Math Notes

Tyler Wilson

Contents

1 Vector Calculus

1.1 Parameterizations of Curves and Surfaces

1.1.1 Parametric Equations of Curves

We have often seen equations in the form of $y = f(x)$. We can express both x and y in terms of a common variable t in order to create a vector form of the function and show how the function changes over time.
A parametric equation will be in the form of $\vec{r}(t) = \langle \vec{x}(t), \vec{y}(t), \vec{z}(t) \rangle$
Ex: What curve is represented by $x = \cos t$ and $y = \sin t$ for $t \in [0, 2\pi]$

$$\sin^2 t + \cos^2 t = 1$$
$$y^2 + x^2 = 1$$

This is the equation of the unit circle. By plugging in points, we can see that it starts at $(1, 0)$ and rotates counterclockwise.
Ex2: What curve is represented by $x = t$ and $y = t^2$
$y = x^2$

The function is an upward opening parabola.
We can also define slightly more complicated functions through parametrics such as a cycloid: $\vec{r}(\theta) = \langle a\theta - a\sin\theta, a - a\cos\theta \rangle$ where a is the radius.
Ex3: Find a parameterization of the curve given by the intersection of $\{z = \sqrt{x^2+y^2}\} \cap \{z = 1+y\}$

$$\begin{aligned}
&\text{try } x = t \\
&z = \sqrt{t^2+y^2},\ z = 1+y \\
&1+y = \sqrt{t^2+y^2} \\
&(1+y)^2 = t^2+y^2 \Rightarrow 1+2y+y^2 = t^2+y^2 \\
&y = \frac{1}{2}(t^2-1) \\
&z = 1+\frac{1}{2}(t^2-1) = \frac{1}{2}(t^2+1) \\
&\vec{r}(t) = \left\langle t, \frac{1}{2}(t^2-1), \frac{1}{2}(t^2+1) \right\rangle
\end{aligned}$$

Derivatives of Parametric Curves
When taking the derivative of a parametric equation, we can either analyze the time derivatives of each component or we can use the chain rule to compare components to each other as we would normally do.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

For the example of the cycloid, we have $\dfrac{d\vec{r}}{d\theta} = \langle a - a\cos\theta, a\sin\theta \rangle$.
We can also get $\dfrac{dy}{dx} = \dfrac{a\sin\theta}{a - a\cos\theta}$
Ex: If $x = 2t^2+3$ and $y = t^4$, find $\frac{dy}{dx}$

$$\begin{aligned}
&\frac{dy}{dt} = 4t^3 \\
&\frac{dx}{dt} = 4t \\
&\frac{dy}{dx} = \frac{4t^3}{4t} = t^2 \\
&x = 2t^2+3 \Rightarrow t^2 = \frac{x-3}{2} \\
&\frac{dy}{dx} = \frac{x-3}{2}
\end{aligned}$$

We can also define higher order derivatives using the same method:

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

The derivatives for linear operations follow naturally from the product rule:

$$\frac{d}{dt}(f(t)\vec{r}(t)) = f'(t)\vec{r}(t) + f(t)\vec{r}'(t)$$

$$\frac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) = \vec{u}'\cdot\vec{v} + \vec{u}\cdot\vec{v}'$$
$$\frac{d}{dt}(\vec{u}(t)\times\vec{v}(t)) = \vec{u}'\times\vec{v} + \vec{u}\times\vec{v}'$$

Motion:
$\vec{r}$ is the position vector, s is the arc length (distance travelled along trajectory), $\vec{v}$ is the velocity, $\|\vec{v}\|$ is the speed, and $\vec{a}$ is the acceleration.
Note that $\vec{v}$ is always the direction tangent to the path of motion.

$$\vec{v} = \frac{d\vec{r}}{dt}$$
$$\vec{a} = \frac{d\vec{v}}{dt}$$
$$\|\vec{v}\| = \frac{ds}{dt}$$
$$\vec{v} = \hat{T}\|\vec{v}\| = \hat{T}\frac{ds}{dt}$$
$$s = \int_{t_0}^{t_1} \|\vec{r}'(t)\|dt$$

Ex: Find the speed of an object travelling around the unit circle.

$$\vec{r} = \langle \cos t, \sin t\rangle$$
$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$
$$\left(\frac{\Delta s}{\Delta t}\right)^2 = \left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2$$
$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$
$$\|\vec{v}\| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

Ex2: Find the circumference of a circle of radius a

$$\vec{r} = \langle a\cos t, a\sin t\rangle$$
$$ds = \sqrt{a^2\sin^2 t + a^2\cos^2 t}dt = adt$$
$$s = \int_0^{2\pi} adt = at\Big|_0^{2\pi} = 2\pi a$$

Kepler's Law:
states that for a planet in an elliptical orbit, the area swept out over time is always constant.
$\|\vec{r}\times\vec{v}\| = \text{constant}$
With this we can prove that $\vec{a}_{\|}\vec{r}$

$$\frac{d}{dt}(\vec{r}\times\vec{v}) = 0$$
$$\frac{d\vec{r}}{dt}\times\vec{v} + \vec{r}\times\frac{d\vec{v}}{dt} = 0$$

$$\vec{v} \times \vec{v} + \vec{r} \times \vec{a} = 0$$
$$\vec{r} \times \vec{a} = 0$$
$$\therefore \vec{a}_{\|} \vec{r}$$

