

Math Notes

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1 Complex Analysis

1.1 Complex Algebra

Complex numbers arise from the roots of polynomials.

Ex: $x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm\sqrt{-1}$. This polynomial has no real roots, however, we can introduce an imaginary number i such that $i^2 = -1$. Then we will have the solution $x = \pm i$

We can introduce *complex numbers* which are numbers in the form $z = x + iy$, where x is the real part of z , $\Re(z)$, and y is the imaginary part of z , $\Im(z)$. These numbers can also be expressed in vector notation along the complex plane.

1.1.1 Complex Arithmetic

Addition, subtraction, and multiplication work the same, just with the addition of the fact $i^2 = -1$. For division, we require what is called the conjugate.

The conjugate of a complex number is the same number, just with the sign of the imaginary component flipped.

$$\bar{z} = x - yi$$

where \bar{z} is the conjugate of z .

Similarly to vectors, we can also define the modulus (length) of a complex number

$$|z|^2 = x^2 + y^2 = z \cdot \bar{z}$$

Using this, we can define the division of a complex number and also define the real and imaginary components of a complex number.

The general expression for division is:

$$\frac{u}{z} = \frac{s + it}{x + iy} = \frac{(s + it)(x - iy)}{(x + iy)(x - iy)} = \frac{u\bar{z}}{x^2 + y^2} = \frac{u\bar{z}}{|z|^2}$$

The real and imaginary components can be computed as

$$z = x + iy$$

$$\bar{z} = x - iy$$

$$\Re(z) = \frac{z + \bar{z}}{2}$$

$$\Im(z) = \frac{z - \bar{z}}{2i}$$

Ex1: Simplify $(1 + 2i)(3 + i)(2 - 3i)$

$$\begin{aligned}(1 + 2i)(3 + i)(2 - 3i) &= (3 + i + 6i - 2)(2 - 3i) = (1 + 7i)(2 - 3i) \\ &= 2 - 3i + 14i + 21 \\ &= 23 + 11i\end{aligned}$$

Ex2: Simplify $\left(\frac{2+i}{1+i}\right)^2$

$$\begin{aligned}\left(\frac{2+i}{1+i}\right)^2 &= \left(\frac{(2+i)(1-i)}{(1+i)(1-i)}\right)^2 = \left(\frac{2+i-2i+1}{2}\right)^2 = \left(\frac{3-i}{2}\right)^2 = \frac{9-6i-1}{4} \\ &= 2 - \frac{3}{2}i\end{aligned}$$

Ex3: Simplify $(1 + 2i)^5$

$$\begin{aligned}(1 + 2i)^5 &= 1^5 + 5(1)^4(2i) + 10(1)^3(2i)^2 + 10(1)^2(2i)^3 + 5(1)(2i)^4 + (2i)^5 \\ &= 1 + 10i + 10(-4) + 10(-8i) + 5(16) + 32i \\ &= 1 + 10i - 40 - 80i + 80 + 32i \\ &= 41 - 38i\end{aligned}$$

Ex4: Prove

$$\text{if } |z| = 1 \text{ then } \Re\left(\frac{1}{1+z}\right) = \frac{1}{2}$$

Proof.

$$\begin{aligned}\Re(w) &= \frac{w + \bar{w}}{2} \\ \overline{\frac{1}{1+z}} &= \frac{1}{1+\bar{z}} \\ \Re\left(\frac{1}{1+z}\right) &= \frac{\frac{1}{1+z} + \frac{1}{1+\bar{z}}}{2} = \frac{1+\bar{z}+1+z}{2(1+z)(1+\bar{z})} = \frac{2+z+\bar{z}}{2(1+z+\bar{z}+|z|^2)} \\ |z| = 1 &\Rightarrow \Re\left(\frac{1}{1+z}\right) = \frac{2+z+\bar{z}}{2(2+z+\bar{z})} = \frac{1}{2}\end{aligned}$$

□

Some properties of \bar{z} :

- $\bar{\bar{z}} = z$
- $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$
- $|z_1 z_2| = |z_1| |z_2|$

Some common inequalities come from the triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof.

$$\begin{aligned}|z_1 + z_2| &\leq |z_1| + |z_2| \\ |z_1 + z_2| &= \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \\ \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} &\leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} \\ |z_1 + z_2| &\leq |z_1| + |z_2|\end{aligned}$$

□

$$\Rightarrow |z_1 \pm z_2| \geq ||z_1| - |z_2||$$

This leads to a general upper and lower bound that can be derived from these inequalities:

$$||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$$

1.1.2 Polar Form of Complex Numbers

Another way to represent complex numbers is through polar form. To use this we must first introduce Euler's identity:

$$e^{i\varphi} = \cos \varphi + i \sin \varphi$$

This then helps us with the equation for the polar form

$$z = a + ib = |z|(\cos \varphi + i \sin \varphi) = |z|e^{i\varphi}$$

For $z = re^{i\varphi}$ We call r the magnitude of the complex number and φ is the argument. This can be especially useful for simplifying some complex numbers.

Ex:

$$\left| \frac{(1 + \sqrt{3}i)^{100}}{(\sqrt{3} - i)^{100}} \right| = \frac{|1 + \sqrt{3}i|^{100}}{|\sqrt{3} - i|^{100}} = \frac{2^{100}}{2^{100}} = 1$$

Note that because sinusoidal functions are periodic every 2π this means that there infinite ways to express a function in polar coordinates.

$$e^{i2\pi k} = 1, \quad k \in \mathbb{Z} \Rightarrow z = re^{i(\varphi+2\pi k)}$$

To get around the issue of having infinite possible polar forms for every complex number we define what's called the principal argument to be

$$\text{Arg}(z) = \varphi \in (-\pi, \pi]$$

We define the regular argument to be

$$\arg(z) = \text{Arg}(z) + 2k\pi, \quad k \in \mathbb{Z}$$

Note that $\text{Arg}(z)$ is singular valued while $\arg(z)$ is multi-valued.

We can define $\text{Arg}(z)$ in terms of the real and imaginary parts (x and y) as

$$\text{Arg}(z) = \arctan\left(\frac{y}{x}\right) \pm k\pi$$

What it is specifically depends on what quadrant of the complex plane the point lies in.

- QI: $\text{Arg}(z) = \arctan\left(\frac{y}{x}\right)$
- QII: $\text{Arg}(z) = \arctan\left(\frac{y}{x}\right) + \pi$
- QIII: $\text{Arg}(z) = \arctan\left(\frac{y}{x}\right) - \pi$
- QIV: $\text{Arg}(z) = \arctan\left(\frac{y}{x}\right)$

Ex:

$$\text{Arg}(-1 - \sqrt{3}i) = \arctan(\sqrt{3}) - \pi = -\frac{2\pi}{3}$$

Ex2:

$$\begin{aligned} \arg(1 - \sqrt{3}i) \\ \text{Arg}(1 - \sqrt{3}i) &= -\frac{\pi}{3} \\ \arg(1 - \sqrt{3}i) &= -\frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z} \end{aligned}$$

Ex3:

$$\arg(-1 + 2i)$$

$$\begin{aligned}\operatorname{Arg}(-1+2i) &= \arctan(-2) + \pi \\ \arg(-1+2i) &= \pi - \arctan(2) + 2k\pi\end{aligned}$$

Ex4: Simplify

$$\begin{aligned}z &= -3 + 3i \\ \operatorname{Arg}(z) &= \arctan\left(\frac{3}{-3}\right) + \pi = \frac{3\pi}{4} \\ |z| &= 3\sqrt{2} \\ z &= 3\sqrt{2}e^{\frac{3\pi}{4}i}\end{aligned}$$

Ex5: Simplify

$$\begin{aligned}z &= -3 - 3i \\ \operatorname{Arg}(z) &= \arctan\left(\frac{-3}{-3}\right) - \pi = -\frac{3\pi}{4} \\ |z| &= 3\sqrt{2} \\ z &= 3\sqrt{2}e^{-\frac{3\pi}{4}i}\end{aligned}$$

Ex6: Simplify

$$\begin{aligned}z &= \frac{1-i}{-\sqrt{3}+i} = \frac{u}{v} \\ \arg(z) &= \arg(u) - \arg(v) = \arctan\left(\frac{-1}{1}\right) - \left(\arctan\left(\frac{1}{-\sqrt{3}}\right) + \pi\right) + 2k\pi \\ &= -\frac{\pi}{4} - \frac{5\pi}{6} + 2k\pi = -\frac{13\pi}{12} + 2k\pi \\ \operatorname{Arg}(z) &= \frac{11\pi}{12} \\ |z| &= \frac{|u|}{|v|} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} \\ z &= \frac{e^{\frac{11\pi}{12}i}}{\sqrt{2}}\end{aligned}$$

Ex7: Simplify

$$\begin{aligned}z &= (\sqrt{3} - i)^2 = w^2 \\ \arg(w) &= \arctan\left(\frac{-1}{\sqrt{3}}\right) + 2k\pi = -\frac{\pi}{6} \\ \Rightarrow \arg(z) &= 2\arg(w) = -\frac{\pi}{3} + 4k\pi \\ \operatorname{Arg}(z) &= -\frac{\pi}{3} \\ |w| &= 2 \Rightarrow |z| = |w|^2 = 4 \\ z &= 4e^{-\frac{\pi}{3}i}\end{aligned}$$

Ex8: Solve for all values of z

$$e^z = -1 - \sqrt{3}i$$

$$e^z = 2e^{-i\frac{2\pi}{3}+2\pi ik} = e^{\ln 2 - i\frac{2\pi}{3}+2\pi ik}$$

$$z = \ln 2 - i\frac{2\pi}{3} + 2\pi ik, \quad \forall k \in \mathbb{Z}$$

Properties of $\text{Arg}(z)$ and $\arg(z)$

- $\text{Arg}(z_1 z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2)$

Proof. Proof by contradiction: assume that $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$ is true.
Take $z_1 = z_2 = -1$.

$$\text{Arg}(z_1) = \text{Arg}(z_2) = \pi$$

$$\Rightarrow \text{Arg}(z_1) + \text{Arg}(z_2) = 2\pi$$

$$\text{Arg}(z_1 z_2) = \text{Arg}(1) = 0$$

$$\text{Arg}(z_1 z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2) \quad \forall z_1, z_2 \neq 0 \in \mathbb{C}$$

□

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\text{Arg}(\bar{z}) \neq -\text{Arg}(z)$

Proof. Proof by contradiction: assume that $\text{Arg}(\bar{z}) = -\text{Arg}(z)$ is true.
Take $z = -1$

$$\bar{z} = z = -1$$

$$\text{Arg}(z) = \pi$$

$$\text{Arg}(\bar{z}) = \pi$$

$$-\text{Arg}(z) = -\pi$$

$$\Rightarrow \text{Arg}(\bar{z}) \neq -\text{Arg}(z) \quad \forall z \in \mathbb{C}$$

□

- $\arg(z) = -\arg(\bar{z})$

Proof.

$$z = |z|e^{i\arg(z)} \quad \forall z \in \mathbb{C}$$

$$\bar{z} = |z|e^{-i\arg(z)}$$

$$\Rightarrow \arg(\bar{z}) = -\arg(z)$$

□

1.1.3 De Moirre's Formula

Using Euler's identity we can derive a powerful formula called De Moirre's Formula as follows:

$$\begin{aligned} e^{iN\varphi} &= \cos(N\varphi) + i \sin(N\varphi) \\ e^{iN\varphi} &= (e^{i\varphi})^N = (\cos \varphi + i \sin \varphi)^N \\ (\cos \varphi + i \sin \varphi)^N &= \cos(N\varphi) + i \sin(N\varphi) \end{aligned}$$

