Math Notes

Tyler Wilson

Contents

1 Complex Analysis

1.1 Complex Algebra

Complex numbers arise from the roots of polynomials.
Ex: $x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm\sqrt{-1}$. This polynomial has no real roots, however, we can introduce an imaginary number i such that $i^2 = -1$. Then we will have the solution $x = \pm i$
We can introduce *complex numbers* which are numbers in the form $z = x + iy$, where x is the real part of z, $\Re(z)$, and y is the imaginary part of z, $\Im(z)$. These numbers can also be expressed in vector notation along the complex plane.

1.1.1 Complex Arithmetic

Addition, subtraction, and multiplication work the same, just with the addition of the fact $i^2 = -1$.
For division, we require what is called the conjugate.
The conjugate of a complex number is the same number, just with the sign of the imaginary component flipped.

$$\overline{z} = x - yi$$

where $\overline{z}$ is the conjugate of z.
Similarly to vectors, we can also define the modulus (length) of a complex number

$$|z|^2 = x^2 + y^2 = z \cdot \overline{z}$$

Using this, we can define the division of a complex number and also define the real and imaginary components of a complex number.
The general expression for division is:

$$\frac{u}{z} = \frac{s+it}{x+iy} = \frac{(s+it)(x-iy)}{(x+iy)(x-iy)} = \frac{u\overline{z}}{x^2+y^2} = \frac{u\overline{z}}{|z|^2}$$

The real and imaginary components can be computed as

$$\begin{aligned}
&z = x + iy \\
&\overline{z} = x - iy \\
&\Re(z) = \frac{z+\overline{z}}{2} \\
&\Im(z) = \frac{z-\overline{z}}{2i}
\end{aligned}$$

Ex1: Simplify $(1+2i)(3+i)(2-3i)$

$$\begin{aligned}
&(1+2i)(3+i)(2-3i) = (3+i+6i-2)(2-3i) = (1+7i)(2-3i) \\
&= 2 - 3i + 14i + 21 \\
&= 23 + 11i
\end{aligned}$$

Ex2: Simplify $\left(\frac{2+i}{1+i}\right)^2$

$$\begin{aligned}
&\left(\frac{2+i}{1+i}\right)^2 = \left(\frac{(2+i)(1-i)}{(1+i)(1-i)}\right)^2 = \left(\frac{2+i-2i+1}{2}\right)^2 = \left(\frac{3-i}{2}\right)^2 = \frac{9-6i-1}{4} \\
&= 2 - \frac{3}{2}i
\end{aligned}$$

Ex3: Simplify $(1+2i)^5$

$$\begin{aligned}
&(1+2i)^5 = 1^5 + 5(1)^4(2i) + 10(1)^3(2i)^2 + 10(1)^2(2i)^3 + 5(1)(2i)^4 + (2i)^5 \\
&= 1 + 10i + 10(-4) + 10(-8i) + 5(16) + 32i \\
&= 1 + 10i - 40 - 80i + 80 + 32i \\
&= 41 - 38i
\end{aligned}$$

Ex4: Prove

$$\text{if } |z| = 1 \text{ then } \Re\left(\frac{1}{1+z}\right) = \frac{1}{2}$$

Proof.

$$\Re(w) = \frac{w+\overline{w}}{2}$$
$$\overline{\frac{1}{1+z}} = \frac{1}{1+\overline{z}}$$
$$\Re\left(\frac{1}{1+z}\right) = \frac{\frac{1}{1+z}+\frac{1}{1+\overline{z}}}{2} = \frac{1+\overline{z}+1+z}{2(1+z)(1+\overline{z})} = \frac{2+z+\overline{z}}{2(1+z+\overline{z}+|z|^2)}$$
$$|z| = 1 \Rightarrow \Re\left(\frac{1}{1+z}\right) = \frac{2+z+\overline{z}}{2(2+z+\overline{z})} = \frac{1}{2}$$

□

Some properties of $\overline{z}$:

- $\overline{\overline{z}} = z$
- $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$
- $|z_1 z_2| = |z_1||z_2|$

Some common inequalities come from the triangle inequality:
$|z_1 + z_2| \leq |z_1| + |z_2|$

Proof.

$$|z_1 + z_2| \leq |z_1| + |z_2|$$
$$|z_1 + z_2| = \sqrt{(x_1+x_2)^2 + (y_1+y_2)^2}$$
$$\sqrt{(x_1+x_2)^2 + (y_1+y_2)^2} \leq \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2}$$
$$|z_1 + z_2| \leq |z_1| + |z_2|$$

□

$$\Rightarrow |z_1 \pm z_2| \geq ||z_1| - |z_2||$$

This leads to a general upper and lower bound that can be derived from these inequalities:

$$||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$$

1.1.2 Polar Form of Complex Numbers

Another way to represent complex numbers is through polar form. To use this we must first introduce Euler's identity:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

This then helps us with the equation for the polar form

$$z = a + ib = |z|(\cos\varphi + i\sin\varphi) = |z|e^{i\varphi}$$

For $z = re^{i\varphi}$ We call r the magnitude of the complex number and φ is the argument. This can be especially useful for simplifying some complex numbers.
Ex:

$$\left|\frac{(1+\sqrt{3}i)^{100}}{(\sqrt{3}-i)^{100}}\right| = \frac{|1+\sqrt{3}i|^{100}}{|\sqrt{3}-i|^{100}} = \frac{2^{100}}{2^{100}} = 1$$

Note that because sinusoidal functions are periodic every 2π this means that there infinite ways to express a function in polar coordinates.

$$e^{i2\pi k} = 1,\ k \in \mathbb{Z} \Rightarrow z = re^{i(\varphi+2\pi k)}$$

To get around the issue of having infinite possible polar forms for every complex number we define what's called the principal argument to be

$$\text{Arg}(z) = \varphi \in (-\pi, \pi]$$

We define the regular argument to be

$$\arg(z) = \text{Arg}(z) + 2k\pi,\ k \in \mathbb{Z}$$

Note that $\text{Arg}(z)$ is singular valued while $\arg(z)$ is multi-valued.
We can define $\text{Arg}(z)$ in terms of the real and imaginary parts (x and y) as

$$\text{Arg}(z) = \arctan(\tfrac{y}{x}) \pm k\pi$$

What it is specifically depends on what quadrant of the complex plane the point lies in.

- QI: $\text{Arg}(z) = \arctan(\frac{y}{x})$
- QII: $\text{Arg}(z) = \arctan(\frac{y}{x}) + \pi$
- QIII: $\text{Arg}(z) = \arctan(\frac{y}{x}) - \pi$
- QIV: $\text{Arg}(z) = \arctan(\frac{y}{x})$

Ex:

$$\text{Arg}(-1-\sqrt{3}i) = \arctan(\sqrt{3}) - \pi = -\frac{2\pi}{3}$$

Ex2:

$$\begin{aligned}&\arg(1-\sqrt{3}i)\\ &\text{Arg}(1-\sqrt{3}i) = -\frac{\pi}{3}\\ &\arg(1-\sqrt{3}) = -\frac{\pi}{3} + 2k\pi,\ k \in \mathbb{Z}\end{aligned}$$

Ex3:

$$\arg(-1+2i)$$

$$\mathrm{Arg}(-1+2i) = \arctan(-2) + \pi$$
$$\arg(-1+2i) = \pi - \arctan(2) + 2k\pi$$

Ex4: Simplify

$$z = -3+3i$$
$$\mathrm{Arg}(z) = \arctan\left(\frac{3}{-3}\right) + \pi = \frac{3\pi}{4}$$
$$|z| = 3\sqrt{2}$$
$$z = 3\sqrt{2}e^{\frac{3\pi}{4}i}$$

Ex5: Simplify

$$z = -3-3i$$
$$\mathrm{Arg}(z) = \arctan\left(\frac{-3}{-3}\right) - \pi = -\frac{3\pi}{4}$$
$$|z| = 3\sqrt{2}$$
$$z = 3\sqrt{2}e^{-\frac{3\pi}{4}i}$$

Ex6: Simplify

$$z = \frac{1-i}{-\sqrt{3}+i} = \frac{u}{v}$$
$$\arg(z) = \arg(u) - \arg(v) = \arctan\left(\frac{-1}{1}\right) - \left(\arctan\left(\frac{1}{-\sqrt{3}}\right) + \pi\right) + 2k\pi$$
$$= -\frac{\pi}{4} - \frac{5\pi}{6} + 2k\pi = -\frac{13\pi}{12} + 2k\pi$$
$$\mathrm{Arg}(z) = \frac{11\pi}{12}$$
$$|z| = \frac{|u|}{|v|} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$
$$z = \frac{e^{\frac{11\pi}{12}i}}{\sqrt{2}}$$

Ex7: Simplify

$$z = (\sqrt{3}-i)^2 = w^2$$
$$\arg(w) = \arctan\left(\frac{-1}{\sqrt{3}}\right) + 2k\pi = -\frac{\pi}{6}$$
$$\Rightarrow \arg(z) = 2\arg(w) = -\frac{\pi}{3} + 4k\pi$$
$$\mathrm{Arg}(z) = -\frac{\pi}{3}$$
$$|w| = 2 \Rightarrow |z| = |w|^2 = 4$$
$$z = 4e^{-\frac{\pi}{3}i}$$

Ex8: Solve for all values of z

$$
\begin{aligned}
&e^z = -1 - \sqrt{3}i \\
&e^z = 2e^{-i\frac{2\pi}{3}+2\pi ik} = e^{\ln 2 - i\frac{2\pi}{3}+2\pi ik} \\
&z = \ln 2 - i\frac{2\pi}{3} + 2\pi ik,\ \forall k \in \mathbb{Z}
\end{aligned}
$$

Properties of $\operatorname{Arg}(z)$ and $\arg(z)$

- $\operatorname{Arg}(z_1 z_2) \neq \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$

  Proof. Proof by contradiction: assume that $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$ is true. Take $z_1 = z_2 = -1$.

  $$
  \begin{aligned}
  &\operatorname{Arg}(z_1) = \operatorname{Arg}(z_2) = \pi \\
  &\Rightarrow \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = 2\pi \\
  &\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(1) = 0 \\
  &\operatorname{Arg}(z_1 z_2) \neq \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)\ \forall z_1, z_2 \neq 0 \in \mathbb{C}
  \end{aligned}
  $$

  $\square$

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\operatorname{Arg}(\overline{z}) \neq -\operatorname{Arg}(z)$

  Proof. Proof by contradiction: assume that $\operatorname{Arg}(\overline{z}) = -\operatorname{Arg}(z)$ is true. Take $z = -1$

  $$
  \begin{aligned}
  &\overline{z} = z = -1 \\
  &\operatorname{Arg}(z) = \pi \\
  &\operatorname{Arg}(\overline{z}) = \pi \\
  &-\operatorname{Arg}(z) = -\pi \\
  &\Rightarrow \operatorname{Arg}(\overline{z}) \neq -\operatorname{Arg}(z)\ \forall z \in \mathbb{C}
  \end{aligned}
  $$

  $\square$

- $\arg(z) = -\arg(\overline{z})$

  Proof.

  $$
  \begin{aligned}
  &z = |z|e^{i\arg(z)}\ \forall z \in \mathbb{C} \\
  &\overline{z} = |z|e^{-i\arg(z)} \\
  &\Rightarrow \arg(\overline{z}) = -\arg(z)
  \end{aligned}
  $$

  $\square$

1.1.3 De Moirre's Formula

Using Euler's identity we can derive a powerful formula called De Moirre's Formula as follows:

$$
\begin{aligned}
&e^{iN\varphi} = \cos(N\varphi) + i\sin(N\varphi) \\
&e^{iN\varphi} = (e^{i\varphi})^N = (\cos\varphi + i\sin\varphi)^N \\
&(\cos\varphi + i\sin\varphi)^N = \cos(N\varphi) + i\sin(N\varphi)
\end{aligned}
$$

Applications of De Moirre's Formula:
Binomial expansion:

$$
\begin{aligned}
&N = 2: \\
&\cos(2\theta) = \cos^2\theta - \sin^2\theta \\
&\sin(2\theta) = 2\cos\theta\sin\theta \\
&N = 3: \\
&(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 \\
&\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta \\
&\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta
\end{aligned}
$$

Ex: Prove

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$$

Proof. Using De Moivre's formula with $N = 3$

$$
\begin{aligned}
&(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta) \\
&(\cos\theta)^3 + 3(\cos\theta)^2(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta) \\
&\cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta) = \cos(3\theta) + i\sin(3\theta) \\
&\Im\left\{\cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)\right\} = \Im\left\{\cos(3\theta) + i\sin(3\theta)\right\} \\
&3\cos^2\theta\sin\theta - \sin^3\theta = \sin(3\theta) \\
&\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \sin^2\theta \\
&3(1 - \sin^2\theta)\sin\theta - \sin^3\theta = \sin(3\theta) \\
&3\sin\theta - 3\sin^3\theta - \sin^3\theta = \sin(3\theta) \\
&3\sin\theta - 4\sin^3\theta = \sin(3\theta)
\end{aligned}
$$

□

Computing trigonometric integrals:
Ex:

$$
\begin{aligned}
&\int_0^{2\pi} \cos^8\varphi d\varphi \\
&e^{i\varphi} = \cos\varphi + i\sin\varphi \\
&e^{-i\varphi} = \cos\varphi - i\sin\varphi
\end{aligned}
$$

$$\cos\varphi = \frac{e^{i\varphi}+e^{-i\varphi}}{2}$$
$$\sin\varphi = \frac{e^{i\varphi}-e^{-i\varphi}}{2i}$$
$$\int_0^{2\pi}\left(\frac{e^{i\varphi}+e^{-i\varphi}}{2}\right)^8 d\varphi = \frac{1}{2^8}\int_0^{2\pi}\left(e^{i\varphi}+e^{-i\varphi}\right)^8 d\varphi$$
$$= \frac{1}{2^8}\int_0^{2\pi}\left(e^{i8\varphi}+{}_1C_8 e^{i7\varphi}e^{-i\varphi}+\cdots+{}_7C_8 e^{i\varphi}e^{-i7\varphi}+e^{-i8\varphi}\right)d\varphi$$
$$= \frac{1}{2^8}\left(0+\cdots+{}_4C_8 2\pi+\cdots+0\right) = \frac{{}_4C_8}{2^7}\pi$$

