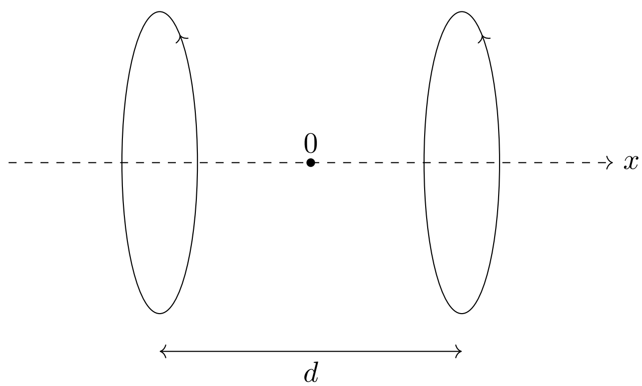
Physics 158 Magnetism Problem Bank

Problem 1

Created by Tyler Wilson 2023

Two current carrying loops, each with current I and radius R, are placed a distance d apart from each other, centered at the points $(-\frac{d}{2}, 0, 0)$ and $(\frac{d}{2}, 0, 0)$



a) Compute the magnitude of the magnetic field at the origin $(0, 0, 0)$ due to this configuration.

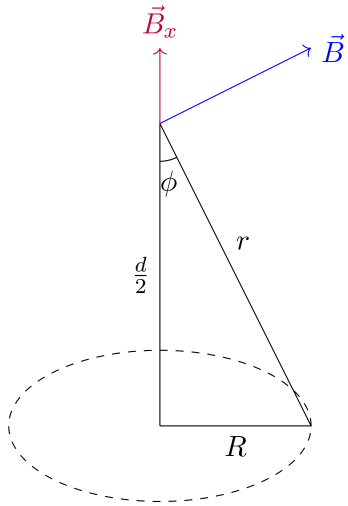
b) Compute the magnitude of the magnetic field for all points on the x-axis, $|\vec{B}(x)|$.

c) Determine the optimal distance d between the two loops such that the magnetic field along the axis of symmetry is as uniform as possible.
Hint: This can be done by choosing d to make as many derivatives of $|\vec{B}(x)$ equal to zero at $x = 0$ as possible.

Solution:

a) We can start by writing the Biot-Savart law for the magnetic field due to a single current carrying loop.

$$B = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{d\vec{l} \times \hat{r}}{r^2}$$



We can see that the magnetic field will only have a component in the x-direction, so we can write the magnetic field as

$$r = \sqrt{R^2 + \frac{d^2}{4}}$$

$$l = R\theta \Rightarrow dl = Rd\theta$$

$$B_x = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{d\vec{l} \times \hat{r}}{r^2} \cdot \hat{i} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{Rd\theta}{R^2 + \frac{d^2}{4}} \sin\phi = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{Rd\theta}{R^2 + \frac{d^2}{4}} \frac{R}{\sqrt{R^2 + \frac{d^2}{4}}}$$

$$B_x = \frac{\mu_0 I R^2}{4\pi}\int_0^{2\pi} \frac{d\theta}{\left(R^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}} = \frac{\mu_0 I R^2}{4\pi} \frac{2\pi}{\left(R^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}} = \frac{\mu_0 I R^2}{2\left(R^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}}$$

This gives us the field for one wire. Using the right-hand-rule we can see that the two fields will add so the total field at the origin is double what we calculated.

$$\boxed{|\vec{B}| = \frac{\mu_0 I R^2}{\left(R^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}}}$$

b) We can use the same method as above to find the field at any point on the x-axis. If we use the same derivation as above but replace $\frac{d}{2}$ with $x + \frac{d}{2}$ for the left ring and $x - \frac{d}{2}$ for the right ring then we get a general expression for the field at any point on the x-axis.

$$\boxed{|\vec{B}|(x) = \frac{\mu_0 I R^2}{2}\left(\frac{1}{\left(R^2 + (x + \frac{d}{2})^2\right)^{3/2}} + \frac{1}{\left(R^2 + (x - \frac{d}{2})^2\right)^{3/2}}\right)}$$

c) We can find the optimal distance by finding the points where the derivative of the field is zero at $z = 0$.

