Physics 158 Electric Fields Problem Bank

Problem 1

Created by Tyler Wilson 2023
A 16 cm long wire of charge $Q = 40\,\mu\mathrm{C}$ is bent into a square.

a) Find the electric field strength 20 cm vertically above the center of the square in the xy-plane.

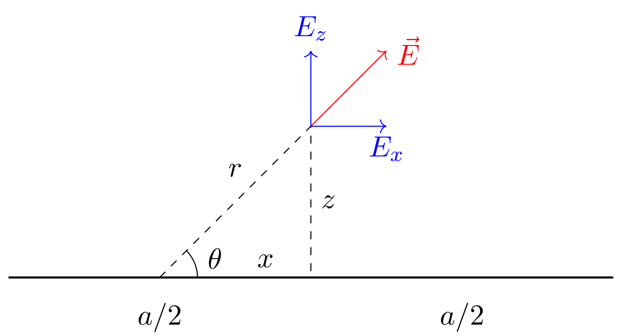
A point charge of mass $m = 100\,\mathrm{g}$ and charge $-5\,\mu\mathrm{C}$ is now placed 20 cm above the center of the square.

b) Find the magnitude and direction of the force acting on the point charge initially when $z = 20\,\mathrm{cm}$.

c) Plot the acceleration of the point charge as a function of it's distance z above the center of the square.

Solution:

a) We can break this problem down into the simpler problem of computing the electric field due to a straight piece of wire. We can then use the superposition principle to find the electric field due to the square.
Electric field due to a straight piece of wire of length a:



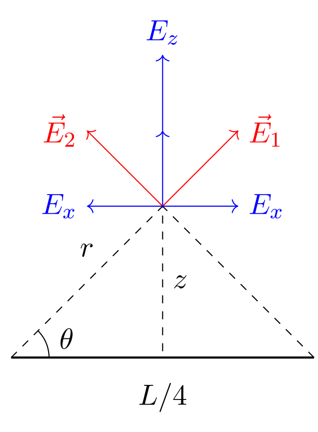
$$d\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{dq}{r^2}\hat{r}$$
$$dq = \lambda dx$$
$$r = \sqrt{x^2 + z^2}$$
$$\vec{E} = \int d\vec{E} = \int_{-a/2}^{a/2} \frac{1}{4\pi\epsilon_0}\frac{\lambda dx}{x^2 + z^2}\hat{r}$$

$$\vec{E} = \frac{\lambda}{4\pi\epsilon_0}\int_{-a/2}^{a/2}\frac{dx}{x^2+z^2}\hat{r}$$

Note that because of symmetry on each side the electric field will only have a vertical component.

$$\hat{r} = \cos(\theta)\hat{x} + \sin(\theta)\hat{z}$$
$$\sin\theta = \frac{z}{r} = \frac{z}{\sqrt{x^2+z^2}}$$
$$E_z = \frac{\lambda}{4\pi\epsilon_0}\int_{-a/2}^{a/2}\frac{dx}{(x^2+z^2)^{3/2}}$$
$$E_z = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{x}{z\sqrt{x^2+z^2}}\right]_{x=-a/2}^{x=a/2}$$
$$E_z = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{a}{z\sqrt{a^2/4+z^2}} - \frac{-a}{z\sqrt{a^2/4+z^2}}\right]$$
$$E_z = \frac{\lambda}{4\pi\epsilon_0}\frac{2a}{z\sqrt{a^2/4+z^2}}$$
$$\vec{E} = E_z\hat{z} = \frac{\lambda}{2\pi\epsilon_0}\frac{a}{z\sqrt{a^2/4+z^2}}\hat{z} = \frac{2k\lambda a}{z\sqrt{a^2/4+z^2}}\hat{z}$$

Let us consider the length of the wire to be $L = 16\,\text{cm}$. We can break this down to four straight pieces of wire of length $a = L/4 = 4\,\text{cm}$. The electric field due to each piece of wire will be the same magnitude and direction due to the symmetry of the problem. The electric field due to the square will be the sum of the electric fields due to each piece of wire.



$$\vec{E}_{\text{total}} = 4E_z\hat{z}$$
$$E_z = \frac{2k\lambda a}{r\sqrt{a^2/4+r^2}}\sin\theta$$
$$r = \sqrt{a^2/4+z^2}$$

$$\sin\theta = \frac{z}{r}$$

$$E_z = \frac{2k\lambda a z}{r^2\sqrt{a^2/4 + r^2}}$$

$$r^2 = \frac{a^2}{4} + z^2$$

$$E_z = \frac{2k\lambda a z}{(a^2/4 + z^2)\sqrt{a^2/2 + z^2}}$$

$$a = \frac{L}{4}, \quad \lambda = \frac{Q}{L}$$

$$E_z = \frac{\frac{1}{2}kQz}{(L^2/64 + z^2)\sqrt{L^2/32 + z^2}}$$

$$\vec{E}_{\text{total}} = 4E_z\hat{z} = \frac{2kQz}{(L^2/64 + z^2)\sqrt{L^2/32 + z^2}}\hat{z}$$