Ex2:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$
$$\int_0^{2\pi}\cos^6\theta d\theta = \int_0^{2\pi}\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)$$
$$= \frac{1}{2^6}\int_0^{2\pi}\sum_{k=0}^{6}\binom{k}{6}e^{i\theta k}e^{-i\theta(6-k)}d\theta$$
$$= \frac{1}{2^6}\sum_{k=0}^{6}\binom{k}{6}\int_0^{2\pi}e^{i\theta(2k-6)}d\theta$$
$$\int_0^{2\pi}e^{ik\theta}d\theta = \left.\frac{e^{ik\theta}}{ik}\right|_0^{2\pi} = \frac{e^{2\pi ik}-1}{ik}$$
$$e^{2\pi ik}=1,\ k\in\mathbb{Z}\Rightarrow\int_0^{2\pi}e^{ik\theta}d\theta = 0,\ k\neq 0\in\mathbb{Z}$$
$$\Rightarrow\int_0^{2\pi}\cos^6\theta d\theta = \frac{1}{2^6}\binom{3}{6}\int_0^{2\pi}d\theta = \frac{(20)(2\pi)}{2^6} = \frac{5\pi}{8}$$

Ex3:

$$\int_0^{2\pi}\sin^6(2\theta)d\theta$$
$$\sin(2\theta) = \frac{e^{2i\theta}-e^{-2i\theta}}{2i}$$
$$\int_0^{2\pi}\sin^6(2\theta)d\theta = \int_0^{2\pi}\left(\frac{e^{2i\theta}-e^{-2i\theta}}{2i}\right)^6 d\theta$$
$$= \frac{1}{(2i)^6}\int_0^{2\pi}\sum_{k=0}^{6}\binom{k}{6}(-1)^{6-k}e^{2ik\theta}e^{-2i\theta(6-k)}d\theta$$
$$= -\frac{1}{2^6}\sum_{k=0}^{6}\binom{k}{6}(-1)^{6-k}\int_0^{2\pi}e^{i\theta(4k-12)}d\theta$$
$$\int_0^{2\pi}e^{ik\theta}d\theta = \left.\frac{e^{ik\theta}}{ik}\right|_0^{2\pi} = \frac{e^{2\pi ik}-1}{ik}$$

$$e^{2\pi ik} = 1,\ k \in \mathbb{Z} \Rightarrow \int_0^{2\pi} e^{ik\theta} d\theta = 0,\ k \neq 0 \in \mathbb{Z}$$
$$\Rightarrow \int_0^{2\pi} \sin^6(2\theta) d\theta = -\frac{1}{2^6}\binom{3}{6}(-1)^3 \int_0^{2\pi} d\theta$$
$$= \frac{5\pi}{8}$$

Ex4: Prove

$$\sum_{k=0}^{n} \cos(k\theta) = \frac{1}{2} + \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}$$

Proof. De Moivre's formula states

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

If we take the conjugate of both sides we get

$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$$

Summing these two equations gives

$$(\cos\theta + i\sin\theta)^k + (\cos\theta - i\sin\theta)^k = 2\cos(k\theta)$$
$$\cos\theta \pm i\sin\theta = e^{\pm i\theta}$$
$$2\cos(k\theta) = (e^{i\theta})^k + (e^{-i\theta})^k$$

We can sum both sides of this to get

$$2\sum_{k=0}^{n} \cos(k\theta) = \sum_{k=0}^{n}(e^{i\theta})^k + \sum_{k=0}^{n}(e^{-i\theta})^k$$

The formula for the geometric sum is

$$\sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}$$

Applying this we get

$$2\sum_{k=0}^{n} \cos(k\theta) = \frac{1-(e^{i\theta})^{n+1}}{1-e^{i\theta}} + \frac{1-(e^{-i\theta})^{n+1}}{1-e^{-i\theta}}$$
$$2\sum_{k=0}^{n} \cos(k\theta) = \frac{(1-e^{i\theta(n+1)})(1-e^{-i\theta}) + (1-e^{-i\theta(n+1)})(1-e^{i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})}$$
$$2\sum_{k=0}^{n} \cos(k\theta) = \frac{1-e^{i\theta(n+1)} - e^{i\theta} + e^{i\theta n} + 1 - e^{-i\theta(n+1)} - e^{-i\theta} + e^{-i\theta n}}{1-e^{i\theta}-e^{-i\theta}+1}$$
$$2\sum_{k=0}^{n} \cos(k\theta) = 1 + \frac{e^{i\theta n} + e^{-i\theta n} - e^{i\theta(n+1)} - e^{-i\theta(n+1)}}{2-e^{i\theta}-e^{-i\theta}}$$

$$2\sum_{k=0}^{n}\cos(k\theta) = 1 + \frac{e^{i\frac{\theta}{2}}\left(e^{-i\theta(n+\frac{1}{2})} - e^{i\theta(n+\frac{1}{2})}\right) + e^{-i\frac{\theta}{2}}\left(e^{i\theta(n+\frac{1}{2})} - e^{i\theta(n+\frac{1}{2})}\right)}{e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}\right) + e^{-i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}\right)}$$

$$2\sum_{k=0}^{n}\cos(k\theta) = 1 + \frac{\left(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}\right)\left(e^{i\theta(n+\frac{1}{2})} - e^{-i\theta(n+\frac{1}{2})}\right)}{-\left(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}}\right)^2}$$

$$2\sum_{k=0}^{n}\cos(k\theta) = 1 + \frac{e^{i\theta(n+\frac{1}{2})} - e^{-i\theta(n+\frac{1}{2})}}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}}$$

$$2\sum_{k=0}^{n}\cos(k\theta) = 1 + \frac{\frac{1}{2i}\left(e^{i\theta(n+\frac{1}{2})} - e^{-i\theta(n+\frac{1}{2})}\right)}{\frac{1}{2i}\left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}\right)}$$

$$\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$$

$$2\sum_{k=0}^{n}\cos(k\theta) = 1 + \frac{\sin\left(\theta(n+\frac{1}{2})\right)}{\sin\left(\frac{\theta}{2}\right)}$$

$$\sum_{k=0}^{n}\cos(k\theta) = \frac{1}{2} + \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}$$

$\square$

1.1.4 Geometry in the Complex Plane

Using the notation such that $z = x + iy$ where $\Re(z) = x$ and $\Im(z) = y$ we can define a circle in the complex plane as

$$(x - x_0)^2 + (y - y_0)^2 = r_0^2$$

This is analogous to writing

$$|z - z_0| = r_0$$

The two can be related as follows:

$$\begin{aligned}
|z - z_0| &= r_0 \\
|x + iy - x_0 - iy_0| &= r_0 \\
\sqrt{(x - x_0)^2 + (y - y_0)^2} &= r_0 \\
(x - x_0)^2 + (y - y_0)^2 &= r_0^2
\end{aligned}$$

Ex: describe the circle formed by $2|z| = |z + 1|$

$$\begin{aligned}
2|z| &= |z + 1| \\
4|z|^2 &= |z + 1|^2 \\
4x^2 + 4y^2 &= (x + 1)^2 + y^2
\end{aligned}$$

$$
\begin{aligned}
&4x^2+4y^2=x^2+2x+1+y^2\\
&3x^2+3y^2-2x-1=0\\
&3x^2-2x+\frac{1}{3}+3y^2-\frac{4}{3}=0\\
&3\left(x-\frac{1}{3}\right)^2+3y^2=\frac{4}{3}\\
&\left(x-\frac{1}{3}\right)^2+y^2=\frac{4}{9}
\end{aligned}
$$

A line in the complex plane can be written as

$$
ax+by=c\longleftrightarrow a\frac{z+\overline{z}}{2}+b\frac{z-\overline{z}}{2i}=c
$$

Ex: describe the line formed by $|z-1+i|=|z-2i|$

$$
\begin{aligned}
&|z-1+i|=|z-2i|\\
&|z-1+i|^2=|z-2i|^2\\
&(x-1)^2+(y+1)^2=x^2+(y-2)^2\\
&x^2-2x+1+y^2+2y+1=x^2+y^2-4y+4\\
&-2x+2+2y=-4y+4\\
&6y=2x+2\\
&y=\frac{1}{3}(x+1)
\end{aligned}
$$

We can define an ellipse in the complex plane as

$$
\frac{x^2}{a^2}+\frac{y^2}{b^2}=1
$$

If we have $a>b$ then we will have a horizontal ellipse and if $b>a$ then we will have a vertical ellipse.
Assuming that $a>b$ then we can define the foci points to be at

$$
\begin{aligned}
&+F=(\sqrt{a^2-b^2},0)\\
&-F=(-\sqrt{a^2-b^2},0)
\end{aligned}
$$

The equation of an ellipse can also be described by

$$
|z-F|+|z+F|=2a
$$

Ex: describe the ellipse formed by $|z-1|+|z+1|=4$

$$
\begin{aligned}
&|z-1|+|z+1|=4\\
&|z-F|+|z+F|=2a\Rightarrow 2a=4\Rightarrow a=2\\
&F=\sqrt{a^2-b^2}=1\Rightarrow 1=4-b^2\Rightarrow b^2=3
\end{aligned}
$$

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

Ex2: describe the ellipse formed by $|z-1| + |z+3| = 6$

$$|z-1| + |z+3| = 6$$

note that this ellipse is not centered at the origin so we need to shift it

$$|z+1-2| + |z+1+2| = 6$$

$$2a = 6 \Rightarrow a = 3$$

$$F = 2 = \sqrt{a^2 - b^2} \Rightarrow 4 = 9 - b^2 \Rightarrow b^2 = 5$$

$$\frac{(x+1)^2}{9} + \frac{y^2}{5} = 1$$

1.1.5 Roots of a Complex Number

Given $z_0 = r_0 e^{i\varphi_0}$, what is $z_0^{\frac{1}{n}}$?
If we let $w = z_0^{\frac{1}{n}}$ then $w^n = z_0$

$$w = re^{i\varphi},\ w_n = r^n e^{in\varphi}$$

$$w^n = z_0 \Rightarrow r^n e^{in\varphi} = r_0 e^{i\varphi_0}$$

$$\Rightarrow r^n = r_0 \Rightarrow r_0^{\frac{1}{n}}$$

$$e^{in\varphi} = e^{i\varphi_0} \Rightarrow n\varphi = \varphi_0 + 2k\pi$$

$$\varphi = \frac{\varphi}{n} + \frac{2k\pi}{n}$$

So all solutions to $w^n = z_0$ are given by

$$w = r_0^{\frac{1}{n}} e^{i\left(\frac{\varphi_0}{n} + \frac{2k\pi}{n}\right)},\ k \in \mathbb{Z}$$

If we normalize $\varphi_0 = \mathrm{Arg}(z_0)$ then k will be in the range $k \in \{0, 1, \ldots, n-1\}$.
Note that the expression

$$w = r_0^{\frac{1}{n}} e^{i\left(\frac{\varphi_0}{n} + \frac{2k\pi}{n}\right)},\ k \in \mathbb{Z}$$

is multi-valued. If we want to avoid this, we can take what's called the principal value which is the value when $k = 0$:

$$z_0^{\frac{1}{n}} = r_0^{\frac{1}{n}} e^{i\frac{\varphi_0}{n}}$$

Ex: Compute $(-1)^{\frac{1}{2}}$

$$z_0 = -1 = 1^{\frac{1}{2}} e^{i\left(\frac{\pi}{2} + \frac{2k\pi}{2}\right)} = e^{i\left(\frac{\pi}{2} + k\pi\right)} = \left\{\ldots, e^{-i\frac{3\pi}{2}}, e^{-i\frac{\pi}{2}}, e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{2}}, e^{i\frac{5\pi}{2}}, \ldots\right\}$$

note that there are only 2 unique values

$$(-1)^{\frac{1}{2}} = e^{i\left(\frac{\pi}{2} + k\pi\right)},\ k \in \{0, 1\}$$

The principal value (when $k=0$) of this equation works out to be i.
Ex2: Find all solutions to

$$z^7 = i-1$$
$$z^7 = \sqrt{2}e^{i(\frac{3\pi}{4}+2\pi k)}$$
$$z = 2^{1/14}e^{i(\frac{3\pi}{28}+\frac{2\pi}{7}k)},\ k\in\{0,1,2,3,4,5,6\}$$

Ex3: Find all solutions to

$$z^5 = \frac{2i}{-1-\sqrt{3}i}$$
$$z^5 = \frac{2e^{i\frac{\pi}{2}}}{2e^{-i\frac{2\pi}{3}}} = e^{i\frac{7\pi}{6}} = e^{-i(\frac{5\pi}{6}+2\pi k)}$$
$$z = e^{-i(\frac{\pi}{6}+\frac{2\pi}{5}k)},\ k\in\{0,1,2,3,4\}$$

Ex4: Find all solutions to

$$\left(\frac{z}{z+1}\right)^2 = i$$
$$\left(\frac{z}{z+1}\right)^2 = e^{i(\frac{\pi}{2}+2\pi k)}$$
$$\frac{z}{z+1} = e^{i(\frac{\pi}{4}+\pi k)},\ k\in\{0,1\}$$
$$e^{i(\frac{\pi}{4}+\pi k)} = \left\{\frac{1}{\sqrt{2}}(1+i), \frac{1}{\sqrt{2}}(-1-i)\right\}$$
$$z = e^{i(\frac{\pi}{4}+k\pi)}(z+1)$$
$$z = \frac{e^{i(\frac{\pi}{4}+k\pi)}}{1-e^{i(\frac{\pi}{4}+k\pi)}} = \left\{\frac{1+i}{\sqrt{2}-1-i}, \frac{-1-i}{\sqrt{2}+1+i}\right\}$$

Ex5: Find all solutions to

$$z^2+4iz+1=0$$
$$(z^2+4iz-4)+4+1=0$$
$$(z+2i)^2+5=0$$
$$(z+2i)^2=-5=5e^{i(\pi+2\pi k)}$$
$$z+2i=\sqrt{5}e^{i(\frac{\pi}{2}+\pi k)} = \left\{\sqrt{5}i, -\sqrt{5}i\right\}$$
$$z = \left\{(\sqrt{5}-2)i, -(\sqrt{5}+2)i\right\}$$

Ex6: Find all solutions to $(z+1)^4 = (1-i)z^4$

$$(1-i)z^4 = \sqrt{2}e^{-i\frac{\pi}{4}+2\pi k i}z^4$$
$$z+1 = 2^{1/8}e^{-i\frac{\pi}{16}+i\frac{\pi k}{2}}z$$
$$z = \frac{-2^{1/8}}{1-e^{-i\frac{\pi}{16}+i\frac{\pi k}{2}}},\ k\in\{0,1,2,3\}$$

1.2 Complex Functions

1.2.1 Mapping Properties of Simple Functions

Similar to how functions with real variables map values to a different set of values, complex functions do the same. The main difference is that with complex functions we're mapping a 2 dimensional set of inputs to a 2 dimensional set of outputs.

