Math Notes

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Contents

1 Complex Analysis

1.1 Complex Algebra

Complex numbers arise from the roots of polynomials.
Ex: $x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm\sqrt{-1}$. This polynomial has no real roots, however, we can introduce an imaginary number i such that $i^2 = -1$. Then we will have the solution $x = \pm i$
We can introduce *complex numbers* which are numbers in the form $z = x + iy$, where x is the real part of z, $\Re(z)$, and y is the imaginary part of z, $\Im(z)$. These numbers can also be expressed in vector notation along the complex plane.

1.1.1 Complex Arithmetic

Addition, subtraction, and multiplication work the same, just with the addition of the fact $i^2 = -1$.
For division, we require what is called the conjugate.
The conjugate of a complex number is the same number, just with the sign of the imaginary component flipped.

$$\overline{z} = x - yi$$

where $\overline{z}$ is the conjugate of z.
Similarly to vectors, we can also define the modulus (length) of a complex number

$$|z|^2 = x^2 + y^2 = z \cdot \overline{z}$$

Using this, we can define the division of a complex number and also define the real and imaginary components of a complex number.
The general expression for division is:

$$\frac{u}{z} = \frac{s+it}{x+iy} = \frac{(s+it)(x-iy)}{(x+iy)(x-iy)} = \frac{u\overline{z}}{x^2+y^2} = \frac{u\overline{z}}{|z|^2}$$

The real and imaginary components can be computed as

$$\begin{aligned}
&z = x + iy \\
&\overline{z} = x - iy \\
&\Re(z) = \frac{z+\overline{z}}{2} \\
&\Im(z) = \frac{z-\overline{z}}{2i}
\end{aligned}$$

Ex1: Simplify $(1+2i)(3+i)(2-3i)$

$$\begin{aligned}
&(1+2i)(3+i)(2-3i) = (3+i+6i-2)(2-3i) = (1+7i)(2-3i) \\
&= 2 - 3i + 14i + 21 \\
&= 23 + 11i
\end{aligned}$$

Ex2: Simplify $\left(\frac{2+i}{1+i}\right)^2$

$$\begin{aligned}
&\left(\frac{2+i}{1+i}\right)^2 = \left(\frac{(2+i)(1-i)}{(1+i)(1-i)}\right)^2 = \left(\frac{2+i-2i+1}{2}\right)^2 = \left(\frac{3-i}{2}\right)^2 = \frac{9-6i-1}{4} \\
&= 2 - \frac{3}{2}i
\end{aligned}$$

Ex3: Simplify $(1+2i)^5$

$$\begin{aligned}
&(1+2i)^5 = 1^5 + 5(1)^4(2i) + 10(1)^3(2i)^2 + 10(1)^2(2i)^3 + 5(1)(2i)^4 + (2i)^5 \\
&= 1 + 10i + 10(-4) + 10(-8i) + 5(16) + 32i \\
&= 1 + 10i - 40 - 80i + 80 + 32i \\
&= 41 - 38i
\end{aligned}$$

Ex4: Prove

$$\text{if } |z| = 1 \text{ then } \Re\left(\frac{1}{1+z}\right) = \frac{1}{2}$$

Proof.

$$\Re(w) = \frac{w + \overline{w}}{2}$$
$$\overline{\frac{1}{1+z}} = \frac{1}{1+\overline{z}}$$
$$\Re\left(\frac{1}{1+z}\right) = \frac{\frac{1}{1+z} + \frac{1}{1+\overline{z}}}{2} = \frac{1+\overline{z}+1+z}{2(1+z)(1+\overline{z})} = \frac{2+z+\overline{z}}{2(1+z+\overline{z}+|z|^2)}$$
$$|z| = 1 \Rightarrow \Re\left(\frac{1}{1+z}\right) = \frac{2+z+\overline{z}}{2(2+z+\overline{z})} = \frac{1}{2}$$

□

Some properties of $\overline{z}$:

- $\overline{\overline{z}} = z$
- $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$
- $|z_1 z_2| = |z_1||z_2|$

Some common inequalities come from the triangle inequality:
$|z_1 + z_2| \leq |z_1| + |z_2|$

Proof.

$$|z_1 + z_2| \leq |z_1| + |z_2|$$
$$|z_1 + z_2| = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$$
$$\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$$
$$|z_1 + z_2| \leq |z_1| + |z_2|$$

□

$$\Rightarrow |z_1 \pm z_2| \geq ||z_1| - |z_2||$$

This leads to a general upper and lower bound that can be derived from these inequalities:

$$||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$$

1.1.2 Polar Form of Complex Numbers

Another way to represent complex numbers is through polar form. To use this we must first introduce Euler's identity:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

This then helps us with the equation for the polar form

$$z = a + ib = |z|(\cos\varphi + i\sin\varphi) = |z|e^{i\varphi}$$

For $z = re^{i\varphi}$ We call r the magnitude of the complex number and φ is the argument. This can be especially useful for simplifying some complex numbers.
Ex:

$$\left|\frac{(1+\sqrt{3}i)^{100}}{(\sqrt{3}-i)^{100}}\right| = \frac{|1+\sqrt{3}i|^{100}}{|\sqrt{3}-i|^{100}} = \frac{2^{100}}{2^{100}} = 1$$

Note that because sinusoidal functions are periodic every 2π this means that there infinite ways to express a function in polar coordinates.

$$e^{i2\pi k} = 1,\ k \in \mathbb{Z} \Rightarrow z = re^{i(\varphi+2\pi k)}$$

To get around the issue of having infinite possible polar forms for every complex number we define what's called the principal argument to be

$$\mathrm{Arg}(z) = \varphi \in (-\pi, \pi]$$

We define the regular argument to be

$$\arg(z) = \mathrm{Arg}(z) + 2k\pi,\ k \in \mathbb{Z}$$

Note that $\mathrm{Arg}(z)$ is singular valued while $\arg(z)$ is multi-valued.
We can define $\mathrm{Arg}(z)$ in terms of the real and imaginary parts (x and y) as

$$\mathrm{Arg}(z) = \arctan(\tfrac{y}{x}) \pm k\pi$$

What it is specifically depends on what quadrant of the complex plane the point lies in.

- QI: $\mathrm{Arg}(z) = \arctan(\frac{y}{x})$
- QII: $\mathrm{Arg}(z) = \arctan(\frac{y}{x}) + \pi$
- QIII: $\mathrm{Arg}(z) = \arctan(\frac{y}{x}) - \pi$
- QIV: $\mathrm{Arg}(z) = \arctan(\frac{y}{x})$

Ex:

$$\mathrm{Arg}(-1-\sqrt{3}i) = \arctan(\sqrt{3}) - \pi = -\frac{2\pi}{3}$$

Ex2:

$$\arg(1-\sqrt{3}i)$$
$$\mathrm{Arg}(1-\sqrt{3}i) = -\frac{\pi}{3}$$
$$\arg(1-\sqrt{3}) = -\frac{\pi}{3} + 2k\pi,\ k \in \mathbb{Z}$$

Ex3:

$$\arg(-1+2i)$$

$$\text{Arg}(-1+2i) = \arctan(-2) + \pi$$
$$\arg(-1+2i) = \pi - \arctan(2) + 2k\pi$$

Ex4: Simplify

$$z = -3 + 3i$$
$$\text{Arg}(z) = \arctan\left(\frac{3}{-3}\right) + \pi = \frac{3\pi}{4}$$
$$|z| = 3\sqrt{2}$$
$$z = 3\sqrt{2}e^{\frac{3\pi}{4}i}$$

Ex5: Simplify

$$z = -3 - 3i$$
$$\text{Arg}(z) = \arctan\left(\frac{-3}{-3}\right) - \pi = -\frac{3\pi}{4}$$
$$|z| = 3\sqrt{2}$$
$$z = 3\sqrt{2}e^{-\frac{3\pi}{4}i}$$

Ex6: Simplify

$$z = \frac{1-i}{-\sqrt{3}+i} = \frac{u}{v}$$
$$\arg(z) = \arg(u) - \arg(v) = \arctan\left(\frac{-1}{1}\right) - \left(\arctan\left(\frac{1}{-\sqrt{3}}\right) + \pi\right) + 2k\pi$$
$$= -\frac{\pi}{4} - \frac{5\pi}{6} + 2k\pi = -\frac{13\pi}{12} + 2k\pi$$
$$\text{Arg}(z) = \frac{11\pi}{12}$$
$$|z| = \frac{|u|}{|v|} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$
$$z = \frac{e^{\frac{11\pi}{12}i}}{\sqrt{2}}$$

Ex7: Simplify

$$z = (\sqrt{3} - i)^2 = w^2$$
$$\arg(w) = \arctan\left(\frac{-1}{\sqrt{3}}\right) + 2k\pi = -\frac{\pi}{6}$$
$$\Rightarrow \arg(z) = 2\arg(w) = -\frac{\pi}{3} + 4k\pi$$
$$\text{Arg}(z) = -\frac{\pi}{3}$$
$$|w| = 2 \Rightarrow |z| = |w|^2 = 4$$
$$z = 4e^{-\frac{\pi}{3}i}$$