1.1.2 Curvature

Curvature is defined to be how "curvy" a curve is. This is done by using a tangent circle to approximate the curve (similar to how a tangent line works).
The circle which best approximates the curve near a point is called the *circle of curvature.*
The radius of this circle is called the *radius of curvature* (represented by ρ) and the center of the circle is called the *center of curvature.*
The value of curvature itself is defined to be

$$\kappa = \frac{1}{\rho}$$

Note as $\kappa \to \infty$ the curve is *very* curvy and as $\kappa \to 0$ the curve is linear.
The equation for curvature is given by

$$\kappa = \left| \frac{\vec{v}(t) \times \vec{a}(t)}{\left(\frac{ds}{dt}\right)^3} \right|$$

which, expressed in cartesian coordinates can simplify to

$$\kappa = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}$$

Additionally, the radius of curvature is given by

$$\rho(t) = \frac{1}{\kappa(t)}$$

and the center of curvature is given by

$$\vec{r}(t) + \rho(t)\hat{N}(t)$$

1.1.3 Parametric Equations of Surfaces

Similar to lines, we can represent a surface with a parameterization $\vec{r}(u, v)$.
There are often many different ways to parameterize a surface but some will result in simpler forms than others.
If your surface has spherical symmetry, try an analog to spherical coordinates where one variable is taken to be a constant (usually ρ).
If your surface has circular symmetry about an axis, try cylindrical coordinates where one variable is taken to be a constant (usually r).
Otherwise, a good option may be to use Cartesian where you set your two variables to x and y and set the z-position to be a function of x and y.

Ex: Parameterize the hemisphere of radius 5 that lies above the xy-plane using 3 different parameterizations

$$S = \left\{x^2 + y^2 + z^2 = 25,\ z \geq 0\right\}$$
$$\vec{r}(x, y) = \left\langle x, y, \sqrt{25 - x^2 - y^2}\right\rangle$$
$$\vec{r}(r, \theta) = \left\langle r\cos\theta, r\sin\theta, \sqrt{25 - r^2}\right\rangle$$
$$\vec{r}(\theta, \phi) = \langle 5\cos\theta\sin\phi, 5\sin\theta\sin\phi, 5\cos\phi\rangle$$

Ex2: Find a parameterization of a torus with larger radius A (from the origin) smaller radius a (internal radius)

circular symmetry so try polar

let θ be the revolution around the central z-axis

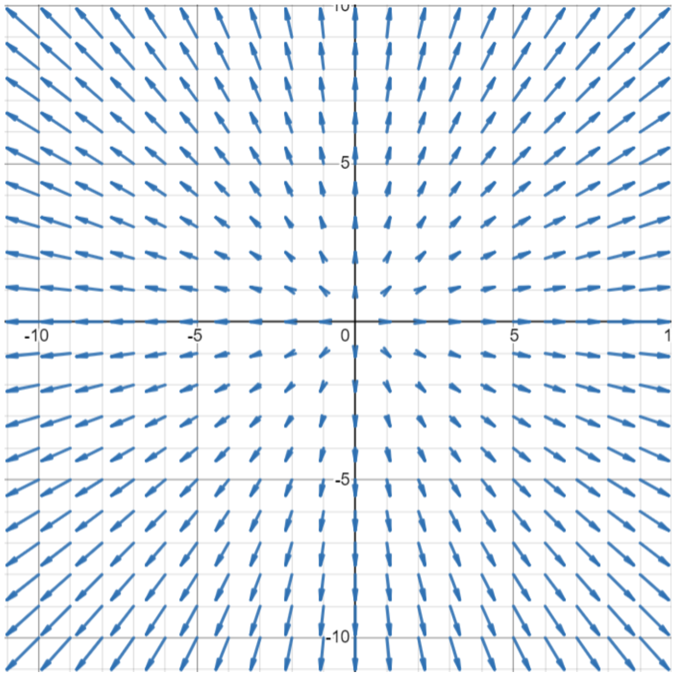
let φ be the revolution about the inside of the torus

$$(r(\varphi), z(\varphi)) = (A + a\sin\varphi, a\cos\varphi)$$
$$x(\theta, \varphi) = r(\varphi)\cos\theta$$
$$y(\theta, \varphi) = r(\varphi)\sin\theta$$
$$z(\theta, \varphi) = z(\varphi)$$
$$\vec{r}(\theta, \varphi) = \langle (A + a\sin\varphi)\cos\theta, (A + a\sin\varphi)\sin\theta, a\cos\varphi\rangle$$

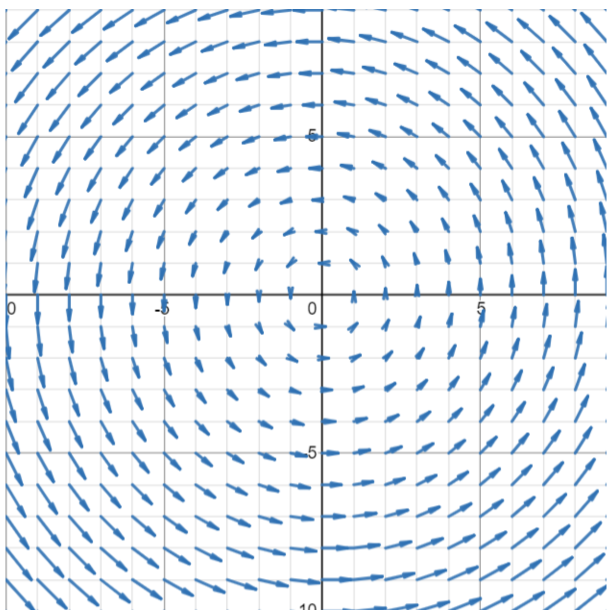
1.2 Integrals in the Plane

1.2.1 Vector Fields

A vector field can be defined as $\vec{F} = \langle P(x, y), Q(x, y)\rangle$ where every point (x, y) maps to a corresponding vector.
Ex: $\vec{F} = \langle x, y\rangle$



