Math Notes

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Contents

1 Foundations

1.1 Numbers

1.1.1 Counting Numbers

Counting numbers are in the order of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ and then increase in place value.

1.1.2 Place Value

Place value refers to how many digits a number has.

ten thousands	thousands	hundreds	tens	ones	tenths	hundredths
10,000	1,000	100	10	1	0.1	0.01

Standard form is how the number is generally written.
Ex: 321

Expanded form is written where the place values are added together.
Ex: $300 + 20 + 1$
Written form is where the number is spelled out.
Ex: three-hundred-and-twenty-one

1.1.3 Rounding

For digits between 0 and 9 for any specific place value, anything below 5 is rounded down to 0 and anything 5 or more is rounded up to the next highest place value.

1.2 Arithmetic

1.2.1 Addition

For an addition statement, $a + b = c$,
a and b are called addends.
The result, c is called the sum.

- Commutative property of addition:
  changing the order of addends does not change the sum.

$$a + b = b + a$$

- Associative property of addition:
  changing the grouping of addends does not change the sum.

$$a + (b + c) = (a + b) + c$$

1.2.2 Subtraction

Subtraction is the opposite operation of addition.
For a subtraction statement, $a - b = c$,
a is called the minuend.
b is called the subtrahend.
The result, c, is called the difference.

1.2.3 Multiplication

Multiplication is analogous to repeated addition.
Ex: $a + a + a = 3a$
For a multiplication statement, $ab = c$,
a and b are called factors.
The result, c, is called the product.
Commutative property of multiplication: changing the order of factors does not change the product.

$$ab = ba$$

Associative property of multiplication: changing the grouping of factors does not change the product.

$$a(bc) = (ab)c$$

Multiplying by 1: any number multiplied by one will just be that number.

$$1a = a$$

Multiplying by 0: any real number multiplied by zero will become zero.

$$0a = 0$$

Long Multiplication:
Ex: $21 \cdot 42$
-Can break it up into 4 multiplication statements and sum the products

$20 \cdot 40 = 800$

$20 \cdot 2 = 40$

$1 \cdot 40 = 40$

$1 \cdot 2 = 2$

1.2.4 Division

Division is the opposite operation of multiplication.
For a division statement, $a \div b = c$ or $\frac{a}{b} = c$
a is called the numerator or the dividend.
b is called the denominator or the divisor
The result, c is called the quotient.
Any amount leftover that doesn't divide evenly is called the remainder.
Dividing by 1: any number divided by 1 is itself.

$$a \div 1 = a$$

Zero divided by any real number is zero.

$$0 \div a = 0$$

Long Division:
Ex: $412 \div 5$

$$\begin{array}{r} 82 \\ 5\overline{)412} \\ \underline{40} \\ 12 \\ \underline{10} \\ 2 \end{array}$$

Remainder is 2
So the answer is $412 \div 5 = 82 + \frac{2}{5}$

1.2.5 Factors

A factor is a whole number that can divide evenly into another number.
Factor pairs are two numbers that multiply to a certain product.
Ex: factor pairs of 8 are {1,8} and {2,4}
Even numbers are those that are evenly divisible by 2 (e.g. 2, 4, 6,8, 10). Odd numbers are numbers that are not even;y divisible by 2. (e.g. 1, 3, 5, 7, 9).
Prime numbers are whole numbers greater than 1 that cannot be exactly divided by any whole number other than itself and 1 (e.g. 2, 3, 5, 7, 11).

Prime Factorization:
Prime factorization is writing a number as a product of all its prime number factors.
Ex: $36 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 2 \cdot 2$

Least Common Multiple (LCM):
It is the smallest whole number that all numbers in a given set can divide evenly into.
Ex: What is the LCM of 12 and 18?
$12 = 3 \cdot 2 \cdot 2$
$18 = 3 \cdot 3 \cdot 2$
The prime factors of the LCM must also contain the prime factors of each number in the set.
$\therefore$ the prime factorization is $3 \cdot 3 \cdot 2 \cdot 2$
$\therefore$ the LCM is 36.

Greatest Common Factor (GCF):
It is the largest number that all numbers in a given set can be divided by.
Ex: What is the GCF of 12 and 18?
$12 = 3 \cdot 2 \cdot 2$
$18 = 3 \cdot 3 \cdot 2$
The GCF can be found by taking the product of the prime factors that all numbers in a set have in common.
12 and 18 have factors 2 and 3 in common,
$\therefore$ the GCF is 6.

Factoring: Factoring is when you take a number or an expression and break it up into factors. This is usually done for the purpose of taking out a GCF in order to simplify, or it is done as a step in solving equations.
Ex: $14 + 6 = 2(7 + 3)$

1.3 Fractional Numbers

1.3.1 Fraction Notation

Fractions are an unsolved division statement expressed as $\frac{a}{b}$.
We usually like to express fractions in their simplest form. To do this, we factor out the GCF from both the numerator and denominator and cancel the GCF.
Ex: $\frac{8}{12} = \frac{4(2)}{4(3)} = \frac{2}{3}$ where $\frac{2}{3}$ is in simplest form.

When a number has a value greater than 1, it can be expressed as either a mixed number or as an improper fraction.
Mixed numbers are a whole number plus the remaining fraction. Ex: $2\frac{1}{2}$
Improper fractions are where the numerator is larger than the denominator. Ex: $\frac{5}{2}$
Note that the value of the two above examples are equivalent.