Now we can plug in the values $L = 0.16\,\mathrm{m}$, $Q = 40\,\mu\mathrm{C}$, and $z = 0.2\,\mathrm{m}$ to get the magnitude of the electric field.

$$\vec{E}_{\text{total}} = \frac{2(8.99\times10^9\,\mathrm{N\cdot m^2/C^2})(40\times10^{-6}\,\mathrm{C})(0.2\,\mathrm{m})}{(0.16^2\,\mathrm{m}^2/64 + (0.2\,\mathrm{m})^2)\sqrt{(0.16^2\,\mathrm{m}^2/32 + (0.2\,\mathrm{m})^2)}}\hat{z} = 1.76\times10^7\,\mathrm{N/C}\hat{z}$$

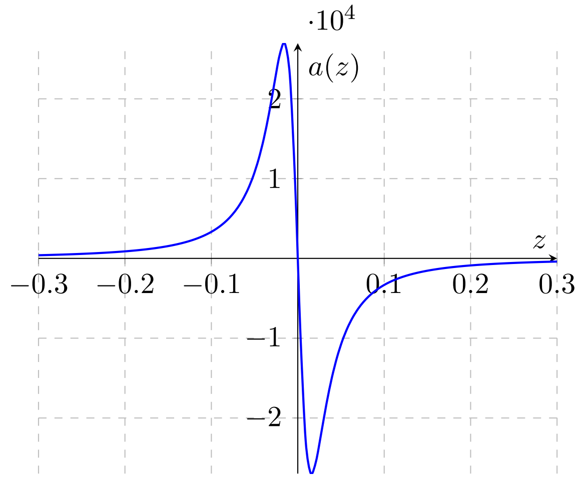
b) The force can be computed as $\vec{F} = q\vec{E}$ and the direction will be toward the center of the square for a negative point charge.

$$\vec{F} = q\vec{E} = -\frac{2kQqz}{(L^2/64 + z^2)\sqrt{L^2/32 + z^2}}\hat{z} = -88\,\mathrm{N}\hat{z}$$

c) We can get an expression for the acceleration from the force

$$\vec{F} = m\vec{a}$$

$$\vec{a} = -\frac{2kQqz}{m(L^2/64 + z^2)\sqrt{L^2/32 + z^2}}\hat{z}$$



Problem 2

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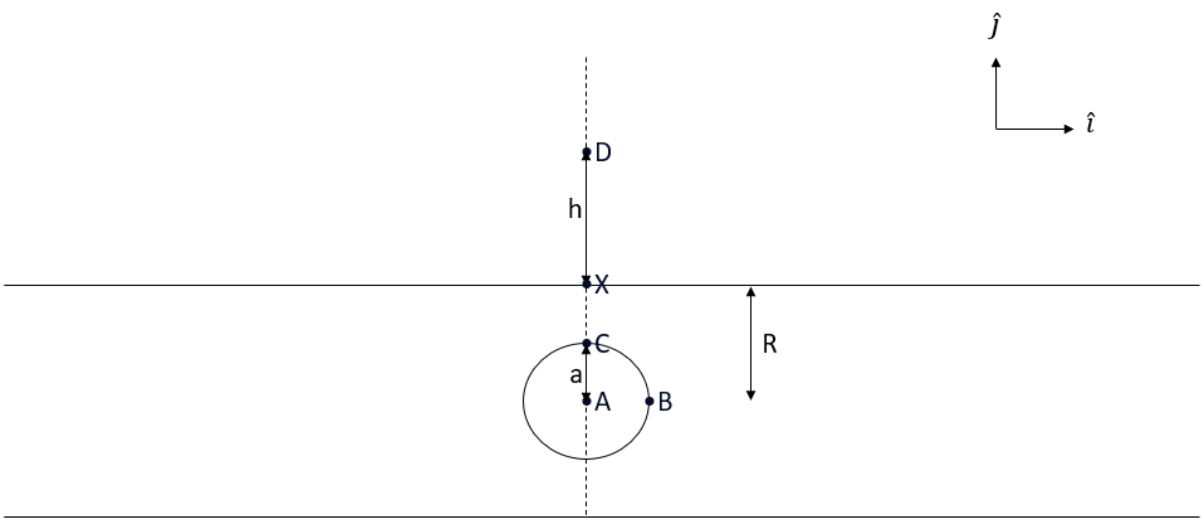
An insulating cylinder of radius R and effectively infinite length in the z-direction contains a uniform charge density of ρ.

a) Find the electric field everywhere in space

If there is now a <u>hollow</u> spherical cavity of radius a located at the center of the cylinder,



b) Find the electric field at the center of the sphere at point A.

c) Find the electric field just outside the sphere at point B.

d) Find the electric field just outside the sphere at point C.

e) Find the electric field outside both objects at point D.