Ex2: $\vec{F} = \langle -y, x \rangle$



A vector field is defined to be conservative if it can be represented as the gradient of a function

$$\vec{F} = \nabla f$$

Ex: Conservation of energy:

$$\vec{F} = m\vec{a}$$

$$\nabla f = m\frac{d\vec{v}}{dt}$$

$$m\vec{v} \cdot \frac{d\vec{v}}{dt} = \vec{v} \cdot \nabla f$$

$$\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \cdot \langle f_x, f_y, f_z \rangle = \frac{1}{2} m \frac{d}{dt} (\vec{v} \cdot \vec{v})$$
$$\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} = \frac{d}{dt} \left(\frac{1}{2} m \|v\|^2 \right)$$
$$\frac{d}{dt} \left(\frac{1}{2} m v^2 - f(x, y, z) \right) = 0$$

A vector field is conservative if the curl is zero and the domain is simply connected (meaning it is continuous and defined everywhere in question).

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Note that just a curl of $\vec{0}$ is not enough to determine if a vector field is conservative (we will see an example of this later).

Divergence and Curl

There are two new operations that can be performed on a vector field. These are divergence and curl.

If we think of a vector filed as a fluid flow, divergence determines how compressible a vector field is. If there is more fluid being created then destroyed in a region, the divergence in that region will be positive (think of a source of a sink). Similarly, if there is more fluid being destroyed then the divergence will be negative (think of a drain of a sink). A divergence of 0 implies that the fluid is incompressible. The restult of the divergence will be a scalar and it is analogous to the dot product.

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = P_x + Q_y + R_z$$

Curl is a measure of how non-conservative a vector field is. A curl of zero implies that that a vector field is irrotational.

Some notes on divergence and curl.

- The image of the gradient are conservative vector fields
- The kernal (null space) of curl is the irrotational vector field
- If the domain is simply connected then the kernel of curl is equal to the image of the gradient.
- When a vector filed is divergence free, every surface is a boundary of a solid.

There are many different product rules that relate to divergence and curl. Here is an example:

$$\begin{aligned} \nabla \cdot (f\vec{F}) &= \nabla \cdot \langle fP, fQ, fR \rangle \\ &= (fP)_x + (fQ)_y + (fR)_z \\ &= f_x P + f_y Q + f_z R + f P_x + f Q_y + f R_z \\ &= \langle f_x, f_y, f_z \rangle \cdot \langle P, Q, R \rangle + f \langle P_x, Q_y, R_z \rangle \\ \nabla \cdot (f\vec{F}) &= (\nabla f) \cdot \vec{F} + f(\nabla \cdot \vec{F}) \end{aligned}$$

Ex: Compute $\nabla \cdot (r^k \vec{r})$ for where $\vec{r} = \langle x, y, z \rangle$ and $r = \|\vec{r}\| = \sqrt{x^2 + y^2 + z^2}$

$$\nabla \cdot (r^k \vec{r}) = (\nabla r^k) \cdot \vec{r} + r^k \nabla \cdot \vec{r}$$
$$\nabla \cdot \vec{r} = \nabla \cdot \langle x, y, z \rangle = 1 + 1 + 1 = 3$$
$$\nabla r^k = k r^{k-1} \nabla r, \;\; \nabla r = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle = \frac{\vec{r}}{r}$$
$$\nabla r^k = k r^{k-2} \vec{r}$$
$$\nabla \cdot (r^k \vec{r}) = k r^{k-2} \vec{r} \cdot \vec{r} + 3 r^k = k r^{k-2} \|\vec{r}\|^2 + 3 r^k$$
$$= (3 + k) r^k$$

1.2.2 Line Integrals

Line integrals over some path C can be computed the same way as with arc length. We first find a parameterization for C and then compute using the following formula,

$$\int_C f ds = \int_{t=a}^{b} f(\vec{r}(t)) \|\vec{r}'(t)\| dt$$

Ex: Integrate $f(x, y) = xy^4$ along the right half of the circle $x^2 + y^2 = 16$

$$C: \;\; \vec{r}(t) = \langle 4\cos t, 4 \sin t \rangle, \;\; -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$
$$\vec{r}'(t) = \langle -4 \sin t, 4 \cos t \rangle$$
$$\|\vec{r}'(t)\| = \sqrt{16 \sin^2 t + 16 \cos^2 t} = 4$$
$$\int_C f ds = \int_{-\pi/2}^{\pi/2} (4 \cos t)(4 \sin t)^4 (4) dt = 4^6 \left[\frac{\sin^5 t}{5} \right]_{-\pi/2}^{\pi/2} = \frac{8192}{5}$$

1.2.3 Work Integrals

We can define a path integral inside a force field using the concept of work. Work is the force of the vector field times the distance travelled in the same direction. So we can define an infinitesimal amount of work to be $dW = \vec{F} \cdot d\vec{r}$ which gives the integral formula