1.3.2 Adding and Subtracting Fractions

Step 1: The denominator of both fractions must be the same. If they are not the same then you will need to find the LCM which will become the new denominator for both fractions.
Step 2: Add or subtract the numerators accordingly, leaving the denominator the same.
Ex: $\frac{3}{12} + \frac{5}{18}$
The LCM is 36 so we can rewrite
$\frac{3}{12}\left(\frac{3}{3}\right) + \frac{5}{18}\left(\frac{2}{2}\right) = \frac{9}{36} + \frac{10}{36} = \frac{19}{36}$

1.3.3 Multiplying and Dividing Fractions

Multiplying fractions is easy. You can simply multiply the numerators and multiply the denominators.
Ex: $\frac{1}{3} \cdot \frac{2}{3} = \frac{2}{18} = \frac{1}{9}$

A reciprocal is defined to be where the numerator and denominator are flipped. So the reciprocal of $\frac{a}{b}$ would be $\frac{b}{a}$.
When dividing fractions, we take the reciprocal of the divisor and then treat it as a multiplication statement.
Ex: $\frac{1}{6} \div \frac{2}{3} = \frac{1}{6} \cdot \frac{3}{2} = \frac{3}{12} = \frac{1}{4}$

1.3.4 Decimals

Decimals are a place value extension for number of place values smaller than one.
The name of each place value (tenths, hundredths, etc.) refers to what that number must be divided by in order to be expressed as a fraction.
For example, 0.25 has a place value of hundredths, $\therefore 0.25 = \frac{25}{100} = \frac{1}{4}$.

A percent is a decimal that is expressed as a fraction over 100. It is meant to represent a part of a whole.
Ex: $25\% = 0.25 \cdot 100\% = 0.25 = \frac{1}{4}$

1.3.5 Ratios and Rates

A ratio is a quantity, a, of one thing compared to a quantity, b of another. Denoted by $a : b$.
A rate is a ratio that compares two quantities of different units of measure, expressed as the number of units of the first quantity for every one unit of the second quantity. A common example of this is velocity which is measured in meters per second.

Unit conversions:
Unit conversions are used to transfer a measure of one type of unit to a different unit that measures the same thing.
Ex: Express $2\,\mathrm{m/s}$ in $\mathrm{km/h}$.
$\frac{2\,\mathrm{m}}{\mathrm{s}} \cdot \frac{60\,\mathrm{s}}{1\,\mathrm{min}} \cdot \frac{60\,\mathrm{min}}{1\,\mathrm{h}} \cdot \frac{1\,\mathrm{km}}{1000\,\mathrm{m}} = 7.2\,\mathrm{km/h}$

1.4 Lines and Angles

1.4.1 Types of Lines

A line extends forever in both directions.
A segment is a part of a line but with a defined starting and stopping point.
A ray is a line with a defined starting point but no defined ending point.
Parallel lines are always the same distance apart from one another. No matter how far they extend, they will never meet.
Perpendicular lines are lines that meet at right angles.

1.4.2 Types of Angles

An angle is formed from intersecting lines. The wider an angle is, the greater its measure.
Types of angles:

Acute:	$0° < \theta < 90°$
Right:	$\theta = 90°$
Obtuse:	$90° < \theta < 180°$
Straight:	$\theta = 180°$
Reflex:	$180° < \theta < 360°$
Circle:	$\theta = 360°$

Complimentary angles are two angles with a sum of $90°$.
$\alpha + \beta = 90°$
Supplementary angles are two angles that sum to $180°$.
$\alpha + \beta = 180°$

1.4.3 Units of Angles

Other units of angles include radians and gradians where radians are the natural unit of measure and gradians are measured with respect to percentages.
The conversion factor is:

$$180° = \pi = 200\%$$

Common conversions:

Degrees	Radians	Gradians
30°	$\pi/6$	$33.\overline{3}\%$
45°	$\pi/4$	50%
60°	$\pi/3$	$66.\overline{6}\%$
90°	$\pi/2$	100%
180°	π	200%
270°	$3\pi/2$	300%
360°	2π	400%

1.5 Negative Numbers

1.5.1 Absolute Values

Negative numbers are less than zero and are exact opposites to their positive counterparts.
This essentially means that the number 2 is the same distance from 0 on the number line as the number -2.
We denote this imaginary distance as the absolute value of the number. This means that we only consider the positive contributions of that number. It is denoted by $|a|$ and has the property that $|a| = |-a|$.
Properties of Absolute Value:

$$\begin{aligned}|a| &= \sqrt{a^2}\\ |ab| &= |a||b|\\ \left|\frac{a}{b}\right| &= \frac{|a|}{|b|}\\ |a-b| &= |b-a|\\ |a+b| &\leq |a|+|b|\end{aligned}$$

1.5.2 Sign of a Number

The sign of a number tells us whether it is positive, negative, or zero. It is denoted by $\text{sgn}(a)$.

$$\text{sgn}(x) = \begin{cases} 1 \text{ if } x > 0\\ 0 \text{ if } x = 0\\ -1 \text{ if } x < 0\end{cases}$$

Some properties of this are the following:

$$\begin{aligned}&\operatorname{sgn} x = \frac{x}{|x|} = \frac{|x|}{x}\\ &\operatorname{sgn} x = \frac{1}{\operatorname{sgn} x},\ x \neq 0\\ &\operatorname{sgn}(ab) = \operatorname{sgn}(a)\operatorname{sgn}(b)\\ &\operatorname{sgn}(ab) = \operatorname{sgn}\left(\frac{a}{b}\right)\\ &\operatorname{sgn}(x^n) = (\operatorname{sgn} x)^n\end{aligned}$$

Ex: $\text{sgn}(-4) = -1$

1.5.3 Adding and Subtracting Negative Numbers

Adding negative numbers is the same thing as subtracting a number and subtracting a negative is the same thing as adding. This can be summed up as follows:
-Adding a positive makes it more positive
-Adding a negative makes it more negative
-Subtracting a positive makes it more negative
-Subtracting a negative makes it more negative

