Math 257 Summary Sheet

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Ordinary Differential Equations

Linear ODEs: $y' + p(x)y = g(x)$ will have a solution of the form

$$\frac{d}{dx}(yr) = rg,\ r = e^{\int p(x)dx}$$

Constant coefficients: $ay'' + by' + cy = 0$. You can write the characteristic equation as $ar^2 + br + c = 0$ and the general solution will be

$$y(x) = \begin{cases} Ae^{r_1 x} + Be^{r_2 x} & r_1 \neq r_2 \in \mathbb{R} \\ Ae^{rx} + Bxe^{rx} & r_1 = r_2 \\ e^{\lambda x}\left(A\sin(\mu x) + B\cos(\mu x)\right) & r = \lambda \pm i\mu \end{cases}$$

Cauchy-Euler: $ax^2y'' + bxy' + cy = 0$. You can write the characteristic equation as $ar(r-1) + br + c = 0$ and the general solution will be

$$y(x) = \begin{cases} Ax^{r_1} + Bx^{r_2} & r_1 \neq r_2 \in \mathbb{R} \\ Ax^r + Bx^r \ln|x| & r_1 = r_2 \\ x^{\lambda}\left(A\sin(\mu\ln|x|) + B\cos(\mu\ln|x|)\right) & r = \lambda \pm i\mu \end{cases}$$

Nonhomogeneous equations: You can write the solution as $y(x) = y_c + y_p$ and can use undetermined coefficients to find y_p

$f(x)$	guess
$e^{\alpha x}$	$ae^{\alpha x}$
$\sin(\omega x)$	$a\cos(\omega x) + b\sin(\omega x)$
$\cos(\omega x)$	$a\cos(\omega x) + b\sin(\omega x)$
t^n	$a_0 + a_1t + a_2t^2 + \cdots + a_nt^n$

If there is any overlap with the complementary solution then you multiply your guess by x

Separation of Variables

For separation of variables, we can write $u(x,t)$ as $u(x,t) = X(x)T(t)$ and solve the eigenvalue problem for X.
The following problems will have the eigenvalue problem $X'' + \lambda X = 0$:

$$u_t = \alpha^2 u_{xx}$$
$$u_t = \alpha^2 u_{xx} - \gamma u$$
$$u_t = \alpha^2 u_{xx} - \gamma(t)u$$
$$u_t = \alpha^2 u_{xx} - \gamma(t)u_t + \eta(t)u$$

$$u_{tt} = c^2 u_{xx}$$
$$u_{tt} = c^2 u_{xx} - \gamma u$$
$$u_{tt} = c^2 u_{xx} - \gamma(t)u$$
$$u_{tt} = c^2 u_{xx} - \gamma u_t + \eta(t)u$$

General solutions for homogeneous boundary conditions:

- $u(0,t) = u(L,t) = 0$ (Dirichlet)

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$
$$X_n = \sin\left(\frac{n\pi x}{L}\right),\ n \geq 1$$

- $u_x(0,t) = u_x(L,t) = 0$ (Neumann)

$$\lambda_n = 0,\ \left(\frac{n\pi}{L}\right)^2$$
$$X_n = 1,\ \cos\left(\frac{n\pi x}{L}\right),\ n \geq 1$$

- $u(0,t) = u(L,t)$ and $u_x(0,t) = u_x(L,t)$ (Periodic)

$$\lambda_n = 0,\ \left(\frac{n\pi}{L}\right)^2$$
$$X_n = 1,\ \sin\left(\frac{n\pi x}{L}\right),\ \cos\left(\frac{n\pi x}{L}\right),\ n \geq 1$$

- $u(0,t) = u_x(L,t) = 0$ (Mixed type 1)

$$\lambda_n = \left(\frac{2n-1}{2L}\pi\right)^2$$
$$X_n = \sin\left(\frac{2n-1}{2L}\pi x\right),\ n \geq 1$$

- $u_x(0,t) = u(L,t)$ (Mixed type 2)

$$\lambda_n = \left(\frac{2n-1}{2L}\pi\right)^2$$
$$X_n = \cos\left(\frac{2n-1}{2L}\pi x\right),\ n \geq 1$$

Fourier Series

The Fourier series is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(\mu_n x) + \sum_{n=1}^{\infty} b_n \sin(\mu_n x)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \quad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(\mu_n x)\,dx \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(\mu_n x)\,dx$$

We can use the Fourier series to create a function identical to our IC, $u(x,0)$ and get the coefficients in our PDE. Another way to do this is by exploiting orthogonality in which case, the following integrals will be of use.

