MATH 152 - Unofficial Formula Sheet

Vectors

Basics

Direction Vector $\vec{ab} = \vec{b} - \vec{a}$

Norm $\|x\| = \sqrt{a_1^2 + a_2^2 + \cdots + a_a^2}$

Unit Vector $\hat{u} = \frac{\vec{u}}{\|u\|}$

Perpendicular $\vec{a}^\perp = \det \begin{bmatrix} \hat{i} & \hat{j} \\ a_1 & a_2 \end{bmatrix}$

Dot Product

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$

$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

$\vec{a} \cdot \vec{b} = \|a\| \|b\| \cos\theta$

$\vec{a} \perp \vec{b}$ if $\vec{a} \cdot \vec{b} = 0$

Cross Product

$\vec{a} \times \vec{b} = \det \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \vec{n}_{\vec{a},\vec{b}}$

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta$

$\vec{a} \parallel \vec{b}$ if $\vec{a} \times \vec{b} = 0$

$\vec{a} \cdot (\vec{b} \times \vec{c}) = \det \begin{bmatrix} -\vec{a}- \\ -\vec{b}- \\ -\vec{c}- \end{bmatrix}$

Projection and Perpendicular

$\text{proj}_{\vec{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{\|\vec{b}\|^2} \vec{b} = (\vec{a} \cdot \hat{b}) \hat{b}$

$\text{perp}_{\vec{b}}(\vec{a}) = \vec{a} - \text{proj}_{\vec{b}}(\vec{a})$

Area and Volume

$A = \|\vec{a} \times \vec{b}\|$

$A = \left| \det \begin{bmatrix} -\vec{a}- \\ -\vec{b}- \end{bmatrix} \right|$

$V = |\vec{a} \cdot (\vec{b} \times \vec{c})| = \left| \det \begin{bmatrix} -\vec{a}- \\ -\vec{b}- \\ -\vec{c}- \end{bmatrix} \right|$

Lines and Planes

Line Equations

Parametric $\vec{x} = \vec{a}t + \vec{p}$

Equation Form in $\mathbb{R}^2$ $\vec{x} \cdot \vec{a}^\perp = x_1 a_1^\perp + x_2 a_2^\perp = d$

Equation Form in $\mathbb{R}^3$ $\begin{cases} \vec{x} \cdot \vec{a}^\perp = x_1 a_1^\perp + x_2 a_2^\perp = d_1 \\ \vec{x} \cdot \vec{b}^\perp = x_1 b_1^\perp + x_2 b_2^\perp = d_2 \end{cases}$

Plane Equations

Parametric $\vec{x} = \vec{a}s + \vec{b}t + \vec{p}$

Equation Form in $\mathbb{R}^3$ $\vec{x} \cdot \vec{n} = d$

Intersection of Objects

Intersection of Planes:

Solve $\begin{bmatrix} n_1 & n_2 & n_3 \\ m_1 & m_2 & m_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

Distance Between Objects

Distance between point $\vec{q}$ and Line $\vec{x} = \vec{p} + \vec{a}t$:

$d = \|\text{perp}_{\vec{a}}(\vec{pq})\| = \|\text{proj}_{\vec{a}^\perp}(\vec{pq})\|$

Distance between point $\vec{q}$ and plane $\vec{x} = \vec{p} + \vec{a}s + \vec{b}t$:

$d = \|\text{proj}_{\vec{n}}(\vec{pq})\|$

General Procedure:

set $d = \|\vec{x}_1 - \vec{x}_2\|$ using parametric forms of $\vec{x}$

solve for $t_1, t_2, \ldots$ using $\vec{\nabla} d = \vec{0}$

plug back in values of $t_1, t_2, \ldots$ and solve for d

Gaussian Elimination

Step 1: Set the top left entry to 1 (or as the LCD of the first column)

Step 2: Use the first row to 'kill off' all other entries in the first column

Step 3: For column 2, use one row to 'kill off' all the other entries in that column

Step 4: Repeat process until finished

Linear Independence

$\vec{a}$, $\vec{b}$, and $\vec{c}$ are linearly dependent if:

$\det \begin{pmatrix} -\vec{a}- \\ -\vec{b}- \\ -\vec{c}- \end{pmatrix} = 0$

or if $\begin{bmatrix} | & | & | \\ \vec{a} & \vec{b} & \vec{c} \\ | & | & | \end{bmatrix}$ has no unique solution.

Linear Transformations

Rules

$T(\vec{x}) = A\vec{x}$

$T(\vec{x} + \vec{y}) = T(\vec{x}) + T(\vec{y})$

$T(s\vec{x}) = sT(\vec{x})$

$(S \circ T)\vec{x} = BA\vec{x}$

$A = [T(\vec{e}_1) | T(\vec{e}_2) | \cdots | T(\vec{e}_n)]$

Rotations

$\text{Rot}_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

$\text{Rot}_\theta^2(\vec{x}) = \text{Rot}_{2\theta}(\vec{x})$

Projections

$\text{Proj}_{\hat{a}} = \begin{bmatrix} a_1^2 & a_1 a_2 \\ a_1 a_2 & a_2^2 \end{bmatrix}$

$\text{Proj}_\theta = \begin{bmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{bmatrix}$

