Introductory Physics for Engineers (PHYS157/158)

Tyler Wilson

Sep 2020 to Apr 2021

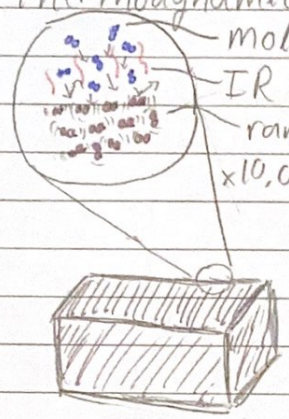
Prof: Steve Dierker, Vesna Sossi

Unit 1 – Thermodynamics	pg. 1-22
1.1 – Temperature and Thermometry	1-3
1.2 – Thermal Expansion, Stress, and Strain	3-8
1.3 – Heat and Temperature/Phase Change	8-9
1.4 – Heat Flow: Conduction, Convection, and Radiation	9-14
1.5 – Thermodynamic Processes, Work, Heat, and Internal Energy	14-17
1.6 – Cyclic Thermodynamic Processes, Heat Engines, and Refrigerators	17-20
1.7 – Entropy and the 2nd Law of Thermodynamics	20-22
Unit 2 – Oscillations and Waves	pg. 22-42
2.1 – Simple Harmonic Oscillators	22-27
2.2 – Damped and Driven Oscillators	27-31
2.3 – Mechanical Wave Interference	31-38
2.4 – Electromagnetic Waves	38-42
Unit 3 – Circuits	pg. 42-57
3.1 – Time-Independent DC Circuits	42-47
3.2 – Time Dependent DC Circuits	47-52
3.3 – AC Circuits	52-57
Unit 4 – Electricity and Magnetism	pg. 58-86
4.1 – Electric Charge and Electric Field	58-62
4.2 – Gauss's Law	62-67
4.3 – Electric Potential and Potential Energy	67-70
4.4 – Capacitance and Dielectrics	70-72
4.5 – Magnetic Force, Torque, and Cyclotron Motion	72-75
4.6 – Magnetic Fields	75-80
4.7 – Electromagnetic Induction	80-84
4.8 – Electromagnetic Waves	84-86

Physics 157 Notes

1.1 – Temperature and Thermometry

Thermodynamics: how to summarize microscopic physics of 10^{23} things



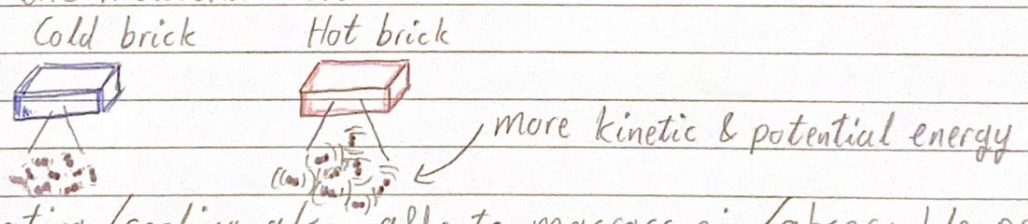
- We can summarize the average properties of lots of molecules by defining "macroscopic variables"
- Thermodynamics lets us understand the behavior of these properties

– Conservation of Energy

In practice:

- Each part of a physical system has a certain amount of energy
- The total energy of an isolated system doesn't change with time
- But: energy can move between different parts and take different forms.
- In thermodynamics, we care about the microscopic kinetic and potential energy of atoms and molecules.

When we heat/cool an object, we are adding/removing energy at the molecular level:

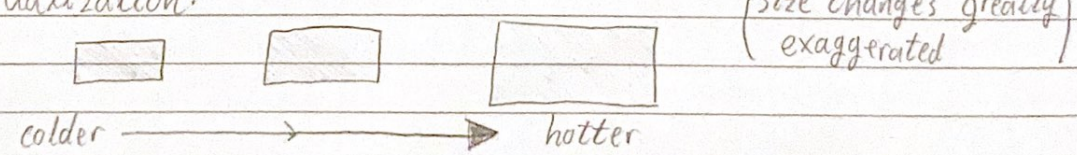


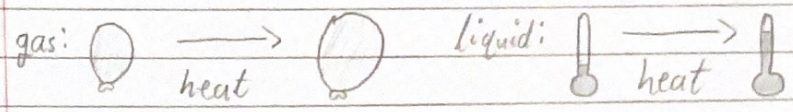
Heating/cooling also affects macroscopic/observable properties

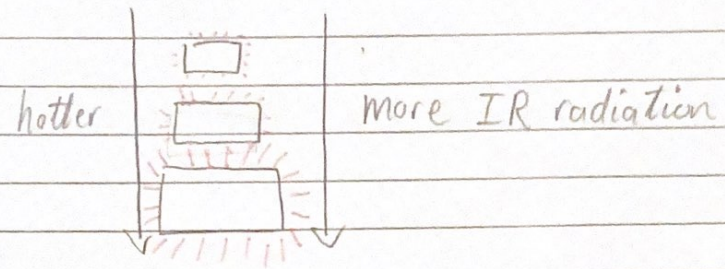
Examples:

- Size, volume, density
- Amount of light/IR radiation emitted at different frequencies
- Electrical conductivity + many others

Visualization:



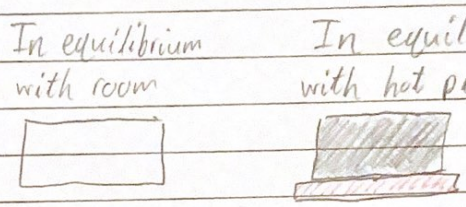




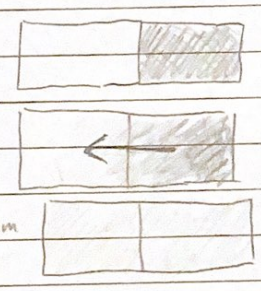
- hotter objects give off more infrared radiation
- hot enough objects emit higher frequency waves and move to the visible spectrum. This is why hot objects appear to glow.

— Heat Transfer

Ex: 2 blocks



- place them in thermal contact
- Heat flows from hot object to cold object
- Blocks are in equilibrium (same temperature and uniform temperature)



Zeroth Law of Thermodynamics:
If A and B are in equilibrium, and B and C are in equilibrium, then A and C are in equilibrium.

*note: temperature ≠ total energy of object
temperature = average kinetic energy of object

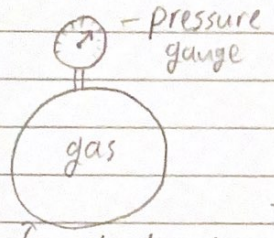
— Extensive vs. Intensive Properties

Extensive properties depend on the amount of matter that is present
Ex: mass, volume, energy, ...

Intensive properties do not depend on the amount of matter present
Ex: temperature, density, colour, taste, melting/boiling point, ...

We can assign a numerical value for different temperatures by using some temperature dependent macroscopic property of a standard object (e.g., volume of liquid in a tube)

Many materials → many possible defenitions of T



- simplest is based on the simplest material: ideal gas
- An ideal gas is one where the only interactions between the molecules are perfectly elastic collisions.
- Look at the changes in pressure for fixed volume

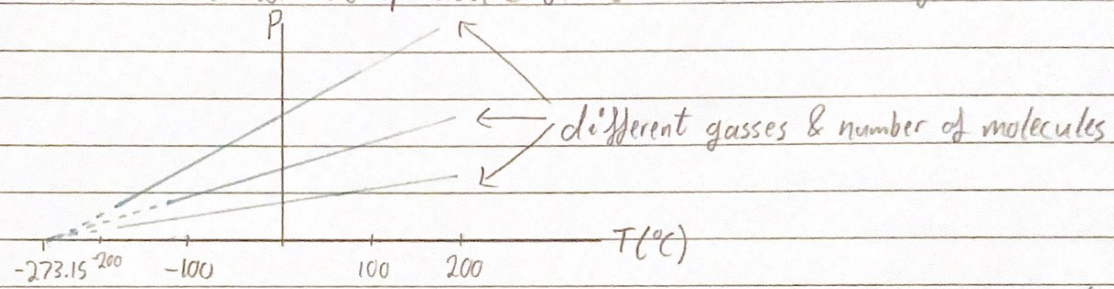
(constant volume gas thermometer

recall: $P = \frac{F}{A}$

so, the temperature is: $T = \text{const.} \times P$

* the constant depends on the particular thermometer

Pressure vs. Celcius temperature for constant volume gas thermometers



* Always extrapolates to P=0 at same T. This called absolute zero (0 Kelvin)

Kelvin scale: $P \propto T$ $\quad T_{kelvin} = T_{celcius} + 273.15$, $\quad T_{Fahrenheit} = \frac{9}{5} T_{celcius} + 32$

$T = \text{const} \times P$ means that $\frac{T_1}{T_2} = \frac{P_1}{P_2}$

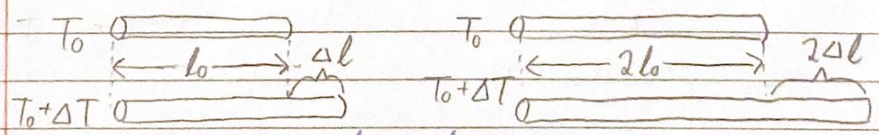
Triple Point of Water:

This is a specific temperature and pressure in which all three states (solid, liquid, gas) can coexist. $T_3 = 273.16K$, $P_3 = 611.66\,Pa$

1.2 — Thermal Expansion, Stress, and Strain

As we know, when objects heat up, they expand.

