

Math Notes

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1 Complex Analysis

1.1 Complex Algebra

Complex numbers arise from the roots of polynomials.

Ex: $x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm\sqrt{-1}$. This polynomial has no real roots, however, we can introduce an imaginary number i such that $i^2 = -1$. Then we will have the solution $x = \pm i$

We can introduce *complex numbers* which are numbers in the form $z = x + iy$, where x is the real part of z , $\Re(z)$, and y is the imaginary part of z , $\Im(z)$. These numbers can also be expressed in vector notation along the complex plane.

1.1.1 Complex Arithmetic

Addition, subtraction, and multiplication work the same, just with the addition of the fact $i^2 = -1$. For division, we require what is called the conjugate.

The conjugate of a complex number is the same number, just with the sign of the imaginary component flipped.

$$\bar{z} = x - yi$$

where \bar{z} is the conjugate of z .

Similarly to vectors, we can also define the modulus (length) of a complex number

$$|z|^2 = x^2 + y^2 = z \cdot \bar{z}$$

Using this, we can define the division of a complex number and also define the real and imaginary components of a complex number.

The general expression for division is:

$$\frac{u}{z} = \frac{s + it}{x + iy} = \frac{(s + it)(x - iy)}{(x + iy)(x - iy)} = \frac{u\bar{z}}{x^2 + y^2} = \frac{u\bar{z}}{|z|^2}$$

The real and imaginary components can be computed as

$$z = x + iy$$

$$\bar{z} = x - iy$$

$$\Re(z) = \frac{z + \bar{z}}{2}$$

$$\Im(z) = \frac{z - \bar{z}}{2i}$$

Ex1: Simplify $(1 + 2i)(3 + i)(2 - 3i)$

$$\begin{aligned}(1 + 2i)(3 + i)(2 - 3i) &= (3 + i + 6i - 2)(2 - 3i) = (1 + 7i)(2 - 3i) \\ &= 2 - 3i + 14i + 21 \\ &= 23 + 11i\end{aligned}$$

Ex2: Simplify $\left(\frac{2+i}{1+i}\right)^2$

$$\begin{aligned}\left(\frac{2+i}{1+i}\right)^2 &= \left(\frac{(2+i)(1-i)}{(1+i)(1-i)}\right)^2 = \left(\frac{2+i-2i+1}{2}\right)^2 = \left(\frac{3-i}{2}\right)^2 = \frac{9-6i-1}{4} \\ &= 2 - \frac{3}{2}i\end{aligned}$$

Ex3: Simplify $(1 + 2i)^5$

$$\begin{aligned}(1 + 2i)^5 &= 1^5 + 5(1)^4(2i) + 10(1)^3(2i)^2 + 10(1)^2(2i)^3 + 5(1)(2i)^4 + (2i)^5 \\ &= 1 + 10i + 10(-4) + 10(-8i) + 5(16) + 32i \\ &= 1 + 10i - 40 - 80i + 80 + 32i \\ &= 41 - 38i\end{aligned}$$

Ex4: Prove

$$\text{if } |z| = 1 \text{ then } \Re\left(\frac{1}{1+z}\right) = \frac{1}{2}$$

Proof.

$$\begin{aligned}\Re(w) &= \frac{w + \bar{w}}{2} \\ \frac{1}{1+z} &= \frac{1}{1+\bar{z}}\end{aligned}$$

$$\Re\left(\frac{1}{1+z}\right) = \frac{\frac{1}{1+z} + \frac{1}{1+\bar{z}}}{2} = \frac{1+\bar{z}+1+z}{2(1+z)(1+\bar{z})} = \frac{2+z+\bar{z}}{2(1+z+\bar{z}+|z|^2)}$$

$$|z|=1 \Rightarrow \Re\left(\frac{1}{1+z}\right) = \frac{2+z+\bar{z}}{2(2+z+\bar{z})} = \frac{1}{2}$$

□

Some properties of \bar{z} :

- $\overline{\bar{z}} = z$
- $\overline{\bar{z}_1 z_2} = \bar{z}_1 \bar{z}_2$
- $|z_1 z_2| = |z_1| |z_2|$

Some common inequalities come from the triangle inequality:

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

Proof.

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

$$|z_1 + z_2| = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$$

$$\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

□

$$\Rightarrow |z_1 \pm z_2| \geq ||z_1| - |z_2||$$

This leads to a general upper and lower bound that can be derived from these inequalities:

$$||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$$

Another way to represent complex numbers is through polar form. To use this we must first introduce Euler's identity:

$$e^{i\varphi} = \cos \varphi + i \sin \varphi$$

This then helps us with the equation for the polar form

$$z = a + ib = |z|(\cos \varphi + i \sin \varphi) = |z|e^{i\varphi}$$

For $z = re^{i\varphi}$ We call r the magnitude of the complex number and φ is the argument. This can be especially useful for simplifying some complex numbers.

Ex:

$$\left| \frac{(1 + \sqrt{3}i)^{100}}{(\sqrt{3} - i)^{100}} \right| = \frac{|1 + \sqrt{3}i|^{100}}{|\sqrt{3} - i|^{100}} = \frac{2^{100}}{2^{100}} = 1$$

Note that because sinusoidal functions are periodic every 2π this means that there infinite ways to express a function in polar coordinates.

$$e^{i2\pi k} = 1, \quad k \in \mathbb{Z} \Rightarrow z = re^{i(\varphi+2\pi k)}$$

To get around the issue of having infinite possible polar forms for every complex number we define what's called the principal argument to be

$$\text{Arg}(z) = \varphi \in (-\pi, \pi]$$

We define the regular argument to be

$$\arg(z) = \text{Arg}(z) + 2k\pi, \quad k \in \mathbb{Z}$$

Note that $\text{Arg}(z)$ is singular valued while $\arg(z)$ is multi-valued.