If the potential at the point X is 0V,

f) What is the potential at point A?

Solution:
Notation: Because we have two types of radii, the cylindrical radius, and the spherical radius, we will use s to represent the variable cylindrical radius and r to represent the variable spherical radius.

a) We can use Gauss's Law to find the electric field everywhere in space. We will use a Gaussian cylinder of radius s and length L.

$$\oiint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enc}}}{\epsilon_0}$$

We will need to break this into two parts: the inside of the cylinder and the outside of the cylinder.
For the inside of the cylinder,

$$\oiint \vec{E}_{\text{in}} \cdot d\vec{A} = E_{\text{in}} A = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$Q_{\text{enc}} = \rho V_{\text{enc}} = \rho \pi s^2 L$$

$$E_{\text{in}} A = \frac{\rho \pi s^2 L}{\epsilon_0}$$

$$2\pi r L E_{\text{in}} = \frac{\rho \pi s^2 L}{\epsilon_0}$$

$$E_{\text{in}} = \frac{\rho s}{2\epsilon_0}$$

$$\boxed{\vec{E}_{\text{in}} = \frac{\rho s}{2\epsilon_0}\hat{s}}$$

For the outside of the cylinder,

$$\oiint \vec{E}_{\text{out}} \cdot d\vec{A} = E_{\text{out}} A = \frac{Q_{\text{enc}}}{\epsilon_0}$$

$$Q_{\text{enc}} = \rho V_{\text{enc}} = \rho \pi R^2 L$$

$$E_{\text{out}} A = \frac{\rho \pi R^2 L}{\epsilon_0}$$

$$2\pi s L E_{\text{out}} = \frac{\rho \pi R^2 L}{\epsilon_0}$$

$$E_{\text{out}} = \frac{\rho R^2}{2\epsilon_0 s}$$

$$\boxed{\vec{E}_{\text{out}} = \frac{\rho R^2}{2\epsilon_0 s}\hat{s}}$$

b) We can use the principle of superposition to find the electric field at point A. We will need to find the electric field due to the cylinder and the electric field due to the cavity and add

the two together to get the total contribution.
The electric field due to the cylinder is given by

$$\vec{E}_{\text{cyl}} = \frac{\rho s}{2\epsilon_0}\hat{s}\bigg|_{s=0} = \vec{0}$$

To find the electric field from the cavity we can treat the cavity like a sphere of radius a with a uniform charge density of $-\rho$. The electric field from this sphere can be computed using Gauss's Law.

$$\oiint \vec{E}\cdot d\vec{A} = EA = \frac{Q_{\text{enc}}}{\epsilon_0}$$

We will also need to break this into two parts: the inside of the sphere and the outside of the sphere.
For the inside of the sphere,

$$\begin{aligned}
Q_{\text{enc}} &= -\rho V_{\text{enc}} = -\rho\frac{4}{3}\pi r^3 \\
E_{\text{in}}A &= E_{\text{in}}(4\pi r^2) = -\frac{\rho\frac{4}{3}\pi r^3}{\epsilon_0} \\
E_{\text{in}} &= -\frac{\rho r}{3\epsilon_0} \\
\vec{E}_{\text{in}} &= E_{\text{in}}\hat{r} = -\frac{\rho r}{3\epsilon_0}\hat{r}
\end{aligned}$$

For the outside of the sphere,

$$\begin{aligned}
Q_{\text{enc}} &= -\rho V_{\text{enc}} = -\rho\frac{4}{3}\pi a^3 \\
E_{\text{out}}A &= E_{\text{out}}(4\pi r^2) = -\frac{\rho\frac{4}{3}\pi a^3}{\epsilon_0} \\
E_{\text{out}} &= -\frac{\rho a^3}{3\epsilon_0 r^2} \\
\vec{E}_{\text{out}} &= E_{\text{out}}\hat{r} = -\frac{\rho a^3}{3\epsilon_0 r^2}\hat{r}
\end{aligned}$$

At point A, the electric field from the cavity will be

$$\vec{E} = -\frac{\rho r}{3\epsilon_0}\hat{r}\bigg|_{r=0} = \vec{0}$$

So the total electric field at point A is

$$\boxed{\vec{E}_A = \vec{0}}$$

c) We will once again use superposition and can reuse the electric field from the cavity that we found in part (b) adn the cylinder in part (a). The electric field from the cylinder is given by

$$\vec{E}_{\text{cyl,B}} = \frac{\rho s}{2\epsilon_0}\hat{s}\bigg|_{s=0} = \vec{0}$$

The electric field from the cavity is given by

$$\vec{E}_{\text{cav,B}} = -\frac{\rho r}{3\epsilon_0}\hat{i}\bigg|_{r=a} = -\frac{\rho a}{3\epsilon_0}\hat{i}$$

