Math Notes

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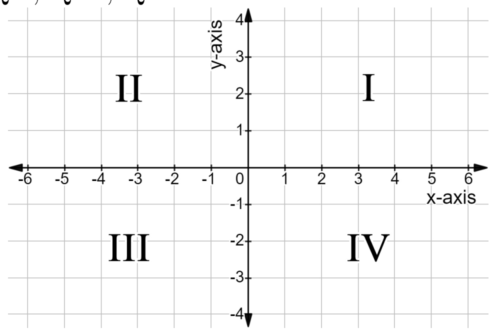
Contents

1 Pre-Calculus

1.1 Functions and Coordinates

1.1.1 Cartesian Coordinate System

The Cartesian coordinate system is a graph to plot points, lines, and functions on a horizontal x-axis and vertical y-axis. The center where the lines meet is called the origin. The axes split the graph into 4 quadrants: QI, QII, QIII, QIV.



Points on the graph are denoted by (x, y) and are called an ordered pair. Depending on if x and y are positive or negative, they will be positioned in different quadrants.
(x, y) $\rightarrow$QI
$(-x, y)$ $\rightarrow$QII
$(-x, -y)$ $\rightarrow$QIII
$(x, -y)$ $\rightarrow$QIV
When graphing equations on the coordinate plane, the dependent variable will usually be on the x-axis and the independent variable will be displayed on the y-axis.

1.1.2 Domain and Range

Domain is the set of all elements that make up the x-coordinates.
Range is the set of all elements that make up the y-coordinates.
The domain and range can be expressed in multiple ways:

- Inequality Description: $a < x < b$
  This means that x is greater than a and less than b.

- Interval Notation: $x \in (a, b)$
  This means that x is between a and b, not including a or b. If we use square brackets, $x \in [a, b]$, then it means x is between a and b while including a and b.
  Note that ∞ and $-\infty$ will always use round parentheses

- Set Notation: $\{x|a < x < b, x \in \mathbb{R}\}$
  This translates to the set of x for which x is greater than a and less then b for where x is an element of all real numbers.

1.1.3 Function Definition

A function is a special type of relation where each element in the domain is associated with exactly one element in the range. (Every x-value can only have one associated y-value).
We can determine if a graph is a function or not using the vertical line test. If you move a vertical line along the graph, the line should intersect no more than one point on the curve.
Function notation, $f(x)$, is used to denote the result after some operation on the independent variable. $f(x)$ and y are used somewhat interchangeably.

1.2 Linear Functions

A linear function is a function whose graph is a straight line. Its degree is 1 or 0. It has a constant rate of change.

1.2.1 Slope

Slope is a measure of how one quantity changes with respect to another, sometimes called the rate of change. For graphs, it is determined by calculating the change in the y-values over the change in x-values and is denoted by m.

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

For example, a line with slope $\frac{3}{2}$ will go up 3 units for every 2 units it goes across.
- Horizontal lines will always have a slope of 0.
- Vertical lines have an undefined slope $(\frac{1}{0})$.
- Parallel lines will have the same slope: $m_{\|} = m$.
- Perpendicular lines will have a slope that is equal to the negative reciprocal of the other. $m_{\perp} = -\dfrac{1}{m}$

1.2.2 Equations of Linear Functions

- Slope Intercept Form: $y = mx + b$
  where m is the slope and b is the y-intercept. Slope intercept form is ideal for graphing the function and is the most commonly used.

- Slope Point Form: $y - y_1 = m(x - x_1)$
  where (x_1, y_1) is some point on the line. This is useful for when you are given a point and the slope and are asked to find an equation.

- General Form: $Ax + By + C = 0$
  where A must be a whole number and B and C are both integers. This is helpful for finding x and y intercepts.

All three forms are interchangeable.