$$w = f(z) = u + iv$$

We define $z \in \mathcal{S}$ the image of $\mathcal{S}$ under w.
Some common mappings:

- The identity map

$$w = f(z) = z$$
$$\begin{cases} u = x \\ v = y \end{cases}$$

- Translation by z_0

$$w = f(z) = z + z_0$$
$$\begin{cases} u = x + x_0 \\ v = y + y_0 \end{cases}$$

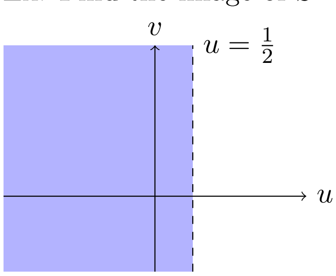
- Stretching ($a > 1$) or contraction ($a < 1$)

$$w = f(z) = az = are^{i\varphi},\ a \in \mathbb{R}$$
$$\begin{cases} u = ax \\ v = ay \end{cases}$$

- Rotation by φ_0

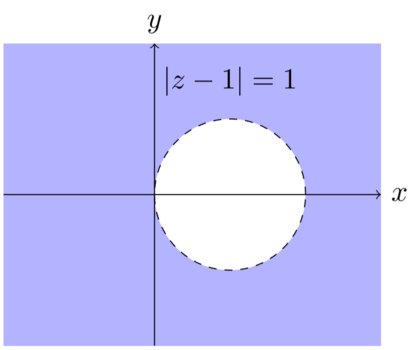
$$w = f(z) = e^{i\varphi_0} z = e^{i(\varphi + \varphi_0)}$$

Using these basic mapping principles we are able to lay the foundation for some more complicated mappings.
Ex: Find the image of $S = \{|z - 1| \geq 1\}$ under the mapping $f(z) = \frac{1}{z}$

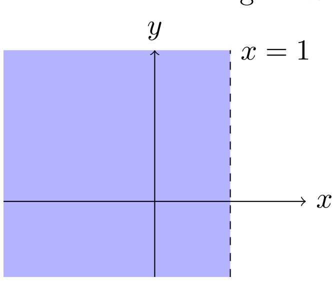


$$z = \frac{1}{w} = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

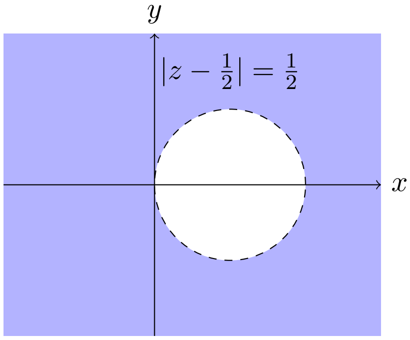
$$x = \frac{u}{u^2+v^2}$$
$$y = -\frac{v}{u^2+v^2}$$
$$|z-1| \geq 1 \Rightarrow \left|\frac{1}{w}-1\right| \geq 1$$
$$\frac{1-w}{w} \geq 1$$
$$|1-w| \geq |w|$$
$$|1-w|^2 \geq |w|^2$$
$$(1-u)^2+v^2 \geq u^2+v^2$$
$$-2u+1 \geq 0$$
$$u \leq \frac{1}{2}$$
$$S' = \left\{u \leq \frac{1}{2}\right\}$$



Ex2: Find the image of $S = \{x \leq 1\}$ under the mapping $f(z) = \frac{1}{z}$



$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$
$$x = \frac{u}{u^2+v^2}$$
$$x \leq 1 \Rightarrow \frac{u}{u^2+v^2} \leq 1 \Rightarrow u^2+v^2 \geq u$$
$$(u-\tfrac{1}{2})^2+v^2 \geq \tfrac{1}{4}$$



We see from the previous two examples that circles map to lines and lines map to circles. Let's see why this is the case.

$$a(x^2+y^2)+bx+cy+d=0$$
$$a|z|^2+b\frac{z+\bar{z}}{2}+c\frac{z-\bar{z}}{2i}+d=0$$

In the case where $a=0$, we have a line. In the case where $a\neq 0$, we have a circle.

$$w=\frac{1}{z}\Rightarrow z=\frac{1}{w}$$
$$z\bar{z}=|z|^2=\frac{1}{|w|^2}=\frac{1}{w\bar{w}}$$
$$a\frac{1}{w\bar{w}}+b\frac{z+\bar{z}}{2}+c\frac{z-\bar{z}}{2i}+d=0$$
$$a\frac{1}{w\bar{w}}+\frac{b}{2}\left(\frac{1}{w}+\frac{1}{\bar{w}}\right)+\frac{c}{2i}\left(\frac{1}{w}-\frac{1}{\bar{w}}\right)+d=0$$
$$a+\frac{b}{2}\left(w+\bar{w}\right)+\frac{c}{2i}\left(w-\bar{w}\right)+d\left(w\bar{w}\right)=0$$

If we have a linear transformation of the form $az+b$ it corresponds to the scaling and translation of the set only. A line will map to a line and a circle will map to a circle.
We can combine this with the $w=\frac{1}{z}$ transformation property to get a more general transformation. We call this the *Mobius transformation*:

$$f(z)=\frac{az+b}{cz+d}=\frac{a}{c}+\frac{b-\frac{ad}{c}}{cz+d}$$

where $a,b,c,d\in\mathbb{C}$
Ex: Find the mapping of $f(z)=\frac{1}{z+1}$ on the set $S=\{\Re(z)>0\}$

$$u+iv=\frac{1}{x+1+iy}\Rightarrow x+1+iy=\frac{1}{u+iv}$$
$$x+1=\frac{u}{u^2+v^2}$$
$$x>0\Rightarrow x+1>1$$
$$\frac{u}{u^2+v^2}>1\Rightarrow u>u^2+v^2$$

$$u^2 + v^2 - u + \frac{1}{4} < \frac{1}{4}$$
$$\left(u - \frac{1}{2}\right)^2 + v^2 < \frac{1}{4}$$
$$S' = \left\{ w = u + iv \middle| \left(u - \frac{1}{2}\right)^2 + v^2 < \left(\frac{1}{2}\right)^2 \right\}$$

Ex2: Find the mapping of $f(z) = \frac{z-i}{z+i}$ on $S = \{|z| < 3\}$

$$wz + iw = z - i \Rightarrow z(w-1) = -i - iw \Rightarrow z = \frac{i(w+1)}{1-w}$$
$$|z| = \frac{|w+1|}{|w-1|} < 3$$
$$|w+1| < 3|w-1| \Rightarrow |w+1|^2 < 9|w-1|^2$$
$$(u+1)^2 + v^2 < 9(u-1)^2 + 9v^2$$
$$u^2 + 2u + 1 + v^2 < 9u^2 - 18u + 9 + 9v^2$$
$$0 < 8u^2 - 20u + 8 + 8v^2 \Rightarrow 0 < u^2 - \frac{5}{2}u + 1 + v^2$$
$$\frac{9}{16} < u^2 - \frac{5}{2}u + \frac{25}{16} + v^2$$
$$\frac{9}{16} < \left(u - \frac{5}{4}\right)^2 + v^2$$
$$S' = \left\{ w = u + iv \middle| \left(u - \frac{5}{4}\right)^2 + v^2 > \left(\frac{3}{4}\right)^2 \right\}$$

Another common mapping is the $f(z) = z^2$ or more generally $f(z) = z^n$ mapping. For $w = z^2$,

$$w = z^2 = r^2 e^{2i\varphi} \Rightarrow \begin{cases} |w| = |z|^2 \\ \arg(w) = 2\arg(z) \end{cases}$$

This mapping scales the magnitude but more notably, it doubles the argument. This means that the mapping of a half circle will now be a full circle.

Ex: Find the mapping of $f(z) = z^2$ on $S = \{0 \le \Re(z) \le 1, \Im(z) = 1\}$

$$w = x^2 + i2xy - y^2$$
$$u = x^2 - y^2 = x^2 - 1 \Rightarrow -1 \le u \le 0$$
$$v = 2xy = 2x \Rightarrow 0 \le v \le 2$$
$$S' = \{w = u + iv | -1 \le u \le 0,\ 0 \le v \le 2\}$$

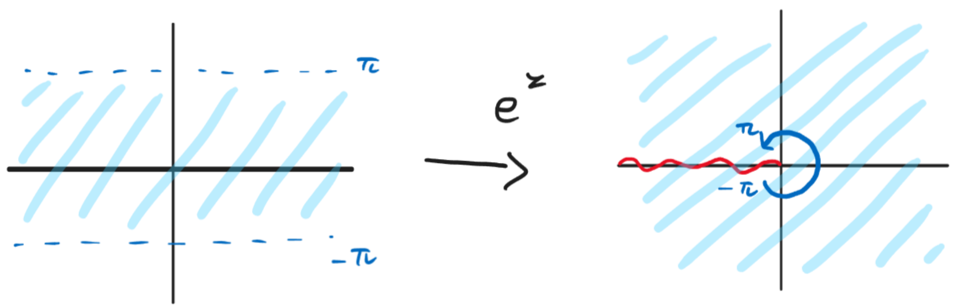
Ex2: Find the mapping of $f(z) = -2z^5$ on $S = \left\{|z| < 1, 0 < \text{Arg}(z) < \frac{\pi}{2}\right\}$

$$z^5 = -\frac{w}{2} \Rightarrow |z|^5 = \frac{|w|}{2} < 1 \Rightarrow |w| < 2$$

$$5\arg(z) = \arg(w) \pm \pi$$
$$0 < \arg(w) \pm \pi < \frac{5\pi}{2}$$
$$-\pi < \arg(w) < \frac{3\pi}{2}$$
$$S' = \{|w| < 2\}$$

Another common mapping is the $f(z) = e^z$ mapping.

$$w = e^z = e^{x+iy} = e^x e^{iy}$$
$$\begin{cases} |w| = e^x \\ \arg(w) = y \end{cases}$$



This mapping has the property that the magnitude is only dependent on x and the argument is exactly y.
Ex: Find the mapping of $f(z) = e^z$ on $S = \{\Re(z) = 1\}$

$$w = e^x e^{iy}$$
$$|w| = e,\ \arg(w) = y$$
$$S' = \{|w| = e\}$$

Ex2: Find the mapping of $f(z) = e^z$ on $S = \left\{0 \le \Im(z) \le \frac{\pi}{4}\right\}$

$$|w| = x$$
$$\arg(w) = y \Rightarrow 0 \le \arg(w) \le \frac{\pi}{4}$$
$$S' = \left\{0 \le \mathrm{Arg}(w) \le \frac{\pi}{4}\right\}$$

Ex3: Find the mapping of $f(z) = e^{iz}$ on $S = \left\{z : -\frac{\pi}{2} \le \Re(z) \le \pi, -1 \le \Im(z) \le 1\right\}$
(Note that multiplying z by i rotates it by 90°)

$$w = e^{iz} = e^{ix} e^{-y}$$
$$|w| = e^{-y} \Rightarrow e^{-1} \le |w| \le e$$
$$\arg(w) = x \Rightarrow -\frac{\pi}{2} \le \arg(w) \le \pi$$

$$S' = \left\{ w \middle| e^{-1} \le |w| \le e, -\frac{\pi}{2} \le \arg(w) \le \pi \right\}$$

Ex4: Prove

$$|e^{-z^3}| \le 1 \ \forall \left\{ z \middle| -\frac{\pi}{6} \le \operatorname{Arg}(z) \le \frac{\pi}{6} \right\}$$

Proof. We can express $-z^3$ as some complex number $a + ib$ where $a = \Re(-z^3)$ and $b = \Im(-z^3)$. Taking the magnitude gives

$$|e^{-z^3}| = |e^{a+ib}| = |e^a e^{ib}| = |e^a||e^{ib}| = |e^a| = |e^{\Re(-z^3)}|$$

z can be written as

$$\begin{aligned}
&z = |z|e^{i\operatorname{Arg}(z)} \\
&z^3 = |z|^3 e^{i3\operatorname{Arg}(z)} = |z|^3 \left(\cos(3\operatorname{Arg}(z)) + i\sin(3\operatorname{Arg}(z))\right) \\
&-z^3 = -|z|^3 \left(\cos(3\operatorname{Arg}(z)) + i\sin(3\operatorname{Arg}(z))\right) \\
&\Re(-z^3) = -|z|^3 \cos(3\operatorname{Arg}(z)) \\
&\operatorname{Arg}(z) \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right] \\
&\Rightarrow 3\operatorname{Arg}(z) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\
&\cos(3\operatorname{Arg}(z)) \in [0, 1] \\
&|z|^3 \in \{x \in \mathbb{R} | x \ge 0\} \\
&\Re(-z^3) = -|z|^3 \cos(3\operatorname{Arg}(z)) \in \{x \in \mathbb{R} | x \le 0\} \\
&e^{\Re(-z^3)} \in [0, 1] \\
&\Rightarrow |e^{-z^3}| \le 1
\end{aligned}$$

□

1.2.2 Calculus of Complex Functions

We define the limit of a complex function to be

$$\begin{aligned}
&w = f(z) = u + iv \\
&\lim_{z \to z_0} f(z) = \lim_{(x,y) \to (x_0,y_0)} u(x, y) + i \lim_{(x,y) \to (x_0,y_0)} v(x, y)
\end{aligned}$$

Note that the notation $(x, y) \to (x_0, y_0)$ means that the limit is taken as (x, y) approaches (x_0, y_0) along *any* path.

The usual limit arithmetic rules are able to be applied as with real numbers.

In order for $\lim_{z \to z_0} f(z)$ to exist, we require that both $\lim_{(x,y) \to (x_0,y_0)} u(x, y)$ and $\lim_{(x,y) \to (x_0,y_0)} v(x, y)$ exist.

If we define $z_0 = x_0 + iy_0$ then we can define the derivative of a complex function as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

If this limit exists then the function is said to be differentiable at z_0.
Ex: $f(z) = z$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{z_0 + \Delta z - z_0}{\Delta z} = 1$$
$$\Rightarrow f'(z_0) = 1$$

Ex2: $f(z) = \bar{z}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$
$$\Delta z = h_1 + ih_2 \Rightarrow \overline{\Delta z} = h_1 - ih_2$$
$$h_2 = 0: \quad \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{h_1 \to 0} \frac{h_1}{h_1} = 1$$
$$h_1 = 0: \quad \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{h_2 \to 0} \frac{-ih_2}{ih_2} = -1$$
$$\lim_{h_1 \to 0} \frac{h_1}{h_1} \neq \lim_{h_2 \to 0} \frac{-ih_2}{ih_2} \quad \therefore \text{ the derivative does not exist}$$

An easy way to determine if a function is differentiable is to use the Cauchy-Riemann equations. Any path that can be taken to approach z_0 can be written as a linear combination of the paths $\Delta z = \Delta x$ and $\Delta z = i\Delta y$ so the derivative must satisfy both of these paths.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$
$$f(z_0) = u(x_0, y_0) + iv(x_0, y_0)$$

Define $\Delta z = \Delta x$

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x}$$
$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$
$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

Define $\Delta z = i\Delta y$

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y}$$
$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y}$$
$$f'(z_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$$
$$\Rightarrow \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$$

Splitting the real and imaginary parts we get that the Cauchy-Riemann equations are

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

If the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous then the function is differentiable.
Some functions are not differentiable everywhere, but are differentiable at a point or a set of points.

