Math Notes

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1 Multivariable Calculus

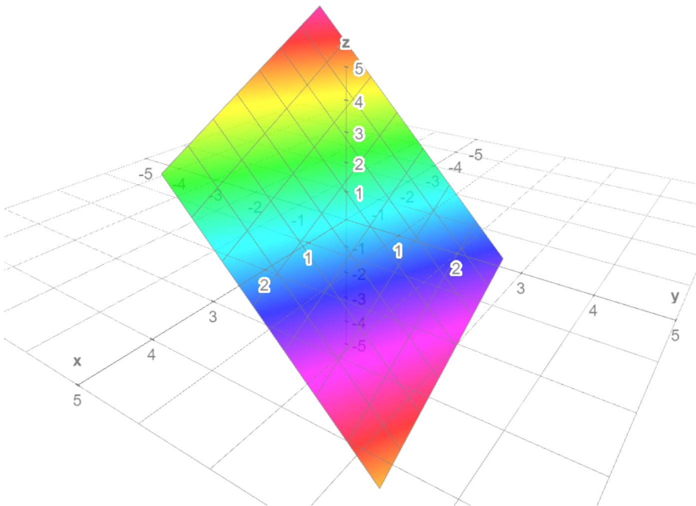
1.1 Functions of Multiple Variables

1.1.1 Classification of 3D Surfaces

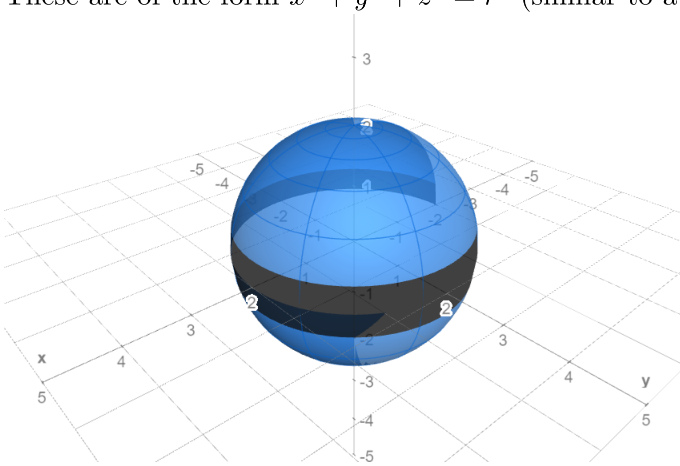
So far, we have only been working in the 2D plane with functions of 2 variables: $y = f(x)$. We can extend the same ideas to higher dimensions: A function $z = f(x, y)$ will form a surface in 3 dimensions and a function $w = f(x, y, z)$ will span 4 dimensions. While we are still able to analyze the behaviors of higher dimensional systems, we will often stick to functions of 3 variables for the sake of visualization.

As with 2D, there are a few common functions that you should recognize and be able to plot.

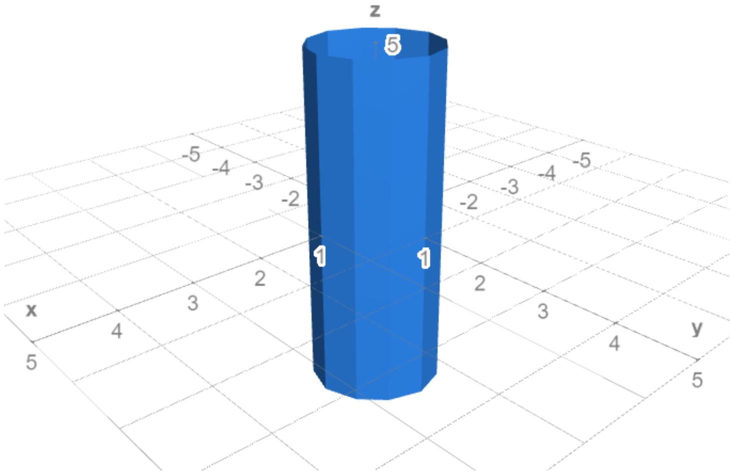
1. Planes
   These are of the form $z = ax + by + c$ (as seen in a previous section)



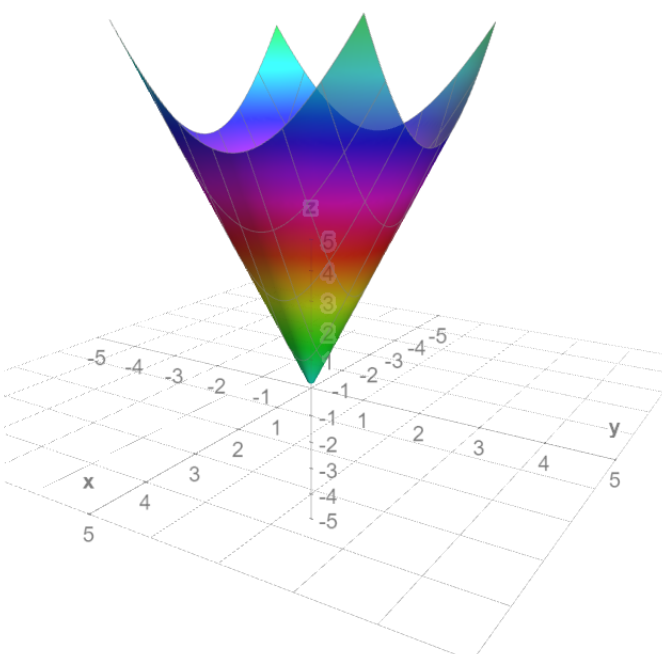
2. Spheres
   These are of the form $x^2 + y^2 + z^2 = r^2$ (similar to a circle in 2D)