1.5.4 Multiplying and Dividing Negative Numbers

The following rules are the same regardless whether you're multiplying or dividing.
For $a > 0$ and $b > 0$,
$(a)(b) = ab$
$(a)(-b) = -ab$
$(-a)(b) = -ab$
$(-a)(-b) = ab$

1.6 Powers and Order of Operations

1.6.1 Exponents

An exponent is a form of repeated multiplication, expressed in the form a^b. It tells us to multiply the base, a, by itself b times.
Ex: $4^3 = 4 \cdot 4 \cdot 4 = 64$

1 to any exponent will be 1.
$1^a = 1$
Any number to exponent 1 will remain the same.
$a^1 = a$
Any number to exponent 0 will be 1.
$a^0 = 1$

1.6.2 Radicals

Radicals are the opposite operation of exponents, in the form $\sqrt[b]{a}$. It gives us a value that, when multiplied by itself as many times as specified by the radical, b, gives us the base, a.
Ex: $\sqrt{9} = 3$
Ex2: $\sqrt[3]{8} = 2$
Note that the root of 1 will always be 1.
$\sqrt[b]{1} = 1$

1.6.3 Order of Operations

This is the order in which to apply different mathematical operations within calculations. The order is as follows:

1. Brackets and grouping symbols

2. Exponents and radicals
3. Multiplication and division
4. Addition and subtraction

1.6.4 Real Number System

Real numbers, $\mathbb{R}$, are all numbers that occur naturally and that we can visualise. They are split into the following sub-categories:
Natural Numbers, $\mathbb{N}$: counting numbers; numbers that do not need to be represented as a fraction or decimal and are larger than zero.
Whole Numbers, $\mathbb{N}_0$: Whole numbers as well as zero.
Integers, $\mathbb{Z}$: All whole numbers and their opposites (negatives).
Rational Numbers, $\mathbb{Q}$: Any number that can be expressed as a fraction. (Includes repeating or terminating decimals).
Irrational Numbers, $\mathbb{A}_{\mathrm{R}}$: All remaining real numbers that cannot be expressed as a fraction.

1.7 Shapes

1.7.1 Polygons

A polygon is a shape with a finite number of straight sides and angles.

3 sides:	triangle
4 sides:	quadrilateral
5 sides:	pentagon
6 sides:	hexagon
7 sides:	heptagon
8 sides:	octagon
9 sides:	nonagon
10 sides:	decagon

The sum of all interior angles of a polygon is expressed with the formula

$$\sum \theta_i = (n-2)180^\circ$$

This means that any triangle will have three angles that add to 180° and any quadrilateral will have 4 angles that sum to 360°.
The area of a shape is defined to be the space enclosed by that shape.
The perimeter of a shape is defined to be the sum of all side lengths of that shape.

1.7.2 Quadrilaterals

Types of quadrilaterals:

- A parallelogram is a quadrilateral with 2 pairs of parallel sides.

  $$A = bh$$

  for where b is the base and h is the hight. This is because a parallelogram can be constructed of two triangles with those same dimensions.

- A trapezoid is a quadrilateral with exactly 1 pair of parallel sides.

$$A = \frac{h}{2}(a + b)$$

  for where a and b are the lengths of the two parallel sides and h is the height, or distance between those parallel sides.

- A kite is a quadrilateral with 2 pairs of congruent (identical length) sides where the congruent sides are next to adjacent (next to another).

$$A = \frac{1}{2}ab$$

  for where a and b are the lengths between opposite vertices of the kite.

- A rectangle is a quadrilateral with 4 right angles.

$$A = lw$$

- A rhombus is a quadrilateral with 4 sides of equal length.

$$A = bh$$

- A square is a quadrilateral with 4 equal sides and 4 right angles.

$$A = l^2$$

1.7.3 Triangles

Types of triangles:

- An acute triangle has 3 angles that each measure less than 90°
- A right triangle has 1 angle that measures exactly 90°
- An obtuse triangle has 1 angle that measures more than 90°
- An equilateral triangle has 3 equal side lengths.
  By definition, an equilateral triangle must also have all three angles be 60°.
- An isosceles triangle has 2 equal sides.
- A scalene triangle has no equal sides.

The area of any triangle can be computed using the base, b, and it's height (or altitude), h.

$$A = \frac{1}{2}bh$$

For any right angle triangle, the side lengths can be related using Pythagorean's Theorem:

$$a^2 + b^2 = c^2$$

for where a and b are two side lengths that form right angles with another and c is the length of hypotenuse (the longest side).
A Pythagorean triple is a set of numbers that satisfies Pythagorean's theorem that are all integer values.

1.7.4 Circles

Every circle has a center (or origin) where the distance from the center of the circle to any part of the edge will always be the same.
This distance is called the radius, r.
The diameter of a circle is the length of a line running through the center that touches two points on the edge of the circle.
The diameter is twice the length of the radius. $d = 2r$
The circumference of a circle is the same as its perimeter.

$$C = 2\pi r$$

The area of a circle is defined as:

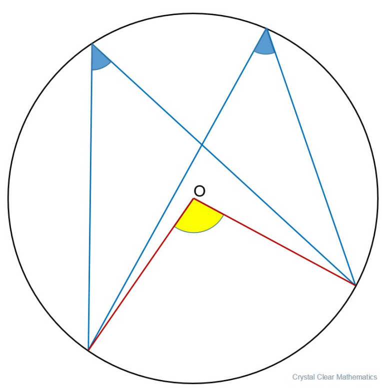
$$A = \pi r^2$$

A tangent line is a line that touches the edge of the circle at exactly 1 point. A tangent line will always be perpendicular to the radius.
A chord is a line that joins two points on a circle. Unlike the diameter, it does not need to pass through the center.
Arc length is some distance travelled along the outside of the circle. It can be computed as:

$$s = r\theta$$

for where θ is measured in radians.
A sector of a circle is a fraction of a circle, like a slice of pizza. Its area is computed as

$$A = \frac{1}{2}r^2\theta$$



In circle geometry, the inscribed angles are shown in blue and the central angle is shown in yellow.
Inscribed angles subtended by the same arc are equal.
The central angle is twice as large as the inscribed angle.

