Math Notes

Tyler Wilson

Contents

1 Complex Analysis

1.1 Complex Algebra

Complex numbers arise from the roots of polynomials.
Ex: $x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm\sqrt{-1}$. This polynomial has no real roots, however, we can introduce an imaginary number i such that $i^2 = -1$. Then we will have the solution $x = \pm i$
We can introduce *complex numbers* which are numbers in the form $z = x + iy$, where x is the real

part of z, $\Re(z)$, and y is the imaginary part of z, $\Im(z)$. These numbers can also be expressed in vector notation along the complex plane.

1.1.1 Complex Arithmetic

Addition, subtraction, and multiplication work the same, just with the addition of the fact $i^2 = -1$. For division, we require what is called the conjugate.
The conjugate of a complex number is the same number, just with the sign of the imaginary component flipped.

$$\overline{z} = x - yi$$

where $\overline{z}$ is the conjugate of z.
Similarly to vectors, we can also define the modulus (length) of a complex number

$$|z|^2 = x^2 + y^2 = z \cdot \overline{z}$$

Using this, we can define the division of a complex number and also define the real and imaginary components of a complex number.
The general expression for division is:

$$\frac{u}{z} = \frac{s+it}{x+iy} = \frac{(s+it)(x-iy)}{(x+iy)(x-iy)} = \frac{u\overline{z}}{x^2+y^2} = \frac{u\overline{z}}{|z|^2}$$

The real and imaginary components can be computed as

$$\begin{aligned}
&z = x + iy \\
&\overline{z} = x - iy \\
&\Re(z) = \frac{z+\overline{z}}{2} \\
&\Im(z) = \frac{z-\overline{z}}{2i}
\end{aligned}$$

Ex1: Simplify $(1+2i)(3+i)(2-3i)$

$$\begin{aligned}
&(1+2i)(3+i)(2-3i) = (3+i+6i-2)(2-3i) = (1+7i)(2-3i) \\
&= 2 - 3i + 14i + 21 \\
&= 23 + 11i
\end{aligned}$$

Ex2: Simplify $\left(\frac{2+i}{1+i}\right)^2$

$$\begin{aligned}
&\left(\frac{2+i}{1+i}\right)^2 = \left(\frac{(2+i)(1-i)}{(1+i)(1-i)}\right)^2 = \left(\frac{2+i-2i+1}{2}\right)^2 = \left(\frac{3-i}{2}\right)^2 = \frac{9-6i-1}{4} \\
&= 2 - \frac{3}{2}i
\end{aligned}$$

Ex3: Simplify $(1+2i)^5$

$$(1+2i)^5 = 1^5 + 5(1)^4(2i) + 10(1)^3(2i)^2 + 10(1)^2(2i)^3 + 5(1)(2i)^4 + (2i)^5$$

$$= 1 + 10i + 10(-4) + 10(-8i) + 5(16) + 32i$$
$$= 1 + 10i - 40 - 80i + 80 + 32i$$
$$= 41 - 38i$$

Ex4: Prove

$$\text{if } |z| = 1 \text{ then } \Re\left(\frac{1}{1+z}\right) = \frac{1}{2}$$

Proof.

$$\Re(w) = \frac{w + \overline{w}}{2}$$
$$\overline{\frac{1}{1+z}} = \frac{1}{1+\overline{z}}$$
$$\Re\left(\frac{1}{1+z}\right) = \frac{\frac{1}{1+z} + \frac{1}{1+\overline{z}}}{2} = \frac{1+\overline{z}+1+z}{2(1+z)(1+\overline{z})} = \frac{2+z+\overline{z}}{2(1+z+\overline{z}+|z|^2)}$$
$$|z| = 1 \Rightarrow \Re\left(\frac{1}{1+z}\right) = \frac{2+z+\overline{z}}{2(2+z+\overline{z})} = \frac{1}{2}$$

□

Some properties of $\overline{z}$:

- $\overline{\overline{z}} = z$
- $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$
- $|z_1 z_2| = |z_1||z_2|$

Some common inequalities come from the triangle inequality:
$|z_1 + z_2| \leq |z_1| + |z_2|$

Proof.

$$|z_1 + z_2| \leq |z_1| + |z_2|$$
$$|z_1 + z_2| = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$$
$$\sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} \leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$$
$$|z_1 + z_2| \leq |z_1| + |z_2|$$

□

$$\Rightarrow |z_1 \pm z_2| \geq ||z_1| - |z_2||$$

This leads to a general upper and lower bound that can be derived from these inequalities:

$$||z_1| - |z_2|| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$$

1.1.2 Polar Form of Complex Numbers

Another way to represent complex numbers is through polar form. To use this we must first introduce Euler's identity:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

This then helps us with the equation for the polar form

$$z = a + ib = |z|(\cos\varphi + i\sin\varphi) = |z|e^{i\varphi}$$

For $z = re^{i\varphi}$ We call r the magnitude of the complex number and φ is the argument. This can be especially useful for simplifying some complex numbers.
Ex:

$$\left|\frac{(1+\sqrt{3}i)^{100}}{(\sqrt{3}-i)^{100}}\right| = \frac{|1+\sqrt{3}i|^{100}}{|\sqrt{3}-i|^{100}} = \frac{2^{100}}{2^{100}} = 1$$

Note that because sinusoidal functions are periodic every 2π this means that there infinite ways to express a function in polar coordinates.

$$e^{i2\pi k} = 1,\ k \in \mathbb{Z} \Rightarrow z = re^{i(\varphi+2\pi k)}$$

To get around the issue of having infinite possible polar forms for every complex number we define what's called the principal argument to be

$$\mathrm{Arg}(z) = \varphi \in (-\pi, \pi]$$

We define the regular argument to be

$$\arg(z) = \mathrm{Arg}(z) + 2k\pi,\ k \in \mathbb{Z}$$

Note that $\mathrm{Arg}(z)$ is singular valued while $\arg(z)$ is multi-valued.
We can define $\mathrm{Arg}(z)$ in terms of the real and imaginary parts (x and y) as

$$\mathrm{Arg}(z) = \arctan(\tfrac{y}{x}) \pm k\pi$$

What it is specifically depends on what quadrant of the complex plane the point lies in.

- QI: $\mathrm{Arg}(z) = \arctan(\frac{y}{x})$
- QII: $\mathrm{Arg}(z) = \arctan(\frac{y}{x}) + \pi$
- QIII: $\mathrm{Arg}(z) = \arctan(\frac{y}{x}) - \pi$
- QIV: $\mathrm{Arg}(z) = \arctan(\frac{y}{x})$

Ex:

$$\mathrm{Arg}(-1-\sqrt{3}i) = \arctan(\sqrt{3}) - \pi = -\frac{2\pi}{3}$$

Ex2:

$$\arg(1-\sqrt{3}i)$$
$$\operatorname{Arg}(1-\sqrt{3}i) = -\frac{\pi}{3}$$
$$\arg(1-\sqrt{3}) = -\frac{\pi}{3} + 2k\pi,\ k \in \mathbb{Z}$$

Ex3:

$$\arg(-1+2i)$$
$$\operatorname{Arg}(-1+2i) = \arctan(-2) + \pi$$
$$\arg(-1+2i) = \pi - \arctan(2) + 2k\pi$$

Ex4: Simplify

$$z = -3+3i$$
$$\operatorname{Arg}(z) = \arctan\left(\frac{3}{-3}\right) + \pi = \frac{3\pi}{4}$$
$$|z| = 3\sqrt{2}$$
$$z = 3\sqrt{2}e^{\frac{3\pi}{4}i}$$

Ex5: Simplify

$$z = -3-3i$$
$$\operatorname{Arg}(z) = \arctan\left(\frac{-3}{-3}\right) - \pi = -\frac{3\pi}{4}$$
$$|z| = 3\sqrt{2}$$
$$z = 3\sqrt{2}e^{-\frac{3\pi}{4}i}$$

Ex6: Simplify

$$z = \frac{1-i}{-\sqrt{3}+i} = \frac{u}{v}$$
$$\arg(z) = \arg(u) - \arg(v) = \arctan\left(\frac{-1}{1}\right) - \left(\arctan\left(\frac{1}{-\sqrt{3}}\right) + \pi\right) + 2k\pi$$
$$= -\frac{\pi}{4} - \frac{5\pi}{6} + 2k\pi = -\frac{13\pi}{12} + 2k\pi$$
$$\operatorname{Arg}(z) = \frac{11\pi}{12}$$
$$|z| = \frac{|u|}{|v|} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$
$$z = \frac{e^{\frac{11\pi}{12}i}}{\sqrt{2}}$$

Ex7: Simplify

$$z = (\sqrt{3} - i)^2 = w^2$$
$$\arg(w) = \arctan\left(\frac{-1}{\sqrt{3}}\right) + 2k\pi = -\frac{\pi}{6}$$
$$\Rightarrow \arg(z) = 2\arg(w) = -\frac{\pi}{3} + 4k\pi$$
$$\operatorname{Arg}(z) = -\frac{\pi}{3}$$
$$|w| = 2 \Rightarrow |z| = |w|^2 = 4$$
$$z = 4e^{-\frac{\pi}{3}i}$$

Ex8: Solve for all values of z

$$e^z = -1 - \sqrt{3}i$$
$$e^z = 2e^{-i\frac{2\pi}{3} + 2\pi ik} = e^{\ln 2 - i\frac{2\pi}{3} + 2\pi ik}$$
$$z = \ln 2 - i\frac{2\pi}{3} + 2\pi ik, \ \forall k \in \mathbb{Z}$$

Properties of $\operatorname{Arg}(z)$ and $\arg(z)$

- $\operatorname{Arg}(z_1 z_2) \neq \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$

  Proof. Proof by contradiction: assume that $\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2)$ is true. Take $z_1 = z_2 = -1$.

  $$\operatorname{Arg}(z_1) = \operatorname{Arg}(z_2) = \pi$$
  $$\Rightarrow \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) = 2\pi$$
  $$\operatorname{Arg}(z_1 z_2) = \operatorname{Arg}(1) = 0$$
  $$\operatorname{Arg}(z_1 z_2) \neq \operatorname{Arg}(z_1) + \operatorname{Arg}(z_2) \ \forall z_1, z_2 \neq 0 \in \mathbb{C}$$

  □

- $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$
- $\operatorname{Arg}(\overline{z}) \neq -\operatorname{Arg}(z)$

  Proof. Proof by contradiction: assume that $\operatorname{Arg}(\overline{z}) = -\operatorname{Arg}(z)$ is true. Take $z = -1$

  $$\overline{z} = z = -1$$
  $$\operatorname{Arg}(z) = \pi$$
  $$\operatorname{Arg}(\overline{z}) = \pi$$
  $$-\operatorname{Arg}(z) = -\pi$$
  $$\Rightarrow \operatorname{Arg}(\overline{z}) \neq -\operatorname{Arg}(z) \ \forall z \in \mathbb{C}$$

  □

- $\arg(z) = -\arg(\overline{z})$

  Proof.
  $$z = |z|e^{i\arg(z)}\ \forall z \in \mathbb{C}$$
  $$\overline{z} = |z|e^{-i\arg(z)}$$
  $$\Rightarrow \arg(\overline{z}) = -\arg(z)$$
  □

1.1.3 De Moirre's Formula

Using Euler's identity we can derive a powerful formula called De Moirre's Formula as follows:

$$e^{iN\varphi} = \cos(N\varphi) + i\sin(N\varphi)$$
$$e^{iN\varphi} = (e^{i\varphi})^N = (\cos\varphi + i\sin\varphi)^N$$
$$(\cos\varphi + i\sin\varphi)^N = \cos(N\varphi) + i\sin(N\varphi)$$

Applications of De Moirre's Formula:
Binomial expansion:

$$N = 2:$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$
$$\sin(2\theta) = 2\cos\theta\sin\theta$$
$$N = 3:$$
$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3$$
$$\cos(3\theta) = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$\sin(3\theta) = 3\cos^2\theta\sin\theta - \sin^3\theta$$

Ex: Prove

$$\sin(3\theta) = 3\sin\theta - 4\sin^3\theta$$

Proof. Using De Moivre's formula with $N = 3$

$$(\cos\theta + i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta)$$
$$(\cos\theta)^3 + 3(\cos\theta)^2(i\sin\theta) + 3\cos\theta(i\sin\theta)^2 + (i\sin\theta)^3 = \cos(3\theta) + i\sin(3\theta)$$
$$\cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta) = \cos(3\theta) + i\sin(3\theta)$$
$$\Im\left\{\cos^3\theta - 3\cos\theta\sin^2\theta + i(3\cos^2\theta\sin\theta - \sin^3\theta)\right\} = \Im\{\cos(3\theta) + i\sin(3\theta)\}$$
$$3\cos^2\theta\sin\theta - \sin^3\theta = \sin(3\theta)$$
$$\sin^2\theta + \cos^2\theta = 1 \Rightarrow \cos^2\theta = 1 - \sin^2\theta$$
$$3(1 - \sin^2\theta)\sin\theta - \sin^3\theta = \sin(3\theta)$$
$$3\sin\theta - 3\sin^3\theta - \sin^3\theta = \sin(3\theta)$$
$$3\sin\theta - 4\sin^3\theta = \sin(3\theta)$$

□

Computing trigonometric integrals:
Ex:

$$\int_0^{2\pi} \cos^8 \varphi d\varphi$$
$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$
$$e^{-i\varphi} = \cos\varphi - i\sin\varphi$$
$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2}$$
$$\sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$
$$\int_0^{2\pi} \left(\frac{e^{i\varphi} + e^{-i\varphi}}{2}\right)^8 d\varphi = \frac{1}{2^8}\int_0^{2\pi} \left(e^{i\varphi} + e^{-i\varphi}\right)^8 d\varphi$$
$$= \frac{1}{2^8}\int_0^{2\pi} \left(e^{i8\varphi} + {}_1C_8 e^{i7\varphi}e^{-i\varphi} + \cdots + {}_7C_8 e^{i\varphi}e^{-i7\varphi} + e^{-i8\varphi}\right) d\varphi$$
$$= \frac{1}{2^8}\left(0 + \cdots + {}_4C_8 2\pi + \cdots + 0\right) = \frac{{}_4C_8}{2^7}\pi$$

Ex2:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$\int_0^{2\pi} \cos^6 \theta d\theta = \int_0^{2\pi} \left(\frac{e^{i\theta} + e^{-i\theta}}{2}\right)$$
$$= \frac{1}{2^6}\int_0^{2\pi} \sum_{k=0}^{6} \binom{k}{6} e^{i\theta k} e^{-i\theta(6-k)} d\theta$$
$$= \frac{1}{2^6}\sum_{k=0}^{6} \binom{k}{6} \int_0^{2\pi} e^{i\theta(2k-6)} d\theta$$
$$\int_0^{2\pi} e^{ik\theta} d\theta = \left.\frac{e^{ik\theta}}{ik}\right|_0^{2\pi} = \frac{e^{2\pi ik} - 1}{ik}$$
$$e^{2\pi ik} = 1,\ k \in \mathbb{Z} \Rightarrow \int_0^{2\pi} e^{ik\theta} d\theta = 0,\ k \neq 0 \in \mathbb{Z}$$
$$\Rightarrow \int_0^{2\pi} \cos^6 \theta d\theta = \frac{1}{2^6}\binom{3}{6}\int_0^{2\pi} d\theta = \frac{(20)(2\pi)}{2^6} = \frac{5\pi}{8}$$

Ex3:

$$\int_0^{2\pi} \sin^6(2\theta) d\theta$$
$$\sin(2\theta) = \frac{e^{2i\theta} - e^{-2i\theta}}{2i}$$
$$\int_0^{2\pi} \sin^6(2\theta) d\theta = \int_0^{2\pi} \left(\frac{e^{2i\theta} - e^{-2i\theta}}{2i}\right)^6 d\theta$$

$$= \frac{1}{(2i)^6} \int_0^{2\pi} \sum_{k=0}^{6} \binom{k}{6} (-1)^{6-k} e^{2ik\theta} e^{-2i\theta(6-k)} d\theta$$

$$= -\frac{1}{2^6} \sum_{k=0}^{6} \binom{k}{6} (-1)^{6-k} \int_0^{2\pi} e^{i\theta(4k-12)} d\theta$$

$$\int_0^{2\pi} e^{ik\theta} d\theta = \frac{e^{ik\theta}}{ik}\bigg|_0^{2\pi} = \frac{e^{2\pi ik} - 1}{ik}$$

$$e^{2\pi ik} = 1,\ k \in \mathbb{Z} \Rightarrow \int_0^{2\pi} e^{ik\theta} d\theta = 0,\ k \neq 0 \in \mathbb{Z}$$

$$\Rightarrow \int_0^{2\pi} \sin^6(2\theta) d\theta = -\frac{1}{2^6} \binom{3}{6} (-1)^3 \int_0^{2\pi} d\theta$$

$$= \frac{5\pi}{8}$$

Ex4: Prove

$$\sum_{k=0}^{n} \cos(k\theta) = \frac{1}{2} + \frac{\sin\left(\left(n + \frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}$$

Proof. De Moivre's formula states

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

If we take the conjugate of both sides we get

$$(\cos\theta - i\sin\theta)^n = \cos(n\theta) - i\sin(n\theta)$$

Summing these two equations gives

$$(\cos\theta + i\sin\theta)^k + (\cos\theta - i\sin\theta)^k = 2\cos(k\theta)$$

$$\cos\theta \pm i\sin\theta = e^{\pm i\theta}$$

$$2\cos(k\theta) = (e^{i\theta})^k + (e^{-i\theta})^k$$

We can sum both sides of this to get

$$2\sum_{k=0}^{n} \cos(k\theta) = \sum_{k=0}^{n} (e^{i\theta})^k + \sum_{k=0}^{n} (e^{-i\theta})^k$$

The formula for the geometric sum is

$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$$

Applying this we get

$$2\sum_{k=0}^{n} \cos(k\theta) = \frac{1 - (e^{i\theta})^{n+1}}{1 - e^{i\theta}} + \frac{1 - (e^{-i\theta})^{n+1}}{1 - e^{-i\theta}}$$

$$
\begin{aligned}
2\sum_{k=0}^{n}\cos(k\theta) &= \frac{(1-e^{i\theta(n+1)})(1-e^{-i\theta})+(1-e^{-i\theta(n+1)})(1-e^{i\theta})}{(1-e^{i\theta})(1-e^{-i\theta})} \\
2\sum_{k=0}^{n}\cos(k\theta) &= \frac{1-e^{i\theta(n+1)}-e^{i\theta}+e^{i\theta n}+1-e^{-i\theta(n+1)}-e^{-i\theta}+e^{-i\theta n}}{1-e^{i\theta}-e^{-i\theta}+1} \\
2\sum_{k=0}^{n}\cos(k\theta) &= 1+\frac{e^{i\theta n}+e^{-i\theta n}-e^{i\theta(n+1)}-e^{-i\theta(n+1)}}{2-e^{i\theta}-e^{-i\theta}} \\
2\sum_{k=0}^{n}\cos(k\theta) &= 1+\frac{e^{i\frac{\theta}{2}}\left(e^{-i\theta(n+\frac{1}{2})}-e^{i\theta(n+\frac{1}{2})}\right)+e^{-i\frac{\theta}{2}}\left(e^{i\theta(n+\frac{1}{2})}-e^{i\theta(n+\frac{1}{2})}\right)}{e^{i\frac{\theta}{2}}\left(e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}\right)+e^{-i\frac{\theta}{2}}\left(e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}\right)} \\
2\sum_{k=0}^{n}\cos(k\theta) &= 1+\frac{\left(e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}\right)\left(e^{i\theta(n+\frac{1}{2})}-e^{-i\theta(n+\frac{1}{2})}\right)}{-\left(e^{-i\frac{\theta}{2}}-e^{i\frac{\theta}{2}}\right)^2} \\
2\sum_{k=0}^{n}\cos(k\theta) &= 1+\frac{e^{i\theta(n+\frac{1}{2})}-e^{-i\theta(n+\frac{1}{2})}}{e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}} \\
2\sum_{k=0}^{n}\cos(k\theta) &= 1+\frac{\frac{1}{2i}\left(e^{i\theta(n+\frac{1}{2})}-e^{-i\theta(n+\frac{1}{2})}\right)}{\frac{1}{2i}\left(e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}\right)} \\
\sin(x) &= \frac{e^{ix}-e^{-ix}}{2i} \\
2\sum_{k=0}^{n}\cos(k\theta) &= 1+\frac{\sin\left(\theta(n+\frac{1}{2})\right)}{\sin\left(\frac{\theta}{2}\right)} \\
\sum_{k=0}^{n}\cos(k\theta) &= \frac{1}{2}+\frac{\sin\left(\left(n+\frac{1}{2}\right)\theta\right)}{2\sin\left(\frac{\theta}{2}\right)}
\end{aligned}
$$

□

1.1.4 Geometry in the Complex Plane

Using the notation such that $z = x + iy$ where $\Re(z) = x$ and $\Im(z) = y$ we can define a circle in the complex plane as

$$(x - x_0)^2 + (y - y_0)^2 = r_0^2$$

This is analogous to writing

$$|z - z_0| = r_0$$

The two can be related as follows:

$$
\begin{aligned}
&|z - z_0| = r_0 \\
&|x + iy - x_0 - iy_0| = r_0 \\
&\sqrt{(x - x_0)^2 + (y - y_0)^2} = r_0
\end{aligned}
$$

$$(x - x_0)^2 + (y - y_0)^2 = r_0^2$$

Ex: describe the circle formed by $2|z| = |z + 1|$

$$2|z| = |z + 1|$$
$$4|z|^2 = |z + 1|^2$$
$$4x^2 + 4y^2 = (x + 1)^2 + y^2$$
$$4x^2 + 4y^2 = x^2 + 2x + 1 + y^2$$
$$3x^2 + 3y^2 - 2x - 1 = 0$$
$$3x^2 - 2x + \frac{1}{3} + 3y^2 - \frac{4}{3} = 0$$
$$3\left(x - \frac{1}{3}\right)^2 + 3y^2 = \frac{4}{3}$$
$$\left(x - \frac{1}{3}\right)^2 + y^2 = \frac{4}{9}$$

A line in the complex plane can be written as

$$ax + by = c \longleftrightarrow a\frac{z + \overline{z}}{2} + b\frac{z - \overline{z}}{2i} = c$$

Ex: describe the line formed by $|z - 1 + i| = |z - 2i|$

$$|z - 1 + i| = |z - 2i|$$
$$|z - 1 + i|^2 = |z - 2i|^2$$
$$(x - 1)^2 + (y + 1)^2 = x^2 + (y - 2)^2$$
$$x^2 - 2x + 1 + y^2 + 2y + 1 = x^2 + y^2 - 4y + 4$$
$$-2x + 2 + 2y = -4y + 4$$
$$6y = 2x + 2$$
$$y = \frac{1}{3}(x + 1)$$

We can define an ellipse in the complex plane as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

If we have $a > b$ then we will have a horizontal ellipse and if $b > a$ then we will have a vertical ellipse.
Assuming that $a > b$ then we can define the foci points to be at

$$+F = (\sqrt{a^2 - b^2}, 0)$$
$$-F = (-\sqrt{a^2 - b^2}, 0)$$

The equation of an ellipse can also be described by

$$|z - F| + |z + F| = 2a$$

Ex: describe the ellipse formed by $|z-1|+|z+1|=4$

$$
\begin{aligned}
&|z-1|+|z+1|=4\\
&|z-F|+|z+F|=2a \Rightarrow 2a=4 \Rightarrow a=2\\
&F=\sqrt{a^2-b^2}=1 \Rightarrow 1=4-b^2 \Rightarrow b^2=3\\
&\frac{x^2}{4}+\frac{y^2}{3}=1
\end{aligned}
$$

Ex2: describe the ellipse formed by $|z-1|+|z+3|=6$

$$|z-1|+|z+3|=6$$

note that this ellipse is not centered at the origin so we need to shift it

$$
\begin{aligned}
&|z+1-2|+|z+1+2|=6\\
&2a=6 \Rightarrow a=3\\
&F=2=\sqrt{a^2-b^2} \Rightarrow 4=9-b^2 \Rightarrow b^2=5\\
&\frac{(x+1)^2}{9}+\frac{y^2}{5}=1
\end{aligned}
$$

1.1.5 Roots of a Complex Number

Given $z_0=r_0e^{i\varphi_0}$, what is $z_0^{\frac{1}{n}}$?
If we let $w=z_0^{\frac{1}{n}}$ then $w^n=z_0$

$$
\begin{aligned}
&w=re^{i\varphi},\ w_n=r^ne^{in\varphi}\\
&w^n=z_0 \Rightarrow r^ne^{in\varphi}=r_0e^{i\varphi_0}\\
&\Rightarrow r^n=r_0 \Rightarrow r_0^{\frac{1}{n}}\\
&e^{in\varphi}=e^{i\varphi_0} \Rightarrow n\varphi=\varphi_0+2k\pi\\
&\varphi=\frac{\varphi}{n}+\frac{2k\pi}{n}
\end{aligned}
$$

So all solutions to $w^n=z_0$ are given by

$$w=r_0^{\frac{1}{n}}e^{i\left(\frac{\varphi_0}{n}+\frac{2k\pi}{n}\right)},\ k\in\mathbb{Z}$$