$$\int_{-L}^{L} \sin(\mu_n x)\sin(\mu_m x)\,dx = \begin{cases} 0 & \mu_m \neq \mu_n \\ L & \mu_m = \mu_n \end{cases}$$

$$\int_{-L}^{L} \cos(\mu_n x)\cos(\mu_m x)\,dx = \begin{cases} 0 & \mu_m \neq \mu_n \\ L & \mu_m = \mu_n \neq 0 \\ 2L & \mu_m = \mu_n = 0 \end{cases}$$

$$\int_{-L}^{L} \sin(\mu_n x)\cos(\mu_m x)\,dx = 0$$

$$\mu_n = \frac{n\pi}{L} \text{ and } \mu_m = \frac{m\pi}{L}$$

$$\text{or } \mu_n = \frac{2n-1}{2L}\pi \text{ and } \mu_m = \frac{2m-1}{2L}\pi$$

Note: in order for orthogonality to apply, we require that the two sinusoids are defined over the same period. For example, with $\int_{-L}^{L}\cos(2x)\cos\left(\frac{2n-1}{2}x\right)dx$, orthogonality would not apply

The following identities may also be useful for simplification:

$$\cos(n\pi) = (-1)^{n+1} \qquad \sin(n\pi) = 0$$

$$\cos\left(\frac{2n-1}{2}\pi\right) = 0 \qquad \sin\left(\frac{2n-1}{2}\pi\right) = (-1)^{n+1}$$

Homogeneous Heat Equation

The heat equation is of the form $u_t = \alpha^2 u_{xx}$

It is solved by first applying seperation of variables and finding the value of X_n and λ_n.

Summary of algorithm for homogeneous heat equation:

1. Write separable form and differentiate: $u(x,t) = X(x)T(t)$
2. Sub into original PDE: $XT' = \alpha^2 X''T \implies \frac{1}{\alpha^2}\frac{T'}{T} = \frac{X''}{X} = -\lambda$
3. Use BCs to prescribe condition for $X(x)$ and solve λ_n, X_n (see Separation of Variables section above)
4. For each λ_n find corresponding $T_n(t)$ via $\frac{1}{\alpha^2}\frac{T'}{T} = -\lambda \implies T_n = e^{-\alpha\lambda_n t}$
5. Recombine into superimposed sum of eigen-solutions to form general solution: $y(x,t) = \sum_{n=0}^{\infty} C_n u_n(x,t) = \sum_{n=0}^{\infty} C_n X_n(x)T_n(t)$
6. Use the IC $u(x,0) = g(x)$ to perform a Fourier analysis and determine C_n
7. Write full solution $y(x,t) = \sum_{n=0}^{\infty} C_n X_n(x)T_n(t)$

Inhomogeneous Heat Equation

We will often decompose $u(x,t)$ into a steady state $w(x)$ or $w(x,t)$ and a time dependent $v(x,t)$ term $u(x,t) = w(x) + v(x,t)$.

Warning: When this technique is properly applied, it should yield a new PDE, BCs, and IC problem for $v(x,t)$. Use substitution $v(x,t) = w(x,t) - u(x,t)$ for BCs and ICs. If the BCs are not homogeneous then reassess if choice of $w(x)$ or $w(x,t)$ is correct.