So the total electric field at point B is

$$\boxed{\vec{E}_B = -\frac{\rho a}{3\epsilon_0}\hat{i}}$$

d) For point C,
The electric field from the cylinder is given by

$$\vec{E}_{\text{cyl,C}} = \frac{\rho s}{2\epsilon_0}\hat{j}\bigg|_{s=a} = \frac{\rho a}{2\epsilon_0}\hat{j}$$

The electric field from the cavity is given by

$$\vec{E}_{\text{cav,C}} = -\frac{\rho r}{3\epsilon_0}\hat{j}\bigg|_{r=a} = -\frac{\rho a}{3\epsilon_0}\hat{j}$$

So the total electric field at point C is

$$\vec{E}_C = \vec{E}_{\text{cyl,C}} + \vec{E}_{\text{cav,C}}$$
$$\vec{E}_C = \frac{\rho a}{2\epsilon_0}\hat{j} - \frac{\rho a}{3\epsilon_0}\hat{j}$$
$$\boxed{\vec{E}_C = \frac{\rho a}{6\epsilon_0}\hat{j}}$$

e) For point D,
The electric field from the cylinder is given by

$$\vec{E}_{\text{cyl,D}} = \frac{\rho R^2}{2\epsilon_0 s}\hat{j}\bigg|_{s=R+h} = \frac{\rho R^2}{2\epsilon_0(R+h)}\hat{j}$$

The electric field from the cavity is given by

$$\vec{E}_{\text{cav,D}} = -\frac{\rho r}{3\epsilon_0}\hat{j}\bigg|_{r=R+h} = -\frac{\rho(R+h)}{3\epsilon_0}\hat{j}$$

So the total electric field at point D is

$$\vec{E}_D = \vec{E}_{\text{cyl,D}} + \vec{E}_{\text{cav,D}}$$
$$\vec{E}_D = \frac{\rho R^2}{2\epsilon_0(R+h)}\hat{j} - \frac{\rho(R+h)}{3\epsilon_0}\hat{j}$$
$$\boxed{\vec{E}_D = \frac{3\rho R^2 - 2\rho(R+h)^2}{6\epsilon_0(R+h)}\hat{j}}$$

f) If the potential at X is zero then to find the potential at point A we can integrate the electric field from X to A.

$$V_{AX} = V_A - V_X = V_A - 0 = V_A$$

Keep in mind that the equations describing the electric field will change at the boundary of the cavity so we will want to break up our solution into two parts:

$$V_{AX} = V_{AC} + V_{CX}$$

For the part from X to C the electric field will be

$$E_{\text{cyl}} = \frac{\rho y}{2\epsilon_0}$$
$$E_{\text{cav}} = -\frac{\rho a^3}{3\epsilon_0 y^2}$$
$$E_{CX} = E_{\text{cyl}} + E_{\text{cav}} = \frac{\rho y}{2\epsilon_0} - \frac{\rho a^3}{3\epsilon_0 y^2}$$

The potential difference will then be

$$V_{CX} = V_C - V_X = V_C - 0 = V_C$$
$$V_C = -\int_R^a \left(\frac{\rho y}{2\epsilon_0} - \frac{\rho a^3}{3\epsilon_0 y^2}\right) dy$$
$$V_C = -\left(\frac{\rho y^2}{4\epsilon_0} + \frac{\rho a^3}{3\epsilon_0 y}\right)\Bigg|_R^a$$
$$V_C = \frac{\rho}{4\epsilon_0}(R^2 - a^2) - \frac{\rho a^3}{3\epsilon_0}\left(\frac{1}{a} - \frac{1}{R}\right)$$

For the part from C to A we get

$$E_{\text{cyl}} = \frac{\rho y}{2\epsilon_0}$$
$$E_{\text{cav}} = -\frac{\rho y}{3\epsilon_0}$$
$$E_{AC} = E_{\text{cyl}} + E_{\text{cav}} = \frac{\rho y}{2\epsilon_0} - \frac{\rho y}{3\epsilon_0} = \frac{\rho y}{6\epsilon_0}$$
$$V_{AC} = -\int_a^0 \frac{\rho y}{6\epsilon_0} dy = -\frac{\rho y^2}{12\epsilon_0}\Bigg|_a^0 = \frac{\rho a^2}{12\epsilon_0}$$
$$V_{AC} = V_A - V_C \Rightarrow V_A = V_{AC} + V_C$$
$$V_A = \frac{\rho a^2}{12\epsilon_0} + \frac{\rho}{4\epsilon_0}(R^2 - a^2) - \frac{\rho a^3}{3\epsilon_0}\left(\frac{1}{a} - \frac{1}{R}\right)$$
$$\boxed{V_A = \frac{\rho a^2}{12\epsilon_0}\left(3\frac{R^2}{a^2} + 4\frac{a}{R} - 6\right)}$$

Problem 3

Created by Tyler Wilson 2023
The potential above some charged Gaussian surface is given by the equation

$$V = \frac{k\sigma}{3y} \text{ Volts}$$

a) What is the equation for the electric field?

b) If $\sigma = 6\,\mu\text{C}$, find the electric field strength and direction at $y = 2\,\text{m}$.

c) What can you say about the electric field in the x-direction?

