$C = C_1 \cup C_2$  where  $C_1$  is the big loop from  $z_1$  to  $z_2$  and  $C_2$  is the small line segment from  $z_2$  to  $z_1$ .



$$\int_{C_1} \frac{dz}{z} = \text{Log}(z_2) - \text{Log}(z_1)$$

$$\left| \int_{C_2} \frac{dz}{z} \right| \leq \max_{z \in C_2} \frac{1}{|z|} \cdot \text{Length}(C_2) \rightarrow 0 \text{ as } z_1 = z_2$$

$$\text{Log } z_2 = \ln |z_2| + i\varphi_2 \rightarrow \ln |z_0| + i\pi$$

$$\text{Log } z_1 = \ln |z_1| + i\varphi_1 \rightarrow \ln |z_0| - i\pi$$

$$\int_C \frac{dz}{z} = \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} \rightarrow 2i\pi + 0$$

$$\int_C \frac{dz}{z} = 2\pi i$$

And so we get

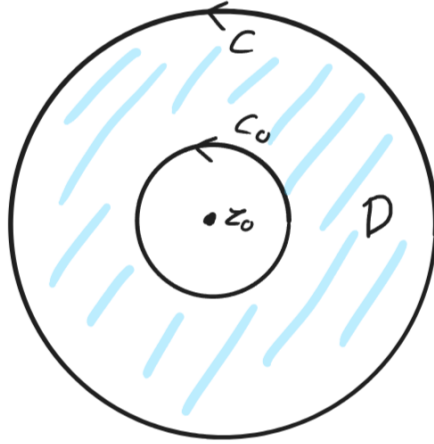
$$\int_C (z - z_0)^n dz = \begin{cases} 0 & \text{if } z_0 \text{ is outside } C \\ 2\pi i & \text{if } z_0 \text{ is inside } C \text{ and } n = -1 \\ 0 & \text{if } z_0 \text{ is inside } C \text{ and } n \neq -1 \end{cases}$$

This means that the result of a loop integral will only be nonzero if there is some singularity inside the loop.

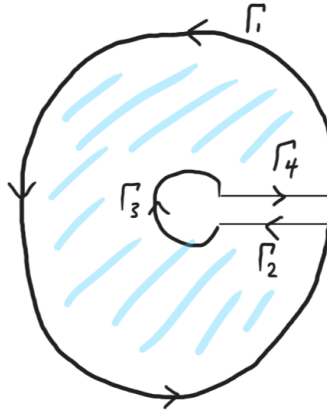
### 1.3.4 Deformation of Path

Suppose  $f(z)$  is analytic in  $D$  then we can show

$$\int_C f(z) dz = \int_{C_0} f(z) dz$$



To show this, we will need a simply connected domain. We can construct the following contour:



We can define a new path,  $C_\epsilon = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ .

Assuming that the function is well defined in the region  $D$  then we get

$$\int_{C_\epsilon} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz + \int_{\Gamma_3} f(z)dz + \int_{\Gamma_4} f(z)dz = 0$$