$$\left.\frac{dB}{dx}\right|_{x=0} = -\frac{\mu_0 I R^2}{2}\frac{3}{2}\left(\frac{2(x + \frac{d}{2})}{\left(R^2 + (x + \frac{d}{2})^2\right)^{5/2}} + \frac{2(x - \frac{d}{2})}{\left(R^2 + (x - \frac{d}{2})^2\right)^{5/2}}\right)\Bigg|_{x=0}$$

$$= \frac{3\mu_0 I R^2}{2}\left(\frac{\frac{d}{2}}{\left(R^2+\frac{d^2}{4}\right)^{5/2}} - \frac{\frac{d}{2}}{\left(R^2+\frac{d^2}{4}\right)^{5/2}}\right) = 0$$

The first derivative didn't tell us anything so now we can look at the second derivative.

$$\left.\frac{d^2B}{dx^2}\right|_{x=0} = -\frac{3\mu_0 I R^2}{2}\frac{d}{dx}\left(\frac{x+\frac{d}{2}}{\left(R^2+(x+\frac{d}{2})^2\right)^{5/2}} + \frac{x-\frac{d}{2}}{\left(R^2+(x-\frac{d}{2})^2\right)^{5/2}}\right)\Bigg|_{x=0} = 0$$

$$\left.\frac{\left(R^2+(x+\frac{d}{2})^2\right)^{5/2} - (x+\frac{d}{2})(\frac{5}{2})(R^2+(x+\frac{d}{2})^2)^{3/2}(2)(x+\frac{d}{2})}{\left(R^2+(x+\frac{d}{2})^2\right)^5}\right|_{x=0}$$

$$+\left.\frac{\left(R^2+(x-\frac{d}{2})^2\right)^{5/2} - (x-\frac{d}{2})(\frac{5}{2})(R^2+(x-\frac{d}{2})^2)^{3/2}(2)(x-\frac{d}{2})}{\left(R^2+(x-\frac{d}{2})^2\right)^5}\right|_{x=0} = 0$$

$$\frac{2(R^2+\frac{d^2}{4})^{5/2} - \frac{5d^2}{2}(R^2+\frac{d^2}{4})^{3/2}}{\left(R^2+\frac{d^2}{4}\right)^5} = 0$$

$$2\left(R^2+\frac{d^2}{4}\right) - \frac{5d^2}{2} = 2R^2 - 2d^2 = 0$$

$$\boxed{d = R}$$

Problem 2

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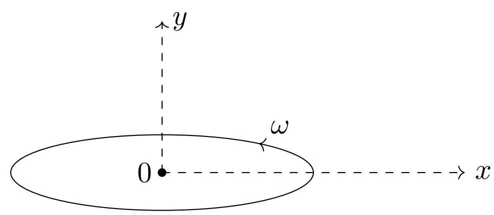
Two current carrying loops, each with current I and radius R, are placed a distance d apart from each other, centered at the points $(-\frac{d}{2},0,0)$ and $(\frac{d}{2},0,0)$



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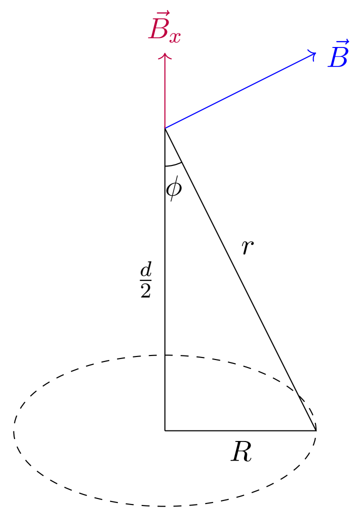

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$$\left.\frac{\left(R^2+(x+\frac{d}{2})^2\right)^{5/2} - (x+\frac{d}{2})(\frac{5}{2})(R^2+(x+\frac{d}{2})^2)^{3/2}(2)(x+\frac{d}{2})}{\left(R^2+(x+\frac{d}{2})^2\right)^5}\right|_{x=0}$$

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