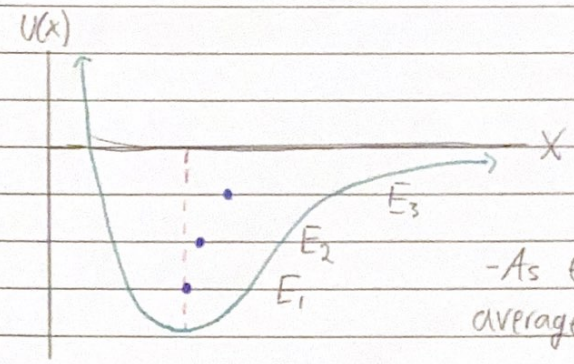
- For moderate temperature changes, $\Delta l \propto \Delta T$, and $\Delta l \propto l_0$



This gives us $\Delta l = \alpha l_0 \Delta T$

wher α = coefficient of linear expansion

Why do objects expand when heated?



x = distance between atoms
• = average distance between atoms

-As energy increases, so does the average distance between atoms.

Coefficients of linear expansion table:

Material	α value (K^{-1})
Aluminum	2.4×10^{-5}
Brass	2.0×10^{-5}
Copper	1.7×10^{-5}
Glass	$0.4\text{-}0.9 \times 10^{-5}$
Invar	0.09×10^{-5}
Quartz (fused)	0.04×10^{-5}
Steel	1.2×10^{-5}

Volume Expansion

Ex: When heated, each side of a cube expands by 0.1%. As a percentage of the original volume, the extra volume after the expansion is:

Original volume: $V_0 = l_0^3$

New volume: $V_0 + \Delta V = (l_0 + \Delta l)^3 = l_0^3 + 3l_0^2\Delta l + 3l_0(\Delta l)^2 + (\Delta l)^3$

$= V_0^3 + 3l_0^2\Delta l + 3l_0(\Delta l)^2 + (\Delta l)^3$

recall: $\Delta l = \alpha l_0 \Delta T$ $\qquad \Delta V = 3l_0^2\Delta l + 3l_0(\Delta l)^2 + (\Delta l)^3$

$\frac{\Delta l}{l_0} = \alpha \Delta T$ $\qquad \frac{\Delta V}{V_0} = 3\frac{\Delta l}{l_0} + 3\left(\frac{\Delta l}{l_0}\right)^2 + \left(\frac{\Delta l}{l_0}\right)^3 = 3(\alpha\Delta T) + 3(\alpha\Delta T)^2 + (\alpha\Delta T)^3$

$\therefore \frac{\Delta V}{V_0} = 3\alpha\Delta T$

these are negligible compared to the first term if $\frac{\Delta l}{l_0}$ is small.

let $3\alpha = \beta$ so,

$\Delta V = \beta V_0 \Delta T$ where $\beta = 3\alpha$

β = coefficient of volume expansion (K^{-1})

*This formula also applies to liquids

*Only works for small ΔT

Stress, Strain, and Young's Modulus

Tensile stress occurs when an object is stretched by forces acting at the ends, such as a guitar string

Compressive stress is similar, just the opposite sign (compression instead of extension)

Stress and Strain

- When you pinch your nose, the force per area that you apply to your nose is called stress.

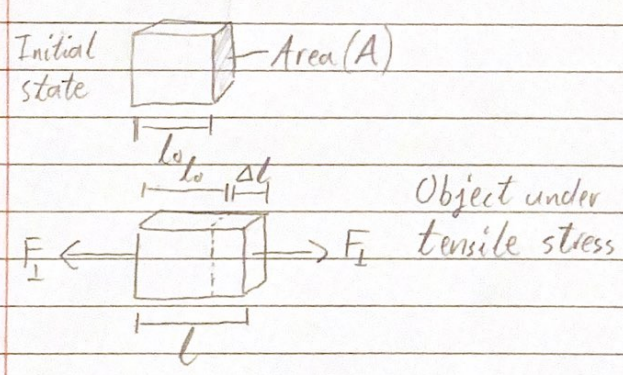
$$\text{stress} = \frac{F}{A} \quad (\text{units: } N/m^2 \text{ or } Pa)$$

- the fractional change in the size of your nose is called strain

$$\text{strain} = \frac{\Delta l}{l_0} \quad (\text{unitless})$$

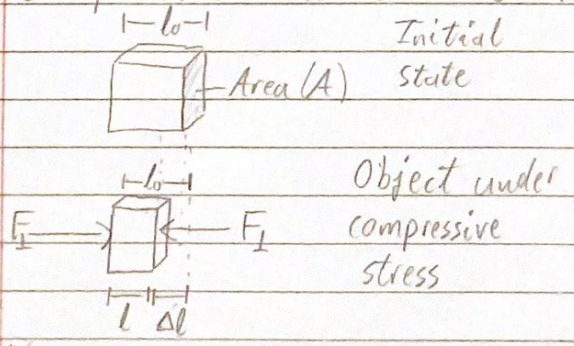
- the deformation is elastic if your nose springs back to its initial size when you stop pinching.

Tensile Stress and Strain



- The net force on the object is 0, but the object deforms
- The tensile stress produces a tensile strain

Compressive Stress and Strain



- the compressive stress and compressive strain are defined in the same way as tensile stress and strain, except that Δl now denotes the distance that the object contracts.

Young's Modulus

- Experiment shows that for a sufficiently small tensile or compressive stress, stress and strain are linearly proportional
- The corresponding elastic modulus is called Young's modulus, Y
- Y is a basic property of a material: resistance to stretching or compressing.

$$Y = \frac{\text{stress}}{\text{strain}} = \frac{(F/A)}{(\Delta l / l_0)} \qquad \frac{F}{A} = Y\frac{\Delta l}{l_0}$$

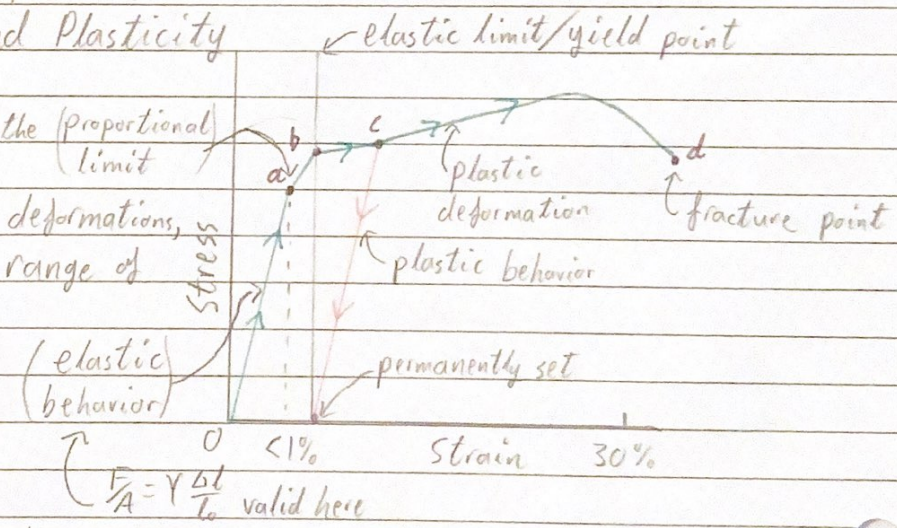
* Y is the same in all cases, it only depends on the material
* The larger the Y, the more force it takes to give some change in l.

Material	Young's Modulus (Pa)
Aluminum	7.0×10^{10}
Brass	9.0×10^{10}
Copper	11×10^{10}
Iron	21×10^{10}
Lead	1.6×10^{10}
Nickel	21×10^{10}
Silicone rubber	0.001×10^{10}
Steel	20×10^{10}
Tendon (typical)	0.12×10^{10}

- Y has units of pressure (Pa)
- It's a measure of how much pressure is required to produce a significant fractional change in length
- For example, a pressure of 0.01% of Y will give 0.01% compression.

