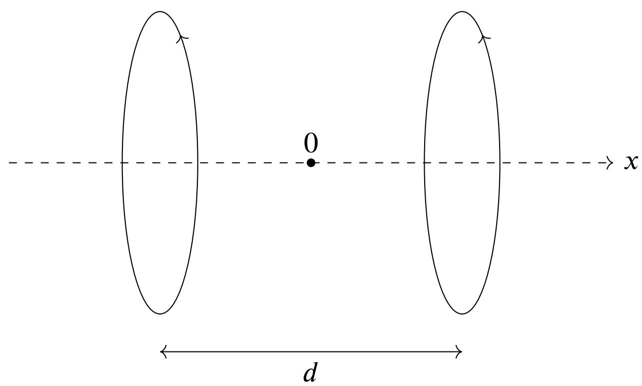
Physics 158 Magnetism Problem Bank

Problem 1

Created by Tyler Wilson 2023
Difficulty: ★★☆
Two current carrying loops, each with current I and radius R, are placed a distance d apart from each other, centered at the points $(-\frac{d}{2}, 0, 0)$ and $(\frac{d}{2}, 0, 0)$



a) Compute the magnitude of the magnetic field at the origin $(0, 0, 0)$ due to this configuration.

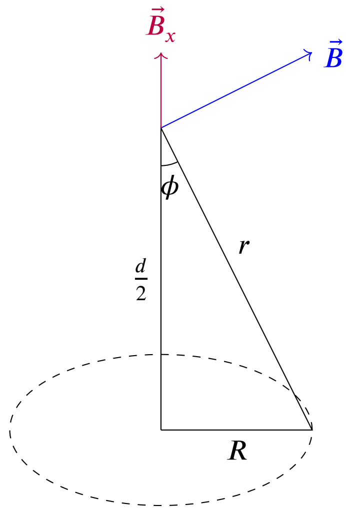
b) Compute the magnitude of the magnetic field for all points on the x-axis, $|\vec{B}(x)|$.

c) Determine the optimal distance d between the two loops such that the magnetic field along the axis of symmetry is as uniform as possible.
Hint: This can be done by choosing d to make as many derivatives of $|\vec{B}(x)$ equal to zero at $x = 0$ as possible.

Solution:

a) We can start by writing the Biot-Savart law for the magnetic field due to a single current carrying loop.

$$B = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{d\vec{l} \times \hat{r}}{r^2}$$



We can see that the magnetic field will only have a component in the x-direction, so we can write the magnetic field as

$$r = \sqrt{R^2 + \frac{d^2}{4}}$$

$$l = R\theta \Rightarrow dl = Rd\theta$$

$$B_x = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{d\vec{l} \times \hat{r}}{r^2} \cdot \hat{i} = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{Rd\theta}{R^2 + \frac{d^2}{4}} \sin\phi = \frac{\mu_0 I}{4\pi}\int_0^{2\pi} \frac{Rd\theta}{R^2 + \frac{d^2}{4}} \frac{R}{\sqrt{R^2 + \frac{d^2}{4}}}$$

$$B_x = \frac{\mu_0 I R^2}{4\pi}\int_0^{2\pi} \frac{d\theta}{\left(R^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}} = \frac{\mu_0 I R^2}{4\pi} \frac{2\pi}{\left(R^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}} = \frac{\mu_0 I R^2}{2\left(R^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}}$$

This gives us the field for one wire. Using the right-hand-rule we can see that the two fields will add so the total field at the origin is double what we calculated.

$$\boxed{|\vec{B}| = \frac{\mu_0 I R^2}{\left(R^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}}}$$

b) We can use the same method as above to find the field at any point on the x-axis. If we use the same derivation as above but replace $\frac{d}{2}$ with $x + \frac{d}{2}$ for the left ring and $x - \frac{d}{2}$ for the right ring then we get a general expression for the field at any point on the x-axis.

$$\boxed{|\vec{B}|(x) = \frac{\mu_0 I R^2}{2}\left(\frac{1}{\left(R^2 + (x + \frac{d}{2})^2\right)^{3/2}} + \frac{1}{\left(R^2 + (x - \frac{d}{2})^2\right)^{3/2}}\right)}$$

c) We can find the optimal distance by finding the points where the derivative of the field is zero at $z = 0$.

$$\left.\frac{dB}{dx}\right|_{x=0} = -\frac{\mu_0 I R^2}{2}\frac{3}{2}\left(\frac{2(x+\frac{d}{2})}{\left(R^2+(x+\frac{d}{2})^2\right)^{5/2}} + \frac{2(x-\frac{d}{2})}{\left(R^2+(x-\frac{d}{2})^2\right)^{5/2}}\right)\Bigg|_{x=0}$$

$$= \frac{3\mu_0 I R^2}{2}\left(\frac{\frac{d}{2}}{\left(R^2+\frac{d^2}{4}\right)^{5/2}} - \frac{\frac{d}{2}}{\left(R^2+\frac{d^2}{4}\right)^{5/2}}\right) = 0$$