Hint:

$$\vec{E} = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k}$$

Solution:

a) The electric field is the negative gradient of the potential. This was defined in the hint above. The electric field would then be

$$\vec{E} = -\frac{\partial}{\partial y}\left(\frac{k\sigma}{3y}\right)\hat{j} = \boxed{\frac{k\sigma}{3y^2}\hat{j}}$$

b) Plugging in these values we would get

$$\vec{E}(y=2) = \frac{k(6\cdot 10^{-6})}{3(2)^2}\hat{j} = \boxed{4495\hat{j}\text{ N/C}}$$

c)

$$\vec{E}_x = -\frac{\partial}{\partial x}\left(\frac{k\sigma}{3y}\right) = 0$$

Therefore, the electric field in the x-direction is 0.

Problem 4

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A rectangular box has dimensions $2\,\text{m} \times 4\,\text{m} \times 8\,\text{m}$. At each corner of the box sits a point charge of value Q.
Leave your answers in terms of k and Q.

a) What is the electric field at the center of the box?

b) What is the potential at the center of the box?

One of the point charges on the corners is removed.

c) What is the magnitude of the new electric field at the center of the box?

d) What is the new potential at the center of the box?

e) What is the work done on the system in removing that point charge?

Solution:

a) Because of the symmetry at the center of the box we find that the electric field from each point charge cancels out with another point charge. We can find the electric field due to each point charge at the center and add the eight vectors together to get the total contribution. In doing this, we will see that the electric field at the center of the box is zero.

b) All of the point charges are the same distance away from the center of the box. So, to compute the potential at the center of the box we can just compute the potential due to one point charge and multiply by eight.

$$r = \sqrt{1^2 + 2^2 + 4^2} = \sqrt{21}$$
$$V_1 = \frac{kQ}{r} = \frac{kQ}{\sqrt{21}}$$
$$V_{\text{total}} = 8V_1 = \boxed{\frac{8kQ}{\sqrt{21}}}$$

c) Originally, the charges diagonally opposite one another were cancelling another out. Once we remove one of the point charges then one of the point charges will no longer be cancelled out so the magnitude of the electric field will be the same as the electric field felt at the center due to one point charge.

$$r = \sqrt{21}$$
$$|\vec{E}| = \frac{kQ}{r^2} = \boxed{\frac{kQ}{21}}$$

d) The potential at the center can be computed in the same manner as before, except now we only have seven point charges contributing to the potential.

$$\boxed{V = \frac{7kQ}{\sqrt{21}}}$$

e) To compute the work done in removing the charge we must compute the potential energy existing between the point that was removed and each of the other charges. This will give the potential difference which we can use to get the work done.

$$W_{12} = \frac{kQ^2}{2}$$
$$W_{13} = \frac{kQ^2}{4}$$

$$W_{14} = \frac{kQ^2}{\sqrt{2^2 + 4^2}} = \frac{kQ^2}{\sqrt{20}}$$
$$W_{15} = \frac{kQ^2}{8}$$
$$W_{16} = \frac{kQ^2}{\sqrt{2^2 + 8^2}} = \frac{kQ^2}{\sqrt{68}}$$
$$W_{17} = \frac{kQ^2}{\sqrt{4^2 + 8^2}} = \frac{kQ^2}{\sqrt{80}}$$
$$W_{18} = \frac{kQ^2}{\sqrt{2^2 + 4^2 + 8^2}} = \frac{kQ^2}{\sqrt{84}}$$
$$W_{\text{total}} = W_{12} + W_{13} + W_{14} + W_{15} + W_{16} + W_{17} + W_{18}$$
$$W_{\text{total}} = \frac{kQ^2}{2} + \frac{kQ^2}{4} + \frac{kQ^2}{\sqrt{20}} + \frac{kQ^2}{8} + \frac{kQ^2}{\sqrt{68}} + \frac{kQ^2}{\sqrt{80}} + \frac{kQ^2}{\sqrt{84}}$$
$$W_{\text{total}} \approx \boxed{1.44kQ^2}$$

Problem 5

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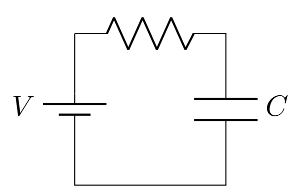
A parallel plate capacitor is hooked up to a battery and resister in series. At time t, the volatge source is doubled and a dielectric is inserted into the capacitor. If the maximum charge on the capacitor remains the same, find the value of the dielectric constant.