Elasticity and Plasticity

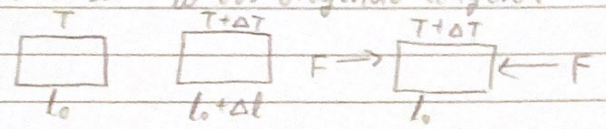
- Hooke's law, the proportionality of stress in elastic deformations, has a limited range of validity



Ex: If 0.2mm diameter nylon fishing line is good for catching fish up to 2kg, what thickness of line would you need to catch a 50kg fish?

$$\frac{F_1}{A_1} = \frac{Y\Delta l}{l_0} = \frac{F_2}{A_2} \qquad \frac{2g}{\pi(0.1)^2} = \frac{50g}{\pi(\frac{d}{2})^2} \Rightarrow \boxed{d = 1mm}$$

Ex2: A steel rod of length l_0 is heated by temperature ΔT. How would you determine how much stress is required to compress the rod back to its original length?

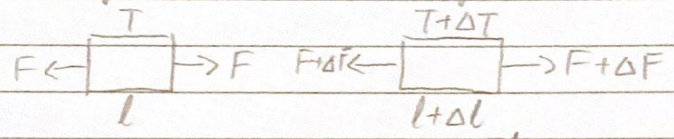


$\Delta l = \alpha l_0 \Delta T$

$\frac{F}{A} = Y\frac{\Delta l}{l_0}$

$$\frac{F}{A} = \frac{Y}{l_0}(\alpha l_0 \Delta T) = \boxed{Y\alpha\Delta T}$$

Ex3: A wire under a tension force of F and at temperature T initially has a length l. If we heat up the wire by ΔT and also change the tension force by ΔF, how would you determine how much the length of the wire changes?

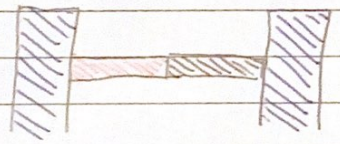


$$\Delta l = \Delta l_T + \Delta l_F = \alpha l_0 \Delta T + \frac{\Delta F l_0}{YA}$$

Ex4: 10m long steel tracks are laid end to end on a winter day (0°C). If no gaps are left between tracks, roughly how much force is generated at the ends when the temperature is 30°C. ($A = 0.01m^2$)

$\Delta F = -YA\alpha\Delta T = -(20\times10^{10})(0.01)(1.2\times10^{-5})(30) = -720\,000\,N$

Ex5: (harder problem)



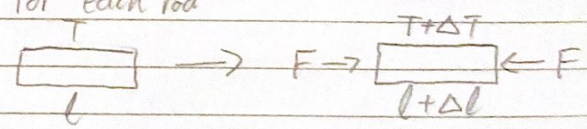
- A compound bar consisting of a copper rod with length 1m and area $2.00cm^2$ and a steel rod with the same dimensions. The rod is between 2 rigid walls with no stress at 20°C. Find force on each wall at 40°C.

$\alpha_s = 12\times10^{-6}$, $\alpha_c = 17\times10^{-6}$, $Y_s = 200\times10^9$, $Y_c = 110\times10^9$

Note: total length is fixed

$\therefore \Delta l_c + \Delta l_s = 0$

For each rod



- the bar is at equilibrium

$\therefore$ all forces on the bar must be equal

For each bar: $\Delta l = \Delta l_{expansion} - \Delta l_{compression}$

$$\Delta l_c = \alpha_c l \Delta T - \frac{Fl}{AY_c} \qquad \Delta l_s = \alpha_s l \Delta T - \frac{Fl}{AY_s}$$

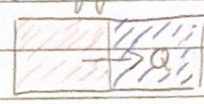
$\Delta l_c + \Delta l_s = 0$

$\alpha_c l \Delta T - \frac{Fl}{AY_c} + \alpha_s l \Delta T - \frac{Fl}{AY_s} = 0$

$\frac{Fl}{A}\left(\frac{1}{Y_c} + \frac{1}{Y_s}\right) = (\alpha_c + \alpha_s) l \Delta T \Rightarrow F = \frac{(\alpha_c + \alpha_s) A \Delta T}{\left(\frac{1}{Y_c} + \frac{1}{Y_s}\right)} = \boxed{8.2 \times 10^3 N}$

1.3 - Heat and Temperature / Phase Change

Q = heat (Joules): amount of energy transferred due to temperature differences



Heat required to raise the temperature of a material of mass m is determined by its specific heat, c:

$Q = mc\Delta T$, Q = heat added, m = mass, c = specific heat, ΔT = change in temperature

Units of c are J/kgK. c is a measure of the energy required to heat 1kg of material by 1K.

Heat required to raise temperature of a material of n moles is determined by its molar heat capacity, C:

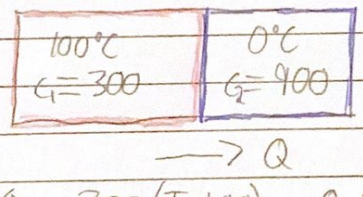
$Q = nC\Delta T$, n = # of moles, C = molar heat capacity (units: J/molK)

C = energy required to heat 1 mole of material by 1K.

Why is heat capacity larger for some materials?

Temperature is proportional to average kinetic energy of molecules, so for more complicated materials, part of energy added goes to rotations/vibrations, so it takes more Q to increase.

Ex: Two objects of mass = 1kg are put in thermal contact. If the initial temperatures are 100°C and 0°C, and the specific heats are 300 J/kgK and 900 J/kgK, calculate the equilibrium temperature.



Energy is conserved, so $Q_{in} = Q_{out}$

$Q_1 + Q_2 = 0$

$Q_1 = 300(T - 100)$, $Q_2 = 900(T - 0)$

$300T - 30000 + 900T = 0$

$\boxed{T = 25°C}$

Phase Changes

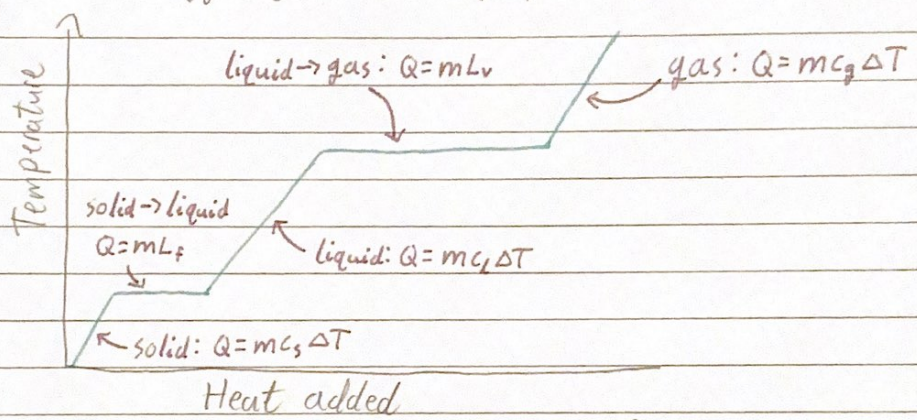
- At transition temperature, transition occurs due to heat added/removed with no temperature change
- Amount of heat required for transition per mass of material is called the latent heat

L_f = latent heat of fusion (freezing/melting) L_v = latent heat of vaporization (boiling/condensing)

Latent Heat

$$Q = mL$$

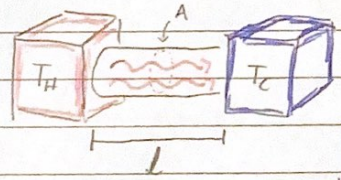
L is the energy required to change phase of 1kg of material



1.4 – Heat Flow: Conduction, Convection, and Radiation

Thermal Conductivity

- heat moves more quickly through some materials than others in response to a temperature gradient



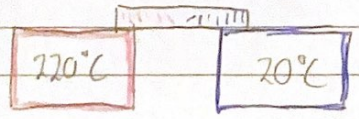
$$H = kA\frac{\Delta T}{l}$$

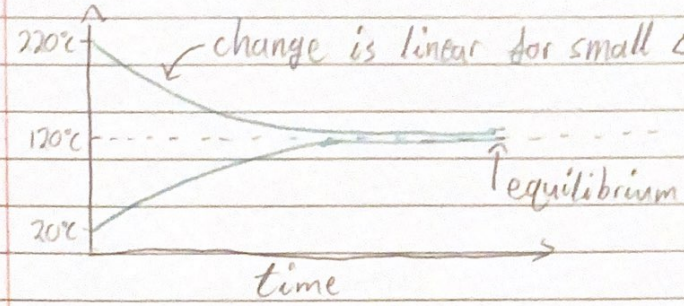
H = heat current $\left(\frac{\Delta Q}{\Delta t}\right)$, A = cross sectional area

k = thermal conductivity

$\frac{\Delta T}{l}$ = temperature gradient $\left(\frac{dT}{dx}\right)$

Ex: A block of aluminum at room temp. ($T_1 = 20°C$) is connected to an identical block with a strip of aluminium.