The first derivative didn't tell us anything so now we can look at the second derivative.

$$\left.\frac{d^2B}{dx^2}\right|_{x=0} = -\frac{3\mu_0 I R^2}{2}\frac{d}{dx}\left(\frac{x+\frac{d}{2}}{\left(R^2+(x+\frac{d}{2})^2\right)^{5/2}} + \frac{x-\frac{d}{2}}{\left(R^2+(x-\frac{d}{2})^2\right)^{5/2}}\right)\Bigg|_{x=0} = 0$$

$$\left.\frac{\left(R^2+(x+\frac{d}{2})^2\right)^{5/2} - (x+\frac{d}{2})(\frac{5}{2})(R^2+(x+\frac{d}{2})^2)^{3/2}(2)(x+\frac{d}{2})}{\left(R^2+(x+\frac{d}{2})^2\right)^5}\right|_{x=0}$$

$$+\left.\frac{\left(R^2+(x-\frac{d}{2})^2\right)^{5/2} - (x-\frac{d}{2})(\frac{5}{2})(R^2+(x-\frac{d}{2})^2)^{3/2}(2)(x-\frac{d}{2})}{\left(R^2+(x-\frac{d}{2})^2\right)^5}\right|_{x=0} = 0$$

$$\frac{2(R^2+\frac{d^2}{4})^{5/2} - \frac{5d^2}{2}(R^2+\frac{d^2}{4})^{3/2}}{\left(R^2+\frac{d^2}{4}\right)^5} = 0$$

$$2\left(R^2+\frac{d^2}{4}\right) - \frac{5d^2}{2} = 2R^2 - 2d^2 = 0$$

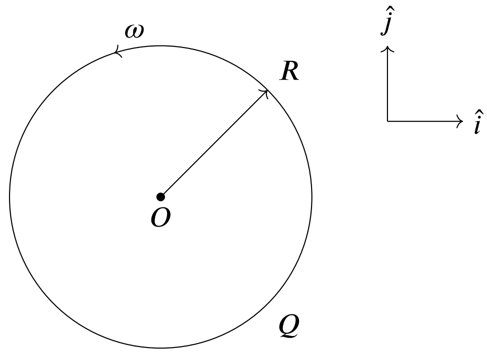
$$\boxed{d = R}$$

Problem 2

Created by Tyler Wilson 2023
Difficulty: ★☆☆
A toy company is planning on rolling out a new product called the magnetic hula-hoop. The hoop is made of a non-conducting material and has a radius of R and has a uniform charge density of λ.

a) If the hoop (depicted below) is spun around the point O with a constant angular velocity of ω, what is the magnetic field at the center of the hoop?

b) If the speed of the ring is no longer constant, but instead is expressed by the function $\omega(t) = \omega_0 e^{-at}$ then what is the direction of the induced emf in the ring? (write your answer as either clockwise or counterclockwise)



Solution:

a) We can recognize that current is simply the rate of change (or flow) of charge. In the case of the spinning ring we are creating a flow of charge so we can write the current as

$$I = \frac{dQ}{dt}$$

We can then use chain rule to rewrite this in terms of the variables that we have.

$$I = \frac{dQ}{dt} = \frac{dQ}{ds}\frac{ds}{dt} = \lambda \cdot v$$
$$v = R\omega \Rightarrow I = \lambda R\omega$$

We can then use the Biot-Savart law to find the magnetic field at the center of the ring.

$$\vec{B} = \frac{\mu_0 I}{4\pi}\int \frac{d\vec{l} \times \hat{r}}{r^2}$$
$$d\vec{l} \times \hat{r} = dl\hat{k}$$
$$dl = Rd\theta$$
$$r = R$$
$$\vec{B} = \frac{\mu_0 \lambda R\omega}{4\pi}\int_0^{2\pi} \frac{Rd\theta}{R^2}\hat{k} = \boxed{\frac{\mu_0 \lambda \omega}{2}\hat{k}}$$

b) We found in part (a) that the magnetic field is pointing out of the page. We also found that the magnetic field is proportional to the angular velocity of the ring so as ω decreases over time then the magnetic field strength will also decrease.

$$\frac{d\vec{B}}{dt} < 0\hat{k}$$
$$\Phi_B = \vec{B} \cdot \vec{A} = BA$$
$$\varepsilon = -\frac{d\Phi_B}{dt} = -\frac{dB}{dt}A > 0(\text{out of page})$$

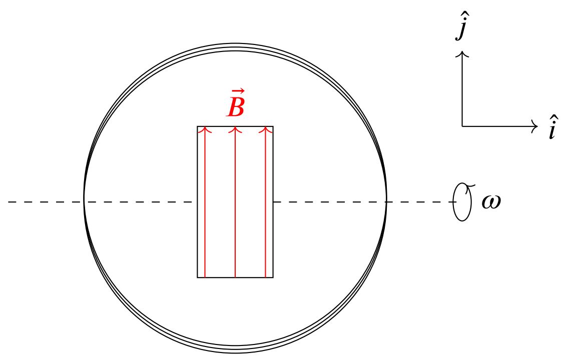
We can see that the induced emf is positive and is pointing out of the page. This means that the induced current will be flowing in the clockwise direction.