- If $f(z)$ is differentiable everywhere in the complex plane then it is said to be **entire**.
- If $f(z)$ is differentiable in some region R then it is said to be **analytic** in R.
  (note that this region cannot be a single point, as the Cauchy-Riemann equations require the partial derivatives to be continuous)

Ex: Show using the Cauchy-Riemann equations that $f(z) = \bar{z}$ is not differentiable anywhere.

$$\overline{z} = x - iy$$
$$u_x = 1 \neq v_y = -1$$

Ex2: Show that $f(z) = z^2$ is entire

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$$
$$u(x, y) = x^2 - y^2$$
$$v(x, y) = 2xy$$
$$u_x = 2x = v_y$$
$$u_y = -2y = -v_x$$

Ex3: Show that $f(z) = \bar{z}$ is differentiable but not analytic at $z_0 = 0$

$$|z|^2 + 2z = x^2 + 2x + y^2 + i2y$$
$$u_x = 2x + 2 = v_y = 2 \Rightarrow x = 0$$
$$u_y = 2y = -v_x = 0 \Rightarrow y = 0$$
differentiable but not analytic on $z = \{0\}$

1.2.3 Conformal Mappings

Using the Cauch-Riemann equations,

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

we can get the Laplacian of u and v,

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$$
$$v_{xx} + v_{yy} = -(u_{yx} + u_{xy}) = 0$$

If the Laplacian of u and v are both zero then the function is said to be **harmonic**.
If $\nabla^2 u = 0$ then we can use the Cauchy-Riemann equations to find its harmonic conjugate v. Ex: Find the harmonic conjugate of $u = xy - x + y$

$$u_x = y - 1 = v_y$$

$$
\begin{aligned}
&v = \int y - 1dy = \frac{y^2}{2} - y + h(x) \\
&u_y = x + 1 = -v_x = -h'(x) \\
&h(x) = \int -x - 1dx = -\frac{x^2}{2} - x + C \\
&v = \frac{y^2}{2} - \frac{x^2}{2} - y - x + C
\end{aligned}
$$

Ex2: Find the harmonic conjugate of $u = \ln \sqrt{x^2 + y^2}$

$$
\begin{aligned}
&u_x = \frac{x}{x^2 + y^2} = v_y \\
&v = \int \frac{x}{x^2 + y^2} dy = \int \frac{1/x}{1 + \frac{y^2}{x^2}} dy = \arctan\left(\frac{y}{x}\right) + h(x) \\
&u_y = \frac{y}{x^2 + y^2} = -v_x = -\frac{y}{1 + \frac{y^2}{x^2}}\left(\frac{-1}{x^2}\right) - h'(x) = \frac{y}{x^2 + y^2} - h'(x) \Rightarrow h'(x) = 0 \\
&h(x) = C \\
&v = \arctan\left(\frac{y}{x}\right) + C \\
&v = \arg(z) + C
\end{aligned}
$$

Ex3: Find the harmonic conjugate of $u = \sin x \cosh y$

$$
\begin{aligned}
&u = \sin x \cosh y \\
&u_x = \cos x \cosh y = v_y \\
&v = \int \cos x \cosh y dy = \cos x \sinh y + h(x) \\
&u_y = \sin x \sinh y = -v_x = -(-\sin x \sinh y) - h'(x) \Rightarrow h'(x) = 0 \\
&h(x) = C \\
&v = \cos x \sinh y + C
\end{aligned}
$$

Another property is that

$$
|f'(z)|^2 = |\nabla u|^2 = |\nabla v|^2
$$

This also implies that

$$
\nabla u \cdot \nabla v = 0
$$

$$
\begin{aligned}
&f'(z) = u_x + iv_x = \frac{1}{i}(u_y + iv_y) \\
&|f'(z)|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2 = |\nabla u|^2 = |\nabla v|^2 \\
&\nabla u \cdot \nabla v = u_x u_y + v_x v_y = u_x v_x + (-v_x)(u_x) = 0
\end{aligned}
$$

A conformal mapping is a mapping between two regions that preserves angles. If we have some function $f(z) = u + iv$ that is analytic then the can create a function of a function as

$$
\Phi(u(x, y), v(x, y)) = \phi(x, y)
$$

where $\phi(x,y)$ is a conformal mapping.
A conformal mapping will have the property that

$$\phi_{xx} + \phi_{yy} = |f'(z)|^2(\Phi_{uu} + \Phi_{vv})$$

where $|f'(z)|^2$ is known as the *conformal factor*.
This can be shown as follows:

$$\begin{aligned}
&\phi_x = \Phi_u u_x + \Phi_v v_x \\
&\phi_{xx} = u_{xx}\Phi_u^2 + 2u_x v_x \Phi_u \Phi_v + v_{xx}\Phi_v^2 + \Phi_u u_{xx} + \Phi_v v_{xx} \\
&\phi_y = \Phi_u u_y + \Phi_v v_y \\
&\phi_{yy} = u_{yy}\Phi_u^2 + 2u_y v_y \Phi_u \Phi_v + v_{yy}\Phi_v^2 + \Phi_u u_{yy} + \Phi_v v_{yy} \\
&\phi_{xx} + \phi_{yy} = \Phi_u \nabla^2 u + \Phi_v \nabla^2 v + \Phi_{uu}|\nabla u|^2 + 2\Phi_u \Phi_v \nabla u \cdot \nabla v + \Phi_{vv}|\nabla v|^2 \\
&\phi_{xx} + \phi_{yy} = |f'(z)|^2(\Phi_{uu} + \Phi_{vv})
\end{aligned}$$

Ex: $f(z) = z^2$ under $\Phi(u,v) = e^u + v^2$

$$\begin{aligned}
&f(z) = x^2 - y^2 + i(2xy) \\
&u = x^2 - y^2 \\
&v = 2xy \\
&\phi(x,y) = e^{x^2-y^2} + (2xy)^2 \\
&\phi_{xx} + \phi_{yy} = 4|z|^2(e^u + 2)
\end{aligned}$$

$f(z)$ is considered a conformal mapping if f is analytic and $f'(z) \neq 0$. As a consequence, if $\phi_{xx} + \phi_{yy} = 0$ then $\Phi_{uu} + \Phi_{vv} = 0$.
If f is a conformal mapping then it will preserve angles.
If we have two curves C_1 and C_2 described by the parametrized functions $z_1(t)$ and $z_2(t)$ that intersect at a point z_0 then the angle between the two curves is given by $\theta = \arg(z_2'(t)) - \arg(z_1'(t))$.
If we then apply a conformal mapping $w = f(z)$ to the curves then we get $w_1(t) = f(z_1(t))$ and $w_2(t) = f(z_2(t))$ and the angle between the two curves is given by $\theta_w = \arg(w_2'(t)) - \arg(w_1'(t))$.

$$\begin{aligned}
&\theta_w = \arg(f'(z_2)z_2'(t)) - \arg(f'(z_1)z_1'(t)) \\
&\theta_w = \arg(f'(z_2)) - \arg(f'(z_1)) + \arg(z_2'(t)) - \arg(z_1'(t)) \\
&\arg(f'(z_1)) = \arg(f'(z_2)) \\
&\Rightarrow \theta_w = \arg(z_2'(t)) - \arg(z_1'(t)) = \theta
\end{aligned}$$

So the angle between the two curves is preserved under a conformal mapping.
Note that $f'(z) \neq 0$ is a necessary condition in this proof.
If we have a nonconformal mapping then the angle between the two curves will not be preserved.
One such case is that of $f(z) = z^2$ at $z_0 = 0$. At this point, $f'(z_0) = 0$ and the angle between the two curves is doubled.

Conformal mappings also have the property that they map Neumann boundary conditions to Neumann boundary conditions.

If we let $\hat{n}$ represent the normal vector to the curve $\phi(x, y)$ then they are related by

$$\frac{\partial \phi}{\partial n} = \nabla\phi \cdot \hat{n}$$

and the mapping is similarly related brcurly

$$\frac{\partial \Phi}{\partial n'} = \nabla\Phi \cdot \hat{n}'$$
$$\frac{\partial \phi}{\partial n} = |f'(z)|\frac{\partial \Phi}{\partial n'}$$

So for Neumann bounday conditions, we will have

$$\frac{\partial \phi}{\partial n} = 0 \Rightarrow \frac{\partial \Phi}{\partial n'} = 0$$

Some examples of conformal mappings come from harmonic functions (having the property that $\nabla^2\phi = 0$).
Some common harmonic functions are:

- $\phi = C$
- $\phi = ax + by + c$
- $\ln\sqrt{x^2 + y^2},\ \mathbb{C}\backslash 0$
- $\phi = \mathrm{Arg}(z),\ \mathbb{C}\backslash(-\infty]$
- $\phi = x^2 - y^2$

1.2.4 Conformal Mapping to Solve Laplace's Equation

Given the useful properties of matching boundary conditions, we can use conformal mappings to help us solve Laplace's equation for a given region.
Ex: Given $\nabla^2\phi = 0$ for $1 < x^2 - y^2 < 4$ and $\phi = 1$ on $x^2 - y^2 = 1$ and $\phi = 3$ on $x^2 - y^2 = 4$, find $\phi(x, y)$.

$$\text{choose } \phi = x^2 - y^2$$
$$\text{choose } \Phi(u, v) = Au + B$$
$$\begin{cases} A(1) + B = 1 \\ A(4) + B = 3 \end{cases} \Rightarrow A = \frac{2}{3},\ B = \frac{1}{3}$$
$$\Phi(u, v) = \frac{2}{3}u + \frac{1}{3}$$
$$u = x^2 - y^2$$
$$\phi(x, y) = \frac{2}{3}(x^2 - y^2) + \frac{1}{3}$$

Ex2: Given $\nabla^2\phi = 0$ within the circular region $\mathcal{D} = \{1 < x^2 + y^2 < 4\}$ with $\phi = 1$ on $x^2 + y^2 = 1$ and $\phi = -2$ on $x^2 + y^2 = 4$, find $\phi(x, y)$.

$$\phi(x, y) = A_1 \ln r + A_2$$

$$r=\sqrt{x^2+y^2}$$
$$r=1:\ \phi=1\Rightarrow A_1\ln(1)+A_2=1$$
$$r=2:\ \phi=-2\Rightarrow A_1\ln(2)+A_2=-2$$
$$\Rightarrow A_2=1,\ A_1=-\frac{3}{\ln 2}$$
$$\phi(x,y)=-\frac{3}{\ln 2}\ln\sqrt{x^2+y^2}+1$$

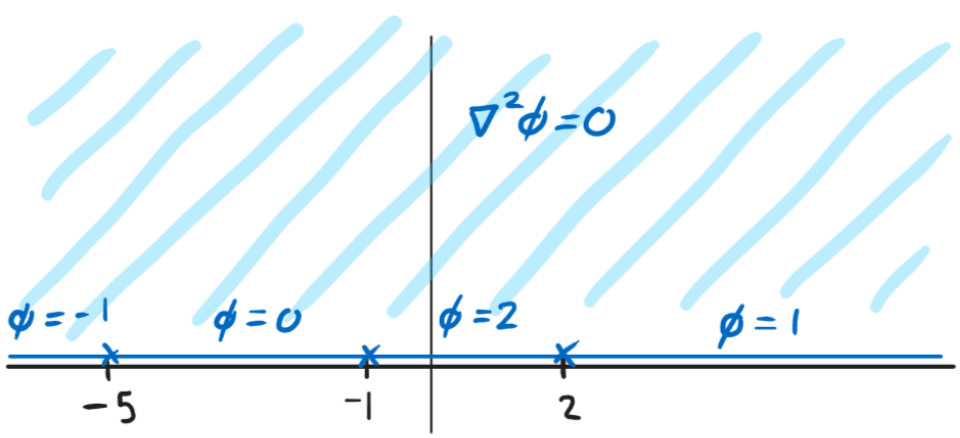
Ex3: Given $\nabla^2=0$ within the strip described by $\{z:-3\leq 3\Re(z)-4\Im(z)\leq 2\}$ with $\phi=0$ for $\{z:-3=3\Re(z)-4\Im(z)\}$ and $\phi=4$ for $\{z:3\Re(z)-4\Im(z)=2\}$ find $\phi(x,y)$

$$u=3x-4y$$
$$u(-3)=0,\ u(2)=4$$
$$\Phi=Au+B$$
$$\Phi(-3)=-3A+B=0\Rightarrow B=3A$$
$$\Phi(2)=2A+3A=5A=4\Rightarrow A=\frac{4}{5}\Rightarrow B=\frac{12}{5}$$
$$\phi(x,y)=\frac{4}{5}(3x-4y)+\frac{12}{5}$$

Ex4: Given $\nabla^2\phi=0$ for the upper half-plane described by $\{y>0\wedge x\in\mathbb{R}\}$ with the boundary conditions along the x-axis given by

$$\phi(x,0)=\begin{cases}-1 & x<-5\\ 0 & -5<x<-1\\ 2 & -1<x<2\\ 1 & x>2\end{cases}$$

find $\phi(x,y)$.



One trick to solve a problem like this is to choose a linear combination of functions of the form $\mathrm{Arg}(z-z_0)$ with a different point z_0 for every place where the boundary condition changes along the x-axis.