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C (\vec{F} \cdot \hat{T}) ds = \int_{t=a}^{b} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt$$

Note that the shorthand notation $d\vec{r} = \|\vec{r}'(t)\| dt$.
Ex: Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle x, x \rangle$ and C is the curve $3x = y^2$ starting at $(0, 0)$ and ending at $(3, 3)$.

$$\vec{r}(t) = \left\langle \frac{t^2}{3}, t \right\rangle$$
$$\vec{r}'(t) = \left\langle \frac{2}{3} t, 1 \right\rangle$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^3 \left\langle \frac{t^2}{3}, \frac{t^2}{3} \right\rangle \cdot \left\langle \frac{2}{3}t, 1 \right\rangle dt = \int_0^3 \left(\frac{2}{9}t^3 + \frac{t^2}{3} \right) dt = \left[\frac{t^4}{18} + \frac{t^3}{9} \right]_0^3 = \frac{15}{2}$$

Fundamental Theorem of Line Integrals:
This states that if $\vec{F}$ is conservative ($\vec{F} = \nabla f$) then

$$\int_C \vec{F}\cdot d\vec{r} = f(P_2) - f(P_1)$$

meaning that the work integral is path independent.
Note that this formula implies that the integral of any closed loop over a conservative vector field is always 0.

$$\oint_C \vec{F}\cdot d\vec{r} = 0$$

$\vec{F}$ is conservative when the curl is zero and the domain is simply connected.
Ex2: Find $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \left\langle e^{-x^2} + 3y^3 - 3x^2y, \arctan(y^3) + 6xy^2 - x^3 \right\rangle$ and C is the closed loop formed by the unit circle in the clockwise direction.

$\nabla \times \vec{F} = \left(6y^2 - 3x^2 - 9y^2 + 6x^2\right)\hat{k} = -3y^2\hat{k}$

we can break this up into two different focre fields $\vec{F}_1$ and $\vec{F}_2$

$\vec{F}_1 = \left\langle e^{-x^2} + 2y^3 - 3x^2y, \arctan(y^3) + 6xy^2 - x^3 \right\rangle, \ \nabla \times \vec{F}_1 = \vec{0}$

$\vec{F}_2 = \left\langle y^3, 0 \right\rangle$

$\vec{F}_1$ is conservative so $\displaystyle\int_C \vec{F}_1 \cdot d\vec{r} = 0$

$\displaystyle\int_C \vec{F}_2 \cdot d\vec{r}$ will require a parameterization

$\vec{r}(t) = \langle -\cos t, \sin t\rangle$

$\vec{r}'(t) = \langle \sin t, \cos t\rangle$

$$\int_C \vec{F}_2 \cdot \ d\vec{r} = \int_0^{2\pi} \langle \sin^3 t, 0\rangle \cdot \langle \sin t, \cos t\rangle\, dt = \int_0^{2\pi} \sin^4 t dt = \frac{3\pi}{4}$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{F}_1 \cdot d\vec{r} + \int_C \vec{F}_2 \cdot d\vec{r} = \frac{3\pi}{4}$$

Ex3: Find $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ and C is the counterclockwise circle of radius R around the origin.

$\nabla \times \vec{F} = \vec{0}$

$\vec{F}$ is undefined at $(0,0)$, meaning it is not simply connected

This means we cannot apply the fundamental theorem of line integrals

$\vec{r}(t) = \langle R\cos t, R\sin t\rangle$

$\vec{r}'(t) = \langle -R\sin t, R\cos t\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left\langle \frac{-R\sin t}{R^2}, \frac{R\cos t}{R^2} \right\rangle \cdot \langle -R\sin t, R\cos t\rangle\, dt = \int_0^{2\pi} (\sin^2 t + \cos^2 t) dt = 2\pi$$

1.2.4 Green's Theorem

Green's Theorem is an extention of the fundamental theorem of line integrals for closed path integrals. Essentially, it transforms a line integral into a double integral over the enclosed region. For a vector field $\vec{F} = \langle P, Q \rangle$

$$\oint_{\partial R} \vec{F} \cdot d\vec{r} = \iint_R (Q_x - P_y) dA$$

Some caveats to this is that the vector field must be defined over the entire region and that the loop travels counterclockwise (it can go counterclockwise but with a change of sign) so that the region will always be to the left of the direction of travel along the loop.

We can see this using some previous examples:

Ex: Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \left\langle e^{-x^2} + 3y^3 - 3x^2y, \arctan(y^3) + 6xy^2 - x^3 \right\rangle$ and C is the closed loop formed by the unit circle in the clockwise direction.

$$Q_x - P_y = \left(6y^2 - 3x^2 - 9y^2 + 6x^2\right) = -3y^2$$
$$\oint_C \vec{F} \cdot d\vec{r} = -\iint_{x^2+y^2=1} -3y^2 dA = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} 3r^3 \sin^2 \theta dr d\theta$$
$$= \frac{3\pi}{4}$$

If a loop is not closed or has a hole in it, we can add another line integral to close the loop so that we can compute with Green's Theorem.

Ex2: Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ and C is the counterclockwise circle of radius R around the origin.