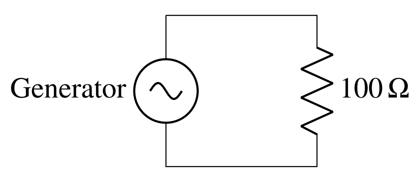
Problem 3

Created by Tyler Wilson 2023
Difficulty: ★☆☆
After learning about magnetic induction, a Physics 158 student decides to try to make their own generator as a personal project. They create a wire loop with N coils and a radius of 0.1 meters. The wires are connected to a light bulb which has a resistance of 100 Ω and requires a power of 2 Watts to light up. Inside the wire loop is a magnet with a magnetic field of 0.1 T. The magnet is fixed on a shaft that rotates at a speed ω.

a) If 500 coils are used, what is the minimum speed the magnet needs to be spinning in order to briefly light up the light bulb?

b) The student now wants to hook up this application to a bicycle. Assume that the bike tires have a radius of 17 cm and that the generator is set up with a gear ratio such that every rotation of the bike tire corresponds to 7 rotations of the magnet. If the student is able to maintain a constant speed of 10 km/h, what is the minimum number of coils needed such that the average power from one revolution of the magnet is enough to light up the light bulb?





Solution:

a) We can compute the emf induced in the loop by using Faraday's law.

$$\varepsilon = -\frac{d\Phi_B}{dt}$$

The magnetic flux through the loop is given by

$$\Phi_B = \vec{A} \cdot \vec{B} = AB\cos\theta$$

Note that in this setup the magnetic field and the area of the loops are both constant and we can take them out of the derivative.

$$\varepsilon = -\frac{d\Phi_B}{dt} = \frac{d}{dt}(AB\cos\theta) = -AB\frac{d}{dt}\cos\theta$$

Note that the derivative of θ is equal to the angular velocity of the magnet.

$$\varepsilon = -AB\frac{d}{dt}\cos\theta = -AB\frac{d}{d\omega}t = AB\sin\theta\frac{d}{d\theta}t = AB\omega\sin\theta$$

We can express the area as the area of a circle with radius r multiplied by the number of coils.

$$A = \pi r^2 N$$
$$\varepsilon = AB\omega\sin\theta = \pi r^2 N B\omega\sin\theta$$

The power generated for this circuit can be computed as

$$P = \frac{\varepsilon^2}{R} = \frac{\pi^2 r^4 N^2 B^2 \omega^2 \sin^2\theta}{R}$$

Rearranging for ω we get

$$\omega = \frac{\sqrt{PR}}{\pi r^2 N B\sin\theta}$$

Depending on the value of θ (which point the magnet is at in its rotation) we will get a different value for ω. We want to find the minimum value of ω so we can take the maximum value of $\sin\theta$ which is 1. This gives us the minimum value of ω as

$$\omega_{min} = \frac{\sqrt{PR}}{\pi r^2 N B} = \frac{\sqrt{2\cdot 100}}{\pi(0.1)^2(500)(0.1)} \approx \boxed{9.00\,\mathrm{rad/s}}$$

b) We can start by finding the angular velocity of the bike tire.

$$v = \frac{10\,\mathrm{km}}{\mathrm{h}}\cdot\frac{1\,\mathrm{h}}{3600\,\mathrm{s}}\cdot\frac{1000\,\mathrm{m}}{1\,\mathrm{km}} = 2.\bar{7}\,\mathrm{m/s}$$
$$\omega_{\text{tire}} = \frac{v}{r} = \frac{2.\bar{7}\,\mathrm{m/s}}{0.17\,\mathrm{m}} \approx 16.3\,\mathrm{rad/s}$$

We are told that every rotation of the bike tire corresponds to 7 rotations of the magnet so we can find the angular velocity of the magnet as

$$\omega_{\text{magnet}} = 7\omega_{\text{tire}} \approx 114.4\,\mathrm{rad/s}$$

We can now use the same equation as in part (a) to compute the average power generated by the magnet.