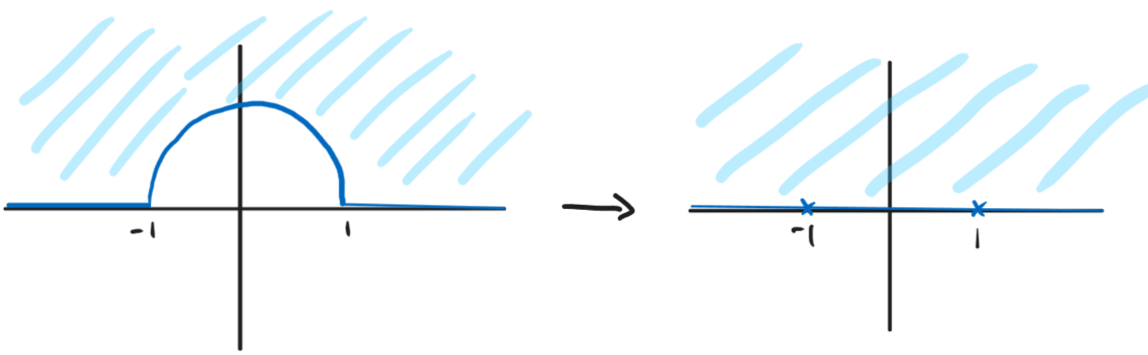
$$\phi=A_1\,\mathrm{Arg}(z+5)+A_2\,\mathrm{Arg}(z+1)+A_3\,\mathrm{Arg}(z-2)+A_4$$

$$\phi(x>2,0)=A_4=1$$
$$\phi(-1<x<2,0)=\pi A_3+1=2\Rightarrow A_3=\frac{1}{\pi}$$
$$\phi(-5<x<-1,0)=\pi A_2+1+1=0\Rightarrow A_2=-\frac{2}{\pi}$$
$$\phi(x<-5,0)=\pi A_1-2+2=-1\Rightarrow A_1=-\frac{1}{\pi}$$
$$\phi(x,y)=-\frac{1}{\pi}\operatorname{Arg}(z+5)-\frac{2}{\pi}\operatorname{Arg}(z+1)+\frac{1}{\pi}\operatorname{Arg}(z-2)+1$$

We can also apply these techniques to other types of boundary conditions in some cases.
Ex5: Given $\nabla^2\phi=0$ in the circular region $\{z:1\le|z|\le 2\}$ with the boundary conditions $\phi=1$ for $|z|=1$ and $\frac{\partial\phi}{\partial r}=2$ for $|z|=2$, find $\phi(x,y)$

$$\phi=A\ln r+B$$
$$\phi(1)=B=1$$
$$\frac{\partial\phi(2)}{\partial r}=\frac{A}{2}=2\Rightarrow A=4$$
$$\phi=4\ln r+1$$
$$\phi(x,y)=4\ln\sqrt{x^2+y^2}+1$$

If we have a region that has a semicircle in it we can use the Joukowski mapping to transform it into a region that is easier to work with.



Ex6: Given $\nabla^2\phi=0$ in the upper region of the plane described by $\{y>0\wedge x^2+y^2>9\}$ with the boundary conditions $\phi=-1$ for $x<3$, $\phi=0$ for $x^2+y^2=9$, and $\phi=2$ for $x>3$, find $\phi(x,y)$.

$$\begin{cases}\phi(x,0)=-1 & x<-3\\ \phi(x,y)=0 & x^2+y^2=9\\ \phi(x,0)=2 & x>3\end{cases}$$
$$w=\frac{1}{2}\left(\frac{z}{3}+\frac{3}{z}\right)=u+iv$$
$$\Phi=A_1\operatorname{Arg}(w+1)+A_2\operatorname{Arg}(w-1)+A_3$$
$$\Phi(u>1,v)=A_3=2$$
$$\Phi(-1<u<1,v)=\pi A_2+2=0\Rightarrow A_2=-\frac{2}{\pi}$$

$$\Phi(u<-1,v)=\pi A_1-2+2=-1 \Rightarrow A_1=-\frac{1}{\pi}$$
$$\Phi=-\frac{1}{\pi}\operatorname{Arg}(w+1)-\frac{2}{\pi}\operatorname{Arg}(w-1)+2$$
$$\phi(z)=-\frac{1}{\pi}\operatorname{Arg}\left(\frac{1}{2}\left(\frac{z}{3}+\frac{3}{z}\right)+1\right)-\frac{2}{\pi}\operatorname{Arg}\left(\frac{1}{2}\left(\frac{z}{3}+\frac{3}{z}\right)-1\right)+2$$

In the case that we have a semicircle not of radius 1 we can apply a scaling before the Joukowski mapping to get the correct radius.

$$w=\frac{1}{2}\left(\frac{z}{a}+\frac{a}{z}\right)$$

1.2.5 Sinusoidal Functions

If we recall Euler's formula $e^{ix}=\cos x+i\sin x$ we can use with complex numbers to get the following identities for complex sinusoidals:

- $\sin z=\frac{e^{iz}-e^{-iz}}{2i}$
- $\cos z=\frac{e^{iz}+e^{-iz}}{2}$
- $\sinh z=\frac{e^{z}-e^{-z}}{2}$
- $\cosh z=\frac{e^{z}+e^{-z}}{2}$
- $\cos z=\sin\left(\frac{\pi}{2}-z\right)$
- $\sinh z=-i\sin(iz)$
- $\cosh z=\cos(iz)=\sin\left(\frac{\pi}{2}-iz\right)$
- $\frac{d}{dz}\sin z=\cos z$
- $\frac{d}{dz}\cos z=-\sin z$
- $\cos^2 z+\sin^2 z=1$
- $\frac{d}{dz}\sinh z=\cosh z$
- $\frac{d}{dz}\cosh z=\sinh z$
- $\cosh^2 z-\sinh^2 z=1$

The most notable difference between the real and complex versions of these functions is that $|\sin z| \not\leq 1$. We will see cases of this soon but it is easy to see once we write out the real and imaginary components of $\sin z$.

$$\sin z = \sin(x+iy) = \frac{e^{i(x+iy)-e^{-i(x+iy)}}}{2i} = \frac{e^{-y}e^{ix} - e^{y}e^{-ix}}{2i}$$
$$\sin z = \frac{e^{-y}\left(\cos x + i\sin x\right) - e^{y}\left(\cos x - i\sin x\right)}{2i} = \frac{e^{-y}-e^{y}}{2}\cos x + \frac{e^{-y}+e^{y}}{2i}\sin x$$
$$\sin z = \sin x \cosh y + i\cos x \sinh y$$

Ex: Show that if $|z| < 1$ then $|\sin z| < 2$.

Proof.

$$|\sin z| < 2, \; |z| < 1$$

$$\sin z = \sin x \cosh y + i\cos x \sinh y$$
$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y}$$
$$\cosh^2 y - \sinh^2 y = 1 \Rightarrow \sinh^2 y = \cosh^2 y - 1$$
$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x(\cosh^2 y - 1)}$$
$$|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \cosh^2 y - \cos^2 x}$$
$$|\sin z| = \sqrt{\cosh^2 y - \cos^2 x} \leq \cosh y$$
$$|z| = \sqrt{x^2+y^2} < 1 \Rightarrow y < 1$$
$$\cosh y < \cosh 1 = \frac{e+e^{-1}}{2} \approx 1.54 < 2$$
$$|\sin z| < 2$$

$\square$

Ex2: Find all solutions to $\sin z = 4i$

$$\sin(z) = 4i = \sin x \cosh y + i\cos x \sinh y$$
$$\sin x \cosh y = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi$$
$$\text{Case 1: } n = 2k: \; \cos x = 1$$
$$4 = \cos(2k\pi)\sinh(y) = \sinh y$$
$$\text{Case 2: } n = 2k+1: \; \cos x = -1$$
$$4 = \cos((2k+1)\pi)\sinh y = -\sinh y$$
$$\sinh y = \frac{e^y - e^{-y}}{2}$$
$$2\sinh y = e^y - e^{-y}$$
$$e^{2y} - 2\sinh y e^y - 1 = 0$$

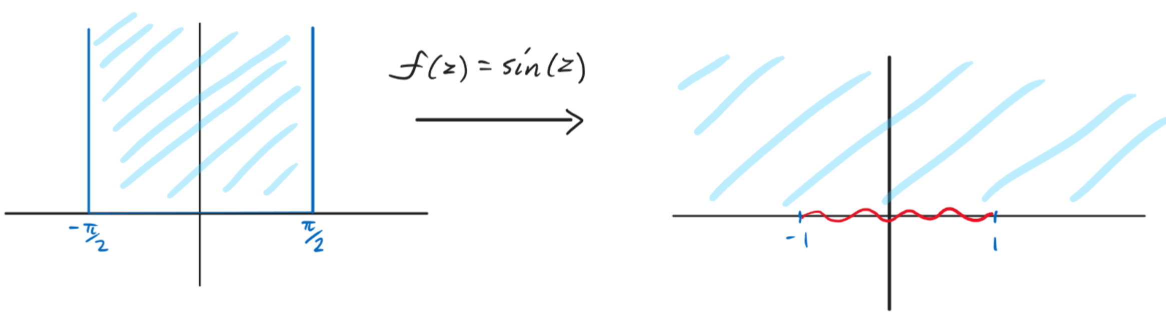
$$
\begin{aligned}
&e^y = \frac{2\sinh y \pm \sqrt{4\sinh^2 y + 4}}{2} = \sinh y \pm \sqrt{\sinh^2 y + 1} \\
&n = 2k: \\
&y = \ln\left(4 \pm \sqrt{17}\right) = \ln\left(4 + \sqrt{17}\right) \\
&n = 2k+1: \\
&y = \ln\left(-4 \pm \sqrt{17}\right) = \ln\left(\sqrt{17} - 4\right) \\
&z = \left\{(x,y) \middle| \left(2k\pi, \ln\left(4+\sqrt{17}\right)\right), \left((2k+1)\pi, \ln\left(\sqrt{17}-4\right), k \in \mathbb{Z}\right)\right\}
\end{aligned}
$$

Ex3: Find all solutions to $\cos z = 0$

$$
\begin{aligned}
&\cos(z^4) = 0 \\
&z^4 = \pi n + \frac{\pi}{2} = \left(\pi n + \frac{\pi}{2}\right) e^{2\pi i l} \\
&z = \left(\pi n + \frac{\pi}{2}\right)^{1/4} e^{i\frac{\pi l}{2}}, \ l = \{0,1,2,3\}, \ n \in \mathbb{Z}
\end{aligned}
$$

Mapping properties of $\sin z$

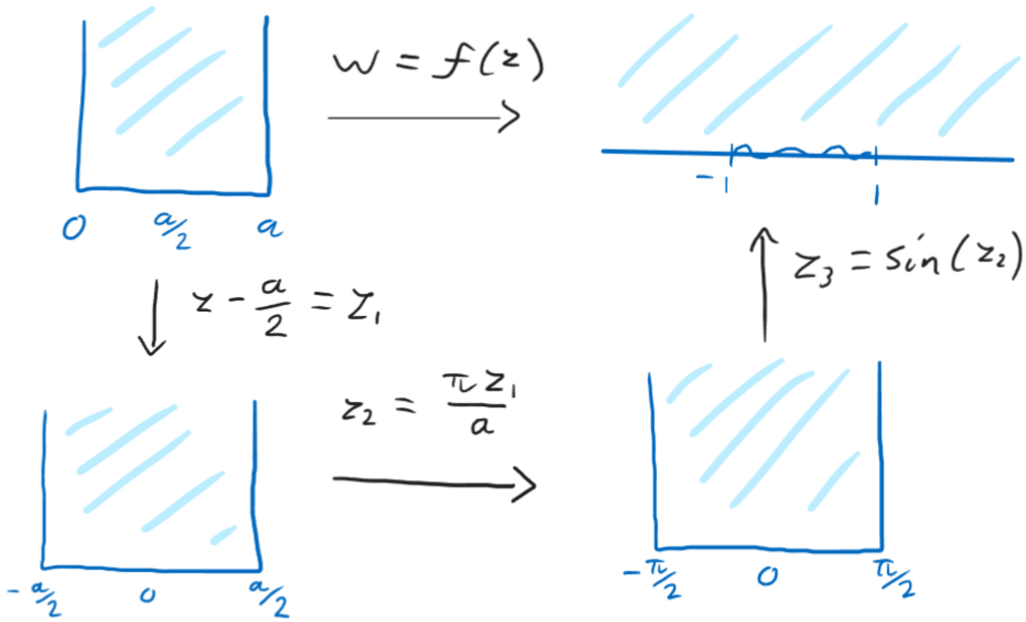
$\sin z$ will map the box $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, $0 \le y < \infty$ to the half plane $v > 0$.



Ex: Find the mapping of $\sin z$ on $S = \left\{-\frac{\pi}{2} < x < \frac{\pi}{2}, 0 < y < 1\right\}$

$$
\begin{aligned}
&u = \sin x \cosh y \\
&\sin x \in (-1, 1) \\
&\cosh y \in (0, \cosh(1)) \\
&u \in (-\cosh(1), \cosh(1)) \\
&v = \cos x \sinh y \\
&\cos x \in (0, 1] \\
&\sinh y \in (0, \sinh(1)) \\
&v \in (0, \sinh(1)) \\
&S' = \{(u,v) | -\cosh(1) < u < \cosh(1), 0 < v < \sinh(1)\}
\end{aligned}
$$

If the box is offcenter we can also apply a composition of mappings to shift it into the usual form.



Ex2: Find the mapping of $\sin z$ on $S = \{-1 < x < 1, y > 0\}$

$$u = \sin x \cosh y$$
$$\sin x \in (-\sin(1), \sin(1))$$
$$\cosh y \in (1, \infty)$$
$$u \in \mathbb{R}$$
$$v = \cos x \sinh y$$
$$\cos x \in (\cos(1), 1]$$
$$\sinh y \in (0, \infty)$$
$$v \in (0, \infty)$$
$$S' = \{(u, v) | v > 0\}$$

Ex3: Solve the Laplace equation $\nabla^2 u = 0$ in the region $S = \{0 \le x \le 2, 0 \le y < \infty\}$ with boundary conditions $\phi = 0$ on $x = 0$ and $\phi = 1$ on $y = 0$ and $\phi = -2$ on $x = 2$.