Applications of De Moirre's Formula:

Binomial expansion:

$$\begin{aligned} N = 2 : \\ \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \\ \sin(2\theta) &= 2 \cos \theta \sin \theta \\ N = 3 : \\ (\cos \theta + i \sin \theta)^3 &= \cos^3 \theta + 3 \cos^2 \theta (i \sin \theta) + 3 \cos \theta (i \sin \theta)^2 + (i \sin \theta)^3 \\ \cos(3\theta) &= \cos^3 \theta - 3 \cos \theta \sin^2 \theta \\ \sin(3\theta) &= 3 \cos^2 \theta \sin \theta - \sin^3 \theta \end{aligned}$$

Ex: Prove

$$\sin(3\theta) = 3 \sin \theta - 4 \sin^3 \theta$$

Proof. Using De Moivre's formula with $N = 3$

$$\begin{aligned} (\cos \theta + i \sin \theta)^3 &= \cos(3\theta) + i \sin(3\theta) \\ (\cos \theta)^3 + 3(\cos \theta)^2(i \sin \theta) + 3 \cos \theta (i \sin \theta)^2 + (i \sin \theta)^3 &= \cos(3\theta) + i \sin(3\theta) \\ \cos^3 \theta - 3 \cos \theta \sin^2 \theta + i(3 \cos^2 \theta \sin \theta - \sin^3 \theta) &= \cos(3\theta) + i \sin(3\theta) \\ \Im \{ \cos^3 \theta - 3 \cos \theta \sin^2 \theta + i(3 \cos^2 \theta \sin \theta - \sin^3 \theta) \} &= \Im \{ \cos(3\theta) + i \sin(3\theta) \} \\ 3 \cos^2 \theta \sin \theta - \sin^3 \theta &= \sin(3\theta) \\ \sin^2 \theta + \cos^2 \theta = 1 &\Rightarrow \cos^2 \theta = 1 - \sin^2 \theta \\ 3(1 - \sin^2 \theta) \sin \theta - \sin^3 \theta &= \sin(3\theta) \\ 3 \sin \theta - 3 \sin^3 \theta - \sin^3 \theta &= \sin(3\theta) \\ 3 \sin \theta - 4 \sin^3 \theta &= \sin(3\theta) \end{aligned}$$

□

Computing trigonometric integrals:

Ex:

$$\begin{aligned} \int_0^{2\pi} \cos^8 \varphi d\varphi \\ e^{i\varphi} &= \cos \varphi + i \sin \varphi \\ e^{-i\varphi} &= \cos \varphi - i \sin \varphi \end{aligned}$$

$$\begin{aligned}
\cos \varphi &= \frac{e^{i\varphi} + e^{-i\varphi}}{2} \\
\sin \varphi &= \frac{e^{i\varphi} - e^{-i\varphi}}{2i} \\
\int_0^{2\pi} \left(\frac{e^{i\varphi} + e^{-i\varphi}}{2} \right)^8 d\varphi &= \frac{1}{2^8} \int_0^{2\pi} (e^{i\varphi} + e^{-i\varphi})^8 d\varphi \\
&= \frac{1}{2^8} \int_0^{2\pi} (e^{i8\varphi} + {}_1C_8 e^{i7\varphi} e^{-i\varphi} + \dots + {}_7C_8 e^{i\varphi} e^{-i7\varphi} + e^{-i8\varphi}) d\varphi \\
&= \frac{1}{2^8} (0 + \dots + {}_4C_8 2\pi + \dots + 0) = \frac{{}_4C_8}{2^7} \pi
\end{aligned}$$

Ex2:

$$\begin{aligned}
\cos \theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\
\int_0^{2\pi} \cos^6 \theta d\theta &= \int_0^{2\pi} \left(\frac{e^{i\theta} + e^{-i\theta}}{2} \right)^6 d\theta \\
&= \frac{1}{2^6} \int_0^{2\pi} \sum_{k=0}^6 \binom{6}{k} e^{ik\theta} e^{-i\theta(6-k)} d\theta \\
&= \frac{1}{2^6} \sum_{k=0}^6 \binom{6}{k} \int_0^{2\pi} e^{i\theta(2k-6)} d\theta \\
\int_0^{2\pi} e^{ik\theta} d\theta &= \frac{e^{ik\theta}}{ik} \Big|_0^{2\pi} = \frac{e^{2\pi ik} - 1}{ik} \\
e^{2\pi ik} &= 1, \quad k \in \mathbb{Z} \Rightarrow \int_0^{2\pi} e^{ik\theta} d\theta = 0, \quad k \neq 0 \in \mathbb{Z} \\
\Rightarrow \int_0^{2\pi} \cos^6 \theta d\theta &= \frac{1}{2^6} \binom{6}{3} \int_0^{2\pi} d\theta = \frac{(20)(2\pi)}{2^6} = \frac{5\pi}{8}
\end{aligned}$$

Ex3:

$$\begin{aligned}
&\int_0^{2\pi} \sin^6(2\theta) d\theta \\
\sin(2\theta) &= \frac{e^{2i\theta} - e^{-2i\theta}}{2i} \\
\int_0^{2\pi} \sin^6(2\theta) d\theta &= \int_0^{2\pi} \left(\frac{e^{2i\theta} - e^{-2i\theta}}{2i} \right)^6 d\theta \\
&= \frac{1}{(2i)^6} \int_0^{2\pi} \sum_{k=0}^6 \binom{6}{k} (-1)^{6-k} e^{2ik\theta} e^{-2i\theta(6-k)} d\theta \\
&= -\frac{1}{2^6} \sum_{k=0}^6 \binom{6}{k} (-1)^{6-k} \int_0^{2\pi} e^{i\theta(4k-12)} d\theta \\
\int_0^{2\pi} e^{ik\theta} d\theta &= \frac{e^{ik\theta}}{ik} \Big|_0^{2\pi} = \frac{e^{2\pi ik} - 1}{ik}
\end{aligned}$$

$$\begin{aligned}
e^{2\pi i k} &= 1, \quad k \in \mathbb{Z} \Rightarrow \int_0^{2\pi} e^{ik\theta} d\theta = 0, \quad k \neq 0 \in \mathbb{Z} \\
&\Rightarrow \int_0^{2\pi} \sin^6(2\theta) d\theta = -\frac{1}{2^6} \binom{3}{6} (-1)^3 \int_0^{2\pi} d\theta \\
&= \frac{5\pi}{8}
\end{aligned}$$

Ex4: Prove

$$\sum_{k=0}^n \cos(k\theta) = \frac{1}{2} + \frac{\sin\left(\left(n + \frac{1}{2}\right)\theta\right)}{2 \sin\left(\frac{\theta}{2}\right)}$$

Proof. De Moivre's formula states

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$$

If we take the conjugate of both sides we get

$$(\cos \theta - i \sin \theta)^n = \cos(n\theta) - i \sin(n\theta)$$

Summing these two equations gives

$$\begin{aligned}
(\cos \theta + i \sin \theta)^k + (\cos \theta - i \sin \theta)^k &= 2 \cos(k\theta) \\
\cos \theta \pm i \sin \theta &= e^{\pm i\theta} \\
2 \cos(k\theta) &= (e^{i\theta})^k + (e^{-i\theta})^k
\end{aligned}$$

We can sum both sides of this to get

$$2 \sum_{k=0}^n \cos(k\theta) = \sum_{k=0}^n (e^{i\theta})^k + \sum_{k=0}^n (e^{-i\theta})^k$$

The formula for the geometric sum is

$$\sum_{k=0}^n z^k = \frac{1 - z^{n+1}}{1 - z}$$

Applying this we get

$$\begin{aligned}
2 \sum_{k=0}^n \cos(k\theta) &= \frac{1 - (e^{i\theta})^{n+1}}{1 - e^{i\theta}} + \frac{1 - (e^{-i\theta})^{n+1}}{1 - e^{-i\theta}} \\
2 \sum_{k=0}^n \cos(k\theta) &= \frac{(1 - e^{i\theta(n+1)})(1 - e^{-i\theta}) + (1 - e^{-i\theta(n+1)})(1 - e^{i\theta})}{(1 - e^{i\theta})(1 - e^{-i\theta})} \\
2 \sum_{k=0}^n \cos(k\theta) &= \frac{1 - e^{i\theta(n+1)} - e^{i\theta} + e^{i\theta n} + 1 - e^{-i\theta(n+1)} - e^{-i\theta} + e^{-i\theta n}}{1 - e^{i\theta} - e^{-i\theta} + 1} \\
2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{e^{i\theta n} + e^{-i\theta n} - e^{i\theta(n+1)} - e^{-i\theta(n+1)}}{2 - e^{i\theta} - e^{-i\theta}}
\end{aligned}$$

$$\begin{aligned}
2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{e^{i\frac{\theta}{2}} \left(e^{-i\theta(n+\frac{1}{2})} - e^{i\theta(n+\frac{1}{2})} \right) + e^{-i\frac{\theta}{2}} \left(e^{i\theta(n+\frac{1}{2})} - e^{-i\theta(n+\frac{1}{2})} \right)}{e^{i\frac{\theta}{2}} \left(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}} \right) + e^{-i\frac{\theta}{2}} \left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}} \right)} \\
2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{\left(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}} \right) \left(e^{i\theta(n+\frac{1}{2})} - e^{-i\theta(n+\frac{1}{2})} \right)}{- \left(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}} \right)^2} \\
2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{e^{i\theta(n+\frac{1}{2})} - e^{-i\theta(n+\frac{1}{2})}}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}} \\
2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{\frac{1}{2i} \left(e^{i\theta(n+\frac{1}{2})} - e^{-i\theta(n+\frac{1}{2})} \right)}{\frac{1}{2i} \left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}} \right)} \\
\sin(x) &= \frac{e^{ix} - e^{-ix}}{2i} \\
2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{\sin \left(\theta \left(n + \frac{1}{2} \right) \right)}{\sin \left(\frac{\theta}{2} \right)} \\
\sum_{k=0}^n \cos(k\theta) &= \frac{1}{2} + \frac{\sin \left(\left(n + \frac{1}{2} \right) \theta \right)}{2 \sin \left(\frac{\theta}{2} \right)}
\end{aligned}$$

□

1.1.4 Geometry in the Complex Plane

Using the notation such that $z = x + iy$ where $\Re(z) = x$ and $\Im(z) = y$ we can define a circle in the complex plane as

$$(x - x_0)^2 + (y - y_0)^2 = r_0^2$$

This is analogous to writing

$$|z - z_0| = r_0$$

The two can be related as follows:

$$\begin{aligned}
|z - z_0| &= r_0 \\
|x + iy - x_0 - iy_0| &= r_0 \\
\sqrt{(x - x_0)^2 + (y - y_0)^2} &= r_0 \\
(x - x_0)^2 + (y - y_0)^2 &= r_0^2
\end{aligned}$$

Ex: describe the circle formed by $2|z| = |z + 1|$

$$\begin{aligned}
2|z| &= |z + 1| \\
4|z|^2 &= |z + 1|^2 \\
4x^2 + 4y^2 &= (x + 1)^2 + y^2
\end{aligned}$$

$$4x^2 + 4y^2 = x^2 + 2x + 1 + y^2$$

$$3x^2 + 3y^2 - 2x - 1 = 0$$

$$3x^2 - 2x + \frac{1}{3} + 3y^2 - \frac{4}{3} = 0$$

$$3\left(x - \frac{1}{3}\right)^2 + 3y^2 = \frac{4}{3}$$

$$\left(x - \frac{1}{3}\right)^2 + y^2 = \frac{4}{9}$$

A line in the complex plane can be written as

$$ax + by = c \longleftrightarrow a\frac{z + \bar{z}}{2} + b\frac{z - \bar{z}}{2i} = c$$

Ex: describe the line formed by $|z - 1 + i| = |z - 2i|$

$$|z - 1 + i| = |z - 2i|$$

$$|z - 1 + i|^2 = |z - 2i|^2$$

$$(x - 1)^2 + (y + 1)^2 = x^2 + (y - 2)^2$$

$$x^2 - 2x + 1 + y^2 + 2y + 1 = x^2 + y^2 - 4y + 4$$

$$-2x + 2 + 2y = -4y + 4$$

$$6y = 2x + 2$$

$$y = \frac{1}{3}(x + 1)$$

We can define an ellipse in the complex plane as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If we have $a > b$ then we will have a horizontal ellipse and if $b > a$ then we will have a vertical ellipse.