If we normalize $\varphi_0=\mathrm{Arg}(z_0)$ then k will be in the range $k\in\{0,1,\ldots,n-1\}$.
Note that the expression

$$w=r_0^{\frac{1}{n}}e^{i\left(\frac{\varphi_0}{n}+\frac{2k\pi}{n}\right)},\ k\in\mathbb{Z}$$

is multi-valued. If we want to avoid this, we can take what's called the principal value which is the value when $k=0$:

$$z_0^{\frac{1}{n}}=r_0^{\frac{1}{n}}e^{i\frac{\varphi_0}{n}}$$

Ex: Compute $(-1)^{\frac{1}{2}}$

$$z_0 = -1 = 1^{\frac{1}{2}} e^{i(\frac{\pi}{2} + \frac{2k\pi}{2})} = e^{i(\frac{\pi}{2} + k\pi)} = \left\{\dots, e^{-i\frac{3\pi}{2}}, e^{-i\frac{\pi}{2}}, e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{2}}, e^{i\frac{5\pi}{2}}, \dots\right\}$$

note that there are only 2 unique values

$$(-1)^{\frac{1}{2}} = e^{i(\frac{\pi}{2} + k\pi)},\ k \in \{0, 1\}$$

The principal value (when $k = 0$) of this equation works out to be i.
Ex2: Find all solutions to

$$\begin{aligned}
&z^7 = i - 1 \\
&z^7 = \sqrt{2} e^{i(\frac{3\pi}{4} + 2\pi k)} \\
&z = 2^{1/14} e^{i(\frac{3\pi}{28} + \frac{2\pi}{7} k)},\ k \in \{0, 1, 2, 3, 4, 5, 6\}
\end{aligned}$$

Ex3: Find all solutions to

$$\begin{aligned}
&z^5 = \frac{2i}{-1 - \sqrt{3}i} \\
&z^5 = \frac{2e^{i\frac{\pi}{2}}}{2e^{-i\frac{2\pi}{3}}} = e^{i\frac{7\pi}{6}} = e^{-i(\frac{5\pi}{6} + 2\pi k)} \\
&z = e^{-i(\frac{\pi}{6} + \frac{2\pi}{5} k)},\ k \in \{0, 1, 2, 3, 4\}
\end{aligned}$$

Ex4: Find all solutions to

$$\begin{aligned}
&\left(\frac{z}{z+1}\right)^2 = i \\
&\left(\frac{z}{z+1}\right)^2 = e^{i(\frac{\pi}{2} + 2\pi k)} \\
&\frac{z}{z+1} = e^{i(\frac{\pi}{4} + \pi k)},\ k \in \{0, 1\} \\
&e^{i(\frac{\pi}{4} + \pi k)} = \left\{\frac{1}{\sqrt{2}}(1 + i), \frac{1}{\sqrt{2}}(-1 - i)\right\} \\
&z = e^{i(\frac{\pi}{4} + k\pi)}(z + 1) \\
&z = \frac{e^{i(\frac{\pi}{4} + k\pi)}}{1 - e^{i(\frac{\pi}{4} + k\pi)}} = \left\{\frac{1 + i}{\sqrt{2} - 1 - i}, \frac{-1 - i}{\sqrt{2} + 1 + i}\right\}
\end{aligned}$$

Ex5: Find all solutions to

$$\begin{aligned}
&z^2 + 4iz + 1 = 0 \\
&(z^2 + 4iz - 4) + 4 + 1 = 0 \\
&(z + 2i)^2 + 5 = 0 \\
&(z + 2i)^2 = -5 = 5e^{i(\pi + 2\pi k)} \\
&z + 2i = \sqrt{5} e^{i(\frac{\pi}{2} + \pi k)} = \left\{\sqrt{5}i, -\sqrt{5}i\right\}
\end{aligned}$$

$$z = \left\{(\sqrt{5}-2)i, -(\sqrt{5}+2)i\right\}$$

Ex6: Find all solutions to $(z+1)^4 = (1-i)z^4$

$$(1-i)z^4 = \sqrt{2}e^{-i\frac{\pi}{4}+2\pi ki}z^4$$
$$z+1 = 2^{1/8}e^{-i\frac{\pi}{16}+i\frac{\pi k}{2}}z$$
$$z = \frac{-2^{1/8}}{1-e^{-i\frac{\pi}{16}+i\frac{\pi k}{2}}},\ k \in \{0,1,2,3\}$$

1.2 Complex Functions

1.2.1 Mapping Properties of Simple Functions

Similar to how functions with real variables map values to a different set of values, complex functions do the same. The main difference is that with complex functions we're mapping a 2 dimensional set of inputs to a 2 dimensional set of outputs.

$$w = f(z) = u + iv$$

We define $z \in \mathcal{S}$ the image of $\mathcal{S}$ under w.
Some common mappings:

- The identity map

$$w = f(z) = z$$
$$\begin{cases} u = x \\ v = y \end{cases}$$

- Translation by z_0

$$w = f(z) = z + z_0$$
$$\begin{cases} u = x + x_0 \\ v = y + y_0 \end{cases}$$

- Stretching $(a > 1)$ or contraction $(a < 1)$

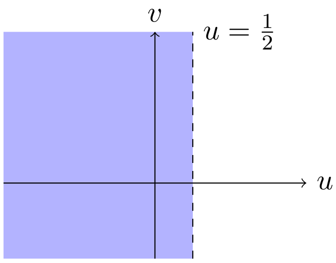
$$w = f(z) = az = are^{i\varphi},\ a \in \mathbb{R}$$
$$\begin{cases} u = ax \\ v = ay \end{cases}$$

- Rotation by φ_0

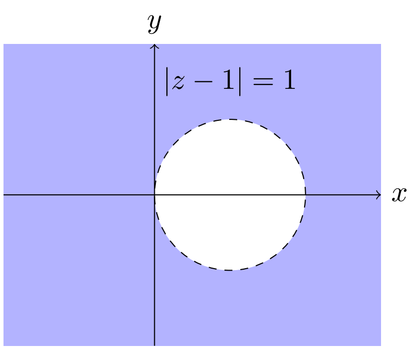
$$w = f(z) = e^{i\varphi_0}z = e^{i(\varphi+\varphi_0)}$$

Using these basic mapping principles we are able to lay the foundation for some more complicated mappings.

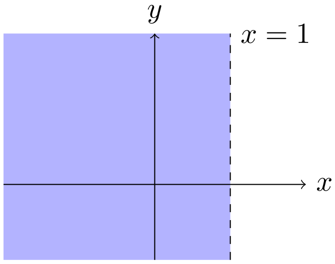
Ex: Find the image of $S = \{|z-1| \geq 1\}$ under the mapping $f(z) = \frac{1}{z}$



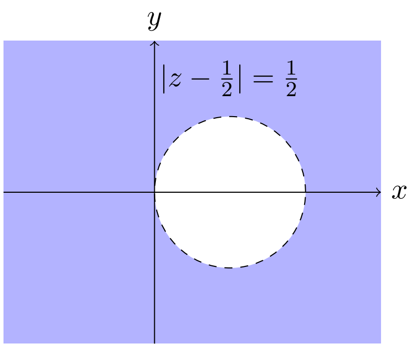
$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$
$$x = \frac{u}{u^2+v^2}$$
$$y = -\frac{v}{u^2+v^2}$$
$$|z-1| \geq 1 \Rightarrow \left|\frac{1}{w} - 1\right| \geq 1$$
$$\frac{1-w}{w} \geq 1$$
$$|1-w| \geq |w|$$
$$|1-w|^2 \geq |w|^2$$
$$(1-u)^2 + v^2 \geq u^2 + v^2$$
$$-2u + 1 \geq 0$$
$$u \leq \frac{1}{2}$$
$$S' = \left\{u \leq \frac{1}{2}\right\}$$



Ex2: Find the image of $S = \{x \leq 1\}$ under the mapping $f(z) = \frac{1}{z}$



$$z = \frac{1}{w} = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$$
$$x = \frac{u}{u^2+v^2}$$
$$x \le 1 \Rightarrow \frac{u}{u^2+v^2} \le 1 \Rightarrow u^2+v^2 \ge u$$
$$(u-\tfrac{1}{2})^2 + v^2 \ge \tfrac{1}{4}$$



We see from the previous two examples that circles map to lines and lines map to circles. Let's see why this is the case.

$$a(x^2+y^2) + bx + cy + d = 0$$
$$a|z|^2 + b\frac{z+\bar{z}}{2} + c\frac{z-\bar{z}}{2i} + d = 0$$

In the case where $a = 0$, we have a line. In the case where $a \neq 0$, we have a circle.

$$w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$$
$$z\bar{z} = |z|^2 = \frac{1}{|w|^2} = \frac{1}{w\bar{w}}$$
$$a\frac{1}{w\bar{w}} + b\frac{z+\bar{z}}{2} + c\frac{z-\bar{z}}{2i} + d = 0$$
$$a\frac{1}{w\bar{w}} + \frac{b}{2}\left(\frac{1}{w} + \frac{1}{\bar{w}}\right) + \frac{c}{2i}\left(\frac{1}{w} - \frac{1}{\bar{w}}\right) + d = 0$$
$$a + \frac{b}{2}(w+\bar{w}) + \frac{c}{2i}(w-\bar{w}) + d(w\bar{w}) = 0$$

If we have a linear transformation of the form $az+b$ it corresponds to the scaling and translation of the set only. A line will map to a line and a circle will map to a circle.

We can combine this with the $w = \frac{1}{z}$ transformation property to get a more general transformation. We call this the *Mobius transformation*:

$$f(z) = \frac{az+b}{cz+d} = \frac{a}{c} + \frac{b - \frac{ad}{c}}{cz+d}$$

where $a, b, c, d \in \mathbb{C}$

Ex: Find the mapping of $f(z) = \frac{1}{z+1}$ on the set $S = \{\Re(z) > 0\}$

$$u + iv = \frac{1}{x+1+iy} \Rightarrow x + 1 + iy = \frac{1}{u+iv}$$
$$x + 1 = \frac{u}{u^2+v^2}$$
$$x > 0 \Rightarrow x + 1 > 1$$
$$\frac{u}{u^2+v^2} > 1 \Rightarrow u > u^2 + v^2$$
$$u^2 + v^2 - u + \frac{1}{4} < \frac{1}{4}$$
$$\left(u - \frac{1}{2}\right)^2 + v^2 < \frac{1}{4}$$
$$S' = \left\{ w = u + iv \middle| \left(u - \frac{1}{2}\right)^2 + v^2 < \left(\frac{1}{2}\right)^2 \right\}$$

Ex2: Find the mapping of $f(z) = \frac{z-i}{z+i}$ on $S = \{|z| < 3\}$

$$wz + iw = z - i \Rightarrow z(w-1) = -i - iw \Rightarrow z = \frac{i(w+1)}{1-w}$$
$$|z| = \frac{|w+1|}{|w-1|} < 3$$
$$|w+1| < 3|w-1| \Rightarrow |w+1|^2 < 9|w-1|^2$$
$$(u+1)^2 + v^2 < 9(u-1)^2 + 9v^2$$
$$u^2 + 2u + 1 + v^2 < 9u^2 - 18u + 9 + 9v^2$$
$$0 < 8u^2 - 20u + 8 + 8v^2 \Rightarrow 0 < u^2 - \frac{5}{2}u + 1 + v^2$$
$$\frac{9}{16} < u^2 - \frac{5}{2}u + \frac{25}{16} + v^2$$
$$\frac{9}{16} < \left(u - \frac{5}{4}\right)^2 + v^2$$
$$S' = \left\{ w = u + iv \middle| \left(u - \frac{5}{4}\right)^2 + v^2 > \left(\frac{3}{4}\right)^2 \right\}$$

Another common mapping is the $f(z) = z^2$ or more generally $f(z) = z^n$ mapping. For $w = z^2$,

$$w = z^2 = r^2 e^{2i\varphi} \Rightarrow \begin{cases} |w| = |z|^2 \\ \arg(w) = 2\arg(z) \end{cases}$$

This mapping scales the magnitude but more notably, it doubles the argument. This means that the mapping of a half circle will now be a full circle.

Ex: Find the mapping of $f(z) = z^2$ on $S = \{0 \le \Re(z) \le 1, \Im(z) = 1\}$

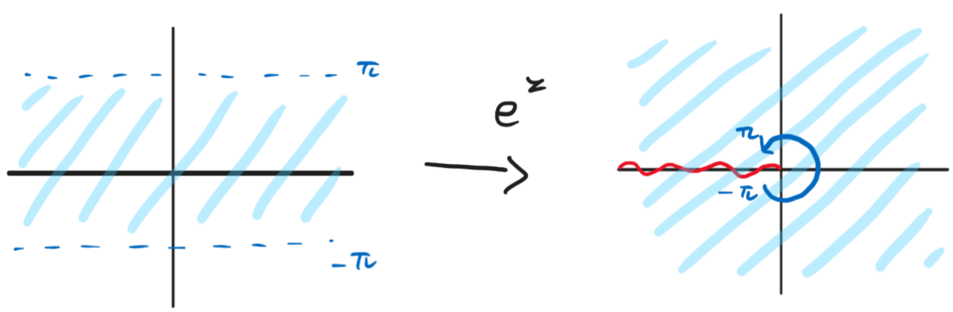
$$w = x^2 + i2xy - y^2$$
$$u = x^2 - y^2 = x^2 - 1 \Rightarrow -1 \le u \le 0$$
$$v = 2xy = 2x \Rightarrow 0 \le v \le 2$$
$$S' = \{w = u + iv | -1 \le u \le 0,\ 0 \le v \le 2\}$$

Ex2: Find the mapping of $f(z) = -2z^5$ on $S = \left\{|z| < 1, 0 < \mathrm{Arg}(z) < \frac{\pi}{2}\right\}$

$$z^5 = -\frac{w}{2} \Rightarrow |z|^5 = \frac{|w|}{2} < 1 \Rightarrow |w| < 2$$
$$5\arg(z) = \arg(w) \pm \pi$$
$$0 < \arg(w) \pm \pi < \frac{5\pi}{2}$$
$$-\pi < \arg(w) < \frac{3\pi}{2}$$
$$S' = \{|w| < 2\}$$

Another common mapping is the $f(z) = e^z$ mapping.

$$w = e^z = e^{x+iy} = e^x e^{iy}$$
$$\begin{cases} |w| = e^x \\ \arg(w) = y \end{cases}$$



This mapping has the property that the magnitude is only dependent on x and the argument is exactly y.
Ex: Find the mapping of $f(z) = e^z$ on $S = \{\Re(z) = 1\}$

$$w = e^x e^{iy}$$
$$|w| = e,\ \arg(w) = y$$
$$S' = \{|w| = e\}$$

Ex2: Find the mapping of $f(z) = e^z$ on $S = \left\{0 \le \Im(z) \le \frac{\pi}{4}\right\}$

$$|w| = x$$

$$\arg(w) = y \Rightarrow 0 \leq \arg(w) \leq \frac{\pi}{4}$$
$$S' = \left\{0 \leq \operatorname{Arg}(w) \leq \frac{\pi}{4}\right\}$$

Ex3: Find the mapping of $f(z) = e^{iz}$ on $S = \left\{z : -\frac{\pi}{2} \leq \Re(z) \leq \pi, -1 \leq \Im(z) \leq 1\right\}$
(Note that multiplying z by i rotates it by $90°$)

$$w = e^{iz} = e^{ix}e^{-y}$$
$$|w| = e^{-y} \Rightarrow e^{-1} \leq |w| \leq e$$
$$\arg(w) = x \Rightarrow -\frac{\pi}{2} \leq \arg(w) \leq \pi$$
$$S' = \left\{w \middle| e^{-1} \leq |w| \leq e, -\frac{\pi}{2} \leq \arg(w) \leq \pi\right\}$$

Ex4: Prove

$$|e^{-z^3}| \leq 1 \ \forall \left\{z \middle| -\frac{\pi}{6} \leq \operatorname{Arg}(z) \leq \frac{\pi}{6}\right\}$$

Proof. We can express $-z^3$ as some complex number $a + ib$ where $a = \Re(-z^3)$ and $b = \Im(-z^3)$. Taking the magnitude gives

$$|e^{-z^3}| = |e^{a+ib}| = |e^a e^{ib}| = |e^a||e^{ib}| = |e^a| = |e^{\Re(-z^3)}|$$

z can be written as

$$z = |z|e^{i \operatorname{Arg}(z)}$$
$$z^3 = |z|^3 e^{i3 \operatorname{Arg}(z)} = |z|^3 \left(\cos(3 \operatorname{Arg}(z)) + i \sin(3 \operatorname{Arg}(z))\right)$$
$$-z^3 = -|z|^3 \left(\cos(3 \operatorname{Arg}(z)) + i \sin(3 \operatorname{Arg}(z))\right)$$
$$\Re(-z^3) = -|z|^3 \cos(3 \operatorname{Arg}(z))$$
$$\operatorname{Arg}(z) \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$$
$$\Rightarrow 3 \operatorname{Arg}(z) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$
$$\cos(3 \operatorname{Arg}(z)) \in [0, 1]$$
$$|z|^3 \in \{x \in \mathbb{R} | x \geq 0\}$$
$$\Re(-z^3) = -|z|^3 \cos(3 \operatorname{Arg}(z)) \in \{x \in \mathbb{R} | x \leq 0\}$$
$$e^{\Re(-z^3)} \in [0, 1]$$
$$\Rightarrow |e^{-z^3}| \leq 1$$

□

1.2.2 Calculus of Complex Functions

We define the limit of a complex function to be

$$w = f(z) = u + iv$$

$$\lim_{z\to z_0} f(z) = \lim_{(x,y)\to(x_0,y_0)} u(x,y) + i \lim_{(x,y)\to(x_0,y_0)} v(x,y)$$

Note that the notation $(x,y) \to (x_0,y_0)$ means that the limit is taken as (x,y) approaches (x_0,y_0) along *any* path.
The usual limit arithmetic rules are able to be applied as with real numbers.
In order for $\lim\limits_{z\to z_0} f(z)$ to exist, we require that both $\lim\limits_{(x,y)\to(x_0,y_0)} u(x,y)$ and $\lim\limits_{(x,y)\to(x_0,y_0)} v(x,y)$ exist.
If we define $z_0 = x_0 + iy_0$ then we can define the derivative of a complex function as

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

If this limit exists then the function is said to be differentiable at z_0.
Ex: $f(z) = z$

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \frac{z_0 + \Delta z - z_0}{\Delta z} = 1$$
$$\Rightarrow f'(z_0) = 1$$

Ex2: $f(z) = \bar{z}$

$$f'(z) = \lim_{\Delta z\to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta z\to 0} \frac{\overline{\Delta z}}{\Delta z}$$
$$\Delta z = h_1 + ih_2 \Rightarrow \overline{\Delta z} = h_1 - ih_2$$
$$h_2 = 0: \lim_{\Delta z\to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{h_1\to 0} \frac{h_1}{h_1} = 1$$
$$h_1 = 0: \lim_{\Delta z\to 0} \frac{\overline{\Delta z}}{\Delta z} = \lim_{h_2\to 0} \frac{-ih_2}{ih_2} = -1$$
$$\lim_{h_1\to 0} \frac{h_1}{h_1} \neq \lim_{h_2\to 0} \frac{-ih_2}{ih_2} \quad \therefore \text{ the derivative does not exist}$$

An easy way to determine if a function is differentiable is to use the Cauchy-Riemann equations. Any path that can be taken to approach z_0 can be written as a linear combination of the paths $\Delta z = \Delta x$ and $\Delta z = i\Delta y$ so the derivative must satisfy both of these paths.

$$f'(z_0) = \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$
$$f(z_0) = u(x_0,y_0) + iv(x_0,y_0)$$
Define $\Delta z = \Delta x$
$$f'(z_0) = \lim_{\Delta x\to 0} \frac{u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0) - u(x_0,y_0) - iv(x_0,y_0)}{\Delta x}$$
$$f'(z_0) = \lim_{\Delta x\to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0,y_0)}{\Delta x} + i \lim_{\Delta x\to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0,y_0)}{\Delta x}$$
$$f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i\frac{\partial v}{\partial x}(x_0,y_0)$$
Define $\Delta z = i\Delta y$

$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y) - u(x_0, y_0) - iv(x_0, y_0)}{i\Delta y}$$
$$f'(z_0) = \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{i\Delta y}$$
$$f'(z_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$$
$$\Rightarrow \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = -i\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0)$$

Splitting the real and imaginary parts we get that the Cauchy-Riemann equations are

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

If the Cauchy-Riemann equations are satisfied and the partial derivatives are continuous then the function is differentiable.
Some functions are not differentiable everywhere, but are differentiable at a point or a set of points.

- If $f(z)$ is differentiable everywhere in the complex plane then it is said to be **entire**.
- If $f(z)$ is differentiable in some region R then it is said to be **analytic** in R.
  (note that this region cannot be a single point, as the Cauchy-Riemann equations require the partial derivatives to be continuous)

Ex: Show using the Cauchy-Riemann equations that $f(z) = \bar{z}$ is not differentiable anywhere.

$$\overline{z} = x - iy$$
$$u_x = 1 \neq v_y = -1$$

Ex2: Show that $f(z) = z^2$ is entire

$$f(z) = z^2 = (x + iy)^2 = x^2 - y^2 + 2ixy$$
$$u(x, y) = x^2 - y^2$$
$$v(x, y) = 2xy$$
$$u_x = 2x = v_y$$
$$u_y = -2y = -v_x$$

Ex3: Show that $f(z) = \bar{z}$ is differentiable but not analytic at $z_0 = 0$

$$|z|^2 + 2z = x^2 + 2x + y^2 + i2y$$
$$u_x = 2x + 2 = v_y = 2 \Rightarrow x = 0$$
$$u_y = 2y = -v_x = 0 \Rightarrow y = 0$$
differentiable but not analytic on $z = \{0\}$

1.2.3 Conformal Mappings

Using the Cauch-Riemann equations,

$$\begin{cases} u_x = v_y \\ u_y = -v_x \end{cases}$$

we can get the Laplacian of u and v,

$$\begin{aligned} u_{xx} + u_{yy} &= v_{yx} - v_{xy} = 0 \\ v_{xx} + v_{yy} &= -(u_{yx} + u_{xy}) = 0 \end{aligned}$$

If the Laplacian of u and v are both zero then the function is said to be **harmonic**.
If $\nabla^2 u = 0$ then we can use the Cauchy-Riemann equations to find its harmonic conjugate v. Ex: Find the harmonic conjugate of $u = xy - x + y$

$$\begin{aligned} u_x &= y - 1 = v_y \\ v &= \int y - 1 dy = \frac{y^2}{2} - y + h(x) \\ u_y &= x + 1 = -v_x = -h'(x) \\ h(x) &= \int -x - 1 dx = -\frac{x^2}{2} - x + C \\ v &= \frac{y^2}{2} - \frac{x^2}{2} - y - x + C \end{aligned}$$

Ex2: Find the harmonic conjugate of $u = \ln\sqrt{x^2 + y^2}$

$$\begin{aligned} u_x &= \frac{x}{x^2 + y^2} = v_y \\ v &= \int \frac{x}{x^2 + y^2} dy = \int \frac{1/x}{1 + \frac{y^2}{x^2}} dy = \arctan\left(\frac{y}{x}\right) + h(x) \\ u_y &= \frac{y}{x^2 + y^2} = -v_x = -\frac{y}{1 + \frac{y^2}{x^2}}\left(\frac{-1}{x^2}\right) - h'(x) = \frac{y}{x^2 + y^2} - h'(x) \Rightarrow h'(x) = 0 \\ h(x) &= C \\ v &= \arctan\left(\frac{y}{x}\right) + C \\ v &= \arg(z) + C \end{aligned}$$

Ex3: Find the harmonic conjugate of $u = \sin x \cosh y$

$$\begin{aligned} u &= \sin x \cosh y \\ u_x &= \cos x \cosh y = v_y \\ v &= \int \cos x \cosh y dy = \cos x \sinh y + h(x) \\ u_y &= \sin x \sinh y = -v_x = -(-\sin x \sinh y) - h'(x) \Rightarrow h'(x) = 0 \\ h(x) &= C \end{aligned}$$

$$v = \cos x \sinh y + C$$

Another property is that
$$|f'(z)|^2 = |\nabla u|^2 = |\nabla v|^2$$

This also implies that
$$\nabla u \cdot \nabla v = 0$$

$$f'(z) = u_x + iv_x = \frac{1}{i}(u_y + iv_y)$$
$$|f'(z)|^2 = u_x^2 + v_x^2 = u_y^2 + v_y^2 = |\nabla u|^2 = |\nabla v|^2$$
$$\nabla u \cdot \nabla v = u_x u_y + v_x v_y = u_x v_x + (-v_x)(u_x) = 0$$

A conformal mapping is a mapping between two regions that preserves angles. If we have some function $f(z) = u + iv$ that is analytic then the can create a function of a function as

$$\Phi(u(x,y), v(x,y)) = \phi(x,y)$$

where $\phi(x,y)$ is a conformal mapping.
A conformal mapping will have the property that

$$\phi_{xx} + \phi_{yy} = |f'(z)|^2(\Phi_{uu} + \Phi_{vv})$$

where $|f'(z)|^2$ is known as the *conformal factor*.
This can be shown as follows:

$$\phi_x = \Phi_u u_x + \Phi_v v_x$$
$$\phi_{xx} = u_{xx}\Phi_u^2 + 2u_x v_x \Phi_u \Phi_v + v_{xx}\Phi_v^2 + \Phi_u u_{xx} + \Phi_v v_{xx}$$
$$\phi_y = \Phi_u u_y + \Phi_v v_y$$
$$\phi_{yy} = u_{yy}\Phi_u^2 + 2u_y v_y \Phi_u \Phi_v + v_{yy}\Phi_v^2 + \Phi_u u_{yy} + \Phi_v v_{yy}$$
$$\phi_{xx} + \phi_{yy} = \Phi_u \nabla^2 u + \Phi_v \nabla^2 v + \Phi_{uu}|\nabla u|^2 + 2\Phi_u \Phi_v \nabla u \cdot \nabla v + \Phi_{vv}|\nabla v|^2$$
$$\phi_{xx} + \phi_{yy} = |f'(z)|^2(\Phi_{uu} + \Phi_{vv})$$

Ex: $f(z) = z^2$ under $\Phi(u,v) = e^u + v^2$

$$f(z) = x^2 - y^2 + i(2xy)$$
$$u = x^2 - y^2$$
$$v = 2xy$$
$$\phi(x,y) = e^{x^2-y^2} + (2xy)^2$$
$$\phi_{xx} + \phi_{yy} = 4|z|^2(e^u + 2)$$

$f(z)$ is considered a conformal mapping if f is analytic and $f'(z) \neq 0$. As a consequence, if $\phi_{xx} + \phi_{yy} = 0$ then $\Phi_{uu} + \Phi_{vv} = 0$.
If f is a conformal mapping then it will preserve angles.
If we have two curves C_1 and C_2 described by the parametrized functions $z_1(t)$ and $z_2(t)$ that intersect at a point z_0 then the angle between the two curves is given by $\theta = \arg(z_2'(t)) - \arg(z_1'(t))$.

