Math Notes

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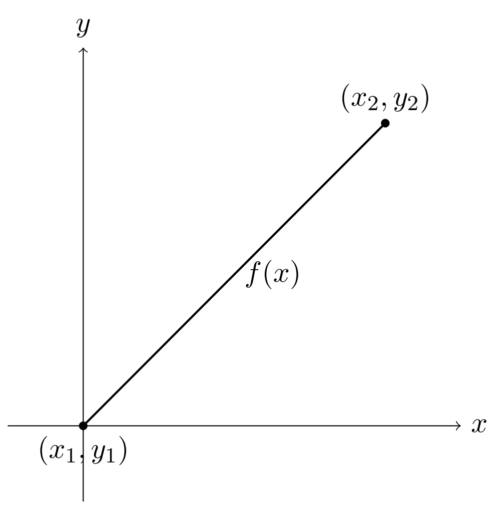
1 Variational Calculus

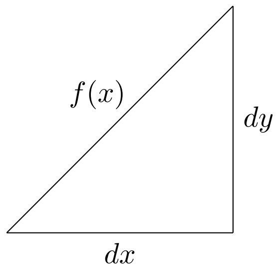
1.1 Variational Derivatives

1.1.1 Definition of the Variational Derivative

An action is a function of a function called a functional. This is best seen by example.
Using variational calculus we can prove that the shortest distance between two points is a straight line.
Let us define two points in $\mathbb{R}^2$, (x_1, y_1) and (x_2, y_2) and some function $f(x)$ that connects these points. This can be generalized to higher dimensions, but for simplicity we will stick to two dimensions for now.



We can define an infinite possible functions that connect these two points. We want to find the specific function that minimizes the distance between these two points. So we will want a function with a minimum arc length.



We can use the Pythagorean theorem to find the arc length of this function.

$$\frac{dy}{dx} = f'(x) \Rightarrow dy = f'(x)dx$$
$$ds^2 = dx^2 + dy^2$$
$$ds = \sqrt{dx^2 + dy^2}$$
$$ds = \sqrt{dx^2 + (f'(x)dx)^2}$$
$$ds = \sqrt{1 + f'(x)^2}dx$$
$$S = \int_{x_1}^{x_2} dx\sqrt{1 + f'(x)^2}, \ \begin{cases} f(x_1) = y_1 \\ f(x_2) = y_2 \end{cases}$$

We define S, the arc length, as the action that we wish to minimize. It is also considered a functional as it is defiend as $S(f(x))$.
We define $\delta f(x)$ as some small change (or wiggle) in the function $f(x)$. It is important to note that $\delta f(x)$ is not some opperation on the original function $f(x)$ but is rather some new function of x we call $\delta f(x)$ that contains very slight variations. We can then define the variation of the action as follows.
Recall that for a regular function $f(x)$, the derivative is defined as

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

We can define the variational derivative in a similar manner.

$$\frac{\delta S}{\delta f(x)} = \lim_{\delta f(x) \to 0} \frac{S(f(x) + \delta f(x)) - S(f(x))}{\delta f(x)}$$

Note that we usually just deal with the numerator of this expression and call it δS. We can also introduce a small parameter ϵ to make the expression easier to follow.

$$\delta S = \lim_{\epsilon \to 0} \frac{S(f(x) + \epsilon\delta f(x)) - S(f(x))}{\epsilon}$$

We can now use this definition to find the variation of the action for our example.

$$\delta S = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{x_1}^{x_2} dx\sqrt{1 + (f'(x) + \epsilon\delta f'(x))^2} - \int_{x_1}^{x_2} dx\sqrt{1 + f'(x)^2}$$
$$\delta S = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{x_1}^{x_2} dx \left(\sqrt{1 + (f'(x) + \epsilon\delta f'(x))^2} - \sqrt{1 + f'(x)^2}\right)$$

$$\delta S = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{x_1}^{x_2} dx \left(\sqrt{1 + f'(x)^2 + 2f'(x)\epsilon\delta f'(x) + \epsilon^2 \delta f'(x)^2} - \sqrt{1 + f'(x)^2} \right)$$

Because ϵ is a very small we can perform a Taylor expansion on the square root.