Assuming that $a > b$ then we can define the foci points to be at

$$+F = (\sqrt{a^2 - b^2}, 0)$$

$$-F = (-\sqrt{a^2 - b^2}, 0)$$

The equation of an ellipse can also be described by

$$|z - F| + |z + F| = 2a$$

Ex: describe the ellipse formed by $|z - 1| + |z + 1| = 4$

$$|z - 1| + |z + 1| = 4$$

$$|z - F| + |z + F| = 2a \Rightarrow 2a = 4 \Rightarrow a = 2$$

$$F = \sqrt{a^2 - b^2} = 1 \Rightarrow 1 = 4 - b^2 \Rightarrow b^2 = 3$$

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

Ex2: describe the ellipse formed by $|z - 1| + |z + 3| = 6$

$$|z - 1| + |z + 3| = 6$$

note that this ellipse is not centered at the origin so we need to shift it

$$|z + 1 - 2| + |z + 1 + 2| = 6$$

$$2a = 6 \Rightarrow a = 3$$

$$F = 2 = \sqrt{a^2 - b^2} \Rightarrow 4 = 9 - b^2 \Rightarrow b^2 = 5$$

$$\frac{(x + 1)^2}{9} + \frac{y^2}{5} = 1$$

1.1.5 Roots of a Complex Number

Given $z_0 = r_0 e^{i\varphi_0}$, what is $z_0^{\frac{1}{n}}$?

If we let $w = z_0^{\frac{1}{n}}$ then $w^n = z_0$

$$w = r e^{i\varphi}, \quad w^n = r^n e^{in\varphi}$$

$$w^n = z_0 \Rightarrow r^n e^{in\varphi} = r_0 e^{i\varphi_0}$$

$$\Rightarrow r^n = r_0 \Rightarrow r_0^{\frac{1}{n}}$$

$$e^{in\varphi} = e^{i\varphi_0} \Rightarrow n\varphi = \varphi_0 + 2k\pi$$

$$\varphi = \frac{\varphi_0}{n} + \frac{2k\pi}{n}$$

So all solutions to $w^n = z_0$ are given by

$$w = r_0^{\frac{1}{n}} e^{i(\frac{\varphi_0}{n} + \frac{2k\pi}{n})}, \quad k \in \mathbb{Z}$$

If we normalize $\varphi_0 = \text{Arg}(z_0)$ then k will be in the range $k \in \{0, 1, \dots, n-1\}$.

Note that the expression

$$w = r_0^{\frac{1}{n}} e^{i(\frac{\varphi_0}{n} + \frac{2k\pi}{n})}, \quad k \in \mathbb{Z}$$

is multi-valued. If we want to avoid this, we can take what's called the principal value which is the value when $k = 0$:

$$z_0^{\frac{1}{n}} = r_0^{\frac{1}{n}} e^{i\frac{\varphi_0}{n}}$$

Ex: Compute $(-1)^{\frac{1}{2}}$

$$z_0 = -1 = 1^{\frac{1}{2}} e^{i(\frac{\pi}{2} + \frac{2k\pi}{2})} = e^{i(\frac{\pi}{2} + k\pi)} = \left\{ \dots, e^{-i\frac{3\pi}{2}}, e^{-i\frac{\pi}{2}}, e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{2}}, e^{i\frac{5\pi}{2}}, \dots \right\}$$

note that there are only 2 unique values

$$(-1)^{\frac{1}{2}} = e^{i(\frac{\pi}{2} + k\pi)}, \quad k \in \{0, 1\}$$

The principal value (when $k = 0$) of this equation works out to be i .

Ex2: Find all solutions to

$$\begin{aligned} z^7 &= i - 1 \\ z^7 &= \sqrt{2}e^{i(\frac{3\pi}{4}+2\pi k)} \\ z &= 2^{1/14}e^{i(\frac{3\pi}{28}+\frac{2\pi}{7}k)}, \quad k \in \{0, 1, 2, 3, 4, 5, 6\} \end{aligned}$$

Ex3: Find all solutions to

$$\begin{aligned} z^5 &= \frac{2i}{-1 - \sqrt{3}i} \\ z^5 &= \frac{2e^{i\frac{\pi}{2}}}{2e^{-i\frac{2\pi}{3}}} = e^{i\frac{7\pi}{6}} = e^{-i(\frac{5\pi}{6}+2\pi k)} \\ z &= e^{-i(\frac{\pi}{6}+\frac{2\pi}{5}k)}, \quad k \in \{0, 1, 2, 3, 4\} \end{aligned}$$

Ex4: Find all solutions to

$$\begin{aligned} \left(\frac{z}{z+1}\right)^2 &= i \\ \left(\frac{z}{z+1}\right)^2 &= e^{i(\frac{\pi}{2}+2\pi k)} \\ \frac{z}{z+1} &= e^{i(\frac{\pi}{4}+\pi k)}, \quad k \in \{0, 1\} \\ e^{i(\frac{\pi}{4}+\pi k)} &= \left\{ \frac{1}{\sqrt{2}}(1+i), \frac{1}{\sqrt{2}}(-1-i) \right\} \\ z &= e^{i(\frac{\pi}{4}+k\pi)}(z+1) \\ z &= \frac{e^{i(\frac{\pi}{4}+k\pi)}}{1 - e^{i(\frac{\pi}{4}+k\pi)}} = \left\{ \frac{1+i}{\sqrt{2}-1-i}, \frac{-1-i}{\sqrt{2}+1+i} \right\} \end{aligned}$$

Ex5: Find all solutions to

$$\begin{aligned} z^2 + 4iz + 1 &= 0 \\ (z^2 + 4iz - 4) + 4 + 1 &= 0 \\ (z + 2i)^2 + 5 &= 0 \\ (z + 2i)^2 &= -5 = 5e^{i(\pi+2\pi k)} \\ z + 2i &= \sqrt{5}e^{i(\frac{\pi}{2}+\pi k)} = \left\{ \sqrt{5}i, -\sqrt{5}i \right\} \\ z &= \left\{ (\sqrt{5}-2)i, -(\sqrt{5}+2)i \right\} \end{aligned}$$

Ex6: Find all solutions to $(z+1)^4 = (1-i)z^4$

$$\begin{aligned} (1-i)z^4 &= \sqrt{2}e^{-i\frac{\pi}{4}+2\pi ki} z^4 \\ z+1 &= 2^{1/8}e^{-i\frac{\pi}{16}+i\frac{\pi k}{2}} z \\ z &= \frac{-2^{1/8}}{1 - e^{-i\frac{\pi}{16}+i\frac{\pi k}{2}}}, \quad k \in \{0, 1, 2, 3\} \end{aligned}$$

1.2 Complex Functions

1.2.1 Mapping Properties of Simple Functions

Similar to how functions with real variables map values to a different set of values, complex functions do the same. The main difference is that with complex functions we're mapping a 2 dimensional set of inputs to a 2 dimensional set of outputs.

$$w = f(z) = u + iv$$

We define $z \in \mathcal{S}$ the image of \mathcal{S} under w .

Some common mappings:

- The identity map

$$w = f(z) = z$$

$$\begin{cases} u = x \\ v = y \end{cases}$$

- Translation by z_0

$$w = f(z) = z + z_0$$

$$\begin{cases} u = x + x_0 \\ v = y + y_0 \end{cases}$$

- Stretching ($a > 1$) or contraction ($a < 1$)

$$w = f(z) = az = are^{i\varphi}, \quad a \in \mathbb{R}$$

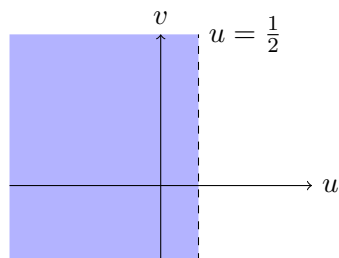
$$\begin{cases} u = ax \\ v = ay \end{cases}$$

- Rotation by φ_0

$$w = f(z) = e^{i\varphi_0} z = e^{i(\varphi + \varphi_0)}$$

Using these basic mapping principles we are able to lay the foundation for some more complicated mappings.

Ex: Find the image of $S = \{|z - 1| \geq 1\}$ under the mapping $f(z) = \frac{1}{z}$



$$z = \frac{1}{w} = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$$x = \frac{u}{u^2 + v^2}$$

$$y = -\frac{v}{u^2 + v^2}$$

$$|z - 1| \geq 1 \Rightarrow \left| \frac{1}{w} - 1 \right| \geq 1$$

$$\frac{1 - w}{w} \geq 1$$

$$|1 - w| \geq |w|$$

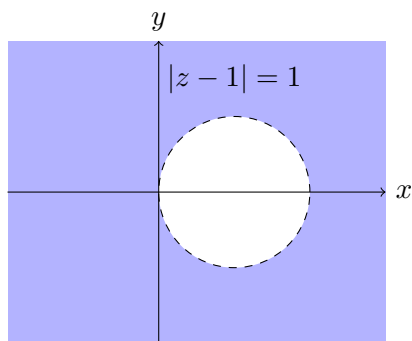
$$|1 - w|^2 \geq |w|^2$$

$$(1 - u)^2 + v^2 \geq u^2 + v^2$$

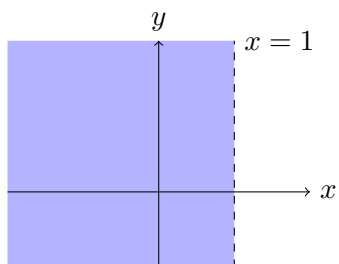
$$-2u + 1 \geq 0$$

$$u \leq \frac{1}{2}$$

$$S' = \left\{ u \leq \frac{1}{2} \right\}$$



Ex2: Find the image of $S = \{x \leq 1\}$ under the mapping $f(z) = \frac{1}{z}$

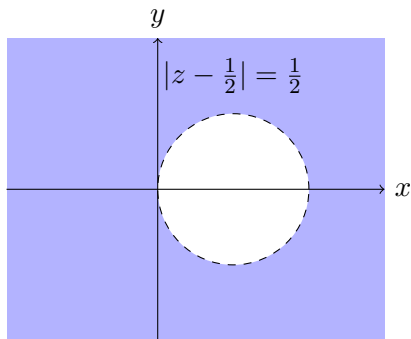


$$z = \frac{1}{w} = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$$x = \frac{u}{u^2 + v^2}$$

$$x \leq 1 \Rightarrow \frac{u}{u^2 + v^2} \leq 1 \Rightarrow u^2 + v^2 \geq u$$

$$(u - \frac{1}{2})^2 + v^2 \geq \frac{1}{4}$$



We see from the previous two examples that circles map to lines and lines map to circles. Let's see why this is the case.

$$a(x^2 + y^2) + bx + cy + d = 0$$

$$a|z|^2 + b\frac{z + \bar{z}}{2} + c\frac{z - \bar{z}}{2i} + d = 0$$

In the case where $a = 0$, we have a line. In the case where $a \neq 0$, we have a circle.