1. Regular Inhomogeneous Boundary Conditions
   Form: $u_t = \alpha^2 u_{xx}$ with $u(0,t) = u_0$ and $u(L,t) = u_1$ (or mixed BCs)
   Guess $w(x) = Ax + B$
   (a) Use the BCs to solve for A and B
   (b) Plug $u = w + v$ into the PDE to find the homogeneous problem for $v(x,t)$ and solve for v
2. Regular Inhomogeneous Boundary Conditions (Neumann)
   Form: $u_t = \alpha^2 u_{xx}$ with $u_x(0,t) = q_0$ and $u_x(L,t) = q_1$ Guess $w(x,t) = Ax^2 + Bx + Ct$
   (a) Use the BCs and plug into the PDE to get $C = 2\alpha^2 A$ and solve for A, B, C
   (b) Plug $u = w + v$ into the PDE to find the homogeneous problem for $v(x,t)$ and solve for v
3. Forcing Function/Irregular Equation
   Form: $u_t = \alpha^2 u_{xx} - \gamma u + g(x)$ with constant boundary conditions
   Guess will be the $w(x)$ that solves the resulting ODE: $\alpha^2 w_{xx} - \gamma w + g(x) = 0$
   (a) Use the BCs to solve for the unknown coefficients coming from the complementary solution of $w(x)$
   (b) Plug $u = w + v$ into the PDE to find the homogeneous problem for $v(x,t)$ and solve for v
4. Time Dependent Boundary Conditions:
   Form: when the 2 boundary conditions are functions of time, such as $u(0,t) = p(t)$ and $u(L,t) = q(t)$
   Guess $w(x,t) = A(t)x + B(t)$
   (a) Use the BCs to find the functions $A(t)$ and $B(t)$
   (b) Plug $u = w + v$ into the PDE to find a new PDE for $v(x,t)$ with time independent BCs. Note that $v(x,t)$ may still be inhomogeneous, requiring you to then solve for v using one of the other methods
5. Eigenfunction Expansion/Time Dependent Source/Sink (General solution of the heat equation)
   Form: $u_t = \alpha^2 u_{xx} + S(x,t)$
   This is the most general solution and will always work
   (a) First use one of the other methods to remove any inhomogeneous boundary conditions. This will provide a new PDE of the form $v_t = \alpha^2 v_{xx} + S_2(x,t)$ with homogeneous boundary conditions.
   (b) Find the general eigenfunctions of X_n for the homogeneous heat equation, $v_t = \alpha^2 v_{xx}$ omitting the source term: $v_t = \alpha^2 v_{xx}$
   (c) Expand the source term in terms of X_n and write the source term as:
   $$S_2(x,t) = \sum_{n=1}^{\infty} S_n(t) X_n(x)$$
   Use the Fourier analysis $S_n(t) = \frac{2}{L}\int_0^L s(x,t)X_n(x)$ or intuition to solve for $S_n(t)$ given that you know $S_2(x,t)$
   (d) Express the solution for v as
   $$v(x,t) = \sum_{n=1}^{\infty} V_n(t) X_n(x)$$
   where V_n is an undetermined function. Express v_t and v_{xx} as an infinite series as well.
   (e) Substitute all values that are now in terms of X_n back into the original PDE.
   $$\sum_{n=1}^{\infty} V_n' X_n = \sum_{n=1}^{\infty} V_n X_n'' + \sum_{n=1}^{\infty} S_n X_n$$
   rearrange to get an expression of the form
   $$\sum_{n=1}^{\infty} [V_n' + \lambda_n V_n - S_n] X_n = 0$$
   (f) Solve the ODE for V_n inside the series: $V_n'(t) + \lambda_n V_n(t) - S_n(t) = 0$ (Integrating factor technique will likely work well)
   (g) The ODE will yield a general solution with an undetermined coefficient for the homogeneous portion of the ODE: $V_n(t) = f(n,t) + C_n g(n,t)$

(h) Use Fourier series and the initial condition $v(x,0) = h(x)$ to solve for the unknown constant that comes from the solution of V_n:

$$v(x,t) = \sum_{n=1}^{\infty} [f(n,t) + C_n g(n,t)] X_n$$

$$v(x,0) = h(x) = \sum_{n=1}^{\infty} \underbrace{[f(n,0) + C_n g(n,0)]}_{D_n} X_n$$

Then $D_n = \frac{2}{L}\int_0^L h(x) X_n(x) dx$, solve for C_n from D_n, then write out full solution.

Wave Equation

The wave equation is of the form $u_{tt} = c^2 u_{xx}$.
The solution method is the exact same as the heat equation (including the methods of dealing with inhomogeneous equations)
The general expression of T_n is $T_n'' + \lambda T_n = 0$, giving

$$T_n(t) = A_n \cos(\mu_n t) + B_n \sin(\mu_n t)$$

Because there are two unknown constants, we are required to match two initial conditions using Fourier series. These are often of the form $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$ but could also be something like $u(x,0) = f(x)$ and $u(x,\tau) = g(x)$ or similar
For where $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$, we can also use D'Alembert's solution to solve the wave equation

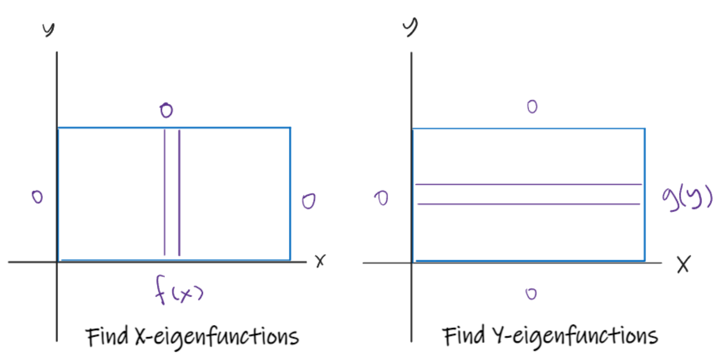
$$u(x,t) = \frac{1}{2}\left(f(\xi) + f(\eta)\right) + \frac{1}{2c}\int_{\xi}^{\eta} g(s)ds$$