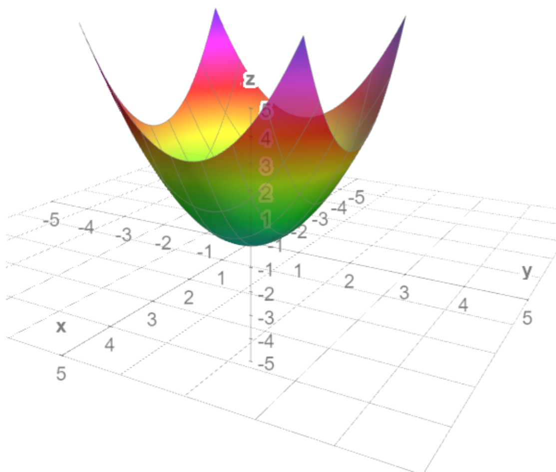
3. Cylinders
   These are of the form $x^2 + y^2 = 1$. Note that this is a function defined in terms of only x and y so it will be the same for all z.



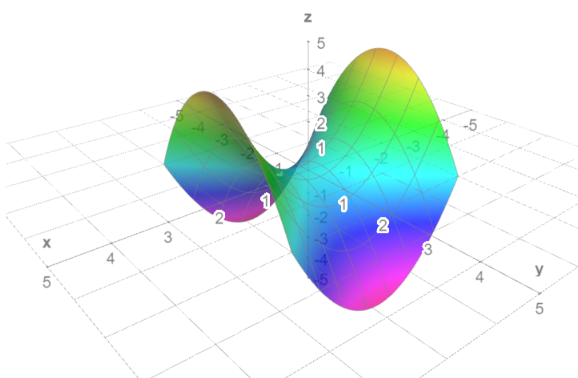
4. Cones
   These are of the form $z = \sqrt{x^2 + y^2}$ for a one-sided cone and $z^2 = x^2 + y^2$ for a two-sided cone.



5. Paraboloids
   These are of the form $z = x^2 + y^2$



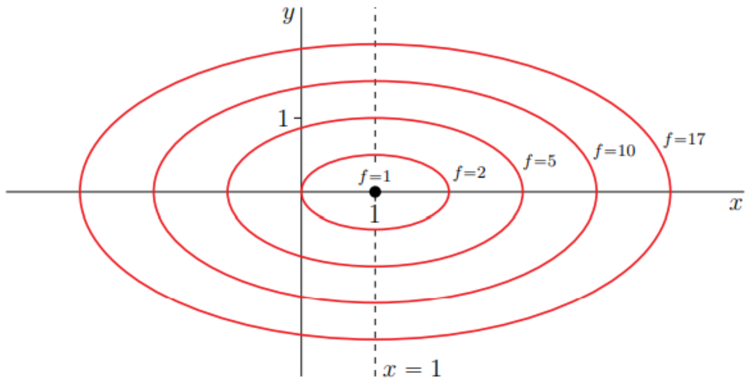
6. Saddles
   These are of the form $z = x^2 - y^2$



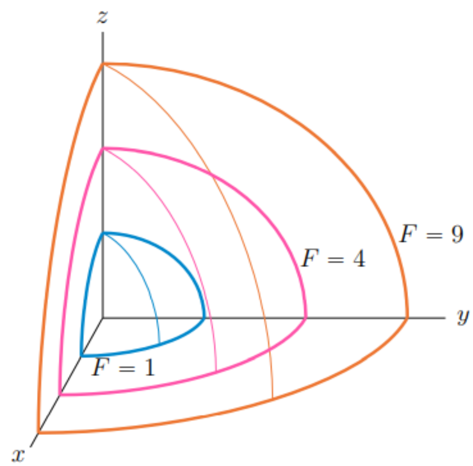
There are many more functions not mentioned such as ellipsoids, hyperboloids, and parabolic cylinders but this should be enough to begin to get an idea of what 3D functions look like.