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$z\bar{z} = |z|^2 = \frac{1}{|w|^2} = \frac{1}{w\bar{w}}$$

$$a\frac{1}{w\bar{w}} + b\frac{z + \bar{z}}{2} + c\frac{z - \bar{z}}{2i} + d = 0$$

$$a\frac{1}{w\bar{w}} + \frac{b}{2}\left(\frac{1}{w} + \frac{1}{\bar{w}}\right) + \frac{c}{2i}\left(\frac{1}{w} - \frac{1}{\bar{w}}\right) + d = 0$$

$$a + \frac{b}{2}(w + \bar{w}) + \frac{c}{2i}(w - \bar{w}) + d(w\bar{w}) = 0$$

If we have a linear transformation of the form $az + b$ it corresponds to the scaling and translation of the set only. A line will map to a line and a circle will map to a circle.

We can combine this with the $w = \frac{1}{z}$ transformation property to get a more general transformation. We call this the *Mobius transformation*:

$$f(z) = \frac{az + b}{cz + d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$

Ex: Find the mapping of $f(z) = \frac{1}{z+1}$ on the set $S = \{\Re(z) > 0\}$

$$u + iv = \frac{1}{x + 1 + iy} \Rightarrow x + 1 + iy = \frac{1}{u + iv}$$

$$x + 1 = \frac{u}{u^2 + v^2}$$

$$x > 0 \Rightarrow x + 1 > 1$$

$$\frac{u}{u^2 + v^2} > 1 \Rightarrow u > u^2 + v^2$$

$$\begin{aligned}
u^2 + v^2 - u + \frac{1}{4} &< \frac{1}{4} \\
\left(u - \frac{1}{2}\right)^2 + v^2 &< \frac{1}{4} \\
S' &= \left\{ w = u + iv \left| \left(u - \frac{1}{2}\right)^2 + v^2 < \left(\frac{1}{2}\right)^2 \right. \right\}
\end{aligned}$$

Ex2: Find the mapping of $f(z) = \frac{z-i}{z+i}$ on $S = \{|z| < 3\}$

$$\begin{aligned}
wz + iw &= z - i \Rightarrow z(w - 1) = -i - iw \Rightarrow z = \frac{i(w + 1)}{1 - w} \\
|z| &= \frac{|w + 1|}{|w - 1|} < 3 \\
|w + 1| &< 3|w - 1| \Rightarrow |w + 1|^2 < 9|w - 1|^2 \\
(u + 1)^2 + v^2 &< 9(u - 1)^2 + 9v^2 \\
u^2 + 2u + 1 + v^2 &< 9u^2 - 18u + 9 + 9v^2 \\
0 &< 8u^2 - 20u + 8 + 8v^2 \Rightarrow 0 < u^2 - \frac{5}{2}u + 1 + v^2 \\
\frac{9}{16} &< u^2 - \frac{5}{2}u + \frac{25}{16} + v^2 \\
\frac{9}{16} &< \left(u - \frac{5}{4}\right)^2 + v^2 \\
S' &= \left\{ w = u + iv \left| \left(u - \frac{5}{4}\right)^2 + v^2 > \left(\frac{3}{4}\right)^2 \right. \right\}
\end{aligned}$$

Another common mapping is the $f(z) = z^2$ or more generally $f(z) = z^n$ mapping. For $w = z^2$,

$$w = z^2 = r^2 e^{2i\varphi} \Rightarrow \begin{cases} |w| = |z|^2 \\ \arg(w) = 2 \arg(z) \end{cases}$$

This mapping scales the magnitude but more notably, it doubles the argument. This means that the mapping of a half circle will now be a full circle.

Ex: Find the mapping of $f(z) = z^2$ on $S = \{0 \leq \Re(z) \leq 1, \Im(z) = 1\}$

$$\begin{aligned}
w &= x^2 + i2xy - y^2 \\
u &= x^2 - y^2 = x^2 - 1 \Rightarrow -1 \leq u \leq 0 \\
v &= 2xy = 2x \Rightarrow 0 \leq v \leq 2 \\
S' &= \{w = u + iv \mid -1 \leq u \leq 0, 0 \leq v \leq 2\}
\end{aligned}$$

Ex2: Find the mapping of $f(z) = -2z^5$ on $S = \{|z| < 1, 0 < \text{Arg}(z) < \frac{\pi}{2}\}$

$$z^5 = -\frac{w}{2} \Rightarrow |z|^5 = \frac{|w|}{2} < 1 \Rightarrow |w| < 2$$

$$5 \arg(z) = \arg(w) \pm \pi$$

$$0 < \arg(w) \pm \pi < \frac{5\pi}{2}$$

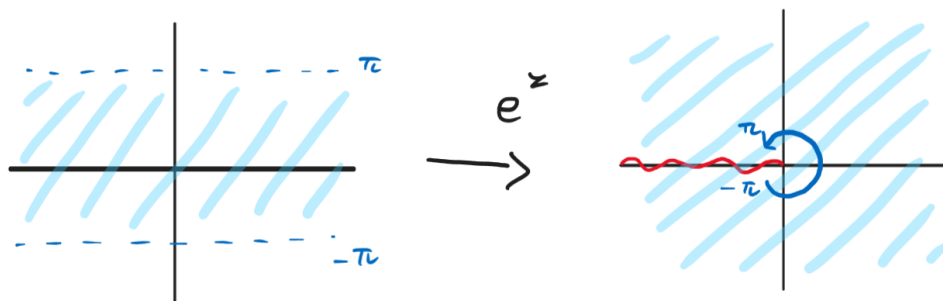
$$-\pi < \arg(w) < \frac{3\pi}{2}$$

$$S' = \{|w| < 2\}$$

Another common mapping is the $f(z) = e^z$ mapping.

$$w = e^z = e^{x+iy} = e^x e^{iy}$$

$$\begin{cases} |w| = e^x \\ \arg(w) = y \end{cases}$$



This mapping has the property that the magnitude is only dependent on x and the argument is exactly y .

Ex: Find the mapping of $f(z) = e^z$ on $S = \{\Re(z) = 1\}$

$$w = e^x e^{iy}$$

$$|w| = e, \arg(w) = y$$

$$S' = \{|w| = e\}$$

Ex2: Find the mapping of $f(z) = e^z$ on $S = \{0 \leq \Im(z) \leq \frac{\pi}{4}\}$

$$|w| = x$$

$$\arg(w) = y \Rightarrow 0 \leq \arg(w) \leq \frac{\pi}{4}$$

$$S' = \left\{0 \leq \arg(w) \leq \frac{\pi}{4}\right\}$$

Ex3: Find the mapping of $f(z) = e^{iz}$ on $S = \{z : -\frac{\pi}{2} \leq \Re(z) \leq \pi, -1 \leq \Im(z) \leq 1\}$

(Note that multiplying z by i rotates it by 90°)

$$w = e^{iz} = e^{ix} e^{-y}$$

$$|w| = e^{-y} \Rightarrow e^{-1} \leq |w| \leq e$$

$$\arg(w) = x \Rightarrow -\frac{\pi}{2} \leq \arg(w) \leq \pi$$

$$S' = \left\{ w \left| e^{-1} \leq |w| \leq e, -\frac{\pi}{2} \leq \arg(w) \leq \pi \right. \right\}$$

Ex4: Prove

$$|e^{-z^3}| \leq 1 \quad \forall \left\{ z \left| -\frac{\pi}{6} \leq \text{Arg}(z) \leq \frac{\pi}{6} \right. \right\}$$

Proof. We can express $-z^3$ as some complex number $a + ib$ where $a = \Re(-z^3)$ and $b = \Im(-z^3)$. Taking the magnitude gives

$$|e^{-z^3}| = |e^{a+ib}| = |e^a e^{ib}| = |e^a| |e^{ib}| = |e^a| = |e^{\Re(-z^3)}|$$

z can be written as

$$\begin{aligned} z &= |z| e^{i \text{Arg}(z)} \\ z^3 &= |z|^3 e^{i 3 \text{Arg}(z)} = |z|^3 (\cos(3 \text{Arg}(z)) + i \sin(3 \text{Arg}(z))) \\ -z^3 &= -|z|^3 (\cos(3 \text{Arg}(z)) + i \sin(3 \text{Arg}(z))) \\ \Re(-z^3) &= -|z|^3 \cos(3 \text{Arg}(z)) \\ \text{Arg}(z) &\in \left[-\frac{\pi}{6}, \frac{\pi}{6} \right] \\ \Rightarrow 3 \text{Arg}(z) &\in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \\ \cos(3 \text{Arg}(z)) &\in [0, 1] \\ |z|^3 &\in \{x \in \mathbb{R} | x \geq 0\} \\ \Re(-z^3) &= -|z|^3 \cos(3 \text{Arg}(z)) \in \{x \in \mathbb{R} | x \leq 0\} \\ e^{\Re(-z^3)} &\in [0, 1] \\ \Rightarrow |e^{-z^3}| &\leq 1 \end{aligned}$$

□

1.2.2 Calculus of Complex Functions

We define the limit of a complex function to be

$$w = f(z) = u + iv \\ \lim_{z \rightarrow z_0} f(z) = \lim_{(x,y) \rightarrow (x_0,y_0)} u(x,y) + i \lim_{(x,y) \rightarrow (x_0,y_0)} v(x,y)$$

Note that the notation $(x, y) \rightarrow (x_0, y_0)$ means that the limit is taken as (x, y) approaches (x_0, y_0) along *any* path.

The usual limit arithmetic rules are able to be applied as with real numbers.

In order for $\lim_{z \rightarrow z_0} f(z)$ to exist, we require that both $\lim_{(x,y) \rightarrow (x_0,y_0)} u(x,y)$ and $\lim_{(x,y) \rightarrow (x_0,y_0)} v(x,y)$ exist.

If we define $z_0 = x_0 + iy_0$ then we can define the derivative of a complex function as

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

If this limit exists then the function is said to be differentiable at z_0 .

Ex: $f(z) = z$

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{z_0 + \Delta z - z_0}{\Delta z} = 1$$

$$\Rightarrow f'(z_0) = 1$$

Ex2: $f(z) = \bar{z}$

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$\Delta z = h_1 + ih_2 \Rightarrow \overline{\Delta z} = h_1 - ih_2$$

$$h_2 = 0 : \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{h_1 \rightarrow 0} \frac{h_1}{h_1} = 1$$

$$h_1 = 0 : \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{h_2 \rightarrow 0} \frac{-ih_2}{ih_2} = -1$$

$$\lim_{h_1 \rightarrow 0} \frac{h_1}{h_1} \neq \lim_{h_2 \rightarrow 0} \frac{-ih_2}{ih_2} \therefore \text{the derivative does not exist}$$

An easy way to determine if a function is differentiable is to use the Cauchy-Riemann equations. Any path that can be taken to approach z_0 can be written as a linear combination of the paths $\Delta z = \Delta x$ and $\Delta z = i\Delta y$ so the derivative must satisfy both of these paths.

