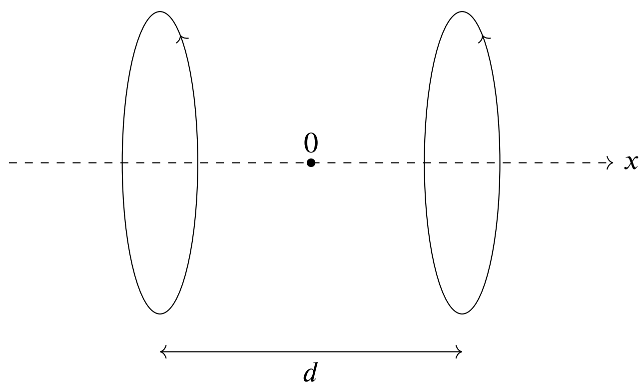
Physics 158 Magnetism Problem Bank

Problem 1

Created by Tyler Wilson 2023
Difficulty: ★★☆
Two current carrying loops, each with current I and radius R, are placed a distance d apart from each other, centered at the points $(-\frac{d}{2}, 0, 0)$ and $(\frac{d}{2}, 0, 0)$



a) Compute the magnitude of the magnetic field at the origin $(0, 0, 0)$ due to this configuration.

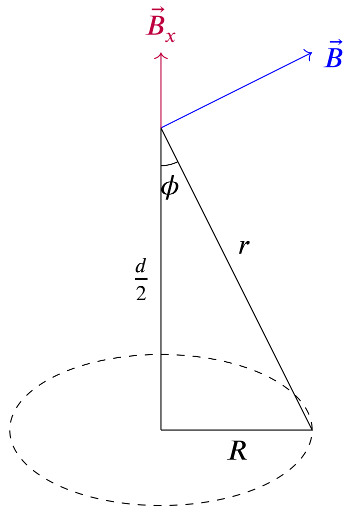
b) Compute the magnitude of the magnetic field for all points on the x-axis, $|\vec{B}(x)|$.

c) Determine the optimal distance d between the two loops such that the magnetic field along the axis of symmetry is as uniform as possible.
Hint: This can be done by choosing d to make as many derivatives of $|\vec{B}(x)$ equal to zero at $x = 0$ as possible.

Solution:

a) We can start by writing the Biot-Savart law for the magnetic field due to a single current carrying loop.

$$B = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{d\vec{l} \times \hat{r}}{r^2}$$



We can see that the magnetic field will only have a component in the x-direction, so we can write the magnetic field as

$$r = \sqrt{R^2 + \frac{d^2}{4}}$$

$$l = R\theta \Rightarrow dl = Rd\theta$$

$$B_x = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{d\vec{l} \times \hat{r}}{r^2} \cdot \hat{i} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{Rd\theta}{R^2 + \frac{d^2}{4}} \sin\phi = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{Rd\theta}{R^2 + \frac{d^2}{4}} \frac{R}{\sqrt{R^2 + \frac{d^2}{4}}}$$

$$B_x = \frac{\mu_0 I R^2}{4\pi}\int_0^{2\pi} \frac{d\theta}{\left(R^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}} = \frac{\mu_0 I R^2}{4\pi} \frac{2\pi}{\left(R^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}} = \frac{\mu_0 I R^2}{2\left(R^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}}$$

This gives us the field for one wire. Using the right-hand-rule we can see that the two fields will add so the total field at the origin is double what we calculated.

$$\boxed{|\vec{B}| = \frac{\mu_0 I R^2}{\left(R^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}}}$$

b) We can use the same method as above to find the field at any point on the x-axis. If we use the same derivation as above but replace $\frac{d}{2}$ with $x + \frac{d}{2}$ for the left ring and $x - \frac{d}{2}$ for the right ring then we get a general expression for the field at any point on the x-axis.

$$\boxed{|\vec{B}|(x) = \frac{\mu_0 I R^2}{2}\left(\frac{1}{\left(R^2 + (x + \frac{d}{2})^2\right)^{3/2}} + \frac{1}{\left(R^2 + (x - \frac{d}{2})^2\right)^{3/2}}\right)}$$

c) We can find the optimal distance by finding the points where the derivative of the field is zero at $z = 0$.

$$\left.\frac{dB}{dx}\right|_{x=0} = -\frac{\mu_0 I R^2}{2}\frac{3}{2}\left(\frac{2(x+\frac{d}{2})}{\left(R^2+(x+\frac{d}{2})^2\right)^{5/2}} + \frac{2(x-\frac{d}{2})}{\left(R^2+(x-\frac{d}{2})^2\right)^{5/2}}\right)\Bigg|_{x=0}$$

$$= \frac{3\mu_0 I R^2}{2}\left(\frac{\frac{d}{2}}{\left(R^2+\frac{d^2}{4}\right)^{5/2}} - \frac{\frac{d}{2}}{\left(R^2+\frac{d^2}{4}\right)^{5/2}}\right) = 0$$