1.1.2 Sketching Surfaces

Sketching surfaces in 3D is often trickier than in 2D but there are a few useful tricks to get an idea what the surface looks like.
One of the most useful ones is level curves. This is where if we have a function $z = f(x, y)$, we set z to be a constant and draw the curve on the xy-plane for varying z (in 2D these are called contour plots).
In short, a contour plot is an overlay of lines of $f(x, y) = k$ plotted on the xy-plane. The rate of greatest change (the gradient) is always perpundicular to the contour lines.
Similarly, a level surface is the overlay of surfaces of $f(x, y, z) = k$ plotted in 3-space (or higher). Ex: $z = x^2 + 4y^2 - 2x + 2$



This becomes increasingly useful for visualizing functions in 4 dimensions as we can define their level curves as 3D functions.
Ex: $F = x^2 + y^2 + z^2$



Sometimes it can be useful to sketch contour plots in multiple planes as it may be easier or give different perspectives.
Other useful tricks in sketching curves that will be seen later may be to take the gradient (as it gives a vector field normal to the level curves at all points) or to find the critical points of the function.

1.2 Partial Derivatives

1.2.1 Partial Derivatives and the Gradient

Notation: $\dfrac{\partial f}{\partial x} = f_x$
Definition:

$$\text{for } z = f(x,y)$$
$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$
$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Partial derivatives are effectively the same but where you hold the rest of the variables to be constants.

$$\text{Ex: } f(x,y) = x^3y + y^2$$
$$\frac{\partial f}{\partial x} = 3x^2y$$
$$\frac{\partial f}{\partial y} = x^3 + 2y$$

Note that the order of differentiation for higher order derivatives does not matter. $f_{xy} = f_{yx}$
Ex2: Find an expression for $\frac{\partial^2 z}{\partial x \partial y}$ for the function $2yz + x^2 + y^2 + z^4 = 18$

$$\frac{\partial}{\partial x}\left(2yz + x^2 + y^2 + z^4 = 18\right)$$
$$2yz_x + 2x + 4z^3z_x = 0 \Rightarrow (2y + 4z^3)z_x = -2x$$
$$z_x = \frac{-x}{y + 2z^3}$$
$$\frac{\partial}{\partial y}\left(2yz + x^2 + y^2 + z^4 = 18\right)$$
$$2z + 2yz_y + 2y + 4z^3z_y = 0$$
$$(y + 2z^3)z_y = -y - z$$
$$z_y = \frac{-y - z}{y + 2z^3}$$
$$z_{xy} = \frac{\partial}{\partial y}\left(\frac{-x}{y + 2z^3}\right) = \frac{x}{(y + 2z^3)^2}(1 + 6z^2z_y) = \frac{x}{(y + 2z^3)^2}\left(1 + 6z^2\left(\frac{-y - z}{y + 2z^3}\right)\right)$$
$$z_{xy} = \frac{xy - 6xyz^2 - 4xz^3}{(y + 2z^3)^3}$$

Chain rule: $z = (x(t), y(t))$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Chain rule for functions of multiple variables: $z(u(s,t), v(s,t))$

$$\begin{bmatrix} \frac{\partial z}{\partial s} \\ \frac{\partial z}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} \end{bmatrix}$$

The above matrix is an example of the Jacobian which essentially maps (x, y) to (s, t).

Directional Derivative:
The directional derivative is when the take the rate of change of f is some arbitrary direction $\hat{u}$.
$x(t) = x_0 + ta,\ \ y(t) = y_0 + tb$ where $\hat{u} = \langle a, b \rangle$
$(D_{\hat{u}}f)(x_0, y_0) = f_x\frac{dx}{dt} + f_y\frac{dy}{dt} = f_x(x_0, y_0)a + f_y(x_0, y_0)b$

$$(D_{\hat{u}}f)(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle$$

This can be simplified by defining the gradient.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Using this, we can simplify the directional derivative to be.

$$(D_{\hat{u}} f)(x_0, y_0) = (\nabla f)(x_0, y_0) \cdot \hat{u}$$

Special note about the gradient: the gradient will always be normal to the surface of the level curve (think a balloon expanding)

1.2.2 Linear Approximation and Tangent Planes

Recall with linear approximations of one variable, they estimate the function at a point using a tangent line. For a 2 variable function, we use the tangent plane as an approximation.
A trick to finding tangent planes is to treat the function as a function of 3 variables as the gradient can be expressed as the normal vector. So we get,

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

for a function of 3 variables.
For functions of 2 variables, we can write it as a functions of 3 variables as $f(x, y, z) = g(x, y) - z = 0$ so $f_z = -1$.
This gives the the equation,

$$f(x, y) \approx f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$$

Ex: Given the cosine law, $c^2 = a^2 + b^2 - 2ab\cos\theta$, find the linear approximation of θ at the point $(a_0, b_0, c_0) = (1/2, 1, \sqrt{3}/2)$