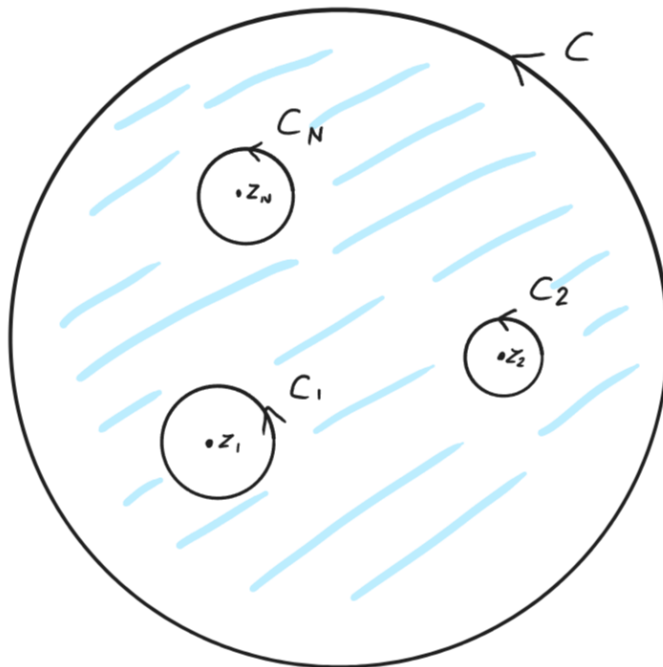
In the limiting case we can see that

$$\begin{aligned} \int_{\Gamma_1} f(z)dz &\rightarrow \int_C f(z)dz \\ \int_{\Gamma_3} f(z)dz &\rightarrow - \int_{C_0} f(z)dz \\ \int_{\Gamma_2} f(z)dz + \int_{\Gamma_4} f(z)dz &\rightarrow 0 \end{aligned}$$

And so we are left with

$$\begin{aligned} \int_{C_\epsilon} f(z)dz &= \int_C f(z)dz - \int_{C_0} f(z)dz = 0 \\ \Rightarrow \int_C f(z)dz &= \int_{C_0} f(z)dz \end{aligned}$$

This means that any integral over a closed path is equal to the sum of closed path integrals around any singularities in the domain.



$$\int_C f(z)dz = \sum_{j=1}^N \int_{C_j} f(z)dz$$

Ex:

$$\int_{|z|=2} \frac{dz}{z(z-1)}$$

$$C_1 : |z-0|=r$$

$$C_2 : |z-1|=r, \text{ } r \text{ is small}$$

$$I = \int_{|z|=2} f(z)dz = \int_{|z-0|=r} \frac{dz}{z(z-1)} + \int_{|z-1|=r} \frac{dz}{z(z-1)}$$

$$\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1} = \frac{A(z-1) + Bz}{z(z-1)} = \frac{(A+B)z - A}{z(z-1)}$$

$$\begin{cases} A+B=0 \\ -A=1 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=1 \end{cases}$$

$$\int_{|z-0|=r} \frac{dz}{z(z-1)} = \int_{|z-0|=r} \left( -\frac{1}{z} + \frac{1}{z-1} \right) dz = - \int_{|z-0|=r} \frac{dz}{z} + \int_{|z-0|=r} \frac{dz}{z-1}$$

$$= - \int_{|z-0|=r} \frac{dz}{z} + 0 = -2\pi i$$

$$\int_{|z-1|=r} \frac{dz}{z(z-1)} = \int_{|z-1|=r} \left( -\frac{1}{z} + \frac{1}{z-1} \right) dz = 0 + \int_{|z-1|=r} \frac{dz}{z-1} = 0 + 2\pi i$$

$$I = -2\pi i + 2\pi i = 0$$



### 1.3.5 Cauchy Integral Formula

Here we relied on using partial fractions. This is a useful technique, however, we are able to develop slightly more general techniques in defining the Cauchy Integral Formula.

If  $f(z)$  is analytic on  $C$  and inside  $C$  then

$$\int_C \frac{f(z)}{z - z_0} = 2\pi i f(z_0)$$

*Proof.*

$$g(z) = \frac{f(z)}{z - z_0}$$

is analytic except at  $z = z_0$ .

By Cauchy Therorm:

$$\begin{aligned} \int_C g(z) dz &= \int_{|z-z_0|=\epsilon} g(z) dz \\ \int_{|z-z_0|=\epsilon} \frac{f(z)}{z - z_0} dz &= \int_{|z-z_0|=\epsilon} \frac{f(z) - f(z_0)}{z - z_0} dz + \int_{|z-z_0|=\epsilon} \frac{f(z_0)}{z - z_0} dz \\ &= \int_{|z-z_0|=\epsilon} \frac{f(z) - f(z_0)}{z - z_0} dz + 2\pi i f(z_0) \\ \left| \int_{|z-z_0|=\epsilon} \frac{f(z) - f(z_0)}{z - z_0} dz \right| &\leq \max_{|z-z_0|=\epsilon} \left| \frac{f(z) - f(z_0)}{z - z_0} \right| 2\pi\epsilon = \max_{|z-z_0|=\epsilon} |f(z) - f(z_0)| \frac{1}{\epsilon} 2\pi\epsilon \\ \left| \int_{|z-z_0|=\epsilon} \frac{f(z) - f(z_0)}{z - z_0} dz \right| &\leq 2\pi \max_{|z-z_0|=\epsilon} |f(z) - f(z_0)| \rightarrow 0 \text{ as } \epsilon \rightarrow 0 \\ \lim_{\epsilon \rightarrow 0} \int_{|z-z_0|=\epsilon} \frac{f(z) - f(z_0)}{z - z_0} dz &= 0 \\ \Rightarrow \int_{|z-z_0|=\epsilon} \frac{f(z)}{z - z_0} dz &= 2\pi i f(z_0) \end{aligned}$$