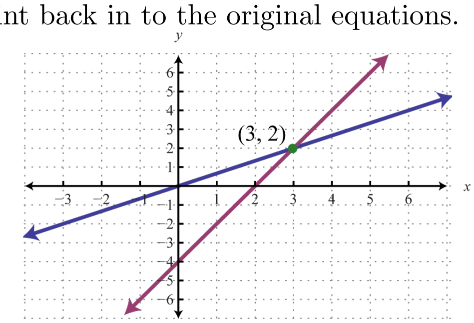
Solving Linear Equations
Often we are interested in when the line crosses the x-axis. We can find this by setting $y = 0$ and solving for x.
Ex: $y = 6x + 4$

$$\begin{aligned} 0 &= 6x + 4 \\ -4 &= 6x \\ x &= -\frac{2}{3} \end{aligned}$$

1.2.3 Systems of Linear Equations

A system of linear equations is composed of two linear equations and is often referred to as a linear system.
A system will have no solutions if the lines are parallel.
A system will have infinite solutions if the lines are identical.
Otherwise, the system will have one solution.

Solving Graphically:
The solution of a linear system can be estimated by graphing both equations. The solution will be where the two lines intersect. This will give a point that satisfies both equations. To verify the solution, you can plug the point back in to the original equations.



Solving by Elimination
Step 1 is to make the value of x or y the same for both equations.
Step 2 is to add or subtract the add or subtract the equations to cancel out one of the variables.
Step 3 is to solve for the remaining variable.
Step 4 is to plug the value back into one of the original equations to solve for the other variable.

Ex: $\begin{cases} 5x - 3y = 18 \\ 4x - 6y = 18 \end{cases}$

$$\begin{cases} 2(5x - 3y = 18) \\ \quad 4x - 6y = 18 \end{cases}$$

$$\begin{cases} 10x - 6y = 36 \\ 4x - 6y = 18 \end{cases}$$

$(10x - 6y) - (4x - 6y) = 36 - 18$

$6x = 18$

$x = 3$

$5(3) - 3y = 18$

$15 - 3y = 18$

$-3y = 3$

$y = -1$

Solution is $(3, -1)$

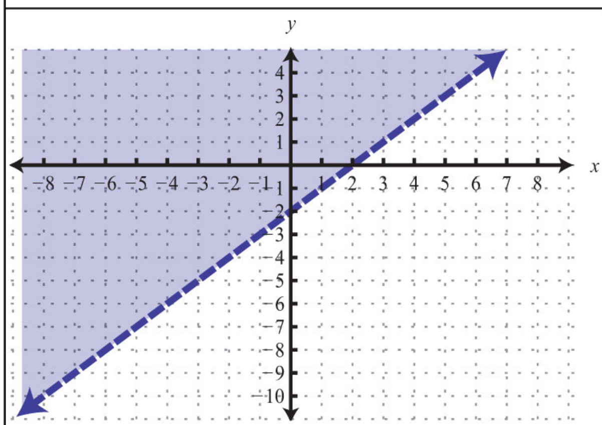
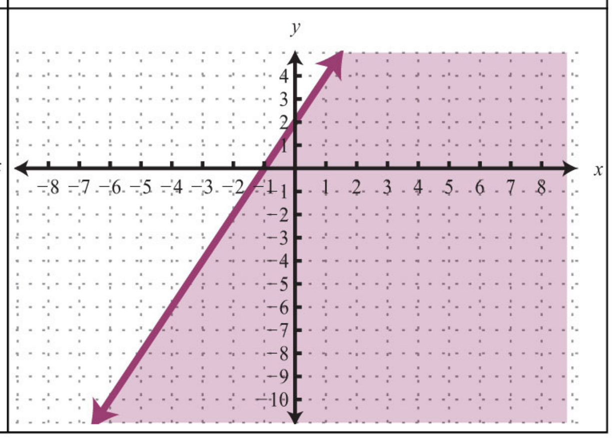
Solving by Substitution:
Step 1 is to isolate one of the variables in one equation.
Step 2 is, in the other equation, replace the variable you isolated for earlier with the equation that

it's in terms of.
Step 3 is to solve for the remaining variable.
Step 4 is to plug the value bck into one of the original equations to solve for the other variable.
Ex: $\begin{cases} x - 3y = 12 \\ 4x + 2y = 8 \end{cases}$

$$x = 3y + 12$$
$$4(3y + 12) + 2y = 8$$
$$48 + 12y + 2y = 8$$
$$14y = -40$$
$$y = -\frac{20}{7}$$
$$x - 3\left(-\frac{20}{7}\right) = 12$$
$$x + \frac{60}{7} = \frac{84}{7}$$
$$x = \frac{24}{7}$$
$$\text{Solution is } \left(\frac{24}{7}, -\frac{20}{7}\right)$$

1.2.4 Linear Inequalities

Inequalities are in the form of $y \geq f(x)$. It creates a boundary, splitting the Cartesian plane into two or more regions. For $y \geq f(x)$, any part above the function is shaded and for $y \leq f(x)$, any part below the function is shaded. If $\geq$ or $\leq$ are used, the boundary will have a solid line. If $>$ or $<$ are used, the boundary will have dashed line.