If we then apply a conformal mapping $w = f(z)$ to the curves then we get $w_1(t) = f(z_1(t))$ and $w_2(t) = f(z_2(t))$ and the angle between the two curves is given by $\theta_w = \arg(w_2'(t)) - \arg(w_1'(t))$.

$$
\begin{aligned}
&\theta_w = \arg(f'(z_2)z_2'(t)) - \arg(f'(z_1)z_1'(t)) \\
&\theta_w = \arg(f'(z_2)) - \arg(f'(z_1)) + \arg(z_2'(t)) - \arg(z_1'(t)) \\
&\arg(f'(z_1)) = \arg(f'(z_2)) \\
&\Rightarrow \theta_w = \arg(z_2'(t)) - \arg(z_1'(t)) = \theta
\end{aligned}
$$

So the angle between the two curves is preserved under a conformal mapping.
Note that $f'(z) \neq 0$ is a necessary condition in this proof.
If we have a nonconformal mapping then the angle between the two curves will not be preserved. One such case is that of $f(z) = z^2$ at $z_0 = 0$. At this point, $f'(z_0) = 0$ and the angle between the two curves is doubled.

Conformal mappings also have the property that they map Neumann boundary conditions to Neumann boundary conditions.
If we let $\hat{n}$ represent the normal vector to the curve $\phi(x, y)$ then they are related by

$$\frac{\partial \phi}{\partial n} = \nabla \phi \cdot \hat{n}$$

and the mapping is similarly related brcurly

$$
\begin{aligned}
&\frac{\partial \Phi}{\partial n'} = \nabla \Phi \cdot \hat{n}' \\
&\frac{\partial \phi}{\partial n} = |f'(z)| \frac{\partial \Phi}{\partial n'}
\end{aligned}
$$

So for Neumann bounday conditions, we will have

$$\frac{\partial \phi}{\partial n} = 0 \Rightarrow \frac{\partial \Phi}{\partial n'} = 0$$

Some examples of conformal mappings come from harmonic functions (having the property that $\nabla^2 \phi = 0$).
Some common harmonic functions are:

- $\phi = C$
- $\phi = ax + by + c$
- $\ln \sqrt{x^2 + y^2},\ \mathbb{C} \backslash 0$
- $\phi = \mathrm{Arg}(z),\ \mathbb{C} \backslash (-\infty]$
- $\phi = x^2 - y^2$

1.2.4 Conformal Mapping to Solve Laplace's Equation

Given the useful properties of matching boundary conditions, we can use conformal mappings to help us solve Laplace's equation for a given region.
Ex: Given $\nabla^2\phi = 0$ for $1 < x^2 - y^2 < 4$ and $\phi = 1$ on $x^2 - y^2 = 1$ and $\phi = 3$ on $x^2 - y^2 = 4$, find $\phi(x, y)$.

$$\text{choose } \phi = x^2 - y^2$$
$$\text{choose } \Phi(u, v) = Au + B$$
$$\begin{cases} A(1) + B = 1 \\ A(4) + B = 3 \end{cases} \Rightarrow A = \frac{2}{3},\ B = \frac{1}{3}$$
$$\Phi(u, v) = \frac{2}{3}u + \frac{1}{3}$$
$$u = x^2 - y^2$$
$$\phi(x, y) = \frac{2}{3}(x^2 - y^2) + \frac{1}{3}$$

Ex2: Given $\nabla^2\phi = 0$ within the circular region $\mathcal{D} = \left\{1 < x^2 + y^2 < 4\right\}$ with $\phi = 1$ on $x^2 + y^2 = 1$ and $\phi = -2$ on $x^2 + y^2 = 4$, find $\phi(x, y)$.

$$\phi(x, y) = A_1 \ln r + A_2$$
$$r = \sqrt{x^2 + y^2}$$
$$r = 1 :\ \phi = 1 \Rightarrow A_1 \ln(1) + A_2 = 1$$
$$r = 2 :\ \phi = -2 \Rightarrow A_1 \ln(2) + A_2 = -2$$
$$\Rightarrow A_2 = 1,\ A_1 = -\frac{3}{\ln 2}$$
$$\phi(x, y) = -\frac{3}{\ln 2} \ln \sqrt{x^2 + y^2} + 1$$

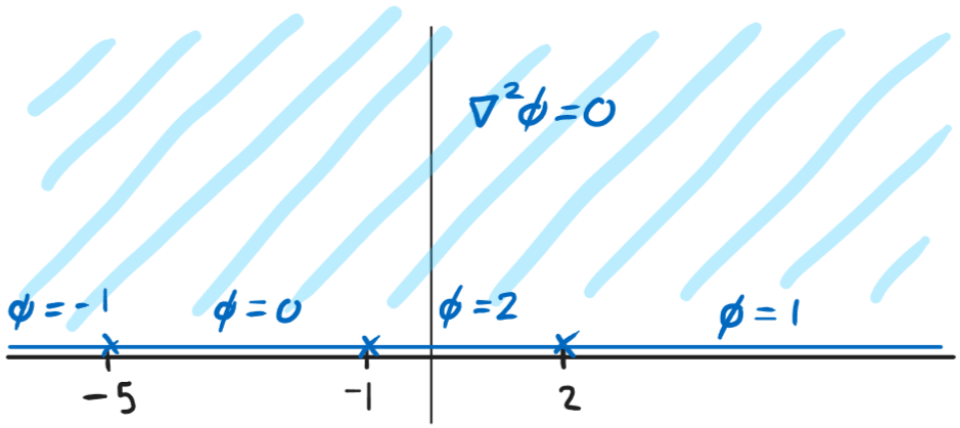
Ex3: Given $\nabla^2 = 0$ within the strip described by $\{z : -3 \le 3\Re(z) - 4\Im(z) \le 2\}$ with $\phi = 0$ for $\{z : -3 = 3\Re(z) - 4\Im(z)\}$ and $\phi = 4$ for $\{z : 3\Re(z) - 4\Im(z) = 2\}$ find $\phi(x, y)$

$$u = 3x - 4y$$
$$u(-3) = 0,\ u(2) = 4$$
$$\Phi = Au + B$$
$$\Phi(-3) = -3A + B = 0 \Rightarrow B = 3A$$
$$\Phi(2) = 2A + 3A = 5A = 4 \Rightarrow A = \frac{4}{5} \Rightarrow B = \frac{12}{5}$$
$$\phi(x, y) = \frac{4}{5}(3x - 4y) + \frac{12}{5}$$

Ex4: Given $\nabla^2\phi = 0$ for the upper half-plane described by $\{y > 0 \wedge x \in \mathbb{R}\}$ with the boundary conditions along the x-axis given by

$$\phi(x, 0) = \begin{cases} -1 & x < -5 \\ 0 & -5 < x < -1 \\ 2 & -1 < x < 2 \\ 1 & x > 2 \end{cases}$$

find $\phi(x, y)$.



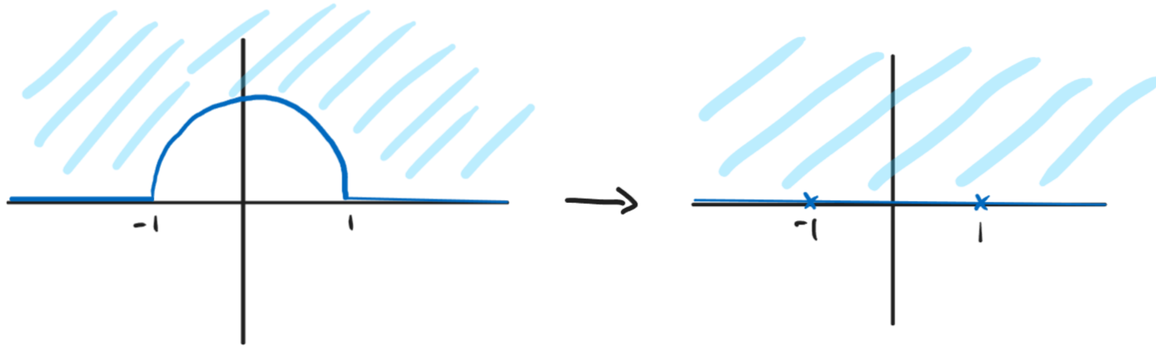
One trick to solve a problem like this is to choose a linear combination of functions of the form $\text{Arg}(z - z_0)$ with a different point z_0 for every place where the boundary condition changes along the x-axis.

$$\phi = A_1 \text{Arg}(z + 5) + A_2 \text{Arg}(z + 1) + A_3 \text{Arg}(z - 2) + A_4$$
$$\phi(x > 2, 0) = A_4 = 1$$
$$\phi(-1 < x < 2, 0) = \pi A_3 + 1 = 2 \Rightarrow A_3 = \frac{1}{\pi}$$
$$\phi(-5 < x < -1, 0) = \pi A_2 + 1 + 1 = 0 \Rightarrow A_2 = -\frac{2}{\pi}$$
$$\phi(x < -5, 0) = \pi A_1 - 2 + 2 = -1 \Rightarrow A_1 = -\frac{1}{\pi}$$
$$\phi(x, y) = -\frac{1}{\pi} \text{Arg}(z + 5) - \frac{2}{\pi} \text{Arg}(z + 1) + \frac{1}{\pi} \text{Arg}(z - 2) + 1$$

We can also apply these techniques to other types of boundary conditions in some cases.
Ex5: Given $\nabla^2\phi = 0$ in the circular region $\{z : 1 \leq |z| \leq 2\}$ with the boundary conditions $\phi = 1$ for $|z| = 1$ and $\frac{\partial\phi}{\partial r} = 2$ for $|z| = 2$, find $\phi(x, y)$

$$\phi = A \ln r + B$$
$$\phi(1) = B = 1$$
$$\frac{\partial\phi(2)}{\partial r} = \frac{A}{2} = 2 \Rightarrow A = 4$$
$$\phi = 4 \ln r + 1$$
$$\phi(x, y) = 4 \ln \sqrt{x^2 + y^2} + 1$$

If we have a region that has a semicircle in it we can use the Joukowski mapping to transform it into a region that is easier to work with.



Ex6: Given $\nabla^2\phi = 0$ in the upper region of the plane described by $\{y > 0 \wedge x^2 + y^2 > 9\}$ with the boundary conditions $\phi = -1$ for $x < 3$, $\phi = 0$ for $x^2 + y^2 = 9$, and $\phi = 2$ for $x > 3$, find $\phi(x, y)$.

$$\begin{cases} \phi(x,0) = -1 & x < -3 \\ \phi(x,y) = 0 & x^2 + y^2 = 9 \\ \phi(x,0) = 2 & x > 3 \end{cases}$$

$$w = \frac{1}{2}\left(\frac{z}{3} + \frac{3}{z}\right) = u + iv$$

$$\Phi = A_1 \operatorname{Arg}(w+1) + A_2 \operatorname{Arg}(w-1) + A_3$$

$$\Phi(u > 1, v) = A_3 = 2$$

$$\Phi(-1 < u < 1, v) = \pi A_2 + 2 = 0 \Rightarrow A_2 = -\frac{2}{\pi}$$

$$\Phi(u < -1, v) = \pi A_1 - 2 + 2 = -1 \Rightarrow A_1 = -\frac{1}{\pi}$$

$$\Phi = -\frac{1}{\pi} \operatorname{Arg}(w+1) - \frac{2}{\pi} \operatorname{Arg}(w-1) + 2$$

$$\phi(z) = -\frac{1}{\pi} \operatorname{Arg}\left(\frac{1}{2}\left(\frac{z}{3} + \frac{3}{z}\right) + 1\right) - \frac{2}{\pi} \operatorname{Arg}\left(\frac{1}{2}\left(\frac{z}{3} + \frac{3}{z}\right) - 1\right) + 2$$

In the case that we have a semicircle not of radius 1 we can apply a scaling before the Joukowski mapping to get the correct radius.

$$w = \frac{1}{2}\left(\frac{z}{a} + \frac{a}{z}\right)$$

1.2.5 Sinusoidal Functions

If we recall Euler's formula $e^{ix} = \cos x + i \sin x$ we can use with complex numbers to get the following identities for complex sinusoidals:

- $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$

- $\cos z = \frac{e^{iz}+e^{-iz}}{2}$
- $\sinh z = \frac{e^{z}-e^{-z}}{2}$
- $\cosh z = \frac{e^{z}+e^{-z}}{2}$
- $\cos z = \sin\left(\frac{\pi}{2} - z\right)$
- $\sinh z = -i \sin(iz)$
- $\cosh z = \cos(iz) = \sin\left(\frac{\pi}{2} - iz\right)$
- $\frac{d}{dz} \sin z = \cos z$
- $\frac{d}{dz} \cos z = -\sin z$
- $\cos^2 z + \sin^2 z = 1$
- $\frac{d}{dz} \sinh z = \cosh z$
- $\frac{d}{dz} \cosh z = \sinh z$
- $\cosh^2 z - \sinh^2 z = 1$

The most notable difference between the real and complex versions of these functions is that $|\sin z| \not\leq 1$. We will see cases of this soon but it is easy to see once we write out the real and imaginary components of $\sin z$.

$$\begin{aligned}
&\sin z = \sin(x+iy) = \frac{e^{i(x+iy)-e^{-i(x+iy)}}}{2i} = \frac{e^{-y}e^{ix} - e^{y}e^{-ix}}{2i} \\
&\sin z = \frac{e^{-y}(\cos x + i\sin x) - e^{y}(\cos x - i\sin x)}{2i} = \frac{e^{-y}-e^{y}}{2}\cos x + \frac{e^{-y}+e^{y}}{2i}\sin x \\
&\sin z = \sin x \cosh y + i \cos x \sinh y
\end{aligned}$$

Ex: Show that if $|z| < 1$ then $|\sin z| < 2$.

Proof.

$$|\sin z| < 2, \ |z| < 1$$

$$\begin{aligned}
&\sin z = \sin x \cosh y + i \cos x \sinh y \\
&|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y} \\
&\cosh^2 y - \sinh^2 y = 1 \Rightarrow \sinh^2 y = \cosh^2 y - 1 \\
&|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x(\cosh^2 y - 1)} \\
&|\sin z| = \sqrt{\sin^2 x \cosh^2 y + \cos^2 x \cosh^2 y - \cos^2 x} \\
&|\sin z| = \sqrt{\cosh^2 y - \cos^2 x} \leq \cosh y
\end{aligned}$$

$$|z| = \sqrt{x^2 + y^2} < 1 \Rightarrow y < 1$$
$$\cosh y < \cosh 1 = \frac{e + e^{-1}}{2} \approx 1.54 < 2$$
$$|\sin z| < 2$$

$\square$

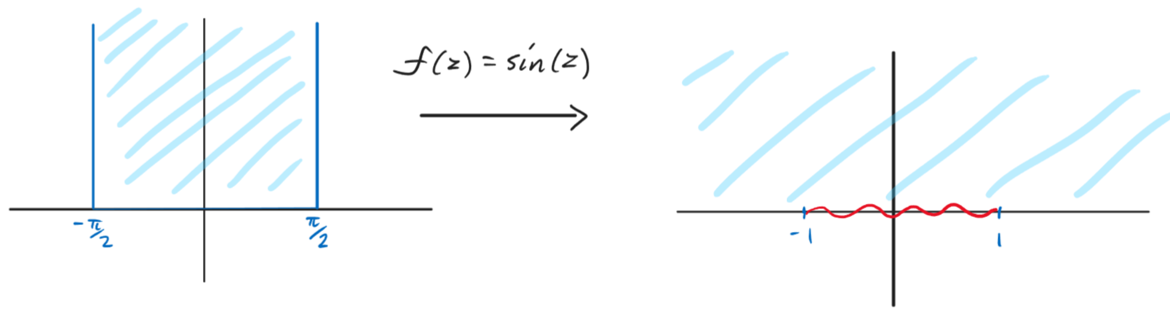
Ex2: Find all solutions to $\sin z = 4i$

$$\sin(z) = 4i = \sin x \cosh y + i \cos x \sinh y$$
$$\sin x \cosh y = 0 \Rightarrow \sin x = 0 \Rightarrow x = n\pi$$
$$\text{Case 1: } n = 2k : \ \cos x = 1$$
$$4 = \cos(2k\pi)\sinh(y) = \sinh y$$
$$\text{Case 2: } n = 2k+1 : \ \cos x = -1$$
$$4 = \cos((2k+1)\pi)\sinh y = -\sinh y$$
$$\sinh y = \frac{e^y - e^{-y}}{2}$$
$$2\sinh y = e^y - e^{-y}$$
$$e^{2y} - 2\sinh y e^y - 1 = 0$$
$$e^y = \frac{2\sinh y \pm \sqrt{4\sinh^2 y + 4}}{2} = \sinh y \pm \sqrt{\sinh^2 y + 1}$$
$$n = 2k :$$
$$y = \ln\left(4 \pm \sqrt{17}\right) = \ln\left(4 + \sqrt{17}\right)$$
$$n = 2k+1 :$$
$$y = \ln\left(-4 \pm \sqrt{17}\right) = \ln\left(\sqrt{17} - 4\right)$$
$$z = \left\{ (x,y) \middle| \left(2k\pi, \ln\left(4 + \sqrt{17}\right)\right), \left((2k+1)\pi, \ln\left(\sqrt{17} - 4\right), k \in \mathbb{Z}\right) \right\}$$

Ex3: Find all solutions to $\cos z = 0$

$$\cos(z^4) = 0$$
$$z^4 = \pi n + \frac{\pi}{2} = \left(\pi n + \frac{\pi}{2}\right) e^{2\pi i l}$$
$$z = \left(\pi n + \frac{\pi}{2}\right)^{1/4} e^{i\frac{\pi l}{2}}, \ l = \{0, 1, 2, 3\}, \ n \in \mathbb{Z}$$

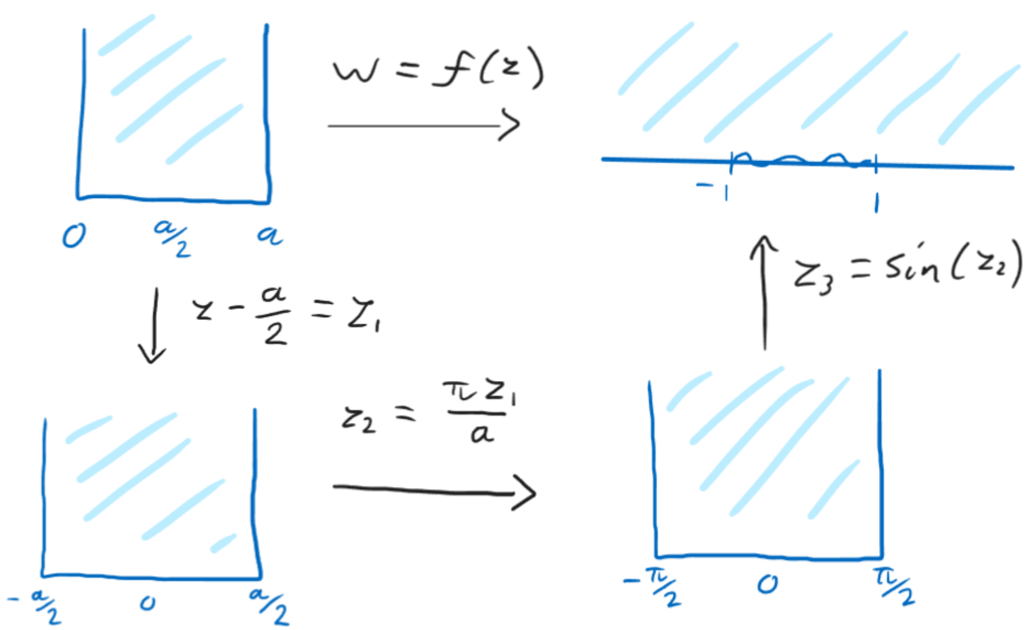
Mapping properties of $\sin z$
$\sin z$ will map the box $-\frac{\pi}{2} \le x \le \frac{\pi}{2}, \ 0 \le y < \infty$ to the half plane $v > 0$.



Ex: Find the mapping of $\sin z$ on $S = \left\{-\frac{\pi}{2} < x < \frac{\pi}{2}, 0 < y < 1\right\}$

$$u = \sin x \cosh y$$
$$\sin x \in (-1, 1)$$
$$\cosh y \in (0, \cosh(1))$$
$$u \in (-\cosh(1), \cosh(1))$$
$$v = \cos x \sinh y$$
$$\cos x \in (0, 1]$$
$$\sinh y \in (0, \sinh(1))$$
$$v \in (0, \sinh(1))$$
$$S' = \{(u, v) | -\cosh(1) < u < \cosh(1), 0 < v < \sinh(1)\}$$

If the box is offcenter we can also apply a composition of mappings to shift it into the usual form.



Ex2: Find the mapping of $\sin z$ on $S = \{-1 < x < 1, y > 0\}$

$$u = \sin x \cosh y$$
$$\sin x \in (-\sin(1), \sin(1))$$

$$
\begin{aligned}
&\cosh y \in (1, \infty) \\
&u \in \mathbb{R} \\
&v = \cos x \sinh y \\
&\cos x \in (\cos(1), 1] \\
&\sinh y \in (0, \infty) \\
&v \in (0, \infty) \\
&S' = \{(u, v) | v > 0\}
\end{aligned}
$$

Ex3: Solve the Laplace equation $\nabla^2 u = 0$ in the region $S = \{0 \le x \le 2, 0 \le y < \infty\}$ with boundary conditions $\phi = 0$ on $x = 0$ and $\phi = 1$ on $y = 0$ and $\phi = -2$ on $x = 2$.

$$
\begin{aligned}
&\text{Map to the half plane using } w = \sin\left(\frac{\pi}{2}(z-1)\right) \\
&\phi(z) = \Phi(w) = A_1 \operatorname{Arg}(w+1) + A_2 \operatorname{Arg}(w-1) + A_3 \\
&u > 1: \;\; \Phi = -2 = A_3 \\
&-1 < u < 1: \;\; \Phi = 1 = A_2\pi - 2 \Rightarrow A_2 = \frac{3}{\pi} \\
&u < -1: \;\; \Phi = 0 = A_1\pi + \frac{3}{\pi}\pi - 2 \Rightarrow A_1 = -\frac{1}{\pi} \\
&\phi(z) = -\frac{1}{\pi} \operatorname{Arg}\left(\sin\left(\frac{\pi}{2}(z-1)\right) + 1\right) + \frac{3}{\pi} \operatorname{Arg}\left(\sin\left(\frac{\pi}{2}(z-1)\right) - 1\right) - 2
\end{aligned}
$$

1.2.6 Logarithmic Functions

$\log z$ is defined as the inverse of e^z.

$$
\begin{aligned}
&e^w = z \Rightarrow w = \log z \\
&w = u + iv \\
&z = re^{i \operatorname{Arg}(z)} \\
&e^u e^{iv} = re^{i \operatorname{Arg}(z)} \\
&e^u = r \Rightarrow u = \ln r \\
&e^{iv} = e^{i \operatorname{Arg}(z)} \Rightarrow v = \operatorname{Arg}(z) + 2k\pi \\
&\log(z) = w = \ln r + i(\operatorname{Arg} z + 2k\pi), \;\; k \in \mathbb{Z} \\
&\log(z) = \{w | e^w = z\} = \ln r + i(\operatorname{Arg} z + 2k\pi) = \ln r + i \arg z \\
&\operatorname{Log}(z) = \ln r + i \operatorname{Arg} z
\end{aligned}
$$

Note that $\log(z) = \ln r + i \arg z$ is a multi-valued function and $\operatorname{Log}(z) = \ln r + i \operatorname{Arg} z$ is the principal value of $\log z$.

Ex: Find all values for $\log 2$ and $\operatorname{Log} 2$.

$$
\begin{aligned}
&z = 2 = 2e^{i0} \\
&\log 2 = \ln 2 + i(0 + 2k\pi) = \ln 2 + 2k\pi i, \;\; k \in \mathbb{Z} \\
&\operatorname{Log} 2 = \ln 2 + i(0) = \ln 2
\end{aligned}
$$

Ex2: Find all values for $\log(-1-\sqrt{3}i)$ and $\operatorname{Log}(-1-\sqrt{3}i)$.

$$z=-1-\sqrt{3}i=2e^{i\frac{2\pi}{3}}$$
$$\log(-1-\sqrt{3}i)=\ln 2+i\left(-\frac{2\pi}{3}+2k\pi\right),\ k\in\mathbb{Z}$$
$$\operatorname{Log}(-1-\sqrt{3}i)=\ln 2+i\left(-\frac{2\pi}{3}\right)$$

Ex3: Find all values for $\log(e^{1+5i})$ and $\operatorname{Log}(e^{1+5i})$.

$$z=e^{1+5i}=ee^{5i}=ee^{i(5+2k\pi)}$$
$$5+2k\pi\in(-\pi,\pi]\Rightarrow k=-1$$
$$\operatorname{Arg}(e^{1+5i})=5-2\pi$$
$$\log(e^{1+5i})=\ln e+i(5-2\pi+2k\pi)=1+i(5-2\pi+2k\pi),\ k\in\mathbb{Z}$$
$$\operatorname{Log}(e^{1+5i})=\ln e+i(5-2\pi)=1+i(5-2\pi)$$

Ex4: Find all values for $\log(-i)$ and $\operatorname{Log}(-i)$.