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$f(z_0) = u(x_0, y_0) + iv(x_0, y_0)$$

Define $\Delta z = \Delta x$

$$f'(z_0) = \lim_{\Delta x \rightarrow 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x}$$

$$f'(z_0) = \lim_{\Delta x \rightarrow 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \rightarrow 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0)$$

Define $\Delta z = i\Delta y$

$$f'(z_0) = \lim_{\Delta y \rightarrow 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y}$$

$$f'(z_0) = \lim_{\Delta y \rightarrow 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \rightarrow 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y}$$

$$f'(z_0) = -i \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$$

$$\Rightarrow \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) = -i \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$$

Splitting the real and imaginary parts we get that the Cauchy-Riemann equations are

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

If the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous then the function is differentiable.

Some functions are not differentiable everywhere, but are differentiable at a point or a set of points.

- If $f(z)$ is differentiable everywhere in the complex plane then it is said to be **entire**.
- If $f(z)$ is differentiable in some region R then it is said to be **analytic** in R .
(note that this region cannot be a single point, as the Cauchy-Riemann equations require the partial derivatives to be continuous)

Ex: Show using the Cauchy-Riemann equations that $f(z) = \bar{z}$ is not differentiable anywhere.

$$\bar{z} = x - iy$$

$$u_x = 1 \neq v_y = -1$$

Ex2: Show that $f(z) = z^2$ is entire

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$$

$$u(x, y) = x^2 - y^2$$

$$v(x, y) = 2xy$$

$$u_x = 2x = v_y$$

$$u_y = -2y = -v_x$$

Ex3: Show that $f(z) = \bar{z}$ is differentiable but not analytic at $z_0 = 0$

$$|z|^2 + 2z = x^2 + 2x + y^2 + i2y$$

$$u_x = 2x + 2 = v_y = 2 \Rightarrow x = 0$$

$$u_y = 2y = -v_x = 0 \Rightarrow y = 0$$

differentiable but not analytic on $z = \{0\}$

1.2.3 Conformal Mappings

Using the Cauchy-Riemann equations,

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

we can get the Laplacian of u and v ,

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$$

$$v_{xx} + v_{yy} = -(u_{yx} + u_{xy}) = 0$$

If the Laplacian of u and v are both zero then the function is said to be **harmonic**.

If $\nabla^2 u = 0$ then we can use the Cauchy-Riemann equations to find its harmonic conjugate v . Ex: Find the harmonic conjugate of $u = xy - x + y$

$$u_x = y - 1 = v_y$$

$$\begin{aligned}
v &= \int y - 1 dy = \frac{y^2}{2} - y + h(x) \\
u_y &= x + 1 = -v_x = -h'(x) \\
h(x) &= \int -x - 1 dx = -\frac{x^2}{2} - x + C \\
v &= \frac{y^2}{2} - \frac{x^2}{2} - y - x + C
\end{aligned}$$

Ex2: Find the harmonic conjugate of $u = \ln \sqrt{x^2 + y^2}$

$$\begin{aligned}
u_x &= \frac{x}{x^2 + y^2} = v_y \\
v &= \int \frac{x}{x^2 + y^2} dy = \int \frac{1/x}{1 + \frac{y^2}{x^2}} dy = \arctan\left(\frac{y}{x}\right) + h(x) \\
u_y &= \frac{y}{x^2 + y^2} = -v_x = -\frac{y}{1 + \frac{y^2}{x^2}} \left(\frac{-1}{x^2}\right) - h'(x) = \frac{y}{x^2 + y^2} - h'(x) \Rightarrow h'(x) = 0 \\
h(x) &= C \\
v &= \arctan\left(\frac{y}{x}\right) + C \\
v &= \arg(z) + C
\end{aligned}$$

Ex3: Find the harmonic conjugate of $u = \sin x \cosh y$

$$\begin{aligned}
u &= \sin x \cosh y \\
u_x &= \cos x \cosh y = v_y \\
v &= \int \cos x \cosh y dy = \cos x \sinh y + h(x) \\
u_y &= \sin x \sinh y = -v_x = -(-\sin x \sinh y) - h'(x) \Rightarrow h'(x) = 0 \\
h(x) &= C \\
v &= \cos x \sinh y + C
\end{aligned}$$

Another property is that

$$|f'(z)|^2 = |\nabla u|^2 = |\nabla v|^2$$

This also implies that

$$\nabla u \cdot \nabla v = 0$$

$$\begin{aligned}
f'(z) &= u_x + iv_x = \frac{1}{i}(u_y + iv_y) \\
|f'(z)|^2 &= u_x^2 + v_x^2 = u_y^2 + v_y^2 = |\nabla u|^2 = |\nabla v|^2 \\
\nabla u \cdot \nabla v &= u_x u_y + v_x v_y = u_x v_x + (-v_x)(u_x) = 0
\end{aligned}$$

A conformal mapping is a mapping between two regions that preserves angles. If we have some function $f(z) = u + iv$ that is analytic then we can create a function of a function as

$$\Phi(u(x, y), v(x, y)) = \phi(x, y)$$

where $\phi(x, y)$ is a conformal mapping.

A conformal mapping will have the property that

$$\phi_{xx} + \phi_{yy} = |f'(z)|^2(\Phi_{uu} + \Phi_{vv})$$

where $|f'(z)|^2$ is known as the *conformal factor*.

This can be shown as follows:

$$\begin{aligned}\phi_x &= \Phi_u u_x + \Phi_v v_x \\ \phi_{xx} &= u_{xx} \Phi_u^2 + 2u_x v_x \Phi_u \Phi_v + v_{xx} \Phi_v^2 + \Phi_u u_{xx} + \Phi_v v_{xx} \\ \phi_y &= \Phi_u u_y + \Phi_v v_y \\ \phi_{yy} &= u_{yy} \Phi_u^2 + 2u_y v_y \Phi_u \Phi_v + v_{yy} \Phi_v^2 + \Phi_u u_{yy} + \Phi_v v_{yy} \\ \phi_{xx} + \phi_{yy} &= \Phi_u \nabla^2 u + \Phi_v \nabla^2 v + \Phi_{uu} |\nabla u|^2 + 2\Phi_u \Phi_v \nabla u \cdot \nabla v + \Phi_{vv} |\nabla v|^2 \\ \phi_{xx} + \phi_{yy} &= |f'(z)|^2(\Phi_{uu} + \Phi_{vv})\end{aligned}$$

Ex: $f(z) = z^2$ under $\Phi(u, v) = e^u + v^2$

$$\begin{aligned}f(z) &= x^2 - y^2 + i(2xy) \\ u &= x^2 - y^2 \\ v &= 2xy \\ \phi(x, y) &= e^{x^2 - y^2} + (2xy)^2 \\ \phi_{xx} + \phi_{yy} &= 4|z|^2(e^u + 2)\end{aligned}$$

$f(z)$ is considered a conformal mapping if f is analytic and $f'(z) \neq 0$. As a consequence, if $\phi_{xx} + \phi_{yy} = 0$ then $\Phi_{uu} + \Phi_{vv} = 0$.

If f is a conformal mapping then it will preserve angles.

If we have two curves C_1 and C_2 described by the parametrized functions $z_1(t)$ and $z_2(t)$ that intersect at a point z_0 then the angle between the two curves is given by $\theta = \arg(z_2'(t)) - \arg(z_1'(t))$. If we then apply a conformal mapping $w = f(z)$ to the curves then we get $w_1(t) = f(z_1(t))$ and $w_2(t) = f(z_2(t))$ and the angle between the two curves is given by $\theta_w = \arg(w_2'(t)) - \arg(w_1'(t))$.

$$\begin{aligned}\theta_w &= \arg(f'(z_2)z_2'(t)) - \arg(f'(z_1)z_1'(t)) \\ \theta_w &= \arg(f'(z_2)) - \arg(f'(z_1)) + \arg(z_2'(t)) - \arg(z_1'(t)) \\ \arg(f'(z_1)) &= \arg(f'(z_2)) \\ \Rightarrow \theta_w &= \arg(z_2'(t)) - \arg(z_1'(t)) = \theta\end{aligned}$$

So the angle between the two curves is preserved under a conformal mapping.

Note that $f'(z) \neq 0$ is a necessary condition in this proof.

If we have a nonconformal mapping then the angle between the two curves will not be preserved. One such case is that of $f(z) = z^2$ at $z_0 = 0$. At this point, $f'(z_0) = 0$ and the angle between the two curves is doubled.

Conformal mappings also have the property that they map Neumann boundary conditions to Neumann boundary conditions.

If we let \hat{n} represent the normal vector to the curve $\phi(x, y)$ then they are related by

$$\frac{\partial \phi}{\partial n} = \nabla \phi \cdot \hat{n}$$

and the mapping is similarly related brcurly

$$\begin{aligned} \frac{\partial \Phi}{\partial n'} &= \nabla \Phi \cdot \hat{n}' \\ \frac{\partial \phi}{\partial n} &= |f'(z)| \frac{\partial \Phi}{\partial n'} \end{aligned}$$

So for Neumann boundary conditions, we will have

$$\frac{\partial \phi}{\partial n} = 0 \Rightarrow \frac{\partial \Phi}{\partial n'} = 0$$

Some examples of conformal mappings come from harmonic functions (having the property that $\nabla^2 \phi = 0$).

Some common harmonic functions are:

- $\phi = C$
- $\phi = ax + by + c$
- $\ln \sqrt{x^2 + y^2}, \mathbb{C} \setminus 0$
- $\phi = \text{Arg}(z), \mathbb{C} \setminus (-\infty]$
- $\phi = x^2 - y^2$

1.2.4 Conformal Mapping to Solve Laplace's Equation

Given the useful properties of matching boundary conditions, we can use conformal mappings to help us solve Laplace's equation for a given region.