Map to the half plane using $w = \sin\left(\frac{\pi}{2}(z-1)\right)$

$$\phi(z) = \Phi(w) = A_1 \operatorname{Arg}(w+1) + A_2 \operatorname{Arg}(w-1) + A_3$$
$$u > 1: \ \Phi = -2 = A_3$$
$$-1 < u < 1: \ \Phi = 1 = A_2\pi - 2 \Rightarrow A_2 = \frac{3}{\pi}$$
$$u < -1: \ \Phi = 0 = A_1\pi + \frac{3}{\pi}\pi - 2 \Rightarrow A_1 = -\frac{1}{\pi}$$
$$\phi(z) = -\frac{1}{\pi} \operatorname{Arg}\left(\sin\left(\frac{\pi}{2}(z-1)\right) + 1\right) + \frac{3}{\pi} \operatorname{Arg}\left(\sin\left(\frac{\pi}{2}(z-1)\right) - 1\right) - 2$$

1.2.6 Logarithmic Functions

$\log z$ is defined as the inverse of e^z.

$$
\begin{aligned}
&e^w = z \Rightarrow w = \log z \\
&w = u + iv \\
&z = re^{i\operatorname{Arg}(z)} \\
&e^u e^{iv} = re^{i\operatorname{Arg}(z)} \\
&e^u = r \Rightarrow u = \ln r \\
&e^{iv} = e^{i\operatorname{Arg}(z)} \Rightarrow v = \operatorname{Arg}(z) + 2k\pi \\
&\log(z) = w = \ln r + i(\operatorname{Arg} z + 2k\pi),\ k \in \mathbb{Z} \\
&\log(z) = \{w | e^w = z\} = \ln r + i(\operatorname{Arg} z + 2k\pi) = \ln r + i \arg z \\
&\operatorname{Log}(z) = \ln r + i \operatorname{Arg} z
\end{aligned}
$$

Note that $\log(z) = \ln r + i \arg z$ is a multi-valued function and $\operatorname{Log}(z) = \ln r + i \operatorname{Arg} z$ is the principal value of $\log z$.
Ex: Find all values for $\log 2$ and $\operatorname{Log} 2$.

$$
\begin{aligned}
&z = 2 = 2e^{i0} \\
&\log 2 = \ln 2 + i(0 + 2k\pi) = \ln 2 + 2k\pi i,\ k \in \mathbb{Z} \\
&\operatorname{Log} 2 = \ln 2 + i(0) = \ln 2
\end{aligned}
$$

Ex2: Find all values for $\log(-1 - \sqrt{3}i)$ and $\operatorname{Log}(-1 - \sqrt{3}i)$.

$$
\begin{aligned}
&z = -1 - \sqrt{3}i = 2e^{i\frac{2\pi}{3}} \\
&\log(-1 - \sqrt{3}i) = \ln 2 + i\left(-\frac{2\pi}{3} + 2k\pi\right),\ k \in \mathbb{Z} \\
&\operatorname{Log}(-1 - \sqrt{3}i) = \ln 2 + i\left(-\frac{2\pi}{3}\right)
\end{aligned}
$$

Ex3: Find all values for $\log(e^{1+5i})$ and $\operatorname{Log}(e^{1+5i})$.

$$
\begin{aligned}
&z = e^{1+5i} = ee^{5i} = ee^{i(5+2k\pi)} \\
&5 + 2k\pi \in (-\pi, \pi] \Rightarrow k = -1 \\
&\operatorname{Arg}(e^{1+5i}) = 5 - 2\pi \\
&\log(e^{1+5i}) = \ln e + i(5 - 2\pi + 2k\pi) = 1 + i(5 - 2\pi + 2k\pi),\ k \in \mathbb{Z} \\
&\operatorname{Log}(e^{1+5i}) = \ln e + i(5 - 2\pi) = 1 + i(5 - 2\pi)
\end{aligned}
$$

Ex4: Find all values for $\log(-i)$ and $\operatorname{Log}(-i)$.

$$
\begin{aligned}
&-i = e^{-i\frac{\pi}{2}} \\
&\arg(-i) = -\frac{\pi}{2} + 2\pi k \\
&\log(-i) = -i\frac{\pi}{2} + i2\pi k
\end{aligned}
$$

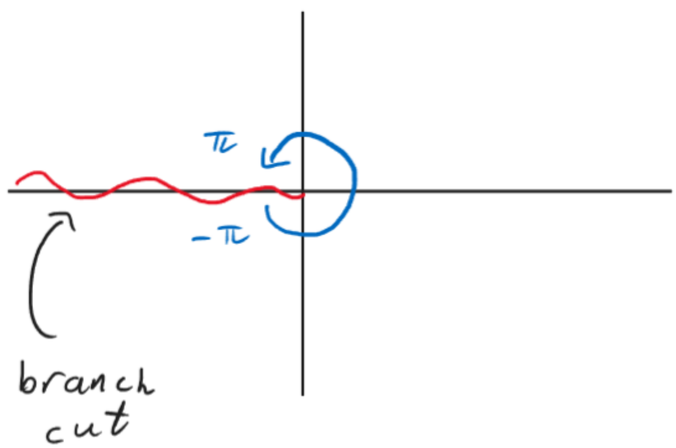
$$\mathrm{Log}(-i) = -i\frac{\pi}{2}$$

Properties of $\log z$ and $\mathrm{Log}\, z$:

- $\log(z_1 z_2) = \log z_1 + \log z_2$
  $\mathrm{Log}(z_1 z_2) \neq \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2$ (same as why $\mathrm{Arg}(z_1 z_2) \neq \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$)
- $e^{\log z} = z$ and $e^{\mathrm{Log}\, z} = z$
  but $\log(e^z) \neq z$ and $\mathrm{Log}(e^z) \neq z$
  Ex: $z = 0$: $\log(e^0) = \log(1) = \ln 1 + i(0 + 2k\pi) = i2k\pi \neq 0$
  $\mathrm{Log}(e^{5i}) = i(5 - 2\pi) \neq 5i$
  rather $z \in \log(e^z)$
- $\log z^n \neq n \log z$

$$z = 1,\ n = 2:\ \log z^2 = \log 1 = i2k\pi$$
$$2\log z = 2\log 1 = 2i(2k')\pi$$

  but $\log z^{\frac{1}{n}} = \frac{1}{n}\log z$
- $\mathrm{Log}\, z$ is analytic in $\mathbb{C}\backslash(-\infty, 0]$



$$\mathrm{Log}\, z = \ln|z| + i\,\mathrm{Arg}\, z,\ -\pi < \mathrm{Arg}\, z < \pi$$

Ex: Find all solutions to $e^z = -1 - i$

$$z = \log(-1 - i)$$
$$-1 - i = \sqrt{2}e^{-i\frac{3\pi}{4}}$$
$$z = \frac{1}{2}\ln(2) - i\frac{3\pi}{4} + i2\pi k$$

Ex2: Find the principal value of $(1 + i)^i = e^{i\,\mathrm{Log}(1+i)}$

$$1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$$

$$\mathrm{Log}(1+i) = \frac{1}{2}\ln(2) + i\frac{\pi}{4}$$
$$(1+i)^i = e^{-\frac{\pi}{4}+i\frac{1}{2}\ln(2)}$$

Ex3: Find the principle value of $z^i = 1+i$

$$1+i = \sqrt{2}e^{i\frac{\pi}{4}+i2\pi k} = e^{\ln(\sqrt{2})+i\frac{\pi}{4}+i2\pi k}$$
$$z = (1+i)^{1/i} = (1+i)^{-i} = e^{\frac{\pi}{4}+2\pi k}e^{-i\ln(\sqrt{2})}$$
$$z = e^{\frac{\pi}{4}}e^{-i\frac{1}{2}\ln(2)}$$

Ex4: Find the principal value of i^i.

$$i^i = (e^{i\frac{\pi}{2}+i2\pi k})^i = e^{-\frac{\pi}{2}-2\pi k} = e^{-\frac{\pi}{2}}$$

Ex5: Find all solutions to $\mathrm{Log}(z^2-1) = \frac{i\pi}{2}$

$$z^2 - 1 = e^{i\frac{\pi}{2}} = i$$
$$z^2 = i + 1 = \sqrt{2}e^{i\frac{\pi}{4}+i2\pi k}$$
$$z = 2^{1/4}e^{i\frac{\pi}{8}+i\pi k}$$

Ex6: Find all solutions to $e^{2z} + e^z + 1 = 0$

$$e^z = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$
$$e^z = e^{i\pi \pm i\frac{\pi}{3}+i2\pi k}$$
$$z = i\pi \pm i\frac{\pi}{3} + i2\pi k$$
$$z = \pm i\frac{2\pi}{3} + i2\pi k$$

Derivative of $\log z$ *and* $\mathrm{Log}\, z$

$$(\mathrm{Log}\, z)' = \frac{1}{z}$$
$$\mathrm{Log}\, z = \ln\sqrt{x^2+y^2} + i\,\mathrm{Arg}\, z = u + iv$$
$$(\mathrm{Log}\, z)' = u_x + iv_x$$
$$\mathrm{Log}\, z = \ln|z| + i\varphi,\ -\pi < \varphi < \pi$$

1.2.7 Branch Cuts

There are many different analytic functions for $\log z$ depending on the branch cut we choose. For example we can choose:

$$\mathrm{Log}\, z = \ln r + i\varphi,\ -\pi < \varphi < \pi,\ \mathbb{C}\backslash(-\infty, 0]$$
$$\log z = \ln r + i\varphi,\ 0 < \varphi < 2\pi,\ \mathbb{C}\backslash[0, \infty)$$
$$\log z = \ln r + i\varphi,\ -\frac{\pi}{2} < \varphi < \frac{5\pi}{2},\ \mathbb{C}\backslash i[0, \infty)$$

Ex: Specify a branch cut for $\log z$ such that it is analytic at $z = -1, -i, 1$

$$\log z = \ln r + i\varphi,\ \frac{\pi}{2} < \varphi < \frac{5\pi}{2}$$

The reason we have branch cuts is so that we can define a single-valued analytic function. This is important for functions such as $\sqrt{z}$ and $\log z$ as one input can correspond to multiple outputs. For example, $\sqrt{1} = 1$ and $\sqrt{1} = -1$. This is why we need to specify a branch cut for $\sqrt{z}$ and $\log z$. Ex: Construct $f(z) = z^{1/2}$ such that it is analytic in $\mathbb{C}\backslash(-\infty, 0]$ and $f(1) = -1$.

$$z^{\frac{1}{2}} = e^{\frac{1}{2}\log z} = e^{\frac{1}{2}\ln r + i\frac{1}{2}\varphi} = r^{\frac{1}{2}} e^{i\frac{\varphi}{2}},\ -\pi < \varphi < \pi$$
$$f(1) = r^{\frac{1}{2}} e^{i\frac{\varphi}{2}} = 1e^{i0} = 1$$
$$z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{i\frac{\varphi}{2}},\ \pi < \varphi < 3\pi$$
$$\to f(1) = 1^{\frac{1}{2}} e^{\frac{2\pi}{2}} = -1$$

Ex2: Find all solutions to $z^{1/2} + 1 - i = 0$ where $z^{1/2}$ is the principal branch of $z^{1/2}$.

$$z^{1/2} = |z|^{1/2} e^{i\frac{\varphi}{2}} = i - 1 = \sqrt{2} e^{i\frac{3\pi}{4} + i2\pi k}$$
$$|z| e^{i\varphi} = 2e^{i\frac{3\pi}{2} + i4\pi k}$$
$$|z| = 2$$
$$\varphi = \frac{3\pi}{2} + 4\pi k \notin (-\pi, \pi]$$
$$\therefore \text{ no solutions}$$

Ex3: Find all solutions to $z^{1/2} + 1 - i = 0$

$$z^{1/2} = |z|^{1/2} e^{i\frac{\varphi}{2} + i\pi k_1} i - 1 = \sqrt{2} e^{i\frac{3\pi}{4} + i2\pi k_2}$$
$$|z| e^{i\varphi + i2\pi k_1} = 2e^{i\frac{3\pi}{2} + i4\pi k_2}$$
$$|z| = 2$$
$$\varphi + 2\pi k_1 = \frac{3\pi}{2} + 4\pi k_2$$
$$\varphi = \frac{3\pi}{2} + 2\pi k_3$$
$$z = 2e^{-i\frac{\pi}{2}} = -2i$$

Ex4: Find where $\mathrm{Log}(1 + z^2)$ is analytic

$$w = 1 + z^2$$
$$\text{not analytic on } \Re(w) \le 0\ \wedge \Im(w) = 0$$
$$x^2 - y^2 + 1 \le 0$$
$$2xy = 0 \Rightarrow x = 0 \vee y = 0$$
$$y = 0:\ x^2 + 1 \le 0 \Rightarrow \nexists x \in \mathbb{R} \text{ s.t. } x^2 \le -1$$

$$x = 0: \; y^2 \geq 1 \Rightarrow |y| \geq 1$$

Analytic on the domain

$$\mathbb{C} \backslash \{(x,y) : x = 0, |y| \geq 1\}$$

Ex5: Find where $\operatorname{Log}\left(\frac{1-z}{1+z}\right)$ is analytic

$$w = \frac{1-z}{1+z}$$

not analytic on $\Re(w) \leq 0 \;\wedge \Im(w) = 0$

$$w = \frac{(1-z)(1+\overline{z})}{(1+z)(1+\overline{z})} = \frac{1-|z|^2 - z + \overline{z}}{1+|z|^2+z+\overline{z}} = \frac{1-x^2-y^2-2iy}{1+x^2+y^2+2x}$$

$y = 0$

$$\frac{1-x^2}{1+x^2+2x} \leq 0 \Rightarrow 1 \leq x^2 \Rightarrow |x| \geq 1$$

Analytic on

$$\mathbb{C} \backslash \{(x,y) : y = 0, |x| \geq 1\}$$

Ex6: Find where $f(z) = \operatorname{Log}(1-z^3)$ is analytic

$$\Re(1-z^3) \leq 0 \qquad \Im(1-z^3) = 0$$

$$z^3 = r^3 e^{3i\varphi} = r^3 = r^3 \left(\cos(3\varphi) + i\sin(3\varphi)\right)$$

$$\Im(1-z^3) = r^3 \sin(3\varphi) = 0 \Rightarrow 3\varphi = n\pi$$

$$\varphi = \frac{n\pi}{3}$$

$$\Re(1-z^3) = 1 - r^3\cos(3\varphi) \leq 0$$

$$n = 2k: \; \cos(3\varphi) = 1 \Rightarrow 1 - r^3 \leq 0 \Rightarrow 1 \leq r$$

$$n = 2k+1: \; \cos(3\varphi) = -1 \Rightarrow 1 + r^3 \leq 0 \Rightarrow 1 \leq -r \Rightarrow -1 \geq r$$

can't have $r \leq -1$ so can't have $n = 2k+1$

$$\Rightarrow \varphi = \frac{2k\pi}{3}, \; 1 \leq r$$

Analytic in $\mathbb{C} \backslash \left\{|z| \geq 1, \; \arg(z) = \dfrac{2k\pi}{3}, \; k \in \mathbb{Z}\right\}$