We can define $\text{Arg}(z)$ in terms of the real and imaginary parts (x and y) as

$$\text{Arg}(z) = \arctan\left(\frac{y}{x}\right) \pm k\pi$$

What it is specifically depends on what quadrant of the complex plane the point lies in.

- QI: $\text{Arg}(z) = \arctan\left(\frac{y}{x}\right)$
- QII: $\text{Arg}(z) = \arctan\left(\frac{y}{x}\right) + \pi$
- QIII: $\text{Arg}(z) = \arctan\left(\frac{y}{x}\right) - \pi$
- QIV: $\text{Arg}(z) = \arctan\left(\frac{y}{x}\right)$

Ex:

$$\text{Arg}(-1 - \sqrt{3}i) = \arctan(\sqrt{3}) - \pi = -\frac{2\pi}{3}$$

Ex2:

$$\begin{aligned} \arg(1 - \sqrt{3}i) \\ \text{Arg}(1 - \sqrt{3}i) &= -\frac{\pi}{3} \\ \arg(1 - \sqrt{3}i) &= -\frac{\pi}{3} + 2k\pi, \quad k \in \mathbb{Z} \end{aligned}$$

Ex3:

$$\begin{aligned} \arg(-1 + 2i) \\ \text{Arg}(-1 + 2i) &= \arctan(-2) + \pi \\ \arg(-1 + 2i) &= \pi - \arctan(2) + 2k\pi \end{aligned}$$

Ex4: Simplify

$$z = -3 + 3i$$

$$\text{Arg}(z) = \arctan\left(\frac{3}{-3}\right) + \pi = \frac{3\pi}{4}$$

$$|z| = 3\sqrt{2}$$

$$z = 3\sqrt{2}e^{\frac{3\pi}{4}i}$$

Ex5: Simplify

$$z = -3 - 3i$$

$$\text{Arg}(z) = \arctan\left(\frac{-3}{-3}\right) - \pi = -\frac{3\pi}{4}$$

$$|z| = 3\sqrt{2}$$

$$z = 3\sqrt{2}e^{-\frac{3\pi}{4}i}$$

Ex6: Simplify

$$z = \frac{1-i}{-\sqrt{3}+i} = \frac{u}{v}$$

$$\arg(z) = \arg(u) - \arg(v) = \arctan\left(\frac{-1}{1}\right) - \left(\arctan\left(\frac{1}{-\sqrt{3}}\right) + \pi\right) + 2k\pi$$

$$= -\frac{\pi}{4} - \frac{5\pi}{6} + 2k\pi = -\frac{13\pi}{12} + 2k\pi$$

$$\text{Arg}(z) = \frac{11\pi}{12}$$

$$|z| = \frac{|u|}{|v|} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$z = \frac{e^{\frac{11\pi}{12}i}}{\sqrt{2}}$$

Ex7: Simplify

$$z = (\sqrt{3} - i)^2 = w^2$$

$$\arg(w) = \arctan\left(\frac{-1}{\sqrt{3}}\right) + 2k\pi = -\frac{\pi}{6}$$

$$\Rightarrow \arg(z) = 2\arg(w) = -\frac{\pi}{3} + 4k\pi$$

$$\text{Arg}(z) = -\frac{\pi}{3}$$

$$|w| = 2 \Rightarrow |z| = |w|^2 = 4$$

$$z = 4e^{-\frac{\pi}{3}i}$$

Properties of $\text{Arg}(z)$ and $\arg(z)$

- $\text{Arg}(z_1 z_2) \neq \text{Arg}(z_1) + \text{Arg}(z_2)$

Proof. Proof by contradiction: assume that $\text{Arg}(z_1 z_2) = \text{Arg}(z_1) + \text{Arg}(z_2)$ is true.

Take $z_1 = z_2 = -1$.

$$\text{Arg}(z_1) = \text{Arg}(z_2) = \pi$$

$$\begin{aligned}\Rightarrow \text{Arg}(z_1) + \text{Arg}(z_2) &= 2\pi \\ \text{Arg}(z_1 z_2) &= \text{Arg}(1) = 0 \\ \text{Arg}(z_1 z_2) &\neq \text{Arg}(z_1) + \text{Arg}(z_2) \quad \forall z_1, z_2 \neq 0 \in \mathbb{C}\end{aligned}$$

□

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\text{Arg}(\bar{z}) \neq -\text{Arg}(z)$

Proof. Proof by contradiction: assume that $\text{Arg}(\bar{z}) = -\text{Arg}(z)$ is true.
Take $z = -1$

$$\begin{aligned}\bar{z} &= z = -1 \\ \text{Arg}(z) &= \pi \\ \text{Arg}(\bar{z}) &= \pi \\ -\text{Arg}(z) &= -\pi \\ \Rightarrow \text{Arg}(\bar{z}) &\neq -\text{Arg}(z) \quad \forall z \in \mathbb{C}\end{aligned}$$

□

- $\arg(z) = -\arg(\bar{z})$

Proof.

$$\begin{aligned}z &= |z|e^{i\arg(z)} \quad \forall z \in \mathbb{C} \\ \bar{z} &= |z|e^{-i\arg(z)} \\ \Rightarrow \arg(\bar{z}) &= -\arg(z)\end{aligned}$$

□

1.1.2 De Moirre's Formula

Using Euler's identity we can derive a powerful formula called De Moirre's Formula as follows:

$$\begin{aligned}e^{iN\varphi} &= \cos(N\varphi) + i\sin(N\varphi) \\ e^{iN\varphi} &= (e^{i\varphi})^N = (\cos \varphi + i\sin \varphi)^N \\ (\cos \varphi + i\sin \varphi)^N &= \cos(N\varphi) + i\sin(N\varphi)\end{aligned}$$