1.7.5 Transformations

- Symmetry:
  A line of symmetry is an imaginary line that divides a shape into two identical parts. Shapes are symmetrical if they have at least one line of symmetry through them.

- Translations:
  Translations are the movement of a figure from one place to another; they don't change their size, arrangement, or direction.

- Rotations:
  Rotations are when objects turn in a circular motion around a fixed pivot point. The figure remains the same, as does each point's distance from the point of rotation

- Reflections:
  A reflection is a transformation that acts like a mirror: it swaps all pairs of points that are on exactly opposite sides of the line of reflection.

- Dilations:
  A dilation is a type of transformation that changes the size of the figure. The scale factor measures how much larger or smaller the image is. The angles remain the same.

- Congruence:
  Shapes are congruent with another if they are the same size and shape (same angles). Transformations can be applied to similar shapes to see if they can become congruent.

- Similarity:
  Two shapes are similar if a series of transformations can be applied to one in order to achieve the other. Similar shapes will always have matching angles.

1.7.6 3D Shapes

The volume of a shape is the 3D space that the object takes up.
The surface area is the sum of the areas of the shape's exterior. Common 3D shapes:

- A cube has all equal side lengths and equal square faces.

$$V = l^3$$

$$SA = 6l^2$$

- A prism has two identical faces on each end, connected by a straight body.

$$V = A_{base}h$$

- A pyramid has the vertices at the base come together to form a point above the base. Note that a cone is a special type of pyramid with a circular base.

$$V = \frac{1}{3}A_{base}$$

- A sphere is entirely circular.

$$V = \frac{4}{3}\pi r^2$$

$$SA = 4\pi r^2$$

1.8 Arithmetic with Variables

1.8.1 Equations and Inequalities

In an equation, one side must of the equation must always be equal to the other side. So what you do to one side, you must do to both sides. The only exception to this rule is if you are doing something to one side that doesn't change the overall value, such as adding 0 or multiplying/dividing by 1. Equations are often used to solve for an unknown.
Ex: $3x - 6 = 7x + 2$

$$3x - 6 - 2 = 7x + 2 - 2$$
$$3x - 8 = 7x$$
$$3x - 8 - 3x = 7x - 3x$$
$$-8 = 4x$$
$$\frac{-8}{4} = \frac{4x}{4}$$
$$-2 = x$$

Inequalities are stating when one side is larger than the other side.
Ex: $3x > 6 \Rightarrow x > 2$
Note that when multiplying both sides by a negative, the sign switches directions.
Ex: $-x > 4 \Rightarrow x < -4$

1.8.2 Like Terms

A term is a group that is linked together through multiplication or division operations. For example, $3x$ would be a term. Terms are broken up by addition and subtraction operations. Like terms are when two terms have the same combination of variables such as $3a$ and $2a$. These can be added or subtracted from each other as they both have the variable a in common. In the case of $3a$ and $4b$, they are not like terms and cannot be combined. Also note that x and x^2 are not like terms, as the one consists of a singular x where as the other one consists of two.
Ex: $6a + 4b + 3a - 2ab + b = 9a + 5b - 2ab$

1.8.3 Distributive Property

This is when you multiply a number or variable on the outside of a pair of parentheses to each term on the inside.
Ex: $3(x + y) = 3x + 3y$
In general,

$$a(b + c) = ab + bc$$

The product of binomials is similar to the distributive property, when you multiply two or more groups in parentheses.

$$(a + b)(c + d) = ac + ad + bc + bd$$

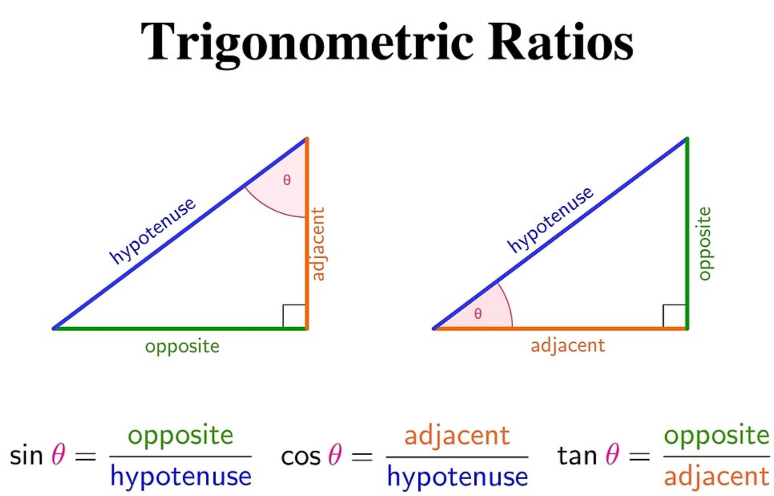
Ex: $(x + 1)(x - 2) = x^2 + x - 2x - 2 = x^2 - x - 2$