$$\frac{3}{4} = \frac{1}{4} + 1 - \cos\theta$$
$$\frac{1}{2} = \cos\theta \Rightarrow \theta = \frac{\pi}{3}$$
$$\frac{\partial}{\partial a}: \ 0 = 2a - 2b\cos\theta + 2ab\sin\theta \cdot \theta_a$$
$$\Rightarrow 1 - 1 + \frac{\sqrt{3}}{2}\theta_a \Rightarrow 0 = \frac{\sqrt{3}}{2}\theta_a \Rightarrow \theta_a = 0$$
$$\frac{\partial}{\partial b}: \ 0 = 2b - 2a\cos\theta + 2ab\sin\theta \cdot \theta_b$$
$$0 = \frac{3}{2} + \frac{\sqrt{3}}{2}\theta_b \Rightarrow \theta_b = -\sqrt{3}$$
$$\frac{\partial}{\partial c}: \ 2c = 2ab\sin\theta \cdot \theta_c$$
$$\sqrt{3} = \frac{\sqrt{3}}{2}\theta_c \Rightarrow \theta_c = 2$$
$$\theta \approx \frac{\pi}{3} - \sqrt{3}(b - 1) + 2\left(c - \frac{\sqrt{3}}{2}\right)$$

1.2.3 Optimization

A critical point of a multivariable function is defined to be where all the partial derivatives equal 0, or more generally, $\nabla f = 0$
When we take the critical point of a 3D function, there are 3 different types of points it could have:
1) A local minimum
2) A local maximum
3) A saddle point (neither max or min)
We can also have a degenerate critical point which is when we have a line that acts as a critical point.
To find out which of these three types it is, we can use the 2nd derivative test.
If (x_0, y_0) is a critical point of f, we can define a discriminant to be,

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

- If $D(x_0, y_0) < 0$ and $f_{xx} > 0$, then it's a local maximum
- If $D(x_0, y_0) > 0$ and $f_{xx} < 0$, then it's a local minimum.
- If $D(x_0, y_0) < 0$ then it's a saddle point
- If $D(x_0, y_0) = 0$ then we can't conclude (degenerate critical point)

Ex: Find and classify the critical points of $f(x, y) = (x^2 + y^2 - 5)(y - 1)$

$$\nabla f = \begin{bmatrix} 2x(y-1) \\ 2y(y-1) + x^2 + y^2 - 5 \end{bmatrix} = \vec{0}$$
$$x = 0: \ 2y(y-1) + y^2 - 5 = 0$$
$$3y^2 - 2y - 5 = 0 \Rightarrow (3y-5)(y+1) = 0 \Rightarrow y = \frac{5}{3}, \ -1$$
$$\left(0, \frac{5}{3}\right), \ (0, -1)$$
$$y = 1: \ x^2 + 1 - 5 = 0 \Rightarrow x = \pm 2$$
$$(-2, 1), \ (2, 1)$$
$$f_{xx} = 2(y-1)$$
$$f_{yy} = 6y - 2$$
$$f_{xy} = 2x$$
$$D = f_{xx}f_{yy} - f_{xy}^2$$
$$D(2, 1) < 0 \rightarrow \text{saddle}$$
$$D(-2, 1) < 0 \rightarrow \text{saddle}$$
$$D\left(0, \frac{5}{3}\right) > 0, \ f_{xx}\left(0, \frac{5}{3}\right) > 0 \rightarrow \text{local min}$$
$$D\left(0, -1\right) > 0, \ f_{xx}(0, -1) < 0 \rightarrow \text{local max}$$

1.2.4 Lagrange Multipliers

In cases where we want to find the max/min of a function over a closed domain, we can use Lagrange multipliers.
Given some constraint equation $g(x, y, z) = 0$, the maximum/minimum value along the boundary will occur where $\nabla f // \nabla g$ or more formally, our critical points will occur where

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$$

note that λ is some scaling constant.
The case where $\lambda = 0$ corresponds the the max/min along the boundary also being a critical point of the function ($\nabla f = 0$). So if we also compute the critical points of the function, we can ignore the $\lambda = 0$ case.

Ex: The plane $x + y + 2z = 2$ intersects the paraboloid $z = x^2 + y^2$ in an ellipse. Find the points on the ellipse nearest and farthest from the origin.

$$D = \sqrt{x^2 + y^2 + z^2}$$

let $D = D^2 = x^2 + y^2 + z^2$

along region $\{x + y + 2z = 2\} \cap \{z = x^2 + y^2\}$

$$\Rightarrow g(x, y) = x + y + 2(x^2 + y^2) - 2 = 0$$

$$f(x, y) = x^2 + y^2 + (x^2 + y^2)$$

$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{cases} \Rightarrow \begin{cases} 2x + 4x(x^2 + y^2) = \lambda(1 + 4x) \\ 2y + 4y(x^2 + y^2) = \lambda(1 + 4y) \\ x + y + 2(x^2 + y^2) = 2 \end{cases} \Rightarrow \begin{cases} 2xy + 4xy(x^2 + y^2) = \lambda y + 4\lambda xy \\ 2xy + 4xy(x^2 + y^2) = \lambda x + 4\lambda xy \\ x + y + 2(x^2 + y^2) = 2 \end{cases}$$

$$\Rightarrow \lambda x = \lambda y$$

$\lambda = 0$ or $x = y$

$\lambda = 0$ case:

$$\begin{cases} 2x + 4x(x^2 + y^2) = 0 \\ 2y + 4y(x^2 + y^2) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$

if $x = y = 0$, $g = x + y + 2(x^2 + y^2) - 2 = -2 \neq 0 \therefore x = y \neq 0$