Notice with this vector field the center is undefined so we can create another infinitesimally small loop around the origin so that the vector field is now defined everywhere.

$$C' = \left\{x^2 + y^2 = \epsilon^2, \ |\epsilon| \ll 1\right\} \text{ travelling clockwise}$$
$$\oint_{\partial R} \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot d\vec{r} + \int_{C'} \vec{F} \cdot d\vec{r}$$
$$Q_x - P_y = 0$$
$$\oint_{\partial R} \vec{F} \cdot d\vec{r} = \iint 0 d\vec{r} = 0$$
$$\vec{r} = \langle -\epsilon \cos t, \epsilon \sin t \rangle$$
$$\vec{r}' = \langle \epsilon \sin t, \epsilon \cos t \rangle$$
$$\int_{C'} \vec{F} \cdot d\vec{r} = \lim_{\epsilon \to 0} \int_0^{2\pi} \left\langle \frac{-\epsilon \sin t}{\epsilon^2}, \frac{\epsilon \cos t}{\epsilon^2} \right\rangle \cdot \langle -\epsilon \sin t, \epsilon \cos t \rangle = 2\pi$$
$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \oint_{\partial R} \vec{F} \cdot d\vec{r} - \int_{C'} \vec{F} \cdot d\vec{r} = 2\pi$$

Ex3: Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\text{need } Q_x - P_y = 1$$
$$\vec{F} = \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle$$

$$\vec{r} = \langle a\cos t, b\sin t\rangle$$
$$\vec{r}' = \langle -a\sin t, b\cos t\rangle$$
$$\oint \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \frac{1}{2}\langle -b\sin t, a\cos t\rangle \cdot \langle -a\sin t, b\cos t\rangle\, dt = \frac{1}{2}\int_0^{2\pi}(ab\sin^2 t + ab\cos^2 t)dt$$
$$= \frac{ab}{2}\int_0^{2\pi} dt = \pi ab$$

1.3 Integrals in Space

1.3.1 Surface Integrals

Similar to line integrals, a surface integral is an integral evaluated over the surface of a region. It is computed by

$$\iint_S f dS = \iint_S f(\vec{r}(u,v))\|\vec{r}_u \times \vec{r}_v\| dA$$

Note that dS decomposes into $dS = \|\vec{r}_u \times \vec{r}_v\| dA$

Ex: Find the surface area of a sphere or radius 2.

$$\vec{r}(\theta,\phi) = \langle 2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi\rangle$$
$$\vec{r}_\theta = \langle -2\sin\theta\sin\phi, 2\cos\theta\sin\phi, 0\rangle$$
$$\vec{r}_\phi = \langle 2\cos\theta\cos\phi, 2\sin\theta\cos\phi, -2\sin\phi\rangle$$
$$\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \end{vmatrix}$$
$$= \left\langle -4\cos\theta\sin^2\phi, -4\sin\theta\sin^2\phi, -4\sin^2\theta\sin\phi\cos\phi - 4\cos^2\theta\sin\phi\cos\phi \right\rangle$$
$$= \left\langle -4\cos\theta\sin^2\phi, -4\sin\theta\sin^2\phi, -4\sin\phi\cos\phi \right\rangle$$
$$= -4\sin\phi\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$$
$$\|\vec{r}_\theta \times \vec{r}_\phi\| = 4\sin\phi\sqrt{\cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi + \cos^2\phi}$$
$$= 4\sin\phi\sqrt{\sin^2\phi + \cos^2\phi}$$
$$= 4\sin\phi$$
$$\iint_S dS = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi} 4\sin\phi = 8\pi\left[-\cos\phi\right]_0^\pi = 16\pi$$

Ex2: Let S be the part of the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$ which lies above the xy-plane and below the plane $z = A$ where A is a positive constant. Compute the integral $\iint_S z dS$.

$$\text{let } \vec{r}(\theta,z) = \left\langle \sqrt{1+z^2}\cos\theta, \sqrt{1+z^2}\sin\theta, z\right\rangle,\ 0\le\theta\le 2\pi,\ 0\le z\le A$$
$$\vec{r}_\theta = \left\langle -\sqrt{1+z^2}\sin\theta, \sqrt{1+z^2}\cos\theta, 0\right\rangle$$
$$\vec{r}_z = \left\langle \frac{z}{\sqrt{1+z^2}}\cos\theta, \frac{z}{\sqrt{1+z^2}}\sin\theta, 1\right\rangle$$
$$\vec{r}_\theta \times \vec{r}_z = \left\langle \sqrt{1+z^2}\cos\theta, \sqrt{1+z^2}\sin\theta, -z\sin^2\theta - z\cos^2\theta\right\rangle$$

$$= \left\langle \sqrt{1+z^2}\cos\theta, \sqrt{1+z^2}\sin\theta, -z \right\rangle$$

$$\|\vec{r}_\theta \times \vec{r}_z\| = \sqrt{(1+z^2)\cos^2\theta + (1+z^2)\sin^2\theta + z^2} = \sqrt{1+z^2+z^2} = \sqrt{1+2z^2}$$

$$\iint_S z dS = \int_{\theta=0}^{2\pi}\int_{z=0}^{A} z\sqrt{1+2z^2}dzd\theta = 2\pi\left[\frac{1}{4}\cdot\frac{2}{3}(1+2z^2)^{3/2}\right] = \frac{\pi}{3}\left((1+2A^2)^{3/2}-1\right)$$

1.3.2 Flux Integrals

Integrating a vector field over a surface in $\mathbb{R}^3$ gives a flux integral. This can be thought of how much the vector field is flowing in or out of the surface.
Note that a flux integral also requires a choice of orientation. Similar to how a curve had a direction, an orientation states which side of the surface is positive.
A flux integral is formed as follows:

$$\iint_S \vec{F}\cdot d\vec{S} = \pm\iint_S \vec{F}\cdot\hat{n}dS = \pm\iint_S \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)dA$$