Ex8: Solve for all values of z

$$\begin{aligned}
&e^z = -1 - \sqrt{3}i \\
&e^z = 2e^{-i\frac{2\pi}{3} + 2\pi ik} = e^{\ln 2 - i\frac{2\pi}{3} + 2\pi ik} \\
&z = \ln 2 - i\frac{2\pi}{3} + 2\pi ik, \ \forall k \in \mathbb{Z}
\end{aligned}$$

Properties of $\operatorname{Arg}(z)$ and $\arg(z)$

- $\operatorname{Arg}(z_1 z_2) \neq \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$

  Proof. Proof by contradiction: assume that $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$ is true. Take $z_1 = z_2 = -1$.

$$\begin{aligned}
&\operatorname{Arg}(z_1) = \operatorname{Arg}(z_2) = \pi \\
&\Rightarrow \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = 2\pi \\
&\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(1) = 0 \\
&\operatorname{Arg}(z_1 z_2) \neq \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) \ \forall z_1, z_2 \neq 0 \in \mathbb{C}
\end{aligned}$$

$\square$

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\operatorname{Arg}(\overline{z}) \neq -\operatorname{Arg}(z)$

  Proof. Proof by contradiction: assume that $\operatorname{Arg}(\overline{z}) = -\operatorname{Arg}(z)$ is true. Take $z = -1$

$$\begin{aligned}
&\overline{z} = z = -1 \\
&\operatorname{Arg}(z) = \pi \\
&\operatorname{Arg}(\overline{z}) = \pi \\
&-\operatorname{Arg}(z) = -\pi \\
&\Rightarrow \operatorname{Arg}(\overline{z}) \neq -\operatorname{Arg}(z) \ \forall z \in \mathbb{C}
\end{aligned}$$

$\square$

- $\arg(z) = -\arg(\overline{z})$

  Proof.

$$\begin{aligned}
&z = |z|e^{i\arg(z)} \ \forall z \in \mathbb{C} \\
&\overline{z} = |z|e^{-i\arg(z)} \\
&\Rightarrow \arg(\overline{z}) = -\arg(z)
\end{aligned}$$

$\square$

1.1.3 De Moirre's Formula

Using Euler's identity we can derive a powerful formula called De Moirre's Formula as follows:

$$
\begin{aligned}
&e^{iN\varphi} = \cos(N\varphi) + i\sin(N\varphi) \\
&e^{iN\varphi} = (e^{i\varphi})^N = (\cos\varphi + i\sin\varphi)^N \\
&(\cos\varphi + i\sin\varphi)^N = \cos(N\varphi) + i\sin(N\varphi)
\end{aligned}
$$

Applications of De Moirre's Formula:
Binomial expansion:

$$
\begin{aligned}
&N = 2: \\
&\cos(2\theta) = \cos^2\theta - \sin^2\theta \\
&\sin(2\theta) = 2\cos\theta\sin\theta \\
&N = 3: \\
&(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 \\
&\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta \\
&\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta
\end{aligned}
$$

Ex: Prove

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$$

Proof. Using De Moivre's formula with $N = 3$

$$
\begin{aligned}
&(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta) \\
&(\cos\theta)^3 + 3(\cos\theta)^2(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta) \\
&\cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta) = \cos(3\theta) + i\sin(3\theta) \\
&\Im\left\{\cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)\right\} = \Im\left\{\cos(3\theta) + i\sin(3\theta)\right\} \\
&3\cos^2\theta\sin\theta - \sin^3\theta = \sin(3\theta) \\
&\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \sin^2\theta \\
&3(1 - \sin^2\theta)\sin\theta - \sin^3\theta = \sin(3\theta) \\
&3\sin\theta - 3\sin^3\theta - \sin^3\theta = \sin(3\theta) \\
&3\sin\theta - 4\sin^3\theta = \sin(3\theta)
\end{aligned}
$$

□

Computing trigonometric integrals:
Ex:

$$
\begin{aligned}
&\int_0^{2\pi} \cos^8\varphi d\varphi \\
&e^{i\varphi} = \cos\varphi + i\sin\varphi \\
&e^{-i\varphi} = \cos\varphi - i\sin\varphi
\end{aligned}
$$

$$\cos\varphi = \frac{e^{i\varphi}+e^{-i\varphi}}{2}$$
$$\sin\varphi = \frac{e^{i\varphi}-e^{-i\varphi}}{2i}$$
$$\int_0^{2\pi}\left(\frac{e^{i\varphi}+e^{-i\varphi}}{2}\right)^8 d\varphi = \frac{1}{2^8}\int_0^{2\pi}\left(e^{i\varphi}+e^{-i\varphi}\right)^8 d\varphi$$
$$= \frac{1}{2^8}\int_0^{2\pi}\left(e^{i8\varphi}+{}_1C_8 e^{i7\varphi}e^{-i\varphi}+\cdots+{}_7C_8 e^{i\varphi}e^{-i7\varphi}+e^{-i8\varphi}\right)d\varphi$$
$$= \frac{1}{2^8}\left(0+\cdots+{}_4C_8 2\pi+\cdots+0\right) = \frac{{}_4C_8}{2^7}\pi$$

Ex2:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$$
$$\int_0^{2\pi}\cos^6\theta d\theta = \int_0^{2\pi}\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)$$
$$= \frac{1}{2^6}\int_0^{2\pi}\sum_{k=0}^{6}\binom{k}{6}e^{i\theta k}e^{-i\theta(6-k)}d\theta$$
$$= \frac{1}{2^6}\sum_{k=0}^{6}\binom{k}{6}\int_0^{2\pi}e^{i\theta(2k-6)}d\theta$$
$$\int_0^{2\pi}e^{ik\theta}d\theta = \left.\frac{e^{ik\theta}}{ik}\right|_0^{2\pi} = \frac{e^{2\pi ik}-1}{ik}$$
$$e^{2\pi ik} = 1,\ k\in\mathbb{Z} \Rightarrow \int_0^{2\pi}e^{ik\theta}d\theta = 0,\ k\neq 0\in\mathbb{Z}$$
$$\Rightarrow \int_0^{2\pi}\cos^6\theta d\theta = \frac{1}{2^6}\binom{3}{6}\int_0^{2\pi}d\theta = \frac{(20)(2\pi)}{2^6} = \frac{5\pi}{8}$$

Ex3:

$$\int_0^{2\pi}\sin^6(2\theta)d\theta$$
$$\sin(2\theta) = \frac{e^{2i\theta}-e^{-2i\theta}}{2i}$$
$$\int_0^{2\pi}\sin^6(2\theta)d\theta = \int_0^{2\pi}\left(\frac{e^{2i\theta}-e^{-2i\theta}}{2i}\right)^6 d\theta$$
$$= \frac{1}{(2i)^6}\int_0^{2\pi}\sum_{k=0}^{6}\binom{k}{6}(-1)^{6-k}e^{2ik\theta}e^{-2i\theta(6-k)}d\theta$$
$$= -\frac{1}{2^6}\sum_{k=0}^{6}\binom{k}{6}(-1)^{6-k}\int_0^{2\pi}e^{i\theta(4k-12)}d\theta$$
$$\int_0^{2\pi}e^{ik\theta}d\theta = \left.\frac{e^{ik\theta}}{ik}\right|_0^{2\pi} = \frac{e^{2\pi ik}-1}{ik}$$

$$e^{2\pi ik} = 1,\ k \in \mathbb{Z} \Rightarrow \int_0^{2\pi} e^{ik\theta} d\theta = 0,\ k \neq 0 \in \mathbb{Z}$$
$$\Rightarrow \int_0^{2\pi} \sin^6(2\theta) d\theta = -\frac{1}{2^6}\binom{3}{6}(-1)^3 \int_0^{2\pi} d\theta$$
$$= \frac{5\pi}{8}$$

Ex4: Prove

$$\sum_{k=0}^{n} \cos(k\theta) = \frac{1}{2} + \frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}$$

Proof. De Moivre's formula states

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

If we take the conjugate of both sides we get

$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$$

Summing these two equations gives

$$(\cos\theta + i\sin\theta)^k + (\cos\theta - i\sin\theta)^k = 2\cos(k\theta)$$
$$\cos\theta \pm i\sin\theta = e^{\pm i\theta}$$
$$2\cos(k\theta) = (e^{i\theta})^k + (e^{-i\theta})^k$$

We can sum both sides of this to get

$$2\sum_{k=0}^{n} \cos(k\theta) = \sum_{k=0}^{n} (e^{i\theta})^k + \sum_{k=0}^{n} (e^{-i\theta})^k$$