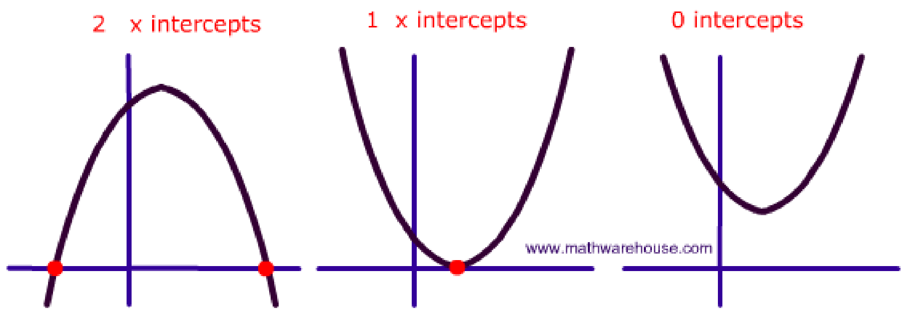
$y > x - 2$	$y \leq 2x + 2$
	

1.3 Quadratic Functions

Quadratic functions are those in the form of $y = x^2$. They have a degree of 2. The shape of the graph is known as a parabola.

1.3.1 Equations and Terminology

Terminology:
The vertex is the lowest or highest point of the parabola (where it changes directions.)
The axis of symmetry is the vertical line that divides the parabola in half, going through the vertex.
A parabola will always have 1 y-intercept and either 0, 1, or 2 x-intercepts.
The domain of a parabola is $x \in \mathbb{R}$
The range of a parabola is dependent on which way it opens and the height of the vertex.



Vertex Form: $y = a(x - p)^2 + q$
a determines how steep the parabola is. Larger values make it steeper and smaller values make it rise more gradually. If a is positive, the parabola opens up. If a is negative, the parabola opens down. p determines how far to the right of the origin the vertex will be located. q determines how far above the origin the vertex will be located.
This form is useful for graphing because you know exactly where the vertex is and how fast the slope is rising.

Standard Form: $y = ax^2 + bx + c$
a determines how steep the parabola is and which way it opens. c is the y-intercept.
This form is useful for solving quadratic equations.

Converting between forms:
To convert vertex form to standard form, you simply need to expand the squared term inside the brackets.
To convert from standard form to vertex form, you can either use the conversion formulas of $p = -\dfrac{b}{2a}$ and $q = c - \dfrac{b^2}{4a} = c - ap^2$ or you can use completing the square method. Completing the square involves adding and subtracting a value in order to create a perfect square trinomial.
Ex: $x^2 + 6x + 5$

$$x^2 + 6x + 5 + 9 - 9$$
$$(x^2 + 6x + 9) - 4$$

$$(x+3)^2-4$$

Ex2: $4x^2-32x-23$

$$\begin{aligned}
&4(x^2-8x)-23\\
&4(x^2-8x+16)-23-4(16)\\
&4(x^2-4)^2-87
\end{aligned}$$

1.3.2 Solving Quadratic Equations

Solving by Factoring: First we must set $y=0$. Then we can factor our quadratic. We will have two terms. At least one of the terms must be equal to zero in order to give an answer of zero so to solve, we can set both terms equal to zero and solve for x.
Ex: $y=x^2+4x-21$

$$\begin{aligned}
0 =&(x+7)(x-3)\\
0 =&(x+7) \Rightarrow x=-7\\
0 =&(x-3) \Rightarrow x=3
\end{aligned}$$

Solving by Completing the Square:
Ex: $y=x^2-6x+7$

$$\begin{aligned}
y =&x^2-6x+9+7-9\\
y =&(x-3)^2-2\\
0 =&(x-3)^2-2\\
2 =&(x-3)^2\\
x-3 =&\pm\sqrt{2}\\
x =&3\pm\sqrt{2}
\end{aligned}$$

