Physics 158 Electric Fields Problem Bank

Problem 1

Created by Tyler Wilson 2023

A 16 cm long wire of charge $Q = 40\,\mu\text{C}$ is bent into a square.

a) Find the electric field strength 20 cm vertically above the center of the square in the xy-plane.

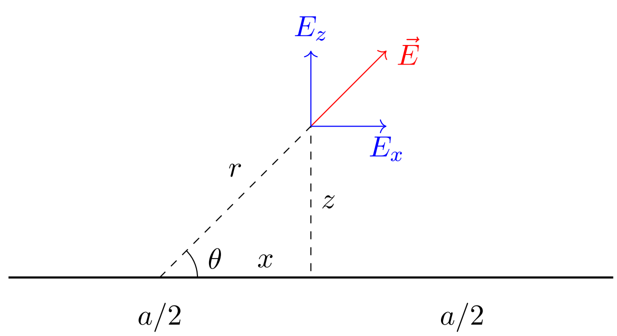
A point charge of mass $m = 100\,\text{g}$ and charge $-5\,\mu\text{C}$ is now placed 20 cm above the center of the square.

b) Find the magnitude and direction of the force acting on the point charge initially when $z = 20\,\text{cm}$.

c) Plot the acceleration of the point charge as a function of it's distance z above the center of the square.

Solution:

a) We can break this problem down into the simpler problem of computing the electric field due to a straight piece of wire. We can then use the superposition principle to find the electric field due to the square.
Electric field due to a straight piece of wire of length a:



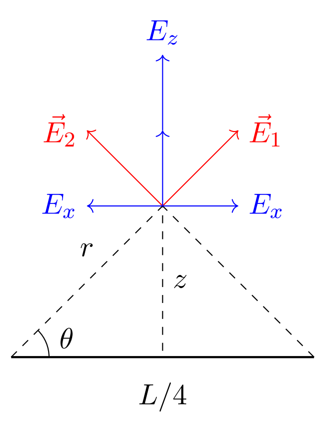
$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\hat{r}$$
$$dq = \lambda dx$$
$$r = \sqrt{x^2 + z^2}$$
$$\vec{E} = \int d\vec{E} = \int_{-a/2}^{a/2} \frac{1}{4\pi\epsilon_0}\frac{\lambda dx}{x^2 + z^2}\hat{r}$$

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0}\int_{-a/2}^{a/2}\frac{dx}{x^2+z^2}\hat{r}$$

Note that because of symmetry on each side the electric field will only have a vertical component.

$$\hat{r} = \cos(\theta)\hat{x} + \sin(\theta)\hat{z}$$
$$\sin\theta = \frac{z}{r} = \frac{z}{\sqrt{x^2+z^2}}$$
$$E_z = \frac{\lambda}{4\pi\epsilon_0}\int_{-a/2}^{a/2}\frac{dx}{(x^2+z^2)^{3/2}}$$
$$E_z = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{x}{z\sqrt{x^2+z^2}}\right]_{x=-a/2}^{x=a/2}$$
$$E_z = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{a}{z\sqrt{a^2/4+z^2}} - \frac{-a}{z\sqrt{a^2/4+z^2}}\right]$$
$$E_z = \frac{\lambda}{4\pi\epsilon_0}\frac{2a}{z\sqrt{a^2/4+z^2}}$$
$$\vec{E} = E_z\hat{z} = \frac{\lambda}{2\pi\epsilon_0}\frac{a}{z\sqrt{a^2/4+z^2}}\hat{z} = \frac{2k\lambda a}{z\sqrt{a^2/4+z^2}}\hat{z}$$

Let us consider the length of the wire to be $L = 16\,\text{cm}$. We can break this down to four straight pieces of wire of length $a = L/4 = 4\,\text{cm}$. The electric field due to each piece of wire will be the same magnitude and direction due to the symmetry of the problem. The electric field due to the square will be the sum of the electric fields due to each piece of wire.