$x = y$ case:

$$2y + 2(2y^2) = 2 \Rightarrow 2y^2 + y - 1 = 0 \Rightarrow (2y - 1)(y + 1) = 0$$

$$x = y = \left\{-1, \frac{1}{2}\right\}, \ z = x^2 + y^2$$

gives points $(-1, -1, 2)$, $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

Ex2: Let (a, b) be a point on the ellipse $x^2 + 3y^2 = 3$ and $(c, 3 - c)$ be a point on the line $x + y = 3$. Find the coordinates of the pair of points which are closest to each other.

let D be the distance squared

$$D = (a-c)^2 + (b-3+c)^2$$
$$g = a^2 + 3b^2 - 3 = 0$$
$$\nabla D = \langle 2(a-c), 2(b-3+c), -2(a-c) \rangle$$
$$\nabla g = \langle 2a, 6b, 0 \rangle$$
$$\begin{cases} a - c = \lambda a \\ b - 3 + c = 3\lambda b \\ a - c = b - 3 + c \\ a^2 + 3b^2 = 3 \end{cases}$$
$$a - c = 3\lambda b$$
$$3\lambda b = \lambda a \Rightarrow \lambda = 0 \text{ or } 3b = a$$
$$\text{if } \lambda = 0, a - c = 0 \Rightarrow a = c$$
$$b - 3 + c = 0 \Rightarrow b = 3 - c$$
$$c^2 + 3(3-c)^2 = 4c^2 - 18c + 27 = 3 \Rightarrow 2c^2 - 9c + 12 = 0$$
$$c = \frac{9 \pm \sqrt{81-96}}{4} \to \text{no real solutions}$$
$$\therefore a = 3b$$
$$9b^2 + 3b^2 = 3 \Rightarrow b^2 = \frac{1}{4} \Rightarrow b = \pm\frac{1}{2} \Rightarrow a = \pm\frac{3}{2}$$
$$3b - c = b - 3 + c \Rightarrow 2b + 3 = 2c \Rightarrow c = b + \frac{3}{2} = \pm 2$$
$$D\left(\frac{3}{2}, \frac{1}{2}, 2\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$
$$D\left(-\frac{3}{2}, -\frac{1}{2}, -2\right) = \frac{1}{4} + \frac{169}{4} = \frac{85}{2}$$
$$\therefore \text{ the closest points are } \left(\frac{3}{2}, \frac{1}{2}\right) \text{ and } (2, 1)$$

1.2.5 Least Squares Interpolation

Given experimental data, (x_1, y_1), (x_2, y_2), $\ldots (x_i, y_i)$, find the best fit line.
Minimize $D = \sum_{i=1}^{n} (y_i - (ax_i + b))^2$
We can find the minimum by finding the critical points.

$$\frac{\partial D}{\partial a} = 0 \Rightarrow \frac{\partial D}{\partial a} = \sum_{i=1}^{n} 2(y_i - (ax_i + b))(-x_i) = 0$$
$$\frac{\partial D}{\partial b} = 0 \Rightarrow \sum_{i=1}^{n} 2(y_i - (ax_i + b))(-1) = 0$$
$$\sum_{i=1}^{n} (x_i^2 a + x_i b - x_i y_i) = 0$$

$$\sum_{i=1}^{n}(x_i a + b - y_i) = 0$$

$$\begin{cases} \left(\sum_{i=1}^{n} x_i^2\right) a + \left(\sum_{i=1}^{n} x_i\right) b = \sum_{i=1}^{n} x_i y_i \\ \left(\sum_{i=1}^{n} x_i\right) a + nb = \sum_{i=1}^{n} y_i \end{cases}$$

$\rightarrow$ gives a 2x2 linear system

$\rightarrow$ solve for a and b

For interpolating non-linear plots, we can linearize the data.
Ex: $y = ce^{ax} \Rightarrow \ln y = \ln c + ax$
For a polynomial function, we can expand the number of coefficients we have.
Ex: for $y = ax^2 + bx + c$, we get:
$D(a, b, c) = \sum\limits_{i=1}^{n}(y_i - (ax_i^2 + bx_i + c))^2$ which gives a 3x3 linear system.