Applications of De Moirre's Formula:

Binomial expansion:

$$\begin{aligned}N &= 2 : \\ \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta \\ \sin(2\theta) &= 2\cos \theta \sin \theta\end{aligned}$$

$N = 3 :$

$$(\cos \theta + i \sin \theta)^3 = \cos^3 \theta + 3 \cos^2 \theta (i \sin \theta) + 3 \cos \theta (i \sin \theta)^2 + (i \sin \theta)^3$$

$$\cos(3\theta) = \cos^3 \theta - 3 \cos \theta \sin^2 \theta$$

$$\sin(3\theta) = 3 \cos^2 \theta \sin \theta - \sin^3 \theta$$

Ex: Prove

$$\sin(3\theta) = 3 \sin \theta - 4 \sin^3 \theta$$

Proof. Using De Moivre's formula with $N = 3$

$$(\cos \theta + i \sin \theta)^3 = \cos(3\theta) + i \sin(3\theta)$$

$$(\cos \theta)^3 + 3(\cos \theta)^2(i \sin \theta) + 3 \cos \theta (i \sin \theta)^2 + (i \sin \theta)^3 = \cos(3\theta) + i \sin(3\theta)$$

$$\cos^3 \theta - 3 \cos \theta \sin^2 \theta + i(3 \cos^2 \theta \sin \theta - \sin^3 \theta) = \cos(3\theta) + i \sin(3\theta)$$

$$\Im \{ \cos^3 \theta - 3 \cos \theta \sin^2 \theta + i(3 \cos^2 \theta \sin \theta - \sin^3 \theta) \} = \Im \{ \cos(3\theta) + i \sin(3\theta) \}$$

$$3 \cos^2 \theta \sin \theta - \sin^3 \theta = \sin(3\theta)$$

$$\sin^2 \theta + \cos^2 \theta = 1 \Rightarrow \cos^2 \theta = 1 - \sin^2 \theta$$

$$3(1 - \sin^2 \theta) \sin \theta - \sin^3 \theta = \sin(3\theta)$$

$$3 \sin \theta - 3 \sin^3 \theta - \sin^3 \theta = \sin(3\theta)$$

$$3 \sin \theta - 4 \sin^3 \theta = \sin(3\theta)$$

□

Computing trigonometric integrals:

Ex:

$$\int_0^{2\pi} \cos^8 \varphi d\varphi$$

$$e^{i\varphi} = \cos \varphi + i \sin \varphi$$

$$e^{-i\varphi} = \cos \varphi - i \sin \varphi$$

$$\cos \varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$

$$\sin \varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$

$$\int_0^{2\pi} \left(\frac{e^{i\varphi} + e^{-i\varphi}}{2} \right)^8 d\varphi = \frac{1}{2^8} \int_0^{2\pi} (e^{i\varphi} + e^{-i\varphi})^8 d\varphi$$

$$= \frac{1}{2^8} \int_0^{2\pi} (e^{i8\varphi} + {}_1C_8 e^{i7\varphi} e^{-i\varphi} + \dots + {}_7C_8 e^{i\varphi} e^{-i7\varphi} + e^{-i8\varphi}) d\varphi$$

$$= \frac{1}{2^8} (0 + \dots + {}_4C_8 2\pi + \dots + 0) = \frac{{}_4C_8}{2^7} \pi$$

Ex2:

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\begin{aligned}
\int_0^{2\pi} \cos^6 \theta d\theta &= \int_0^{2\pi} \left(\frac{e^{i\theta} + e^{-i\theta}}{2} \right) \\
&= \frac{1}{2^6} \int_0^{2\pi} \sum_{k=0}^6 \binom{6}{k} e^{i\theta k} e^{-i\theta(6-k)} d\theta \\
&= \frac{1}{2^6} \sum_{k=0}^6 \binom{6}{k} \int_0^{2\pi} e^{i\theta(2k-6)} d\theta \\
\int_0^{2\pi} e^{ik\theta} d\theta &= \frac{e^{ik\theta}}{ik} \Big|_0^{2\pi} = \frac{e^{2\pi ik} - 1}{ik} \\
e^{2\pi ik} &= 1, \quad k \in \mathbb{Z} \Rightarrow \int_0^{2\pi} e^{ik\theta} d\theta = 0, \quad k \neq 0 \in \mathbb{Z} \\
\Rightarrow \int_0^{2\pi} \cos^6 \theta d\theta &= \frac{1}{2^6} \binom{6}{3} \int_0^{2\pi} d\theta = \frac{(20)(2\pi)}{2^6} = \frac{5\pi}{8}
\end{aligned}$$

Ex3:

$$\begin{aligned}
&\int_0^{2\pi} \sin^6(2\theta) d\theta \\
\sin(2\theta) &= \frac{e^{2i\theta} - e^{-2i\theta}}{2i} \\
\int_0^{2\pi} \sin^6(2\theta) d\theta &= \int_0^{2\pi} \left(\frac{e^{2i\theta} - e^{-2i\theta}}{2i} \right)^6 d\theta \\
&= \frac{1}{(2i)^6} \int_0^{2\pi} \sum_{k=0}^6 \binom{6}{k} (-1)^{6-k} e^{2ik\theta} e^{-2i\theta(6-k)} d\theta \\
&= -\frac{1}{2^6} \sum_{k=0}^6 \binom{6}{k} (-1)^{6-k} \int_0^{2\pi} e^{i\theta(4k-12)} d\theta \\
\int_0^{2\pi} e^{ik\theta} d\theta &= \frac{e^{ik\theta}}{ik} \Big|_0^{2\pi} = \frac{e^{2\pi ik} - 1}{ik} \\
e^{2\pi ik} &= 1, \quad k \in \mathbb{Z} \Rightarrow \int_0^{2\pi} e^{ik\theta} d\theta = 0, \quad k \neq 0 \in \mathbb{Z} \\
\Rightarrow \int_0^{2\pi} \sin^6(2\theta) d\theta &= -\frac{1}{2^6} \binom{6}{3} (-1)^3 \int_0^{2\pi} d\theta \\
&= \frac{5\pi}{8}
\end{aligned}$$