- Temperature will change more quickly at first, and more slowly as they approach equilibrium.

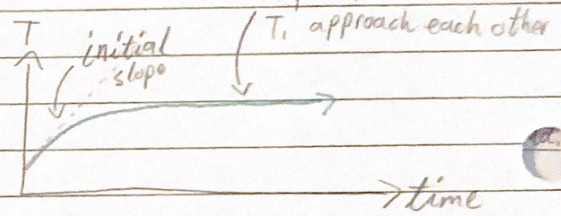
What determines the initial slope of the temperature vs. time?

$Q = mc\,dT_1 \qquad H = \frac{Q}{dt}$

$dT = \frac{Q}{mc} \qquad Q = H\,dt$

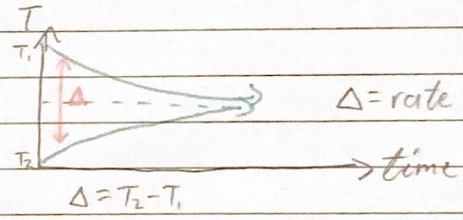
$dT = \frac{H\,dt}{mc} \qquad H = kA\frac{T_2 - T_1}{l}$

$\frac{dT_1}{dt} = \frac{H}{mc} = \frac{kA}{mcl}(T_2 - T_1)$



and $\frac{dT_2}{dt} = \frac{-kA}{mcl}(T_2 - T_1)$

so $\frac{d(T_2 - T_1)}{dt} = \frac{d\Delta}{dt} = \frac{-2kA}{mcl}\Delta$



Δ = rate of decrease

$\int \frac{d\Delta}{\Delta} = \int \frac{-2kA}{mcl}dt$

$\ln|\Delta| = \frac{-2kA}{mcl}t + C$

$\Delta = e^{\frac{-2kA}{mcl}t + C}$

τ = time constant

$\tau = \frac{mcl}{2kA}$

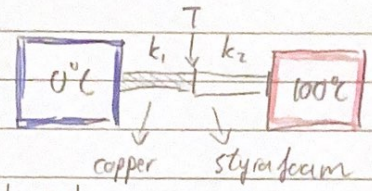
$\underline{\Delta(t) = \Delta_{t=0}e^{-t/\tau}}$ for small times, $\Delta(t) \approx -\Delta_{t=0}\, t/\tau$

Thermal Resistance: measures effectiveness of insulation layer. $R = \frac{l}{k}$ (units: $ft^2\,°F\,\frac{hrs}{BTU}$) Larger is better

Similarity to Ohms law

$\frac{H}{A} = \frac{T_H - T_C}{R} \qquad I = \frac{V_2 - V_1}{R}$

Similar to electrical resistance, multiple layers of materials add R-values (i.e. $R = R_1 + R_2 + R_3$)



would T feel warmer or colder than 50°C?
- colder.

$k_1 > k_2 \longleftarrow H$

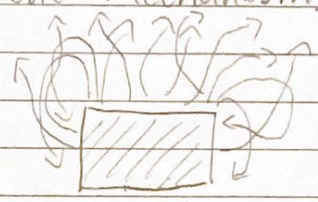
$H_1 = H_2$ due to conservation of energy

$$H_1 = k_1 A \frac{(T-0)}{l} = H_2 = k_2 A \frac{(100-T)}{l}$$

$$T = 100 \frac{k_2}{k_1 + k_2}$$

If k_1 is much larger than k_2 then T will be closer to 0°C.

Other mechanisms for heat transfer



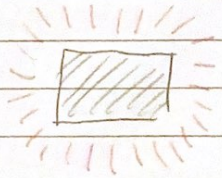
- Heat transfer via macroscopic motion of fluids
- Very complicated fluid dynamics to understand.

Convection

Free Convection
- Fluids have different densities at different temperatures
- Hot fluids rise/cold fluids sink
- Fluid circulates and carries heat with it.

Forced convection
- Forcing hot/cold fluids through tubing to transfer heat
- Seen in heating a house or cooling an engine.

Radiation

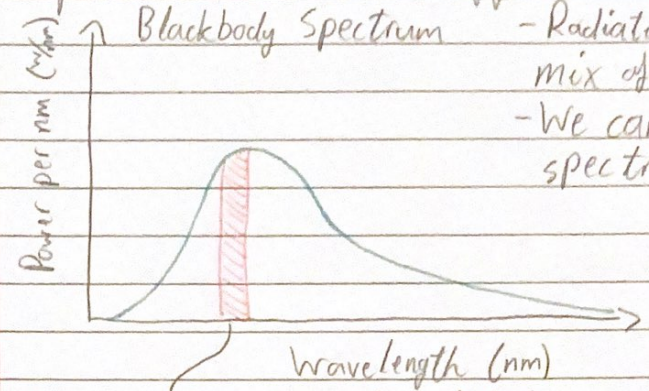


- All objects give off electromagnetic radiation (light, IR, etc)
- This carries energy away from the object
- They can also absorb radiation and gain energy.

Light carries energy
colour is determined by wavelength
Intensity/brightness is determined by amplitude

Thermal Radiation from an Object

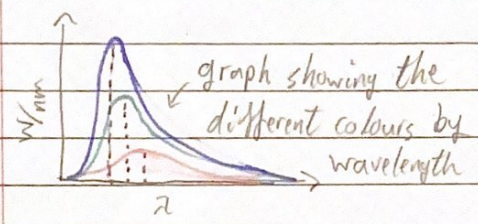
- Typically in IR / visible spectrum
- Spectrum measures energy current for various wavelengths



- Radiation from an object comes in a mix of wavelengths
- We can describe this by graphing the spectrum

→ area gives total power emitted in indicated range of wavelenths

The wavelength correlating to the peak of the graph usually indicates which colour we see.



Wien Displacement Law

$\lambda_{max} = \frac{b}{T}$, $b = 2.89772$ mmK

Stefan-Boltzmann Law

$H = A\varepsilon\sigma T^4$, A = Surface area, ε = emissivity

σ = Stefan-Boltzmann constant

$\sigma = \frac{2\pi^5 k_b^4}{15h^3c^2} = 5.67 \times 10^{-8} \frac{W}{m^2K^4}$

$\varepsilon = 1$ = perfect absorber (black)

$\varepsilon = 0$ = perfect reflector (mirror)

Emissivity

$\varepsilon = \frac{H}{H_{blackbody}}$

- perfect absorber = black body
- In general, ε depends on the material properties but also on temperature and wavelength

Net heat current from thermal radiation

$\Sigma H = A\varepsilon\sigma(T^4 - T_s^4)$, T_s = temperature of surroundings

- while an object radiates heat to its surroundings ($H_{emitted}$), it also absorbs heat from its surroundings ($H_{absorbed}$).
- In equilibrium, $T = T_s$, so $\Sigma H = 0$
- For positive H, heat flows out of the object, so $T > T_s$ in this case.

Ex: Estimate the surface temperature of Earth.

Earth is in a steady state heat flow: $H_{in} = H_{out}$

H_{in} = absorbed sunlight

H_{out} = IR radiation note ε of Earth is close to 1

First note:

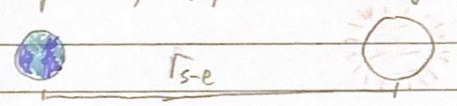
If a light bulb producing 100W of radiation is placed at the center of a sphere of $\varepsilon = 1$, and radius r. A hole is cut into the sphere, removing an area A of material. The rate of energy flow is: $100W \cdot \frac{A_{circle}}{SA_{sphere}} = \frac{100 \cdot A}{4\pi r^2}$

Albedo

- Albedo is the overall average reflection coefficient for solar radiation incident on an object
- Albedo of Earth is about 0.3 which means that the Earth as a whole reflects 30% of solar radiation.

So, if $\varepsilon_{sun} \approx 1$, then $H_{sun} = A_{sun}\sigma T_{sun}^4 = 4\pi r_s^2 \sigma T_s^4$

The power, H_{in}, absorbed by Earth is



Cross-sectional area of Earth is πr_e^2 $H_{in} = H_{sun}\frac{\pi r_e^2}{4\pi r_{s-e}^2}$

Fraction $(1-a_e)$ is absorbed

The Solar Constant

- At Earth's orbit, the power per unit area, or intensity of sunlight is:

$$I_{sc} = \frac{H_s}{4\pi r_{s-e}^2} = 1367\ W/m^2$$

I_{sc} = solar constant

so, $H_{in} = \pi r_e^2 (1-a_e) I_{sc}$ while $H_{out} = A_e \varepsilon_e \sigma T_e^4$

$$H_{in} = H_{out} = \pi r_e^2 (1-a_e) I_{sc} = 4\pi r_e^2 \varepsilon_e \sigma T_e^4$$

$$T_e = \sqrt[4]{\frac{(1-a_e) I_{sc}}{4\varepsilon_e \sigma}} \approx -18°C$$

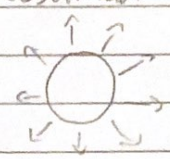
- this doesn't seem right

The actual surface temperature is larger due to the greenhouse effect

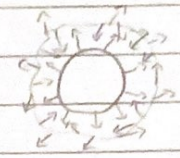
Greenhouse effect: some IR radiation is absorbed by greenhouse

gases and re-emitted back to earth.