Solution:



$$Q_{\text{max}} = CV$$
$$Q_0 = Q_f \Rightarrow C_0V_0 = C_fV_f$$
$$V_f = 2V_0 \Rightarrow C_0V_0 = 2C_fV_0 \Rightarrow C_0 = 2C_f$$
$$C_f = \kappa C_0 \Rightarrow C_0 = 2\kappa C_0 \Rightarrow 1 = 2\kappa$$
$$\boxed{\kappa = \frac{1}{2}}$$

Problem 6

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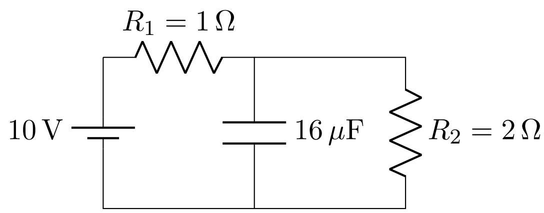
A resistor ($R_2 = 2\,\Omega$) and a capacitor ($C = 16\,\mu\text{F}$) are connected in parallel to a 10 V battery with another resistor ($R_1 = 1\,\Omega$) as shown.



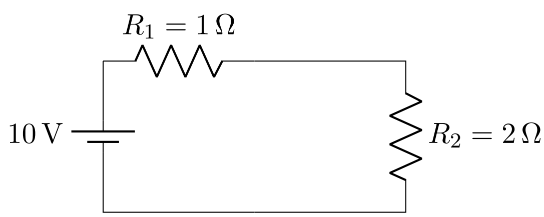
a) What is the current flowing through R_2 after a long period of time?

At time $t = t_0$ a dielectric of $\kappa = 4$ is inserted into the capacitor.

b) What is the new current flowing through R_2 at the instant just after the dielectric is inserted (at $t = t_0^+$)?

Solution:

a) The capacitor will act as a short so we are left with a series circuit



$$R_{eq} = R_1 + R_2 = 3\,\Omega$$
$$I = \frac{V}{R_{eq}} = \frac{10\,\mathrm{V}}{3\,\Omega} = \boxed{3.33\,\mathrm{A}}$$

b) At the instant just before the dielectric is inserted the voltage across R_1 is

$$V_{R_1} = IR_1 = 3.33\,\mathrm{A} \cdot 1\,\Omega = 3.33\,\mathrm{V}$$

This means that there must be $10\,\mathrm{V} - 3.33\,\mathrm{V} = 6.67\,\mathrm{V}$ across the capacitor just before the dielectric is inserted.
The charge on the capacitor just before the dielectric is inserted is

$$Q = CV = 16\,\mu\mathrm{F} \cdot 6.67\,\mathrm{V} = 106.67\,\mu\mathrm{C}$$

The new capacitance value with the dielectric will be

$$C = \kappa C_0 = 4 \cdot 16\,\mu\mathrm{C} = 48\,\mu\mathrm{C}$$

The charge that was accumulated on the capacitor cannot instantaneously jump in value so we know that the charge on the capacitor just after the dielectric is inserted is the same as the charge just before the dielectric is inserted ($q(t_0^-) = q(t_0^+)$). This means that the voltage across the capacitor must have changed when the dielectric was inserted.

$$V = \frac{Q}{C} = \frac{106.67\,\mu\text{C}}{48\,\mu\text{C}} = 2.22\,\text{V}$$

Because R_2 is in parallel with the capacitor, it must have the same voltage. We can use this to compute the current.

$$V_{R_2} = 2.22\,\text{V}$$
$$I_{R_2} = \frac{V_{R_2}}{R_2} = \frac{2.22\,\text{V}}{2\,\Omega} = \boxed{1.11\,\text{A}}$$

Problem 7

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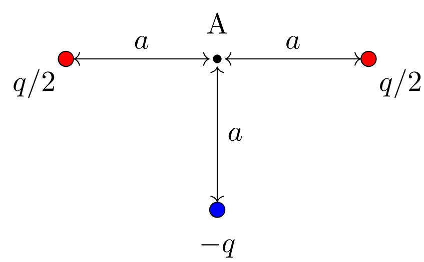
A very rough approximation for a water molecule can be represented by the following diagram: (a negative oxygen atom (blue) with two positive hydrogen atoms (red) attached to it)

Assume $V(\infty) = 0$



a) What is the total electrical energy stored in this molecule in terms of charge q and distance a?

b) Interpret the sign of your answer to part (a). What does it mean in terms of the work required to assemble the molecule?

c) Assuming all the charges are fixed in place, if you bring a positive test charge (q') from infinity to the point A, how much external work would be required?
(Simplify your answer as much as possible)

d) After bringing the point charge to the origin in part (c), what would the new total energy stored in the system?