Ex4: Prove

$$\sum_{k=0}^n \cos(k\theta) = \frac{1}{2} + \frac{\sin\left(\left(n + \frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}$$

Proof. De Moivre's formula states

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$$

If we take the conjugate of both sides we get

$$(\cos \theta - i \sin \theta)^n = \cos(n\theta) - i \sin(n\theta)$$

Summing these two equations gives

$$\begin{aligned} (\cos \theta + i \sin \theta)^k + (\cos \theta - i \sin \theta)^k &= 2 \cos(k\theta) \\ \cos \theta \pm i \sin \theta &= e^{\pm i\theta} \\ 2 \cos(k\theta) &= (e^{i\theta})^k + (e^{-i\theta})^k \end{aligned}$$

We can sum both sides of this to get

$$2 \sum_{k=0}^n \cos(k\theta) = \sum_{k=0}^n (e^{i\theta})^k + \sum_{k=0}^n (e^{-i\theta})^k$$

The formula for the geometric sum is

$$\sum_{k=0}^n z^k = \frac{1 - z^{n+1}}{1 - z}$$

Applying this we get

$$\begin{aligned} 2 \sum_{k=0}^n \cos(k\theta) &= \frac{1 - (e^{i\theta})^{n+1}}{1 - e^{i\theta}} + \frac{1 - (e^{-i\theta})^{n+1}}{1 - e^{-i\theta}} \\ 2 \sum_{k=0}^n \cos(k\theta) &= \frac{(1 - e^{i\theta(n+1)})(1 - e^{-i\theta}) + (1 - e^{-i\theta(n+1)})(1 - e^{i\theta})}{(1 - e^{i\theta})(1 - e^{-i\theta})} \\ 2 \sum_{k=0}^n \cos(k\theta) &= \frac{1 - e^{i\theta(n+1)} - e^{i\theta} + e^{i\theta n} + 1 - e^{-i\theta(n+1)} - e^{-i\theta} + e^{-i\theta n}}{1 - e^{i\theta} - e^{-i\theta} + 1} \\ 2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{e^{i\theta n} + e^{-i\theta n} - e^{i\theta(n+1)} - e^{-i\theta(n+1)}}{2 - e^{i\theta} - e^{-i\theta}} \\ 2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{e^{i\frac{\theta}{2}} \left(e^{-i\theta(n+\frac{1}{2})} - e^{i\theta(n+\frac{1}{2})} \right) + e^{-i\frac{\theta}{2}} \left(e^{i\theta(n+\frac{1}{2})} - e^{-i\theta(n+\frac{1}{2})} \right)}{e^{i\frac{\theta}{2}} \left(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}} \right) + e^{-i\frac{\theta}{2}} \left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}} \right)} \\ 2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{\left(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}} \right) \left(e^{i\theta(n+\frac{1}{2})} - e^{-i\theta(n+\frac{1}{2})} \right)}{- \left(e^{-i\frac{\theta}{2}} - e^{i\frac{\theta}{2}} \right)^2} \\ 2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{e^{i\theta(n+\frac{1}{2})} - e^{-i\theta(n+\frac{1}{2})}}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}} \\ 2 \sum_{k=0}^n \cos(k\theta) &= 1 + \frac{\frac{1}{2i} \left(e^{i\theta(n+\frac{1}{2})} - e^{-i\theta(n+\frac{1}{2})} \right)}{\frac{1}{2i} \left(e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}} \right)} \\ \sin(x) &= \frac{e^{ix} - e^{-ix}}{2i} \end{aligned}$$

$$2 \sum_{k=0}^n \cos(k\theta) = 1 + \frac{\sin(\theta(n + \frac{1}{2}))}{\sin(\frac{\theta}{2})}$$

$$\sum_{k=0}^n \cos(k\theta) = \frac{1}{2} + \frac{\sin((n + \frac{1}{2})\theta)}{2 \sin(\frac{\theta}{2})}$$

□

1.1.3 Geometry in the Complex Plane

Using the notation such that $z = x + iy$ where $\Re(z) = x$ and $\Im(z) = y$ we can define a circle in the complex plane as

$$(x - x_0)^2 + (y - y_0)^2 = r_0^2$$

This is analogous to writing

$$|z - z_0| = r_0$$

The two can be related as follows:

$$|z - z_0| = r_0$$

$$|x + iy - x_0 - iy_0| = r_0$$

$$\sqrt{(x - x_0)^2 + (y - y_0)^2} = r_0$$

$$(x - x_0)^2 + (y - y_0)^2 = r_0^2$$

Ex: describe the circle formed by $2|z| = |z + 1|$

$$2|z| = |z + 1|$$

$$4|z|^2 = |z + 1|^2$$

$$4x^2 + 4y^2 = (x + 1)^2 + y^2$$

$$4x^2 + 4y^2 = x^2 + 2x + 1 + y^2$$

$$3x^2 + 3y^2 - 2x - 1 = 0$$

$$3x^2 - 2x + \frac{1}{3} + 3y^2 - \frac{4}{3} = 0$$

$$3\left(x - \frac{1}{3}\right)^2 + 3y^2 = \frac{4}{3}$$

$$\left(x - \frac{1}{3}\right)^2 + y^2 = \frac{4}{9}$$

A line in the complex plane can be written as

$$ax + by = c \longleftrightarrow a \frac{z + \bar{z}}{2} + b \frac{z - \bar{z}}{2i} = c$$

Ex: describe the line formed by $|z - 1 + i| = |z - 2i|$

$$|z - 1 + i| = |z - 2i|$$

$$\begin{aligned}
|z - 1 + i|^2 &= |z - 2i|^2 \\
(x - 1)^2 + (y + 1)^2 &= x^2 + (y - 2)^2 \\
x^2 - 2x + 1 + y^2 + 2y + 1 &= x^2 + y^2 - 4y + 4 \\
-2x + 2 + 2y &= -4y + 4 \\
6y &= 2x + 2 \\
y &= \frac{1}{3}(x + 1)
\end{aligned}$$

We can define an ellipse in the complex plane as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If we have $a > b$ then we will have a horizontal ellipse and if $b > a$ then we will have a vertical ellipse.