$$\vec{E}_{\text{total}} = 4E_z\hat{z}$$
$$E_z = \frac{2k\lambda a}{r\sqrt{a^2/4+r^2}}\sin\theta$$
$$r = \sqrt{a^2/4+z^2}$$

$$\sin\theta = \frac{z}{r}$$

$$E_z = \frac{2k\lambda a z}{r^2\sqrt{a^2/4 + r^2}}$$

$$r^2 = \frac{a^2}{4} + z^2$$

$$E_z = \frac{2k\lambda a z}{(a^2/4 + z^2)\sqrt{a^2/2 + z^2}}$$

$$a = \frac{L}{4}, \quad \lambda = \frac{Q}{L}$$

$$E_z = \frac{\frac{1}{2}kQz}{(L^2/64 + z^2)\sqrt{L^2/32 + z^2}}$$

$$\vec{E}_{\text{total}} = 4E_z\hat{z} = \frac{2kQz}{(L^2/64 + z^2)\sqrt{L^2/32 + z^2}}\hat{z}$$

Now we can plug in the values $L = 0.16\,\mathrm{m}$, $Q = 40\,\mu\mathrm{C}$, and $z = 0.2\,\mathrm{m}$ to get the magnitude of the electric field.

$$\vec{E}_{\text{total}} = \frac{2(8.99\times10^9\,\mathrm{N\cdot m^2/C^2})(40\times10^{-6}\,\mathrm{C})(0.2\,\mathrm{m})}{(0.16^2\,\mathrm{m}^2/64 + (0.2\,\mathrm{m})^2)\sqrt{(0.16^2\,\mathrm{m}^2/32 + (0.2\,\mathrm{m})^2)}}\hat{z} = 1.76\times10^7\,\mathrm{N/C}\hat{z}$$

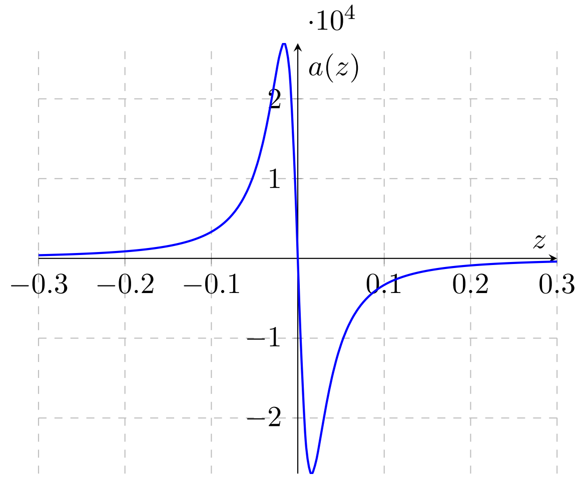
b) The force can be computed as $\vec{F} = q\vec{E}$ and the direction will be toward the center of the square for a negative point charge.

$$\vec{F} = q\vec{E} = -\frac{2kQqz}{(L^2/64 + z^2)\sqrt{L^2/32 + z^2}}\hat{z} = -88\,\mathrm{N}\hat{z}$$