The Quadratic Formula:
Derivation:
If there is no horizontal shift, a quadratic will take the form $y=ax^2+q$ and solving the quadratic is easy. We can make any quadratic take this form by applying a horizontal translation, shifting the quadratic by the location of the vertex. So we set $x=x_s-\frac{b}{2a}$. This will allow us to solve for any quadratic.

$$\begin{aligned}
0 =&ax^2+bx+c\\
0 =&a\left(x_s-\frac{b}{2s}\right)^2+b\left(x_s-\frac{b}{2a}\right)+c\\
0 =&\left(x_s-\frac{b}{2s}\right)^2+\frac{b}{a}\left(x_s-\frac{b}{2a}\right)+\frac{c}{a}\\
0 =&x_s^2-\frac{b}{a}x_s+\frac{b^2}{4a^2}+\frac{b}{a}x_s+\frac{c}{a}
\end{aligned}$$

$$
\begin{aligned}
x_s^2 =& \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2} \\
x_s =& \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x =& x_s - \frac{b}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\
x =& \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}
$$

*A similar derivation could be done using the complete the square method.
This formula allows us to solve any quadratic equation. It also has a property that allows us to deduce how many solutions a quadratic will have without actually solving it. This is done using the discriminant (the part under the square root).
If $b^2 - 4ac > 0$ there are 2 real solutions
If $b^2 - 4ac = 0$ there is 1 real solution
If $b^2 - 4ac < 0$ there are no real solutions

Systems of Quadratic Equations:
To find the point(s) where two quadratics intersect, we can use substitution for y and set the two equations equal to each other. This gives a new quadratic and will give the points of intersection.
Ex: $\begin{cases} y = 3x^2 - x - 2 \\ y = 6x^2 + 4x - 4 \end{cases}$

$$
\begin{aligned}
3x^2 - x - 2 =& 6x^2 + 4x - 4 \\
0 =& 3x^2 + 5x - 2 \\
0 =& (3x - 1)(x + 2) \\
0 =& (3x - 1) \Rightarrow x = \frac{1}{3} \\
0 =& (x + 2) \Rightarrow x = -2
\end{aligned}
$$

1.3.3 Quadratic Inequalities

Because a quadratic can have two x-intercepts, it can have two places it switches from positive to negative so quadratic inequalities are often bounded by two numbers.
We can break it up into a series of linear cases based on factors.
Ex: $(x + 3)(x - 2) \geq 0$
$x + 3 \geq 0 \Rightarrow x \geq -3$

$x - 2 \geq 0 \Rightarrow x \geq 2$

$(x + 3)(x - 2) \geq 0 \Rightarrow x \leq -3 \,\text{or}\, x \geq 2$

This method of lining up the number line graphs works for all polynomials and even rational expressions to solve for difficult inequalities.

1.4 Transforming Functions

1.4.1 Transformations

Transformations are when the graph of a relation is shifted or changed. An image point is the point that results from a transformation. Mapping relates one set of points to the corresponding points in the image.

Vertical translations move the graph up or down, specified by the value k.

$$f(x) \to f(x) + k$$

Mapping notation: $(x, y) \to (x, y + k)$

Horizontal translations move the graph left or right specified by the value h.

$$f(x) \to f(x - h)$$

Mapping notation: $(x, y) \to (x - h, y)$

Reflections across the x-axis:

$$f(x) \to -f(x)$$

Mapping notation: $(x, y) \to (x, -y)$

Reflections across the y-axis:

$$f(x) \to f(-x)$$

Mapping notation: $(x, y) \to (-x, y)$

Vertical stretches multiplies all y-coordinates by a factor of a.

$$f(x) \to af(x)$$

Mapping notation: $(x, y) \to (x, ay)$

Horizontal stretches multiply all x-coordinates by a factor of b.

$$f(x) \to f\left(\frac{1}{b}x\right)$$

Mapping notation: $(x, y) \to (bx, y)$

All together, we can use transformations to manipulate ordinary functions into various forms. The order transformations are applied is stretches and reflections are applied first and then translations after.

$$f(x) \to af\left(\frac{x}{b} - h\right) + k$$

Mapping notation: $(x, y) \to (bx - h, ay + k)$
Points that do not change given a series of transformations are called invariant points.