Assuming that $a > b$ then we can define the foci points to be at

$$\begin{aligned}
+F &= (\sqrt{a^2 - b^2}, 0) \\
-F &= (-\sqrt{a^2 - b^2}, 0)
\end{aligned}$$

The equation of an ellipse can also be described by

$$|z - F| + |z + F| = 2a$$

Ex: describe the ellipse formed by $|z - 1| + |z + 1| = 4$

$$\begin{aligned}
|z - 1| + |z + 1| &= 4 \\
|z - F| + |z + F| &= 2a \Rightarrow 2a = 4 \Rightarrow a = 2 \\
F = \sqrt{a^2 - b^2} &= 1 \Rightarrow 1 = 4 - b^2 \Rightarrow b^2 = 3 \\
\frac{x^2}{4} + \frac{y^2}{3} &= 1
\end{aligned}$$

Ex2: describe the ellipse formed by $|z - 1| + |z + 3| = 6$

$$\begin{aligned}
|z - 1| + |z + 3| &= 6 \\
\text{note that this ellipse is not centered at the origin so we need to shift it} \\
|z + 1 - 2| + |z + 1 + 2| &= 6 \\
2a = 6 &\Rightarrow a = 3 \\
F = 2 = \sqrt{a^2 - b^2} &\Rightarrow 4 = 9 - b^2 \Rightarrow b^2 = 5 \\
\frac{(x + 1)^2}{9} + \frac{y^2}{5} &= 1
\end{aligned}$$

1.1.4 Roots of a Complex Number

Given $z_0 = r_0 e^{i\varphi_0}$, what is $z_0^{\frac{1}{n}}$?

If we let $w = z_0^{\frac{1}{n}}$ then $w^n = z_0$

$$w = r e^{i\varphi}, \quad w^n = r^n e^{in\varphi}$$

$$\begin{aligned}
w^n &= z_0 \Rightarrow r^n e^{in\varphi} = r_0 e^{i\varphi_0} \\
&\Rightarrow r^n = r_0 \Rightarrow r_0^{\frac{1}{n}} \\
e^{in\varphi} &= e^{i\varphi_0} \Rightarrow n\varphi = \varphi_0 + 2k\pi \\
\varphi &= \frac{\varphi_0}{n} + \frac{2k\pi}{n}
\end{aligned}$$

So all solutions to $w^n = z_0$ are given by

$$w = r_0^{\frac{1}{n}} e^{i(\frac{\varphi_0}{n} + \frac{2k\pi}{n})}, \quad k \in \mathbb{Z}$$

If we normalize $\varphi_0 = \text{Arg}(z_0)$ then k will be in the range $k \in \{0, 1, \dots, n-1\}$.
Note that the expression

$$w = r_0^{\frac{1}{n}} e^{i(\frac{\varphi_0}{n} + \frac{2k\pi}{n})}, \quad k \in \mathbb{Z}$$

is multi-valued. If we want to avoid this, we can take what's called the principal value which is the value when $k = 0$:

$$z_0^{\frac{1}{n}} = r_0^{\frac{1}{n}} e^{i\frac{\varphi_0}{n}}$$

Ex: Compute $(-1)^{\frac{1}{2}}$

$$z_0 = -1 = 1^{\frac{1}{2}} e^{i(\frac{\pi}{2} + \frac{2k\pi}{2})} = e^{i(\frac{\pi}{2} + k\pi)} = \left\{ \dots, e^{-i\frac{3\pi}{2}}, e^{-i\frac{\pi}{2}}, e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{2}}, e^{i\frac{5\pi}{2}}, \dots \right\}$$

note that there are only 2 unique values

$$(-1)^{\frac{1}{2}} = e^{i(\frac{\pi}{2} + k\pi)}, \quad k \in \{0, 1\}$$

The principal value (when $k = 0$) of this equation works out to be i .

Ex2: Find all solutions to

$$\begin{aligned}
z^7 &= i - 1 \\
z^7 &= \sqrt{2} e^{i(\frac{3\pi}{4} + 2\pi k)} \\
z &= 2^{1/14} e^{i(\frac{3\pi}{28} + \frac{2\pi k}{7})}, \quad k \in \{0, 1, 2, 3, 4, 5, 6\}
\end{aligned}$$

Ex3: Find all solutions to

$$\begin{aligned}
z^5 &= \frac{2i}{-1 - \sqrt{3}i} \\
z^5 &= \frac{2e^{i\frac{\pi}{2}}}{2e^{-i\frac{2\pi}{3}}} = e^{i\frac{7\pi}{6}} = e^{-i(\frac{5\pi}{6} + 2\pi k)} \\
z &= e^{-i(\frac{\pi}{6} + \frac{2\pi k}{5})}, \quad k \in \{0, 1, 2, 3, 4\}
\end{aligned}$$

