Physics 158 Written Homework 1

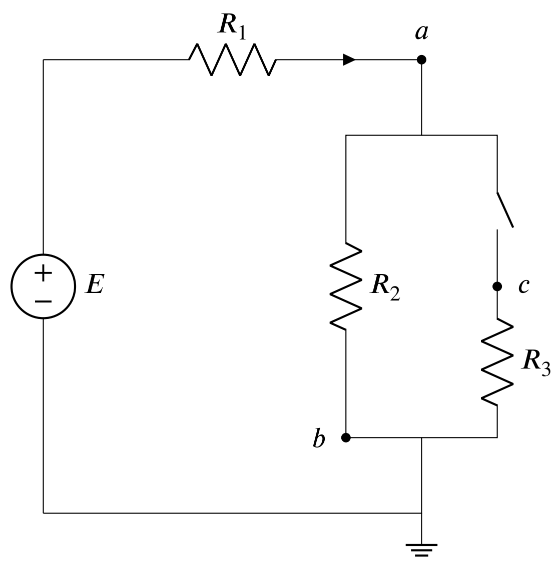
Difficulty: ★☆☆
If the switch is open,

a) Find all currents and potentials at the labelled points.

If the switch is then closed,

b) Find all currents and potentials at the labelled points.

$E = 12\,\mathrm{V}$, $R_1 = 7\,\Omega$, $R_2 = 4\,\Omega$, $R_3 = 10\,\Omega$



Solution:

a) When the switch is open, there is a short circuit in the branch with R_3 so the circuit will act as a series circuit containing R_1, R_2, and E. The potential and voltage at point c will be 0.
The equivalent resistance will be $R_{\mathrm{eq}} = R_1 + R_2$. The current can be computed from Ohm's law:

$$E = I_a R_{\mathrm{eq}}$$

$$I_a = I_b = \frac{E}{R_{\mathrm{eq}}} = \frac{E}{R_1 + R_2} = \frac{12\,\mathrm{V}}{11\,\Omega} = \boxed{1.09\,\mathrm{A}}$$

Point b is connected to ground so the potential of point b will be $0\,\mathrm{V}$. The potential of point a can be computed as the voltage drop across R_2.

$$V_{R_2} = V_{ab} = V_a - V_b = V_a$$

$$V_{R_2} = I_a R_2 = \frac{12}{11}\,\mathrm{A} \cdot 4\,\Omega = \boxed{4.36\,\mathrm{V}}$$

b) When the switch is closed, R_2 and R_3 will be in parallel. We can compute the equivalent resistance for the circuit as

$$\frac{1}{R_{23}} = \frac{1}{R_2} + \frac{1}{R_3} = \frac{1}{4} + \frac{1}{10} = \frac{7}{20} \Rightarrow R_{23} = \frac{20}{7}\,\Omega = 2.67\,\Omega$$
$$R_{eq} = R_1 + R_{23} = 7 + \frac{20}{7} = \frac{69}{7}\,\Omega$$

Then we can compute the total current as

$$E = I_a R_{eq} \Rightarrow I_a = \frac{E}{R_{eq}} = \frac{12\,\text{V}}{\frac{69}{7}\,\Omega} = \boxed{1.22\,\text{A}}$$

We can use this to compute the volatge across R_1

$$V_{R_1} = I_a R_1 = 8.52\,\text{V}$$

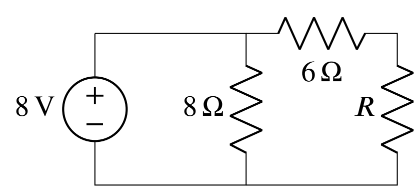
Then we can use Kirchoff's voltage law to find the remaining voltages

$$V_{R_2} = V_{R_3}$$
$$V_E = V_{R_1} + V_{R_2} \Rightarrow V_{R_2} = V_{R_3} = V_E - V_{R_1} = 3.48\,\text{V}$$

From this we can get that the potential at c and the potential at a will be $V_a = V_c = \boxed{3.48\,\text{V}}$ while the potential at b will be 0 V as in part a.
We can also use the voltages across the resistors to compute the currents in each branch.

$$I_b = \frac{V_{R_2}}{R_2} = \frac{3.48}{4} = \boxed{0.87\,\text{A}}$$
$$I_c = \frac{V_{R_3}}{R_3} = \frac{3.48}{10} = \boxed{0.35\,\text{A}}$$

Difficulty: ★☆☆
If the total power dissipated in the circuit is 15W, what is the value of R?