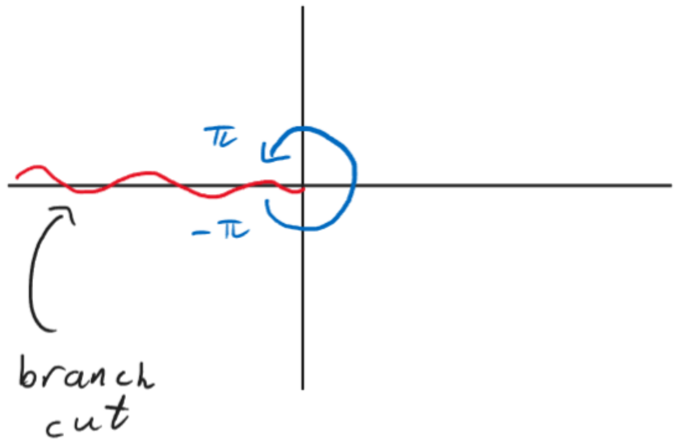
$$-i=e^{-i\frac{\pi}{2}}$$
$$\arg(-i)=-\frac{\pi}{2}+2\pi k$$
$$\log(-i)=-i\frac{\pi}{2}+i2\pi k$$
$$\operatorname{Log}(-i)=-i\frac{\pi}{2}$$

Properties of $\log z$ and $\operatorname{Log} z$:

- $\log(z_1z_2)=\log z_1+\log z_2$
  $\operatorname{Log}(z_1z_2)\neq\operatorname{Log} z_1+\operatorname{Log} z_2$ (same as why $\operatorname{Arg}(z_1z_2)\neq\operatorname{Arg}(z_1)+\operatorname{Arg}(z_2)$)
- $e^{\log z}=z$ and $e^{\operatorname{Log} z}=z$
  but $\log(e^z)\neq z$ and $\operatorname{Log}(e^z)\neq z$
  Ex: $z=0$: $\log(e^0)=\log(1)=\ln 1+i(0+2k\pi)=i2k\pi\neq 0$
  $\operatorname{Log}(e^{5i})=i(5-2\pi)\neq 5i$
  rather $z\in\log(e^z)$
- $\log z^n\neq n\log z$

  $$z=1,\ n=2:\ \log z^2=\log 1=i2k\pi$$
  $$2\log z=2\log 1=2i(2k')\pi$$

  but $\log z^{\frac{1}{n}}=\frac{1}{n}\log z$
- $\operatorname{Log} z$ is analytic in $\mathbb{C}\backslash(-\infty,0]$



$$\operatorname{Log} z = \ln|z| + i \operatorname{Arg} z, \ -\pi < \operatorname{Arg} z < \pi$$

Ex: Find all solutions to $e^z = -1 - i$

$$z = \log(-1-i)$$
$$-1-i = \sqrt{2}e^{-i\frac{3\pi}{4}}$$
$$z = \frac{1}{2}\ln(2) - i\frac{3\pi}{4} + i2\pi k$$

Ex2: Find the principal value of $(1+i)^i = e^{i\operatorname{Log}(1+i)}$

$$1+i = \sqrt{2}e^{i\frac{\pi}{4}}$$
$$\operatorname{Log}(1+i) = \frac{1}{2}\ln(2) + i\frac{\pi}{4}$$
$$(1+i)^i = e^{-\frac{\pi}{4}+i\frac{1}{2}\ln(2)}$$

Ex3: Find the principle value of $z^i = 1+i$

$$1+i = \sqrt{2}e^{i\frac{\pi}{4}+i2\pi k} = e^{\ln(\sqrt{2})+i\frac{\pi}{4}+i2\pi k}$$
$$z = (1+i)^{1/i} = (1+i)^{-i} = e^{\frac{\pi}{4}+2\pi k}e^{-i\ln(\sqrt{2})}$$
$$z = e^{\frac{\pi}{4}}e^{-i\frac{1}{2}\ln(2)}$$

Ex4: Find the principal value of i^i.

$$i^i = (e^{i\frac{\pi}{2}+i2\pi k})^i = e^{-\frac{\pi}{2}-2\pi k} = e^{-\frac{\pi}{2}}$$

Ex5: Find all solutions to $\operatorname{Log}(z^2-1) = \frac{i\pi}{2}$

$$z^2 - 1 = e^{i\frac{\pi}{2}} = i$$
$$z^2 = i+1 = \sqrt{2}e^{i\frac{\pi}{4}+i2\pi k}$$
$$z = 2^{1/4}e^{i\frac{\pi}{8}+i\pi k}$$

Ex6: Find all solutions to $e^{2z} + e^z + 1 = 0$

$$e^z = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$
$$e^z = e^{i\pi \pm i\frac{\pi}{3} + i2\pi k}$$
$$z = i\pi \pm i\frac{\pi}{3} + i2\pi k$$
$$z = \pm i\frac{2\pi}{3} + i2\pi k$$

Derivative of $\log z$ *and* $\operatorname{Log} z$

$$(\operatorname{Log} z)' = \frac{1}{z}$$
$$\operatorname{Log} z = \ln\sqrt{x^2+y^2} + i\operatorname{Arg} z = u + iv$$
$$(\operatorname{Log} z)' = u_x + iv_x$$
$$\operatorname{Log} z = \ln|z| + i\varphi,\ -\pi < \varphi < \pi$$

1.2.7 Branch Cuts

There are many different analytic functions for $\log z$ depending on the branch cut we choose. For example we can choose:

$$\operatorname{Log} z = \ln r + i\varphi,\ -\pi < \varphi < \pi,\ \mathbb{C}\backslash(-\infty, 0]$$
$$\log z = \ln r + i\varphi,\ 0 < \varphi < 2\pi,\ \mathbb{C}\backslash[0, \infty)$$
$$\log z = \ln r + i\varphi,\ -\frac{\pi}{2} < \varphi < \frac{5\pi}{2},\ \mathbb{C}\backslash i[0, \infty)$$

Ex: Specify a branch cut for $\log z$ such that it is analytic at $z = -1, -i, 1$

$$\log z = \ln r + i\varphi,\ \frac{\pi}{2} < \varphi < \frac{5\pi}{2}$$

The reason we have branch cuts is so that we can define a single-valued analytic function. This is important for functions such as $\sqrt{z}$ and $\log z$ as one input can correspond to multiple outputs. For example, $\sqrt{1} = 1$ and $\sqrt{1} = -1$. This is why we need to specify a branch cut for $\sqrt{z}$ and $\log z$. Ex: Construct $f(z) = z^{1/2}$ such that it is analytic in $\mathbb{C}\backslash(-\infty, 0]$ and $f(1) = -1$.

$$z^{\frac{1}{2}} = e^{\frac{1}{2}\log z} = e^{\frac{1}{2}\ln r + i\frac{1}{2}\varphi} = r^{\frac{1}{2}} e^{i\frac{\varphi}{2}},\ -\pi < \varphi < \pi$$
$$f(1) = r^{\frac{1}{2}} e^{i\frac{\varphi}{2}} = 1e^{i0} = 1$$
$$z^{\frac{1}{2}} = r^{\frac{1}{2}} e^{i\frac{\varphi}{2}},\ \pi < \varphi < 3\pi$$
$$\to f(1) = 1^{\frac{1}{2}} e^{\frac{2\pi}{2}} = -1$$

Ex2: Find all solutions to $z^{1/2} + 1 - i = 0$ where $z^{1/2}$ is the principal branch of $z^{1/2}$.

$$z^{1/2} = |z|^{1/2} e^{i\frac{\varphi}{2}} = i - 1 = \sqrt{2}e^{i\frac{3\pi}{4} + i2\pi k}$$
$$|z|e^{i\varphi} = 2e^{i\frac{3\pi}{2} + i4\pi k}$$

$$|z| = 2$$
$$\varphi = \frac{3\pi}{2} + 4\pi k \notin (-\pi, \pi]$$
$$\therefore \text{ no solutions}$$

Ex3: Find all solutions to $z^{1/2} + 1 - i = 0$

$$z^{1/2} = |z|^{1/2} e^{i\frac{\varphi}{2} + i\pi k_1} i - 1 = \sqrt{2} e^{i\frac{3\pi}{4} + i2\pi k_2}$$
$$|z| e^{i\varphi + i2\pi k_1} = 2e^{i\frac{3\pi}{2} + i4\pi k_2}$$
$$|z| = 2$$
$$\varphi + 2\pi k_1 = \frac{3\pi}{2} + 4\pi k_2$$
$$\varphi = \frac{3\pi}{2} + 2\pi k_3$$
$$z = 2e^{-i\frac{\pi}{2}} = -2i$$

Ex4: Find where $\mathrm{Log}(1 + z^2)$ is analytic

$$w = 1 + z^2$$
$$\text{not analytic on } \Re(w) \leq 0 \ \wedge \Im(w) = 0$$
$$x^2 - y^2 + 1 \leq 0$$
$$2xy = 0 \Rightarrow x = 0 \vee y = 0$$
$$y = 0: \ x^2 + 1 \leq 0 \Rightarrow \nexists x \in \mathbb{R} \text{ s.t. } x^2 \leq -1$$
$$x = 0: \ y^2 \geq 1 \Rightarrow |y| \geq 1$$

Analytic on the domain

$$\mathbb{C} \setminus \{(x, y) : x = 0, |y| \geq 1\}$$

Ex5: Find where $\mathrm{Log}\left(\frac{1-z}{1+z}\right)$ is analytic

$$w = \frac{1 - z}{1 + z}$$
$$\text{not analytic on } \Re(w) \leq 0 \ \wedge \Im(w) = 0$$
$$w = \frac{(1 - z)(1 + \overline{z})}{(1 + z)(1 + \overline{z})} = \frac{1 - |z|^2 - z + \overline{z}}{1 + |z|^2 + z + \overline{z}} = \frac{1 - x^2 - y^2 - 2iy}{1 + x^2 + y^2 + 2x}$$
$$y = 0$$
$$\frac{1 - x^2}{1 + x^2 + 2x} \leq 0 \Rightarrow 1 \leq x^2 \Rightarrow |x| \geq 1$$

Analytic on

$$\mathbb{C} \setminus \{(x, y) : y = 0, |x| \geq 1\}$$

Ex6: Find where $f(z) = \text{Log}(1-z^3)$ is analytic

$$\Re(1-z^3) \le 0 \qquad \Im(1-z^3) = 0$$
$$z^3 = r^3 e^{3i\varphi} = r^3 = r^3\left(\cos(3\varphi) + i\sin(3\varphi)\right)$$
$$\Im(1-z^3) = r^3\sin(3\varphi) = 0 \Rightarrow 3\varphi = n\pi$$
$$\varphi = \frac{n\pi}{3}$$
$$\Re(1-z^3) = 1 - r^3\cos(3\varphi) \le 0$$
$$n = 2k: \ \cos(3\varphi) = 1 \Rightarrow 1 - r^3 \le 0 \Rightarrow 1 \le r$$
$$n = 2k+1: \ \cos(3\varphi) = -1 \Rightarrow 1 + r^3 \le 0 \Rightarrow 1 \le -r \Rightarrow -1 \ge r$$

can't have $r \le -1$ so can't have $n = 2k+1$

$$\Rightarrow \varphi = \frac{2k\pi}{3}, \ 1 \le r$$

Analytic in $\mathbb{C}\backslash\left\{|z| \ge 1, \ \arg(z) = \dfrac{2k\pi}{3}, \ k \in \mathbb{Z}\right\}$

Ex7: Find a branch cut for $f(z) = \sqrt{z(z-1)}$ such that it is analytic for $\{(x,y)|y=0, x<0\}$ such that $f(2) = \sqrt{2}$

$$\sqrt{z(z-1)} = |z|^{1/2}e^{i\frac{\varphi_1}{2}}|z-1|^{1/2}e^{i\frac{\varphi_2}{2}} = (|z||z-1|)^{1/2}e^{i\frac{\varphi_1+\varphi_2}{2}}$$
$$\varphi_1 \in (-\pi,\pi), \ \varphi_2 \in (-\pi,\pi)$$
$$\lim_{\varphi_1\to\pi}\lim_{\varphi_2\to\pi} e^{i\frac{\varphi_1+\varphi_2}{2}} = e^{\pi i}$$
$$\lim_{\varphi_1\to-\pi}\lim_{\varphi_2\to-\pi} e^{i\frac{\varphi_1+\varphi_2}{2}} = e^{-\pi i} = e^{\pi i}$$

$\therefore$ continuous for $\{(x,y)|y=0, x<0\}$

$$z = 2: \ \varphi_1 = 0, \ \varphi_2 = 0 \Rightarrow \sqrt{z(z-1)} = \sqrt{2}$$

Ex8: Find a branch cut for $f(z) = \sqrt{z(z-1)}$ such that it is analytic for $\{(x,y)|y=0, x<0\}$ such that $f(2) = -\sqrt{2}$

$$\sqrt{z(z-1)} = |z|^{1/2}e^{i\frac{\varphi_1}{2}}|z-1|^{1/2}e^{i\frac{\varphi_2}{2}} = (|z||z-1|)^{1/2}e^{i\frac{\varphi_1+\varphi_2}{2}}$$
$$\varphi_1 \in (-\pi,\pi), \ \varphi_2 \in (-3\pi,-\pi)$$
$$\lim_{\varphi_1\to\pi}\lim_{\varphi_2\to-\pi} e^{i\frac{\varphi_1+\varphi_2}{2}} = e^{-2\pi i}$$
$$\lim_{\varphi_1\to-\pi}\lim_{\varphi_2\to-3\pi} e^{i\frac{\varphi_1+\varphi_2}{2}} = e^{0} = e^{-2\pi i}$$

$\therefore$ continuous for $\{(x,y)|y=0, x<0\}$

$$z = 2: \ \varphi_1 = 0, \ \varphi_2 = -2\pi \Rightarrow \sqrt{z(z-1)} = \sqrt{2}e^{-i\pi} = -\sqrt{2}$$

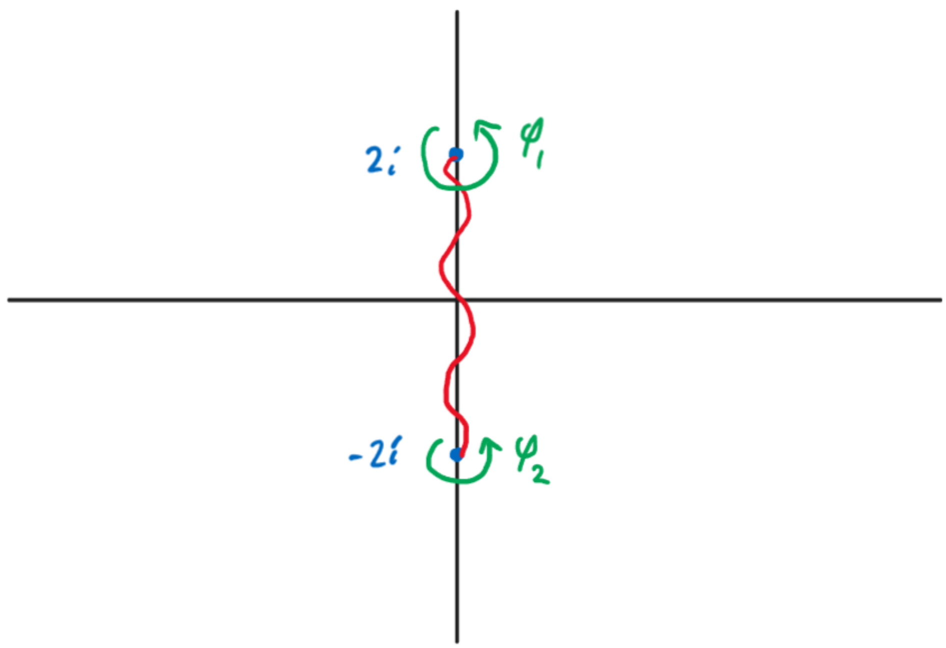
Ex9: Find a branch cut for $f(z) = \log(z^2+2z+2)$ such that it is analytic for $\{(x,y)|x>-1\}$ such that $\frac{d}{dz}f(-1) = 0$

$$\log(z^2+2z+2)$$

$$z = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$$
$$\log(z^2+2z+2) = \ln(|z-z_1||z-z_2|) + i(\varphi_1+\varphi_2)$$
$$-\pi < \varphi_1 < \pi,\ -\pi < \varphi_2 < \pi$$
$$z=-1:\ \varphi_1 = -\frac{\pi}{2},\ \varphi_2 = \frac{\pi}{2}$$
$$\log(z^2+2z+2)\Big|_{z=-1} = 0 + i\left(-\frac{\pi}{2}+\frac{\pi}{2}\right) = 0$$
$$\frac{d}{dz}\log(z^2+2z+2)\Big|_{z=-1} = \frac{2z+2}{z^2+2z+2}\Big|_{z=-1} = 0$$

If we have a function such as $f(z) = \sqrt{z^2-1}$ then we will have two branch points at $z = \pm 1$. If two of these branch cuts overlap then they have the property of cancelling out. Note that if we have a cube root then we require three overlapping branch cuts to cancel out. For an nth root we require n overlapping branch cuts to cancel out.
Ex: Find a branch cut for $f(z) = \sqrt{4+z^2}$ such that it is analytic for $\mathbb{C}\setminus\{x=0, -2 \le y \le 2\}$.

$$(4+z^2)^{1/2},\ \mathbb{C}\setminus\{x=0,-2\le y\le 2\}$$
$$4+z^2 = 0 \Rightarrow z = \pm 2i$$
$$(4+z^2)^{1/2} = |z+2i|^{1/2}|z-2i|^{1/2}e^{i\left(\frac{\varphi_1+\varphi_2}{2}\right)}$$

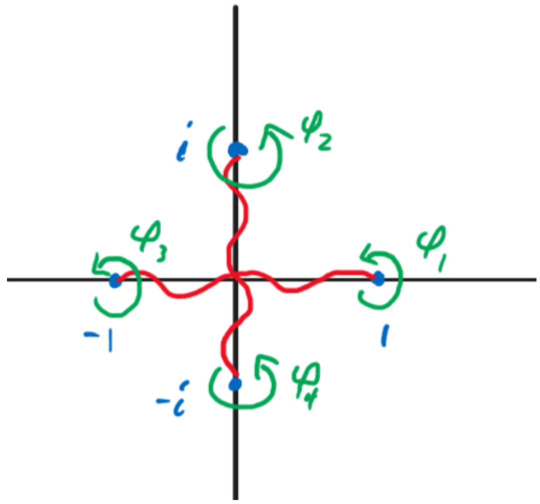


Choose $\frac{\pi}{2} < \varphi_1 < \frac{5\pi}{2}$ and $\frac{\pi}{2} < \varphi_2 < \frac{5\pi}{2}$
Check continuity at $y > 2$:

	φ_1	φ_2	$e^{i\left(\frac{\varphi_1+\varphi_2}{2}\right)}$
A	$\frac{5\pi}{2}$	$\frac{5\pi}{2}$	$e^{i\frac{5\pi}{2}} = i$
A’	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$e^{i\frac{\pi}{2}} = i$

Ex2: Find a branch cut for $f(z) = (z^4 - 1)^{1/2}$ such that it is analytic for $|z| > 1$.

$$(z^4 - 1)^{1/2}, \ \{|z| > 1\}$$
$$z^4 - 1 = 0 \Rightarrow z = \pm 1, \pm i$$
$$(z^4 - 1)^{1/2} = |z+1|^{1/2}|z-1|^{1/2}|z+i|^{1/2}|z-i|^{1/2}e^{i\left(\frac{\varphi_1+\varphi_2+\varphi_3+\varphi_4}{2}\right)}$$



Choose $-\pi < \varphi_1 < \pi$, $\frac{\pi}{2} < \varphi_2 < \frac{5\pi}{2}$, $-\pi < \varphi_3 < \pi$, and $\frac{\pi}{2} < \varphi_4 < \frac{5\pi}{2}$
Check continuity at $y > 1$:

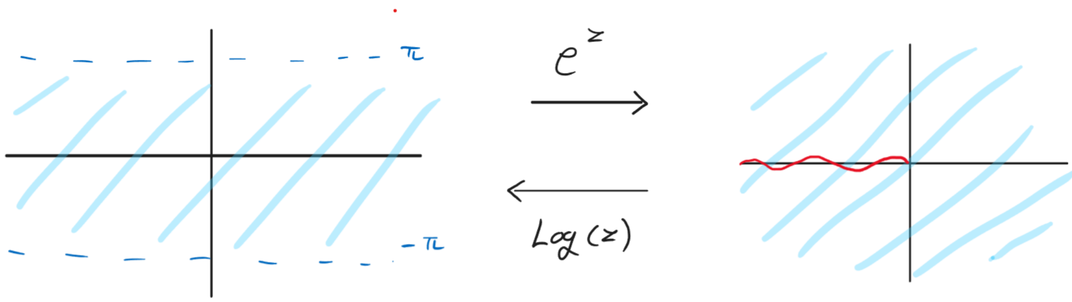
	φ_2	φ_4	$e^{i\left(\frac{\varphi_1+\varphi_2+\varphi_3+\varphi_4}{2}\right)}$
A	$\frac{5\pi}{2}$	$\frac{5\pi}{2}$	$e^{i\frac{5\pi}{2}}e^{i\left(\frac{\varphi_1+\varphi_3}{2}\right)} = ie^{i\left(\frac{\varphi_1+\varphi_3}{2}\right)}$
A'	$\frac{\pi}{2}$	$\frac{\pi}{2}$	$e^{i\frac{\pi}{2}}e^{i\left(\frac{\varphi_1+\varphi_3}{2}\right)} = ie^{i\left(\frac{\varphi_1+\varphi_3}{2}\right)}$

Check continuity at $x < 1$:

	φ_1	φ_3	$e^{i\left(\frac{\varphi_1+\varphi_2+\varphi_3+\varphi_4}{2}\right)}$
B	π	π	$e^{i\pi}e^{i\left(\frac{\varphi_2+\varphi_4}{2}\right)} = -e^{i\left(\frac{\varphi_2+\varphi_4}{2}\right)}$
B'	$-\pi$	$-\pi$	$e^{-i\pi} = -e^{i\left(\frac{\varphi_2+\varphi_4}{2}\right)}$

1.2.8 Inverse Functions

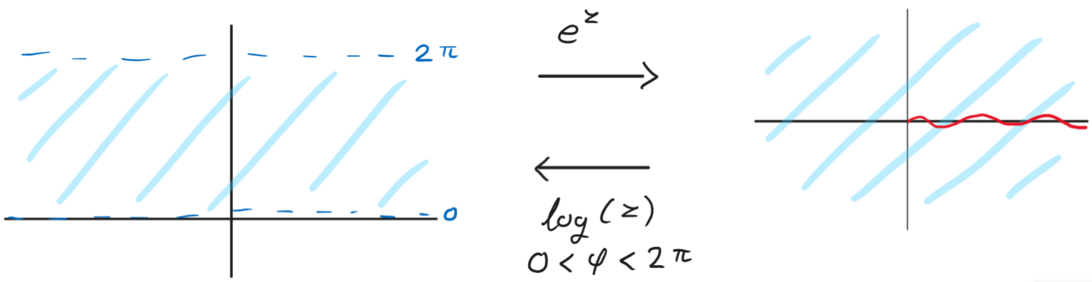
To get a one-to-one function we often need to restrict the domain of the function.
Inverse of e^z is $z = \text{Log}(w)$ or a branch cut of $\log(w)$



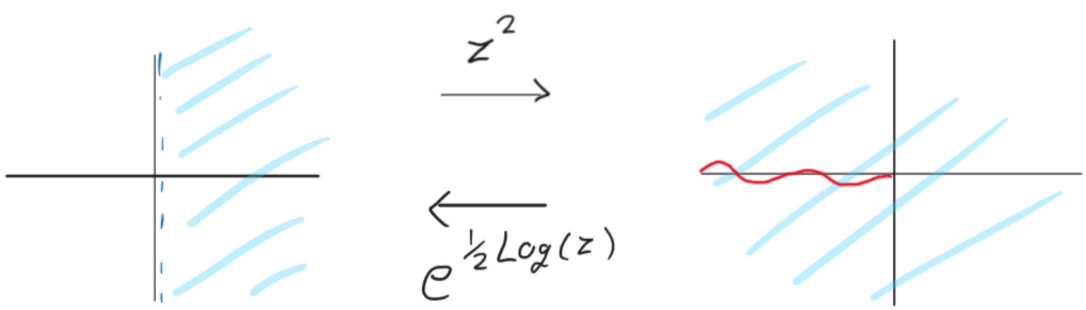
$e^z: \ \{-\pi < y < \pi\} \to \mathbb{C}\backslash(-\infty, 0]$

$\mathrm{Log}(z): \ \mathbb{C}\backslash(-\infty, 0] \to \{-\pi < v < \pi\}$

The inverse of e^z over $\{0 < y < 2\pi\}$ is $\log z = \ln r + i\varphi, \ 0 < \varphi < 2\pi$.