Ex4: Find all solutions to

$$\left(\frac{z}{z+1} \right)^2 = i$$

$$\begin{aligned} \left(\frac{z}{z+1}\right)^2 &= e^{i(\frac{\pi}{2}+2\pi k)} \\ \frac{z}{z+1} &= e^{i(\frac{\pi}{4}+\pi k)}, \quad k \in \{0, 1\} \\ e^{i(\frac{\pi}{4}+\pi k)} &= \left\{ \frac{1}{\sqrt{2}}(1+i), \frac{1}{\sqrt{2}}(-1-i) \right\} \\ z &= e^{i(\frac{\pi}{4}+k\pi)}(z+1) \\ z &= \frac{e^{i(\frac{\pi}{4}+k\pi)}}{1 - e^{i(\frac{\pi}{4}+k\pi)}} = \left\{ \frac{1+i}{\sqrt{2}-1-i}, \frac{-1-i}{\sqrt{2}+1+i} \right\} \end{aligned}$$

Ex5: Find all solutions to

$$\begin{aligned} z^2 + 4iz + 1 &= 0 \\ (z^2 + 4iz - 4) + 4 + 1 &= 0 \\ (z + 2i)^2 + 5 &= 0 \\ (z + 2i)^2 &= -5 = 5e^{i(\pi+2\pi k)} \\ z + 2i &= \sqrt{5}e^{i(\frac{\pi}{2}+\pi k)} = \left\{ \sqrt{5}i, -\sqrt{5}i \right\} \\ z &= \left\{ (\sqrt{5}-2)i, -(\sqrt{5}+2)i \right\} \end{aligned}$$

1.2 Complex Functions

1.2.1 Mapping Properties of Simple Functions

Similar to how functions with real variables map values to a different set of values, complex functions do the same. The main difference is that with complex functions we're mapping a 2 dimensional set of inputs to a 2 dimensional set of outputs.

$$w = f(z) = u + iv$$

We define $z \in \mathcal{S}$ the image of \mathcal{S} under w .

Some common mappings:

- The identity map

$$\begin{aligned} w = f(z) &= z \\ \begin{cases} u = x \\ v = y \end{cases} \end{aligned}$$

- Translation by z_0

$$\begin{aligned} w = f(z) &= z + z_0 \\ \begin{cases} u = x + x_0 \\ v = y + y_0 \end{cases} \end{aligned}$$

- Stretching ($a > 1$) or contraction ($a < 1$)

$$w = f(z) = az = are^{i\varphi}, \quad a \in \mathbb{R}$$

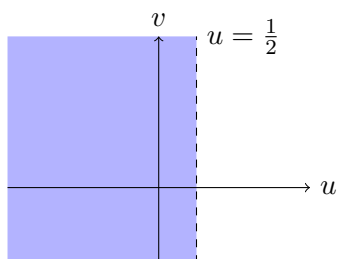
$$\begin{cases} u = ax \\ v = ay \end{cases}$$

- Rotation by φ_0

$$w = f(z) = e^{i\varphi_0} z = e^{i(\varphi+\varphi_0)}$$

Using these basic mapping principles we are able to lay the foundation for some more complicated mappings.

Ex: Find the image of $S = \{|z - 1| \geq 1\}$ under the mapping $f(z) = \frac{1}{z}$



$$z = \frac{1}{w} = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$$x = \frac{u}{u^2 + v^2}$$

$$y = -\frac{v}{u^2 + v^2}$$

$$|z - 1| \geq 1 \Rightarrow \left| \frac{1}{w} - 1 \right| \geq 1$$

$$\left| \frac{1 - w}{w} \right| \geq 1$$

$$|1 - w| \geq |w|$$

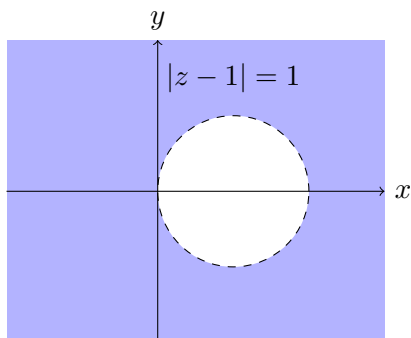
$$|1 - w|^2 \geq |w|^2$$

$$(1 - u)^2 + v^2 \geq u^2 + v^2$$

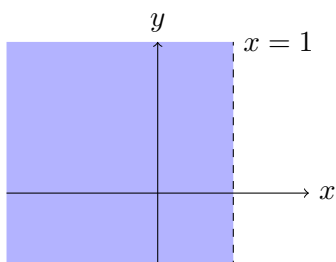
$$-2u + 1 \geq 0$$

$$u \leq \frac{1}{2}$$

$$S' = \left\{ u \leq \frac{1}{2} \right\}$$



Ex2: Find the image of $S = \{x \leq 1\}$ under the mapping $f(z) = \frac{1}{z}$

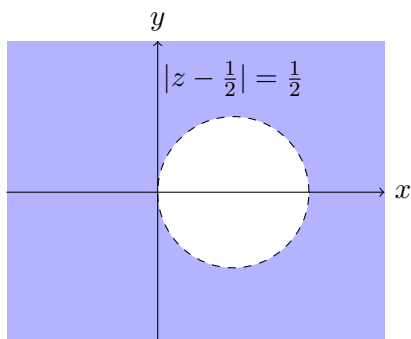


$$z = \frac{1}{w} = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

$$x = \frac{u}{u^2 + v^2}$$

$$x \leq 1 \Rightarrow \frac{u}{u^2 + v^2} \leq 1 \Rightarrow u^2 + v^2 \geq u$$

$$(u - \frac{1}{2})^2 + v^2 \geq \frac{1}{4}$$



We see from the previous two examples that circles map to lines and lines map to circles. Let's see why this is the case.

$$a(x^2 + y^2) + bx + cy + d = 0$$

$$a|z|^2 + b\frac{z + \bar{z}}{2} + c\frac{z - \bar{z}}{2i} + d = 0$$

In the case where $a = 0$, we have a line. In the case where $a \neq 0$, we have a circle.