□

Ex:

$$C : \frac{x^2}{4} + y^2 = 1$$

$$\int_C \frac{dz}{(z-1)^2}$$

$$z = 1$$

$$\int_C \frac{dz}{(z-1)^2} = \lim_{r \rightarrow 0} \int_{|z-1|=r} \frac{dz}{(z-1)^2} = -\frac{1}{z-1} \Big|_{|z-1|=r} = 0$$

Ex2:

$$C : \frac{x^2}{4} + y^2 = 1$$

$$\int_C \frac{e^z}{z(z-1)} dz$$

$$z = 0, 1$$

$$\int_C \frac{e^z}{z(z-1)} dz = \int_{|z|=r} \frac{e^z}{z(z-1)} dz + \int_{|z-1|=r} \frac{e^z}{z(z-1)} dz$$

$$\int_{|z|=r} \frac{e^z}{z(z-1)} dz = \int_{|z|=r} \frac{1}{z} f_1(z) dz, \quad f_1(z) = \frac{e^z}{z-1}$$

$$f_1(0) = \frac{e^0}{0-1} = -1$$

$$\int_{|z|=r} \frac{1}{z} f_1(z) dz = 2\pi i f_1(0) = -2\pi i$$

$$\int_{|z-1|=r} \frac{e^z}{z(z-1)} dz = \int_{|z-1|=r} \frac{1}{z-1} f_2(z) dz, \quad f_2(z) = \frac{e^z}{z}$$

$$f_2(1) = e$$

$$\int_{|z-1|=r} \frac{1}{z-1} f_2(z) dz = 2\pi i e$$

$$\int_C \frac{e^z}{z(z-1)} dz = -2\pi i + 2\pi i e = 2\pi i(e-1)$$

Ex3:

$$C : \frac{x^2}{4} + y^2 = 1$$

$$\int_C \frac{dz}{z(z^2-1)}$$

$$z = 0, \pm 1$$

$$\int_C \frac{dz}{z(z^2-1)} = \int_{|z|=r} \frac{1}{z} f_1(z) dz + \int_{|z+1|=r} \frac{1}{z+1} f_2(z) dz + \int_{|z-1|=r} \frac{1}{z-1} f_3(z) dz$$

$$f_1(z) = \frac{1}{z^2-1} \Rightarrow f_1(0) = -1$$

$$f_2(z) = \frac{1}{z(z-1)} \Rightarrow f_2(-1) = \frac{1}{-(-2)} = \frac{1}{2}$$

$$f_3(z) = \frac{1}{z(z+1)} \Rightarrow f_3(1) = \frac{1}{2}$$

$$\int_C \frac{dz}{z(z^2-1)} = -2\pi i + \frac{1}{2} 2\pi i + \frac{1}{2} 2\pi i = 0$$

Ex4:

$$C : \frac{x^2}{4} + y^2 = 1$$

$$\begin{aligned} & \int_C \frac{dz}{2z^2 + 1} \\ & z = \pm \frac{i}{\sqrt{2}} \\ & \int_C \frac{dz}{2z^2 + 1} = \int_{|z+\frac{i}{\sqrt{2}}|} \frac{1}{z + \frac{i}{\sqrt{2}}} f_1(z) dz + \int_{|z-\frac{i}{\sqrt{2}}|} \frac{1}{z - \frac{i}{\sqrt{2}}} f_2(z) dz \\ & f_1(z) = \frac{2}{z - \frac{i}{\sqrt{2}}} \Rightarrow f_1\left(-\frac{i}{\sqrt{2}}\right) = \frac{2}{-\frac{2i}{\sqrt{2}}} = \sqrt{2}i \\ & f_2(z) = \frac{2}{z + \frac{i}{\sqrt{2}}} \Rightarrow f_2\left(\frac{i}{\sqrt{2}}\right) = \frac{2}{\frac{2i}{\sqrt{2}}} = -\sqrt{2}i \\ & \int_C \frac{dz}{2z^2 + 1} = 2\pi i \sqrt{2}i - 2\pi i \sqrt{2}i = 0 \end{aligned}$$