The formula for the geometric sum is

$$\sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}$$

Applying this we get

$$2\sum_{k=0}^{n} \cos(k\theta) = \frac{1-(e^{i\theta})^{n+1}}{1-e^{i\theta}} + \frac{1-(e^{-i\theta})^{n+1}}{1-e^{-i\theta}}$$
$$2\sum_{k=0}^{n} \cos(k\theta) = \frac{(1-e^{i\theta(n+1)})(1-e^{-i\theta}) + (1-e^{-i\theta(n+1)})(1-e^{i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})}$$
$$2\sum_{k=0}^{n} \cos(k\theta) = \frac{1-e^{i\theta(n+1)} - e^{i\theta} + e^{i\theta n} + 1 - e^{-i\theta(n+1)} - e^{-i\theta} + e^{-i\theta n}}{1-e^{i\theta}-e^{-i\theta}+1}$$
$$2\sum_{k=0}^{n} \cos(k\theta) = 1 + \frac{e^{i\theta n} + e^{-i\theta n} - e^{i\theta(n+1)} - e^{-i\theta(n+1)}}{2-e^{i\theta}-e^{-i\theta}}$$

$$
\begin{aligned}
2\sum_{k=0}^{n}\cos(k\theta) &= 1+\frac{e^{i\frac{\theta}{2}}\left(e^{-i\theta(n+\frac{1}{2})}-e^{i\theta(n+\frac{1}{2})}\right)+e^{-i\frac{\theta}{2}}\left(e^{i\theta(n+\frac{1}{2})}-e^{i\theta(n+\frac{1}{2})}\right)}{e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}\right)+e^{-i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}\right)}\\
2\sum_{k=0}^{n}\cos(k\theta) &= 1+\frac{\left(e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}\right)\left(e^{i\theta(n+\frac{1}{2})}-e^{-i\theta(n+\frac{1}{2})}\right)}{-\left(e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}\right)^2}\\
2\sum_{k=0}^{n}\cos(k\theta) &= 1+\frac{e^{i\theta(n+\frac{1}{2})}-e^{-i\theta(n+\frac{1}{2})}}{e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}}\\
2\sum_{k=0}^{n}\cos(k\theta) &= 1+\frac{\frac{1}{2i}\left(e^{i\theta(n+\frac{1}{2})}-e^{-i\theta(n+\frac{1}{2})}\right)}{\frac{1}{2i}\left(e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}\right)}\\
\sin(x) &= \frac{e^{ix}-e^{-ix}}{2i}\\
2\sum_{k=0}^{n}\cos(k\theta) &= 1+\frac{\sin\left(\theta(n+\frac{1}{2})\right)}{\sin\left(\frac{\theta}{2}\right)}\\
\sum_{k=0}^{n}\cos(k\theta) &= \frac{1}{2}+\frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}
\end{aligned}
$$

□

1.1.4 Geometry in the Complex Plane

Using the notation such that $z = x + iy$ where $\Re(z) = x$ and $\Im(z) = y$ we can define a circle in the complex plane as

$$(x-x_0)^2+(y-y_0)^2 = r_0^2$$

This is analogous to writing

$$|z-z_0| = r_0$$

The two can be related as follows:

$$
\begin{aligned}
|z-z_0| &= r_0\\
|x+iy-x_0-iy_0| &= r_0\\
\sqrt{(x-x_0)^2+(y-y_0)^2} &= r_0\\
(x-x_0)^2+(y-y_0)^2 &= r_0^2
\end{aligned}
$$

Ex: describe the circle formed by $2|z| = |z+1|$

$$
\begin{aligned}
2|z| &= |z+1|\\
4|z|^2 &= |z+1|^2\\
4x^2+4y^2 &= (x+1)^2+y^2
\end{aligned}
$$

$$4x^2 + 4y^2 = x^2 + 2x + 1 + y^2$$
$$3x^2 + 3y^2 - 2x - 1 = 0$$
$$3x^2 - 2x + \frac{1}{3} + 3y^2 - \frac{4}{3} = 0$$
$$3\left(x - \frac{1}{3}\right)^2 + 3y^2 = \frac{4}{3}$$
$$\left(x - \frac{1}{3}\right)^2 + y^2 = \frac{4}{9}$$

A line in the complex plane can be written as

$$ax + by = c \longleftrightarrow a\frac{z + \overline{z}}{2} + b\frac{z - \overline{z}}{2i} = c$$

Ex: describe the line formed by $|z - 1 + i| = |z - 2i|$

$$|z - 1 + i| = |z - 2i|$$
$$|z - 1 + i|^2 = |z - 2i|^2$$
$$(x - 1)^2 + (y + 1)^2 = x^2 + (y - 2)^2$$
$$x^2 - 2x + 1 + y^2 + 2y + 1 = x^2 + y^2 - 4y + 4$$
$$-2x + 2 + 2y = -4y + 4$$
$$6y = 2x + 2$$
$$y = \frac{1}{3}(x + 1)$$

We can define an ellipse in the complex plane as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If we have $a > b$ then we will have a horizontal ellipse and if $b > a$ then we will have a vertical ellipse.
Assuming that $a > b$ then we can define the foci points to be at

$$+F = (\sqrt{a^2 - b^2}, 0)$$
$$-F = (-\sqrt{a^2 - b^2}, 0)$$

The equation of an ellipse can also be described by

$$|z - F| + |z + F| = 2a$$

Ex: describe the ellipse formed by $|z - 1| + |z + 1| = 4$

$$|z - 1| + |z + 1| = 4$$
$$|z - F| + |z + F| = 2a \Rightarrow 2a = 4 \Rightarrow a = 2$$
$$F = \sqrt{a^2 - b^2} = 1 \Rightarrow 1 = 4 - b^2 \Rightarrow b^2 = 3$$

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

Ex2: describe the ellipse formed by $|z-1| + |z+3| = 6$

$$|z-1| + |z+3| = 6$$

note that this ellipse is not centered at the origin so we need to shift it

$$|z+1-2| + |z+1+2| = 6$$

$$2a = 6 \Rightarrow a = 3$$

$$F = 2 = \sqrt{a^2 - b^2} \Rightarrow 4 = 9 - b^2 \Rightarrow b^2 = 5$$

$$\frac{(x+1)^2}{9} + \frac{y^2}{5} = 1$$

1.1.5 Roots of a Complex Number

Given $z_0 = r_0 e^{i\varphi_0}$, what is $z_0^{\frac{1}{n}}$?

If we let $w = z_0^{\frac{1}{n}}$ then $w^n = z_0$

$$w = re^{i\varphi},\ w_n = r^n e^{in\varphi}$$

$$w^n = z_0 \Rightarrow r^n e^{in\varphi} = r_0 e^{i\varphi_0}$$

$$\Rightarrow r^n = r_0 \Rightarrow r_0^{\frac{1}{n}}$$

$$e^{in\varphi} = e^{i\varphi_0} \Rightarrow n\varphi = \varphi_0 + 2k\pi$$

$$\varphi = \frac{\varphi}{n} + \frac{2k\pi}{n}$$

So all solutions to $w^n = z_0$ are given by

$$w = r_0^{\frac{1}{n}} e^{i\left(\frac{\varphi_0}{n} + \frac{2k\pi}{n}\right)},\ k \in \mathbb{Z}$$

If we normalize $\varphi_0 = \mathrm{Arg}(z_0)$ then k will be in the range $k \in \{0, 1, \ldots, n-1\}$.

Note that the expression

$$w = r_0^{\frac{1}{n}} e^{i\left(\frac{\varphi_0}{n} + \frac{2k\pi}{n}\right)},\ k \in \mathbb{Z}$$

is multi-valued. If we want to avoid this, we can take what's called the principal value which is the value when $k = 0$:

$$z_0^{\frac{1}{n}} = r_0^{\frac{1}{n}} e^{i\frac{\varphi_0}{n}}$$

Ex: Compute $(-1)^{\frac{1}{2}}$

$$z_0 = -1 = 1^{\frac{1}{2}} e^{i\left(\frac{\pi}{2} + \frac{2k\pi}{2}\right)} = e^{i\left(\frac{\pi}{2} + k\pi\right)} = \left\{\ldots, e^{-i\frac{3\pi}{2}}, e^{-i\frac{\pi}{2}}, e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{2}}, e^{i\frac{5\pi}{2}}, \ldots\right\}$$

note that there are only 2 unique values

$$(-1)^{\frac{1}{2}} = e^{i\left(\frac{\pi}{2} + k\pi\right)},\ k \in \{0, 1\}$$

The principal value (when $k = 0$) of this equation works out to be i.
Ex2: Find all solutions to

$$z^7 = i - 1$$
$$z^7 = \sqrt{2}e^{i(\frac{3\pi}{4}+2\pi k)}$$
$$z = 2^{1/14}e^{i(\frac{3\pi}{28}+\frac{2\pi}{7}k)},\ k \in \{0, 1, 2, 3, 4, 5, 6\}$$

Ex3: Find all solutions to

$$z^5 = \frac{2i}{-1-\sqrt{3}i}$$
$$z^5 = \frac{2e^{i\frac{\pi}{2}}}{2e^{-i\frac{2\pi}{3}}} = e^{i\frac{7\pi}{6}} = e^{-i(\frac{5\pi}{6}+2\pi k)}$$
$$z = e^{-i(\frac{\pi}{6}+\frac{2\pi}{5}k)},\ k \in \{0, 1, 2, 3, 4\}$$