1.8.4 Factoring

Factoring is the opposite of the distributive property. It's where you take out a common factor from a set of numbers and put them in brackets.
Ex: $3x + 6 = 3(x + 2)$
The first step is to factor out a GCF if possible. Sometimes this is all that can be done, as with binomials.
Trinomials can prove tricky.
Ex: $x^2 + 5x + 6$
You need to find two numbers that add to get the middle term and multiply to get the constant (3rd term).
To do this, we can find the factor pairs of the constant.
So for this example, they are $\{1, 6\}$, $\{-1, -6\}$, $\{2, 3\}$, $\{-2, -3\}$
We can then deduce that the pair $\{2, 3\}$ will work, so the answer will be
$x^2 + 5x + 6 = (x + 2)(x + 3)$
If the coefficient of the first term is greater than 1, we instead find two numbers that multiply to get the constant times said coefficient and add to get the middle term.
Ex: $2x^2 + x - 3$
We find factor pairs of -6 which are $\{-1, 6\}$, $\{1, -6\}$, $\{2, -3\}$, $\{-2, 3\}$
The pair is $\{-2, 3\}$ so our answer is
$2x^2 + x - 3 = (2x + 3)(x - 1)$

1.9 Trigonometric Ratios

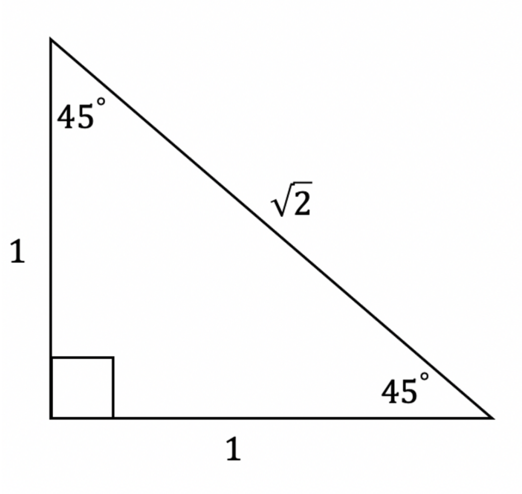
1.9.1 Right Angle Trigonometry

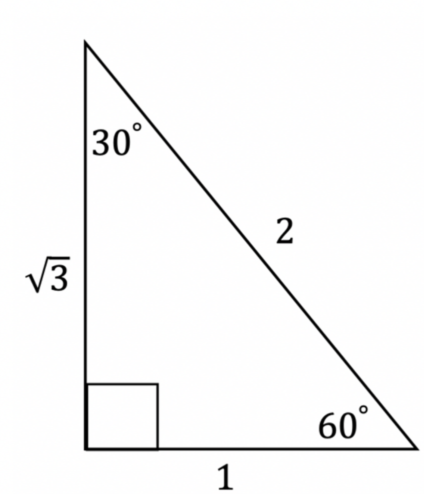


Hypotenuse is the lonest end of the triangle.
Opposite is the side across from the angle in question.
Adjacent is the side next to the angle in question.
Note these ratios only work for right angle triangles.

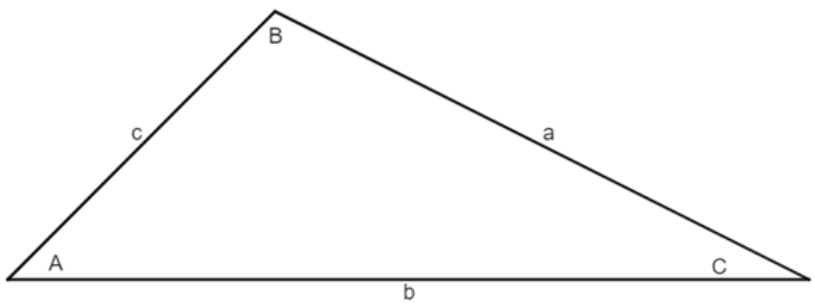
Special Triangles:
Often we need a calculator to determine the numeric value of the side lengths. There are two useful

triangles that we use that allows us to compute the trig ratios without using a calculator. These are called special triangles.





1.9.2 Sine and Cosine Law



The cosine law is an extension of Pythagorean's theorem for all triangles (not just right angled ones).

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Note that the usage of a, b, and c is interchangeable.

The sine law is a relationship between the ratio of sides and angles in any triangle.

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$$

1.10 Sequences and Series

1.10.1 Summation Notation

Summation notation is a shorthand was to express the sum of a set of numbers. The bottom of the sigma tells you what index to start at and the top of the sigma tells you which number to end at.

Ex: $\sum\limits_{x=1}^{5} x = 1 + 2 + 3 + 4 + 5 = 15$

We can use similar notation for repeated multiplication as well. For this we use an upper case pi and use the same notation, using the starting point, stopping point, and index.
Ex: $\prod_{x=2}^{4} = 2 \cdot 3 \cdot 4 = 24$

1.10.2 Arithmetic Sequence

A sequence is an ordered list of numbers. Each number in the list is referred to as an element or term. Each new term follows a pattern or rule to determine the next term in the sequence.
An arithmetic sequence is an ordered list of terms in which the difference between consecutive terms is constant. It is generally expressed as $\{a, a+d, a+2d, a+3d, \ldots\}$
The formula for the nth term of the sequence is given by,

$$t_n = a + (n-1)d$$

where a is the first term, d is the common difference between consecutive terms, and n is the term number.

1.10.3 Arithmetic Series

An arithmetic series is the sum of all terms that form an arithmetic sequence. S_n represents the sum of the first n terms of a series.

$$S_N = \frac{N}{2}(a + t_N) = \frac{N}{2}(2a + (N-1)d) = \sum_{n=1}^{N}(a + (n-1)d)$$

1.10.4 Geometric Sequence

Geometric sequences are sequences in which the ratio of consecutive terms is constant. The common ratio, r, can be found by taking any term, except for the first, and dividing that term by the preceding term.
The general geometric sequence is $\{a, ar, ar^2, ar^3, \ldots\}$
The general term for a geometric sequence is,

$$t_n = ar^{n-1}$$

where a is the first term and r is the common ratio.