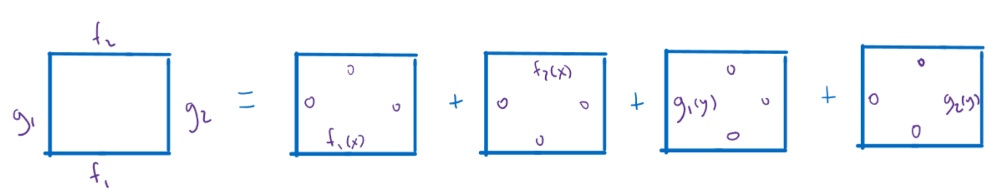
$u_t t = c^2 u_{xx}$	$u(x,0) = f(x)$	$u_t(x,0) = g(x)$
$\eta = x + ct$	$\xi = x - ct$	

η and ξ are essentially space time coordinates indexing the invariant state of the two wave-forms that are moving in opposite directions. Notice that η is a wave moving to the right, since it is the transposition of x coordinates at a velocity of c, similar to ξ moving to the left. If $\int g(s)ds = G(s)$ then $u(\xi,\eta) = \frac{1}{2c}[(cf(\xi) + G(\xi)) + (cf(\eta) - G(\eta))]$ We can see that the general case is always two waves moving at the same velocity in opposite directions. On a space-time graph, constant ξ or η are steady states for each wave form. The overall solution at any point in time is the superposition of the two waves.

Laplace Equation

The Laplace equation is of the form $\nabla^2 u = u_{xx} + u_{yy} = 0$ The solution method involves splitting the problem up into 4 subproblems with each subproblem having a solution similar to the wave equation.





1. Draw out the domain and BCs
2. Decompose into up to 4 subproblems
3. Assume $u(x,y) = X(x)Y(y)$ so that $\frac{Y''}{Y} = \frac{-X''}{X} = \pm\lambda$
4. Solve for eigenfunctions in the completely homogeneous axis first (see above). Here we cover the homogeneous X case.
5. Apply homogeneous boundary conditions to X to fix X_n, λ_n identical to eigenfunctions of heat equation boundary conditions.
6. Y_n will take the form of $Y_n(y) = A_n \cosh(\mu_n y) + B_n \sinh(\mu_n y)$
7. Use the remaining homogeneous boundary to eliminate a coefficient or reduce the expression to $A_n = C\,B_n$ for some constant C and rewrite the equation in terms of a single constant Q_n. Using hyperbolic identities, the 8 common half-homogeneous boundary scenarios are listed below:

Common Laplace Equation Boundary Solutions

If $u(x,0) = 0$

$$u^{\text{top}} = \sum_{n=1}^{\infty} Q_n \sinh(\mu_n y) X_n$$

If $u(x,b) = 0$

$$u^{\text{bottom}} = \sum_{n=1}^{\infty} Q_n \sinh(\mu_n (y-b)) X_n$$

If $u(0,y) = 0$

$$u^{\text{right}} = \sum_{n=1}^{\infty} Q_n \sinh(\mu_n x) Y_n$$

If $u(a,y) = 0$

$$u^{\text{left}} = \sum_{n=1}^{\infty} Q_n \sinh(\mu_n (x-a)) Y_n$$

If $u_y(x,0) = 0$

$$u^{\text{top}} = \sum_{n=1}^{\infty} Q_n \cosh(\mu_n y) X_n$$

If $u_y(x,b) = 0$

$$u^{\text{bottom}} = \sum_{n=1}^{\infty} Q_n \cosh(\mu_n (y-b)) X_n$$

If $u_x(0,y) = 0$

$$u^{\text{right}} = \sum_{n=1}^{\infty} Q_n \cosh(\mu_n x) Y_n$$

If $u_x(a,y) = 0$

$$u^{\text{left}} = \sum_{n=1}^{\infty} Q_n \cosh(\mu_n (x-a)) Y_n$$

8. Use the final in-homogenous boundary condition to find a Fourier relation to determine the coefficients Q_n
9. Solve the remaining subproblems, the final solution is:

$$u(x,y) = u(x,y)^{\text{left}} + u(x,y)^{\text{right}} + u(x,y)^{\text{top}} + u(x,y)^{\text{bottom}}$$

* Some of the sub-problems have trivial solutions if there is no in-homogenous boundary condition for that side.