Ex: Given $\nabla^2 \phi = 0$ for $1 < x^2 - y^2 < 4$ and $\phi = 1$ on $x^2 - y^2 = 1$ and $\phi = 3$ on $x^2 - y^2 = 4$, find $\phi(x, y)$.

$$\text{choose } \phi = x^2 - y^2$$

$$\text{choose } \Phi(u, v) = Au + B$$

$$\begin{cases} A(1) + B = 1 \\ A(4) + B = 3 \end{cases} \Rightarrow A = \frac{2}{3}, B = \frac{1}{3}$$

$$\Phi(u, v) = \frac{2}{3}u + \frac{1}{3}$$

$$u = x^2 - y^2$$

$$\phi(x, y) = \frac{2}{3}(x^2 - y^2) + \frac{1}{3}$$

Ex2: Given $\nabla^2 \phi = 0$ within the circular region $\mathcal{D} = \{1 < x^2 + y^2 < 4\}$ with $\phi = 1$ on $x^2 + y^2 = 1$ and $\phi = -2$ on $x^2 + y^2 = 4$, find $\phi(x, y)$.

$$\phi(x, y) = A_1 \ln r + A_2$$

$$r = \sqrt{x^2 + y^2}$$

$$r = 1 : \phi = 1 \Rightarrow A_1 \ln(1) + A_2 = 1$$

$$r = 2 : \phi = -2 \Rightarrow A_1 \ln(2) + A_2 = -2$$

$$\Rightarrow A_2 = 1, A_1 = -\frac{3}{\ln 2}$$

$$\phi(x, y) = -\frac{3}{\ln 2} \ln \sqrt{x^2 + y^2} + 1$$

Ex3: Given $\nabla^2 \phi = 0$ within the strip described by $\{z : -3 \leq 3\Re(z) - 4\Im(z) \leq 2\}$ with $\phi = 0$ for $\{z : -3 = 3\Re(z) - 4\Im(z)\}$ and $\phi = 4$ for $\{z : 3\Re(z) - 4\Im(z) = 2\}$ find $\phi(x, y)$

$$u = 3x - 4y$$

$$u(-3) = 0, u(2) = 4$$

$$\Phi = Au + B$$

$$\Phi(-3) = -3A + B = 0 \Rightarrow B = 3A$$

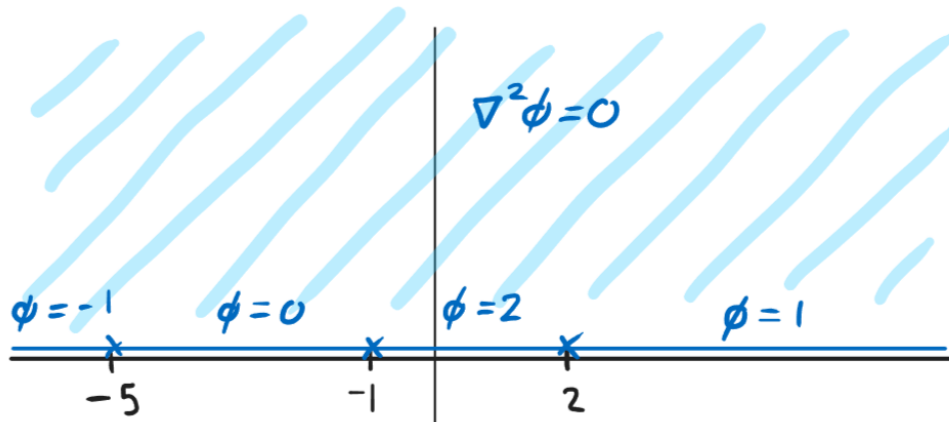
$$\Phi(2) = 2A + 3A = 5A = 4 \Rightarrow A = \frac{4}{5} \Rightarrow B = \frac{12}{5}$$

$$\phi(x, y) = \frac{4}{5}(3x - 4y) + \frac{12}{5}$$

Ex4: Given $\nabla^2 \phi = 0$ for the upper half-plane described by $\{y > 0 \wedge x \in \mathbb{R}\}$ with the boundary conditions along the x-axis given by

$$\phi(x, 0) = \begin{cases} -1 & x < -5 \\ 0 & -5 < x < -1 \\ 2 & -1 < x < 2 \\ 1 & x > 2 \end{cases}$$

find $\phi(x, y)$.



One trick to solve a problem like this is to choose a linear combination of functions of the form $\text{Arg}(z - z_0)$ with a different point z_0 for every place where the boundary condition changes along the x-axis.

$$\phi = A_1 \text{Arg}(z + 5) + A_2 \text{Arg}(z + 1) + A_3 \text{Arg}(z - 2) + A_4$$

$$\phi(x > 2, 0) = A_4 = 1$$

$$\phi(-1 < x < 2, 0) = \pi A_3 + 1 = 2 \Rightarrow A_3 = \frac{1}{\pi}$$

$$\phi(-5 < x < -1, 0) = \pi A_2 + 1 + 1 = 0 \Rightarrow A_2 = -\frac{2}{\pi}$$

$$\phi(x < -5, 0) = \pi A_1 - 2 + 2 = -1 \Rightarrow A_1 = -\frac{1}{\pi}$$

$$\phi(x, y) = -\frac{1}{\pi} \text{Arg}(z + 5) - \frac{2}{\pi} \text{Arg}(z + 1) + \frac{1}{\pi} \text{Arg}(z - 2) + 1$$

We can also apply these techniques to other types of boundary conditions in some cases.

Ex5: Given $\nabla^2 \phi = 0$ in the circular region $\{z : 1 \leq |z| \leq 2\}$ with the boundary conditions $\phi = 1$ for $|z| = 1$ and $\frac{\partial \phi}{\partial r} = 2$ for $|z| = 2$, find $\phi(x, y)$

$$\phi = A \ln r + B$$

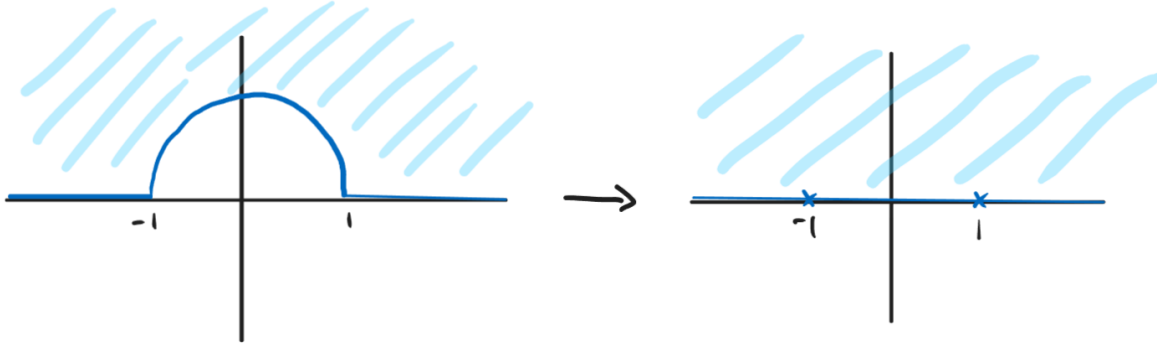
$$\phi(1) = B = 1$$

$$\frac{\partial \phi(2)}{\partial r} = \frac{A}{2} = 2 \Rightarrow A = 4$$

$$\phi = 4 \ln r + 1$$

$$\phi(x, y) = 4 \ln \sqrt{x^2 + y^2} + 1$$

If we have a region that has a semicircle in it we can use the Joukowski mapping to transform it into a region that is easier to work with.



Ex6: Given $\nabla^2 \phi = 0$ in the upper region of the plane described by $\{y > 0 \wedge x^2 + y^2 > 9\}$ with the boundary conditions $\phi = -1$ for $x < -3$, $\phi = 0$ for $x^2 + y^2 = 9$, and $\phi = 2$ for $x > 3$, find $\phi(x, y)$.

$$\begin{cases} \phi(x, 0) = -1 & x < -3 \\ \phi(x, y) = 0 & x^2 + y^2 = 9 \\ \phi(x, 0) = 2 & x > 3 \end{cases}$$

$$w = \frac{1}{2} \left(\frac{z}{3} + \frac{3}{z} \right) = u + iv$$

$$\Phi = A_1 \text{Arg}(w + 1) + A_2 \text{Arg}(w - 1) + A_3$$

$$\Phi(u > 1, v) = A_3 = 2$$

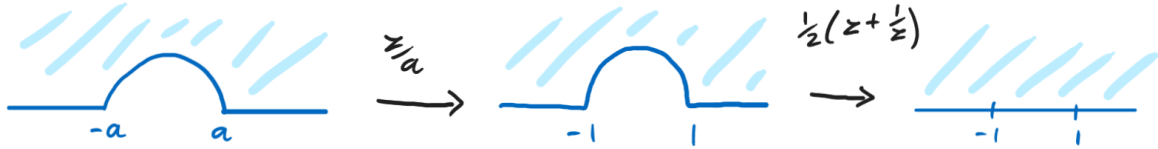
$$\Phi(-1 < u < 1, v) = \pi A_2 + 2 = 0 \Rightarrow A_2 = -\frac{2}{\pi}$$

$$\Phi(u < -1, v) = \pi A_1 - 2 + 2 = -1 \Rightarrow A_1 = -\frac{1}{\pi}$$

$$\Phi = -\frac{1}{\pi} \operatorname{Arg}(w+1) - \frac{2}{\pi} \operatorname{Arg}(w-1) + 2$$

$$\phi(z) = -\frac{1}{\pi} \operatorname{Arg} \left(\frac{1}{2} \left(\frac{z}{3} + \frac{3}{z} \right) + 1 \right) - \frac{2}{\pi} \operatorname{Arg} \left(\frac{1}{2} \left(\frac{z}{3} + \frac{3}{z} \right) - 1 \right) + 2$$

In the case that we have a semicircle not of radius 1 we can apply a scaling before the Joukowski mapping to get the correct radius.



$$w = \frac{1}{2} \left(\frac{z}{a} + \frac{a}{z} \right)$$

1.2.5 Sinusoidal Functions

If we recall Euler's formula $e^{ix} = \cos x + i \sin x$ we can use with complex numbers to get the following identities for complex sinusoids:

- $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$
- $\cos z = \frac{e^{iz} + e^{-iz}}{2}$
- $\sinh z = \frac{e^z - e^{-z}}{2}$
- $\cosh z = \frac{e^z + e^{-z}}{2}$
- $\cos z = \sin \left(\frac{\pi}{2} - z \right)$
- $\sinh z = -i \sin (iz)$
- $\cosh z = \cos (iz) = \sin \left(\frac{\pi}{2} - iz \right)$
- $\frac{d}{dz} \sin z = \cos z$
- $\frac{d}{dz} \cos z = -\sin z$
- $\cos^2 z + \sin^2 z = 1$
- $\frac{d}{dz} \sinh z = \cosh z$
- $\frac{d}{dz} \cosh z = \sinh z$
- $\cosh^2 z - \sinh^2 z = 1$

The most notable difference between the real and complex versions of these functions is that $|\sin z| \not\leq 1$. We will see cases of this soon but it is easy to see once we write out the real and imaginary components of $\sin z$.

$$\begin{aligned}\sin z &= \sin(x + iy) = \frac{e^{i(x+iy)} - e^{-i(x+iy)}}{2i} = \frac{e^{-y}e^{ix} - e^ye^{-ix}}{2i} \\ \sin z &= \frac{e^{-y}(\cos x + i \sin x) - e^y(\cos x - i \sin x)}{2i} = \frac{e^{-y} - e^y}{2} \cos x + \frac{e^{-y} + e^y}{2i} \sin x \\ \sin z &= \sin x \cosh y + i \cos x \sinh y\end{aligned}$$