c) We can get an expression for the acceleration from the force

$$\vec{F} = m\vec{a}$$

$$\vec{a} = -\frac{2kQqz}{m(L^2/64 + z^2)\sqrt{L^2/32 + z^2}}\hat{z}$$

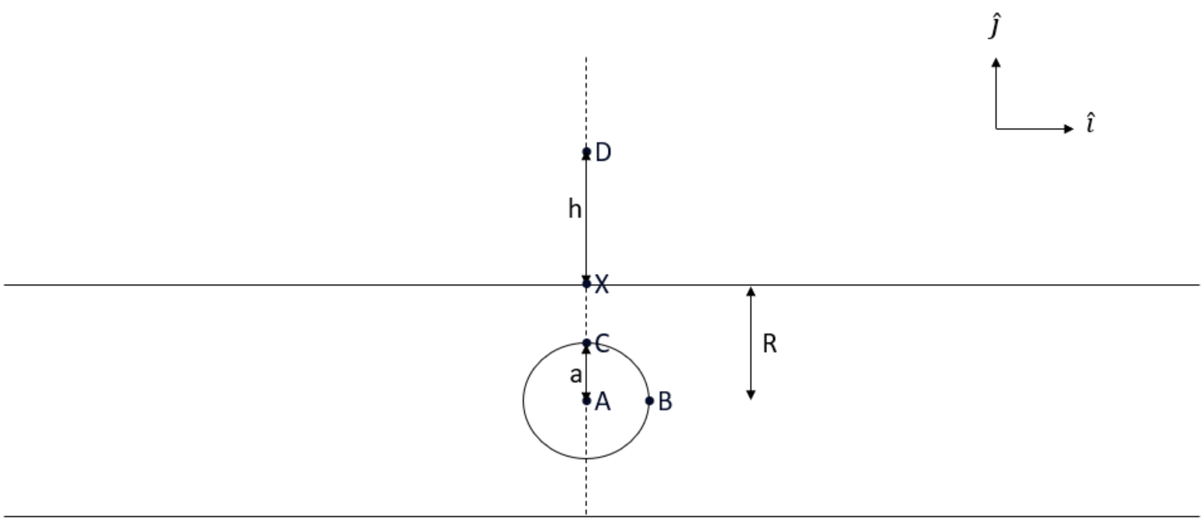


Problem 2

Created by Tyler Wilson 2023
An insulating cylinder of radius R and effectively infinite length in the z-direction contains a uniform charge density of ρ.

a) Find the electric field everywhere in space

If there is now a <u>hollow</u> spherical cavity of radius a located at the center of the cylinder,



b) Find the electric field at the center of the sphere at point A.

c) Find the electric field just outside the sphere at point B.

d) Find the electric field just outside the sphere at point C.

e) Find the electric field outside both objects at point D.

If the potential at the point X is 0V,

f) What is the potential at point A?

Solution:
Notation: Because we have two types of radii, the cylindrical radius, and the spherical radius, we will use s to represent the variable cylindrical radius and r to represent the variable spherical radius.

a) We can use Gauss's Law to find the electric field everywhere in space. We will use a Gaussian cylinder of radius s and length L.

$$\oiint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

We will need to break this into two parts: the inside of the cylinder and the outside of the cylinder.
For the inside of the cylinder,

$$\oiint \vec{E}_{\text{in}} \cdot d\vec{A} = E_{\text{in}} A = \frac{Q_{\text{enc}}}{\epsilon_0}$$
$$Q_{\text{enc}} = \rho V_{\text{enc}} = \rho \pi s^2 L$$
$$E_{\text{in}} A = \frac{\rho \pi s^2 L}{\epsilon_0}$$
$$2\pi r L E_{\text{in}} = \frac{\rho \pi s^2 L}{\epsilon_0}$$
$$E_{\text{in}} = \frac{\rho s}{2\epsilon_0}$$
$$\boxed{\vec{E}_{\text{in}} = \frac{\rho s}{2\epsilon_0} \hat{s}}$$

For the outside of the cylinder,

$$\oiint \vec{E}_{\text{out}} \cdot d\vec{A} = E_{\text{out}} A = \frac{Q_{\text{enc}}}{\epsilon_0}$$
$$Q_{\text{enc}} = \rho V_{\text{enc}} = \rho \pi R^2 L$$
$$E_{\text{out}} A = \frac{\rho \pi R^2 L}{\epsilon_0}$$
$$2\pi s L E_{\text{out}} = \frac{\rho \pi R^2 L}{\epsilon_0}$$
$$E_{\text{out}} = \frac{\rho R^2}{2\epsilon_0 s}$$
$$\boxed{\vec{E}_{\text{out}} = \frac{\rho R^2}{2\epsilon_0 s} \hat{s}}$$

b) We can use the principle of superposition to find the electric field at point A. We will need to find the electric field due to the cylinder and the electric field due to the cavity and add

the two together to get the total contribution.
The electric field due to the cylinder is given by

$$\vec{E}_{\text{cyl}} = \frac{\rho s}{2\epsilon_0}\hat{s}\bigg|_{s=0} = \vec{0}$$

To find the electric field from the cavity we can treat the cavity like a sphere of radius a with a uniform charge density of $-\rho$. The electric field from this sphere can be computed using Gauss's Law.