Ex4: Find all solutions to

$$\left(\frac{z}{z+1}\right)^2 = i$$
$$\left(\frac{z}{z+1}\right)^2 = e^{i(\frac{\pi}{2}+2\pi k)}$$
$$\frac{z}{z+1} = e^{i(\frac{\pi}{4}+\pi k)},\ k \in \{0, 1\}$$
$$e^{i(\frac{\pi}{4}+\pi k)} = \left\{\frac{1}{\sqrt{2}}(1+i), \frac{1}{\sqrt{2}}(-1-i)\right\}$$
$$z = e^{i(\frac{\pi}{4}+k\pi)}(z+1)$$
$$z = \frac{e^{i(\frac{\pi}{4}+k\pi)}}{1-e^{i(\frac{\pi}{4}+k\pi)}} = \left\{\frac{1+i}{\sqrt{2}-1-i}, \frac{-1-i}{\sqrt{2}+1+i}\right\}$$

Ex5: Find all solutions to

$$z^2 + 4iz + 1 = 0$$
$$(z^2 + 4iz - 4) + 4 + 1 = 0$$
$$(z + 2i)^2 + 5 = 0$$
$$(z + 2i)^2 = -5 = 5e^{i(\pi+2\pi k)}$$
$$z + 2i = \sqrt{5}e^{i(\frac{\pi}{2}+\pi k)} = \left\{\sqrt{5}i, -\sqrt{5}i\right\}$$
$$z = \left\{(\sqrt{5}-2)i, -(\sqrt{5}+2)i\right\}$$

Ex6: Find all solutions to $(z + 1)^4 = (1 - i)z^4$

$$(1 - i)z^4 = \sqrt{2}e^{-i\frac{\pi}{4}+2\pi k i}z^4$$
$$z + 1 = 2^{1/8}e^{-i\frac{\pi}{16}+i\frac{\pi k}{2}}z$$
$$z = \frac{-2^{1/8}}{1-e^{-i\frac{\pi}{16}+i\frac{\pi k}{2}}},\ k \in \{0, 1, 2, 3\}$$

1.2 Complex Functions

1.2.1 Mapping Properties of Simple Functions

Similar to how functions with real variables map values to a different set of values, complex functions do the same. The main difference is that with complex functions we're mapping a 2 dimensional set of inputs to a 2 dimensional set of outputs.

$$w = f(z) = u + iv$$

We define $z \in \mathcal{S}$ the image of $\mathcal{S}$ under w.
Some common mappings:

- The identity map

$$w = f(z) = z$$
$$\begin{cases} u = x \\ v = y \end{cases}$$

- Translation by z_0

$$w = f(z) = z + z_0$$
$$\begin{cases} u = x + x_0 \\ v = y + y_0 \end{cases}$$

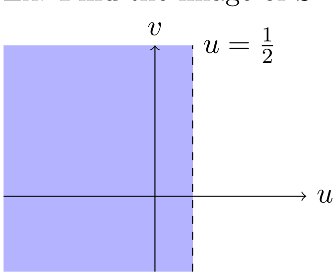
- Stretching $(a > 1)$ or contraction $(a < 1)$

$$w = f(z) = az = are^{i\varphi},\ a \in \mathbb{R}$$
$$\begin{cases} u = ax \\ v = ay \end{cases}$$

- Rotation by φ_0

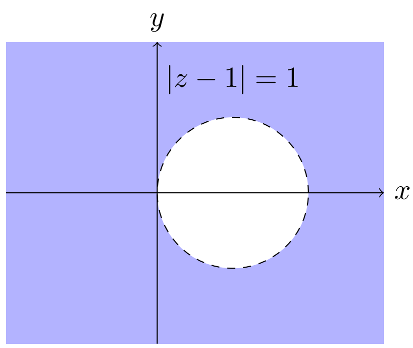
$$w = f(z) = e^{i\varphi_0} z = e^{i(\varphi + \varphi_0)}$$

Using these basic mapping principles we are able to lay the foundation for some more complicated mappings.
Ex: Find the image of $S = \{|z - 1| \geq 1\}$ under the mapping $f(z) = \frac{1}{z}$

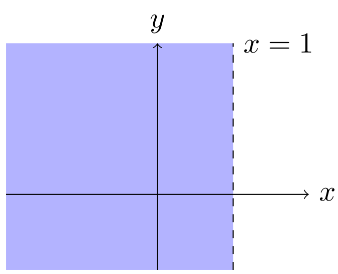


$$z = \frac{1}{w} = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$

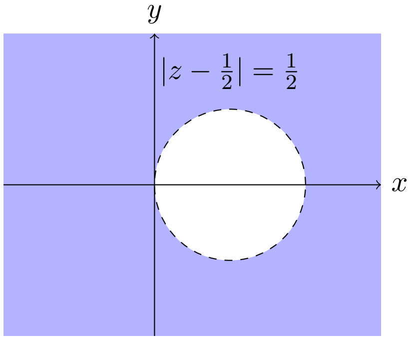
$$x = \frac{u}{u^2 + v^2}$$
$$y = -\frac{v}{u^2 + v^2}$$
$$|z - 1| \geq 1 \Rightarrow \left| \frac{1}{w} - 1 \right| \geq 1$$
$$\frac{1 - w}{w} \geq 1$$
$$|1 - w| \geq |w|$$
$$|1 - w|^2 \geq |w|^2$$
$$(1 - u)^2 + v^2 \geq u^2 + v^2$$
$$-2u + 1 \geq 0$$
$$u \leq \frac{1}{2}$$
$$S' = \left\{ u \leq \frac{1}{2} \right\}$$



Ex2: Find the image of $S = \{x \leq 1\}$ under the mapping $f(z) = \frac{1}{z}$



$$z = \frac{1}{w} = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$$
$$x = \frac{u}{u^2 + v^2}$$
$$x \leq 1 \Rightarrow \frac{u}{u^2 + v^2} \leq 1 \Rightarrow u^2 + v^2 \geq u$$
$$(u - \tfrac{1}{2})^2 + v^2 \geq \tfrac{1}{4}$$



We see from the previous two examples that circles map to lines and lines map to circles. Let's see why this is the case.

$$\begin{aligned}
&a(x^2+y^2)+bx+cy+d=0\\
&a|z|^2+b\frac{z+\bar{z}}{2}+c\frac{z-\bar{z}}{2i}+d=0
\end{aligned}$$

In the case where $a=0$, we have a line. In the case where $a\neq 0$, we have a circle.

$$\begin{aligned}
&w=\frac{1}{z}\Rightarrow z=\frac{1}{w}\\
&z\bar{z}=|z|^2=\frac{1}{|w|^2}=\frac{1}{w\bar{w}}\\
&a\frac{1}{w\bar{w}}+b\frac{z+\bar{z}}{2}+c\frac{z-\bar{z}}{2i}+d=0\\
&a\frac{1}{w\bar{w}}+\frac{b}{2}\left(\frac{1}{w}+\frac{1}{\bar{w}}\right)+\frac{c}{2i}\left(\frac{1}{w}-\frac{1}{\bar{w}}\right)+d=0\\
&a+\frac{b}{2}\left(w+\bar{w}\right)+\frac{c}{2i}\left(w-\bar{w}\right)+d\left(w\bar{w}\right)=0
\end{aligned}$$

If we have a linear transformation of the form $az+b$ it corresponds to the scaling and translation of the set only. A line will map to a line and a circle will map to a circle.
We can combine this with the $w=\frac{1}{z}$ transformation property to get a more general transformation. We call this the *Mobius transformation*:

$$f(z)=\frac{az+b}{cz+d}=\frac{a}{c}+\frac{b-\frac{ad}{c}}{cz+d}$$

where $a,b,c,d\in\mathbb{C}$
Ex: Find the mapping of $f(z)=\frac{1}{z+1}$ on the set $S=\{\Re(z)>0\}$

$$\begin{aligned}
&u+iv=\frac{1}{x+1+iy}\Rightarrow x+1+iy=\frac{1}{u+iv}\\
&x+1=\frac{u}{u^2+v^2}\\
&x>0\Rightarrow x+1>1\\
&\frac{u}{u^2+v^2}>1\Rightarrow u>u^2+v^2
\end{aligned}$$

$$u^2 + v^2 - u + \frac{1}{4} < \frac{1}{4}$$
$$\left(u - \frac{1}{2}\right)^2 + v^2 < \frac{1}{4}$$
$$S' = \left\{ w = u + iv \middle| \left(u - \frac{1}{2}\right)^2 + v^2 < \left(\frac{1}{2}\right)^2 \right\}$$

Ex2: Find the mapping of $f(z) = \frac{z-i}{z+i}$ on $S = \{|z| < 3\}$

$$wz + iw = z - i \Rightarrow z(w-1) = -i - iw \Rightarrow z = \frac{i(w+1)}{1-w}$$
$$|z| = \frac{|w+1|}{|w-1|} < 3$$
$$|w+1| < 3|w-1| \Rightarrow |w+1|^2 < 9|w-1|^2$$
$$(u+1)^2 + v^2 < 9(u-1)^2 + 9v^2$$
$$u^2 + 2u + 1 + v^2 < 9u^2 - 18u + 9 + 9v^2$$
$$0 < 8u^2 - 20u + 8 + 8v^2 \Rightarrow 0 < u^2 - \frac{5}{2}u + 1 + v^2$$
$$\frac{9}{16} < u^2 - \frac{5}{2}u + \frac{25}{16} + v^2$$
$$\frac{9}{16} < \left(u - \frac{5}{4}\right)^2 + v^2$$
$$S' = \left\{ w = u + iv \middle| \left(u - \frac{5}{4}\right)^2 + v^2 > \left(\frac{3}{4}\right)^2 \right\}$$

Another common mapping is the $f(z) = z^2$ or more generally $f(z) = z^n$ mapping. For $w = z^2$,

$$w = z^2 = r^2 e^{2i\varphi} \Rightarrow \begin{cases} |w| = |z|^2 \\ \arg(w) = 2\arg(z) \end{cases}$$

This mapping scales the magnitude but more notably, it doubles the argument. This means that the mapping of a half circle will now be a full circle.