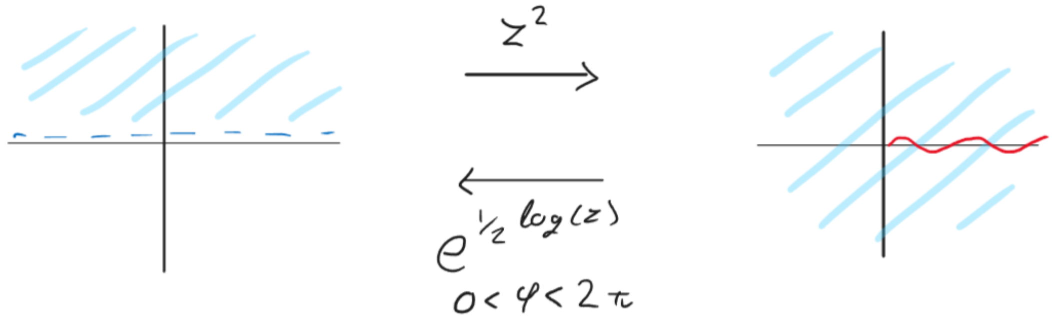


The inverse function for z^2 over $\{x > 0\}$ is $z^{\frac{1}{2}} = e^{\frac{1}{2}\mathrm{Log}(z)}$



$z^2: \ \{x > 0\} \to \mathbb{C}\backslash(-\infty, 0]$

The inverse function for z^2 over $\{y > 0\}$ is $z^{\frac{1}{2}} = e^{\frac{1}{2}\log(z)},\ 0 < \varphi < 2\pi$



$$z^2 : \ \{y > 0\} \to \mathbb{C} \backslash [0, \infty)$$

Ex: Find the inverse function of e^z on the domain $\left\{\frac{\pi}{4} < y < \frac{9\pi}{4}\right\}$

$$w = e^x e^{iy}$$
$$\log(w) = \ln(e^x) + iy = x + iy = z$$
$$\log(w)$$
$$f^{-1}(z) = \log(z), \ \left\{\frac{\pi}{4} < \arg(z) < \frac{9\pi}{4}\right\}$$

Ex2: Find the inverse function of z^4 on the domain $\{x > 0, y > 0\}$

$$z = w^{1/4} = |w|^{1/4} e^{i\frac{\arg(w)}{4}}$$
$$0 < \arg(z) < \frac{\pi}{2} \Rightarrow 0 < \frac{\arg(w)}{4} < \frac{\pi}{2} \Rightarrow 0 < \arg(w) < 2\pi$$
$$f^{-1}(z) = z^{1/4}, \ \{0 < \arg(z) < 2\pi\}$$

The inverse of $\sin z$ is a little bit less obvious. Let us find all inverse functions for $\sin z$.

$$\sin w = z$$
$$\frac{e^{iw} - e^{-iw}}{2i} = z$$
$$e^{iw} - e^{-iw} = 2iz$$
$$e^{2iw} - 2ize^{iw} - 1 = 0$$
$$e^{iw} = \frac{2iz \pm \sqrt{(-2iz)^2 + 4}}{2}$$
$$e^{iw} = iz \pm \sqrt{1 - z^2}$$
$$e^{iw} = iz + i(z^2 - 1)^{\frac{1}{2}}$$
$$iw = \log\left(iz + i(z^2 - 1)^{\frac{1}{2}}\right)$$
$$w = -i\log\left(iz + i(z^2 - 1)^{\frac{1}{2}}\right)$$

The general solutions to $\sin w = z$ are given by the above expression. This expression contains two multivalued parts, log and $(z^2-1)^{\frac{1}{2}}$.
Ex: Find all solutions to $\sin w = -i$

$$\begin{aligned}
&\sin(w) = -i\\
&\sin x \cosh y + i \cos x \sinh y = -i\\
&\sin x \cosh y = 0 \Rightarrow x = n\pi\\
&\cos x \sinh y = -1\\
&n = 2k:\\
&\sinh y = -1\\
&n = 2k+1:\\
&-\sinh y = -1 \Rightarrow \sinh y = 1\\
&\sinh y = \frac{e^y - e^{-y}}{2}\\
&e^{2y} - 2\sinh y e^y - 1 = 0\\
&e^y = \frac{2\sinh y \pm \sqrt{4\sinh^2 y + 4}}{2} = \sinh y \pm \sqrt{\sinh^2 y + 1}\\
&y = \ln\left(\sinh y \pm \sqrt{\sinh^2 y + 1}\right)\\
&n = 2k:\\
&y = \ln\left(-1 \pm \sqrt{2}\right) = \ln\left(\sqrt{2} - 1\right)\\
&n = 2k+1:\\
&y = \ln\left(1 \pm \sqrt{2}\right) = \ln\left(1 + \sqrt{2}\right)\\
&w = \left\{2k_1\pi + i\ln\left(\sqrt{2}-1\right), (2k_2+1)\pi + i\ln\left(1+\sqrt{2}\right), k_1, k_2 \in \mathbb{Z}\right\}
\end{aligned}$$

Ex2: Find all solutions to $\cos w = -i$

$$\begin{aligned}
&\cos(w) = -i\\
&\cos(w) = \cos x \cosh y - i \sin x \sinh y\\
&\cos x \cosh y = 0 \Rightarrow x = \frac{\pi}{2} + n\pi\\
&n = 2k:\\
&-\sinh y = -1 \Rightarrow \sinh y = 1\\
&n = 2k+1:\\
&\sinh y = -1\\
&\sinh y = \frac{e^y - e^{-y}}{2}\\
&e^{2y} - 2\sinh y e^y - 1 = 0\\
&e^y = \frac{2\sinh y \pm \sqrt{4\sinh^2 y + 4}}{2} = \sinh y \pm \sqrt{\sinh^2 y + 1}
\end{aligned}$$

$$y = \ln\left(\sinh y \pm \sqrt{\sinh^2 y + 1}\right)$$
$$n = 2k :$$
$$y = \ln\left(1 \pm \sqrt{2}\right) = \ln\left(1 + \sqrt{2}\right)$$
$$n = 2k + 1 :$$
$$y = \ln\left(-1 \pm \sqrt{2}\right) = \ln\left(\sqrt{2} - 1\right)$$
$$z = \left\{\frac{\pi}{2} + 2k_1\pi + i\ln\left(1 + \sqrt{2}\right), \frac{\pi}{2} + (2k_2 + 1)\pi + i\ln\left(\sqrt{2} - 1\right),\ k_1, k_2 \in \mathbb{Z}\right\}$$

The principal value of the inverse function of $\sin z$ is called $f(z) = \arcsin z$ and is defined such that $f(0) = 0$.
The principal value of the inverse function of $\cos z$ is called $f(z) = \arccos z$ and is defined such that $f(0) = \frac{\pi}{2}$.
These functions are defined as

$$\arcsin z = -i\,\mathrm{Log}\left(iz + i(z^2 - 1)^{\frac{1}{2}}\right)$$
$$\text{where } (z^2 - 1)^{\frac{1}{2}} = |z - 1|^{\frac{1}{2}}|z + 1|^{\frac{1}{2}}e^{i\frac{\arg(z-1)+\arg(z+1)}{2}}$$
$$\text{such that } 0 < \arg(z - 1) < 2\pi \text{ and } \pi < \arg(z + 1) < 3\pi$$
$$\arccos z = -i\,\mathrm{Log}\left(z + (z^2 - 1)^{\frac{1}{2}}\right)$$
$$\text{where } (z^2 - 1)^{\frac{1}{2}} = |z - 1|^{\frac{1}{2}}|z + 1|^{\frac{1}{2}}e^{i\frac{\arg(z-1)+\arg(z+1)}{2}}$$

Ex: Find all solutions to $w = \arcsin(-i)$

$$w = \arcsin(-i)$$
$$\arcsin(z) = -i\,\mathrm{Log}(iz + i(z^2 - 1)^{1/2})$$
$$(z^2 - 1)^{1/2} = |z + 1|^{1/2}|z - 1|^{1/2}e^{i\left(\frac{\varphi_1+\varphi_2}{2}\right)},\ \varphi_1 \in (0, 2\pi),\ \varphi_2 \in (\pi, 3\pi)$$
$$(z^2 - 1)^{1/2}\Big|_{z=-i} \Rightarrow \varphi_1 = \frac{5\pi}{4},\ \varphi_2 = \frac{3\pi}{4} + \pi$$
$$(z^2 - 1)^{1/2}\Big|_{z=-i} = \sqrt{2}e^{i\frac{3\pi}{2}} = -\sqrt{2}i$$
$$\arcsin(-i) = -i\,\mathrm{Log}(1 + i(-\sqrt{2}i)) = -i\,\mathrm{Log}(1 + \sqrt{2})$$
$$\arcsin(-i) = -i\ln\left(1 + \sqrt{2}\right)$$

Ex2: Find all solutions to $w = \arccos(-i)$

$$w = \arccos(-i)$$
$$w = -i\,\mathrm{Log}(z + (z^2 - 1)^{1/2})$$
$$(z^2 - 1)^{1/2} = |z + 1|^{1/2}|z - 1|^{1/2}e^{i\left(\frac{\varphi_1+\varphi_2}{2}\right)},\ \varphi_1 \in (0, 2\pi),\ \varphi_2 \in (-\pi, \pi)$$

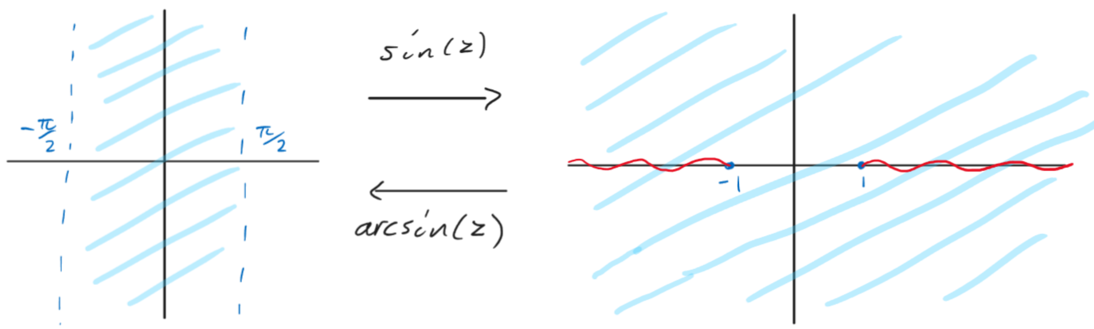
$$\left.(z^2-1)^{1/2}\right|_{z=-i} \Rightarrow \varphi_1 = \frac{5\pi}{4},\ \varphi_2 = -\frac{\pi}{4}$$

$$\left.(z^2-1)^{1/2}\right|_{z=-i} = \sqrt{2}e^{i\frac{\pi}{2}} = \sqrt{2}i$$

$$w = -i\,\mathrm{Log}(-i+\sqrt{2}i) = -i\,\mathrm{Log}((\sqrt{2}-1)e^{i\frac{\pi}{2}}) = -i\left(\ln(\sqrt{2}-1)+i\frac{\pi}{2}\right)$$

$$w = \frac{\pi}{2} - i\ln\left(\sqrt{2}-1\right)$$

Mapping properties of $\sin z$ and $\arcsin z$:



We can also apply this mapping to solve Laplace's equation.
Ex:

$$\Delta\phi = 0,\ \{x>0, y>0\}$$

$$\text{BCs}: \begin{cases} \phi = 1 & x=0, y>0 \\ \phi_y = 0 & 0<x<1, y=0 \\ \phi = 2 & x>1, y=0 \end{cases}$$

$$w = \arcsin(z)$$

$$\Phi = Au + B$$

$$\Phi(u=0) = 1 \Rightarrow B = 1$$

$$\Phi\left(u=\frac{\pi}{2}\right) = 2 = A\frac{\pi}{2}+1 \Rightarrow A = \frac{2}{\pi}$$

$$\Phi = \frac{2}{\pi}u + 1$$

$$u + iv = \arcsin(x+iy)$$

$$\sin(u+iv) = x+iy$$

$$\begin{cases} x = \sin u \cosh v \\ y = \cos u \sinh v \end{cases} \Rightarrow \begin{cases} \cosh v = \frac{x}{\sin u} \\ \sinh v = \frac{y}{\cos u} \end{cases} \Rightarrow \cosh^2 v - \sinh^2 v = \frac{x^2}{\sin^2 u} - \frac{y^2}{\cos^2 u} = 1$$

$$t = \sin^2 u$$

$$\sin^2 u + \cos^2 u = 1 \Rightarrow \cos^2 u = 1-t$$

$$\frac{x^2}{t} - \frac{y^2}{1-t} = 1$$

$$t(1-t) = (1-t)x^2 - ty^2$$

$$t - t^2 = x^2 - tx^2 - ty^2$$
$$t^2 + t(-x^2 - y^2 - 1) + x^2$$
$$t = \frac{x^2 + y^2 + 1 \pm \sqrt{(x^2 + y^2 + 1)^2 - 4x^2}}{2}$$
$$0 \le \sin^2 u \le 1 \Rightarrow 0 \le t \le 1 \Rightarrow t = \frac{x^2 + y^2 + 1 - \sqrt{(x^2 + y^2 + 1)^2 - 4x^2}}{2}$$
$$\sin^2 u = \frac{x^2 + y^2 + 1 - \sqrt{(x^2 + y^2 + 1)^2 - 4x^2}}{2}$$
$$0 < u < \frac{\pi}{2} \Rightarrow 0 < \sin u < 1$$
$$\sin u = \sqrt{\frac{x^2 + y^2 + 1 - \sqrt{(x^2 + y^2 + 1)^2 - 4x^2}}{2}}$$
$$u = \arcsin\left(\sqrt{\frac{x^2 + y^2 + 1 - \sqrt{(x^2 + y^2 + 1)^2 - 4x^2}}{2}}\right)$$
$$\phi = \frac{2}{\pi}\arcsin\left(\sqrt{\frac{x^2 + y^2 + 1 - \sqrt{(x^2 + y^2 + 1)^2 - 4x^2}}{2}}\right) + 1$$

We can also define inverse functions for $\sinh z$ and $\cosh z$ using the same method as for $\sin z$ and $\cos z$.
Ex: Find the inverse of $\sinh(z)$ such that $f^{-1}(0) = \ln(1)$.

$$w = \sinh(z) = \frac{e^z - e^{-z}}{2}$$
$$2w = e^z - e^{-z}$$
$$e^{2z} - 2we^z - 1$$
$$e^z = \frac{2w \pm \sqrt{4w^2 + 4}}{2} = w \pm \sqrt{w^2 + 1}$$
$$|e^z| > 0 \Rightarrow |w \pm \sqrt{w^2 + 1}|$$
$$\log(e^z) = \ln(e^x) + i(y + 2\pi k) = \log\left(w \pm (w^2 + 1)^{1/2}\right)$$
$$z = \log\left(w + (w^2 + 1)^{1/2}\right)$$
$$(w^2 + 1)^{1/2} = |w + i|^{1/2}|w - i|^{1/2}e^{i\left(\frac{\varphi_1 + \varphi_2}{2}\right)},\ \varphi_1 \in \left(\frac{\pi}{2}, \frac{5\pi}{2}\right),\ \varphi_2 \in \left(\frac{3\pi}{2}, \frac{7\pi}{2}\right)$$
$$(w^2 + 1)^{1/2}\Big|_{w=0} \Rightarrow \varphi_1 = \frac{3\pi}{2},\ \varphi_2 = \frac{5\pi}{2}$$
$$(w^2 + 1)^{1/2}\Big|_{w=0} = e^{i2\pi} = 1$$
$$0 = \log(1) = \ln(1) + i2\pi k$$
$$f^{-1}(z) = \mathrm{Log}\left(z + |z + i|^{1/2}|z - i|^{1/2}e^{i\left(\frac{\varphi_1 + \varphi_2}{2}\right)}\right),\ \varphi_1 \in \left(\frac{\pi}{2}, \frac{5\pi}{2}\right),\ \varphi_2 \in \left(\frac{3\pi}{2}, \frac{7\pi}{2}\right)$$

1.3 Integration of Complex Functions

1.3.1 Path Integrals

Given that complex functions exist in a two-dimensional space, it is natural to consider integration over a path in the complex plane.
We can come up with a parameterization to express this path as a function of a real variable t:

$$w(t) = x(t) + iy(t)$$

$$\int_a^b w(t)dt = \int_a^b x(t)dt + i\int_a^b y(t)dt$$

Complex integrals will have the same properties as real integrals, such as linearity. One new property for complex integrals is:

$$\left|\int_a^b w(t)dt\right| \leq \left|\int_a^b |w(t)|dt\right|$$

Proof.

$$\int_a^b w(t)dt = \rho e^{i\varphi}$$

$$\rho = \left|\int_a^b w(t)dt\right|$$

$$\rho = e^{-i\varphi}\int_a^b w(t)dt = \int_a^b w(t)e^{-i\varphi}dt$$

$$\rho = \left|\int_a^b w(t)dt\right| = \int_a^b e^{-it}w(t)dt = \Re\left(\int_a^b e^{-i\varphi}w(t)dt\right)$$

$$\rho = \int_a^b \Re\left(e^{-i\varphi}w(t)\right)dt \leq \int_a^b \left|e^{-i\varphi}w(t)\right|dt = \int_a^b |w(t)|dt$$

□

A path or contour C is a piecewise smooth curve in the complex plane.

$$C = z(t) = x(t) + iy(t),\ a \leq t \leq b$$

A path also has a direction, which is the direction of increasing t.

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

$$dz = z'(t)dt = (x'(t) + iy'(t))dt$$

Ex: $f(x, y) = x - 2xyi$

$$\int_C (x - 2xyi)dz,\ C = \left\{y = x^2, 0 \leq x \leq 1\right\} \cup \{y = 1, -1 \leq x \leq 1\}$$

$$f_1 = x - 2x^3 i$$
$$z = x + iy = x + ix^2 \Rightarrow dz = (1 + 2xi)dx$$
$$\int_0^1 (x - 2x^3 i)(1 + 2xi)dx = \int_0^1 (x + 2x^2 i - 2x^3 i + 4x^4)dx = \frac{1}{2} + \frac{2}{3}i - \frac{2}{4}i + \frac{4}{5}$$
$$f_2 = x - 2xi$$
$$z = x + iy = x + i \Rightarrow dz = dx$$
$$\int_{-1}^1 (x - 2xi)dx = \left[\frac{x^2}{2} - x^2 i\right]_{-1}^1 = 0$$
$$\int_C (x - 2xyi)dz = \frac{13}{10} + \frac{1}{6}i$$

A closed path (has the same start and end points) is called a contour. It is the same as a path integral but we will see later that contours have special properties with complex functions.

Ex2: $f(z) = \bar{z}^2$ on the contour C that is the box with vertices $0, 1, 1+i, i, 0$ in the counterclockwise direction.

$$\bar{z}^2 = (x - iy)^2 = x^2 - 2xyi - y^2$$
$$y = 0: \ f_1 = x^2$$
$$z = x + iy = x \Rightarrow dz = dx$$
$$\int_{C_1} f_1 dz = \int_0^1 x^2 dx = \frac{1}{3}$$
$$x = 1: \ f_2 = 1 - 2yi - y^2$$
$$z = x + iy = 1 + iy \Rightarrow dz = idy$$
$$\int_{C_2} f_2 dz = \int_0^1 (1 - 2yi - y^2)idy = \left[iy + y^2 - \frac{y^3}{3}i\right]_0^1 = i + 1 - \frac{i}{3} = \frac{2}{3}i + 1$$
$$y = 1: \ f_3 = x^2 - 2xi - 1$$
$$z = x + iy = x + i \Rightarrow dz = dx$$
$$\int_{C_3} f_3 dz = -\int_0^1 (x^2 - 2xi - 1)dx = -\frac{1}{3} + i + 1 = i + \frac{2}{3}$$
$$x = 0: \ f_4 = -y^2$$
$$z = x + iy = iy \Rightarrow dz = idy$$
$$\int_{C_4} f_4 dz = -\int_0^1 (-y^2)idy = \frac{i}{3}$$
$$\int_C \bar{z}^2 dz = 2 + 2i$$

Ex3: $f(z) = \mathrm{Log}(z)$

$$C = \{|z| = 1, \Re(z) \geq 0\}$$
$$z = e^{it}, \ -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$$
$$dz = ie^{it}dt$$

$$\int_C \mathrm{Log}(z)dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathrm{Log}(e^{it})\left(ie^{it}\right)dt = -\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} te^{it}dt$$

$$= -\frac{te^{it}}{i}\bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^{it}}{i}dt = -e^{it}\bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = -2i$$

1.3.2 Fundamental Theorem of Calculus

A neat property of complex integrals is that they are path independent. This means that the value of the integral is the same for any path between the start and end points. This is not true for real integrals.
Ex: Given the function $f(z) = z^2 + 1$ we will compute three different integrals between the points $z_1 = -i$ and $z_2 = 1$ and show that they are all equal.
Path 1:

$$C = \{y = x - 1\}$$

$$z^2 + 1 = (x + iy)^2 + 1 = x^2 + 2xyi - y^2 + 1$$

$$\int_C (x^2 - y^2 + 1 + 2xyi)dz$$

$$z = x + iy = x + i(x - 1) \Rightarrow dz = (1 + i)dx$$

$$\int_0^1 (x^2 - (x - 1)^2 + 1 + 2x(x - 1)i)(1 + i)dx$$

$$= \int_0^1 (2x + 2x^2 i - 2xi)(1 + i)dx = \left[x^2 + \frac{2x^3}{3}i - x^2 i\right]_0^1 (1 + i)$$

$$= \left(1 - \frac{i}{3}\right)(1 + i) = \frac{4}{3} + \frac{2}{3}i$$

Path 2:

$$C = \{x = 0, -1 \le y \le 0\} \cup \{y = 0, 0 \le x \le 1\}$$

$$C_1 : \int_{-1}^0 (-y^2 + 1)idy = \left[y - \frac{y^3}{3}\right]_{-1}^0 i = \frac{2}{3}i$$

$$C_2 : \int_0^1 (x^2 + 1)dx = \left[\frac{x^3}{3} + x\right]_0^1 = \frac{4}{3}$$

$$\int_C (z^2 + 1)dz = \frac{4}{3} + \frac{2}{3}i$$

Path 3:

$$z = e^{it},\ -\frac{\pi}{2} \le t \le 0$$

$$z^2 + 1 = e^{2it} + 1$$

$$dz = ie^{it}dt$$

$$\int_{-\frac{\pi}{2}}^0 \left(e^{2it} + 1\right) ie^{it}dt = i\int_{-\frac{\pi}{2}}^0 \left(e^{3it} + e^{it}\right)dt = i\left[\frac{e^{3it}}{3i} + \frac{e^{it}}{i}\right]_{-\frac{\pi}{2}}^0 = \frac{4}{3} - \frac{e^{-i\frac{3\pi}{2}}}{3} - e^{-i\frac{\pi}{2}}$$

$$= \frac{4}{3} + \frac{2}{3}i$$

This finding leads us to the Fundamental Theorem of Calculus for complex integrals.

$$\int_C f(z)dz = F(z_2) - F(z_1)$$

where $F'(z) = f(z)$.
When an antiderivative of $f(z)$ exits it becomes far easier to compute the integral.
Ex:

$$\int_C \sin z dz,\ C:\ e^{it},\ -\frac{\pi}{2} \leq t \leq \frac{5\pi}{4}$$
$$\int_C \sin z dz = -\cos z_1 + \cos z_0$$
$$z_0 = e^{-i\frac{\pi}{2}} = -i$$
$$z_1 = e^{i\frac{5\pi}{4}} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$$
$$\cos(z) = \cos(x)\cosh(y) - i\sin(x)\sinh(y)$$
$$\cos(z_0) = \cosh(1)$$
$$\cos(z_1) = \cos\left(\frac{1}{\sqrt{2}}\right)\cosh\left(\frac{1}{\sqrt{2}}\right) - i\sin\left(\frac{1}{\sqrt{2}}\right)\sinh\left(\frac{1}{\sqrt{2}}\right)$$
$$\int_C \sin z dz = \cos(-i) - \cos\left(-\frac{1}{\sqrt{2} - \frac{1}{\sqrt{2}}i}\right) = \cosh(1) - \cos\left(\frac{1}{\sqrt{2}}\right)\cosh\left(\frac{1}{\sqrt{2}}\right) + i\sin\left(\frac{1}{\sqrt{2}}\right)\sinh\left(\frac{1}{\sqrt{2}}\right)$$

Ex2:

$$\int_C \frac{dz}{z},\ C:\ \frac{x^2}{4} + y^2 = 1,\ y \geq 0$$
$$\int_C \frac{dz}{z} = \log^*(z) = \ln|z| + i\varphi\Big|_{z_0}^{z_1},\ \varphi \in \left(-\frac{\pi}{2}, \frac{3\pi}{2}\right)$$
$$y = 0:\ \frac{x^2}{4} = 1 \Rightarrow x = \pm 2$$
$$z_0 = 2,\ z_1 = -2$$
$$\int_C \frac{dz}{z} = \ln 2 + \pi i - \ln 2 = \pi i$$

Ex3:

$$\int_C \frac{dz}{z^2},\ C:\ \frac{x^2}{4} + y^2 = 1,\ y \geq 0$$
$$\int_C \frac{dz}{z^2} = -\frac{1}{z_1} + \frac{1}{z_0}$$
$$z_0 = 2,\ z_1 = -2$$
$$\int_C \frac{dz}{z^2} = \frac{1}{2} + \frac{1}{2} = 1$$