1.10.5 Geometric Series

A geometric series is the expression for the sum of the first N terms of a geometric sequence. It is expressed as,

$$S_N = \frac{a(r^n - 1)}{r-1} = \frac{rt_n - a}{r-1} = \sum_{n=1}^{N} ar^{n-1}$$

As the number of terms in the series gets increasingly large, it will either approach an infinite value or a fixed and finite value. The series comes to a fixed value for $|r| < 1$. This infinite series can be computed as,

$$S_\infty = \frac{a}{1-r}$$

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An arithmetic sequence is an ordered list of terms in which the difference between consecutive terms is constant. It is generally expressed as $\{a, a + d, a + 2d, a + 3d, \ldots\}$
The formula for the nth term of the sequence is given by,

$$t_n = a + (n - 1)d$$

where a is the first term, d is the common difference between consecutive terms, and n is the term number.

1.11.3 Arithmetic Series

An arithmetic series is the sum of all terms that form an arithmetic sequence. S_n represents the sum of the first n terms of a series.

$$S_N = \frac{N}{2}(a + t_N) = \frac{N}{2}(2a + (N - 1)d) = \sum_{n=1}^{N}(a + (n - 1)d)$$

1.11.4 Geometric Sequence

Geometric sequences are sequences in which the ratio of consecutive terms is constant. The common ratio, r, can be found by taking any term, except for the first, and dividing that term by the preceding term.
The general geometric sequence is $\{a, ar, ar^2, ar^3, \ldots\}$
The general term for a geometric sequence is,

$$t_n = ar^{n-1}$$

where a is the first term and r is the common ratio.

1.11.5 Geometric Series

A geometric series is the expression for the sum of the first N terms of a geometric sequence. It is expressed as,

$$S_N = \frac{a(r^n - 1)}{r - 1} = \frac{rt_n - a}{r - 1} = \sum_{n=1}^{N} ar^{n-1}$$

As the number of terms in the series gets increasingly large, it will either approach an infinite value or a fixed and finite value. The series comes to a fixed value for $|r| < 1$. This infinite series can be computed as,

$$S_\infty = \frac{a}{1 - r}$$

1.12 Analyzing Data

1.12.1 Displaying Data

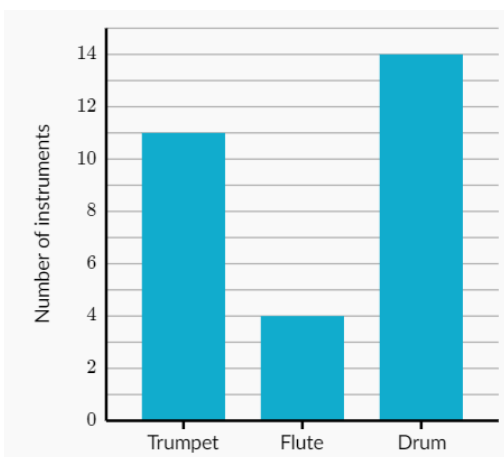
If we have the table:

Drink	Type	Calories	Sugar (g)	Caffeine (mg)
coffee	hot	4	0	260
latte	hot	100	14	75
mocha	hot	170	27	95

The *individuals* are defined to be the drinks in this case (the things we are collecting data from). There are also 4 variables. The variable "Type" is *categorical*, meaning that the data it holds fits into one of some number of categories (digital vs. analog).

Some types of graphs we can use to represent data includes pictographs, bar graphs, frequency graphs, pie graphs, and many more.

Ex: Bar graph



Data that relates to itself, we can represent with a two-way table.

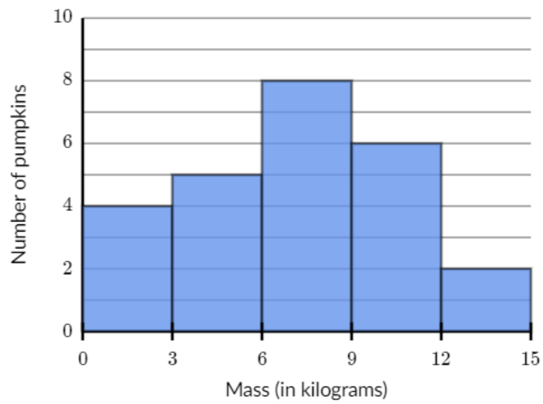
Ex: Say we have 10 candies, 6 chocolate, 3 chocolate-coconut, 1 coconut, and 2 plain. We can express this as

	Coconut	No Coconut
Chocolate	3	6
No Chocolate	1	2

A two-way table with relative frequencies is when the columns are normalized to equal 1.

	Coconut	No Coconut
Chocolate	75%	75%
No Chocolate	25%	25%

Conditional distribution is when you analyze a single row/column or a subset of the table.
Marginal distribution is when you analyze the totals of the rows/columns.
Frequency plots are similar to bar graphs but are more all-encompassing. We can use frequency graphs to represent data that is more analog than discrete, called a histogram.
Ex:



With histograms we can classify different behaviors of the data.
The maximum value of the histogram is the *peak*.
If there are groupings of data points in a similar area, we call those *clusters*
If there are a small number of data points far away from the majority of the data, we call those *outliers*.
We can also represent frequency data using a stem-leaf plot.
Ex:

Stem	Leaf
0	0, 0, 2, 4, 7, 7, 9
1	1, 1, 3, 8
2	0

maps to the data 0, 0, 2, 4, 7, 7, 9, 11, 11, 13, 18, 20.
The stem maps the tens place value and the leaf maps the ones place value

1.12.2 Center and Spread

The average of a set of data is a measure of central tendency of that data and there are many ways to express the average.
The *mean* is the arithmetic sum of the data divided by the number of data points

$$\mu = \frac{1}{N}\sum_{i=1}^{N} x_i$$

The *median* is the middle value of the data set. If there are an even number of data points, the median is the mean of the two middle data points.
The *mode* is the most commonly occurring value.