Ex: Find the mapping of $f(z) = z^2$ on $S = \{0 \leq \Re(z) \leq 1, \Im(z) = 1\}$

$$w = x^2 + i2xy - y^2$$
$$u = x^2 - y^2 = x^2 - 1 \Rightarrow -1 \leq u \leq 0$$
$$v = 2xy = 2x \Rightarrow 0 \leq v \leq 2$$
$$S' = \{w = u + iv | -1 \leq u \leq 0,\ 0 \leq v \leq 2\}$$

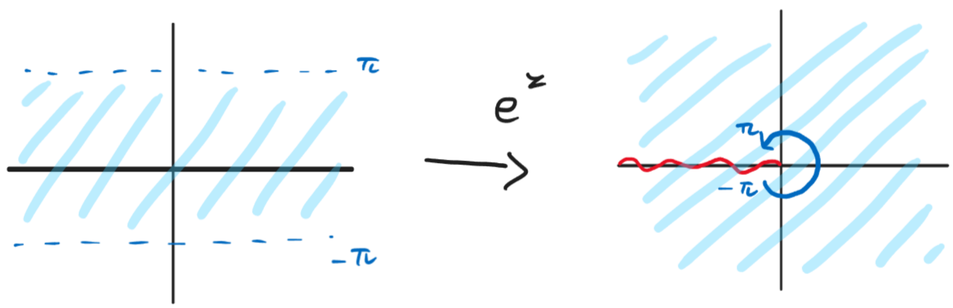
Ex2: Find the mapping of $f(z) = -2z^5$ on $S = \left\{|z| < 1, 0 < \mathrm{Arg}(z) < \frac{\pi}{2}\right\}$

$$z^5 = -\frac{w}{2} \Rightarrow |z|^5 = \frac{|w|}{2} < 1 \Rightarrow |w| < 2$$

$$
\begin{aligned}
&5\arg(z) = \arg(w) \pm \pi \\
&0 < \arg(w) \pm \pi < \frac{5\pi}{2} \\
&-\pi < \arg(w) < \frac{3\pi}{2} \\
&S' = \{|w| < 2\}
\end{aligned}
$$

Another common mapping is the $f(z) = e^z$ mapping.

$$
\begin{aligned}
&w = e^z = e^{x+iy} = e^x e^{iy} \\
&\begin{cases} |w| = e^x \\ \arg(w) = y \end{cases}
\end{aligned}
$$



This mapping has the property that the magnitude is only dependent on x and the argument is exactly y.
Ex: Find the mapping of $f(z) = e^z$ on $S = \{\Re(z) = 1\}$

$$
\begin{aligned}
&w = e^x e^{iy} \\
&|w| = e, \ \arg(w) = y \\
&S' = \{|w| = e\}
\end{aligned}
$$

Ex2: Find the mapping of $f(z) = e^z$ on $S = \left\{0 \le \Im(z) \le \frac{\pi}{4}\right\}$

$$
\begin{aligned}
&|w| = x \\
&\arg(w) = y \Rightarrow 0 \le \arg(w) \le \frac{\pi}{4} \\
&S' = \left\{0 \le \operatorname{Arg}(w) \le \frac{\pi}{4}\right\}
\end{aligned}
$$

Ex3: Find the mapping of $f(z) = e^{iz}$ on $S = \left\{z : -\frac{\pi}{2} \le \Re(z) \le \pi, -1 \le \Im(z) \le 1\right\}$
(Note that multiplying z by i rotates it by $90°$)

$$
\begin{aligned}
&w = e^{iz} = e^{ix}e^{-y} \\
&|w| = e^{-y} \Rightarrow e^{-1} \le |w| \le e \\
&\arg(w) = x \Rightarrow -\frac{\pi}{2} \le \arg(w) \le \pi
\end{aligned}
$$

$$S' = \left\{ w \middle| e^{-1} \leq |w| \leq e, -\frac{\pi}{2} \leq \arg(w) \leq \pi \right\}$$

Ex4: Prove

$$|e^{-z^3}| \leq 1 \; \forall \left\{ z \middle| -\frac{\pi}{6} \leq \operatorname{Arg}(z) \leq \frac{\pi}{6} \right\}$$

Proof. We can express $-z^3$ as some complex number $a + ib$ where $a = \Re(-z^3)$ and $b = \Im(-z^3)$. Taking the magnitude gives

$$|e^{-z^3}| = |e^{a+ib}| = |e^a e^{ib}| = |e^a||e^{ib}| = |e^a| = |e^{\Re(-z^3)}|$$

z can be written as

$$\begin{aligned}
&z = |z|e^{i\operatorname{Arg}(z)} \\
&z^3 = |z|^3 e^{i3\operatorname{Arg}(z)} = |z|^3 \left(\cos(3\operatorname{Arg}(z)) + i\sin(3\operatorname{Arg}(z))\right) \\
&-z^3 = -|z|^3 \left(\cos(3\operatorname{Arg}(z)) + i\sin(3\operatorname{Arg}(z))\right) \\
&\Re(-z^3) = -|z|^3 \cos(3\operatorname{Arg}(z)) \\
&\operatorname{Arg}(z) \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right] \\
&\Rightarrow 3\operatorname{Arg}(z) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \\
&\cos(3\operatorname{Arg}(z)) \in [0, 1] \\
&|z|^3 \in \{x \in \mathbb{R} | x \geq 0\} \\
&\Re(-z^3) = -|z|^3 \cos(3\operatorname{Arg}(z)) \in \{x \in \mathbb{R} | x \leq 0\} \\
&e^{\Re(-z^3)} \in [0, 1] \\
&\Rightarrow |e^{-z^3}| \leq 1
\end{aligned}$$

□

1.2.2 Calculus of Complex Functions

We define the limit of a complex function to be

$$\begin{aligned}
&w = f(z) = u + iv \\
&\lim_{z \to z_0} f(z) = \lim_{(x,y) \to (x_0,y_0)} u(x, y) + i \lim_{(x,y) \to (x_0,y_0)} v(x, y)
\end{aligned}$$

Note that the notation $(x, y) \to (x_0, y_0)$ means that the limit is taken as (x, y) approaches (x_0, y_0) along *any* path.
The usual limit arithmetic rules are able to be applied as with real numbers.
In order for $\lim_{z \to z_0} f(z)$ to exist, we require that both $\lim_{(x,y) \to (x_0,y_0)} u(x, y)$ and $\lim_{(x,y) \to (x_0,y_0)} v(x, y)$ exist.
If we define $z_0 = x_0 + iy_0$ then we can define the derivative of a complex function as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

If this limit exists then the function is said to be differentiable at z_0.
Ex: $f(z) = z$

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{z_0 + \Delta z - z_0}{\Delta z} = 1$$
$$\Rightarrow f'(z_0) = 1$$

Ex2: $f(z) = \bar{z}$

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$
$$\Delta z = h_1 + ih_2 \Rightarrow \overline{\Delta z} = h_1 - ih_2$$
$$h_2 = 0: \quad \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{h_1 \to 0} \frac{h_1}{h_1} = 1$$
$$h_1 = 0: \quad \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{h_2 \to 0} \frac{-ih_2}{ih_2} = -1$$
$$\lim_{h_1 \to 0} \frac{h_1}{h_1} \neq \lim_{h_2 \to 0} \frac{-ih_2}{ih_2} \quad \therefore \text{ the derivative does not exist}$$

An easy way to determine if a function is differentiable is to use the Cauchy-Riemann equations. Any path that can be taken to approach z_0 can be written as a linear combination of the paths $\Delta z = \Delta x$ and $\Delta z = i\Delta y$ so the derivative must satisfy both of these paths.

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$
$$f(z_0) = u(x_0, y_0) + iv(x_0, y_0)$$

Define $\Delta z = \Delta x$

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{\Delta x}$$
$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$
$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0)$$

Define $\Delta z = i\Delta y$

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y}$$
$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y}$$
$$f'(z_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$$
$$\Rightarrow \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$$

Splitting the real and imaginary parts we get that the Cauchy-Riemann equations are

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

If the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous then the function is differentiable.
Some functions are not differentiable everywhere, but are differentiable at a point or a set of points.