Ex4:

$$
\begin{aligned}
&\int_C \frac{dz}{z},\ C:\ z=e^{it},\ 0\le t\le \frac{3\pi}{2}\\
&\int_C \frac{dz}{z} = \ln|z| + i\varphi\Big|_{z_0}^{z_1},\ \varphi\in\left(-\frac{\pi}{4},\frac{7\pi}{4}\right)\\
&z_0 = e^0 = 1\\
&z_1 = e^{i\frac{3\pi}{2}} = -i\\
&\int_C \frac{dz}{z} = \left(\ln(1)+\frac{3\pi}{2}i\right) - (\ln(1)+0) = \frac{3\pi}{2}i
\end{aligned}
$$

Ex5:

$$
\begin{aligned}
&\int_C z^{1/3}dz,\ C:\ r=2\cos\left(\frac{\theta}{2}\right),\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\\
&\int_C z^{1/3}dz = \frac{3}{4}z^{4/3}\Big|_{z_0}^{z_1} = \frac{3}{4}|z|^{4/3}e^{i\varphi}\Big|_{z_0}^{z_1},\ \varphi\in(-\pi,\pi)\\
&r\left(\pm\frac{\pi}{2}\right) = 2\cos\left(\frac{\pi}{4}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}\\
&z_0 = -\sqrt{2}i,\ z_1 = \sqrt{2}i\\
&\int_C z^{1/3}dz = \frac{3}{4}(\sqrt{2})^{4/3}\left(e^{i\frac{2\pi}{3}} - e^{-i\frac{2\pi}{3}}\right)\\
&\int_C z^{1/3}dz = \frac{3}{2^{1/3}}\frac{i}{2i}\left(e^{i\frac{2\pi}{3}} - e^{-i\frac{2\pi}{3}}\right) = \frac{3i}{2^{1/3}}\sin\left(\frac{2\pi}{3}\right)
\end{aligned}
$$

Ex6:

$$
\begin{aligned}
&\int_C \mathrm{Log}(z)dz,\ C:\ r=2\cos\left(\frac{\theta}{2}\right),\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}\\
&z_0 = -\sqrt{2}i,\ z_1 = \sqrt{2}i\\
&\int_C \mathrm{Log}\, z dz = [z\,\mathrm{Log}\, z - z]_{-\sqrt{2}i}^{\sqrt{2}i} = \sqrt{2}i\left(\ln\sqrt{2}+i\frac{\pi}{2}-1\right) + \sqrt{2}i\left(\ln\sqrt{2}-i\frac{\pi}{2}-1\right)\\
&= \left(2\sqrt{2}\ln\sqrt{2} - 2\sqrt{2}\right)i
\end{aligned}
$$

Another property of complex integrals we will make use of often is the inequality

$$
\left|\int_C f(z)dz\right| \le \max_{z\in C}|f(z)|\cdot \mathrm{Length}(C)
$$

Proof.

$$
\begin{aligned}
\left|\int_C f(z)dz\right| &= \left|\int_a^b f(z(t))z'(t)dt\right|\\
&\le \left|\int_a^b |f(z(t))||z'(t)|dt\right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \max_{a \leq t \leq b} |f(z(t))| \cdot \int_a^b \sqrt{x^2+y^2}dt \\
&\leq \max_{z \in C} |f(z)| \cdot \text{Length}(C)
\end{aligned}
$$

□

Ex: Prove the inequality

$$\left|\int_C \frac{dz}{z^2+i}\right| \leq \frac{3\pi}{4}$$

C is the circle $|z| = 3$ traversed once

Proof.

$$
\begin{aligned}
&\left|\int_C f(z)dz\right| \leq \max_{z \in C} |f(z)| \text{Length}(C) \\
&\text{Length}(C) = 6\pi \\
&\left|\int_C \frac{dz}{z^2+i}\right| \leq 6\pi \max_{z \in C} \left|\frac{1}{z^2+i}\right| \\
&\max_{z \in C} \left|\frac{1}{z^2+i}\right| = \frac{1}{\min_{z \in C} |z^2+i|} \\
&|z^2+i| \geq \big||z^2| - |i|\big| = |9-1| = 8 \\
&\min_{z \in C} |z^2+i| = 8 \\
&\max_{z \in C} \left|\frac{1}{z^2+i}\right| = \frac{1}{8} \\
&\left|\int_C \frac{dz}{z^2+i}\right| \leq \frac{6\pi}{8} = \frac{3\pi}{4}
\end{aligned}
$$

□

Ex2: Prove the inequality

$$\left|\int_C \text{Log}(z)dz\right| \leq \frac{\pi^2}{2}$$

where C is described by e^{it}, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$

Proof. Note that the curve is a semicircle of radius 1 where $x \geq 0$.

$$
\begin{aligned}
&\left|\int_C f(z)dz\right| \leq \max_{z \in C} |f(z)| \text{Length}(C) \\
&\text{Length}(C) = \pi \\
&f(z) = \operatorname{Log} z = \ln|z| + i\varphi,\ \varphi \in (-\pi, \pi) \\
&\max_{z \in C} |f(z)| = \max_{z \in C} |\operatorname{Log} z| = \max_{z \in C} |\ln|z| + i\varphi|
\end{aligned}
$$

$$z = e^{it}$$
$$|z| = 1$$
$$\varphi = t$$
$$\max_{z\in C} |f(z)| = \max_{-\frac{\pi}{2}\le t\le \frac{\pi}{2}} |it| = \max_{-\frac{\pi}{2}\le t\le \frac{\pi}{2}} |t| = \frac{\pi}{2}$$
$$\left|\int_C \operatorname{Log}(z)dz\right| \le \frac{\pi}{2}\cdot\pi = \frac{\pi^2}{2}$$

□

Ex3: Prove the inequality

$$\left|\int_C \frac{e^{3z}}{e^z+1}dz\right| \le \frac{2\pi e^{3R}}{e^R-1}$$

where C is the vertical line segment from $z = R$ to $z = R + 2\pi i$ such that $R \in \mathbb{R}$ and $R > 0$.

Proof.

$$\left|\int_C f(z)dz\right| \le \max_{z\in C} |f(z)|\text{Length}(C)$$
$$\text{Length}(C) = 2\pi$$
$$f(z) = \frac{e^{3z}}{e^z+1}$$

Let us describe C by

$$C:\ z = R + it,\ 0 \le t \le 2\pi$$
$$\max_{z\in C} |f(z)| = \max_{z\in C}\left|\frac{e^{3z}}{e^z+1}\right| = \max_{0\le t\le 2\pi}\left|\frac{e^{3R}e^{3it}}{e^Re^{it}+1}\right| = \max_{0\le t\le 2\pi}\frac{e^{3R}}{|e^Re^{it}+1|}$$
$$\max_{0\le t\le 2\pi}\frac{1}{|e^Re^{it}+1|} = \frac{1}{\min\limits_{0\le t\le 2\pi}|e^Re^{it}+1|}$$
$$\min_{0\le t\le 2\pi}|e^Re^{it}+1| = \min_{0\le t\le 2\pi}|e^R(\cos(t)+i\sin(t))+1| = \min_{0\le t\le 2\pi}\sqrt{(e^R\cos(t)+1)^2+e^{2R}\sin^2(t)}$$
$$= \min_{0\le t\le 2\pi}\sqrt{e^{2R}\cos^2 t+e^{2R}\sin^2 t+2e^R\cos t+1} = \min_{0\le t\le 2\pi}\sqrt{e^{2R}+2e^R\cos t+1}$$
$$\min_{0\le t\le 2\pi}\cos t = -1$$
$$\min_{0\le t\le 2\pi}|e^Re^{it}+1| = \sqrt{e^{2R}-2e^R+1} = \sqrt{(e^R-1)^2} = |e^R-1|$$
$$R > 0 \Rightarrow e^R > 1 \Rightarrow |e^R-1| = e^R-1$$
$$\max_{z\in C}\left|\frac{e^{3z}}{e^z+1}\right| = \frac{e^{3R}}{e^R-1}$$
$$\left|\int_C \frac{e^{3z}}{e^z+1}dz\right| \le \frac{2\pi e^{3R}}{e^R-1}$$

□

1.3.3 Loop Integrals

A loop, C, is a path such that $z_2 = z_1$ (the start and end points are the same).
A simple closed loop is a loop with no other intersections. For example, a figure 8 is not a simple closed loop.
Using the FTC (Fundamental Theorem of Calculus) we can see that if $f(z)$ is continuous and has an antiderivative $F(z)$ in $\mathcal{D}$ then

$$\int_C f(z)dz = 0$$

for any loops in $\mathcal{D}$.
Ex:

$$f(z) = \frac{\cos z}{z^2 + 6z + 10}$$
$$z = \frac{-6 \pm \sqrt{36 - 40}}{2} = -3 \pm i$$

$f(z)$ is analytic on $|z| \leq 2$

$$\therefore \int_{|z|=2} f(z)dz = 0$$

Ex2:

$$f(z) = \text{Log}(2z + 5)$$
$$w = 2z + 5$$
$$\text{not analytic where } \Re(w) < 0,\ \Im(w) = 0$$
$$w = 2x + 5 + iy$$
$$y = 0$$
$$2x + 5 < 0 \Rightarrow x < -\frac{5}{2}$$

$f(z)$ is analytic on $|z| \leq 2$

$$\therefore \int_{|z|=2} f(z)dz = 0$$

Ex3:

$$f(z) = \arcsin\left(\frac{z}{3}\right)$$
$$w = \frac{z}{3}$$
$$\text{not analytic where } |\Re(w)| > 1,\ \Im(w) = 0$$
$$w = \frac{x}{3} + i\frac{y}{3}$$
$$y = 0$$

$$\frac{|x|}{3} > 1 \Rightarrow |x| > 3$$

$f(z)$ is analytic on $|z| \leq 2$

$$\therefore \int_{|z|=2} f(z)dz = 0$$

Ex4:

$$\begin{aligned}
&f(z) = \tan\left(\frac{z}{2}\right) = \frac{\sin\left(\frac{z}{2}\right)}{\cos\left(\frac{z}{2}\right)} \\
&\cos\left(\frac{z}{2}\right) = 0 \\
&\frac{z}{2} = \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z} \\
&z = \pi + 2\pi n = \{\cdots, -\pi, \pi, \cdots\}
\end{aligned}$$

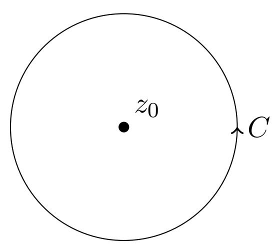
$f(z)$ is analytic on $|z| \leq 2$

$$\therefore \int_{|z|=2} f(z)dz = 0$$

Let us look at the slightly more general case of

$$\int_C (z-z_0)^n dz$$

where C is a simple closed loop and z_0 is a point inside C.



If $n \geq 0$,

$$\begin{aligned}
&\left(\frac{1}{n+1}(z-z_0)^{n+1}\right)' = (z-z_0)^n \\
&\Rightarrow \int_C (z-z_0)^n dz = 0
\end{aligned}$$

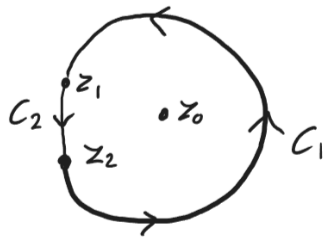
If $n < 0$ and $n \neq -1$,

$$\begin{aligned}
&\left(\frac{1}{n+1}(z-z_0)^{n+1}\right)' = (z-z_0)^n \\
&\Rightarrow \int_C (z-z_0)^n dz = 0
\end{aligned}$$

If $n = -1$,

$$\int_C \frac{dz}{z - z_0}$$

Let us assume that $z_0 = 0$ by shifting everything over by z_0.
$C = C_1 \cup C_2$ where C_1 is the big loop from z_1 to z_2 and C_2 is the small line segment from z_2 to z_1.



$$\int_{C_1} \frac{dz}{z} = \mathrm{Log}(z_2) - \mathrm{Log}(z_1)$$
$$\left| \int_{C_2} \frac{dz}{z} \right| \le \max_{z \in C_2} \frac{1}{|z|} \cdot \mathrm{Length}(C_2) \to 0 \text{ as } z_1 = z_2$$
$$\mathrm{Log}\, z_2 = \ln|z_2| + i\varphi_2 \to \ln|z_0| + i\pi$$
$$\mathrm{Log}\, z_1 = \ln|z_1| + i\varphi_1 \to \ln|z_0| - i\pi$$
$$\int_C \frac{dz}{z} = \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} \to 2i\pi + 0$$
$$\int_C \frac{dz}{z} = 2\pi i$$

And so we get

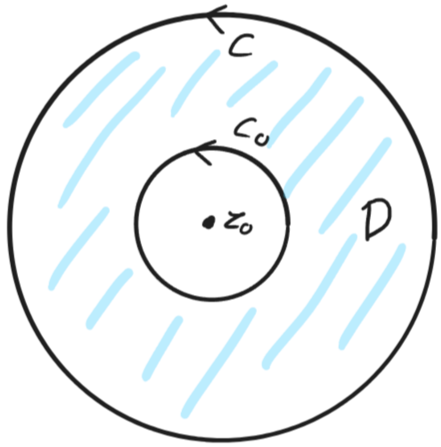
$$\int_C (z - z_0)^n dz = \begin{cases} 0 & \text{if } z_0 \text{ is outside } C \\ 2\pi i & \text{if } z_0 \text{ is inside } C \text{ and } n = -1 \\ 0 & \text{if } z_0 \text{ is inside } C \text{ and } n \neq -1 \end{cases}$$

This means that the result of a loop integral will only be nonzero if there is some singularity inside the loop.

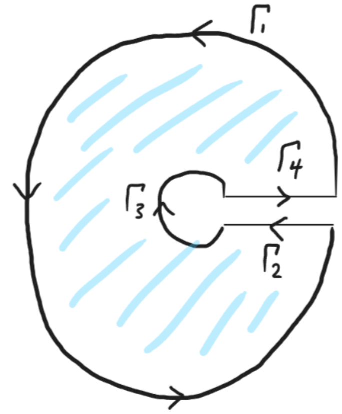
1.3.4 Deformation of Path

Suppose $f(z)$ is analytic in D then we can show

$$\int_C f(z) dz = \int_{C_0} f(z) dz$$



To show this, we will need a simply connected domain. We can construct the following contour:



We can define a new path, $C_\epsilon = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.
Assuming that the function is well defined in the region D then we get

$$\int_{C_\epsilon} f(z)dz = \int_{\Gamma_1} f(z)dz + \int_{\Gamma_2} f(z)dz + \int_{\Gamma_3} f(z)dz + \int_{\Gamma_4} f(z)dz = 0$$

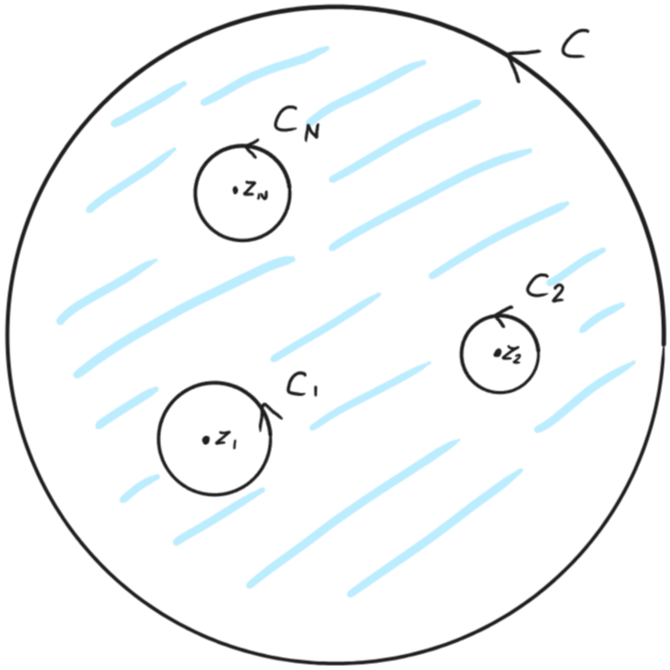
In the limiting case we can see that

$$\int_{\Gamma_1} f(z)dz \to \int_C f(z)dz$$
$$\int_{\Gamma_3} f(z)dz \to -\int_{C_0} f(z)dz$$
$$\int_{\Gamma_2} f(z)dz + \int_{\Gamma_4} f(z)dz \to 0$$

And so we are left with

$$\int_{C_\epsilon} f(z)dz = \int_C f(z)dz - \int_{C_0} f(z)dz = 0$$
$$\Rightarrow \int_C f(z)dz = \int_{C_0} f(z)dz$$

This means that any integral over a closed path is equal to the sum of closed path integrals around any singularities in the domain.



$$\int_C f(z)dz = \sum_{j=1}^{N} \int_{C_j} f(z)dz$$

Ex:

$$\int_{|z|=2} \frac{dz}{z(z-1)}$$

$C_1 : \ |z-0| = r$

$C_2 : \ |z-1| = r, \ r$ is small

$$I = \int_{|z|=2} f(z)dz = \int_{|z-0|=r} \frac{dz}{z(z-1)} + \int_{|z-1|=r} \frac{dz}{z(z-1)}$$

$$\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1} = \frac{A(z-1)+Bz}{z(z-1)} = \frac{(A+B)z-A}{z(z-1)}$$

$$\begin{cases} A+B=0 \\ -A=1 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=1 \end{cases}$$

$$\int_{|z-0|=r} \frac{dz}{z(z-1)} = \int_{|z-0|=r} \left(-\frac{1}{z} + \frac{1}{z-1}\right) dz = -\int_{|z-0|=r} \frac{dz}{z} + \int_{|z-0|=r} \frac{dz}{z-1}$$

$$= -\int_{|z-0|=r} \frac{dz}{z} + 0 = -2\pi i$$

$$\int_{|z-1|=r} \frac{dz}{z(z-1)} = \int_{|z-1|=r} \left(-\frac{1}{z} + \frac{1}{z-1}\right) dz = 0 + \int_{|z-1|=r} \frac{dz}{z-1} = 0 + 2\pi i$$

$I = -2\pi i + 2\pi i = 0$

1.3.5 Cauchy Integral Formula

Here we relied on using partial fractions. This is a useful technique, however, we are able to develop slightly more general techniques in defining the Cauchy Integral Formula.
If $f(z)$ is analytic on C and inside C then

$$\int_C \frac{f(z)}{z-z_0} = 2\pi i f(z_0)$$

Proof.

$$g(z) = \frac{f(z)}{z-z_0}$$

is analytic except at $z = z_0$.
By Cauchy Therorm:

$$\int_C g(z)dz = \int_{|z-z_0|=\epsilon} g(z)dz$$

$$\int_{|z-z_0|=\epsilon} \frac{f(z)}{z-z_0}dz = \int_{|z-z_0|=\epsilon} \frac{f(z)-f(z_0)}{z-z_0}dz + \int_{|z-z_0|=\epsilon} \frac{f(z_0)}{z-z_0}dz$$

$$= \int_{|z-z_0|=\epsilon} \frac{f(z)-f(z_0)}{z-z_0}dz + 2\pi i f(z_0)$$

$$\left| \int_{|z-z_0|=\epsilon} \frac{f(z)-f(z_0)}{z-z_0}dz \right| \leq \max_{|z-z_0|=\epsilon} \left| \frac{f(z)-f(z_0)}{z-z_0} \right| 2\pi\epsilon = \max_{|z-z_0|=\epsilon} |f(z)-f(z_0)| \frac{1}{\epsilon} 2\pi\epsilon$$

$$\left| \int_{|z-z_0|=\epsilon} \frac{f(z)-f(z_0)}{z-z_0}dz \right| \leq 2\pi \max_{|z-z_0|=\epsilon} |f(z)-f(z_0)| \to 0 \text{ as } \epsilon \to 0$$

$$\lim_{\epsilon\to 0} \int_{|z-z_0|=\epsilon} \frac{f(z)-f(z_0)}{z-z_0}dz = 0$$

$$\Rightarrow \int_{|z-z_0|=\epsilon} \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$$

□

Ex:

$$C: \ \frac{x^2}{4} + y^2 = 1$$

$$\int_C \frac{dz}{(z-1)^2}$$

$$z = 1$$

$$\int_C \frac{dz}{(z-1)^2} = \lim_{r\to 0} \int_{|z-1|=r} \frac{dz}{(z-1)^2} = -\frac{1}{z-1}\bigg|_{|z-1|=r} = 0$$

Ex2:

$$C: \ \frac{x^2}{4} + y^2 = 1$$

$$\int_C \frac{e^z}{z(z-1)} dz$$
$$z = 0, \ 1$$
$$\int_C \frac{e^z}{z(z-1)} dz = \int_{|z|=r} \frac{e^z}{z(z-1)} dz + \int_{|z-1|=r} \frac{e^z}{z(z-1)} dz$$
$$\int_{|z|=r} \frac{e^z}{z(z-1)} dz = \int_{|z|=r} \frac{1}{z} f_1(z) dz, \ f_1(z) = \frac{e^z}{z-1}$$
$$f(0) = \frac{e^0}{0-1} = -1$$
$$\int_{|z|=r} \frac{1}{z} f_1(z) dz = 2\pi i f_1(0) = -2\pi i$$
$$\int_{|z-1|=r} \frac{e^z}{z(z-1)} dz = \int_{|z-1|=r} \frac{1}{z-1} f_2(z) dz, \ f_2(z) = \frac{e^z}{z}$$
$$f_2(1) = e$$
$$\int_{|z-1|=r} \frac{1}{z-1} f_2(z) dz = 2\pi i e$$
$$\int_C \frac{e^z}{z(z-1)} dz = -2\pi i + 2\pi e i = 2\pi i(e-1)$$

Ex3:

$$C: \ \frac{x^2}{4} + y^2 = 1$$

$$\int_C \frac{dz}{z(z^2-1)}$$
$$z = 0, \ \pm 1$$
$$\int_C \frac{dz}{z(z^2-1)} = \int_{|z|=r} \frac{1}{z} f_1(z) dz + \int_{|z+1|} \frac{1}{z+1} f_2(z) dz + \int_{|z-1|=r} \frac{1}{z-1} f_3(z) dz$$
$$f_1(z) = \frac{1}{z^2-1} \Rightarrow f_1(0) = -1$$
$$f_2(z) = \frac{1}{z(z-1)} \Rightarrow f_2(-1) = \frac{1}{-(-2)} = \frac{1}{2}$$
$$f_3(z) = \frac{1}{z(z+1)} \Rightarrow f_3(1) = \frac{1}{2}$$
$$\int_C \frac{dz}{z(z^2-1)} = -2\pi i + \frac{1}{2} 2\pi i + \frac{1}{2} 2\pi i = 0$$

Ex4:

$$C:\ \frac{x^2}{4}+y^2=1$$

$$\int_C \frac{dz}{2z^2+1}$$
$$z=\pm\frac{i}{\sqrt{2}}$$
$$\int_C \frac{dz}{2z^2+1}=\int_{|z+\frac{i}{\sqrt{2}}|}\frac{1}{z+\frac{i}{\sqrt{2}}}f_1(z)dz+\int_{|z-\frac{i}{\sqrt{2}}|}\frac{1}{z-\frac{i}{\sqrt{2}}}f_2(z)dz$$
$$f_1(z)=\frac{2}{z-\frac{i}{\sqrt{2}}}\Rightarrow f_1\left(-\frac{i}{\sqrt{2}}\right)=\frac{2}{-\frac{2i}{\sqrt{2}}}=\sqrt{2}i$$
$$f_2(z)=\frac{2}{z+\frac{i}{\sqrt{2}}}\Rightarrow f_2\left(\frac{i}{\sqrt{2}}\right)=\frac{2}{\frac{2i}{\sqrt{2}}}=-\sqrt{2}i$$
$$\int_C \frac{dz}{2z^2+1}=2\pi i\sqrt{2}i-2\pi i\sqrt{2}i=0$$

Ex5:

$$\int_{|z|=2}\frac{dz}{z^2+2z+2}$$
$$z=\frac{-2\pm\sqrt{4-8}}{2}=-1\pm i$$
$$C_1:\ \lim_{r\to 0}|z-(-1+i)|=r$$
$$C_2:\ \lim_{r\to 0}|z-(-1-i)|=r$$
$$\int_{|z|=2}\frac{dz}{z^2-2z+3}=\int_{C_1}\frac{1}{z-(-1+i)}f_1(-1+i)dz+\int_{C_2}\frac{1}{z-(-1-i)}f_2(-1-i)dz$$
$$f_1(z)=\frac{1}{z-(-1-i)}\Rightarrow f_1(-1+i)=\frac{1}{2i}$$
$$f_2(z)=\frac{1}{z-(-1+i)}\Rightarrow f_2(-1-i)=\frac{1}{-2i}$$
$$\int_{|z|=2}\frac{dz}{z^2+2z+2}=2\pi i\frac{1}{2i}-2\pi i\frac{1}{2i}=0$$

Ex6:

$$\int_{|z|=2}\frac{dz}{z^2-2z-3}$$
$$z^2-2z-3=(z-3)(z+1)\Rightarrow z=3,\ -1$$
$$C_1:\ \lim_{r\to 0}|z+1|=r$$
$$\int_{|z|=2}\frac{dz}{z^2-2z-3}=\int_{C_1}\frac{1}{z+1}f_1(-1)dz$$

$$f_1(z) = \frac{1}{z-3} \Rightarrow f_1(-1) = -\frac{1}{4}$$
$$\int_{|z|=2} \frac{dz}{z^2-2z-3} = 2\pi i\left(-\frac{1}{4}\right) = -\frac{\pi}{2}i$$

So, from deformation of path, we got the Formula

$$\int_C f(z)dz = \sum_{j=1}^{N} \int_{C_j} f(z)dz$$

where C is a closed path and C_j are closed paths around singularities.
Now, using the Cauchy Integral Formula, we can get the following formula, assuming that $f(z) = \frac{1}{z-z_j} f_j(z)$ and $f(z)$ has singularities $z_1, \ldots, z_m$