We assumed: Actual:



$\varepsilon = 1$



$\varepsilon \approx 0.6$

- The CO_2 in the atmosphere re-emits radiation in all directions, includind back toward the earth and into space.

Using $\varepsilon = 0.6$ in the same calculation gives $T \cong 14.5°C$

- Global Warming
  - greenhouse gasses, especially CO_2, absorb IR radiation emitted by the the earth and re-emit it back to the earth, effectively reducing the earth's emissivity.
  - In addition, as Earth warms, white, sunlight reflecting ice in the Arctic dissappears, lowering our albedo and further increasing the temperature

Ex: (Temperature of Venus)

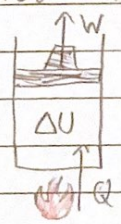
- Venus has albedo 0.750, effective $\varepsilon = 0.01$, radius 6052km, and is 0.72 times as close to the sun as Earth. Find the temperatue

$$I_{s-v} = \frac{H_s}{4\pi r_{s-v}^2} = \frac{r_{s-e}^2}{r_{s-v}^2}\frac{H_s}{4\pi r_{s-e}^2} = \frac{r_{s-e}^2}{r_{s-v}^2} I_{sc} = \frac{1367}{(0.72)^2} = 2637\ W/m^2$$

$$T_v = \sqrt[4]{\frac{(1-a_v) I_{sc-v}}{4\varepsilon_v \sigma}} = 464°C$$

1.5 - Thermodynamic Processes, Work, Heat, and Internal Energy

First Law of Thermodynamics



$\Delta U = Q - W$

ΔU = change in internal energy

$Q > 0$, system absorbs heat

$Q < 0$, system releases heat

$W > 0$, work done by the system

$W < 0$, work done on the system

Ideal Gas Pressure

→ double density ⇒ $2 \times P$

→ double mass ⇒ $2 \times P$

→ double velocity ⇒ $4 \times P$ (doubles collisions and twice as much force per impact)

$P = \text{const} \cdot \frac{n}{V} m\langle v \rangle = \text{const} \cdot \frac{n}{V} \langle K \rangle$

$P \propto \langle K \rangle$ and $P \propto T \Rightarrow T = \text{const} \cdot \langle K \rangle \Rightarrow P = \text{const} \cdot \frac{n}{V} T$

$\frac{n}{V}$ = molecular density, $\langle K \rangle$ = avg. kinetic energy per molecule

$PV = nRT$, $R = 8.31 \frac{J}{mol\,K}$ n = # of moles, R = gas constant

- tells us how much force a gas exerts on the wall

Work = energy transferred by a mechanical process

$W = F\Delta x$ → displacement in direction of force (constant F)

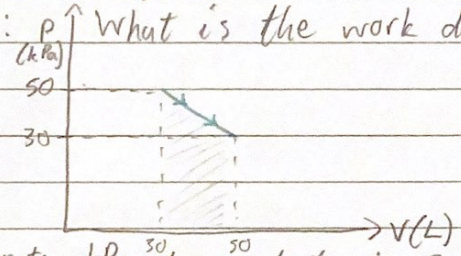
↳ force exerted

$P = F/A$, $\Delta V = A\Delta x$

so, $W = P\Delta V$ where P is constant

- Work is the area under a P-V graph

Ex: What is the work done



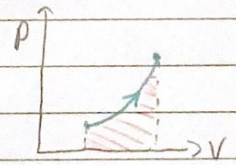
Area $= \frac{1}{2}(20)(20) + 30(20) = 800\,J$

* note kPa · L cancels to give J.

For a changing pressure, work can be expressed as:

$$W = \int_{V_1}^{V_2} P(V)\,dV$$

Ex: An ideal gas is heated and expands from 1L to 2L where the pressure is $P = aV^2$, $a = 100\,kPa/L^2$. How much work is done?



$$W = \int_1^2 aV^2\,dV = \frac{a}{3}V^3\Big|_1^2 = \frac{100}{3}(2^3 - 1^3) = 233\,J$$

The internal energy of a gas

- is a sum of: kinetic energy → (vroom!)

rotational energy

vibrational energy

electrostatic potential energy

$\Delta U = nC_v\Delta T$, C_v = molar specific heat (larger for more complex molecules)

Energy of a monatomic ideal gas:

U = total K of molecules
$U = N\langle K\rangle$, N = # of molecules
$\langle K\rangle = \text{const} \cdot T$, $\text{const} = \frac{3}{2}\frac{R}{N_A}$

$U = \frac{3}{2}nRT$ this also implies $C_v = \frac{3}{2}R$ for a monatomic ideal gas.

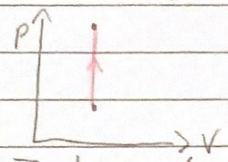
U is a state variable, meaning it only depends on the final and initial state, not the path to get there.

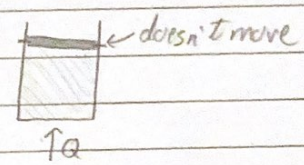
- Isochoric (constant volume)

$PV = nRT \rightarrow P \propto T$ so $\frac{T_2}{T_1} = \frac{P_2}{P_1}$

$W = P\Delta V = 0$

$\therefore \Delta U = Q$, $Q = nC_v\Delta T = \Delta U$

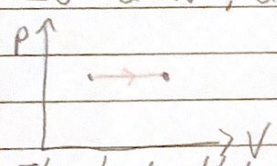


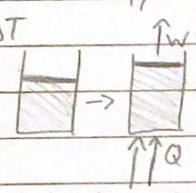


- Isobaric (constant Pressure)

$PV = nRT \rightarrow V \propto T$ so $\frac{T_2}{T_1} = \frac{V_2}{V_1}$

$\Delta U = Q - W$, $Q = nC_p\Delta T$





aside: $Q = \Delta U + W$, $W = P\Delta V = nR\Delta T$

$nC_p\Delta T = nC_v\Delta T + nR\Delta T$

$C_p = C_v + R$

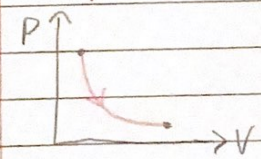
The heat added uses a heat capicity for constant pressure ($C_p > C_v$) as it has to equal the work and the internal energy.

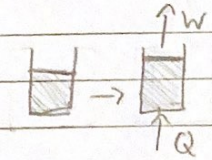
- Isothermal (constant temperature)

$PV = nRT \rightarrow PV = \text{const} \rightarrow P \propto \frac{1}{V}$

T is const so $\Delta U = 0 = nC_v\Delta T$

$\therefore W = Q$





$PV = nRT$

$P = \frac{nRT}{V}$

$W = \int_{V_i}^{V_f} P\,dV = \int_{V_i}^{V_f} \frac{nRT}{V}\,dV$

$W = (nRT\ln V)\Big|_{V_i}^{V_f}$

$W = nRT\ln(V_i) - nRT\ln(V_f)$

$W = nRT\ln\left(\frac{V_f}{V_i}\right) = Q$

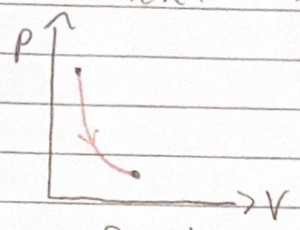
- Adiabatic (no heat transfer)

this happens in two cases

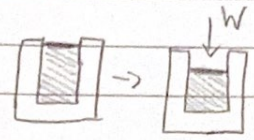
1) gas is well insulated
2) process happens very quickly (faster than H)

$PV = nRT \rightarrow PV \propto T \rightarrow \frac{PV}{T} = const$

$\Delta U = -W$



$P \propto \frac{1}{V^{\gamma}}$



$\Delta U = -W$ $\quad PV = nRT$

$nC_v dT = -PdV$ $\quad P = \frac{nRT}{V}$

$nC_v dT = -\frac{nRTdV}{V}$

$\frac{dT}{T} = \frac{R}{C_v}\frac{dV}{V}$

let $\gamma = \frac{C_p}{C_v}$ $\quad \frac{R}{C_v} = \frac{C_p - C_v}{C_v} = \gamma - 1$

$\ln T = -(\gamma-1)\ln V + const$

$\ln T + \ln V^{\gamma-1} = const$

$\ln TV^{\gamma-1} = const$

$TV^{\gamma-1} = const$

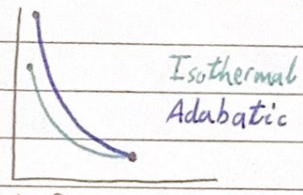
$TV^{\gamma-1} = const$

$\frac{PV}{nR}\frac{V^{\gamma}}{V} = const$

$PV^{\gamma} = const$

* for monatomic ideal gas, $\gamma = \frac{5}{3}$, for diatomic gas: $\gamma \approx 1.40$

Isothermal vs. Adabatic



Compression: adabatic does greater work on gas
Expansion: isothermal does more work

Ex: Gas with $C_v = 3R$, at STP, is compressed rapidly. The compression ratio is 15. What's the final temperature of the gas?