$$\oiint \vec{E} \cdot d\vec{A} = EA = \frac{Q_{\text{enc}}}{\epsilon_0}$$

We will also need to break this into two parts: the inside of the sphere and the outside of the sphere.
For the inside of the sphere,

$$\begin{aligned}
Q_{\text{enc}} &= -\rho V_{\text{enc}} = -\rho\frac{4}{3}\pi r^3 \\
E_{\text{in}}A = E_{\text{in}}(4\pi r^2) &= -\frac{\rho\frac{4}{3}\pi r^3}{\epsilon_0} \\
E_{\text{in}} &= -\frac{\rho r}{3\epsilon_0} \\
\vec{E}_{\text{in}} = E_{\text{in}}\hat{r} &= -\frac{\rho r}{3\epsilon_0}\hat{r}
\end{aligned}$$

For the outside of the sphere,

$$\begin{aligned}
Q_{\text{enc}} &= -\rho V_{\text{enc}} = -\rho\frac{4}{3}\pi a^3 \\
E_{\text{out}}A = E_{\text{out}}(4\pi r^2) &= -\frac{\rho\frac{4}{3}\pi a^3}{\epsilon_0} \\
E_{\text{out}} &= -\frac{\rho a^3}{3\epsilon_0 r^2} \\
\vec{E}_{\text{out}} = E_{\text{out}}\hat{r} &= -\frac{\rho a^3}{3\epsilon_0 r^2}\hat{r}
\end{aligned}$$

At point A, the electric field from the cavity will be

$$\vec{E} = -\frac{\rho r}{3\epsilon_0}\hat{r}\bigg|_{r=0} = \vec{0}$$

So the total electric field at point A is

$$\boxed{\vec{E}_A = \vec{0}}$$

c) We will once again use superposition and can reuse the electric field from the cavity that we found in part (b) adn the cylinder in part (a). The electric field from the cylinder is given by

$$\vec{E}_{\text{cyl,B}} = \frac{\rho s}{2\epsilon_0}\hat{s}\bigg|_{s=0} = \vec{0}$$

The electric field from the cavity is given by

$$\vec{E}_{\text{cav,B}} = -\frac{\rho r}{3\epsilon_0}\hat{i}\bigg|_{r=a} = -\frac{\rho a}{3\epsilon_0}\hat{i}$$

So the total electric field at point B is

$$\boxed{\vec{E}_B = -\frac{\rho a}{3\epsilon_0}\hat{i}}$$

d) For point C,
The electric field from the cylinder is given by

$$\vec{E}_{\text{cyl,C}} = \frac{\rho s}{2\epsilon_0}\hat{j}\bigg|_{s=a} = \frac{\rho a}{2\epsilon_0}\hat{j}$$

The electric field from the cavity is given by

$$\vec{E}_{\text{cav,C}} = -\frac{\rho r}{3\epsilon_0}\hat{j}\bigg|_{r=a} = -\frac{\rho a}{3\epsilon_0}\hat{j}$$

So the total electric field at point C is

$$\vec{E}_C = \vec{E}_{\text{cyl,C}} + \vec{E}_{\text{cav,C}}$$
$$\vec{E}_C = \frac{\rho a}{2\epsilon_0}\hat{j} - \frac{\rho a}{3\epsilon_0}\hat{j}$$
$$\boxed{\vec{E}_C = \frac{\rho a}{6\epsilon_0}\hat{j}}$$

e) For point D,
The electric field from the cylinder is given by

$$\vec{E}_{\text{cyl,D}} = \frac{\rho R^2}{2\epsilon_0 s}\hat{j}\bigg|_{s=R+h} = \frac{\rho R^2}{2\epsilon_0(R+h)}\hat{j}$$

The electric field from the cavity is given by

$$\vec{E}_{\text{cav,D}} = -\frac{\rho r}{3\epsilon_0}\hat{j}\bigg|_{r=R+h} = -\frac{\rho(R+h)}{3\epsilon_0}\hat{j}$$

So the total electric field at point D is

$$\vec{E}_D = \vec{E}_{\text{cyl,D}} + \vec{E}_{\text{cav,D}}$$
$$\vec{E}_D = \frac{\rho R^2}{2\epsilon_0(R+h)}\hat{j} - \frac{\rho(R+h)}{3\epsilon_0}\hat{j}$$
$$\boxed{\vec{E}_D = \frac{3\rho R^2 - 2\rho(R+h)^2}{6\epsilon_0(R+h)}\hat{j}}$$

f) If the potential at X is zero then to find the potential at point A we can integrate the electric field from X to A.