Ex: Let S be the part of the surface $z = 4 - x^2 - y^2$ lying above the xy-plane, oriented upward. Let $\vec{F} = \langle x(x^2+y^2), y(x^2+y^2), z\rangle$. Compute $\iint_S \vec{F}\cdot d\vec{S}$.

$$\vec{r}(\theta,z) = \left\langle \sqrt{4-z}\cos\theta, \sqrt{4-z}\sin\theta, z\right\rangle,\ 0\le\theta\le 2\pi,\ 0\le z\le 4$$

$$\vec{F}(\vec{r}) = \left\langle \sqrt{4-z}\cos\theta(4-z), \sqrt{4-z}\sin\theta(4-z), z\right\rangle$$

$$\vec{r}_z = \left\langle \frac{-1}{2\sqrt{4-z}}\cos\theta, \frac{-1}{2\sqrt{4-z}}\sin\theta, 1\right\rangle$$

$$\vec{r}_\theta = \left\langle -\sqrt{4-z}\sin\theta, \sqrt{4-z}\cos\theta, 0\right\rangle$$

$$\vec{r}_z\times\vec{r}_\theta = \left\langle -\sqrt{4-z}\cos\theta, -\sqrt{4-z}\sin\theta, -\frac{1}{2}(\cos^2\theta+\sin^2\theta)\right\rangle$$

$$\iint_S \vec{F}\cdot d\vec{S} = -\int_{\theta=0}^{2\pi}\int_{z=0}^{4}\left(-(4-z)^2(\cos^2\theta+\sin^2\theta) - \frac{z}{2}\right)dzd\theta$$

$$= 2\pi\left[-\frac{1}{3}(4-z)^3 + \frac{z^2}{4}\right]_0^4$$

$$= 2\pi\left(4+\frac{4^3}{3}\right) = \frac{152\pi}{3}$$

Ex2: Let S be the disk of radius 3, oriented upward on the xy-plane centered at $(15,16,0)$ and let $\vec{F} = \left\langle e^{\tan\sqrt{z}} + yz, \frac{x\ln(z+1)}{y^6}, 3+2\cos(z^2)\right\rangle$.

note $z = 0$ and $x, y > 0$ for all points on S

$$\vec{F} = \langle 1, 0, 5\rangle$$

$$\hat{n} = \hat{k} = \langle 0,0,1\rangle$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iint_S \langle 1,0,5\rangle\cdot\langle 0,0,1\rangle\, dS = \iint_S 5dS = 5(9\pi) = 45\pi$$

1.3.3 The Divergence Theorem

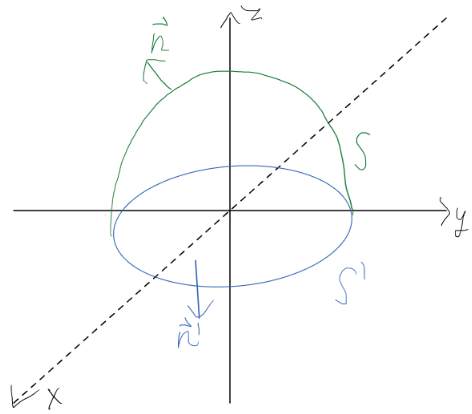
Divergence Theorem is given by

$$\oiint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E (\nabla \cdot \vec{F}) dV$$

where E is a solid region in $\mathbb{R}^3$ and ∂E is the boundary of E oriented outward.
This formula can be thought of as the rate that fluid escapes E is equal to the total rate of fluid being produced.
Ex: Find the flux from the vector field $\vec{F} = \langle x, y, z \rangle$ out of the cube of side length 2 centered at the origin.

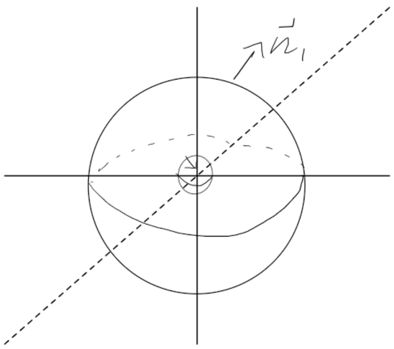
$$\nabla \cdot \vec{F} = 3$$
$$\iiint_E 3dV = 3 \cdot \text{Vol}(E) = 24$$

Ex2: Let S be the hemisphere of radius 1 located above the xy-plane, oriented outward and let $\vec{F} = \left\langle z^2 x, \frac{1}{3}y^3 + \tan\sqrt{z}, x^2 z + y^2 \right\rangle$. Find $\iint_S \vec{F} \cdot d\vec{S}$.