$$P = \frac{\pi^2 r^4 N^2 B^2 \omega^2 \sin^2\theta}{R}$$
$$P_{avg} = \int_0^{2\pi} \frac{\pi^2 r^4 N^2 B^2 \omega^2 \sin^2\theta}{R}\,d\theta = \frac{\pi^2 r^4 N^2 B^2 \omega^2}{R}\int_0^{2\pi}\sin^2\theta\,d\theta = \frac{\pi^2 r^4 N^2 B^2 \omega^2}{R}\pi$$

We can then rearrange for N to find the minimum number of coils needed.

$$N = \sqrt{\frac{P_{avg}R}{\pi^3 r^4 B^2 \omega^2}} = \sqrt{\frac{2\cdot 100}{\pi^3(0.17)^4(0.1)^2(114.4)^2}} \approx 7.68\,\text{coils}$$

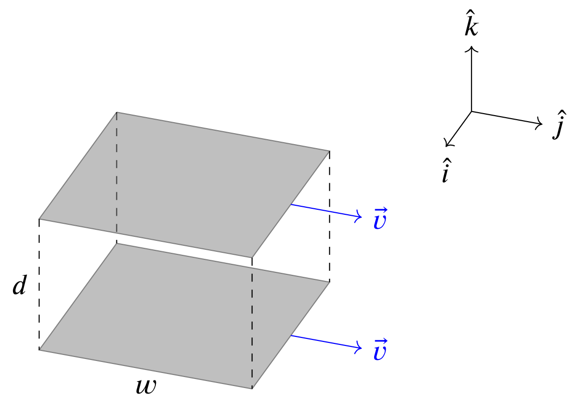
Rounding up to the nearest whole number, we get that we require $\boxed{8}$ coils.

Problem 4

Created by Tyler Wilson 2023
Difficulty: ★★☆
Two identical square plates have side length w. The top plate has a surface charge density of σ while the bottom plate has a surface charge density of $-\sigma$. The plates are separated by a distance d. The two plates are moved at some constant velocity $\vec{v}$ in the direction shown.
Assume that $w \gg d$ so that the plates can be treated as infinite planes.

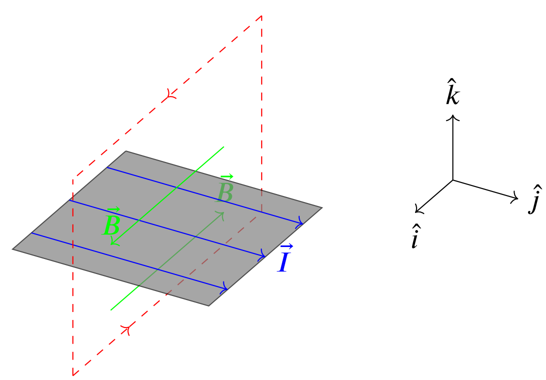
a) Find the magnetic field everywhere in space.

b) What would be different if we didn't make the assumption that $w \gg d$?



Solution:

a) We can start by solving this problem with one plate. The plate being in motion means that we will have moving charged particles which will resemble a current in the direction of the velocity. Using the right-hand-rule, we can find the direction of the magnetic field and create an Amperian loop.



Ampere's law tells us that

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}$$

With the dot product, the left side of the equation works out to be

$$\oint \vec{B} \cdot d\vec{l} = 2wB$$

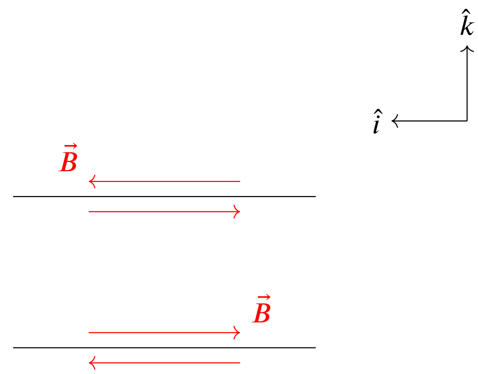
The enclosed current can be computed as follows:

$$I = \frac{dq}{dt} = \frac{dq}{dA}\frac{dA}{dt} = \sigma\frac{dA}{dt}$$
$$A = wy \Rightarrow \frac{dA}{dt} = w\frac{dy}{dt} = wv$$
$$I = \sigma wv$$

And so we get

$$2wB = \mu_0\sigma wv$$
$$B = \frac{\mu_0\sigma v}{2}$$

If we combine both plates then we can see that the magnetic field will be doubled in the center and will cancel on the outside.



And so we get that the magnetic field is

$$\vec{B} = -\mu_0\sigma v\hat{i}$$

between the plates and zero everywhere else.

b) In assuming that $w \gg d$ we are assuming that the plates are infinite and so there will be no edge effects.

If we account for the edge effects then we will no longer have $\vec{B} \cdot d\vec{l} = 0$ for the vertical parts of our loop and so we would no longer be able to solve for B using Ampere's law. Instead, we would have to use the Biot-Savart law to find the magnetic field everywhere in space which would be much more difficult.