1.3 Multiple Integrals

1.3.1 Double Integrals

By definition, we get

$$\iint_R f(x, y)dA = \lim_{M,N\to\infty} \sum_{i=1}^{M}\sum_{j=1}^{N} f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y$$

where $dA = dydx$
The double integral can be interpreted as volume under the curve $f(x, y)$.
We compute the double integral by first integrating with respect to one variable and then the other.
A double integral will be computed in either of the following forms:

$$\iint_R f(x, y) = \int_{x=a}^{x=b} \int_{y=y_1(x)}^{y=y_2(x)} f(x, y)dydx$$

$$\iint_R f(x, y) = \int_{y=a}^{y=b} \int_{x=x_1(y)}^{x=x_2(y)} f(x, y)dxdy$$

Ex: Find $\iint_R ydA$ over the region bounded by $x = y$ and $x = 2 - y^2$

intersections:

$y = 2 - y^2 \Rightarrow y^2 + y - 2 = 0 \Rightarrow (y + 2)(y - 1) = 0 \Rightarrow y = 1,\ -2$

$$\iint ydA = \int_{y=-2}^{1} \int_{x=y}^{x=2-y^2} ydydx = \int_{-2}^{1} y\,[x]_y^{2-y^2}\, dy$$

$$= \int_{-2}^{1} y(2 - y^2 - y)dy = \int_{-2}^{1}(2y - y^3 - y^2)dy$$

$$= \left[y^2 - \frac{y^4}{4} - \frac{y^3}{3}\right]_{-2}^{1} = 1 - \frac{1}{4} - \frac{1}{3} - \left(4 - 4 + \frac{8}{3}\right) = -\frac{9}{4}$$

Note that the rules for even/odd functions are the same as with single integrals. If you have an odd function over a symmetric region, the entire integral will be zero. If you have an even function over a symmetric region, you can simplify the region and double the integral.
Applications of double integrals:
Area of a region:

$$\text{Area}(R) = \iint_R dA$$

The average value of a function can be computed as:

$$\overline{f(x,y)} = \frac{1}{\text{Area}(R)} \iint_R f(x,y)dA$$

Average height of a region:

$$\overline{y} = \frac{1}{\text{Area}(R)} \iint_R ydA$$

Surface Area:

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2}dA$$

Center of mass:

$$x_{CM} = \frac{1}{\text{Mass}(R)} \iint_R x\rho(x,y)dA$$

Moment of inertia about an axis a where $D(x,y)$ is the distance from the axis.

$$I_a = \iint_R (D(x,y))^2\rho(x,y)dA$$

1.3.2 Polar Coordinates

Polar coordinates uses the variables r and θ to describe functions.
r is the distance from the origin
θ is the angle from the x-axis to the line formed by r.
We can convert between polar coordinates and rectangular coordinates using the following conversions:

$$r = \sqrt{x^2 + y^2}$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$

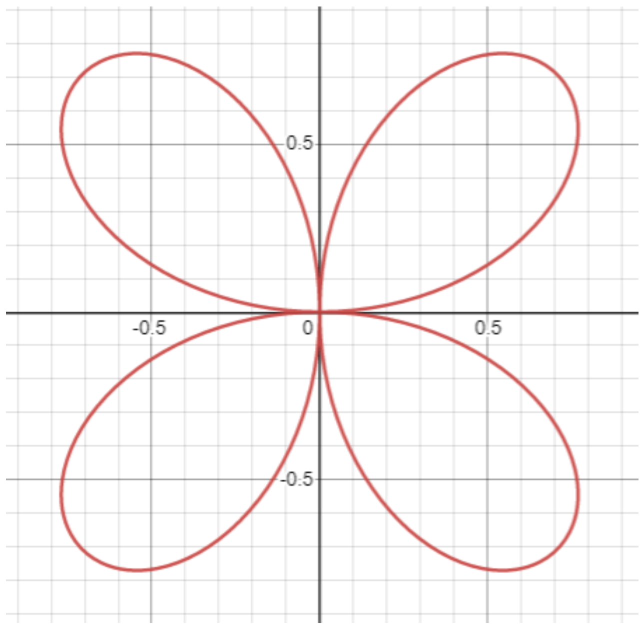
Some common expressions in polar coordinates are:

circle: $r = a$
ray: $\theta = a$

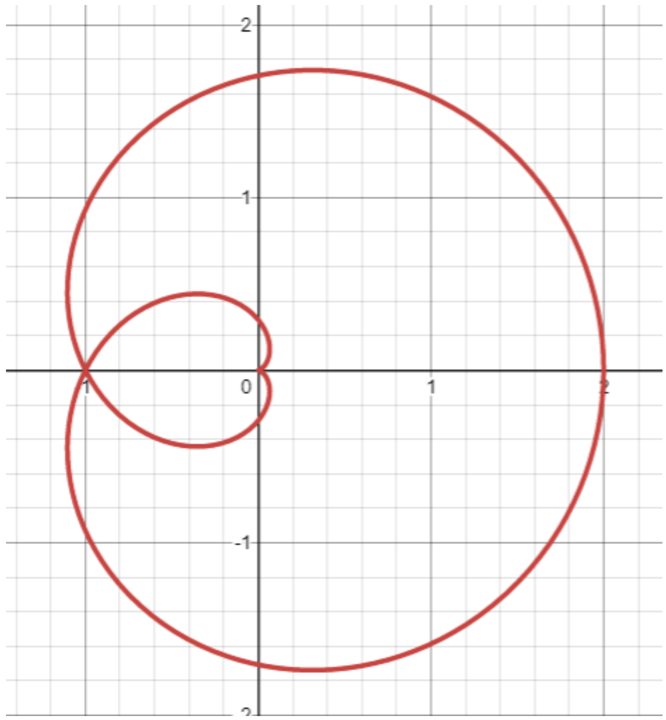
Ex: Find the equation of an off-center circle in polar coordinates