Solution:

a) The total electrical energy stored in this molecule is the sum of the potential energy of each pair of charges. The general expression for the potential energy between two point charges is

$$W = -\frac{kq_1q_2}{r}$$

For the energy between either hydrogen atom and the oxygen atom we get

$$r = \sqrt{a^2 + a^2} = a\sqrt{2}$$
$$W = -\frac{k(-q)(\frac{q}{2})}{a\sqrt{2}} = \frac{kq^2}{2a\sqrt{2}}$$

For the energy between the two hydrogen atoms we get

$$r = 2a$$
$$W = -\frac{k(\frac{q}{2})(\frac{q}{2})}{2a} = -\frac{kq^2}{8a}$$

Summing all of the energies together we get

$$W = \frac{kq^2}{2a\sqrt{2}} + \frac{kq^2}{2a\sqrt{2}} - \frac{kq^2}{8a}$$
$$W = \frac{kq^2}{a\sqrt{2}} - \frac{kq^2}{8a}$$
$$\boxed{W = \frac{kq^2}{a}\left(\frac{1}{\sqrt{2}} - \frac{1}{8}\right)}$$

b) $W > 0$ which means that energy was added in order to create the system. If the system were released then energy would be released.

c) We know that the test charge starts at infinity and ends at point A. The work done on the point charge can be computed as

$$W = q'V$$

This means to solve the problem we just need to compute the potential difference. To compute the potential, we can compute the potential difference between infinity and point A.
We already know that the potential at infinity is zero. We can compute the potential at point A by summing the potential due to each charge.

$$V = \frac{kq}{r}$$
$$V(A) = \frac{k(\frac{q}{2})}{a} + \frac{k(\frac{q}{2})}{a} + \frac{k(-q)}{a}$$
$$V(A) = 0$$
$$V(A) - V(\infty) = 0$$

We get that the charge at point A is the same as that at infinity. Therefore, no work is required to bring the charge from infinity to point A.

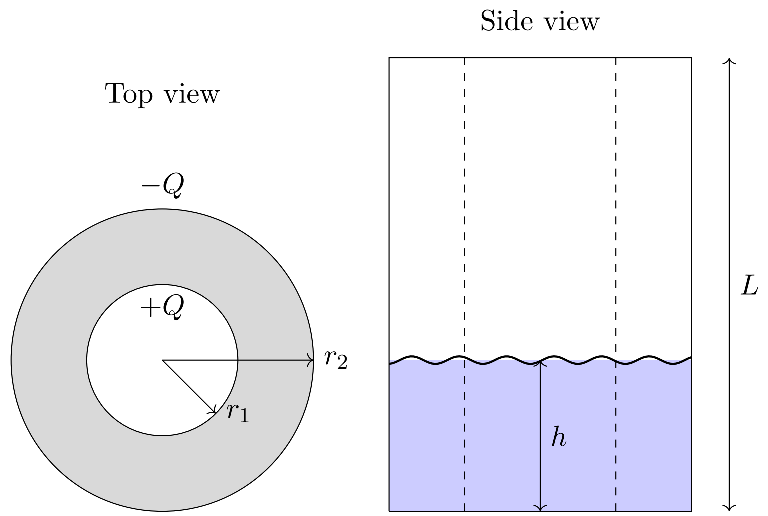
d) This is a bit of a trick question. Becasue there was no work done in bringing the charge into the system, the total energy stored in the system is the same as it was before.

$$\boxed{W = \frac{kq^2}{a}\left(\frac{1}{\sqrt{2}} - \frac{1}{8}\right)}$$

Problem 8

Created by Tyler Wilson 2023

A cylindrical capacitor has two concentric cylinders of radius r_1 and r_2 with length L. The inner cylinder is held at a charge $+Q$ and the outer cylinder is held at a charge $-Q$. Between these two cylinders is a liquid dielectric with dielectric constant κ that comes up to a height h with $h \leq L$.



a) First let's assume that there is no dielectric between the two cylinders ($h = 0$ case). In this case, find the capacitance of the cylinder in terms of the given variables ($r_1,\ r_2,\ L$).

b) Now we will look at the case where $h \neq 0$. Find the capacitance of the cylinder in this case. *Hint: It may be helpful to consider the system as two seperate capacitors being connected. You can use the result from part (a) to help you.*

c) Find an expression for the energy in this capacitor.

d) Compute the force acting on the dielectric due to the electric field. *Hint: You can get the force from the energy by taking the negative gradient:* $\vec{F} - \nabla U$. *(In our case this will be* $\vec{F} = \frac{\partial U}{\partial h}\hat{j}$*)*

e) If the dielectric has a mass density of ρ, compute the equilibrium height of the dielectric.