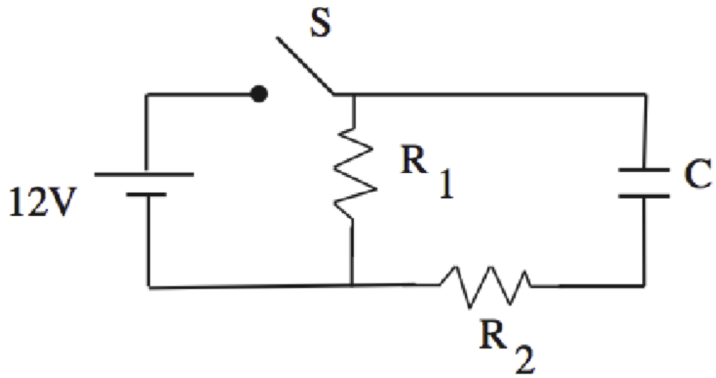
Solution:
The total power dissipated in the circuit will be the sum of the power dissipated in each resistor. We know that the power dissipated across a resistor is $P = IV$. We also know $V = IR$ and can rearrange to get $P = \frac{V^2}{R}$.
The voltage across the $8\,\Omega$ resistor is 8 V so power across that resistor will be

$$P_1 = \frac{(8\,\text{V})^2}{8\,\Omega} = 8\,\text{W}$$

The sum of the power across the remaining two resistors will be $P_{\text{total}} - P_1 = 15\,\text{W} - 8\,\text{W} = 7\,\text{W}$.
We can combine resistor R and the $6\,\Omega$ to get an equivalent resistor with value $R + 6$. The volatge across this equivalent resistor will be 8 V and will have a power usage of 7 W. We can then solve for R as

$$7\,\text{W} = \frac{(8\,\text{V})^2}{R + 6\,\Omega}$$
$$R = \frac{(8\,\text{V})^2}{7\,\text{W}} - 6\,\Omega = \boxed{3.14\,\Omega}$$

Difficulty: ★★☆
The circuit below has the switch S is opened for a long time. $R_1 = 2\,\Omega$, $R_2 = 4\,\Omega$, $C = 2\,\text{F}$



a) The switch S is now closed. Find all currents just after the switch is closed.

b) Find all currents after the switch has been closed for a very long time.

c) After the switch was closed for a very long time it is opened again find the current through R_2 as a function of time.

Solution:

a) The capacitor will want to initially act as a wire so we can analyze the circuit as two resistors in parallel. Due to Kirchoff's loop law, we can say that each resistor must have a voltage drop of 12 V and we can get the current of each from Ohm's law:

$$I_1 = \frac{\varepsilon}{R_1} = \frac{12}{2} = \boxed{6\,\text{A}}$$
$$I_2 = \frac{\varepsilon}{R_2} = \frac{12}{4} = \boxed{3\,\text{A}}$$

b) After the switch has been closed for a long time, the capacitor will be fully charged and act as a short circuit. The circuit can then be analyzed as the loop going through the battery and R_1

$$I_1 = \frac{\varepsilon}{R_1} = \frac{12}{2} = \boxed{6\,\text{A}}$$
$$\boxed{I_2 = 0\,\text{A}}$$

c) After the switch is opened the current will flow through the loop containing R_1, R_2, and C. We can write the voltage loop equation as

$$0 = V_C + V_{R_1} + V_{R_2}$$
$$0 = \frac{q}{C} + iR_1 + iR_2$$

We know that $i = \frac{dq}{dt}$ and can take the derivative of both sides to get a 1st order differential equation and solve for $i(t)$