$$\begin{aligned}
&\sqrt{1+x}\bigg|_{x\approx 0} = 1 + \frac{x}{2} + \mathcal{O}(x^2) \\
&\sqrt{1 + f'(x)^2 + 2f'(x)\epsilon\delta f(x) + \epsilon^2 f'(x)^2} = \sqrt{1 + f'(x)^2}\sqrt{1 + \frac{2f'(x)\epsilon\delta f(x) + \epsilon^2 f'(x)^2}{1 + f'(x)^2}} \\
&= \sqrt{1 + f'(x)^2}\left(1 + \frac{1}{2}\frac{2f'(x)\epsilon\delta f(x) + \epsilon^2 f'(x)^2}{1 + f'(x)^2}\right) + \mathcal{O}\left(\left(\frac{2f'(x)\epsilon\delta f(x) + \epsilon^2 f'(x)^2}{1 + f'(x)^2}\right)^2\right) \\
&= \sqrt{1 + f'(x)^2}\left(1 + \epsilon\frac{f'(x)\delta f(x)}{1 + f'(x)^2} + \mathcal{O}(\epsilon^2)\right) \\
&= \sqrt{1 + f'(x)^2} + \epsilon\frac{f'(x)\delta f(x)}{\sqrt{1 + f'(x)^2}} + \mathcal{O}(\epsilon^2)
\end{aligned}$$

Plugging this back into the original expression we get

$$\begin{aligned}
\delta S &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{x_1}^{x_2} dx \left(\sqrt{1 + f'(x)^2} + \epsilon\frac{f'(x)\delta f'(x)}{\sqrt{1 + f'(x)^2}} + \mathcal{O}(\epsilon^2) - \sqrt{1 + f'(x)^2} \right) \\
\delta S &= \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{x_1}^{x_2} dx \left(\epsilon\frac{f'(x)\delta f'(x)}{\sqrt{1 + f'(x)^2}} + \mathcal{O}(\epsilon^2) \right) \\
\delta S &= \lim_{\epsilon \to 0} \int_{x_1}^{x_2} dx \left(\frac{f'(x)\delta f'(x)}{\sqrt{1 + f'(x)^2}} + \mathcal{O}(\epsilon) \right) \\
\delta S &= \lim_{\epsilon \to 0} \int_{x_1}^{x_2} dx \frac{f'(x)\delta f'(x)}{\sqrt{1 + f'(x)^2}} + \lim_{\epsilon \to 0} \int_{x_1}^{x_2} dx \mathcal{O}(\epsilon)
\end{aligned}$$

If we take the limit as $\epsilon \to 0$ then the $\mathcal{O}(\epsilon)$ term goes to zero and we are left with

$$\delta S = \int_{x_1}^{x_2} dx \frac{f'(x)\delta f'(x)}{\sqrt{1 + f'(x)^2}}$$

Using integration by parts we can rewrite this as

$$\delta S = \delta f(x) \frac{f'(x)}{\sqrt{1 + f'(x)^2}}\bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} dx \delta x \left(\frac{f'(x)}{\sqrt{1 + f'(x)^2}} \right)'$$

Note that the first term goes to zero because $\delta f(x_1) = \delta f(x_2) = 0$ as we cannot have any variation in the start and end points. We can then rewrite this as

$$\delta S = -\int_{x_1}^{x_2} dx \delta x \left(\frac{f'(x)}{\sqrt{1 + f'(x)^2}} \right)'$$

Because we are trying to find a minimum for the action, we can set $\delta S = 0$ which gives us

$$\int_{x_1}^{x_2} dx \delta x \left(\frac{f'(x)}{\sqrt{1+f'(x)^2}} \right)' = 0$$

$$\frac{d}{dx} \left(\frac{f'(x)}{\sqrt{1+f'(x)^2}} \right) = 0$$

We now have an expression only in terms of $f(x)$ which we can use to solve for the function that minimizes the action S.

$$\frac{f'(x)}{\sqrt{1+f'(x)^2}} = C$$

$$f'(x) = C\sqrt{1+f'(x)^2}$$

$$f'(x)^2 = C^2(1+f'(x)^2)$$

$$f'(x)^2 - C^2 f'(x)^2 = C^2$$

$$(1-C^2)f'(x)^2 = C^2$$

$$f'(x) = \pm\frac{C}{\sqrt{1-C^2}} = C_1$$

$$f(x) = C_1 x + C_2$$

And so we have proved that $f(x)$ is a linear function and so the shortest path between two points in $\mathbb{R}^2$ is a straight line.

1.1.2 Functions of Many Variables

Intuitively the previous example should hold in higher dimensions as well (we know this to be true in $\mathbb{R}^3$). To show this we will use the same method as before but with a function of many variables. We will start with the action