1.4.2 Absolute Value of a Function

Given the property of the absolute value, the function will have all y-values be positive with the negative values being reflected across the x-axis. The range of an absolute value function cannot contain negative values.

$$y = |f(x)|$$

Mapping notation: $(x, y) \to (x, |y|)$
Absolute value functions can be described in two different ways: The root of a square,

$$|f(x)| = \sqrt{(f(x))^2}$$

or as a piecewise function:

$$|f(x)| = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

Piecewise functions are not limited to absolute value functions. They are functions comprised of multiple parts or functions over different domains.

Solving Absolute Value Equations:
When solving for absolute value equations, we must consider both cases, $f(x)$ and $-f(x)$, and check our restrictions after.
Ex: $|2x - 5| = 5 - 3x$

Case 1: $|2x - 5| = 2x - 5$ for $x \geq \frac{5}{2}$

$2x - 5 = 5 - 3x$

$5x = 10$

$x = 2$

Because $x \ngeq \frac{5}{2}$, the solution is extraneous.

Case 2: $|2x - 5| = 5 - 2x$ for $x < \frac{5}{2}$

$5 - 2x = 5 - 3x$

$x = 0$

Check $x < \frac{5}{2}$ $\therefore$ it is a solution

Ex: $|x^2 - 2x| = -1$

Case 1: $|x^2 - 2x| = x^2 - 2x$ for $x \leq 0$ or $x \geq 2$

$x^2 - 2x + 1 = 0$

$(x - 1)^2 = 0$

$x - 1 = 0$

$x = 1$

Check restrictions: $x = 1$ is extraneous

Case 2: $|x^2 - 2x| = 2x - x^2$ for $0 < x < 2$
$2x - x^2 = -1$
$x^2 - 2x - 1 = 0$
$x = \dfrac{2 \pm \sqrt{8}}{2} = 1 \pm \sqrt{2}$
Check restrictions: both extraneous
$\therefore$ no real solutions

Absolute Value Inequalities:

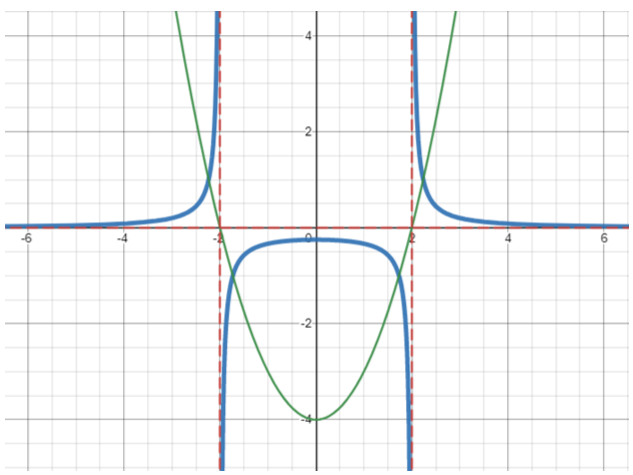
$$|f(x)| < k \Rightarrow -k < f(x) < k$$

1.4.3 Reciprocal of a Function

The reciprocal of a function introduced a few interesting new aspects.

$$y = \frac{1}{f(x)}$$

Mapping notation: $(x, y) \rightarrow (x, \frac{1}{y}$
The reciprocal will have vertical asymptotes where $f(x) = 0$ in the original function and a horizontal asymptote at $y = 0$. An asymptote is an imaginary line that the function comes infinitely close to but never reaches. The reciprocal will have invariant points any place where $y = 1$ in the original function.
Ex: $y = \frac{1}{x^2-4}$



1.4.4 Inverse Functions

The inverse of a function reverses the process represented by that function, interchanging x and y coordinates. The result is a reflection along the line $y = x$. Inverse notation is $f^{-1}(x)$ or $x = f(y)$.
Mapping notation: $(x, y) \rightarrow (y, x)$
To solve for the inverse of a function, switch x and y and then solve for y.