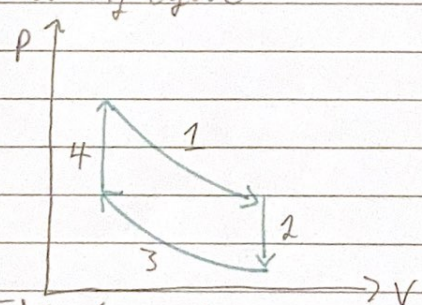
* assume adiabatic

$TV^{\gamma-1} = const \therefore T_1V_2^{\gamma-1} = T_2V_2^{\gamma-1}$ $\quad \gamma = \frac{C_p}{C_v} = \frac{C_v + R}{C_v} = \frac{4R}{3R} = 4/3$

$T_2 = T_1\left(\frac{V_1}{V_2}\right)^{\gamma-1} = T_1\left(\frac{V_1}{V_2}\right)^{1/3} = 723K$

1.6 - Cyclic Thermodynamic Processes

Stirling Cycle



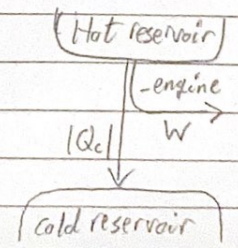
1. isothermal expansion ($\Delta U = 0$, $Q = W > 0$)
2. isochoric cooling ($W = 0$, $Q = \Delta U = nC_v\Delta T$)
3. isothermal compression ($\Delta U = 0$, $Q = W < 0$)
4. isochoric heating ($W = 0$, $Q = \Delta U = nC_v\Delta T$)

Efficiency of an engine

Q_H = heat absorbed by gas/cycle

Q_c = heat expelled by gas/cycle

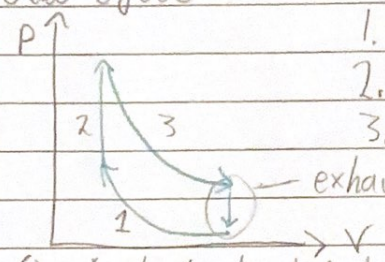
W = net work done/cycle



$W = Q_H + Q_c$

efficiency $= e = \frac{W}{Q_H} = \frac{\text{work we get out}}{\text{heat supplied}}$ $\quad e = 1 - \frac{|Q_c|}{|Q_H|}$

Otto cycle



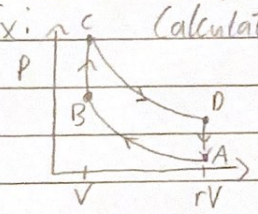
1. adiabatic compression ("compression stroke")
2. combustion of gas (heat at constant V)
3. adiabatic expansion ("power stroke")

exhaust stroke, then intake stroke (where the gas gets replaced)

Q_H is heat absorbed by the engine -> goes into work

Q_c is heat rejected by the engine

Ex: Calculate the efficiency in terms of T_A, r, $\frac{P_c}{P_B}$, $\frac{P_D}{P_A}$



$$T_B V_B^{\gamma-1} = T_A V_A^{\gamma-1}$$

$$T_B = T_A \left(\frac{V_A}{V_B}\right)^{\gamma-1} = T_A r^{\gamma-1}$$

$$\frac{T_c}{P_c} = \frac{T_B}{P_B} \Rightarrow T_c = \frac{P_c}{P_B} T_B = \frac{P_c}{P_B} T_A r^{\gamma-1}$$

Similarly, $T_c = T_D r^{\gamma-1} \Rightarrow T_D = \frac{P_D}{P_A} T_A$

$W_{C-D} = -\Delta U = -nC_v(T_D - T_c)$ $\qquad Q_H = \Delta U = nC_v(T_c - T_B)$

$W_{A-B} = -\Delta U = -nC_v(T_B - T_A)$

$W_{net} = nC_v(T_c - T_D + T_A - T_B)$

$$e = \frac{W_{net}}{Q_H} = \frac{T_c - T_D + T_A - T_B}{T_c - T_B} = 1 - \frac{(T_D - T_A)}{(T_c - T_B)} = 1 - \frac{1}{r^{\gamma-1}}$$

So the efficiency for an otto cycle is

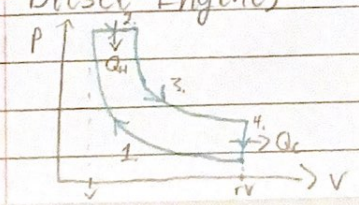
$$e = 1 - \frac{1}{r^{\gamma-1}}$$

meaning a higher efficiency for larger compression ratio $r = \frac{V_{max}}{V_{min}}$

But gas will spontaneously ignite is r is too large.

For real engines $r \sim 8\text{-}10$ and $\gamma \sim 1.22 \Rightarrow e \sim 38\%$

- Diesel Engines



1. Adiabatic compression
2. Fuel ignition
3. Adiabatic expansion
4. Cooling at constant volume

Because there is no fuel in ①, the compression stroke, pre-ignition cannot occur and r can be much higher.
This improves efficiency.
For diesel engines, $r \sim 15\text{-}23$ and $\gamma \sim 1.22 \Rightarrow e \sim 45\text{-}50\%$

Ex: In a diesel cycle, the combustion heat added is 3000 J and the heat expelled is 1800 J. What is the efficiency?

$$W_{net} = Q_{net} = 3000 - 1800 = 1200$$

$$e = \frac{W}{Q_H} = \frac{1200}{3000} = \underline{0.4}$$

Ex2: In a diesel engine, the net work is 1200 J. If a car with a 6-cylinder engine is running at 3000 rpm, how many horsepower is the engine? (1 kW = 1.33 hp)

Note: 1 cycle corresponds to 2 revolutions
(1 rev = heat transfer, work done)
- useful work only done on power stroke.

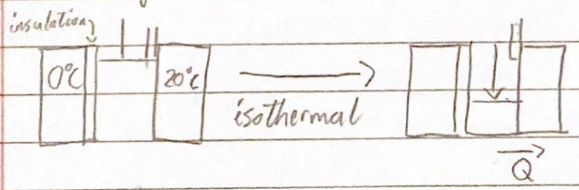
$*Q_H = W + Q_c$

$$P = \frac{W}{t} = 6 \cdot \frac{3000}{2} = 180000 \text{ kW} = \underline{240 \text{ hp}}.$$

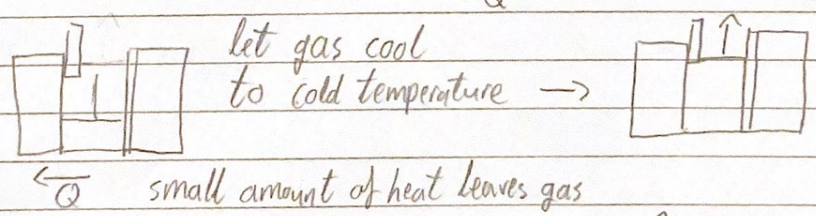
- Refrigerators
  * basically a reverse heat engine
  - inputs work to remove heat from a cold place

Coefficient of performance is $K = \frac{|Q_c|}{|W|} = \frac{\text{heat removed}}{\text{work put in}}$

Q_H is rejected to the outside air



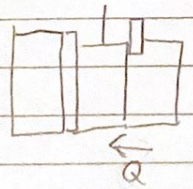
heat flows out of gas (large amount)



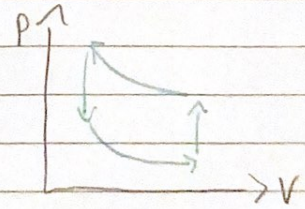
let gas cool to cold temperature →

gas expands isothermally

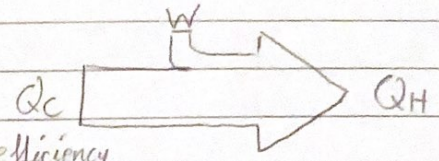
small amount of heat leaves gas



let gas reach hot temperature again



real systems have COP ~ 3 to 4



- Pros/Cons of Stirling Engines.