$$\nabla \cdot \vec{F} = z^2 + y^2 + x^2 = \rho^2$$
$$\partial E = S + S'$$
$$E = \left\{x^2 + y^2 + z^2 \leq 1,\ z \geq 0\right\}$$
$$S' = \left\{x^2 + y^2 = 1\ z = 0\right\}$$
$$\iiint_E \rho^2 dV = \iint_S \vec{F} \cdot d\vec{S} + \iint_{S'} \vec{F} \cdot d\vec{S}$$
$$\iiint_E \rho^2 dV = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/2} \int_{\rho=0}^{1} \rho^4 \sin\phi d\rho d\phi d\theta = 2\pi \left[-\cos\phi\right]_0^{\pi/2} \left[\frac{\rho^5}{5}\right]_0^1 = \frac{2\pi}{5}$$
$$\iint_{S'} \vec{F} \cdot d\vec{S} = \left\langle z^2 x, \frac{1}{3}y^3 + \tan\sqrt{z}, x^2 z + y^2 \right\rangle \cdot \langle 0, 0, -1 \rangle dS = -\iint_{S'} x^2 z + y^2 dS$$
$$\iint_{S'} \vec{F} \cdot d\vec{S} = -\iint_{S'} y^2 dS = -\int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^3 \sin^2\theta dr d\theta = -\pi \left[\frac{r^4}{4}\right]_0^1 = -\frac{\pi}{4}$$
$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \rho^2 dV - \iint_{S'} \vec{F} \cdot d\vec{S} = \frac{5\pi}{2} + \frac{\pi}{4} = \frac{13\pi}{20}$$

It is also important to note that to apply the divergence theorem, the vector field must be defined within the region.
Ex3: Find the flux from the vector field $\vec{F} = \frac{\vec{r}}{r^3}$ out of the surface ∂E where E is the sphere of radius 2 centered about the origin.
Note that because the vector field is undefined at the origin, we need to compute the flux about the origin separately.
We can define the sphere S_ϵ to be the sphere of radius ϵ about the origin for where $\epsilon \to 0$ and subtract its flux from the total flux of the region.



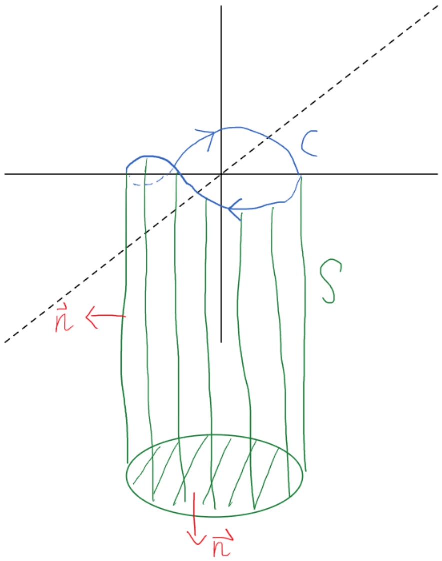
$$\iiint_{\partial E} (\nabla \cdot \vec{F}) dV = \iint_S \vec{F} \cdot d\vec{S} + \iint_{S_\epsilon} \vec{F} \cdot d\vec{S}$$
$$\vec{F} = \left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$
$$\nabla \cdot \vec{F} = 0$$
$$\hat{n}_2 = -\frac{\vec{r}}{r}$$
$$\iint_{S_\epsilon} \vec{F} \cdot d\vec{S} = \iint_{S_\epsilon} \frac{\vec{r}}{r^3} \cdot \frac{-\vec{r}}{r} dS = -\iint_{S_\epsilon} \frac{1}{r^2} dS = -\frac{1}{\epsilon^2} \iint_{S_\epsilon} dS = -\frac{1}{\epsilon^2} \cdot 4\pi\epsilon^2 = -4\pi$$
$$\iint_S \vec{F} \cdot d\vec{S} = -\iint_{S_\epsilon} \vec{F} \cdot d\vec{S} = 4\pi$$

1.3.4 Stokes' Theorem

Stokes' Theorem is a more general case of Green's Theorem and is given by

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

Ex: Compute $\oint_C \vec{F} \cdot d\vec{r}$ where C is given by $\vec{r}(t) = \langle \sin t, \cos t, -\sin 2t \rangle$
and $\vec{F} = \left\langle \tan\sqrt{1+x^4}, zy + e^{y^3}, \frac{x^2}{2} + \sqrt[3]{\sin(z^2)} \right\rangle$



$$\nabla \times \vec{F} = \langle -y, -x, 0 \rangle$$
$$S_1 = \left\{x^2 + y^2 = 1\right\}$$
$$S_2 = \{z = -A\}$$
$$\oint \vec{F} \cdot d\vec{r} = \iint_{S_1} (\nabla \times \vec{F}) cdot d\vec{S} + \iint_{S_2} (\nabla \times \vec{F}) \cdot d\vec{S}$$
$$\iint_{S_2} \langle -y, -x, 0 \rangle \cdot \langle 0, 0, -1 \rangle \, dS = 0$$
$$S_1 : \ \vec{r}(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle \,, \ 0 \le \theta \le 2\pi, \ -A \le z \le -\sin 2\theta$$
$$\vec{r}_z = \langle 0, 0, 1 \rangle$$
$$\vec{r}_\theta = \langle -\sin\theta, \cos\theta, 0 \rangle$$
$$\vec{r}_z \times \vec{r}_\theta = \langle -\cos\theta, -\sin\theta, 0 \rangle$$
$$\iint_{S_1} (\nabla \times \vec{F}) \cdot d\vec{S} = -\int_{\theta=0}^{2\pi} \int_{z=-A}^{-\sin 2\theta} \langle -\sin\theta, -\cos\theta, 0 \rangle \cdot \langle -\cos\theta, -\sin\theta, 0 \rangle \, dz d\theta$$
$$= \int_0^{2\pi} \int_{-A}^{-\sin 2\theta} -2 \sin\theta \cos\theta dz d\theta = \int_0^{2\pi} -2 \sin\theta \cos\theta \left(-\sin 2\theta + A\right) d\theta$$
$$= \int_0^{2\pi} (\sin^2(2\theta) - A \sin(2\theta)) d\theta = \pi$$
$$\Rightarrow \oint \vec{F} \cdot d\vec{r} = \pi$$

1.3.5 Differential Forms

Differential forms are the unifying language of derivatives and integrals.
Let $x_1, \ldots, x_n$ be coordinates on $\mathbb{R}^n$. We can introduce the *wedge product* such that $dx_i \wedge dx_j = -dx_j \wedge dx_i$. Now let $U \subset \mathbb{R}^n$ be an open set. A differential k-form is a linear combination of k-fold products of dx's with functions on U as coefficients.