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{m} f_j(z_j)$$

In some cases, it may be hard to find all the singularities. Take the example

$$\int_{|z|=2} \frac{z+1}{z^3+z+3}dz$$

Finding the roots of z^3+z+3 is not easy, however, if we can show that all of the zeros are inside some region, then we can use the deformation of path and compute the intergral

$$\lim_{R\to\infty} \int_{|z|=R} \frac{z+1}{z^3+z+3}dz$$

instead and it will give the same result.
Giving this a try we get

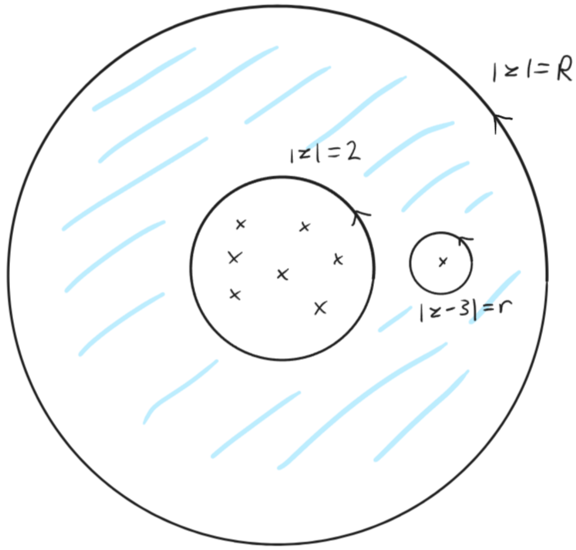
$$\text{assume } |z^3+z+3| > 0 \text{ on } |z| > 2$$
$$|z^3+z+3| \geq ||z|^3 - |z| - 3| > |8-2-3| = 3$$
$$\therefore \text{ all 0s inside region}$$
$$\int_{|z|=2} \frac{z+1}{z^3+z+3}dz = \lim_{R\to\infty} \int_{|z|=R} \frac{z+1}{z^3+z+3}dz$$

Now to compute the integral we can use the inequality formula

$$\left|\int_{|z|=R} \frac{z+1}{z^3+z+3}dz\right| \leq \max_{|z|=R} \left|\frac{z+1}{z^3+z+3}\right| 2\pi R = \frac{(R+1)2\pi R}{R^3-R-3} \to 0$$
$$\int_{|z|=2} \frac{z+1}{z^3+z+3}dz = 0$$

A more general extension of this trick can be seen with the example

$$\int_{|z|=2} \frac{dz}{(z-3)(z^7+z+1)}$$



assume $|z^7+z+1|>0$ on $|z|>2$

$|z^7+z+1| \geq ||z|^7-|z|-1| > |128-2-1| = 125$

$\therefore$ all 0s from z^7+z+1 inside region

$$\lim_{R\to\infty}\int_{|z|=R}\frac{dz}{(z-3)(z^7+z+1)} = \int_{|z|=2}\frac{dz}{(z-3)(z^7+z+1)} + \lim_{r\to 0}\int_{|z-3|=r}\frac{dz}{(z-3)(z^7+z+1)}$$

$$\int_{|z|=2}\frac{dz}{(z-3)(z^7+z+1)} = \lim_{R\to\infty}\int_{|z|=R}\frac{dz}{(z-3)(z^7+z+1)} - \lim_{r\to 0}\int_{|z-3|=r}\frac{dz}{(z-3)(z^7+z+1)}$$

$$\lim_{r\to 0}\int_{|z-3|=r}\frac{dz}{(z-3)(z^7+z+1)} = \lim_{r\to 0}\int_{|z-3|=r}\frac{1}{z-3}f_1(3)dz$$

$$f_1(z) = \frac{1}{z^7+z+1} \Rightarrow f_1(3) = \frac{1}{2191}$$

$$\lim_{r\to 0}\int_{|z-3|=r}\frac{dz}{(z-3)(z^7+z+1)} = \frac{2\pi i}{2191}$$

$$\left|\int_{|z|=R}\frac{dz}{(z-3)(z^7+z+1)}\right| \leq \max_{|z|=R}\left|\frac{1}{(z-3)(z^7+z+1)}\right| 2\pi R = \frac{2\pi R}{(R-3)(R^7-R-1)} \to 0$$

$$\int_{|z|=2}\frac{dz}{(z-3)(z^7+z+1)} = -\frac{2\pi i}{2191}$$

Extension of the Cauchy Integral Formula:
If we have a pole (singularity) of order 2 at z_0 then we can extend the Cauchy Integral Formula

$$\int_C \frac{f(z)}{(z-z_0)}dz = 2\pi i f(z_0)$$

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)}dz$$

$$f(z_0+\Delta z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-(z_0+\Delta z_0)}dz$$

$$\frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z_0} = \frac{1}{2\pi i}\int_C \frac{f(z)}{\Delta z_0}\left(\frac{1}{z-(z_0+\Delta z_0)} - \frac{1}{z-z_0}\right)dz$$
$$\frac{1}{z-(z_0+\Delta z_0)} - \frac{1}{z-z_0} = \frac{\Delta z_0}{(z-(z_0+\Delta z_0))(z-z_0)}$$
$$\frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z_0} = \frac{1}{2\pi i}\int_C \frac{f(z)}{\Delta z_0}\frac{\Delta z_0}{(z-(z_0+\Delta z_0))(z-z_0)}dz = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-(z_0+\Delta z_0))(z-z_0)}dz$$
$$\lim_{\Delta z_0\to 0}\frac{f(z_0+\Delta z_0)-f(f_0)}{\Delta z_0} = \frac{1}{2\pi i}\lim_{\Delta z_0\to 0}\int_C \frac{f(z)}{(z-(z_0+\Delta z_0))(z-z_0)}dz$$
$$f'(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^2}dz$$

Using the same technique we can generalize this to poles of order $m+1$

$$f^{(m)}(z_0) = \frac{m!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{m+1}}dz$$

For N poles, $z_1,\ldots,z_j$ of order $m_1+1,\ldots,m_j+1$ respectively, with $f(z) = \frac{1}{(z-z_j)^{m_j+1}}f_j(z)$ we get

$$\int_C f(z)dz = 2\pi i\sum_{j=1}^{N}\frac{f^{(m_j)}(z_j)}{m_j!}$$

Ex:

$$\int_{|z|=3}\frac{e^z}{(z^2+1)^2}dz$$
$$z = \pm i$$
$$f^{(m)}(z_0) = \frac{m!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{m+1}}dz$$
$$z_1 = i$$
$$f_1(z) = \frac{e^z}{(z+i)^2}$$
$$f_1'(z) = \frac{e^z}{(z+i)^2} - \frac{2e^z}{(z+i)^3}$$
$$f_1'(z_1) = \frac{e^i}{(2i)^2} - \frac{2e^i}{(2i)^3} = e^i\left(\frac{1}{-4} - \frac{2}{-8i}\right) = \frac{e^i}{4}(-1-i)$$
$$z_2 = -i$$
$$f_2(z) = \frac{e^z}{(z-i)^2}$$
$$f_2'(z) = \frac{e^z}{(z-i)^2} - \frac{2e^z}{(z-i)^3}$$
$$f_2'(z_2) = \frac{e^{-i}}{(-2i)^2} - \frac{2e^{-i}}{(-2i)^3} = e^{-i}\left(\frac{1}{-4} - \frac{2}{8i}\right) = \frac{e^{-i}}{4}(-1+i)$$
$$\int_{|z|=3}\frac{e^z}{(z^2+1)^2}dz = 2\pi i f_1'(z_1) + 2\pi i f_2'(z_2)$$

$$= \frac{\pi}{2}\left(e^{i}(1-i)+e^{-i}(-1-i)\right)$$
$$e^{i} = \cos(1)+i\sin(1)$$
$$e^{-i} = \cos(1)-i\sin(1)$$
$$I = \frac{\pi}{2}\left(\cos(1)(1-i)+i\sin(1)(1-i)+\cos(1)(-1-i)-i\sin(1)(-1-i)\right)$$
$$= \frac{\pi}{2}\left(-2i\cos(1)+2i\sin(1)\right)$$
$$= \pi i(\sin(1)-\cos(1))$$

Ex2:

$$\int_{|z|=2}\frac{\cos(z)}{z^3(z-1)}dz$$
$$z_1 = 0,\ \ z_2 = 1$$
$$\int_{|z|=2}\frac{\cos(z)}{z^3(z-1)}dz = \frac{2\pi i}{2}f_1''(z_1)+2\pi i f_2(z_2)$$
$$f_1(z) = \frac{\cos(z)}{z-1}$$
$$f_1'(z) = -\frac{\sin(z)}{z-1}-\frac{\cos(z)}{(z-1)^2}$$
$$f_1''(z) = -\frac{\cos(z)}{z-1}+\frac{2\sin(z)}{(z-1)^2}+\frac{2\cos(z)}{(z-1)^3}$$
$$f_1''(z_1) = -\frac{1}{-1}+0+\frac{2}{(-1)^3} = -1$$
$$f_2(z) = \frac{\cos(z)}{z^3}$$
$$f_z(z_2) = \frac{\cos(1)}{1} = \cos(1)$$
$$I = -\pi i + 2\pi i\cos(1)$$

Ex3:

$$\int_{|z|=2}\frac{z^2+1}{(z-1)^3}dz$$
$$z_0 = 1$$
$$\int_{|z|=2}\frac{z^2+1}{(z-1)^3}dz = \frac{2\pi i}{2}f''(z_0)$$
$$f(z) = z^2+1$$
$$f'(z) = 2z$$
$$f''(z) = 2 = f''(z_0)$$
$$I = 2\pi i$$

Ex4:

$$\int_{|z-2|=1}\frac{\mathrm{Log}(z)}{(z^2-4)^2}dz$$

$z = \pm 2 \Rightarrow z_0 = 2$

$$\int_{|z-2|=1} \frac{\mathrm{Log}(z)}{(z^2-4)^2} dz = 2\pi i f'(z_0)$$

$$f(z) = \frac{\mathrm{Log}(z)}{(z+2)^2}$$

$$f'(z) = \frac{1/z}{(z+2)^2} - \frac{2\,\mathrm{Log}(z)}{(z+2)^3}$$

$$f'(2) = \frac{1/2}{4^2} - \frac{2\ln 2}{4^3} = \frac{1}{32} - \frac{\ln 2}{32} = \frac{1}{32}(1-\ln 2)$$

$$I = \frac{\pi i}{16}(1-\ln 2)$$

Ex5:

$$\int_{|z|=2} \frac{e^{1/z}}{z^4+3z+1} dz$$

$|z^4+3z+1| > 0,\ |z| > 2$

$|z^4+3z+1| \geq ||z|^4 - 3|z| - 1| > |16-6-1| = 9$

$$\int_{|z|=R} f(z)dz = \int_{|z|=2} f(z)dz$$

$$\left|\int_{|z|=R} \frac{e^{1/z}}{z^4+3z+1} dz\right| \leq \max_{|z|=R}\left|\frac{e^{1/z}}{z^4+3z+1}\right| 2\pi R$$

$$\frac{1}{z} = \frac{\overline{z}}{|z|^2}$$

$$|e^{1/z}| = e^{\frac{x}{|z|^2}}$$

$$\max_{|z|=R} e^{\frac{x}{|z|^2}} = e^{\frac{R}{R^2}} = e^{\frac{1}{R}}$$

$$\left|\int_{|z|=R} \frac{e^{1/z}}{z^4+3z+1} dz\right| \leq \frac{2\pi R e^{\frac{1}{R}}}{R^4-3R-1} \to 0 \text{ as } R \to \infty$$

$I = 0$

Ex6:

$$\int_{|z|=2} \frac{z}{(z-3)^2(z^3+z+1)} dz$$

$|z^3+z+1| > 0,\ |z| > 2$

$|z^3+z+1| \geq ||z|^3 - |z| - 1| > |8-2-1| = 5$

$z_0 = 3$

$$\int_{|z|=R} f(z)dz = \int_{|z|=2} f(z)dz + \int_{|z-3|=r} f(z)dz$$

$$\left|\int_{|z|=R} \frac{z}{(z-3)^2(z^3+z+1)} dz\right| \leq \max_{|z|=R}\left|\frac{z}{(z-3)^2(z^3+z+1)}\right| 2\pi R = \frac{2\pi R^2}{(R-3)^2(R^3-R-1)} \to 0$$

$$\int_{|z-3|=r} \frac{z}{(z-3)^2(z^3+z+1)}dz = \int_{|z-3|=r} \frac{f_0(z)}{(z-3)^2}dz = 2\pi i f_0'(z_0)$$
$$f_0(z) = \frac{z}{z^3+z+1}$$
$$f_0'(z) = \frac{z^3+z+1-z(3z^2+1)}{(z^3+z+1)^2}$$
$$f_0'(z_0) = \frac{3^3+3+1-3(3^3+1)}{(3^3+3+1)^2} = -\frac{53}{961}$$
$$I = \frac{106}{961}\pi i$$

We can also use this formula to compute some real integrals as well.
Ex:

$$\int_0^{2\pi} \frac{d\varphi}{3+\sin\varphi}$$
$$z = e^{i\varphi} \Rightarrow dz = ie^{i\varphi}d\varphi = izd\varphi \Rightarrow d\varphi = \frac{dz}{iz}$$
$$\sin\varphi = \frac{z-z^{-1}}{2i}$$
$$I = \int_{|z|=1} \frac{\frac{dz}{iz}}{3+\frac{z-z^{-1}}{2i}} = 2\int_{|z|=1} \frac{dz}{6iz+z^2-1}$$
$$z_{1,2} = \frac{-6i \pm \sqrt{-36+4}}{2} = \frac{-6i \pm 4\sqrt{2}i}{2} = -3i \pm 2\sqrt{2}i$$
$$|z_1| = |-3i+2\sqrt{2}| \leq 1$$
$$I = 2\int_{|z-z_1|} \frac{dz}{(z-z_1)(z-z_2)} = \frac{4\pi i}{z_1-z_2} = \frac{4\pi i}{4\sqrt{2}i} = \frac{\pi}{\sqrt{2}}$$

Ex2:

$$\int_0^{\pi} \frac{d\varphi}{2+\cos\varphi}$$
$$f(-\varphi) = \frac{1}{2+\cos(-\varphi)} = f(\varphi)$$
$$I = \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\varphi}{2+\cos\varphi}$$
$$z = e^{i\varphi} \Rightarrow dz = ie^{i\varphi}d\varphi = izd\varphi \Rightarrow d\varphi = \frac{dz}{iz}$$
$$\cos\varphi = \frac{z+z^{-1}}{2}$$
$$I = \frac{1}{i}\int_{|z|=1} \frac{dz}{4z+z^2+1}$$
$$z_{1,2} = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$$
$$|z_1| = |-2+\sqrt{3}| \leq 1$$

$$I = \frac{1}{i}\int_{|z-z_1|=r}\frac{dz}{(z-z_1)(z-z_2)} = \frac{1}{i}2\pi i\frac{1}{z_1-z_2} = \frac{2\pi}{2\sqrt{3}} = \frac{\pi}{\sqrt{3}}$$

Ex3:

$$\int_0^\pi \frac{d\varphi}{(2+\cos\varphi)^2} = \frac{1}{2}\int_{-\pi}^{\pi}\frac{d\varphi}{(2+\cos\varphi)^2}$$

$$z = e^{i\varphi} \Rightarrow dz = ie^{i\varphi}d\varphi = izd\varphi \Rightarrow d\varphi = \frac{dz}{iz}$$

$$\cos\varphi = \frac{z+z^{-1}}{2}$$

$$I = \frac{2}{i}\int_{|z|=1}\frac{dz}{z(4+z+z^{-1})^2} = \frac{2}{i}\int_{|z|=1}\frac{zdz}{(4z+z^2+1)^2}$$

$$z_{1,2} = -2\pm\sqrt{3}$$

$$|z_1| = |-2+\sqrt{3}| \leq 1$$

$$f_1(z) = \frac{z}{(z-z_2)^2}$$

$$f_1'(z) = \frac{(z-z_2)^2 - 2z(z-z_2)}{(z-z_2)^4} = \frac{z-z_2-2z}{(z-z_2)^3}$$

$$f_1'(z_1) = \frac{-z_1-z_2}{(z_1-z_2)^3} = \frac{4}{(2\sqrt{3})^3} = \frac{1}{2\sqrt{27}}$$

$$I = \frac{2}{i}2\pi i f_1'(z_1) = \frac{2\pi}{3\sqrt{3}}$$

1.3.6 Applications of the Cauchy Integral Formula

Pointwise Estimate:

We can estimate the maximum value of a point z_0 inside a closed path C.
Let us define a region enclosed by $|z-z_0| = R$ and start with the Cauchy Integral Formula

$$f^{(m)}(z_0) = \frac{m!}{2\pi i}\int_{|z-z_0|=R}\frac{f(z)}{(z-z_0)^{m+1}}dz$$

$$\left|f^{(m)}(z_0)\right| = \frac{m!}{2\pi}\left|\int_{|z-z_0|=R}\frac{f(z)}{(z-z_0)^{m+1}}dz\right| \leq \frac{m!}{2\pi}\max_{|z-z_0|=R}\left|\frac{f(z)}{(z-z_0)^{m+1}}\right|2\pi R$$

$$\max_{|z-z_0|=R}|(z-z_0)^{m+1}| = R^{m+1}$$

$$\left|f^{(m)}(z_0)\right| \leq \frac{m!}{2\pi}\max_{|z-z_0|=R}|f(z)|\frac{2\pi R}{R^{m+1}}$$

$$\left|f^{(m)}(z_0)\right| \leq \frac{m!}{R^m}\max_{|z-z_0|=R}|f(z)|$$

Ex: If $|f(z)| \leq 1$ on $|z| \leq 1$ estimate $f(0)$

$$|f^{(m)}(z_0)| \leq \frac{m!}{R^m}\max_{|z-z_0|=R}|f(z)|$$

$$z_0 = 0,\ R = 1$$

$$|f''(0)| \le \frac{2!}{1^2} \max_{|z|=1} |f(z)|$$

$$|f''(0)| \le 2$$

Ex2: If $|f(z)| \le 1$ on $|z| \le 1$ estimate $f(\frac{1}{2})$

$$|f^{(m)}(z_0)| \le \frac{m!}{R^m} \max_{|z-z_0|=R} |f(z)|$$

$$z_0 = \frac{1}{2},\ R = \frac{1}{2}$$

$$\max_{|z-\frac{1}{2}|=\frac{1}{2}} |f(z)| = \max_{|z-\frac{1}{2}|\le\frac{1}{2}} |f(z)| = 1$$

$$|f''(\tfrac{1}{2})| \le \frac{2!}{(\frac{1}{2})^2} \max_{|z-\frac{1}{2}|=\frac{1}{2}} |f(z)|$$

$$|f''(\tfrac{1}{2})| \le 8$$

Liouville's Theorem:
Recall that if $f(z)$ is entire then $f(z)$ is analytic in C. Liouville's Theorem states that if $f(z)$ is entire and bounded (i.e. $|f(z)| \le M$) then $f(z)$ is constant.

Proof. Starting off from the pointwise estimate formula with $m = 1$ we have

$$\left|f'(z_0)\right| \le \frac{1}{R} \max_{|z-z_0|=R} |f(z)|$$

Because $f(z)$ is bounded we have

$$\max_{|z-z_0|=R} |f(z)| = M$$

So we have

$$\left|f'(z_0)\right| \le \frac{M}{R}$$

Now because f is entire we can take the limit as $R \to \infty$ and we get

$$|f'(z_0)| \le 0 \Rightarrow f'(z) = 0$$

So $f(z)$ is constant. $\square$

We can also extend this to mth order derivatives in a more general manner.
Starting off from the pointwise estimate formula we got above we have

$$\left|f^{(m)}(z_0)\right| \le \frac{m!}{R^m} \max_{|z-z_0|=R} |f(z)|$$

Because $f(z)$ is bounded we have

$$\max_{|z-z_0|=R} |f(z)| = M$$

So we have

$$\left|f^{(m)}(z_0)\right| \leq \frac{m!M}{R^m}$$

Now because f is entire we can take the limit as $R \to \infty$ and we get

$$|f^{(m)}(z_0)| \leq 0 \Rightarrow f^{(m)}(z) = 0$$

And so we can determine the degree of the polynomial $f(z)$ assuming we are provided information about which derivative is bounded. Ex: Show that if $f(z)$ is entire and $|f(z)| \leq C(1+|z|)^4$ then $f(z)$ is a polynomial of at most degree 4.

$$|f^{(5)}(z_0)| \leq \frac{5!}{R^5} \max_{|z-z_0|=R} |f(z)|$$
$$|f(z)| \leq 2 \max_{|z-z_0|=R} (1+|z|)^4$$
$$|f^{(5)}(z_0)| \leq \frac{240}{R^5} \max_{|z-z_0|=R} (1+|z|)^4$$
$$|z| \leq |z-z_0| + |z_0| \leq R + |z_0|$$
$$|f^{(5)}(z_0)| \leq \frac{240}{R^5}(1+R+|z_0|)^4 \to 0 \text{ as } R \to \infty$$
$$|f^{(5)}(z_0)| = 0 \Rightarrow f''''(z) = C_1 \Rightarrow f'''(z) = C_1 z + C_2 \Rightarrow f''(z) = C_1 z^2 + C_2$$
$$\Rightarrow f'(z) = C_1 z^3 + C_2 z^2 + C_3 z + C_4 \Rightarrow f(z) = C_1 z^4 + C_2 z^3 + C_3 z^2 + C_4 z + C_5$$
$$\text{degree}(f(z)) \leq 4$$

We can also extend this further and say that for $f = u + iv$ if some linear combination of u and v is bounded then $f(z)$ is constant.
Ex: $u + v \geq 0$

$$f = u + iv$$
$$u + v \geq 0$$
$$\Re((\alpha_1 + i\alpha_2)(u+iv)) = \alpha_1 u - \alpha_2 v$$
$$\alpha = 1 - i$$
$$g = e^{-\alpha f}$$
$$g \text{ is entire because } f \text{ is entire}$$
$$|g(z)| = |e^{\alpha f}| = |e^{-u-v}|$$
$$u + v \geq 0 \Rightarrow e^{-u-v} \leq 1$$
$$|g(z)| \leq 1$$
$$|g'(z_0)| \leq \frac{1}{R} \max_{|z-z_0|=R} |g(z)| \to 0$$
$$g'(z) = 0 \Rightarrow g(z) \equiv C$$
$$\Rightarrow f(z) \equiv C$$

Maximum Principle:
If $f(z)$ is analytic in a region $D \cup \partial D$ then the maximum value on that region is equal to the maximum on the boundary

$$\max_{D\cup\partial D} |f(z)| = \max_{\partial D} |f(z)|$$

Proof.

$$M = \max_{D\cup\partial D} |f(z)| = |f(z_0)|$$

for some $z_0 \in D \cup \partial D$ Case 1: if $z_0 \in \partial D$ then

$$M = |f(z_0)| \leq \max_{\partial D} |f(z)| \leq \max_{D\cup\partial D} |f(z)| = M$$

Case 2: $z_0 \in D$

$$\begin{aligned}
&\{|z - z_0| \leq r\} \subset D \\
&f(z_0) = \frac{1}{2\pi i} \int_{|z-z_0|=r} \frac{f(z)}{z - z_0} dz \\
&|f(z_0)| = M \leq \frac{1}{2\pi} \int_{|z-z_0|=r} \frac{|f(z)|}{|z - z_0|} dz = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta \\
&M = |f(z_0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| d\theta \\
&\leq \frac{1}{2\pi} \int_0^{2\pi} \max_{D\cup\partial D} |f(z)| d\theta \\
&\leq \frac{1}{2\pi} M \cdot 2\pi = M
\end{aligned}$$

□

We can also state the minimum principle which is very similar to the maximum principle.
Either

$$\min_{D\cup\partial D} |f(z)| = \min_{\partial D} |f(z)|$$

or

$$\min_{D\cup\partial D} |f(z)| = 0$$

Ex: Find the maximum and minimum values of $(z+1)^2 e^{-z}$ on $|z| \leq 1$

$$\begin{aligned}
&\max_{|z|\leq 1} |f(z)| = \max_{|z|=1} |f(z)| \\
&z = e^{i\varphi} \\
&|(z+1)^2 e^{-z}| = |z+1|^2 |e^{-z}| = ((\cos\varphi + 1)^2 + \sin^2\varphi) e^{-\cos\varphi} = (2\cos\varphi + 2) e^{-\cos\varphi} = h(\varphi) \\
&h'(\varphi) = e^{-\cos\varphi} ((2\cos\varphi + 2)\sin\varphi - 2\sin\varphi) = 0
\end{aligned}$$

$$\sin\varphi\cos\varphi = 0 \Rightarrow \varphi = \frac{n\pi}{2},\ n \in \mathbb{Z}$$
$$\cos\varphi = \{0, -1, 1\}$$
$$\text{CPs} = \left\{2, 4e^{-1}, 0\right\}$$
$$\max_{|z|\le 1} |(z+1)^2 e^{-z}| = 2$$
$$\min_{|z|\le 1} |(z+1)^2 e^{-z}| = 0$$

Ex2: Find the maximum and minimum values of $z^{100} + 3z^{50} - 1$ on $|z| \le 1$

$$\max_{|z|\le 1} |f(z)| = \max_{|z|=1} |f(z)|$$
$$z = e^{i\varphi}$$
$$|z^{100} + 3z^{50} - 1| = |z^{50}||z^{50} + 3 - z^{-50}| = |e^{i50\varphi} + 3 - e^{-i50\varphi}| = |2i\sin(50\varphi) + 3| = \sqrt{4\sin^2(50\varphi) + 9}$$
$$\max_{\varphi} \sqrt{4\sin^2(50\varphi) + 9} = \sqrt{13}$$

Argument Principle:
This involves finding the zeros of a polynomial inside a given region (typically the positive real axis). Applications of this is used to determine the number of positive real roots of a polynomial which can be used to determine the stability of solutions to differential equations.
If $f(z)$ is analytic on C and inside C with $f(z) \neq 0$ on C, $f(z) = 0$ has finitely many roots inside C. Each root has multiplicity $n_j,\ j = 1, \ldots, k$. Then,

$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i N$$

where N is the number of zeros of $f(z)$ inside C (counting multiplicities) and is defined by

$$N = \sum_{j=1}^{k} n_j$$

$$C:\ z = z(t)$$
$$\int_C \frac{f'(z)}{f(z)} dz = \int_a^b \frac{f'(z(t))}{f(z(t))} z'(t) dt$$
$$w(t) = f(z(t)) \Rightarrow w'(t) = f'(z(t))z'(t)$$
$$\int_C \frac{f'(z)}{f(z)} dz = \int_a^b \frac{w'(t)}{w(t)} dt = \int_{w(a)}^{w(b)} \frac{dw}{w} = \int_{f(C)} \frac{dw}{w} = i\arg(w(t))\Big|_{f(C)}$$
$$\int_C \frac{f'(z)}{f(z)} dz = i\arg(f(z))\Big|_{f(C)}$$
$$\int_C \frac{f'(z)}{f(z)} dz = 2\pi i N = i\arg(f(z))\Big|_{f(C)}$$

This leads us to the general form of the argument principle

$$N = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = \frac{\left.\mathrm{Arg}(f(z))\right|_{f(C)}}{2\pi}$$

We can break this down into common applications.
Rouche's Theorem:
If $f(z) = g(z) + h(z)$ and $|h(z)| = |f(z) - g(z)| < |g(z)|$ on C then the number of zeros in f is the same as the number of zeros in g ($N_f = N_g$).