$$0 = \frac{i}{C} + \frac{di}{dt}(R_1 + R_2)$$
$$\frac{di}{dt} = -\frac{i}{(R_1 + R_2)C}$$
$$\frac{di}{i} = -\frac{dt}{(R_1 + R_2)C}$$
$$\int \frac{di}{i} = -\int \frac{dt}{(R_1 + R_2)C}$$
$$\ln|i| = -\frac{t}{(R_1 + R_2)C} + \text{Constant}$$
$$i = i_0 e^{-\frac{t}{(R_1+R_2)C}}$$

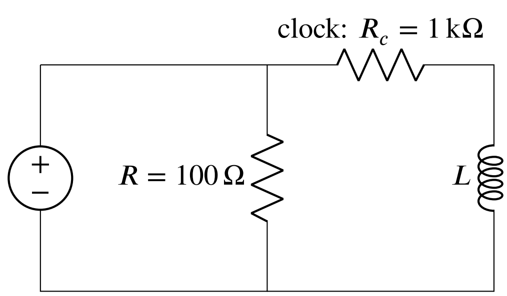
We can solve for the initial current by using our same voltage loop equation and knowing that the initial voltage across the capacitor is 12 V from the charge stored on it. The capacitor will be discharging so the potential in the equation can be thought of as negative.

$$0 = V_C + i_0(R_1 + R_2)$$
$$0 = -12 + 6i_0 \Rightarrow i_0 = 2\,\text{A}$$

Plugging this all in we get,

$$\boxed{i(t) = 2e^{-\frac{t}{12}}\ \text{Amps}}$$

Difficulty: ★★☆
Using their newfound knowledge of LR circuits, a Phys 158 student came up with a clever idea for a prank. They want to design an alarm clock that will continue to ring for 10 seconds after the battery is removed. The alarm clock can be thought of as a 1 kΩ resistor which requires at least 1 Watt to operate. They designed the following circuit to achieve this.



a) What value should the battery be such that the power supplied to the alarm clock does not exceed 3 Watts?

b) What value of inductor should they use so that the alarm clock remains on for 10 seconds after the battery is disconnected?

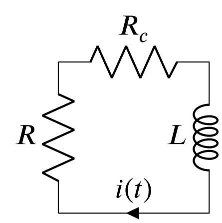
Solution:

a) For maximum power to be supplied to the clock, $P_c = 3\,\mathrm{W}$

$$P_C = i_C^2 R_C \Rightarrow i_C = \sqrt{\frac{P_C}{R_C}}$$

$$\varepsilon = V_C = i_C R_C = \sqrt{\frac{P_C}{R_C}} \cdot R_C = \sqrt{P_C R_C} = \sqrt{(3\,\mathrm{W})(1000\,\Omega)} = 54.8\,\mathrm{V}$$

b) We can start by writing out Kirchoff's loop voltage law for the circuit and then solving the resulting differential equation by separation of variables to get an expression for the current as a function of time:



$$iR + iR_C + L\frac{di}{dt} = 0$$
$$L\frac{di}{dt} = -i(R + R_C)$$
$$\frac{di}{dt} = -\frac{R + R_C}{L} i$$
$$\frac{di}{i} = -\frac{R + R_C}{L} dt$$
$$\int \frac{di}{i} = -\frac{R + R_C}{L} \int dt$$
$$\ln |i| = -\frac{R + R_C}{L} t + \text{Constant}$$
$$i(t) = e^{-\frac{R+R_C}{L} t + \text{Constant}} = i_0 e^{-\frac{R+R_C}{L} t}$$

Alternatively, we can get the same expression by thinking about it conceptually and computing the time constant.
We know that the current will initially want to stay the same because of the inductor and it will slowly decay to 0 so we know that the equation of the current should look like exponential decay and be of the form

$$i(t) = i_0 e^{-t/\tau}$$

We can then compute the time constant for an RL circuit as

$$\tau = \frac{L}{R_{eq}} = \frac{L}{R + R_C}$$

Plugging this in will yield the same expression as above.
The initial current, i_0, will be the current that was initially flowing through the inductor. We computed this in part (a) to be

$$i_0 = i_C = \sqrt{\frac{P_C}{R_C}} = \sqrt{\frac{3\,\mathrm{W}}{1000\,\Omega}} = 54.8\,\mathrm{mA}$$