Solution:

a) The equation for capacitance is

$$C = \frac{Q}{V}$$

We have the charge Q so we just need to find the potential difference V between the two capacitor plates. We can find the potential difference by integrating the electric field from the inner cylinder to the outer cylinder.
We can get the electric field between the plates using Gauss's law for a cylinder.

$$\oiint \vec{E} \cdot d\vec{A} = \frac{q_{\text{enc}}}{\epsilon_0}$$

$$q_{\text{enc}} = Q$$
$$EA = \frac{Q}{\epsilon_0}$$
$$E = \frac{Q}{\epsilon_0 A} = \frac{Q}{\epsilon_0(2\pi r L)} = \frac{Q}{2\pi\epsilon_0 r L}$$

We can then integrate the electric field from the inner cylinder to the outer cylinder to get the potential difference.

$$\Delta V = -\int_{r_2}^{r_1} E dr$$
$$\Delta V = -\int_{r_2}^{r_1} \frac{Q}{2\pi\epsilon_0 r L} dr$$
$$\Delta V = -\frac{Q}{2\pi\epsilon_0 L}\int_{r_2}^{r_1} \frac{dr}{r}$$
$$\Delta V = -\frac{Q}{2\pi\epsilon_0 L}\ln(r)\Big|_{r_2}^{r_1}$$
$$\Delta V = -\frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{r_1}{r_2}\right)$$
$$\Delta V = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{r_2}{r_1}\right)$$

Now we can find the capacitance.

$$C = \frac{Q}{\Delta V}$$
$$C = \frac{Q}{\frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{r_2}{r_1}\right)}$$
$$\boxed{C = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{r_2}{r_1}\right)}}$$

b) We can analyse this problem by considering the system as two seperate capacitors. The first capacitor is the same as the one from part (a) but with length $L - h$. The second capacitor is the one formed by the dielectric and will be of length h. These two capacitors are connected side by side so we can imagine that they are connected in parallel and so the total capacitance can be gotten by adding the two capacitors in parallel.

$$C_1 = \frac{2\pi\epsilon_0(L-h)}{\ln\left(\frac{r_2}{r_1}\right)}$$
$$C_2 = \frac{2\pi\epsilon_0 h\kappa}{\ln\left(\frac{r_2}{r_1}\right)}$$
$$C_{\text{total}} = C_1 + C_2$$

$$C_{\text{total}} = \frac{2\pi\epsilon_0(L-h)}{\ln\left(\frac{r_2}{r_1}\right)} + \frac{2\pi\epsilon_0 h\kappa}{\ln\left(\frac{r_2}{r_1}\right)}$$

$$\boxed{C_{\text{total}} = \frac{2\pi\epsilon_0}{\ln\left(\frac{r_2}{r_1}\right)}(L + h(\kappa - 1))}$$

c) The energy in a capacitor is given by

$$U = \frac{1}{2}CV^2$$

We can rearrange this equation to get it in terms of C and Q

$$C = \frac{Q}{V} \Rightarrow V = \frac{Q}{C}$$

$$U = \frac{1}{2}CV^2 = \frac{1}{2}C\left(\frac{Q}{C}\right)^2 = \frac{1}{2}CQ^2$$

We can then plug in the values we computed earlier to get the energy.

$$U = \frac{1}{2}CQ^2$$

$$U = \frac{1}{2}\left(\frac{2\pi\epsilon_0}{\ln\left(\frac{r_2}{r_1}\right)}(L + h(\kappa - 1))\right)Q^2$$

$$\boxed{U = \frac{\pi\epsilon_0 Q^2}{\ln\left(\frac{r_2}{r_1}\right)}(L + h(\kappa - 1))}$$

d) The force can be computed using the derivative of the potential energy with respect to the height.

$$\vec{F} = \frac{\partial U}{\partial h}\hat{j}$$

$$\vec{F} = \frac{\partial}{\partial h}\left(\frac{\pi\epsilon_0 Q^2}{\ln\left(\frac{r_2}{r_1}\right)}(L + h(\kappa - 1))\right)\hat{j}$$

$$\boxed{\vec{F} = \frac{\pi\epsilon_0 Q^2}{\ln\left(\frac{r_2}{r_1}\right)}(\kappa - 1)\hat{j}}$$

e) At equilibrium the net force on the dielectric will be zero. There is a force from the electric field pulling the dielectric up into the cylinder and the opposing force of gravity pulling it downward. We can set these two forces equal to each other and solve for the height.

$$F_g = F_E$$

$$F_E = \frac{\pi\epsilon_0 Q^2}{\ln\left(\frac{r_2}{r_1}\right)}(\kappa - 1)$$

$$F_g = mg$$

$$\rho = \frac{m}{V} \Rightarrow m = \rho V$$

$$F_g = \rho V g$$

$$V = \pi(r_2^2 - r_1^2)h$$

$$F_g = \rho\pi(r_2^2 - r_1^2)hg$$

$$\frac{\pi\epsilon_0 Q^2}{\ln\left(\frac{r_2}{r_1}\right)}(\kappa - 1) = \rho\pi(r_2^2 - r_1^2)hg$$

$$h = \frac{\frac{\pi\epsilon_0 Q^2}{\ln\left(\frac{r_2}{r_1}\right)}(\kappa - 1)}{\rho\pi(r_2^2 - r_1^2)g}$$

$$\boxed{h = \frac{\epsilon_0 Q^2(\kappa - 1)}{\rho\ln\left(\frac{r_2}{r_1}\right)(r_2^2 - r_1^2)g}}$$