For example, a 1-form corresponds to $Pdx + Qdy + Rdz$ and a 2-form can be written as $Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$
Notice that with wedge products there is also the rule $dx \wedge dx = 0$.
Zero forms are just functions.
A set of k-forms are written as $\Omega^k(U)$.

k-form	expression	geometric meaning
$\Omega^0(U)$	f	function
$\Omega^1(U)$	$F_1dx + F_2dy + F_3dz$	vector field
$\Omega^2(U)$	$F_1dy \wedge dz + F_2dz \wedge dx + F_3dx \wedge dy$	vector field
$\Omega^3(U)$	$fdx \wedge dy \wedge dz$	function

A wedge product is computed as

$$\Omega^k(U) \wedge \Omega^l(U) \to \Omega^{k+l}(U)$$

This allows us to express vector operations in terms of wedge products in a more compact notation.
Ex: The cross product is given by $\Omega^1(U) \wedge \Omega^1(U) \to \Omega^2(U)$

$$\begin{aligned}
&F, G \in \Omega^1(\mathbb{R}^3) \\
&F \wedge G = (F_1dx + F_2dy + F_3dz) \wedge (G_1dx + G_2dy + G_3dz) \\
&= F_1G_2dx \wedge dy + F_1G_3dx \wedge dz + F_2G_1dy \wedge dx + F_2G_3dy \wedge dz + F_3G_1dz \wedge dx + F_3G_2dz \wedge dy \\
&= (F_2G_3 - F_3G_2)dy \wedge dz + (F_3G_1 - F_1G_3)dz \wedge dx + (F_1G_2 - F_2G_1)dx \wedge dy \\
&= \vec{F} \times \vec{G}
\end{aligned}$$

Ex2: The dot product is given by $\Omega^1(U) \wedge \Omega^2(U) \to \Omega^3(U)$

$$\begin{aligned}
&F \in \Omega^1(\mathbb{R}^3) \\
&G \in \Omega^2(\mathbb{R}^3) \\
&F \wedge G = (F_1dx + F_2dy + F_3dz) \wedge (G_1dy \wedge dz + G_2dz \wedge dx + G_3dx \wedge dy) \\
&= F_1G_1dx \wedge dy \wedge dz + F_2G_2dy \wedge dz \wedge dx + F_3G_3dz \wedge dx \wedge dy \\
&= (F_1G_1 + F_2G_2 + F_3G_3)dx \wedge dy \wedge dz \\
&= \vec{F} \cdot \vec{G}
\end{aligned}$$

Differential forms also allows us to generalize the types of derivatives (gradient, curl, divergence) into one derivative.

$$d\Omega^k(U) \to \Omega^{k+1}(U)$$

$\Omega^k(U)$	derivative transformation	Type of derivative
$\Omega^0(U)$	scalar to vector field	gradient (∇f)
$\Omega^1(U)$	vector field to vector field	curl ($\nabla \times \vec{F}$)
$\Omega^2(U)$	vector field to scalar	divergence ($\nabla \cdot \vec{F}$)

In general, taking a derivative twice gives 0. $d(d\Omega^k(U)) = 0$
Differential forms also allows us to easily define product rules using the following formula.
If $\alpha \in \Omega^k(U)$ and $\beta \in \Omega^l(U)$ then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

Ex: $\nabla \times (f\vec{F})$

$f \in \Omega^0(U),\ \vec{F} \in \Omega^1(U)$

$\nabla \times (f\vec{F}) = \nabla f \times \vec{F} + f(\nabla \times \vec{F})$

Ex2: $\nabla \cdot (\vec{A} \times \vec{B})$

$\vec{A}, \vec{B} \in \Omega^1(U)$

$\nabla \cdot (\vec{A} \times \vec{B}) = (\nabla \times \vec{A}) \cdot \vec{B} - \vec{A} \cdot (\nabla \times \vec{B})$

Similar to derivatives, we can go in the opposite direction and generalize multiple integration using differential forms.
If M is a (k+1)-manifold, ∂M is a k-manifold, α is a k-form, and $d\alpha$ is a (k+1)-form then we can write the generalized Stokes' Theorem as

$$\int_{\partial M} \alpha = \int_M d\alpha$$

Differential forms also behave nicely under a change of variables.

Ex: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$dx = \dfrac{\partial x}{\partial r}dr + \dfrac{\partial x}{\partial \theta}d\theta = \cos\theta dr - r\sin\theta d\theta$

$dy = \dfrac{\partial y}{\partial r}dr + \dfrac{\partial y}{\partial \theta}d\theta = \sin\theta dr + r\cos\theta d\theta$

$dx \wedge dy = (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta)$

$= r\cos^2\theta dr \wedge d\theta + r\sin^2\theta dr \wedge d\theta = r dr \wedge d\theta$