Ex7: Find a branch cut for $f(z) = \sqrt{z(z-1)}$ such that it is analytic for $\{(x,y)|y=0, x<0\}$ such that $f(2) = \sqrt{2}$

$$\sqrt{z(z-1)} = |z|^{1/2} e^{i\frac{\varphi_1}{2}} |z-1|^{1/2} e^{i\frac{\varphi_2}{2}} = (|z||z-1|)^{1/2} e^{i\frac{\varphi_1+\varphi_2}{2}}$$

$$\varphi_1 \in (-\pi, \pi), \; \varphi_2 \in (-\pi, \pi)$$

$$\lim_{\varphi_1 \to \pi} \lim_{\varphi_2 \to \pi} e^{i\frac{\varphi_1+\varphi_2}{2}} = e^{\pi i}$$

$$\lim_{\varphi_1 \to -\pi} \lim_{\varphi_2 \to -\pi} e^{i\frac{\varphi_1+\varphi_2}{2}} = e^{-\pi i} = e^{\pi i}$$

$\therefore$ continuous for $\{(x,y)|y=0, x<0\}$

$z=2: \ \varphi_1 = 0, \ \varphi_2 = 0 \Rightarrow \sqrt{z(z-1)} = \sqrt{2}$

Ex8: Find a branch cut for $f(z) = \sqrt{z(z-1)}$ such that it is analytic for $\{(x,y)|y=0, x<0\}$ such that $f(2) = -\sqrt{2}$

$$\sqrt{z(z-1)} = |z|^{1/2} e^{i\frac{\varphi_1}{2}} |z-1|^{1/2} e^{i\frac{\varphi_2}{2}} = (|z||z-1|)^{1/2} e^{i\frac{\varphi_1+\varphi_2}{2}}$$

$\varphi_1 \in (-\pi, \pi), \ \varphi_2 \in (-3\pi, -\pi)$

$$\lim_{\varphi_1 \to \pi} \lim_{\varphi_2 \to -\pi} e^{i\frac{\varphi_1+\varphi_2}{2}} = e^{-2\pi i}$$

$$\lim_{\varphi_1 \to -\pi} \lim_{\varphi_2 \to -3\pi} e^{i\frac{\varphi_1+\varphi_2}{2}} = e^{0} = e^{-2\pi i}$$

$\therefore$ continuous for $\{(x,y)|y=0, x<0\}$

$z=2: \ \varphi_1 = 0, \ \varphi_2 = -2\pi \Rightarrow \sqrt{z(z-1)} = \sqrt{2}e^{-i\pi} = -\sqrt{2}$

Ex9: Find a branch cut for $f(z) = \log(z^2+2z+2)$ such that it is analytic for $\{(x,y)|x>-1\}$ such that $\frac{d}{dz}f(-1) = 0$

$$\log(z^2+2z+2)$$

$$z = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$\log(z^2+2z+2) = \ln(|z-z_1||z-z_2|) + i(\varphi_1+\varphi_2)$$

$$-\pi < \varphi_1 < \pi, \ -\pi < \varphi_2 < \pi$$

$$z=-1: \ \varphi_1 = -\frac{\pi}{2}, \ \varphi_2 = \frac{\pi}{2}$$

$$\log(z^2+2z+2)\Big|_{z=-1} = 0 + i\left(-\frac{\pi}{2} + \frac{\pi}{2}\right) = 0$$

$$\frac{d}{dz}\log(z^2+2z+2)\Big|_{z=-1} = \frac{2z+2}{z^2+2z+2}\Big|_{z=-1} = 0$$

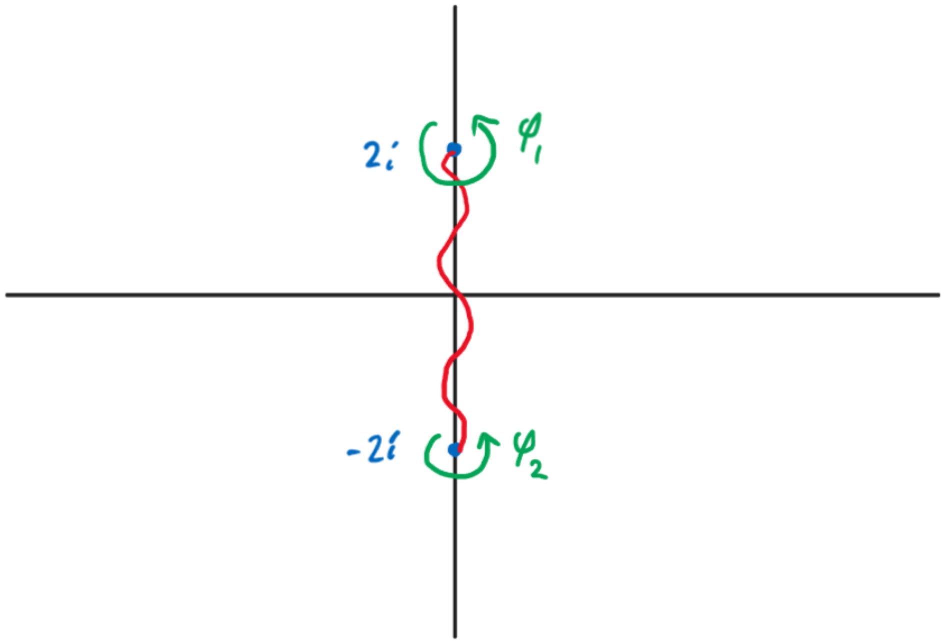
If we have a function such as $f(z) = \sqrt{z^2-1}$ then we will have two branch points at $z = \pm 1$. If two of these branch cuts overlap then they have the property of cancelling out. Note that if we have a cube root then we require three overlapping branch cuts to cancel out. For an nth root we require n overlapping branch cuts to cancel out.

Ex: Find a branch cut for $f(z) = \sqrt{4+z^2}$ such that it is analytic for $\mathbb{C} \setminus \{x=0, -2 \le y \le 2\}$.

$$(4+z^2)^{1/2}, \ \mathbb{C} \setminus \{x=0, -2 \le y \le 2\}$$

$$4+z^2 = 0 \Rightarrow z = \pm 2i$$

$$(4+z^2)^{1/2} = |z+2i|^{1/2}|z-2i|^{1/2} e^{i\left(\frac{\varphi_1+\varphi_2}{2}\right)}$$



Choose $\frac{\pi}{2} < \varphi_1 < \frac{5\pi}{2}$ and $\frac{\pi}{2} < \varphi_2 < \frac{5\pi}{2}$
Check continuity at $y > 2$:

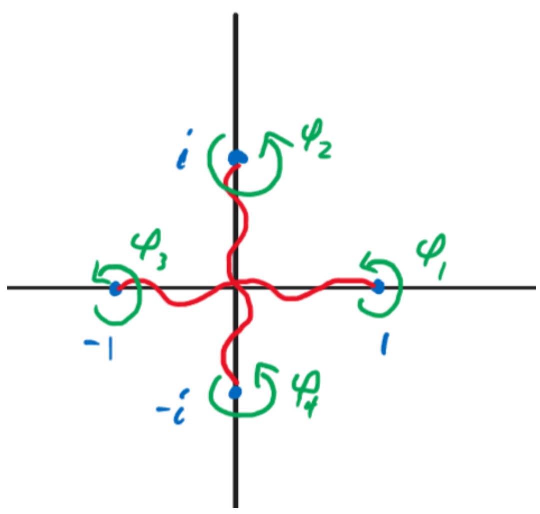
	φ_1	φ_2	$e^{i\left(\frac{\varphi_1+\varphi_2}{2}\right)}$
A	$\frac{5\pi}{2}$	$\frac{5\pi}{2}$	$e^{i\frac{5\pi}{2}} = i$
A'	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$e^{i\frac{\pi}{2}} = i$

Ex2: Find a branch cut for $f(z) = (z^4 - 1)^{1/2}$ such that it is analytic for $|z| > 1$.

$$(z^4 - 1)^{1/2},\ \{|z| > 1\}$$

$$z^4 - 1 = 0 \Rightarrow z = \pm 1, \pm i$$

$$(z^4 - 1)^{1/2} = |z+1|^{1/2}|z-1|^{1/2}|z+i|^{1/2}|z-i|^{1/2}e^{i\left(\frac{\varphi_1+\varphi_2+\varphi_3+\varphi_4}{2}\right)}$$



Choose $-\pi < \varphi_1 < \pi$, $\frac{\pi}{2} < \varphi_2 < \frac{5\pi}{2}$, $-\pi < \varphi_3 < \pi$, and $\frac{\pi}{2} < \varphi_4 < \frac{5\pi}{2}$

Check continuity at $y > 1$:

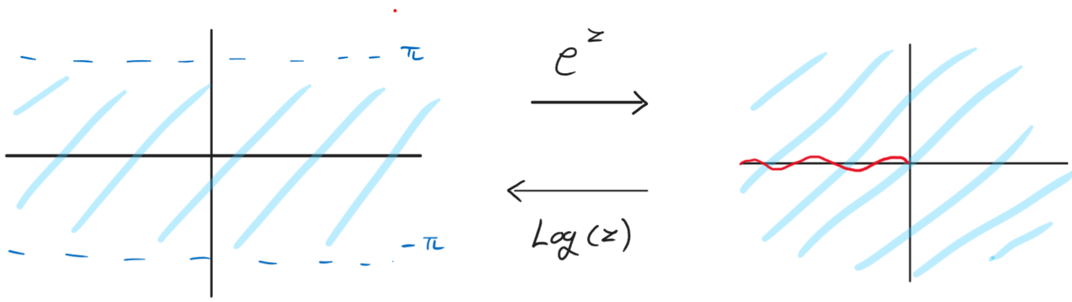
	φ_2	φ_4	$e^{i\left(\frac{\varphi_1+\varphi_2+\varphi_3+\varphi_4}{2}\right)}$
A	$\frac{5\pi}{2}$	$\frac{5\pi}{2}$	$e^{i\frac{5\pi}{2}}e^{i\left(\frac{\varphi_1+\varphi_3}{2}\right)} = ie^{i\left(\frac{\varphi_1+\varphi_3}{2}\right)}$
A’	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$e^{i\frac{\pi}{2}}e^{i\left(\frac{\varphi_1+\varphi_3}{2}\right)} = ie^{i\left(\frac{\varphi_1+\varphi_3}{2}\right)}$

Check continuity at $x < 1$:

	φ_1	φ_3	$e^{i\left(\frac{\varphi_1+\varphi_2+\varphi_3+\varphi_4}{2}\right)}$
B	π	π	$e^{i\pi}e^{i\left(\frac{\varphi_2+\varphi_4}{2}\right)} = -e^{i\left(\frac{\varphi_2+\varphi_4}{2}\right)}$
B’	$-\pi$	$-\pi$	$e^{-i\pi} = -e^{i\left(\frac{\varphi_2+\varphi_4}{2}\right)}$

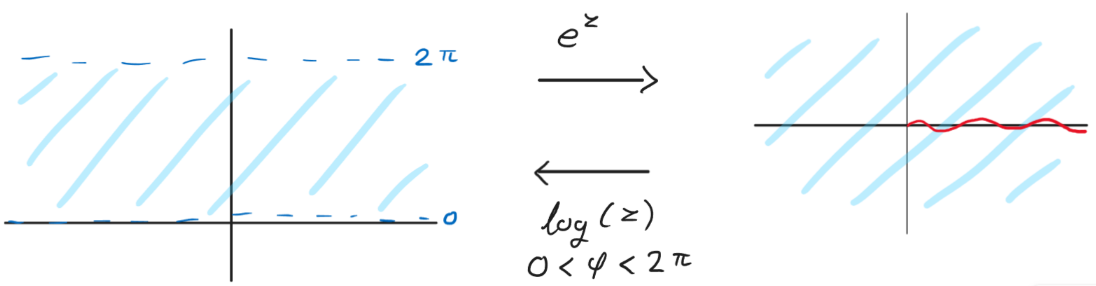
1.2.8 Inverse Functions

To get a one-to-one function we often need to restrict the domain of the function. Inverse of e^z is $z = \mathrm{Log}(w)$ or a branch cut of $\log(w)$

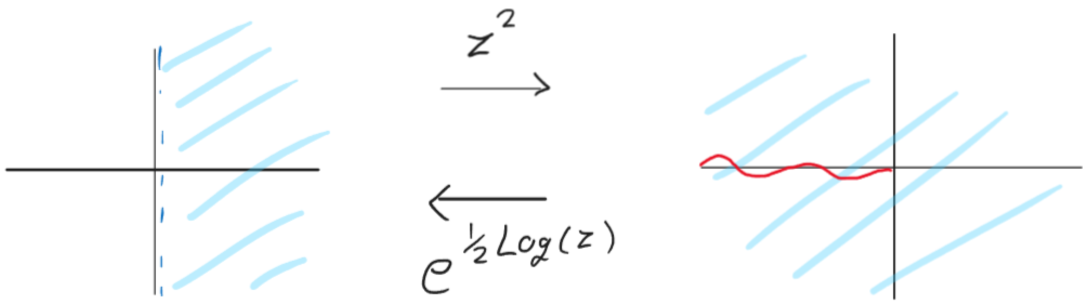


$$e^z: \ \{-\pi < y < \pi\} \to \mathbb{C}\backslash(-\infty, 0]$$
$$\mathrm{Log}(z): \ \mathbb{C}\backslash(-\infty, 0] \to \{-\pi < v < \pi\}$$

The inverse of e^z over $\{0 < y < 2\pi\}$ is $\log z = \ln r + i\varphi, \ 0 < \varphi < 2\pi$.