Circular Laplace Equation

The Laplacian in polar coordinates is given by $\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$. When we apply separation of variables, we get

$$-\left(r^2 \frac{R''}{R} + r\frac{R'}{R}\right) = \frac{\Theta''}{\Theta} = -\lambda$$

Along with the given boundary conditions, we also have the following implied boundary conditions:

- If the domain contains $r = 0$ then $\lim_{r\to 0} u(r,\theta)$ must be finite
- If the domain contains $r = \infty$ then $\lim_{r\to\infty} u(r,\theta)$ must be finite
- If the domain forms a full circle/ring then the boundary conditions in θ are periodic

Important notes: you cannot use superposition on a circular domain like you can on a rectangular domain. We will also always start by solving for Θ first in this class.
Solution method:

1. The first thing to do is to make sure the boundary conditions in θ are homogeneous. If not then use the methods mentioned earlier to make them homogeneous.

2. Solve the eigenvalue problem in Θ to get λ_n and Θ_n

3. Plug in λ_n and solve the ODE, $r^2R_n''+rR_n'-\lambda_nR_n=0$ to find R_n. The solution should be of the form $R_n = A_nr^{\mu_n} + B_nr^{-\mu_n}$

4. Write out $u(r,\theta) = a_0\Theta_0R_0 + \sum_{n=1}^{\infty}\Theta_nR_n$ and impose the initial conditions in r to solve for the unknown constants, a_0, A_n, B_n

Finite Difference Approximations

Want to solve to get u_i^{k+1} in terms of u^k terms so we can solve for the next time step.
Formulas:

$$\text{Forward: } f'(x_0) = \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} + \mathcal{O}(\Delta x)$$
$$\text{Backward: } f'(x_0) = \frac{f(x_0)-f(x_0-\Delta x)}{\Delta x} + \mathcal{O}(\Delta x)$$
$$\text{Centre: } f'(x_0) = \frac{f(x_0+\Delta x)-f(x_0-\Delta x)}{2\Delta x} + \mathcal{O}(\Delta x^2)$$
$$\text{2nd Order: } f''(x_0) = \frac{f(x_0+\Delta x)-2f(x_0)+f(x_0-\Delta x)}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

More formulas can be derived using Taylor series as a starting point.
Index notation: we write $x = i\Delta x$ and $t = k\Delta t$ and so $u_i^k = u(i\Delta x, k\Delta t)$ where i is the step in x and k is the time step.
Method: We use the above formulas to write expressions for u_t and u_{xx}, plug them into our PDE, and solve for u_i^{k+1}. The expression with $\mathcal{O}(\Delta x^2, \Delta t)$ is given by

$$u_i^{k+1} = \alpha^2\frac{\Delta t}{\Delta x^2}\left(u_{i+1}^k - 2u_i^k + u_{i-1}^k\right) + u_i^k$$

From here we use our IC to get points for u_i^0 and use the BCs to get information about the points at the edges. For example,

$$u(0,t) = 0 \Rightarrow u_0^k = 0\ \forall k$$
$$u_x(0,t) = 0 \Rightarrow \frac{u_1^k - u_{-1}^k}{2\Delta x} = 0 \Rightarrow u_{-1}^k = u_1^k$$

Trigonometric Identities and General Formulas

Taylor expansion:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''}{2!}(x-a)^3 + \frac{f'''(x)}{3!}(x-a)^3 + \cdots$$
$$f(x+\Delta x) = f(x) - f'(x)\Delta x + \frac{f''(x)}{2!}\Delta x^2 - \frac{f'''(x)}{3!}\Delta x^3 + \mathcal{O}(\Delta x^4)$$

Trigonometric identities:

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sin(A\pm B) = \sin A\cos B \pm \sin B\cos A$$
$$\cos(A\pm B) = \cos A\cos B \mp \sin A\sin B$$
$$\sin(2\theta) = 2\sin\theta\cos\theta$$
$$\cos(2\theta) = \cos^2\theta - \sin^2\theta$$

Hyperbolic trigonometric identities:

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh(A\pm B) = \sinh A\cosh B \pm \sinh B\cosh A$$
$$\cosh(A\pm B) = \cosh A\cosh B \pm \sinh A\sinh B$$
$$\sinh(2x) = 2\sinh x\cosh x$$
$$\cosh(2x) = \cosh^2 x + \sinh^2 x$$

Inverse Matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$