The first derivative didn't tell us anything so now we can look at the second derivative.

$$\left.\frac{d^2B}{dx^2}\right|_{x=0} = -\frac{3\mu_0 I R^2}{2}\frac{d}{dx}\left(\frac{x+\frac{d}{2}}{\left(R^2+(x+\frac{d}{2})^2\right)^{5/2}} + \frac{x-\frac{d}{2}}{\left(R^2+(x-\frac{d}{2})^2\right)^{5/2}}\right)\Bigg|_{x=0} = 0$$

$$\left.\frac{\left(R^2+(x+\frac{d}{2})^2\right)^{5/2} - (x+\frac{d}{2})(\frac{5}{2})(R^2+(x+\frac{d}{2})^2)^{3/2}(2)(x+\frac{d}{2})}{\left(R^2+(x+\frac{d}{2})^2\right)^5}\right|_{x=0}$$

$$+\left.\frac{\left(R^2+(x-\frac{d}{2})^2\right)^{5/2} - (x-\frac{d}{2})(\frac{5}{2})(R^2+(x-\frac{d}{2})^2)^{3/2}(2)(x-\frac{d}{2})}{\left(R^2+(x-\frac{d}{2})^2\right)^5}\right|_{x=0} = 0$$

$$\frac{2(R^2+\frac{d^2}{4})^{5/2} - \frac{5d^2}{2}(R^2+\frac{d^2}{4})^{3/2}}{\left(R^2+\frac{d^2}{4}\right)^5} = 0$$

$$2\left(R^2+\frac{d^2}{4}\right) - \frac{5d^2}{2} = 2R^2 - 2d^2 = 0$$

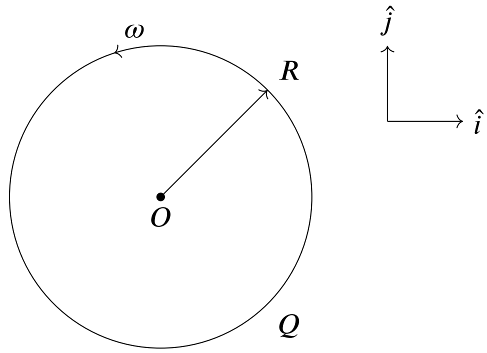
$$\boxed{d = R}$$

Problem 2

Created by Tyler Wilson 2023
Difficulty: ★☆☆
A toy company is planning on rolling out a new product called the magnetic hula-hoop. The hoop is made of a non-conducting material and has a radius of R and has a uniform charge density of λ.

a) If the hoop (depicted below) is spun around the point O with a constant angular velocity of ω, what is the magnetic field at the center of the hoop?

b) If the speed of the ring is no longer constant, but instead is expressed by the function $\omega(t) = \omega_0 e^{-at}$ then what is the direction of the induced emf in the ring? (write your answer as either clockwise or counterclockwise)



Solution:

a) We can recognize that current is simply the rate of change (or flow) of charge. In the case of the spinning ring we are creating a flow of charge so we can write the current as

$$I = \frac{dQ}{dt}$$

We can then use chain rule to rewrite this in terms of the variables that we have.

$$I = \frac{dQ}{dt} = \frac{dQ}{ds}\frac{ds}{dt} = \lambda \cdot v$$
$$v = R\omega \Rightarrow I = \lambda R\omega$$

We can then use the Biot-Savart law to find the magnetic field at the center of the ring.

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l} \times \hat{r}}{r^2}$$
$$d\vec{l} \times \hat{r} = dl\hat{k}$$
$$dl = Rd\theta$$
$$r = R$$
$$\vec{B} = \frac{\mu_0 \lambda R\omega}{4\pi}\int_0^{2\pi} \frac{Rd\theta}{R^2}\hat{k} = \boxed{\frac{\mu_0 \lambda\omega}{2}\hat{k}}$$

b) We found in part (a) that the magnetic field is pointing out of the page. We also found that the magnetic field is proportional to the angular velocity of the ring so as ω decreases over time then the magnetic field strength will also decrease.

$$\frac{d\vec{B}}{dt} < 0\hat{k}$$
$$\Phi_B = \vec{B} \cdot \vec{A} = BA$$
$$\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{dB}{dt}A > 0(\text{out of page})$$

We can see that the induced emf is positive and is pointing out of the page. This means that the induced current will be flowing in the clockwise direction.