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$

$$z\bar{z} = |z|^2 = \frac{1}{|w|^2} = \frac{1}{w\bar{w}}$$

$$\begin{aligned}
a \frac{1}{w\bar{w}} + b \frac{z + \bar{z}}{2} + c \frac{z - \bar{z}}{2i} + d &= 0 \\
a \frac{1}{w\bar{w}} + \frac{b}{2} \left(\frac{1}{w} + \frac{1}{\bar{w}} \right) + \frac{c}{2i} \left(\frac{1}{w} - \frac{1}{\bar{w}} \right) + d &= 0 \\
a + \frac{b}{2} (w + \bar{w}) + \frac{c}{2i} (w - \bar{w}) + d (w\bar{w}) &= 0
\end{aligned}$$

If we have a linear transformation of the form $az + b$ it corresponds to the scaling and translation of the set only. A line will map to a line and a circle will map to a circle.

We can combine this with the $w = \frac{1}{z}$ transformation property to get a more general transformation. We call this the *Mobius transformation*:

$$f(z) = \frac{az + b}{cz + d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz + d}$$

where $a, b, c, d \in \mathbb{C}$

Ex: Find the mapping of $f(z) = \frac{1}{z+1}$ on the set $S = \{\Re(z) > 0\}$

$$\begin{aligned}
u + iv &= \frac{1}{x + 1 + iy} \Rightarrow x + 1 + iy = \frac{1}{u + iv} \\
x + 1 &= \frac{u}{u^2 + v^2} \\
x > 0 &\Rightarrow x + 1 > 1 \\
\frac{u}{u^2 + v^2} &> 1 \Rightarrow u > u^2 + v^2 \\
u^2 + v^2 - u + \frac{1}{4} &< \frac{1}{4} \\
\left(u - \frac{1}{2}\right)^2 + v^2 &< \frac{1}{4} \\
S' &= \left\{ w = u + iv \mid \left(u - \frac{1}{2}\right)^2 + v^2 < \left(\frac{1}{2}\right)^2 \right\}
\end{aligned}$$

Ex2: Find the mapping of $f(z) = \frac{z-i}{z+i}$ on $S = \{|z| < 3\}$

$$\begin{aligned}
wz + iw &= z - i \Rightarrow z(w - 1) = -i - iw \Rightarrow z = \frac{i(w + 1)}{1 - w} \\
|z| &= \frac{|w + 1|}{|w - 1|} < 3 \\
|w + 1| &< 3|w - 1| \Rightarrow |w + 1|^2 < 9|w - 1|^2 \\
(u + 1)^2 + v^2 &< 9(u - 1)^2 + 9v^2 \\
u^2 + 2u + 1 + v^2 &< 9u^2 - 18u + 9 + 9v^2 \\
0 &< 8u^2 - 20u + 8 + 8v^2 \Rightarrow 0 < u^2 - \frac{5}{2}u + 1 + v^2 \\
\frac{9}{16} &< u^2 - \frac{5}{2}u + \frac{25}{16} + v^2
\end{aligned}$$

$$\frac{9}{16} < \left(u - \frac{5}{4}\right)^2 + v^2$$

$$S' = \left\{ w = u + iv \mid \left(u - \frac{5}{4}\right)^2 + v^2 > \left(\frac{3}{4}\right)^2 \right\}$$

Another common mapping is the $f(z) = z^2$ or more generally $f(z) = z^n$ mapping. For $w = z^2$,

$$w = z^2 = r^2 e^{2i\varphi} \Rightarrow \begin{cases} |w| = |z|^2 \\ \arg(w) = 2 \arg(z) \end{cases}$$

This mapping scales the magnitude but more notably, it doubles the argument. This means that the mapping of a half circle will now be a full circle.

Ex: Find the mapping of $f(z) = z^2$ on $S = \{0 \leq \Re(z) \leq 1, \Im(z) = 1\}$

$$w = x^2 + i2xy - y^2$$

$$u = x^2 - y^2 = x^2 - 1 \Rightarrow -1 \leq u \leq 0$$

$$v = 2xy = 2x \Rightarrow 0 \leq v \leq 2$$

$$S' = \{w = u + iv \mid -1 \leq u \leq 0, 0 \leq v \leq 2\}$$

Ex2: Find the mapping of $f(z) = -2z^5$ on $S = \{|z| < 1, 0 < \text{Arg}(z) < \frac{\pi}{2}\}$

$$z^5 = -\frac{w}{2} \Rightarrow |z|^5 = \frac{|w|}{2} < 1 \Rightarrow |w| < 2$$

$$5 \arg(z) = \arg(w) \pm \pi$$

$$0 < \arg(w) \pm \pi < \frac{5\pi}{2}$$

$$-\pi < \arg(w) < \frac{3\pi}{2}$$

$$S' = \{|w| < 2\}$$

Another common mapping is the $f(z) = e^z$ mapping.

$$w = e^z = e^{x+iy} = e^x e^{iy}$$

$$\begin{cases} |w| = e^x \\ \arg(w) = y \end{cases}$$

This mapping has the property that the magnitude is only dependent on x and the argument is exactly y .