Ex: Inverse of $y = \frac{1}{2x+5}$

$$\begin{aligned} x =& \frac{1}{2y+5} \\ x(2y+5) =& 1 \\ 2y+5 =& \frac{1}{x} \\ 2y =& \frac{1}{x} - 5 \\ y =& \frac{1}{2x} - \frac{5}{2} \end{aligned}$$

Ex2: Inverse of $y = x^2 + 2x - 2$

$$\begin{aligned} x =& y^2 + 2y - 2 \\ x =& y^2 + 2y + 1 - 2 + 1 \\ x =& (y+1)^2 - 1 \\ x + 1 =& (y+1)^2 \\ \pm\sqrt{x+1} =& y + 1 \\ y = \pm& \sqrt{y+1} - 1 \end{aligned}$$

1.4.5 Square Root of a Function

A square root function is the inverse of a quadratic. Because the inverse of a parabola does not meet the requirements of a function, we can restrict the range in order to make it a function. In doing this, we state that the square root must always be positive and this goes for all functions.

$$y = \sqrt{f(x)}$$

Mapping notation: $(x, y) \rightarrow (x, \sqrt{y})$
The square root of a function will travel in the same direction as the original function but the shape and value will be different. For $f(x) < 0$, $\sqrt{f(x)}$ is undefined
For $0 < f(x) < 1$, $\sqrt{f(x)} > f(x)$
For $f(x) > 1$, $\sqrt{f(x)} < f(x)$
Note that we must restrict the domain of x such that the value of $f(x) > 0$
An interesting thing to note is that the square root of any downward opening parabola forms a semicircle.

Solving radical equations:
Ex: $\sqrt{x-2} = 4 - x$

Restrictions: $x - 2 \geq 0 \Rightarrow x \geq 2$

and $x - 4 \geq 0 \Rightarrow x \leq 4$

$(\sqrt{x-2})^2 = (4-x)^2$

$x - 2 = 16 - 8x + x^2$

$x^2 - 9x + 18 = 0$
$(x-3)(x-6) = 0$
$x = 3, x = 6$
$x = 6 \nleq 4 \therefore$ not a solution
$\Rightarrow x = 3$

1.4.6 Rational Functions

Rational functions are an algebraic fraction with a numerator and denominator that are polynomials. In other words, it is a function divided by a function.
Ex: $\dfrac{x-7}{x^2-4x+3}$

The graphs of rational functions can have a variety of features including asymptotes and holes. Asymptotes and holes occur at points where the function is not defined (divided by 0). These values for which the denominator is equal to 0 are called non-permissible value (NPVs).
Ex: NPVs of $\dfrac{x-7}{(x-3)(x-1)}$ are $x \neq 3$ and $x \neq 1$

Solving Rational Functions:
The x-intercepts of a rational function occur where the numerator is equal to 0 (given the denominator is not also 0).
Ex: $\dfrac{6-x}{2x} \Rightarrow 6 - x \Longrightarrow x = 6$

Simplifying Rational Expressions:
Some rational expressions can be factored and simplified.
Ex: $\dfrac{6-2x}{x^2-9} = \dfrac{-2(x-3)}{(x+3)(x-3)}$
The $(x-3)$ terms cancel, leaving $\dfrac{-2}{x+3}$.
Note that you must include all original NPVs. Otherwise, the simplified function would not be the same as the original. This new, simplified function will have a hole (removable discontinuity) at the point $x = 3$, the term that was removed.
So, our final answer is $\dfrac{-2}{x+3}$ where $x \neq \pm 3$