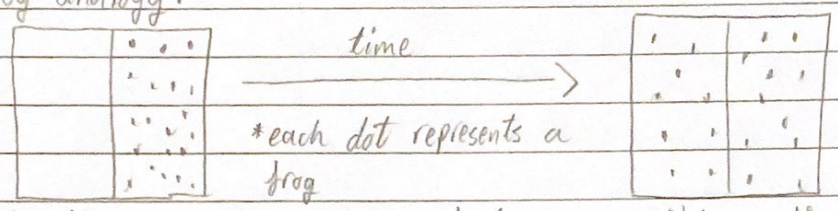
Pros: quiet, uses any heat source, high efficiency

Cons: high cost, large size, start/stops slow.

1.7 - Entropy and the 2nd Law of Thermodynamics

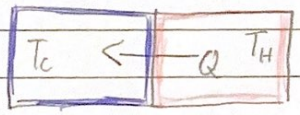
Why does heat always flow from hot objects to colder objects?

Frog analogy:



- As time passes, we move between possible configurations of frogs
- There is more possible combinations with the frogs spread out, meaning the frogs spread from a high density to a lower density

The frogs are analagous to energy.



Heat energy will flow from higher temp (higher energy density) to lower temp (lower energy density)

If there are 100 frogs on one side with 10^5 possible locations for each frog on that side, there are 10^{500} possible arrangments (or microstates)

If we double the area (both sides) with the same number of frogs, there are 2×10^5 locations/frog or 10^{530} possible microstates.

So, it is 10^{30} times more likely for the frogs to be spread out.

Entropy is a measure of how many possible microscopic configurations there are for a specified set of macroscopic variables.

$S = k_B \ln(N)$, N = # of microstates

→ derivation: for k molecules and E units of energy and $E \gg k \gg 0$

$$\binom{E+k-1}{k-1} = \frac{(E+k-1)!}{(k-1)!\,E!} \approx \frac{(E+k)!}{k!\,E!}$$

$$S = \ln\left(\frac{(E+k)!}{k!\,E!}\right) \qquad \text{note } \ln(x!) \approx x\ln x - x$$

$$S \approx (E+k)\ln(E+k) - (E+k) - k\ln k + k - E\ln E + E$$

$$S \approx E\ln(E+k) + k\ln(E+k) - k\ln k - E\ln E$$

$$S \approx E\ln\left(1+\frac{k}{E}\right) + k\ln\left(1+\frac{E}{k}\right) \approx k\ln\left(\frac{E}{k}\right)$$

Note: $\frac{E}{k}$ is energy per molecule which is temperature

so, $S = k \ln T$

$dS = k \cdot \frac{1}{T} dT = \frac{dQ}{T}$

$dS = \frac{dQ}{T}$, $Q = \int T dS$

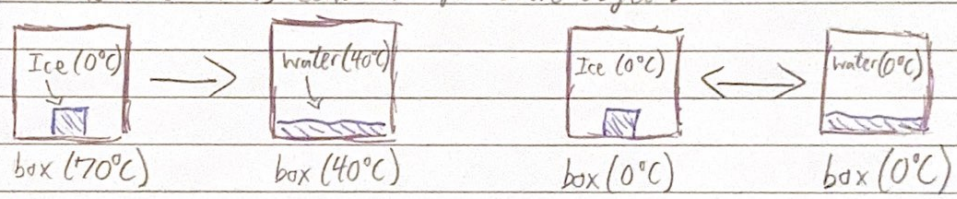
Hot $\longrightarrow$ Cold Cold $\longrightarrow$ Hot

↳ doesn't happen because much fewer microstates

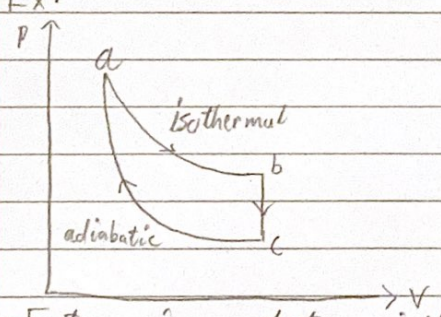
Ex: Suppose 1J of energy flows from a 100K object to a 400K object. What's the entropy change?

$$dS = dS_1 + dS_2 = \frac{dQ_1}{T_1} + \frac{dQ_2}{T_2} = \frac{-1}{100} + \frac{1}{400} = -0.0075 \text{ J/K}$$

- Thermodynamic processes are mostly irreversible
- ΔS is either 0 or positive. – total entropy never decreases
- Only two kinds of reversible processes
  1.) Adiabatic: no heat transfer ∴ no ΔS
  2.) Isothermal: T is constant for both objects



Ex:



ΔS for $c \to a$ is: $dQ = 0 \Rightarrow dS = 0$

ΔS for $a \to b$ is: $dS = \frac{dQ}{T} = \frac{Q}{T}$

ΔS for $b \to c$ is: $S_c = S_a$ because $\Delta S_{c \to a} = 0$

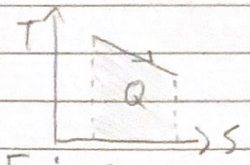
so, $\Delta S_{b \to c} = -\Delta S_{a-b} = -\frac{Q}{T}$

- Entropy is a state variable

  so, ΔS is 0 for a complete cycle, however, S for environment usually increases.
- Entropy is additive ("extensive") $S_{total} = S_1 + S_2$

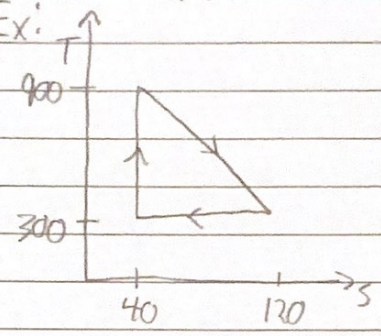
2nd Law of Thermodynamics: Total entropy of a system never decreases.

Temperature–Entropy Graphs:



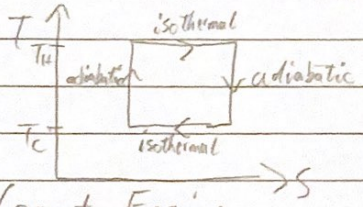
$dQ = TdS \Rightarrow Q = \int TdS$

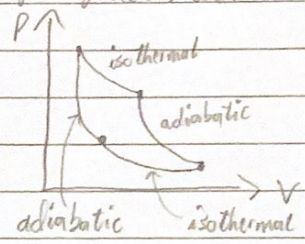
Ex:



- What is the net heat?
  $Q_{net} = \frac{1}{2}(600)(80) - (300)(80) = 24kJ$
- What is the net work?
  $\Delta U = 0$ for cycle $\therefore W = Q = 24kJ$
- What is the efficiency?
  $Q_{in} = \frac{1}{2}(600)(80) + (300)(80) = 48kJ$
  $e = \frac{W}{Q_{in}} = \frac{24}{48} = 0.5$

- Carnot Cycle
- Maximum possible efficiency for fixed max and min temperatures





Carnot Engine

$e = \frac{T_H - T_C}{T_H}$, $\kappa = \frac{T_C}{T_H - T_C}$

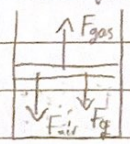
Efficiency is maximized by making T_H as hot as possible and T_C as cold as possible.

Unit 2 - Oscillations and Waves.

2.1 - Simple Harmonic Oscillators

Mechanical Equilibrium occurs when forces (and torques) on each part of the system add to zero.

Ex:



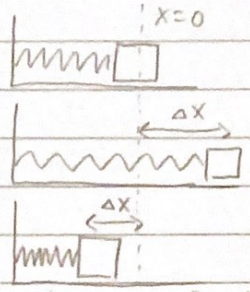
- piston is in equilibrium

If a system results in a net force, it is no longer in equilibrium and the system will fail.

This happens easily in an unstable equilibrium

Restoring Forces: For a stable equilibrium, a displacement in one direction leads to a net force in the other direction.

Ex: Mass on a spring.

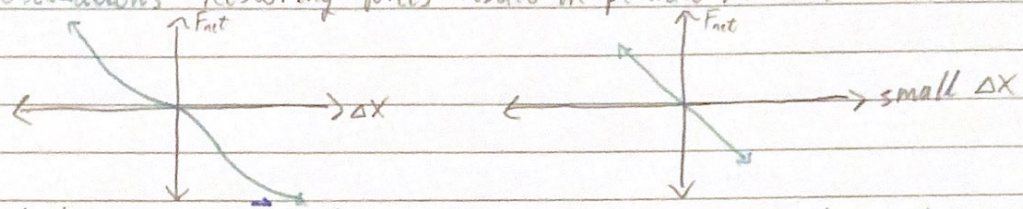


in equilibrium: $\Delta x = 0$

stretched: $\Delta x > 0$, $F_s < 0$

compressed: $\Delta x < 0$, $F_s > 0$

Oscillations: Restoring forces result in periodic motion.