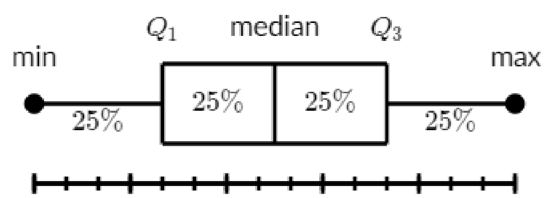
The *midrange* is the mean of the difference between the maximum and minimum values in the data set.

Mean is the standard measure for average in most cases, though it is important to note that outliers can sometimes have a large, unwanted, effect on the mean so in these cases, it may be best to use median as the average.

Data sets can have the same averages but may look very different. To represent this, we introduce spread. Spread is how spread out the data points are.
The most simple measure of spread is the *range* which is simply the difference between the maximum and minimum values in the data set.
A slightly more elaborate method of calculating spread is the *mean absolute deviation* (MAD). This is the mean of how far away a number is from the mean.

$$MAD = \frac{1}{N}\sum_{i=1}^{N}|x_i - \mu|$$

The *interquartile range* (IQR) is analogous to the median of a data set. The IQR is the difference between the 1st and 3rd quartiles in a data set. The first quartile is the median of the lower half of the data and the 3rd quartile is the median of the higher half of the data.
Ex: For data of 1,3,3,3,4,4,4,6,6,
Median= 4
$Q_1 = \text{median}\,\{1, 3, 3, 3\} = 3$
$Q_3 = \text{median}\,\{4, 4, 6, 6\} = 5$
IQR$= 5 - 3 = 2$
We can use the IQR to represent data as a box and whisker plot:



We can also use the IQR to define outliers. We can use the general rule of 1.5 times the IQR. Any outliers can be defined to be less than $Q_1 - 1.5 \cdot IQR$ or greater than $Q_3 + 1.5 \cdot IQR$
Variance is the mean of the squared distance between the value and the mean, represented by σ^2

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2$$

By expanding the quadratic and using the definition of the mean, we can also write this as

$$\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}x_i - \mu^2$$

The *standard deviation* is the square root of the variance.

$$\sigma = \sqrt{\frac{1}{N}\sum_{i=1}^{N}(x_i - \mu)^2}$$

We can use the standard deviation to calculate Z-scores. A Z-score is how many standard deviations a data point is from the mean.

$$Z_i = \frac{x_i - \mu}{\sigma}$$

In an ordered list of data, we can calculate the percentile by dividing the position of the data point by the number of data points in the sample size. Note that the median corresponds to the 50th percentile. Z-scores are particularly useful in calculating percentiles as we can use the aid of Z-tables to compute percentiles along a normal distribution.

1.13 Probability and Combinatorics

1.13.1 Probability

Probability is the likelihood of an event occurring. (the number of possibilities that satisfy a given constraint over the number of possible outcomes)
Probability is always expressed as a number between 0 and 1.
The probability of an event, A, is denoted by $P(A)$.
The sum of all possibilities of an event is equal to 1.
Relative frequency is the frequency of an event occurring in a trial to provide an estimate of the probability. Note that the more trials, the better the estimate will be.

Definitions:

- Two events are mutually exclusive if they cannot occur at the same time.
- The probability of event A, given event B, is called conditional probability. $P(A|B)$ is conditional probability of A, given B.
- The intersection of two events is the likelihood of both events occurring. It is the set of elements that are common to both events. Denoted by $P(A \cap B)$
- The union of two events is the probability of at least one of the events occurring. Denoted by $P(A \cup B)$
- Independent Events: The probability of A does not affect the probability of B.
- Dependent Events: Occurrence of event A changes the probability of event B occurring.
- The compliment of an event is the case of an event not occurring. $P(A')$ is the probability that A does not occur.

Rules of Probability:

$P(A) + P(A') = 1$
$P(A \cap B) = P(A)P(B|A)$
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

1.13.2 Combinatorics

Consider a task made up of several stages. The number of choices for the first stage is a, the number for the second stage is b, then c and so on. The number of ways the task can be completed (or the number of possible outcomes) is: $a \cdot b \cdot c \cdot d \cdots$
Ex: using the letters A, B, C, D, E, F, how many 4 letter combinations can be made if:
a) letters can be repeated?
b) letters can not be repeated?
a) $6 \cdot 6 \cdot 6 \cdot 6 = 6^4 = 1296$
b) $6 \cdot 5 \cdot 4 \cdot 3 = 360$

Factorial Notation:
The symbol '!' indicates factorial and means the product of decreasing natural numbers

$$n! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1 = \prod_{i=1}^{n} i$$

Ex: Simplify $\frac{25!}{23!5!}$

$$\frac{25 \cdot 24 \cdot 23!}{23!5 \cdot 4 \cdot 3 \cdot 2 \cdot 1} = \frac{25 \cdot 24}{5 \cdot 4 \cdot 3 \cdot 2} = 5$$

Ex2: Simplify $\frac{(n+1)!}{(n-1)!}$

$$\frac{n+1)(n)(n-1)!}{(n-1)!} = n(n+1)$$

Permutations:
A permutation is an arrangement of distinct objects in which the order is important.
The number of permutations of n distinct objects is $n!$
The number of permutations of n distinct objects taken r at a time is:

$${}_nP_r = \frac{n!}{(n-r)!}$$

Ex: licence plates have 3 digits followed by 3 letters. How many different plates can be made?