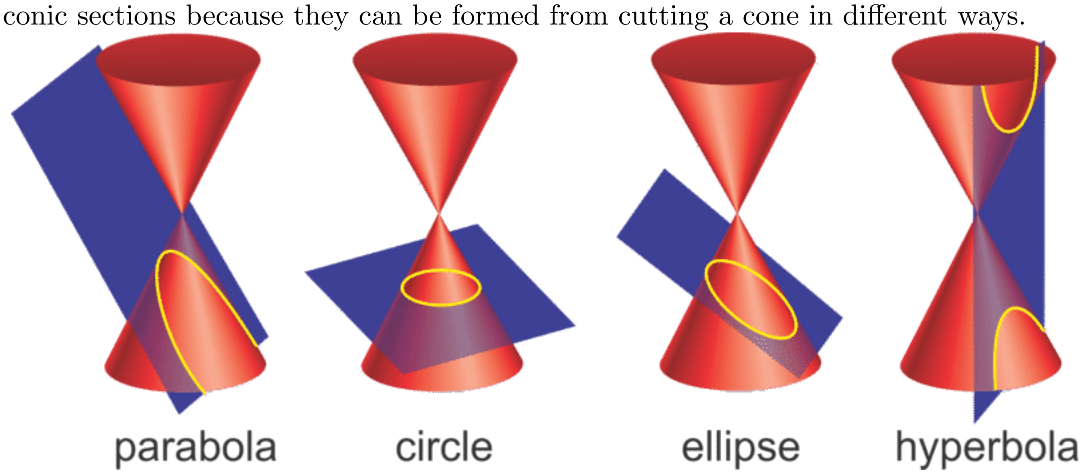
Operations with Rational Expressions:
When dividing two rational expressions, we must take the NPVs of both expressions at the start and then once more after dividing.
Ex: $\dfrac{3x^2}{y^2} \div \dfrac{x}{y} \rightarrow$ NPVs: $y \neq 0$
$\dfrac{3x^2y}{xy^2} \rightarrow$ NPVs: $y \neq 0$, $x \neq 0$
When adding and subtracting rational epressions, you must find the LCM and multiply the expressions accordingly as you do with regular addition and subtraction with fractions.
Ex: $\dfrac{5x}{x+2} + \dfrac{2x-3}{x-1} = \dfrac{5x(x-1)+(2x-3)(x+2)}{(x+2)(x-1)}$ where $x \neq -2$, $x \neq 1$

Asymptotes:
Vertical asymptotes occur at points where the function has NVPs that are not holes. For example, $\frac{1}{x}$ has a vertical asymptote at $x = 0$. At these asymptotes, the function will curl up or down towards $y = \infty$ or $y = -\infty$.
Horizontal asymptotes are the end behavior of rational functions. As x approaches positive or negative infinity, the function will either diverge (go to infinity) or approach a finite value as a horizontal asymptote.

- If the degree of the numerator is greater than the degree of the denominator, $f(x) \to \pm\infty$
- If the degree of the numerator is the same as the degree of the denominator, $f(x) \to L$ where L is some constant.
- If the degree of the numerator is less than the degree of the denominator, $f(x) \to 0$

1.5 Conic Sections

Conic sections include four distinct shapes: circles, ellipses, parabolas, and hyperbolas. They are called conic sections because they can be formed from cutting a cone in different ways.



1.5.1 Circles

The standard equation for a circle is $(x + h)^2 + (y - k)^2 = r^2$ where the point (h, k) is where the center of the circle is located and r is the radius of the circle.
If given the equation of a circle in expanded form, you can complete the square for both x and y to change it to standard form.

Ex: $x^2 - 2x + y^2 + 4y - 4 = 0$

$(x^2 + 2x + 1) + (y^2 + 4y + 4) - 4 = 1 + 4$

$(x - 1)^2 + (y + 2)^2 = 9$

1.5.2 Ellipses

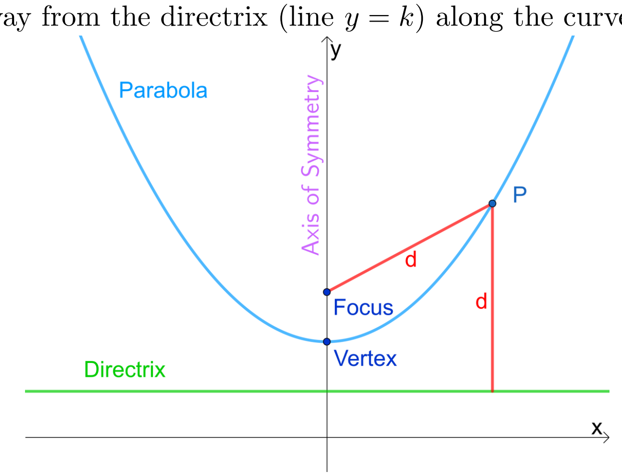
An ellipse is a circular object with two radii of different length.
Terminology:

- The major axis is the longer axis (twice the length of the longer radius)
- The minor axis is the shorter axis (twice the length of the shorter radius)
- The vertices are the two points on either end of the major axis
- The co-vertices are the two points on either end of the minor axis
- The foci of an ellipse are two whose sum of distances from any of point on the ellipse is always the same. They lie on the major axis and are equidistant away from the origin.
- The distance between each focus and the center is called the focal length f where $f^2 = p^2 - q^2$ where p is the major radius and q is the minor radius.