$$V_{AX} = V_A - V_X = V_A - 0 = V_A$$

Keep in mind that the equations describing the electric field will change at the boundary of the cavity so we will want to break up our solution into two parts:

$$V_{AX} = V_{AC} + V_{CX}$$

For the part from X to C the electric field will be

$$E_{\text{cyl}} = \frac{\rho y}{2\epsilon_0}$$
$$E_{\text{cav}} = -\frac{\rho a^3}{3\epsilon_0 y^2}$$
$$E_{CX} = E_{\text{cyl}} + E_{\text{cav}} = \frac{\rho y}{2\epsilon_0} - \frac{\rho a^3}{3\epsilon_0 y^2}$$

The potential difference will then be

$$V_{CX} = V_C - V_X = V_C - 0 = V_C$$
$$V_C = -\int_R^a \left(\frac{\rho y}{2\epsilon_0} - \frac{\rho a^3}{3\epsilon_0 y^2}\right) dy$$
$$V_C = -\left(\frac{\rho y^2}{4\epsilon_0} + \frac{\rho a^3}{3\epsilon_0 y}\right)\Bigg|_R^a$$
$$V_C = \frac{\rho}{4\epsilon_0}(R^2 - a^2) - \frac{\rho a^3}{3\epsilon_0}\left(\frac{1}{a} - \frac{1}{R}\right)$$

For the part from C to A we get

$$E_{\text{cyl}} = \frac{\rho y}{2\epsilon_0}$$
$$E_{\text{cav}} = -\frac{\rho y}{3\epsilon_0}$$
$$E_{AC} = E_{\text{cyl}} + E_{\text{cav}} = \frac{\rho y}{2\epsilon_0} - \frac{\rho y}{3\epsilon_0} = \frac{\rho y}{6\epsilon_0}$$
$$V_{AC} = -\int_a^0 \frac{\rho y}{6\epsilon_0} dy = -\frac{\rho y^2}{12\epsilon_0}\Bigg|_a^0 = \frac{\rho a^2}{12\epsilon_0}$$
$$V_{AC} = V_A - V_C \Rightarrow V_A = V_{AC} + V_C$$
$$V_A = \frac{\rho a^2}{12\epsilon_0} + \frac{\rho}{4\epsilon_0}(R^2 - a^2) - \frac{\rho a^3}{3\epsilon_0}\left(\frac{1}{a} - \frac{1}{R}\right)$$
$$\boxed{V_A = \frac{\rho a^2}{12\epsilon_0}\left(3\frac{R^2}{a^2} + 4\frac{a}{R} - 6\right)}$$

Problem 3

Created by Tyler Wilson 2023

The potential above some charged Gaussian surface is given by the equation

$$V = \frac{k\sigma}{3y} \text{ Volts}$$

a) What is the equation for the electric field?

b) If $\sigma = 6\,\mu\text{C}$, find the electric field strength and direction at $y = 2\,\text{m}$.

c) What can you say about the electric field in the x-direction?

Hint:

$$\vec{E} = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k}$$

Solution:

a) The electric field is the negative gradient of the potential. This was defined in the hint above. The electric field would then be

$$\vec{E} = -\frac{\partial}{\partial y}\left(\frac{k\sigma}{3y}\right)\hat{j} = \boxed{\frac{k\sigma}{3y^2}\hat{j}}$$

b) Plugging in these values we would get

$$\vec{E}(y=2) = \frac{k(6 \cdot 10^{-6})}{3(2)^2}\hat{j} = \boxed{4495\hat{j}\text{ N/C}}$$

c)

$$\vec{E}_x = -\frac{\partial}{\partial x}\left(\frac{k\sigma}{3y}\right) = 0$$

Therefore, the electric field in the x-direction is 0.

Problem 4

Created by Tyler Wilson 2023

A rectangular box has dimensions $2\,\text{m} \times 4\,\text{m} \times 8\,\text{m}$. At each corner of the box sits a point charge of value Q.