$${}_{10}P_3\,{}_{26}P_3 = \frac{10!}{(10-3)!}\frac{26!}{(26-3)!} = 11232000$$

Ex2: Solve for n if ${}_nP_2 = 30$

$$\frac{n!}{(n-2)!} = 30$$
$$\frac{n(n-1)(n-2)!}{(n-2)!} = 30$$

$n^2 - n = 30$
$n^2 - n - 30 = 0$
$(n-6)(n+5) = 0$
$\Rightarrow n = 6,\ n = -5$
$n \neq -5$ because $n \in \mathbb{N}$
$\therefore n = 6$

Some permutations have restrictions where certain objects must be grouped. If objects are grouped together, the number of possible outcomes can be calculated by multiplying:
(total number of groups or individuals)!(number in group 1)!(number in group 2)!$\cdots$
Ex: How many ways can "PICTURE" be arranged if:
a) two vowels must be together?
b) A vowel cannot be beside another vowel?

a) This forms 6 groups with one group of two so $6!2! = 1440$
b) For this one we can calculate it by taking the total arrangements, 7!, and subtracting the arrangements that don't fit the criteria.
3 vowels together: 5!3!
2 vowels together: 6!2!
So the answer will be: $7! - 5!3! - 6!2! = 280$

If there are repetitions, the number of permutations can be expressed as $\dfrac{n!}{a!b!c!\cdots}$ where a, b, and c are the number of repetitions for each identical object
Ex: a true-false quiz has 10 questions. How many answer keys are possible if 4 are true and 6 are false?
$\dfrac{10!}{6!4!} = 210$

Combinations:
A combination is an arrangement of distinct objects in which the order of the objects is not important.

$${}_nC_r = \frac{{}_nP_r}{r!} = \frac{n!}{r!(n-r)!} = \binom{n}{r}$$

Ex: Looking at a 5-card hand in a deck of cards:
a) how many 5 card hands are there?
b) how many with all hearts?
c) how many with 3 hearts and 2 spades?
d) how many with at most 3 hearts?

a) ${}_{52}C_5 = 2598960$
b) ${}_{13}C_5\,{}_{39}C_0 = 1287$
c) ${}_{13}C_3\,{}_{13}C_2\,{}_{39}C_0$
d) We can sum all the different possibilities

3 hearts: ${}_{13}C_3\,{}_{39}C_2$

2 hearts: ${}_{13}C_2\,{}_{39}C_3$
1 heart: ${}_{13}C_1\,{}_{39}C_4$
0 hearts: ${}_{13}C_0\,{}_{39}C_5$
Total is 2569788

1.13.3 Pascal's Triangle

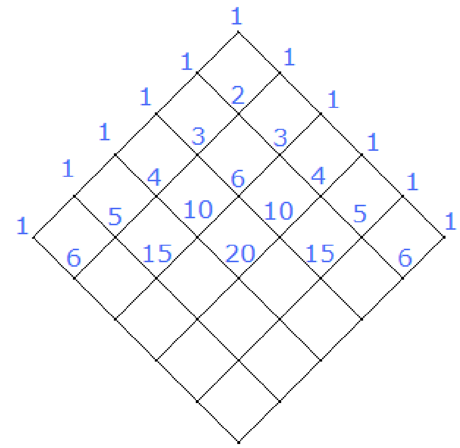
1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 200 252 200 120 45 10 1

It can also be written using combinations for where n is the row number and r is the term number where we start counting from 0.

$$\binom{0}{0}$$
$$\binom{1}{0} \quad \binom{1}{1}$$
$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$
$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$
$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$
$$\binom{5}{0} \quad \binom{5}{1} \quad \binom{5}{2} \quad \binom{5}{3} \quad \binom{5}{4} \quad \binom{5}{5}$$

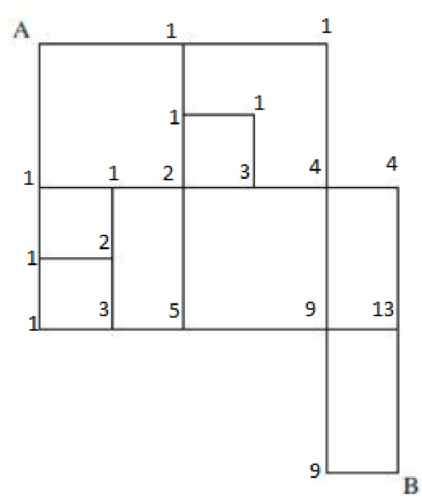
The sum of each row can be expresses by the geometric sequence $t_n = 2^{n-1}$ where n is the row number and t_n is the sum of that row.

Pathway Problems:
Another occurrence of Pascal's triangle is if we want to find the number of different possible pathways to travel to a point on a grid.



For a simple grid as shown, we can find the number of pathways using permutations.
For the 5x5 grid, we go down 5 moves and right 5 moves for a total of 10 moves. So we can express it as $\frac{10!}{5!5!} = 252$ moves.
Note that this only works for perfect grids. A more difficult example, we need to use the general process of Pascal's triangle and sum adjacent corners to come to the answer.
Ex:

Find the number of pathways from A to B if you can only travel to the right and down.



So there will be a total of 22 different paths between A and B.

The Binomial Theorem:

This is used to expand any power of a binomial expression $(a+b)^n$
It follows the patterns of Pascal's triangle and can be expressed as:

$$(a+b)^n = {}_nC_0(a)^n(b)^0 + {}_nC_1(a)^{n-1}(b)^1 + {}_nC_2(a)^{n-2}(b)^2 + \cdots + {}_nC_{n-1}(a)^1(b)^{n-1} + {}_nC_n(a)^0(b)^n$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$