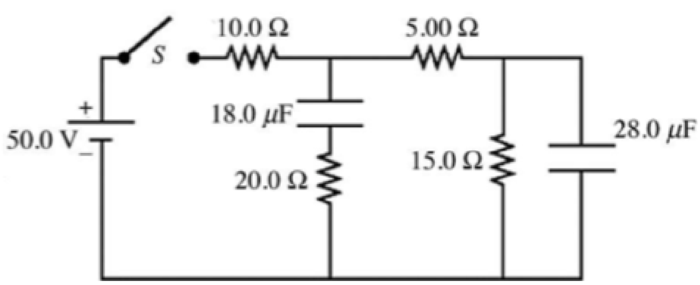
Now we have a complete expression for the current as a function of time. We can get the power as a function of time as

$$P(t) = i^2 R_C = i_0^2 R_C e^{-\frac{2(R+R_C)}{L}t}$$
$$i_0^2 R_C = \frac{P_C}{R_C} \cdot R_C = P_C$$
$$P(t) = P_C e^{-\frac{2(R+R_C)}{L}t}$$

We are told that the clock must have a minimum of 1 Watt and we want it to last for 10 seconds so we can set $P = 1$ and $t = 10$ and solve for L.

$$\frac{P}{P_C} = e^{-\frac{2(R+R_C)}{L}t}$$
$$\ln\left(\frac{P}{P_C}\right) = -\frac{2(R+R_C)}{L}t$$
$$L = -\frac{2(R+R_C)t}{\ln\left(\frac{P}{P_C}\right)} = -\frac{2(100\,\Omega + 1000\,\Omega)(10\,\mathrm{s})}{\ln\left(\frac{1\,\mathrm{W}}{3\,\mathrm{W}}\right)} = 20,025\,\mathrm{H}$$

Difficulty: ★★☆
The circuit shown below initially has no charge on the capacitors and the switch S is originally open.



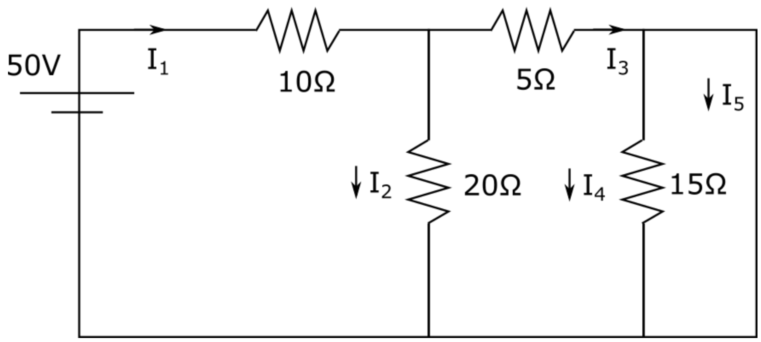
a) Just after closing the switch S, find all the currents.

b) After the switch has been closed for a very long time, find all the currents.

c) After the switch S has been closed for a very long time, find the potential difference across the 28.0 μF capacitor.

Solution:

a) just after we close the switch there is NO charge on the Capacitors. Hence we have the following circuit:

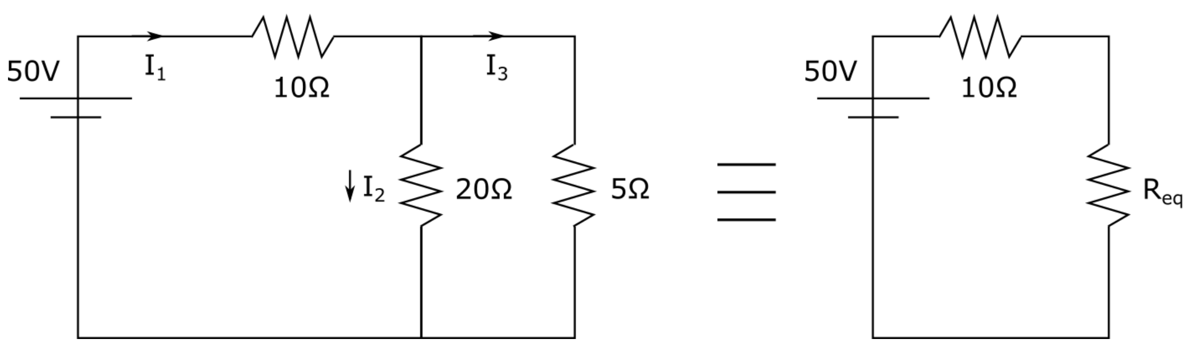


Note the Capacitors act like a wire when uncharged

$K_1 \implies$ Hence $\underline{I_4 = 0}, I + 1 = I_2 + I_3, I_3 = I_4 + I_5 = I_5$
$K_2 \implies 50 - 10I_1 - 20I_2 = 0$
$0 = -5I_3 + 20I_2$

You now have 3 equations and 3 unknowns. Solve

EQUIVALENT Circuit



$$\frac{1}{R_{eq}} = \frac{1}{20} + \frac{1}{5} = \frac{1}{20} + \frac{4}{20} = \frac{5}{20} = \frac{1}{4}$$
$$\rightarrow R_{eq} = 4\Omega$$

Hence $I_1 = \frac{50V}{14\Omega}$, Since $I_1 = I_2 + I_3$ and $20I_2 = 5I_3$ we have

$$\underline{I_1 = \frac{50}{14}\text{ Amps}} = 3.57\,A = I_2 + 4I_2 = 5I_2$$

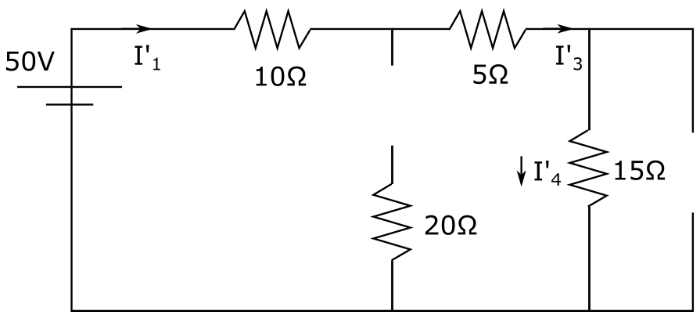
$$\therefore \underline{I_2 = \frac{10}{14}\text{ Amps}} = 0.714\,A$$

and finally

$$I_3 = 4I_2 = \underline{\frac{40}{14}\text{ Amps}} = 2.86\,A$$

b) after a long time the Capacitors are fully charged

$$\therefore I_2' = 0, I_5' = 0 \implies I_1' = I_3' = I_4'$$

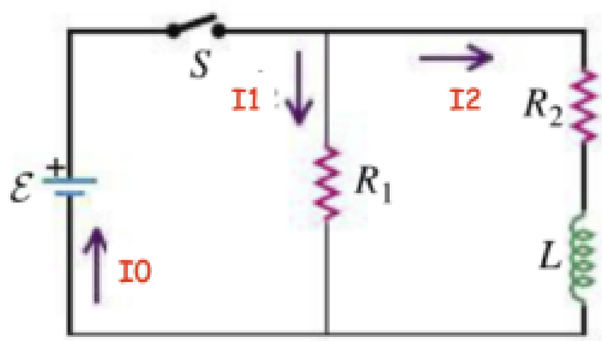


$$\therefore \underline{\frac{50V}{30\Omega} = I_1' = \frac{5}{3}\text{Amp}}$$

c) $V_{28\mu F} = V_{15\Omega} = I_4'(15\Omega) = \frac{5}{3}(15) = \underline{25V}$

Difficulty: ★★☆

The circuit below has been open for a very long time and then the switch is closed at $t = 0$.