$$(x-a)^2+y^2=a^2$$
$$x^2-2ax+a^2+y^2=a^2$$
recall $r^2=x^2+y^2$
$$\Rightarrow r^2-2ax=0$$
$$r^2-2ar\cos\theta=0$$
$$r^2=2ar\cos\theta$$
$$r=2a\cos\theta$$

Graphing in Polar Coordinates:
We can create some interesting graphs using polar coordinates. Here are a few examples:
Ex1: $r=\sin(2\theta)$



Ex2: $r=1+\cos\left(\frac{\theta}{2}\right)$



For regions with circular symmetry, it is often easier to integrate in polar coordinates. The differential area is given by:

$$dA = r dr d\theta$$

Note: This distortion factor of r in $dA = r dr d\theta$ comes from the arc length.
Ex: Area of a circle of radius R

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=R} r dr d\theta$$

$$\int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^R d\theta = \int_0^{2\pi} \frac{R^2}{2} d\theta = \pi R^2$$

Ex2: $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx$

$$\text{let } I = \int_{-\infty}^{\infty} e^{-x^2} dx,\ I^2 = \int_{-\infty}^{\infty} e^{-x_1^2} dx_1 \int_{-\infty}^{\infty} e^{-x_2^2} dx_2 = \int_{x_1=-\infty}^{\infty} \int_{x_2=-\infty}^{\infty} e^{-x_1^2} e^{-x_2^2} dx_2 dx_1$$

$$I^2 = \iint_{\mathbb{R}^2} e^{-x_1^2 - x_2^2} dA$$

$$r^2 = x_1^2 + x_2^2$$

$$I^2 = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-r^2} r dr d\theta = 2\pi \int_0^{\infty} r e^{-r^2} dr = 2\pi \left[-\frac{e^{-r^2}}{2}\right]_0^{\infty} = 2\pi \left(\frac{1}{2}\right)$$

$$I^2 = \pi$$

$$I = \sqrt{\pi}$$

1.3.3 Triple Integrals

By definition, we get

$$\iiint f(x,y,z)dV = \lim_{\Delta x,\Delta y,\Delta z \to 0} \sum_{i=1}^{L}\sum_{j=1}^{M}\sum_{k=1}^{N} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta x\Delta y\Delta z$$

These are computed in the same manner as double integrals, just with an additional step.

Ex: $\iiint_E 2xdV$, where E is the region in the first octant bounded by $2x+3y+z=6$

$I = \int_{x=0}^{3}\int_{y=0}^{y=-\frac{2}{3}x+2}\int_{z=0}^{z=6-2x-3y} 2xdzdydx$

$I = \int_{x=0}^{3}\int_{y=0}^{y=-\frac{2}{3}x+2} 2x(6-2x-3y)dydx$

$I = \int_{0}^{3} \left[12xy - 4x^2y - 3xy^2\right]_{y=0}^{y=-\frac{2}{3}x+2} dx$

$I = \int_{0}^{3} \left(\frac{4}{3}x^3 - 8x^2 + 12x\right) dx$

$I = 9$

Ex2: Find the volume of intersection between the cylinders $x^2+y^2=R^2$ and $x^2+z^2=R^2$

$V = \iiint dV = \int_{x=-R}^{R}\int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{z=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dzdydx$

$V = \int_{x=-R}^{R}\int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 2\sqrt{R^2-x^2}dydx$

$V = \int_{x=-R}^{R} 4(R^2-x^2)dx$

$V = 4\left[R^2x - \frac{x^3}{3}\right]_{-R}^{R} = 8\left(R^3 - \frac{R^3}{3}\right) = \frac{16R^3}{3}$

1.3.4 Change of Coordinate Systems

Two common change of coordinates in triple integrals are cylindrical coordinates and spherical coordinates.
Cylindrical coordinates is an extention of polar coordinates where the conversions are as follows:

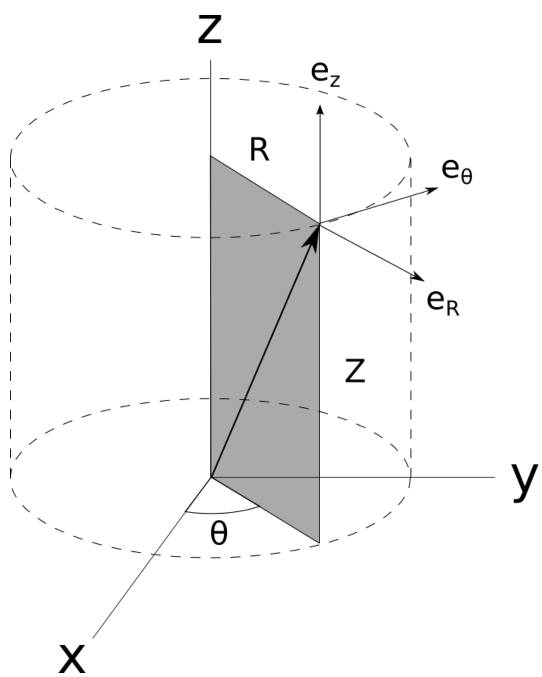
$r = \sqrt{x^2+y^2}$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$dV = rdzdrd\theta$



Ex: Find the volume of the region below the paraboloid $z = 5 - x^2 - y^2$ and above the plane $z = 1$

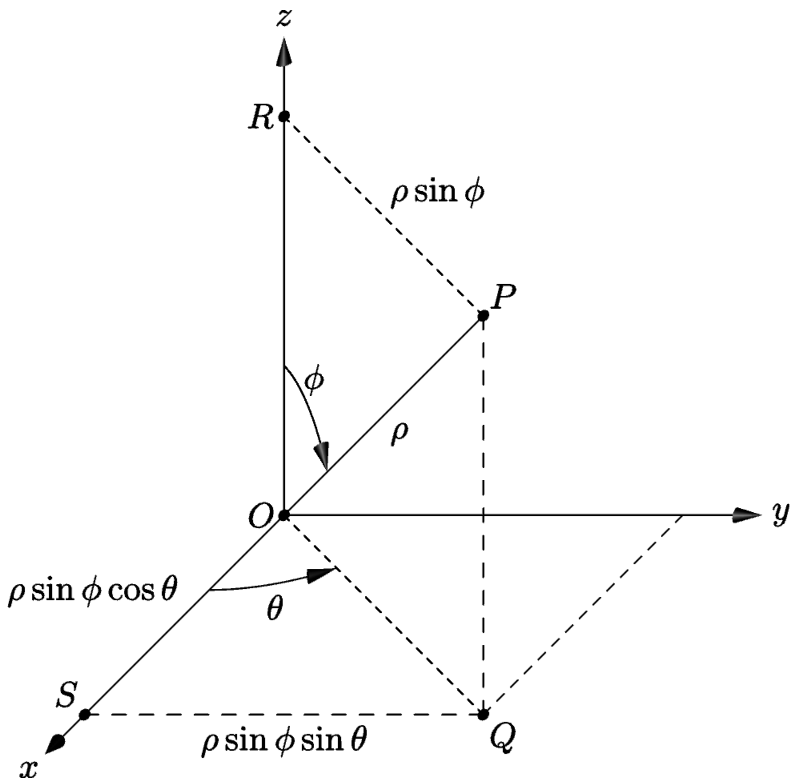
Projected area is $\{z = 5 - x^2 - y^2\} \cap \{z = 1\} \Rightarrow x^2 + y^2 = 4$

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=1}^{5-r^2} r dz dr d\theta = 2\pi \int_0^2 (4r - r^3) dr = 2\pi \left[2r^2 - \frac{r^4}{4}\right]_0^2$$

$$V = 2\pi\,(8 - 4) = 8\pi$$

Spherical coordinates is given by

$$\rho^2 = x^2 + y^2 + z^2$$
$$x = \rho \cos\theta \sin\phi$$
$$y = \rho \sin\theta \sin\phi$$
$$z = \cos\phi$$
$$dV = \rho^2 \sin\phi d\rho d\phi d\theta$$



Ex: Find the volume below the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$

$$V = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{2} \rho^2 \sin\phi d\rho d\phi d\theta = 2\pi \int_0^{\pi/4} \frac{8}{3} \sin\phi d\phi = \frac{16\pi}{3} \left[-\cos\phi\right]_0^{\pi/4}$$

$$V = \frac{16\pi}{3} \left(1 - \frac{1}{\sqrt{2}}\right) = \frac{8\pi}{3}(2 - \sqrt{2})$$

We can define an arbitrary change of coordinates in the following way:
If we have the equations

$$\begin{cases} x = x(u, v, w) \\ y = y(u, v, w) \\ z = z(u, v, w) \end{cases}$$

we can represent the distortion factor as the determinant of the Jacobian.

$$\det J = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

Ex: Find the volume of the ellipsoid enclosed by the surface $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$

$$\text{let } \begin{cases} x = au \\ y = bv \\ z = cw \end{cases}$$

$$\det J = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$V = \iiint_{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\leq 1} dV = \iiint_{u^2+v^2+w^2\leq 1} abc du dv dw$$

$$V = abc \cdot \text{Volume(unit sphere)} = \frac{4\pi abc}{3}$$