- If $f(z)$ is differentiable everywhere in the complex plane then it is said to be **entire**.
- If $f(z)$ is differentiable in some region R then it is said to be **analytic** in R.
  (note that this region cannot be a single point, as the Cauchy-Riemann equations require the partial derivatives to be continuous)

Ex: Show using the Cauchy-Riemann equations that $f(z) = \bar{z}$ is not differentiable anywhere.

$$\overline{z} = x - iy$$
$$u_x = 1 \neq v_y = -1$$

Ex2: Show that $f(z) = z^2$ is entire

$$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy$$
$$u(x,y) = x^2 - y^2$$
$$v(x,y) = 2xy$$
$$u_x = 2x = v_y$$
$$u_y = -2y = -v_x$$

Ex3: Show that $f(z) = \bar{z}$ is differentiable but not analytic at $z_0 = 0$

$$|z|^2 + 2z = x^2 + 2x + y^2 + i2y$$
$$u_x = 2x + 2 = v_y = 2 \Rightarrow x = 0$$
$$u_y = 2y = -v_x = 0 \Rightarrow y = 0$$

differentiable but not analytic on $z = \{0\}$

1.2.3 Conformal Mappings

Using the Cauch-Riemann equations,

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

we can get the Laplacian of u and v,

$$u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$$
$$v_{xx} + v_{yy} = -(u_{yx} + u_{xy}) = 0$$

If the Laplacian of u and v are both zero then the function is said to be **harmonic**.
If $\nabla^2 u = 0$ then we can use the Cauchy-Riemann equations to find its harmonic conjugate v. Ex: Find the harmonic conjugate of $u = xy - x + y$

$$u_x = y - 1 = v_y$$

$$
\begin{aligned}
&v = \int y - 1 dy = \frac{y^2}{2} - y + h(x) \\
&u_y = x + 1 = -v_x = -h'(x) \\
&h(x) = \int -x - 1 dx = -\frac{x^2}{2} - x + C \\
&v = \frac{y^2}{2} - \frac{x^2}{2} - y - x + C
\end{aligned}
$$

Ex2: Find the harmonic conjugate of $u = \ln \sqrt{x^2 + y^2}$

$$
\begin{aligned}
&u_x = \frac{x}{x^2 + y^2} = v_y \\
&v = \int \frac{x}{x^2 + y^2} dy = \int \frac{1/x}{1 + \frac{y^2}{x^2}} dy = \arctan\left(\frac{y}{x}\right) + h(x) \\
&u_y = \frac{y}{x^2 + y^2} = -v_x = -\frac{y}{1 + \frac{y^2}{x^2}} \left(\frac{-1}{x^2}\right) - h'(x) = \frac{y}{x^2 + y^2} - h'(x) \Rightarrow h'(x) = 0 \\
&h(x) = C \\
&v = \arctan\left(\frac{y}{x}\right) + C \\
&v = \arg(z) + C
\end{aligned}
$$

Ex3: Find the harmonic conjugate of $u = \sin x \cosh y$

$$
\begin{aligned}
&u = \sin x \cosh y \\
&u_x = \cos x \cosh y = v_y \\
&v = \int \cos x \cosh y dy = \cos x \sinh y + h(x) \\
&u_y = \sin x \sinh y = -v_x = -(-\sin x \sinh y) - h'(x) \Rightarrow h'(x) = 0 \\
&h(x) = C \\
&v = \cos x \sinh y + C
\end{aligned}
$$

Another property is that

$$|f'(z)|^2 = |\nabla u|^2 = |\nabla v|^2$$

This also implies that

$$\nabla u \cdot \nabla v = 0$$

$$
\begin{aligned}
&f'(z) = u_x + iv_x = \frac{1}{i}(u_y + iv_y) \\
&|f'(z)|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2 = |\nabla u|^2 = |\nabla v|^2 \\
&\nabla u \cdot \nabla v = u_x u_y + v_x v_y = u_x v_x + (-v_x)(u_x) = 0
\end{aligned}
$$

A conformal mapping is a mapping between two regions that preserves angles. If we have some function $f(z) = u + iv$ that is analytic then the can create a function of a function as

$$\Phi(u(x, y), v(x, y)) = \phi(x, y)$$

where $\phi(x,y)$ is a conformal mapping.
A conformal mapping will have the property that

$$\phi_{xx} + \phi_{yy} = |f'(z)|^2(\Phi_{uu} + \Phi_{vv})$$

where $|f'(z)|^2$ is known as the *conformal factor*.
This can be shown as follows:

$$\begin{aligned}
&\phi_x = \Phi_u u_x + \Phi_v v_x \\
&\phi_{xx} = u_{xx}\Phi_u^2 + 2u_x v_x \Phi_u \Phi_v + v_{xx}\Phi_v^2 + \Phi_u u_{xx} + \Phi_v v_{xx} \\
&\phi_y = \Phi_u u_y + \Phi_v v_y \\
&\phi_{yy} = u_{yy}\Phi_u^2 + 2u_y v_y \Phi_u \Phi_v + v_{yy}\Phi_v^2 + \Phi_u u_{yy} + \Phi_v v_{yy} \\
&\phi_{xx} + \phi_{yy} = \Phi_u \nabla^2 u + \Phi_v \nabla^2 v + \Phi_{uu}|\nabla u|^2 + 2\Phi_u \Phi_v \nabla u \cdot \nabla v + \Phi_{vv}|\nabla v|^2 \\
&\phi_{xx} + \phi_{yy} = |f'(z)|^2(\Phi_{uu} + \Phi_{vv})
\end{aligned}$$

Ex: $f(z) = z^2$ under $\Phi(u,v) = e^u + v^2$

$$\begin{aligned}
&f(z) = x^2 - y^2 + i(2xy) \\
&u = x^2 - y^2 \\
&v = 2xy \\
&\phi(x,y) = e^{x^2-y^2} + (2xy)^2 \\
&\phi_{xx} + \phi_{yy} = 4|z|^2(e^u + 2)
\end{aligned}$$

$f(z)$ is considered a conformal mapping if f is analytic and $f'(z) \neq 0$. As a consequence, if $\phi_{xx} + \phi_{yy} = 0$ then $\Phi_{uu} + \Phi_{vv} = 0$.
If f is a conformal mapping then it will preserve angles.
If we have two curves C_1 and C_2 described by the parametrized functions $z_1(t)$ and $z_2(t)$ that intersect at a point z_0 then the angle between the two curves is given by $\theta = \arg(z_2'(t)) - \arg(z_1'(t))$.
If we then apply a conformal mapping $w = f(z)$ to the curves then we get $w_1(t) = f(z_1(t))$ and $w_2(t) = f(z_2(t))$ and the angle between the two curves is given by $\theta_w = \arg(w_2'(t)) - \arg(w_1'(t))$.

$$\begin{aligned}
&\theta_w = \arg(f'(z_2)z_2'(t)) - \arg(f'(z_1)z_1'(t)) \\
&\theta_w = \arg(f'(z_2)) - \arg(f'(z_1)) + \arg(z_2'(t)) - \arg(z_1'(t)) \\
&\arg(f'(z_1)) = \arg(f'(z_2)) \\
&\Rightarrow \theta_w = \arg(z_2'(t)) - \arg(z_1'(t)) = \theta
\end{aligned}$$

So the angle between the two curves is preserved under a conformal mapping.
Note that $f'(z) \neq 0$ is a necessary condition in this proof.
If we have a nonconformal mapping then the angle between the two curves will not be preserved. One such case is that of $f(z) = z^2$ at $z_0 = 0$. At this point, $f'(z_0) = 0$ and the angle between the two curves is doubled.

Conformal mappings also have the property that they map Neumann boundary conditions to Neumann boundary conditions.

If we let $\hat{n}$ represent the normal vector to the curve $\phi(x, y)$ then they are related by

$$\frac{\partial \phi}{\partial n} = \nabla \phi \cdot \hat{n}$$

and the mapping is similarly related brcurly

$$\frac{\partial \Phi}{\partial n'} = \nabla \Phi \cdot \hat{n}'$$
$$\frac{\partial \phi}{\partial n} = |f'(z)| \frac{\partial \Phi}{\partial n'}$$

So for Neumann bounday conditions, we will have

$$\frac{\partial \phi}{\partial n} = 0 \Rightarrow \frac{\partial \Phi}{\partial n'} = 0$$

Some examples of conformal mappings come from harmonic functions (having the property that $\nabla^2 \phi = 0$).
Some common harmonic functions are:

- $\phi = C$
- $\phi = ax + by + c$
- $\ln \sqrt{x^2 + y^2},\ \mathbb{C} \backslash 0$
- $\phi = \mathrm{Arg}(z),\ \mathbb{C} \backslash (-\infty]$
- $\phi = x^2 - y^2$

1.2.4 Conformal Mapping to Solve Laplace's Equation

Given the useful properties of matching boundary conditions, we can use conformal mappings to help us solve Laplace's equation for a given region.
Ex: Given $\nabla^2 \phi = 0$ for $1 < x^2 - y^2 < 4$ and $\phi = 1$ on $x^2 - y^2 = 1$ and $\phi = 3$ on $x^2 - y^2 = 4$, find $\phi(x, y)$.

$$\text{choose } \phi = x^2 - y^2$$
$$\text{choose } \Phi(u, v) = Au + B$$
$$\begin{cases} A(1) + B = 1 \\ A(4) + B = 3 \end{cases} \Rightarrow A = \frac{2}{3},\ B = \frac{1}{3}$$
$$\Phi(u, v) = \frac{2}{3}u + \frac{1}{3}$$
$$u = x^2 - y^2$$
$$\phi(x, y) = \frac{2}{3}(x^2 - y^2) + \frac{1}{3}$$