Leave your answers in terms of k and Q.

a) What is the electric field at the center of the box?

b) What is the potential at the center of the box?

One of the point charges on the corners is removed.

c) What is the magnitude of the new electric field at the center of the box?

d) What is the new potential at the center of the box?

e) What is the work done on the system in removing that point charge?

Solution:

a) Because of the symmetry at the center of the box we find that the electric field from each point charge cancels out with another point charge. We can find the electric field due to each point charge at the center and add the eight vectors together to get the total contribution. In doing this, we will see that the electric field at the center of the box is zero.

b) All of the point charges are the same distance away from the center of the box. So, to compute the potential at the center of the box we can just compute the potential due to one point charge and multiply by eight.

$$r = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{21}$$
$$V_1 = \frac{kQ}{r} = \frac{kQ}{\sqrt{21}}$$
$$V_{\text{total}} = 8V_1 = \boxed{\frac{8kQ}{\sqrt{21}}}$$

c) Originally, the charges diagonally opposite one another were cancelling another out. Once we remove one of the point charges then one of the point charges will no longer be cancelled out so the magnitude of the electric field will be the same as the electric field felt at the center due to one point charge.

$$r = \sqrt{21}$$
$$|\vec{E}| = \frac{kQ}{r^2} = \boxed{\frac{kQ}{21}}$$

d) The potential at the center can be computed in the same manner as before, except now we only have seven point charges contributing to the potential.

$$\boxed{V = \frac{7kQ}{\sqrt{21}}}$$

e) To compute the work done in removing the charge we must compute the potential energy existing between the point that was removed and each of the other charges. This will give the potential difference which we can use to get the work done.

$$W_{12} = \frac{kQ^2}{2}$$
$$W_{13} = \frac{kQ^2}{4}$$

$$W_{14} = \frac{kQ^2}{\sqrt{2^2+4^2}} = \frac{kQ^2}{\sqrt{20}}$$
$$W_{15} = \frac{kQ^2}{8}$$
$$W_{16} = \frac{kQ^2}{\sqrt{2^2+8^2}} = \frac{kQ^2}{\sqrt{68}}$$
$$W_{17} = \frac{kQ^2}{\sqrt{4^2+8^2}} = \frac{kQ^2}{\sqrt{80}}$$
$$W_{18} = \frac{kQ^2}{\sqrt{2^2+4^2+8^2}} = \frac{kQ^2}{\sqrt{84}}$$
$$W_{\text{total}} = W_{12} + W_{13} + W_{14} + W_{15} + W_{16} + W_{17} + W_{18}$$
$$W_{\text{total}} = \frac{kQ^2}{2} + \frac{kQ^2}{4} + \frac{kQ^2}{\sqrt{20}} + \frac{kQ^2}{8} + \frac{kQ^2}{\sqrt{68}} + \frac{kQ^2}{\sqrt{80}} + \frac{kQ^2}{\sqrt{84}}$$
$$W_{\text{total}} \approx \boxed{1.44kQ^2}$$

Problem 5

Created by Tyler Wilson 2023
A parallel plate capacitor is hooked up to a battery in series. At time t, a dielectric is inserted into the capacitor and the volatge source is doubled. If the charge on the capacitor remains the same, find the dielectric constant.

Problem 6

Created by Tyler Wilson 2023
A resistor ($R = 2\,\Omega$) and a capacitor ($C = 16\,\mu$F) are connected in parallel to a 10 V battery.

a) What is the current flowing through the resistor after a long period of time?

At time $t = t_0$ a dielectric of $\kappa = 4$ is inserted into the capacitor.

a) What is the new current flowing through the resistor at the instant just after the dielectric is inserted (at $t = t_0$)?

b) What is the new current flowing through the resistor after a long time has passed?

Problem 7

Created by Tyler Wilson 2023
A resistor ($R = 6\,\Omega$) and a capacitor of unknown value are connected in series to a 12 V battery at time $t = 0$. The current through the resistor 0.1 seconds after the battery is connected is measured to be 1 A.
A dielectric of unknown value κ is then inserted in the capacitor. and the current through the resistor after 0.1 seconds is measured to be 0.75 A.
What are the values of both the capacitor and the dielectric?