Ex: Find all solutions to $\sin z = 4i$

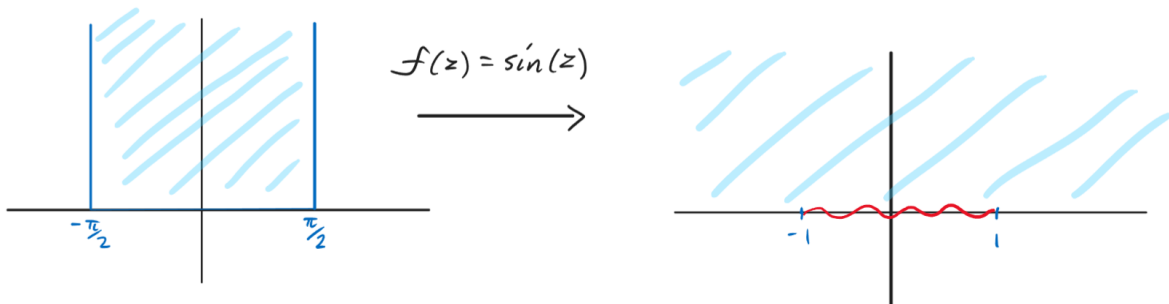
$$\begin{aligned}\sin(z) &= 4i = \sin x \cosh y + i \cos x \sinh y \\ \sin x \cosh y &= 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi \\ \text{Case 1: } n &= 2k : \cos x = 1 \\ 4 &= \cos(2k\pi) \sinh(y) = \sinh y \\ \text{Case 2: } n &= 2k + 1 : \cos x = -1 \\ 4 &= \cos((2k + 1)\pi) \sinh y = -\sinh y \\ \sinh y &= \frac{e^y - e^{-y}}{2} \\ 2 \sinh y &= e^y - e^{-y} \\ e^{2y} - 2 \sinh y e^y - 1 &= 0 \\ e^y &= \frac{2 \sinh y \pm \sqrt{4 \sinh^2 y + 4}}{2} = \sinh y \pm \sqrt{\sinh^2 y + 1} \\ n &= 2k : \\ y &= \ln(4 \pm \sqrt{17}) = \ln(4 + \sqrt{17}) \\ n &= 2k + 1 : \\ y &= \ln(-4 \pm \sqrt{17}) = \ln(\sqrt{17} - 4) \\ z &= \left\{ (x, y) \mid \left(2k\pi, \ln(4 + \sqrt{17}) \right), \left((2k + 1)\pi, \ln(\sqrt{17} - 4) \right), k \in \mathbb{Z} \right\}\end{aligned}$$

Ex2: Find all solutions to $\cos z = 0$

$$\begin{aligned}\cos(z^4) &= 0 \\ z^4 &= \pi n + \frac{\pi}{2} = \left(\pi n + \frac{\pi}{2} \right) e^{2\pi i l} \\ z &= \left(\pi n + \frac{\pi}{2} \right)^{1/4} e^{i \frac{\pi l}{2}}, \quad l = \{0, 1, 2, 3\}, \quad n \in \mathbb{Z}\end{aligned}$$

Mapping properties of $\sin z$

$\sin z$ will map the box $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$, $0 \leq y < \infty$ to the half plane $v > 0$.



Ex: Find the mapping of $\sin z$ on $S = \{-\frac{\pi}{2} < x < \frac{\pi}{2}, 0 < y < 1\}$

$$u = \sin x \cosh y$$

$$\sin x \in (-1, 1)$$

$$\cosh y \in (0, \cosh(1))$$

$$u \in (-\cosh(1), \cosh(1))$$

$$v = \cos x \sinh y$$

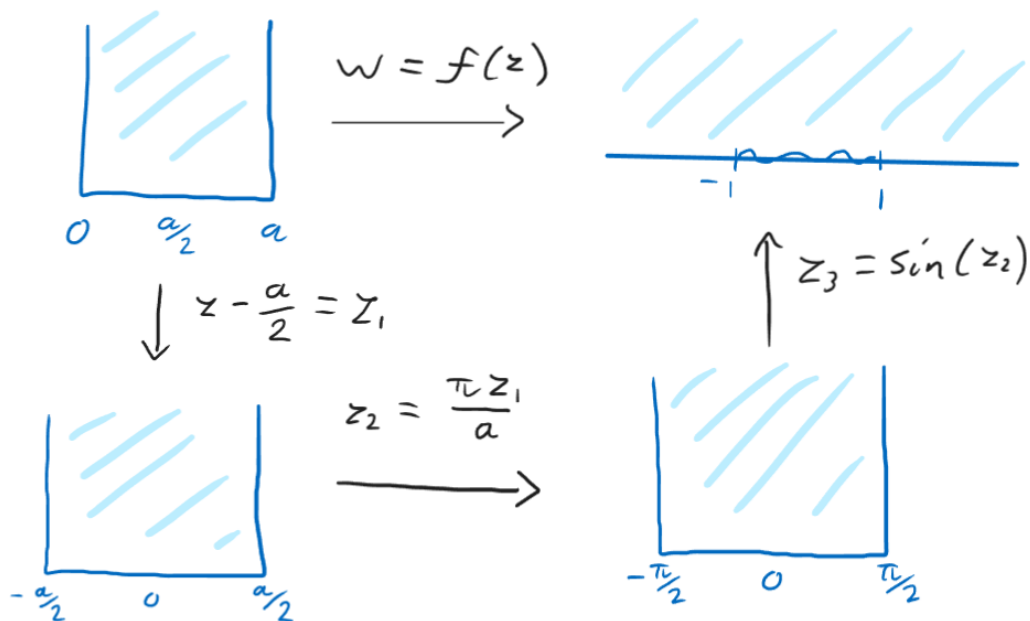
$$\cos x \in (0, 1]$$

$$\sinh y \in (0, \sinh(1))$$

$$v \in (0, \sinh(1))$$

$$S' = \{(u, v) | -\cosh(1) < u < \cosh(1), 0 < v < \sinh(1)\}$$

If the box is offcenter we can also apply a composition of mappings to shift it into the usual form.



Ex2: Find the mapping of $\sin z$ on $S = \{-1 < x < 1, y > 0\}$

$$u = \sin x \cosh y$$

$$\sin x \in (-\sin(1), \sin(1))$$

$$\begin{aligned}
&\cosh y \in (1, \infty) \\
&u \in \mathbb{R} \\
&v = \cos x \sinh y \\
&\cos x \in (\cos(1), 1] \\
&\sinh y \in (0, \infty) \\
&v \in (0, \infty) \\
&S' = \{(u, v) | v > 0\}
\end{aligned}$$

Ex3: Solve the Laplace equation $\nabla^2 u = 0$ in the region $S = \{0 \leq x \leq 2, 0 \leq y < \infty\}$ with boundary conditions $\phi = 0$ on $x = 0$ and $\phi = 1$ on $y = 0$ and $\phi = -2$ on $x = 2$.

$$\begin{aligned}
&\text{Map to the half plane using } w = \sin\left(\frac{\pi}{2}(z-1)\right) \\
&\phi(z) = \Phi(w) = A_1 \operatorname{Arg}(w+1) + A_2 \operatorname{Arg}(w-1) + A_3 \\
&u > 1 : \Phi = -2 = A_3 \\
&-1 < u < 1 : \Phi = 1 = A_2\pi - 2 \Rightarrow A_2 = \frac{3}{\pi} \\
&u < -1 : \Phi = 0 = A_1\pi + \frac{3}{\pi}\pi - 2 \Rightarrow A_1 = -\frac{1}{\pi} \\
&\phi(z) = -\frac{1}{\pi} \operatorname{Arg}\left(\sin\left(\frac{\pi}{2}(z-1)\right) + 1\right) + \frac{3}{\pi} \operatorname{Arg}\left(\sin\left(\frac{\pi}{2}(z-1)\right) - 1\right) - 2
\end{aligned}$$

1.2.6 Logarithmic Functions

$\log z$ is defined as the inverse of e^z .

$$\begin{aligned}
&e^w = z \Rightarrow w = \log z \\
&w = u + iv \\
&z = re^{i \operatorname{Arg}(z)} \\
&e^u e^{iv} = re^{i \operatorname{Arg}(z)} \\
&e^u = r \Rightarrow u = \ln r \\
&e^{iv} = e^{i \operatorname{Arg}(z)} \Rightarrow v = \operatorname{Arg}(z) + 2k\pi \\
&\log(z) = w = \ln r + i(\operatorname{Arg} z + 2k\pi), \quad k \in \mathbb{Z} \\
&\log(z) = \{w | e^w = z\} = \ln r + i(\operatorname{Arg} z + 2k\pi) = \ln r + i \arg z \\
&\operatorname{Log}(z) = \ln r + i \operatorname{Arg} z
\end{aligned}$$

Note that $\log(z) = \ln r + i \arg z$ is a multi-valued function and $\operatorname{Log}(z) = \ln r + i \operatorname{Arg} z$ is the principal value of $\log z$.

Ex: Find all values for $\log 2$ and $\operatorname{Log} 2$.

$$\begin{aligned}
&z = 2 = 2e^{i0} \\
&\log 2 = \ln 2 + i(0 + 2k\pi) = \ln 2 + 2k\pi i, \quad k \in \mathbb{Z} \\
&\operatorname{Log} 2 = \ln 2 + i(0) = \ln 2
\end{aligned}$$

Ex2: Find all values for $\log(-1 - \sqrt{3}i)$ and $\text{Log}(-1 - \sqrt{3}i)$.

$$z = -1 - \sqrt{3}i = 2e^{i\frac{2\pi}{3}}$$

$$\log(-1 - \sqrt{3}i) = \ln 2 + i \left(-\frac{2\pi}{3} + 2k\pi \right), \quad k \in \mathbb{Z}$$

$$\text{Log}(-1 - \sqrt{3}i) = \ln 2 + i \left(-\frac{2\pi}{3} \right)$$

Ex3: Find all values for $\log(e^{1+5i})$ and $\text{Log}(e^{1+5i})$.

$$z = e^{1+5i} = e e^{5i} = e e^{i(5+2k\pi)}$$

$$5 + 2k\pi \in (-\pi, \pi] \Rightarrow k = -1$$

$$\text{Arg}(e^{1+5i}) = 5 - 2\pi$$

$$\log(e^{1+5i}) = \ln e + i(5 - 2\pi + 2k\pi) = 1 + i(5 - 2\pi + 2k\pi), \quad k \in \mathbb{Z}$$

$$\text{Log}(e^{1+5i}) = \ln e + i(5 - 2\pi) = 1 + i(5 - 2\pi)$$

Ex4: Find all values for $\log(-i)$ and $\text{Log}(-i)$.

$$-i = e^{-i\frac{\pi}{2}}$$

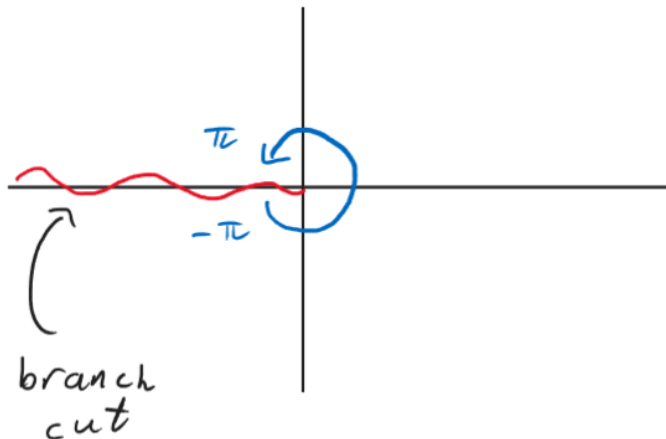
$$\arg(-i) = -\frac{\pi}{2} + 2\pi k$$

$$\log(-i) = -i\frac{\pi}{2} + i2\pi k$$

$$\text{Log}(-i) = -i\frac{\pi}{2}$$

Properties of $\log z$ and $\text{Log } z$:

- $\log(z_1 z_2) = \log z_1 + \log z_2$
 $\text{Log}(z_1 z_2) \neq \text{Log } z_1 + \text{Log } z_2$ (same as why $\text{Arg}(z_1 z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2)$)
- $e^{\log z} = z$ and $e^{\text{Log } z} = z$
but $\log(e^z) \neq z$ and $\text{Log}(e^z) \neq z$
Ex: $z = 0$: $\log(e^0) = \log(1) = \ln 1 + i(0 + 2k\pi) = i2k\pi \neq 0$
 $\text{Log}(e^{5i}) = i(5 - 2\pi) \neq 5i$
rather $z \in \log(e^z)$
- $\log z^n \neq n \log z$
 $z = 1, n = 2$: $\log z^2 = \log 1 = i2k\pi$
 $2 \log z = 2 \log 1 = 2i(2k')\pi$
but $\log z^{\frac{1}{n}} = \frac{1}{n} \log z$
- $\text{Log } z$ is analytic in $\mathbb{C} \setminus (-\infty, 0]$



$$\text{Log } z = \ln |z| + i \text{Arg } z, \quad -\pi < \text{Arg } z < \pi$$

Ex: Find all solutions to $e^z = -1 - i$

$$z = \log(-1 - i)$$

$$-1 - i = \sqrt{2}e^{-i\frac{3\pi}{4}}$$

$$z = \frac{1}{2}\ln(2) - i\frac{3\pi}{4} + i2\pi k$$

Ex2: Find the principal value of $(1 + i)^i = e^{i \text{Log}(1+i)}$

$$1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$$

$$\text{Log}(1 + i) = \frac{1}{2}\ln(2) + i\frac{\pi}{4}$$

$$(1 + i)^i = e^{-\frac{\pi}{4} + i\frac{1}{2}\ln(2)}$$

Ex3: Find the principle value of $z^i = 1 + i$

$$1 + i = \sqrt{2}e^{i\frac{\pi}{4} + i2\pi k} = e^{\ln(\sqrt{2}) + i\frac{\pi}{4} + i2\pi k}$$

$$z = (1 + i)^{1/i} = (1 + i)^{-i} = e^{\frac{\pi}{4} + 2\pi k} e^{-i \ln(\sqrt{2})}$$

$$z = e^{\frac{\pi}{4}} e^{-i\frac{1}{2}\ln(2)}$$

Ex4: Find the principal value of i^i .

$$i^i = (e^{i\frac{\pi}{2} + i2\pi k})^i = e^{-\frac{\pi}{2} - 2\pi k} = e^{-\frac{\pi}{2}}$$

Ex5: Find all solutions to $\text{Log}(z^2 - 1) = \frac{i\pi}{2}$

$$z^2 - 1 = e^{i\frac{\pi}{2}} = i$$

$$z^2 = i + 1 = \sqrt{2}e^{i\frac{\pi}{4} + i2\pi k}$$

$$z = 2^{1/4} e^{i\frac{\pi}{8} + i\pi k}$$

Ex6: Find all solutions to $e^{2z} + e^z + 1 = 0$

$$e^z = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$e^z = e^{i\pi \pm i\frac{\pi}{3} + i2\pi k}$$

$$z = i\pi \pm i\frac{\pi}{3} + i2\pi k$$

$$z = \pm i\frac{2\pi}{3} + i2\pi k$$

Derivative of $\log z$ and $\text{Log } z$

$$(\text{Log } z)' = \frac{1}{z}$$

$$\text{Log } z = \ln \sqrt{x^2 + y^2} + i \text{Arg } z = u + iv$$

$$(\text{Log } z)' = u_x + iv_x$$

$$\text{Log } z = \ln |z| + i\varphi, \quad -\pi < \varphi < \pi$$

1.2.7 Branch Cuts

There are many different analytic functions for $\log z$ depending on the branch cut we choose. For example we can choose:

$$\text{Log } z = \ln r + i\varphi, \quad -\pi < \varphi < \pi, \quad \mathbb{C} \setminus (-\infty, 0]$$

$$\log z = \ln r + i\varphi, \quad 0 < \varphi < 2\pi, \quad \mathbb{C} \setminus [0, \infty)$$

$$\log z = \ln r + i\varphi, \quad -\frac{\pi}{2} < \varphi < \frac{5\pi}{2}, \quad \mathbb{C} \setminus i[0, \infty)$$

Ex: Specify a branch cut for $\log z$ such that it is analytic at $z = -1, -i, 1$

$$\log z = \ln r + i\varphi, \quad \frac{\pi}{2} < \varphi < \frac{5\pi}{2}$$

The reason we have branch cuts is so that we can define a single-valued analytic function. This is important for functions such as \sqrt{z} and $\log z$ as one input can correspond to multiple outputs. For example, $\sqrt{1} = 1$ and $\sqrt{1} = -1$. This is why we need to specify a branch cut for \sqrt{z} and $\log z$. Ex: Construct $f(z) = z^{1/2}$ such that it is analytic in $\mathbb{C} \setminus (-\infty, 0]$ and $f(1) = 1$.

$$z^{\frac{1}{2}} = e^{\frac{1}{2} \log z} = e^{\frac{1}{2} \ln r + i\frac{1}{2}\varphi} = r^{\frac{1}{2}} e^{i\frac{\varphi}{2}}, \quad -\pi < \varphi < \pi$$

$$f(1) = r^{\frac{1}{2}} e^{i\frac{\varphi}{2}} = 1e^{i0} = 1$$

$$z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{i\frac{\varphi}{2}}, \quad \pi < \varphi < 3\pi$$

$$\rightarrow f(1) = 1^{\frac{1}{2}} e^{i\frac{2\pi}{2}} = -1$$

Ex2: Find all solutions to $z^{1/2} + 1 - i = 0$ where $z^{1/2}$ is the principal branch of $z^{1/2}$.

$$z^{1/2} = |z|^{1/2} e^{i\frac{\varphi}{2}} = i - 1 = \sqrt{2} e^{i\frac{3\pi}{4} + i2\pi k}$$

$$|z| e^{i\varphi} = 2 e^{i\frac{3\pi}{2} + i4\pi k}$$

$$|z| = 2$$

$$\varphi = \frac{3\pi}{2} + 4\pi k \notin (-\pi, \pi]$$

\therefore no solutions

Ex3: Find all solutions to $z^{1/2} + 1 - i = 0$

$$z^{1/2} = |z|^{1/2} e^{i\frac{\varphi}{2} + i\pi k_1} i - 1 = \sqrt{2} e^{i\frac{3\pi}{4} + i2\pi k_2}$$

$$|z| e^{i\varphi + i2\pi k_1} = 2 e^{i\frac{3\pi}{2} + i4\pi k_2}$$

$$|z| = 2$$

$$\varphi + 2\pi k_1 = \frac{3\pi}{2} + 4\pi k_2$$

$$\varphi = \frac{3\pi}{2} + 2\pi k_3$$

$$z = 2e^{-i\frac{\pi}{2}} = -2i$$

Ex4: Find where $\text{Log}(1 + z^2)$ is analytic

$$w = 1 + z^2$$

$$\text{not analytic on } \Re(w) \leq 0 \wedge \Im(w) = 0$$

$$x^2 - y^2 + 1 \leq 0$$

$$2xy = 0 \Rightarrow x = 0 \vee y = 0$$

$$y = 0 : x^2 + 1 \leq 0 \Rightarrow \nexists x \in \mathbb{R} \text{ s.t. } x^2 \leq -1$$

$$x = 0 : y^2 \geq 1 \Rightarrow |y| \geq 1$$

Analytic on the domain

$$\mathbb{C} \setminus \{(x, y) : x = 0, |y| \geq 1\}$$

Ex5: Find where $\text{Log}\left(\frac{1-z}{1+z}\right)$ is analytic

$$w = \frac{1-z}{1+z}$$

$$\text{not analytic on } \Re(w) \leq 0 \wedge \Im(w) = 0$$

$$w = \frac{(1-z)(1+\bar{z})}{(1+z)(1+\bar{z})} = \frac{1-|z|^2 - z + \bar{z}}{1+|z|^2 + z + \bar{z}} = \frac{1-x^2-y^2-2iy}{1+x^2+y^2+2x}$$

$$y = 0$$

$$\frac{1-x^2}{1+x^2+2x} \leq 0 \Rightarrow 1 \leq x^2 \Rightarrow |x| \geq 1$$

Analytic on

$$\mathbb{C} \setminus \{(x, y) : y = 0, |x| \geq 1\}$$

Ex6: Find a branch cut for $f(z) = \sqrt{z(z-1)}$ such that it is analytic for $\{(x, y)|y = 0, x < 0\}$ such that $f(2) = \sqrt{2}$

$$\sqrt{z(z-1)} = |z|^{1/2} e^{i\frac{\varphi_1}{2}} |z-1|^{1/2} e^{i\frac{\varphi_2}{2}} = (|z||z-1|)^{1/2} e^{i\frac{\varphi_1+\varphi_2}{2}}$$

$$\varphi_1 \in (-\pi, \pi), \varphi_2 \in (-\pi, \pi)$$

$$\lim_{\varphi_1 \rightarrow \pi} \lim_{\varphi_2 \rightarrow \pi} e^{i\frac{\varphi_1+\varphi_2}{2}} = e^{\pi i}$$

$$\lim_{\varphi_1 \rightarrow -\pi} \lim_{\varphi_2 \rightarrow -\pi} e^{i\frac{\varphi_1+\varphi_2}{2}} = e^{-\pi i} = e^{\pi i}$$

$$\therefore \text{continuous for } \{(x, y)|y = 0, x < 0\}$$

$$z = 2: \varphi_1 = 0, \varphi_2 = 0 \Rightarrow \sqrt{z(z-1)} = \sqrt{2}$$

Ex7: Find a branch cut for $f(z) = \sqrt{z(z-1)}$ such that it is analytic for $\{(x, y)|y = 0, x < 0\}$ such that $f(2) = -\sqrt{2}$

$$\sqrt{z(z-1)} = |z|^{1/2} e^{i\frac{\varphi_1}{2}} |z-1|^{1/2} e^{i\frac{\varphi_2}{2}} = (|z||z-1|)^{1/2} e^{i\frac{\varphi_1+\varphi_2}{2}}$$

$$\varphi_1 \in (-\pi, \pi), \varphi_2 \in (-3\pi, -\pi)$$

$$\lim_{\varphi_1 \rightarrow \pi} \lim_{\varphi_2 \rightarrow -\pi} e^{i\frac{\varphi_1+\varphi_2}{2}} = e^{-2\pi i}$$

$$\lim_{\varphi_1 \rightarrow -\pi} \lim_{\varphi_2 \rightarrow -3\pi} e^{i\frac{\varphi_1+\varphi_2}{2}} = e^0 = e^{-2\pi i}$$

$$\therefore \text{continuous for } \{(x, y)|y = 0, x < 0\}$$

$$z = 2: \varphi_1 = 0, \varphi_2 = -2\pi \Rightarrow \sqrt{z(z-1)} = \sqrt{2}e^{-i\pi} = -\sqrt{2}$$

Ex8: Find a branch cut for $f(z) = \log(z^2 + 2z + 2)$ such that it is analytic for $\{(x, y)|x > -1\}$ such that $\frac{d}{dz}f(-1) = 0$

$$\log(z^2 + 2z + 2)$$

$$z = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$

$$\log(z^2 + 2z + 2) = \ln(|z - z_1||z - z_2|) + i(\varphi_1 + \varphi_2)$$

$$-\pi < \varphi_1 < \pi, -\pi < \varphi_2 < \pi$$

$$z = -1: \varphi_1 = -\frac{\pi}{2}, \varphi_2 = \frac{\pi}{2}$$

$$\log(z^2 + 2z + 2) \Big|_{z=-1} = 0 + i \left(-\frac{\pi}{2} + \frac{\pi}{2} \right) = 0$$

$$\frac{d}{dz} \log(z^2 + 2z + 2) \Big|_{z=-1} = \frac{2z+2}{z^2+2z+2} \Big|_{z=-1} = 0$$