Ex: Find the mapping of $f(z) = e^z$ on $S = \{\Re(z) = 1\}$

$$w = e^x e^{iy}$$

$$|w| = e, \arg(w) = y$$

$$S' = \{|w| = e\}$$

Ex2: Find the mapping of $f(z) = e^z$ on $S = \{0 \leq \Im(z) \leq \frac{\pi}{4}\}$

$$|w| = x$$

$$\arg(w) = y \Rightarrow 0 \leq \arg(w) \leq \frac{\pi}{4}$$

$$S' = \left\{ 0 \leq \text{Arg}(w) \leq \frac{\pi}{4} \right\}$$

Ex3: Prove

$$|e^{-z^3}| \leq 1 \vee \left\{ z \mid -\frac{\pi}{6} \leq \text{Arg}(z) \leq \frac{\pi}{6} \right\}$$

Proof. We can express $-z^3$ as some complex number $a + ib$ where $a = \Re(-z^3)$ and $b = \Im(-z^3)$. Taking the magnitude gives

$$|e^{-z^3}| = |e^{a+ib}| = |e^a e^{ib}| = |e^a| |e^{ib}| = |e^a| = |e^{\Re(-z^3)}|$$

z can be written as

$$\begin{aligned} z &= |z|e^{i\text{Arg}(z)} \\ z^3 &= |z|^3 e^{i3\text{Arg}(z)} = |z|^3 (\cos(3\text{Arg}(z)) + i\sin(3\text{Arg}(z))) \\ -z^3 &= -|z|^3 (\cos(3\text{Arg}(z)) + i\sin(3\text{Arg}(z))) \\ \Re(-z^3) &= -|z|^3 \cos(3\text{Arg}(z)) \\ \text{Arg}(z) &\in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right] \\ \Rightarrow 3\text{Arg}(z) &\in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\ \cos(3\text{Arg}(z)) &\in [0, 1] \\ |z|^3 &\in \{x \in \mathbb{R} | x \geq 0\} \\ \Re(-z^3) &= -|z|^3 \cos(3\text{Arg}(z)) \in \{x \in \mathbb{R} | x \leq 0\} \\ e^{\Re(-z^3)} &\in [0, 1] \\ \Rightarrow |e^{-z^3}| &\leq 1 \end{aligned}$$

□

1.2.2 Calculus of Complex Functions

We define the limit of a complex function to be

$$w = f(z) = u + iv$$

$$\lim_{z \rightarrow z_0} f(z) = \lim_{(x,y) \rightarrow (x_0,y_0)} u(x,y) + i \lim_{(x,y) \rightarrow (x_0,y_0)} v(x,y)$$

Note that the notation $(x, y) \rightarrow (x_0, y_0)$ means that the limit is taken as (x, y) approaches (x_0, y_0) along *any* path.

The usual limit arithmetic rules are able to be applied as with real numbers.

In order for $\lim_{z \rightarrow z_0} f(z)$ to exist, we require that both $\lim_{(x,y) \rightarrow (x_0,y_0)} u(x,y)$ and $\lim_{(x,y) \rightarrow (x_0,y_0)} v(x,y)$ exist.

If we define $z_0 = x_0 + iy_0$ then we can define the derivative of a complex function as

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

If this limit exists then the function is said to be differentiable at z_0 .

Ex: $f(z) = z$

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{z_0 + \Delta z - z_0}{\Delta z} = 1$$

$$\Rightarrow f'(z_0) = 1$$

Ex2: $f(z) = \bar{z}$

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z}$$

$$\Delta z = h_1 + ih_2 \Rightarrow \overline{\Delta z} = h_1 - ih_2$$

$$h_2 = 0 : \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{h_1 \rightarrow 0} \frac{h_1}{h_1} = 1$$

$$h_1 = 0 : \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{h_2 \rightarrow 0} \frac{-ih_2}{ih_2} = -1$$

$$\lim_{h_1 \rightarrow 0} \frac{h_1}{h_1} \neq \lim_{h_2 \rightarrow 0} \frac{-ih_2}{ih_2} \therefore \text{the derivative does not exist}$$

An easy way to determine if a function is differentiable is to use the Cauchy-Riemann equations. Any path that can be taken to approach z_0 can be written as a linear combination of the paths $\Delta z = \Delta x$ and $\Delta z = i\Delta y$ so the derivative must satisfy both of these paths.

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

$$f(z_0) = u(x_0, y_0) + iv(x_0, y_0)$$

Define $\Delta z = \Delta x$

$$f'(z_0) = \lim_{\Delta x \rightarrow 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x}$$

$$f'(z_0) = \lim_{\Delta x \rightarrow 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \rightarrow 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0)$$

Define $\Delta z = i\Delta y$

$$f'(z_0) = \lim_{\Delta y \rightarrow 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y}$$

$$f'(z_0) = \lim_{\Delta y \rightarrow 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \rightarrow 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y}$$

$$f'(z_0) = -i \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$$

$$\Rightarrow \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0) = -i \frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$$

Splitting the real and imaginary parts we get that the Cauchy-Riemann equations are

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

If the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous then the function is differentiable.

Some functions are not differentiable everywhere, but are differentiable at a point or a set of points.

- If $f(z)$ is differentiable everywhere in the complex plane then it is said to be **entire**.
- If $f(z)$ is differentiable in some region R then it is said to be **analytic** in R .
(note that this region cannot be a single point, as the Cauchy-Riemann equations require the partial derivatives to be continuous)

Ex: Show using the Cauchy-Riemann equations that $f(z) = \bar{z}$ is not differentiable anywhere.

$$\bar{z} = x - iy$$

$$u_x = 1 \neq v_y = -1$$

Ex2: Show that $f(z) = z^2$ is entire

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$$

$$u(x, y) = x^2 - y^2$$

$$v(x, y) = 2xy$$

$$u_x = 2x = v_y$$

$$u_y = -2y = -v_x$$

Ex3: Show that $f(z) = \bar{z}$ is differentiable but not analytic at $z_0 = 0$

$$|z|^2 + 2z = x^2 + 2x + y^2 + i2y$$

$$u_x = 2x + 2 = v_y = 2 \Rightarrow x = 0$$

$$u_y = 2y = -v_x = 0 \Rightarrow y = 0$$

differentiable but not analytic on $z = \{0\}$

1.3 Conformal Mappings