$\text{Proj}^2(\vec{x}) = \text{Proj}(\vec{x})$

Reflections

$\text{Ref} = 2\text{Proj} - I$

$\text{Ref}_\theta = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$

$\text{Ref}_{\hat{a}} = \begin{bmatrix} 2a_1^2 - 1 & 2a_1 a_2 \\ 2a_1 a_2 & 2a_2^2 - 1 \end{bmatrix}$

$\text{Ref}^2(\vec{x}) = \vec{x}$

Transpose

Identities

$(A^T)^T = A$
$(A+B)^T = A^T + B^T$
$(kA)^T = kA^T$
$(AB)^T = B^T A^T$

Decomposition of a Matrix

jth column $= A\vec{e}_j$
ith row $= \vec{e}_i^T A$
Entry $a_{ij} = \vec{e}_i^T A \vec{e}_j$

Dot Product

$\vec{a}\cdot\vec{b} = \vec{a}^T\vec{b}$

Determinants

2x2 and 3x3 Determinants

$\det(A) = |A|$

$$\det\begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - cb$$

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

General

$\det(A) = \det(A^T)$
$\det(A^{-1}) = \frac{1}{\det(A)}$
$\det(A^x) = \det(A)^x$
$\det(AB) = \det(A)\det(B)$
$\det(kA) = k^n \det(A)$ where n is the matrix size
$\det(A) = \lambda_1\lambda_2\cdots\lambda_n$

Simplifying Determinants

Swap Rows	$\det(B) = -\det(A)$
Multiply Row by k:	$\det(B) = k\det(A)$
Add Factor of a Row	$\det(B) = \det(A)$

The determinant of a triangular matrix is the product of the diagonals

Inverse

A is invertible iff $\det(A) \neq 0$
If A is invertible then:
$A^{-1}A = AA^{-1} = I$
$(A^{-1})^{-1} = A$
$(kA)^{-1} = \frac{1}{k}A^{-1}$
$(AB)^{-1} = B^{-1}A^{-1}$
$(A^T)^{-1} = (A^{-1})^T$

Complex Numbers

Definitions

Imaginary Number	$i^2 = -1$
Complex Number	$z = a + ib$
Conjugate	$\overline{z} = a - ib$
Real Part	$\Re(z) = \frac{z+\overline{z}}{2}$
Imaginary Part	$\Im(z) = \frac{z-\overline{z}}{2}$
Norm	$|z| = \sqrt{a^2+b^2}$

Operations and Identities

$\overline{zu} = \overline{z}\cdot\overline{u}$
$\overline{(z\pm u)} = \overline{z}\pm\overline{u}$
$z\cdot\overline{z} = |z|^2$
$|zu| = |z||u|$
$\frac{u}{z} = \frac{u\overline{z}}{|z|^2}$

Polar Form

$e^{i\theta} = \cos\theta + i\sin\theta =$
$z = |z|e^{i\theta}$
$\arg(z) = \theta = \arctan(\frac{b}{a})$

Eigen-Analysis

General Formulas (to be updated by Friday)

$\det(A - I\lambda) = 0$
$A\vec{v} = \lambda\vec{v}$

Solving Method

Step 1:	Use $\det(A - I\lambda) = 0$ to solve for λ
Step 2:	Use $[A - I\lambda_i][\vec{v}_i] = \vec{0}$ to solve for each eigenvector $\vec{v}_i$
Step 3:	To find $A^n\vec{x}$, set $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots$
Step 4:	Solve $\vec{v}_1\ \vec{v}_2\ \cdots][\vec{c}] = \vec{x}$ to solve for coefficients
Step 5:	Set $A^n\vec{x} = c_1\lambda_1^n\vec{v}_2 + c_2\lambda_2^n\vec{v}_2 + \cdots$
(Step 6:)	Each conjugate pair can be expressed as $2\Re(c_i\lambda_i^n\vec{v}_i)$

Differential Equations

Solving Method

Step 1:	Set $\frac{d\vec{x}}{dt} = A\vec{x}$
Step 2:	Find eigenvalues and eigenvectors of A
Step 3:	Write $\vec{x}(t) = c_1e^{\lambda_1 t}\vec{v}_1 + c_2e^{\lambda_2 t}\vec{v}_2 + \cdots$ as the general form of the solution.
Step 4:	If imaginary, express conjugate pairs as $c_1\vec{P}(t) + c_2\vec{Q}(t)$ Where $\vec{P}(t) = \Re(e^{\alpha t}\vec{v}_1(\cos(\beta t) + i\sin(\beta t))$ and $\vec{Q}(t) = \Im(e^{\alpha t}\vec{v}_1(\cos(\beta t) + i\sin(\beta t))$
Step 5:	Use initial conditions to solve for coefficients

LCR Circuits

Step 1:	Write voltage and current equations
Step 2:	Express i and E in terms of I and V
Step 3:	Write differential equations of form $\frac{dV}{dt} = \pm\frac{i}{C}$ and $\frac{dI}{dt} = \pm\frac{E}{L}$
Step 4:	Express system of differential equations in matrix form
Step 5:	Solve differential equation for various $I(t)$ and $V(t)$

Compiled by Tyler Wilson 2021