Proof.

$$\begin{aligned}
&f_\alpha(z) = g(z) + \alpha h(z),\ 0 \le \alpha \le 1\\
&f_0(z) = g(z),\ f_1(z) = f(z)\\
&\frac{1}{2\pi i}\int_C \frac{f'_\alpha(z)}{f_\alpha(z)}dz = N(\alpha) = N_{f_\alpha}\\
&\frac{1}{2\pi i}\int_C \frac{f'_\alpha(z)}{f_\alpha(z)}dz \text{ is continuous in } \alpha\\
&N(\alpha) \text{ is continuous in } \alpha\\
&\Rightarrow N(\alpha) \text{ is constant in } \alpha\\
&N(0) = N(1) \Rightarrow N_g = N_f
\end{aligned}$$

□

Ex: Find the number of zeros of $z^5 + 2z^2 - 1$ in $|z| \le 1$

$$\begin{aligned}
&p(z) = z^5 + 2z^2 + 5z - 1,\ |z| \le 1\\
&g(z) = 5z\\
&|g(z)| = 5\\
&|f(z) - g(z)| = |z^5 + 2z^2 - 1| \le |z|^5 + 2|z|^2 + 1 = 4 < 5\\
&N_f = N_g = 1
\end{aligned}$$

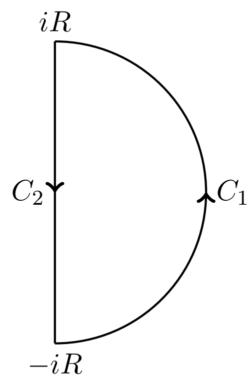
Ex2: Find the number of zeros of $8z^5 + 3z^4 + 4z^2 - 3$ in $|z| < 1$

$$\begin{aligned}
&p(z) = 8z^5 + 3z^4 + 4z^2 - 3,\ |z| < 1\\
&g(z) = 8z^5\\
&|g(z)| = 8\\
&|f(z) - g(z)| = |3z^4 + 4z^2 - 3| = |z|^2|3z^2 - 3z^{-2} + 4| = |3z^2 - 3z^{-2} + 4|\\
&z = e^{i\varphi}\\
&|f(z) - g(z)| = |3e^{2i\varphi} - 3e^{-2i\varphi} + 4| = |6i\sin(2\varphi) + 4| = 2\sqrt{9\sin^2(2\varphi) + 4} \le 2\sqrt{13} < 8\\
&N_f = N_g = 5
\end{aligned}$$

Ex3: Find the number of zeros of $z^5 + 7z^2 + 2$ in $1 \le |z| \le 2$

$$
\begin{aligned}
&p(z) = z^5 + 7z^2 + 2,\ 1 \le |z| \le 2 \\
&|z| = 2: \\
&g(z) = z^5 \\
&|g(z)| = 32 \\
&|f(z) - g(z)| = |7z^2 + 2| \le 7|z|^2 + 2 = 30 < 32 \\
&N_f = N_g = 5 \\
&|z| = 1: \\
&g(z) = 7z^2 \\
&|g(z)| = 7 \\
&|f(z) - g(z)| = |z^5 + 2| \le |z|^5 + 2 = 3 < 7 \\
&N_f = N_g = 2 \\
&\Rightarrow N = 5 - 2 = 3
\end{aligned}
$$

Ex4: Find the number of zeros of $z^2 - 4 + 3e^{-z}$ in $\Re(z) > 0$



$$
\begin{aligned}
&f(z) = z^2 - 4 + 3e^{-z},\ \Re(z) > 0 \\
&g(z) = z^2 - 4 \\
&h(z) = 3e^{-z} \\
&C = C_1 \cup C_2 \\
&C_1:\ Re^{it},\ -\frac{\pi}{2} \le t \le \frac{\pi}{2} \\
&C_2:\ iy,\ -R \le y \le R \\
&C_1: \\
&|h(z)| = 3|e^{-z}| = 3e^{-x} \le 3 \\
&|g(z)| = |z^2 - 4| \ge |z|^2 - 4 = R^2 - 4 \\
&R \to \infty:\ |g(z)| > |h(z)| \\
&C_2:
\end{aligned}
$$

$$|h(z)| = 3e^{-x} = 3$$
$$|g(z)| = |z^2 - 4| = |-y^2 - 4| = y^2 + 4 \geq 4$$
$$|g(z)| > |h(z)|$$
$$g(z) = z^2 - 4 = 0 \Rightarrow z = \pm 2 \Rightarrow N_g = 1$$
$$N_f = N_g = 1$$

Finding the number of zeros in the positive real part was a little complicated using Rouche's Theorem. Fortunately, there is a better method at solving this problem.

Nyquist Criteria:

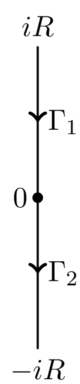
From the argument principle we have

$$N_p = \frac{1}{2\pi} \arg(p(z))\Big|_{p(C)}$$

Because we have C broken into two parts, $C = C_1 \cup C_2$, we get

$$\arg(p(z))\Big|_{p(C)} = \arg(p)\Big|_{p(C_1)} + \arg(p)\Big|_{p(C_2)}$$
$$C_1 : \; z = Re^{it}, \; p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots = a_n R e^{int} + a_{n-1} R^{n-1} e^{i(n-1)t} + \cdots$$
$$-\pi \leq t \leq \pi \Rightarrow \arg(p)\Big|_{C_1} = n\pi$$

For C_2, we can break it up into two paths for the positive and negative parts:



$$\arg(p)\Big|_{p(\Gamma_1 \cup \Gamma_2)} = \arg(p)\Big|_{p(\Gamma_1)} + \arg(p)\Big|_{\Gamma_2}$$
$$= \arg(p(i\infty)) - \arg(p(0)) + \arg(p(0)) - \arg(p(-i\infty))$$
$$= \arg(p(i\infty)) - \arg(p(-i\infty))$$
$$p(-iy) = \overline{p(iy)} \Rightarrow p(-i\infty) = \overline{p(i\infty)}$$
$$\arg(p(-i\infty)) = \arg(\overline{p(i\infty)}) = -\arg(p(i\infty))$$
$$\arg(p)\Big|_{p(\Gamma_1 \cup \Gamma_2)} = 2\arg(p)\Big|_{p(\Gamma_1)}$$

And so the number of zeros in the positive real part is

$$N = \frac{1}{2\pi}\left(n\pi + 2\arg(p)\Big|_{p(\Gamma_1)}\right)$$

where $\Gamma_1: \; z = iy, \; 0 < y < \infty$

Ex: Find the number of zeros of $z^3 + 2z^2 + 4$ in $\Re(z) > 0$

$$p(z) = z^3 + 2z^2 + 4, \; \Re(z) > 0$$

$$N = \frac{1}{2\pi}\left(n\pi + 2\arg(p)\Big|_{p(\Gamma_1)}\right), \; n = 3$$

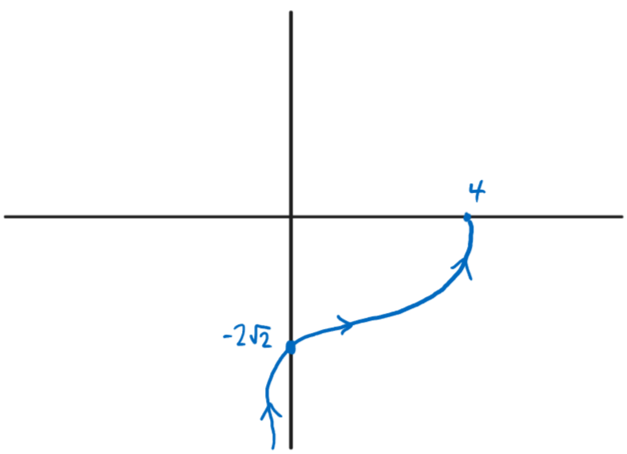
$$\Gamma_1: \; z = iy, \; 0 < y < \infty$$

$$p_R = -2y^2 + 4$$

$$p_I = -y^3$$

$$p_R = 0 \Rightarrow -2y^2 + 4 = 0 \Rightarrow y^2 = 2 \Rightarrow y = \sqrt{2}$$

y	p_R	p_I
∞	$-\infty$	$-\infty$
$\sqrt{2}$	0	$-2\sqrt{2}$
0	4	0



$$\arg(p)\Big|_{p(\Gamma_1)} = \frac{\pi}{2}$$

$$N = \frac{1}{2\pi}\left(3\pi + 2\frac{\pi}{2}\right) = 2$$

Ex2: Find the number of zeros of $z^3 + 2z^2 + 4z + 2$ in $\Re(z) > 0$

$$p(z) = z^3 + 2z^2 + 4z + 2$$

$$N = \frac{1}{2\pi}\left(n\pi + 2\arg(p)\Big|_{p(\Gamma_1)}\right), \; n = 3$$

$\Gamma_1: \; z = iy, \; 0 < y < \infty$

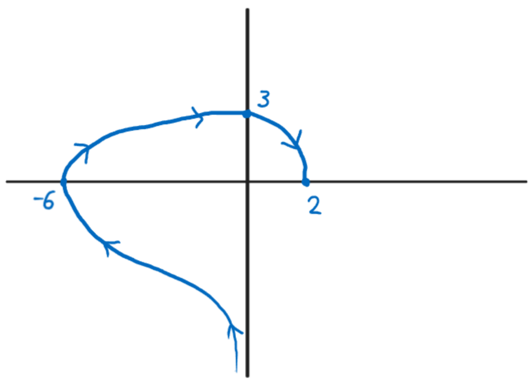
$p_R = -2y^2 + 2$

$p_I = -y^3 + 4y$

$p_R = 0 \Rightarrow y^2 = 1 \Rightarrow y = 1$

$p_I = 0 \Rightarrow y(-y^2+4) = 0 \Rightarrow y = 0, \; y^2 = 4 \Rightarrow y = 0, 2$

y	p_R	p_I
∞	$-\infty$	$-\infty$
2	-6	0
1	0	3
0	2	0



$$\arg(p)\Big|_{p(\Gamma_1)} = -\frac{3\pi}{2}$$

$$N = \frac{1}{2\pi}\left(3\pi + 2\left(-\frac{3\pi}{2}\right)\right) = 0$$

Ex3: Find the number of zeros of $z^3 + z^2 + 4z + 1$ in $\Re(z) > 0$

$p(z) = z^3 + z^2 + 4z + 1$

$$N = \frac{1}{2\pi}\left(n\pi + 2\arg(p)\Big|_{p(\Gamma_1)}\right), \; n = 3$$

$\Gamma_1: \; z = iy, \; 0 < y < \infty$

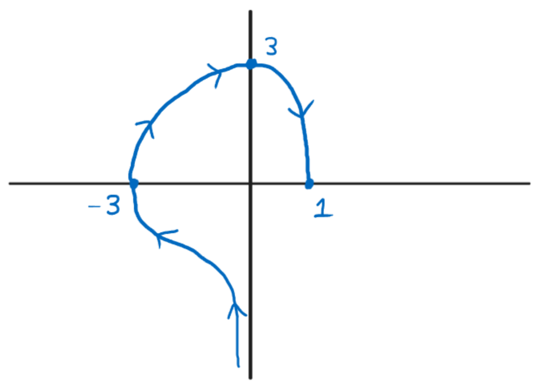
$p_R = -y^2 + 1$

$p_I = -y^3 + 4y$

$p_R = 0 \Rightarrow y = 1$

$p_I = 0 \Rightarrow y(-y^2+4) = 0 \Rightarrow y = 0, 2$

y	p_R	p_I
∞	$-\infty$	$-\infty$
2	-3	0
1	0	3
0	1	0



$$
\begin{aligned}
\arg(p)\Big|_{p(\Gamma_1)} &= -\frac{3\pi}{2} \\
N &= \frac{1}{2\pi}\left(3\pi + 2\left(-\frac{3\pi}{2}\right)\right) = 0
\end{aligned}
$$

1.3.7 Laurent Series

Recall the Taylor series of a function is defined by

$$
f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n
$$

The Taylor series will converge if $|z - z_0| < r$ where r is the radius of convergence. The radius of convergence can be found to be

$$
r = \min |z_0 - \text{other singularities of } f(z)|
$$

Some common Taylor series are

$$
\begin{aligned}
e^z &= \sum_{n=0}^{\infty} \frac{z^n}{n!} \\
\sin z &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} \\
\cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \\
\frac{1}{1-z} &= \sum_{n=0}^{\infty} z^n, \ |z| < 1 \\
\mathrm{Log}(1+z) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}, \ |z| < 1
\end{aligned}
$$

The Laurent series is a generalization of the Taylor series that extends the Taylor series to negative exponentials in the series. If we compare the two we get,

Taylor Series:
f is analytic in $|z-z_0|<r$

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

$$a_n=\frac{f^{(n)}(z_0)}{n!}=\frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{f(\eta)}{(\eta-z_0)^{n+1}}d\eta$$

Laurent Series:
f is analytic in $r_1<|z-z_0|<r_2$

$$f(z)=\sum_{n=-\infty}^{\infty}a_n(z-z_0)^n$$

$$a_n=\frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{f(\eta)}{(\eta-z_0)^{n+1}}d\eta$$

Ex: Find the Laurent series of $f(z)=\frac{z}{(z+1)(z-2)}$ in each of $|z|<1$, $1<|z|<2$, and $|z|>2$

$$f(z)=\frac{z}{(z+1)(z-2)}$$
$$f(z)=\frac{A}{z+1}+\frac{B}{z-2}$$
$$A+B=1$$
$$-2A+B=0\Rightarrow 2A=B$$
$$3A=1\Rightarrow A=\frac{1}{3}\Rightarrow B=\frac{2}{3}$$
$$f(z)=\frac{1}{3}\left(\frac{1}{z+1}+\frac{2}{z-2}\right)$$

$|z|<1$

$$\frac{1}{1+z}=1-z+z^2-z^3+\cdots=\sum_{n=0}^{\infty}(-1)^nz^n$$
$$\frac{1}{1-z}=1+z+z^2+z^3+\cdots=\sum_{n=0}^{\infty}z^n$$
$$f(z)=\frac{1}{3}\left(\frac{1}{z+1}-\frac{1}{1-\frac{z}{2}}\right)$$
$$f(z)=\frac{1}{3}\sum_{n=0}^{\infty}\left((-1)^n-\frac{1}{2^n}\right)z^n$$

$|z|>2$

$$\frac{1}{z+1} = \frac{1}{z}\frac{1}{1-(-\frac{1}{z})} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^n$$
$$\frac{1}{z-2} = \frac{1}{z}\frac{1}{1-\frac{2}{z}} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{2}{z}\right)^n$$
$$f(z) = \frac{1}{3}\sum_{n=0}^{\infty}\left((-1)^n + 2^{n+1}\right)\frac{1}{z^{n+1}}$$

$1 < |z| < 2$
$$\frac{1}{z+1} = \frac{1}{z}\frac{1}{1-(-\frac{1}{z})} = \frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{-1}{z}\right)^n$$
$$\frac{1}{z-2} = -\frac{1}{2}\frac{1}{1-\frac{z}{2}} = -\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$$
$$f(z) = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{(-1)^n}{z^{n+1}} - \frac{z^n}{2^n}\right)$$

Big O Notation:
When n gets very large then $\mathcal{O}(n)$ determines how fast the function grows.

$$\mathcal{O}(n) \Leftrightarrow |\mathcal{O}(n)| \leq C|n|$$

When n gets very small then $o(n)$ determines how fast the function shrinks.

$$o(n) \Leftrightarrow \frac{o(n)}{n} \to 0 \text{ as } n \to 0$$

Ex2: Find the first few terms of the Laurent series for $f(z) = \frac{z}{\operatorname{Log} z},\ |z-1| < 1$

$$f(z) = \frac{z}{\operatorname{Log} z},\ |z-1| < 1$$
$$\operatorname{Log}(z) = \operatorname{Log}(z-1+1)$$
$$w = z-1$$
$$\operatorname{Log}(z) = \operatorname{Log}(w+1) = w - \frac{w^2}{2} + \frac{w^3}{3} - \mathcal{O}(w^4) = w\left(1 - \frac{w}{2} + \frac{w^2}{3} - \mathcal{O}(w^3)\right)$$
$$\frac{1}{\operatorname{Log}(z)} = \frac{1}{w}\frac{1}{1-\frac{w}{2}+\frac{w^2}{3}-\mathcal{O}(w^3)}$$
$$w_1 = \frac{w}{2} - \frac{w^2}{3} + \mathcal{O}(w^3)$$
$$\frac{1}{1-w_1} = 1 + w_1 + w_1^2 + \mathcal{O}(w_1^3)$$
$$\frac{1}{1-w_1} = 1 + \frac{w}{2} - \frac{w^2}{3} + \frac{w^2}{4} + \mathcal{O}(w^3) = 1 + \frac{w}{2} - \frac{w^2}{12} + \mathcal{O}(w^3)$$
$$\frac{1}{\operatorname{Log}(z)} = \frac{1}{w} + \frac{1}{2} - \frac{w}{12} + \mathcal{O}(w^2)$$

$$\frac{z}{\mathrm{Log}(z)} = (w+1)\left(\frac{1}{w} + \frac{1}{2} - \frac{w}{12} + \mathcal{O}(w^2)\right)$$
$$\frac{z}{\mathrm{Log}(z)} = \frac{1}{w} + \frac{1}{2} - \frac{w}{12} + 1 + \frac{w}{2} + \mathcal{O}(w^2) = \frac{1}{w} + \frac{3}{2} + \frac{5}{12}w + \mathcal{O}(w^2)$$
$$\frac{z}{\mathrm{Log}(z)} = \frac{1}{z-1} + \frac{3}{2} + \frac{5}{12}(z-1) + \mathcal{O}((z-1)^2)$$

Ex3: Find the Laurent series for $f(z) = \frac{1}{z(z-2)},\ 1 < |z+1| < 3$

$$f(z) = \frac{1}{z(z-2)},\ 1 < |z+1| < 3$$
$$f(z) = \frac{1}{(z+1-1)(z+1-3)}$$
$$w = z+1$$
$$f(z) = \frac{1}{(w-1)(w-3)}$$
$$= \frac{A}{w-1} + \frac{B}{w-3}$$
$$A + B = 0 \Rightarrow A = -B$$
$$-3A - B = 1 \Rightarrow 3B - B = 1 \Rightarrow 2B = 1 \Rightarrow B = \frac{1}{2} \Rightarrow A = -\frac{1}{2}$$
$$f(z) = \frac{1}{2}\left(\frac{1}{w-3} - \frac{1}{w-1}\right)$$
$$= \frac{1}{2}\left(-\frac{1}{3}\frac{1}{1-\frac{w}{3}} - \frac{1}{w}\frac{1}{1-\frac{1}{w}}\right)$$
$$= -\frac{1}{2}\left(\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{w}{3}\right)^n + \frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{1}{w}\right)^n\right)$$
$$= -\frac{1}{6} - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{w^n}{3^{n+1}} + \frac{1}{w^n}\right)$$
$$f(z) = -\frac{1}{6} - \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{(z+1)^n}{3^{n+1}} + \frac{1}{(z+1)^n}\right)$$

Ex4: Find the Laurent series for $f(z) = \frac{1}{z^2+1},\ |z-i| > 2$

$$f(z) = \frac{z}{z^2+1},\ |z-i| > 2$$
$$z^2 + 1 = (z+i)(z-i) = (z-i+2i)(z-i)$$
$$f(z) = \frac{z-i+i}{(z-i)(z-i+2i)}$$
$$w = z-i$$
$$f(z) = \frac{w+i}{w(w+2i)} = \frac{1}{w+2i} + \frac{i}{w}\frac{1}{w+2i}$$

$$\frac{1}{w+2i} = \frac{1}{w}\frac{1}{1+\frac{2i}{w}} = \frac{1}{w}\sum_{n=0}^{\infty}\left(\frac{-2i}{w}\right)^n$$

$$f(z) = \sum_{n=0}^{\infty}\left(\frac{(-2i)^n}{w^{n+1}} + \frac{(-1)^n 2^n i^{n+1}}{w^{n+2}}\right)$$

$$= \frac{1}{w} + \sum_{n=0}^{\infty}\left(\frac{(-2i)^{n+1}}{w^{n+2}} + \frac{(-1)^n 2^n i^{n+1}}{w^{n+2}}\right) = \frac{1}{w} + \sum_{n=0}^{\infty}\frac{2^n i^{n+1}}{w^{n+2}}(-2+1)$$

$$f(z) = \frac{1}{z-i} - \sum_{n=0}^{\infty}\frac{(-2)^n i^{n+1}}{(z-i)^{n+2}}$$

Ex5: Find the Laurent series for $\sqrt{z}$ about $z_0 = 1$

$$\sqrt{z},\ z_0 = 1$$

$$w = z - z_0 = z - 1$$

$$\sqrt{z} = \sqrt{z+1-1} = \sqrt{w+1}$$

$$(1+w)^r = \sum_{n=0}^{\infty}\binom{r}{n} w^n$$

$$\sqrt{w+1} = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n} w^n$$

$$\sqrt{z} = \sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}(z-1)^n \approx 1 + \frac{1}{2}(z-1) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(z-1)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(z-1)^3 + \cdots$$

Ex6: Find the Laurent series for $f(z) = \frac{e^z}{1-z}$ about $z_0 = 0$

$$\frac{e^z}{1-z},\ z_0 = 0$$

$$e^z = \sum_{n=0}^{\infty}\frac{z^n}{n!}$$

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

$$f(z) = \sum_{m=0}^{\infty} z^m \sum_{n=0}^{\infty}\frac{z^n}{n!} = (1 + z + z^2 + \mathcal{O}(z^3))\left(1 + z + \frac{z^2}{2!} + \mathcal{O}(z^3)\right)$$

$$f(z) = \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{1}{m!}\right) z^n \approx 1 + \left(1 + \frac{1}{1!}\right) z + \left(1 + \frac{1}{1!} + \frac{1}{2!}\right) z^2 + \cdots$$

1.3.8 Cauchy Residue Theorem

Classification of Singularities:
For a Laurent series of $f(z)$ in $r_1 < |z - z_0| < r_2$ with $f(z) = a_0 + a_1(z-z_0) + \cdots + a_n(z-z_0)^n + a_{-1}(z-z_0)^{-1} + \cdots + a_{-n}(z-z_0)^{-n}$ we can classify singularities as follows:

- If $a_{-1} = a_{-2} = \cdots = a_{-n} = 0$ (all negative terms) then z_0 is a removable singularity
- If $a_{-n} = 0,\ n \geq m+1$ (finite negative terms) then z_0 is a pole of order m
  If $m = 1$ then it is called a simple pole
- If $a_{-n} \neq 0$ for all n then z_0 is an essential singularity

The *residue* of $f(z)$ at z_0 is defined to be the a_{-1} coefficient of the Laurent series of $f(z)$ about z_0.

$$a_{-1} = \text{Res}(f; z_0)$$

Ex: Find the residue of $f(z) = \frac{1+2z^2}{z^5+z^3}$ about $z_0 = 0$

$$f(z) = \frac{1+2z^2}{z^5+z^3},\ z_0 = 0$$
$$f(z) = \frac{1}{z^3}\left(1+2z^2\right)\left(z^2+1\right)^{-1}$$
$$(z^2+1)^{-1} = 1 - z^2 + \frac{-1(-1-1)}{2}z^4 + \mathcal{O}(z^6)$$
$$f(z) = \frac{1}{z^3}\left(1+2z^2\right)\left(1 - z^2 + z^4 + \mathcal{O}(z^6)\right) = \frac{1}{z^3}\left(1 - z^2 + z^4 + 2z^2 - 2z^4 + \mathcal{O}(z^6)\right)$$
$$f(z) = \frac{1}{z^3} + \frac{1}{z} - z + \mathcal{O}(z^3)$$
$$\text{Res}\left(f(z); 0\right) = 1$$