The standard equation for an ellipse is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ where t (h, k) is the center point, a is the horizontal axis, b is the vertical radius.

1.5.3 Parabolas

Parabolas can be viewed as the set of all points whose distance from a certain point (the focus) is equal to their distance from a certain line called the directrix. The focus (point (a, b)) will always be the same distance away from the directrix (line $y = k$) along the curve of the parabola.



The equation of a parabola can be derived from the location of the focus and directrix using the equation $y = \frac{(x-a)^2}{2(b-k)} + \frac{b+k}{2}$

The location of the focus and directrix can be found from the equation of a quadratic by working backwards with the same formula.

Ex: $y = 2(x-1)^2 + 4 \Rightarrow a = 1$

$$2 = \frac{1}{2(b-k)}$$

$$b - k = \frac{1}{4}$$

$$\frac{1}{4} = 8 - 2k$$

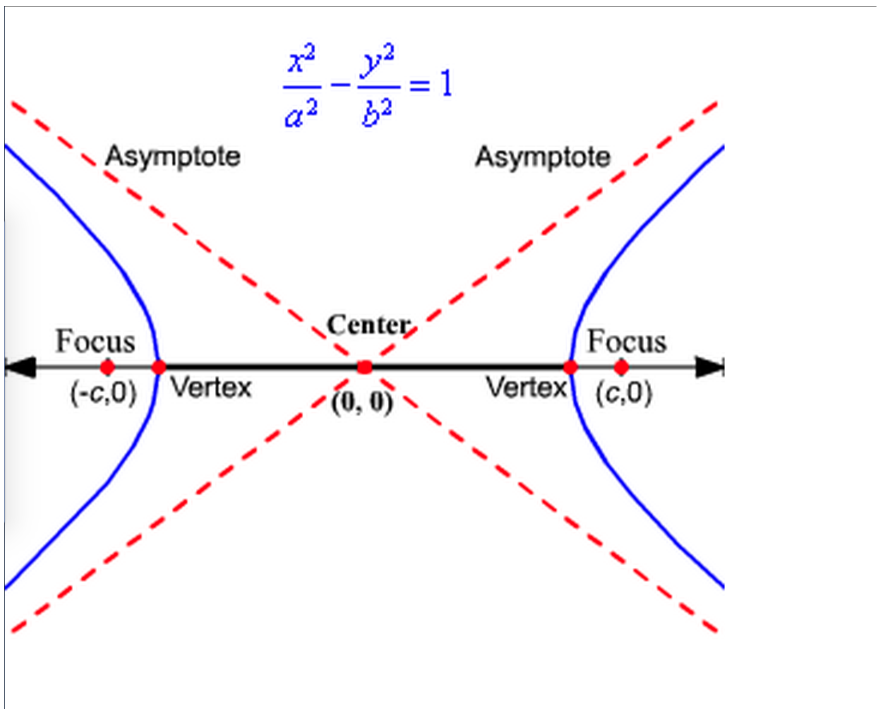
$$4 = \frac{b+k}{2}$$

$$b = 8 - k$$

$$k = \frac{31}{8}$$
$$8 - \frac{31}{8} = b$$
$$b = \frac{33}{8}$$

1.5.4 Hyperbolas

There are two cases of hyperbolas defined by two slightly different equations: $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ or $\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1$
If the x-term is positive, the hyperbola will open up to the side. if the y-term is positive, the hyperbola will open up.
a is always associated with the x-term and b is always associated with the y-term.
The point (h, k) will be the center of the hyperbolas and the point a or b determines how far away the vertices of the hyperbola will be from its center. Each hyperbola will have 2 diagonal asymptotes that meet in its center. The slopes of the asymptotes will always be $m_{asym} = \pm\frac{b}{a}$



The difference from each focus to a point on the hyperbola will always be a constant $|d_1 - d_2| = C$. If it opens horizontally, it will be $|d_1 - d_2| = 2a$ and if it opens vertically, it will be $|d_1 - d_2| = 2b$. The distance from each focus to the center of the hyperbola is $f^2 = a^2 + b^2$.