Ex2: Given $\nabla^2 \phi = 0$ within the circular region $\mathcal{D} = \left\{1 < x^2 + y^2 < 4\right\}$ with $\phi = 1$ on $x^2 + y^2 = 1$ and $\phi = -2$ on $x^2 + y^2 = 4$, find $\phi(x, y)$.

$$\phi(x, y) = A_1 \ln r + A_2$$

$$r = \sqrt{x^2+y^2}$$
$$r = 1: \ \phi = 1 \Rightarrow A_1 \ln(1) + A_2 = 1$$
$$r = 2: \ \phi = -2 \Rightarrow A_1 \ln(2) + A_2 = -2$$
$$\Rightarrow A_2 = 1, \ A_1 = -\frac{3}{\ln 2}$$
$$\phi(x,y) = -\frac{3}{\ln 2} \ln \sqrt{x^2+y^2} + 1$$

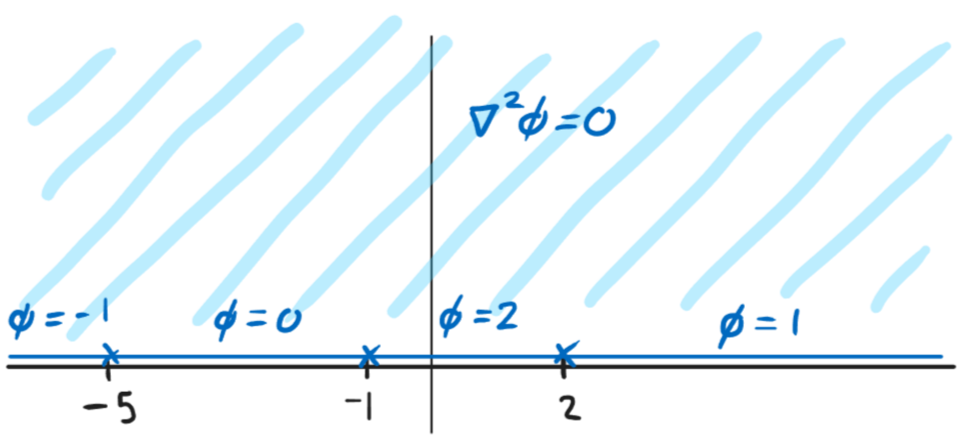
Ex3: Given $\nabla^2 = 0$ within the strip described by $\{z : -3 \leq 3\Re(z) - 4\Im(z) \leq 2\}$ with $\phi = 0$ for $\{z : -3 = 3\Re(z) - 4\Im(z)\}$ and $\phi = 4$ for $\{z : 3\Re(z) - 4\Im(z) = 2\}$ find $\phi(x,y)$

$$u = 3x - 4y$$
$$u(-3) = 0, \ u(2) = 4$$
$$\Phi = Au + B$$
$$\Phi(-3) = -3A + B = 0 \Rightarrow B = 3A$$
$$\Phi(2) = 2A + 3A = 5A = 4 \Rightarrow A = \frac{4}{5} \Rightarrow B = \frac{12}{5}$$
$$\phi(x,y) = \frac{4}{5}(3x - 4y) + \frac{12}{5}$$

Ex4: Given $\nabla^2\phi = 0$ for the upper half-plane described by $\{y > 0 \wedge x \in \mathbb{R}\}$ with the boundary conditions along the x-axis given by

$$\phi(x,0) = \begin{cases} -1 & x < -5 \\ 0 & -5 < x < -1 \\ 2 & -1 < x < 2 \\ 1 & x > 2 \end{cases}$$

find $\phi(x,y)$.



One trick to solve a problem like this is to choose a linear combination of functions of the form $\mathrm{Arg}(z - z_0)$ with a different point z_0 for every place where the boundary condition changes along the x-axis.

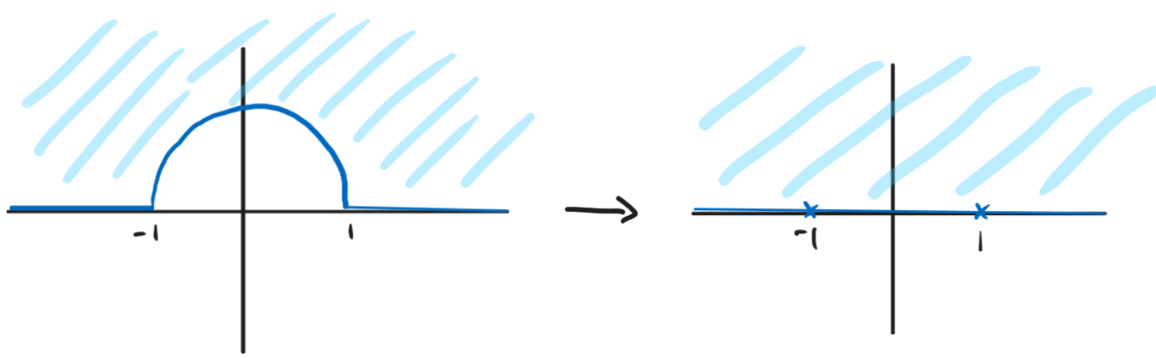
$$\phi = A_1 \,\mathrm{Arg}(z+5) + A_2 \,\mathrm{Arg}(z+1) + A_3 \,\mathrm{Arg}(z-2) + A_4$$

$$\phi(x > 2, 0) = A_4 = 1$$
$$\phi(-1 < x < 2, 0) = \pi A_3 + 1 = 2 \Rightarrow A_3 = \frac{1}{\pi}$$
$$\phi(-5 < x < -1, 0) = \pi A_2 + 1 + 1 = 0 \Rightarrow A_2 = -\frac{2}{\pi}$$
$$\phi(x < -5, 0) = \pi A_1 - 2 + 2 = -1 \Rightarrow A_1 = -\frac{1}{\pi}$$
$$\phi(x, y) = -\frac{1}{\pi}\operatorname{Arg}(z+5) - \frac{2}{\pi}\operatorname{Arg}(z+1) + \frac{1}{\pi}\operatorname{Arg}(z-2) + 1$$

We can also apply these techniques to other types of boundary conditions in some cases.
Ex5: Given $\nabla^2\phi = 0$ in the circular region $\{z : 1 \le |z| \le 2\}$ with the boundary conditions $\phi = 1$ for $|z| = 1$ and $\frac{\partial\phi}{\partial r} = 2$ for $|z| = 2$, find $\phi(x, y)$

$$\phi = A \ln r + B$$
$$\phi(1) = B = 1$$
$$\frac{\partial\phi(2)}{\partial r} = \frac{A}{2} = 2 \Rightarrow A = 4$$
$$\phi = 4 \ln r + 1$$
$$\phi(x, y) = 4 \ln \sqrt{x^2 + y^2} + 1$$

If we have a region that has a semicircle in it we can use the Joukowski mapping to transform it into a region that is easier to work with.



Ex6: Given $\nabla^2\phi = 0$ in the upper region of the plane described by $\{y > 0 \wedge x^2 + y^2 > 9\}$ with the boundary conditions $\phi = -1$ for $x < 3$, $\phi = 0$ for $x^2 + y^2 = 9$, and $\phi = 2$ for $x > 3$, find $\phi(x, y)$.

$$\begin{cases} \phi(x,0) = -1 & x < -3 \\ \phi(x,y) = 0 & x^2 + y^2 = 9 \\ \phi(x,0) = 2 & x > 3 \end{cases}$$
$$w = \frac{1}{2}\left(\frac{z}{3} + \frac{3}{z}\right) = u + iv$$
$$\Phi = A_1 \operatorname{Arg}(w+1) + A_2 \operatorname{Arg}(w-1) + A_3$$
$$\Phi(u > 1, v) = A_3 = 2$$
$$\Phi(-1 < u < 1, v) = \pi A_2 + 2 = 0 \Rightarrow A_2 = -\frac{2}{\pi}$$

$$\Phi(u < -1, v) = \pi A_1 - 2 + 2 = -1 \Rightarrow A_1 = -\frac{1}{\pi}$$
$$\Phi = -\frac{1}{\pi} \operatorname{Arg}(w+1) - \frac{2}{\pi} \operatorname{Arg}(w-1) + 2$$
$$\phi(z) = -\frac{1}{\pi} \operatorname{Arg}\left(\frac{1}{2}\left(\frac{z}{3} + \frac{3}{z}\right) + 1\right) - \frac{2}{\pi} \operatorname{Arg}\left(\frac{1}{2}\left(\frac{z}{3} + \frac{3}{z}\right) - 1\right) + 2$$

In the case that we have a semicircle not of radius 1 we can apply a scaling before the Joukowski mapping to get the correct radius.

$$w = \frac{1}{2}\left(\frac{z}{a} + \frac{a}{z}\right)$$

1.2.5 Sinusoidal Functions

If we recall Euler's formula $e^{ix} = \cos x + i \sin x$ we can use with complex numbers to get the following identities for complex sinusoidals:

- $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$
- $\cos z = \frac{e^{iz} + e^{-iz}}{2}$
- $\sinh z = \frac{e^{z} - e^{-z}}{2}$
- $\cosh z = \frac{e^{z} + e^{-z}}{2}$
- $\cos z = \sin\left(\frac{\pi}{2} - z\right)$
- $\sinh z = -i \sin(iz)$
- $\cosh z = \cos(iz) = \sin\left(\frac{\pi}{2} - iz\right)$
- $\frac{d}{dz} \sin z = \cos z$
- $\frac{d}{dz} \cos z = -\sin z$
- $\cos^2 z + \sin^2 z = 1$
- $\frac{d}{dz} \sinh z = \cosh z$
- $\frac{d}{dz} \cosh z = \sinh z$
- $\cosh^2 z - \sinh^2 z = 1$

The most notable difference between the real and complex versions of these functions is that $|\sin z| \nleq 1$. We will see cases of this soon but it is easy to see once we write out the real and imaginary components of $\sin z$.

$$\sin z = \sin(x+iy) = \frac{e^{i(x+iy) - e^{-i(x+iy)}}}{2i} = \frac{e^{-y}e^{ix} - e^{y}e^{-ix}}{2i}$$
$$\sin z = \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i} = \frac{e^{-y} - e^{y}}{2}\cos x + \frac{e^{-y}+e^{y}}{2i}\sin x$$
$$\sin z = \sin x \cosh y + i\cos x \sinh y$$

Ex: Find all solutions to $\sin z = 4i$

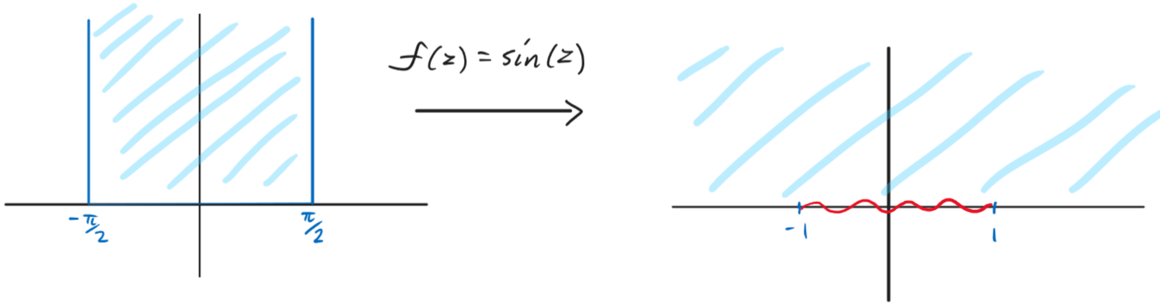
$$\sin(z) = 4i = \sin x \cosh y + i \cos x \sinh y$$
$$\sin x \cosh y = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi$$
$$\text{Case 1: } n = 2k: \ \cos x = 1$$
$$4 = \cos(2k\pi)\sinh(y) = \sinh y$$
$$\text{Case 2: } n = 2k+1: \ \cos x = -1$$
$$4 = \cos((2k+1)\pi)\sinh y = -\sinh y$$
$$\sinh y = \frac{e^{y} - e^{-y}}{2}$$
$$2\sinh y = e^{y} - e^{-y}$$
$$e^{2y} - 2\sinh y e^{y} - 1 = 0$$
$$e^{y} = \frac{2\sinh y \pm \sqrt{4\sinh^2 y + 4}}{2} = \sinh y \pm \sqrt{\sinh^2 y + 1}$$
$$n = 2k:$$
$$y = \ln\left(4 \pm \sqrt{17}\right) = \ln\left(4 + \sqrt{17}\right)$$
$$n = 2k+1:$$
$$y = \ln\left(-4 \pm \sqrt{17}\right) = \ln\left(\sqrt{17} - 4\right)$$
$$z = \left\{ (x,y) \middle| \left(2k\pi, \ln\left(4+\sqrt{17}\right)\right), \left((2k+1)\pi, \ln\left(\sqrt{17}-4\right), k \in \mathbb{Z}\right) \right\}$$

Ex2: Find all solutions to $\cos z = 0$

$$\cos(z^4) = 0$$
$$z^4 = \pi n + \frac{\pi}{2} = \left(\pi n + \frac{\pi}{2}\right) e^{2\pi i l}$$
$$z = \left(\pi n + \frac{\pi}{2}\right)^{1/4} e^{i\frac{\pi l}{2}}, \ l = \{0,1,2,3\}, \ n \in \mathbb{Z}$$

Mapping properties of $\sin z$

$\sin z$ will map the box $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}, \ 0 \leq y < \infty$ to the half plane $v > 0$.



Ex: Find the mapping of $\sin z$ on $S = \left\{-\frac{\pi}{2} < x < \frac{\pi}{2}, 0 < y < 1\right\}$

$u = \sin x \cosh y$

$\sin x \in (-1, 1)$

$\cosh y \in (0, \cosh(1))$

$u \in (-\cosh(1), \cosh(1))$

$v = \cos x \sinh y$

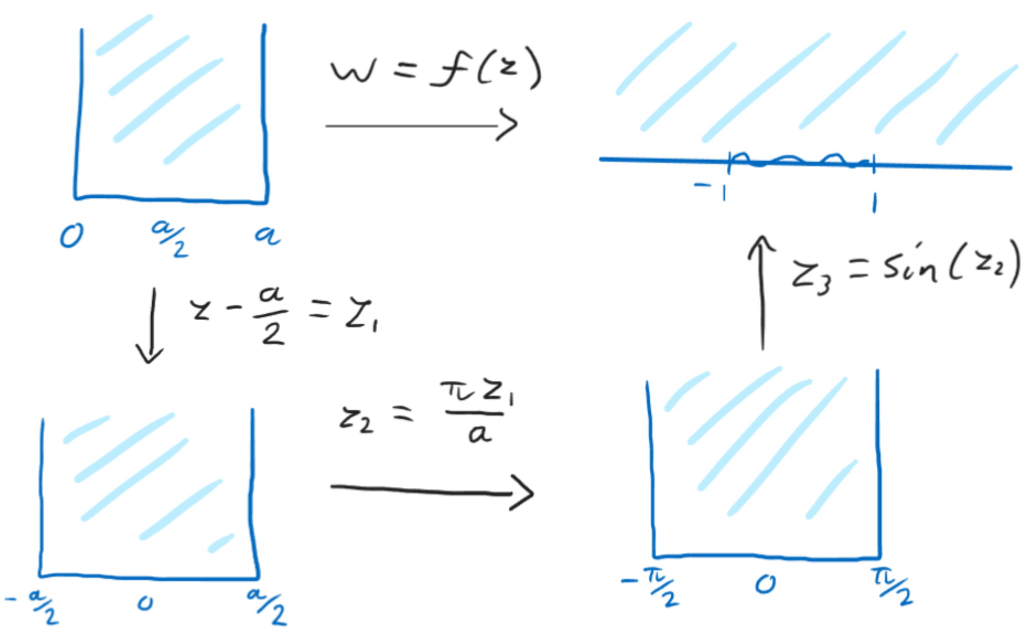
$\cos x \in (0, 1]$

$\sinh y \in (0, \sinh(1))$

$v \in (0, \sinh(1))$

$S' = \{(u, v)| -\cosh(1) < u < \cosh(1), 0 < v < \sinh(1)\}$

If the box is offcenter we can also apply a composition of mappings to shift it into the usual form.



Ex2: Find the mapping of $\sin z$ on $S = \{-1 < x < 1, y > 0\}$

$u = \sin x \cosh y$

$\sin x \in (-\sin(1), \sin(1))$

$\cosh y \in (1, \infty)$
$u \in \mathbb{R}$
$v = \cos x \sinh y$
$\cos x \in (\cos(1), 1]$
$\sinh y \in (0, \infty)$
$v \in (0, \infty)$
$S' = \{(u, v) | v > 0\}$

Ex3: Solve the Laplace equation $\nabla^2 u = 0$ in the region $S = \{0 \leq x \leq 2, 0 \leq y < \infty\}$ with boundary conditions $\phi = 0$ on $x = 0$ and $\phi = 1$ on $y = 0$ and $\phi = -2$ on $x = 2$.

Map to the half plane using $w = \sin\left(\frac{\pi}{2}(z-1)\right)$

$\phi(z) = \Phi(w) = A_1 \operatorname{Arg}(w+1) + A_2 \operatorname{Arg}(w-1) + A_3$

$u > 1: \ \Phi = -2 = A_3$

$-1 < u < 1: \ \Phi = 1 = A_2\pi - 2 \Rightarrow A_2 = \frac{3}{\pi}$

$u < -1: \ \Phi = 0 = A_1\pi + \frac{3}{\pi}\pi - 2 \Rightarrow A_1 = -\frac{1}{\pi}$

$$\phi(z) = -\frac{1}{\pi} \operatorname{Arg}\left(\sin\left(\frac{\pi}{2}(z-1)\right) + 1\right) + \frac{3}{\pi} \operatorname{Arg}\left(\sin\left(\frac{\pi}{2}(z-1)\right) - 1\right) - 2$$

1.2.6 Logarithmic Functions