Ex5:

$$\begin{aligned} & \int_{|z|=2} \frac{dz}{z^2 + 2z + 2} \\ & z = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i \\ & C_1 : \lim_{r \rightarrow 0} |z - (-1 + i)| = r \\ & C_2 : \lim_{r \rightarrow 0} |z - (-1 - i)| = r \\ & \int_{|z|=2} \frac{dz}{z^2 - 2z + 3} = \int_{C_1} \frac{1}{z - (-1 + i)} f_1(-1 + i) dz + \int_{C_2} \frac{1}{z - (-1 - i)} f_2(-1 - i) dz \\ & f_1(z) = \frac{1}{z - (-1 - i)} \Rightarrow f_1(-1 + i) = \frac{1}{2i} \\ & f_2(z) = \frac{1}{z - (-1 + i)} \Rightarrow f_2(-1 - i) = \frac{1}{-2i} \\ & \int_{|z|=2} \frac{dz}{z^2 + 2z + 2} = 2\pi i \frac{1}{2i} - 2\pi i \frac{1}{2i} = 0 \end{aligned}$$

Ex6:

$$\begin{aligned} & \int_{|z|=2} \frac{dz}{z^2 - 2z - 3} \\ & z^2 - 2z - 3 = (z - 3)(z + 1) \Rightarrow z = 3, -1 \\ & C_1 : \lim_{r \rightarrow 0} |z + 1| = r \\ & \int_{|z|=2} \frac{dz}{z^2 - 2z - 3} = \int_{C_1} \frac{1}{z + 1} f_1(-1) dz \end{aligned}$$

$$f_1(z) = \frac{1}{z-3} \Rightarrow f_1(-1) = -\frac{1}{4}$$

$$\int_{|z|=2} \frac{dz}{z^2 - 2z - 3} = 2\pi i \left(-\frac{1}{4}\right) = -\frac{\pi}{2}i$$

So, from deformation of path, we got the Formula

$$\int_C f(z)dz = \sum_{j=1}^N \int_{C_j} f(z)dz$$

where  $C$  is a closed path and  $C_j$  are closed paths around singularities.

Now, using the Cauchy Integral Formula, we can get the following formula, assuming that  $f(z) = \frac{1}{z-z_j} f_j(z)$  and  $f(z)$  has singularities  $z_1, \dots, z_m$

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^m f_j(z_j)$$

In some cases, it may be hard to find all the singularities. Take the example

$$\int_{|z|=2} \frac{z+1}{z^3+z+3} dz$$

Finding the roots of  $z^3 + z + 3$  is not easy, however, if we can show that all of the zeros are inside some region, then we can use the deformation of path and compute the integral

$$\lim_{R \rightarrow \infty} \int_{|z|=R} \frac{z+1}{z^3+z+3} dz$$

instead and it will give the same result.

Giving this a try we get

$$\text{assume } |z^3 + z + 3| > 0 \text{ on } |z| > 2$$

$$|z^3 + z + 3| \geq ||z|^3 - |z| - 3| > |8 - 2 - 3| = 3$$

$$\therefore \text{ all 0s inside region}$$

$$\int_{|z|=2} \frac{z+1}{z^3+z+3} dz = \lim_{R \rightarrow \infty} \int_{|z|=R} \frac{z+1}{z^3+z+3} dz$$

Now to compute the integral we can use the inequality formula

$$\left| \int_{|z|=R} \frac{z+1}{z^3+z+3} dz \right| \leq \max_{|z|=R} \left| \frac{z+1}{z^3+z+3} \right| 2\pi R = \frac{(R+1)2\pi R}{R^3 - R - 3} \rightarrow 0$$

$$\int_{|z|=2} \frac{z+1}{z^3+z+3} dz = 0$$