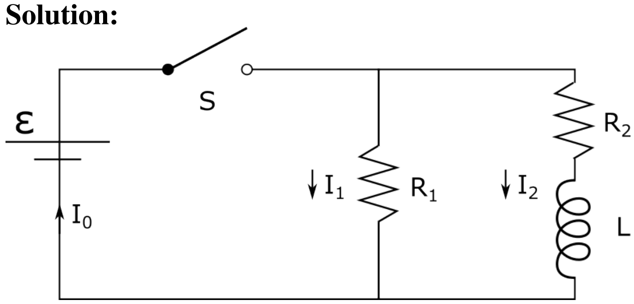


a) Find all of the currents at $t = 0^+$.

b) Find all of the currents after a very long time.

c) Find all of the currents as a function of time

Solution:

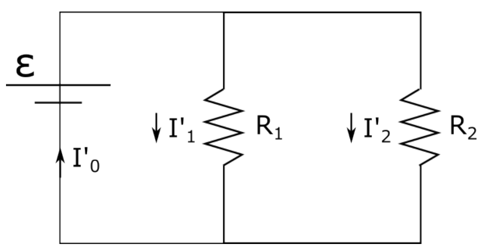


a) Close switch at $t = 0$, then $I_2(0) = 0$ and the Inductor acts like a Battery

$I_0(0) = I_1(0)$

$$\mathcal{E} - I_1 R_1 = 0 \implies \underline{I_1 = \frac{\mathcal{E}}{R_1} = I_0}$$

b) As $t \to \infty$ the Inductor voltage $\to 0$ (becomes a wire)



Now $I_0' = I_1' + I_2'$:

$$I_1' R_1 = I_2' R_2 = \mathcal{E} = I$$

$$\underline{\therefore I_1' = \frac{\mathcal{E}}{R_1},\ I_2' = \frac{\mathcal{E}}{R_2},\ I_0' = \mathcal{E}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)}$$

c) At any time t we have

$K_1 \implies I_0(t) = I_1(t) + I_2(t)$

$$K_2 \implies \begin{cases} \mathcal{E} - I_1 R_1 = 0 \\ \mathcal{E} - I_2 R_2 - L\frac{dI_2}{dt} = 0 \end{cases}$$

This last equation looks like a regular RL circuit, so its solution is $I_2(t) = A\exp(-t/\tau) + B$ with $\tau =$ time constant

$$I_2(\infty) \to \frac{\mathcal{E}}{R_2} = B,\ I_2(0) = A + B = 0 \implies A = \frac{\mathcal{E}}{R_2}$$

so $I_2(t) = -\frac{\mathcal{E}}{R_2}\exp(-t/\tau) + \frac{\mathcal{E}}{R_2} = \frac{\mathcal{E}}{R_2}(1 - \exp(t/\tau)$.

To find τ we use the aforementioned equation that looks like a regular RL circuit

$$\mathcal{E} - R_2\left(\frac{\mathcal{E}}{R_2}(1 - \exp(-t/\tau)\right) - \frac{L\mathcal{E}}{R_2}\left(\frac{1}{\tau}\exp(-t/\tau)\right) = 0.$$

$$\therefore \mathcal{E} - \mathcal{E} + \mathcal{E}\exp(-t/\tau) - \frac{L\mathcal{E}}{R_2}\frac{1}{\tau}\exp(-t/\tau) = 0,$$

and all $\mathcal{E}$ can be divide out, resulting in

$$\left(1 - \frac{L}{R_2}\frac{1}{\tau}\right) = 0 \implies \tau = \frac{L}{R_2}$$

$$\implies \underline{I_2 = \frac{\mathcal{E}}{R_2}(1 - \exp(-tR_2/\tau))}$$

$$\underline{I_1 = \frac{\mathcal{E}}{R_1} \;\; I_0 = \frac{\mathcal{E}}{R_1} + \frac{\mathcal{E}}{R_2}\left(1 - \exp(-tR_2/L)\right)}$$