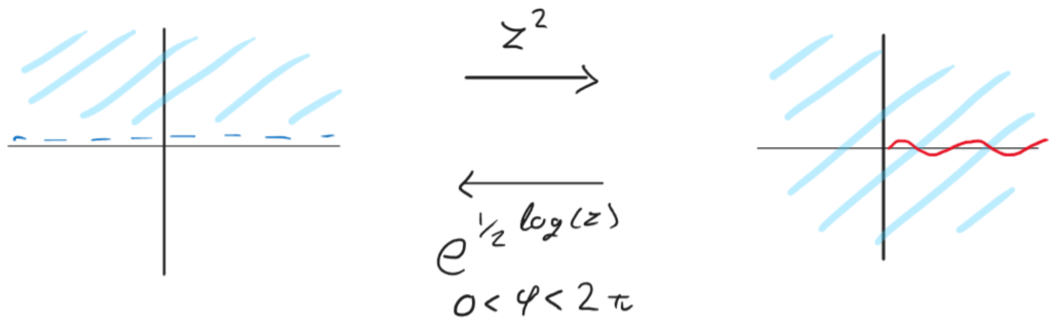


The inverse function for z^2 over $\{x > 0\}$ is $z^{\frac{1}{2}} = e^{\frac{1}{2}\,\mathrm{Log}(z)}$



$$z^2: \ \{x > 0\} \to \mathbb{C}\backslash(-\infty, 0]$$

The inverse function for z^2 over $\{y > 0\}$ is $z^{\frac{1}{2}} = e^{\frac{1}{2}\log(z)}, \ 0 < \varphi < 2\pi$



$$z^2: \ \{y > 0\} \to \mathbb{C}\backslash[0, \infty)$$

Ex: Find the inverse function of e^z on the domain $\left\{\frac{\pi}{4} < y < \frac{9\pi}{4}\right\}$

$$w = e^x e^{iy}$$
$$\log(w) = \ln(e^x) + iy = x + iy = z$$
$$\log(w)$$
$$f^{-1}(z) = \log(z), \ \left\{\frac{\pi}{4} < \arg(z) < \frac{9\pi}{4}\right\}$$

Ex2: Find the inverse function of z^4 on the domain $\{x > 0, y > 0\}$

$$z = w^{1/4} = |w|^{1/4} e^{i\frac{\arg(w)}{4}}$$
$$0 < \arg(z) < \frac{\pi}{2} \Rightarrow 0 < \frac{\arg(w)}{4} < \frac{\pi}{2} \Rightarrow 0 < \arg(w) < 2\pi$$
$$f^{-1}(z) = z^{1/4}, \ \{0 < \arg(z) < 2\pi\}$$

The inverse of $\sin z$ is a little bit less obvious. Let us find all inverse functions for $\sin z$.

$$\sin w = z$$

$$
\begin{aligned}
&\frac{e^{iw}-e^{-iw}}{2i}=z\\
&e^{iw}-e^{-iw}=2iz\\
&e^{2iw}-2ize^{iw}-1=0\\
&e^{iw}=\frac{2iz\pm\sqrt{(-2iz)^2+4}}{2}\\
&e^{iw}=iz\pm\sqrt{1-z^2}\\
&e^{iw}=iz+i(z^2-1)^{\frac{1}{2}}\\
&iw=\log\left(iz+i(z^2-1)^{\frac{1}{2}}\right)\\
&w=-i\log\left(iz+i(z^2-1)^{\frac{1}{2}}\right)
\end{aligned}
$$

The general solutions to $\sin w = z$ are given by the above expression. This expression contains two multivalued parts, log and $(z^2-1)^{\frac{1}{2}}$.
Ex: Find all solutions to $\sin w = -i$

$$
\begin{aligned}
&\sin(w)=-i\\
&\sin x\cosh y+i\cos x\sinh y=-i\\
&\sin x\cosh y=0\Rightarrow x=n\pi\\
&\cos x\sinh y=-1\\
&n=2k:\\
&\sinh y=-1\\
&n=2k+1:\\
&-\sinh y=-1\Rightarrow\sinh y=1\\
&\sinh y=\frac{e^y-e^{-y}}{2}\\
&e^{2y}-2\sinh y e^y-1=0\\
&e^y=\frac{2\sinh y\pm\sqrt{4\sinh^2 y+4}}{2}=\sinh y\pm\sqrt{\sinh^2 y+1}\\
&y=\ln\left(\sinh y\pm\sqrt{\sinh^2 y+1}\right)\\
&n=2k:\\
&y=\ln\left(-1\pm\sqrt{2}\right)=\ln\left(\sqrt{2}-1\right)\\
&n=2k+1:\\
&y=\ln\left(1\pm\sqrt{2}\right)=\ln\left(1+\sqrt{2}\right)\\
&w=\left\{2k_1\pi+i\ln\left(\sqrt{2}-1\right),(2k_2+1)\pi+i\ln\left(1+\sqrt{2}\right),k_1,k_2\in\mathbb{Z}\right\}
\end{aligned}
$$

Ex2: Find all solutions to $\cos w = -i$

$$\cos(w)=-i$$

$$
\begin{aligned}
&\cos(w) = \cos x \cosh y - i \sin x \sinh y \\
&\cos x \cosh y = 0 \Rightarrow x = \frac{\pi}{2} + n\pi \\
&n = 2k: \\
&-\sinh y = -1 \Rightarrow \sinh y = 1 \\
&n = 2k+1: \\
&\sinh y = -1 \\
&\sinh y = \frac{e^y - e^{-y}}{2} \\
&e^{2y} - 2\sinh y e^y - 1 = 0 \\
&e^y = \frac{2\sinh y \pm \sqrt{4\sinh^2 y + 4}}{2} = \sinh y \pm \sqrt{\sinh^2 y + 1} \\
&y = \ln\left(\sinh y \pm \sqrt{\sinh^2 y + 1}\right) \\
&n = 2k: \\
&y = \ln\left(1 \pm \sqrt{2}\right) = \ln\left(1 + \sqrt{2}\right) \\
&n = 2k+1: \\
&y = \ln\left(-1 \pm \sqrt{2}\right) = \ln\left(\sqrt{2} - 1\right) \\
&z = \left\{\frac{\pi}{2} + 2k_1\pi + i\ln\left(1+\sqrt{2}\right), \frac{\pi}{2} + (2k_2+1)\pi + i\ln\left(\sqrt{2}-1\right),\ k_1, k_2 \in \mathbb{Z}\right\}
\end{aligned}
$$

The principal value of the inverse function of $\sin z$ is called $f(z) = \arcsin z$ and is defined such that $f(0) = 0$.
The principal value of the inverse function of $\cos z$ is called $f(z) = \arccos z$ and is defined such that $f(0) = \frac{\pi}{2}$.
These functions are defined as

$$
\begin{aligned}
&\arcsin z = -i \operatorname{Log}\left(iz + i(z^2-1)^{\frac{1}{2}}\right) \\
&\text{where } (z^2-1)^{\frac{1}{2}} = |z-1|^{\frac{1}{2}}|z+1|^{\frac{1}{2}} e^{i\frac{\arg(z-1)+\arg(z+1)}{2}} \\
&\text{such that } 0 < \arg(z-1) < 2\pi \text{ and } \pi < \arg(z+1) < 3\pi \\
&\arccos z = -i \operatorname{Log}\left(z + (z^2-1)^{\frac{1}{2}}\right) \\
&\text{where } (z^2-1)^{\frac{1}{2}} = |z-1|^{\frac{1}{2}}|z+1|^{\frac{1}{2}} e^{i\frac{\arg(z-1)+\arg(z+1)}{2}}
\end{aligned}
$$

Ex: Find all solutions to $w = \arcsin(-i)$

$$
\begin{aligned}
&w = \arcsin(-i) \\
&\arcsin(z) = -i \operatorname{Log}(iz + i(z^2-1)^{1/2}) \\
&(z^2-1)^{1/2} = |z+1|^{1/2}|z-1|^{1/2} e^{i\left(\frac{\varphi_1+\varphi_2}{2}\right)},\ \varphi_1 \in (0, 2\pi),\ \varphi_2 \in (\pi, 3\pi)
\end{aligned}
$$

$$(z^2-1)^{1/2}\Big|_{z=-i} \Rightarrow \varphi_1 = \frac{5\pi}{4},\ \varphi_2 = \frac{3\pi}{4} + \pi$$

$$(z^2-1)^{1/2}\Big|_{z=-i} = \sqrt{2}e^{i\frac{3\pi}{2}} = -\sqrt{2}i$$

$$\arcsin(-i) = -i\,\mathrm{Log}(1 + i(-\sqrt{2}i)) = -i\,\mathrm{Log}(1+\sqrt{2})$$

$$\arcsin(-i) = -i\ln\left(1+\sqrt{2}\right)$$

Ex2: Find all solutions to $w = \arccos(-i)$

$$w = \arccos(-i)$$

$$w = -i\,\mathrm{Log}(z + (z^2-1)^{1/2})$$

$$(z^2-1)^{1/2} = |z+1|^{1/2}|z-1|^{1/2}e^{i\left(\frac{\varphi_1+\varphi_2}{2}\right)},\ \varphi_1 \in (0, 2\pi),\ \varphi_2 \in (-\pi, \pi)$$

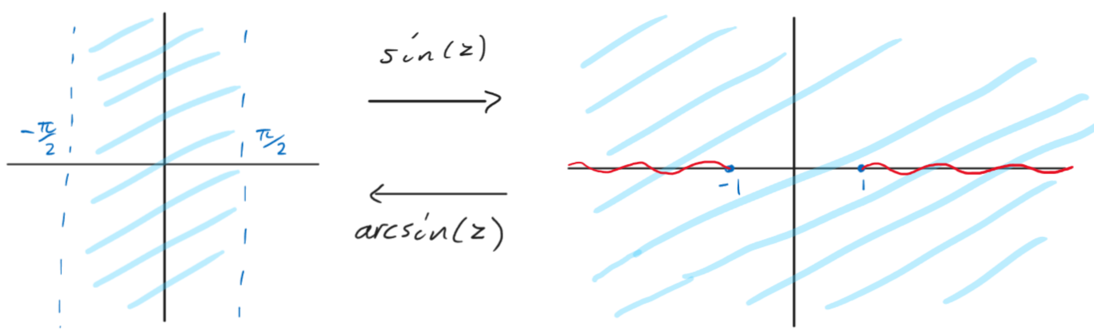
$$(z^2-1)^{1/2}\Big|_{z=-i} \Rightarrow \varphi_1 = \frac{5\pi}{4},\ \varphi_2 = -\frac{\pi}{4}$$

$$(z^2-1)^{1/2}\Big|_{z=-i} = \sqrt{2}e^{i\frac{\pi}{2}} = \sqrt{2}i$$

$$w = -i\,\mathrm{Log}(-i+\sqrt{2}i) = -i\,\mathrm{Log}((\sqrt{2}-1)e^{i\frac{\pi}{2}}) = -i\left(\ln(\sqrt{2}-1) + i\frac{\pi}{2}\right)$$

$$w = \frac{\pi}{2} - i\ln\left(\sqrt{2}-1\right)$$

Mapping properties of $\sin z$ and $\arcsin z$:



We can also apply this mapping to solve Laplace's equation.
Ex:

$$\Delta\phi = 0,\ \{x>0, y>0\}$$

$$\text{BCs}: \begin{cases} \phi = 1 & x=0, y>0 \\ \phi_y = 0 & 0<x<1, y=0 \\ \phi = 2 & x>1, y=0 \end{cases}$$

$$w = \arcsin(z)$$

$$\Phi = Au + B$$

$$\Phi(u=0) = 1 \Rightarrow B = 1$$

$$\Phi\left(u=\frac{\pi}{2}\right)=2=A\frac{\pi}{2}+1\Rightarrow A=\frac{2}{\pi}$$

$$\Phi=\frac{2}{\pi}u+1$$

$$u+iv=\arcsin(x+iy)$$

$$\sin(u+iv)=x+iy$$

$$\begin{cases}x=\sin u\cosh v\\ y=\cos u\sinh v\end{cases}\Rightarrow\begin{cases}\cosh v=\frac{x}{\sin u}\\ \sinh v=\frac{y}{\cos u}\end{cases}\Rightarrow\cosh^2 v-\sinh^2 v=\frac{x^2}{\sin^2 u}-\frac{y^2}{\cos^2 u}=1$$

$$t=\sin^2 u$$

$$\sin^2 u+\cos^2 u=1\Rightarrow\cos^2 u=1-t$$

$$\frac{x^2}{t}-\frac{y^2}{1-t}=1$$

$$t(1-t)=(1-t)x^2-ty^2$$

$$t-t^2=x^2-tx^2-ty^2$$

$$t^2+t(-x^2-y^2-1)+x^2$$

$$t=\frac{x^2+y^2+1\pm\sqrt{(x^2+y^2+1)^2-4x^2}}{2}$$

$$0\le\sin^2 u\le 1\Rightarrow 0\le t\le 1\Rightarrow t=\frac{x^2+y^2+1-\sqrt{(x^2+y^2+1)^2-4x^2}}{2}$$

$$\sin^2 u=\frac{x^2+y^2+1-\sqrt{(x^2+y^2+1)^2-4x^2}}{2}$$

$$0<u<\frac{\pi}{2}\Rightarrow 0<\sin u<1$$

$$\sin u=\sqrt{\frac{x^2+y^2+1-\sqrt{(x^2+y^2+1)^2-4x^2}}{2}}$$

$$u=\arcsin\left(\sqrt{\frac{x^2+y^2+1-\sqrt{(x^2+y^2+1)^2-4x^2}}{2}}\right)$$

$$\phi=\frac{2}{\pi}\arcsin\left(\sqrt{\frac{x^2+y^2+1-\sqrt{(x^2+y^2+1)^2-4x^2}}{2}}\right)+1$$

We can also define inverse functions for $\sinh z$ and $\cosh z$ using the same method as for $\sin z$ and $\cos z$.
Ex: Find the inverse of $\sinh(z)$ such that $f^{-1}(0)=\ln(1)$.

$$w=\sinh(z)=\frac{e^z-e^{-z}}{2}$$

$$2w=e^z-e^{-z}$$

$$e^{2z}-2we^z-1$$

$$e^z=\frac{2w\pm\sqrt{4w^2+4}}{2}=w\pm\sqrt{w^2+1}$$

$$|e^z| > 0 \Rightarrow |w \pm \sqrt{w^2+1}|$$

$$\log(e^z) = \ln(e^x) + i(y + 2\pi k) = \log\left(w \pm (w^2+1)^{1/2}\right)$$

$$z = \log\left(w + (w^2+1)^{1/2}\right)$$

$$(w^2+1)^{1/2} = |w+i|^{1/2}|w-i|^{1/2}e^{i\left(\frac{\varphi_1+\varphi_2}{2}\right)},\ \varphi_1 \in \left(\frac{\pi}{2}, \frac{5\pi}{2}\right),\ \varphi_2 \in \left(\frac{3\pi}{2}, \frac{7\pi}{2}\right)$$

$$(w^2+1)^{1/2}\Big|_{w=0} \Rightarrow \varphi_1 = \frac{3\pi}{2},\ \varphi_2 = \frac{5\pi}{2}$$

$$(w^2+1)^{1/2}\Big|_{w=0} = e^{i2\pi} = 1$$

$$0 = \log(1) = \ln(1) + i2\pi k$$

$$f^{-1}(z) = \mathrm{Log}\left(z + |z+i|^{1/2}|z-i|^{1/2}e^{i\left(\frac{\varphi_1+\varphi_2}{2}\right)}\right),\ \varphi_1 \in \left(\frac{\pi}{2}, \frac{5\pi}{2}\right),\ \varphi_2 \in \left(\frac{3\pi}{2}, \frac{7\pi}{2}\right)$$