Hooke's Law: $\vec{F}_r = -k\vec{x}$, F_r = restoring force, x = displacement

$F = -kx = ma$

$$a = \frac{d^2x}{dt^2} = \frac{-k}{m}x = -\omega^2 x \quad \text{where } \omega = \sqrt{\frac{k}{m}} = \text{angular frequency}$$

solving this gives $x(t) = A\cos(\omega t + \phi)$, A = amplitude, ϕ = phase

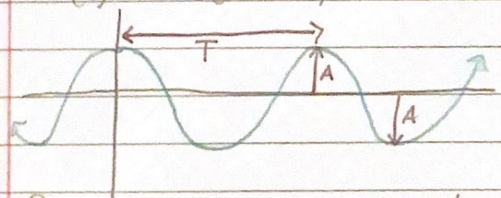
Periodic motion is any type of motion that repeats itself.

Simple Harmonic Motion:

A special case of periodic motion where the force is:

1) restoring back to an equilibrium position
2) proportional to the distance from equilibrium.

$x(t) = A\cos(\omega t + \phi)$



Period: T = time for 1 full cycle

Frequency: f = oscillations per second

$\omega = 2\pi f = \frac{2\pi}{T}$

Relationships between T, f, and ω

- Period, T: time from max → max
- Frequency, f, and period are related: $f = \frac{1}{T}$
- The units of frequency are cycles per second. A cycle is unitless, so the

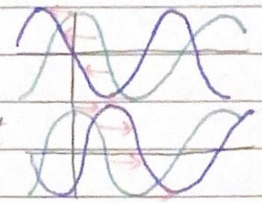
units of frequency are just inverse seconds, or Hertz (Hz)

- Angular frequency, ω, isn't necessarily related to the angular velocity; it is the time rate change of the phase measured in radians per second.

- Phase Shift, ϕ

Positive ϕ shifts left

$\phi = \pm 2\pi \cdot \frac{\text{shift}}{\text{period}}$



$x(t) = A\cos(\omega t + \phi)$

Negative ϕ shifts right

*Shift of 2π is a period.

The graph will shift by a magnitude of $\frac{\phi}{\omega}$

Velocity and Acceleration in SHM

$\vec{x}(t) = A\cos(\omega t + \phi_0)$

$\vec{v}(t) = -\omega A\sin(\omega t + \phi_0)$

$\vec{a}(t) = -\omega^2 A\cos(\omega t + \phi_0) = -\omega^2 x(t)$ = defenition of SHM.

How to find ϕ

$$\phi(\text{rad}) = \pm 2\pi \cdot \frac{\text{shift (sec)}}{\text{Period (sec)}}$$

at $t=0$ we know $x_0 = A\cos\phi$ and $v_0 = -\omega A\sin\phi$

this gives $\frac{v_0}{x_0} = -\omega\tan\phi \Rightarrow \phi = \arctan\left(\frac{-v_0}{\omega x_0}\right)$

similarly, we can do $x_0^2 = A^2\cos^2\phi$ and $v_0^2 = \omega^2 A^2 \sin^2\phi$

$$x_0^2 + \left(\frac{v_0}{\omega}\right)^2 = A^2(\cos^2\phi + \sin^2\phi) = A^2$$

so, $A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega}\right)^2}$

Ex: A 0.5kg mass on a spring has velocity $v(t) = -3.6\frac{m}{s}\sin\left(4.71t - \frac{\pi}{2}\right)$

Find: a) T, b) A, c) a_{max}, d) k

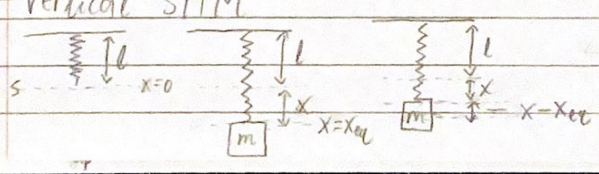
a) $T = \frac{2\pi}{\omega} = \frac{2\pi}{4.71} = 1.33s$

b) $-A\omega = -0.036 \Rightarrow A = \frac{0.036}{4.71} = 0.0076m$

c) $a_{max} = \omega A^2 = 0.17\ m/s^2$

d) $\omega = \sqrt{\frac{k}{m}} \Rightarrow k = \omega^2 m = 4.71^2 \cdot 0.5 = 11.1\ N/m$

- Vertical SHM

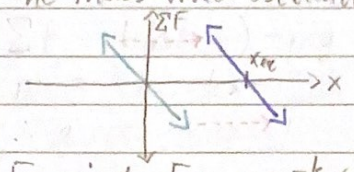


If we attach a mass, it stretches until $\Sigma F = F_{el} + F_g = 0$

$F_{el} = -kx$ and $F_g = mg \Rightarrow \Sigma F = mg - kx$

for $\Sigma F = 0 \Rightarrow mg = kx_{eq} \Rightarrow x_{eq} = \frac{mg}{k}$

The mass will oscillate about this new orgin $x = x_{eq}$



slope $= -k \propto \omega$

For just F_{el}, $a = -\frac{k}{m}x$ and $a = -A\omega^2 x$, so $\omega = \sqrt{\frac{k}{m}}$

For F_e and F_g, $\Sigma F = mg - kx \Rightarrow -\frac{dF}{dx} = k_{effective} = k$

k = constant and m = constant $\therefore$ ω = constant

- You will always have SHM when the force is:
  - restoring back to an equilibrium position
  - proportional to the distance from the equilibrium.

or $F_{SHM} = -\text{constant}(x - x_{eq})$

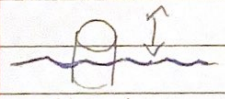
* For other systems, the dependence of the force on displacement can be different and this leads to different expressions for ω.

But you can still find the effective spring constant from $k = -\frac{dF}{dx}$ and $\omega = \sqrt{\frac{k}{m}}$

- General approach for finding ω

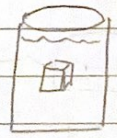
1. Find ΣF as a function of x
2. Find x_{eq} by solving $\Sigma F = 0$
3. Solve for k_{ef} by $k_{ef} = -\frac{dF(x_{eq})}{dx}$
4. $\omega = \sqrt{\frac{k_{ef}}{m}}$

Ex: Vertical bobbing



Will it SHM?

- The buoyant force



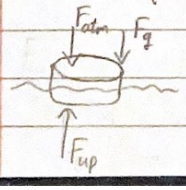
$P_{up} = P_{down}$ $\quad P_{down} = \frac{F_g}{A} + P_{atm}$ $\quad m = \rho V$

$= \frac{mg}{A} + P_{atm}$

$= \frac{\rho V g}{A} + P_{atm} = \rho g h + P_{atm}$

$P_{up} = \frac{F_{up}}{A} = \rho g h + P_{atm}$

$F_{up} = \rho g V_D + P_{atm} A$



note: in the diagram F_{atm} cancels so we can rewrite $F_g = F_b = \rho g V_D$

where V_D is the volume of fluid displaced.

so $F_b = \rho g V_D$

For vertical bobbing, we have:

$\Sigma F = mg - \rho g A h$

$mg = \rho g A h_{eq}$

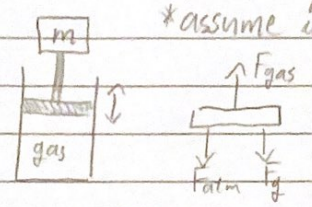
$h_{eq} = \frac{m}{\rho A}$

$k_{eff} = \frac{-dF(h_{eq})}{dx} = g\rho A$

$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{g\rho A}{m}}$

Ex2: Air Leg

*assume isothermal expansions



$F_{gas} = P_{gas} A$, $P_{gas} = \frac{nRT}{V} = \frac{nRT}{Ah}$

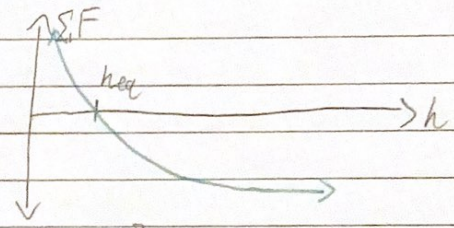
$F_{gas} = \frac{nRT}{h}$

$F_g = mg$, $F_{atm} = P_{atm} A$

$\Sigma F = \frac{nRT}{h} - P_{atm}A - mg$

$\frac{nRT}{h} = P_{atm}A + mg$

$h_{eq} = \frac{nRT}{P_{atm}A + mg}$

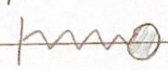


$k = \frac{-dF(h_{eq})}{dx} = \frac{nRT}{h_{eq}^2} = \frac{(P_{atm}A + mg)^2}{nRT}$

$\omega = \sqrt{\frac{k}{m}} = \frac{P_{atm}A + mg}{\sqrt{nRTm}}$

- Kinetic Energy in Simple Harmonic Motion

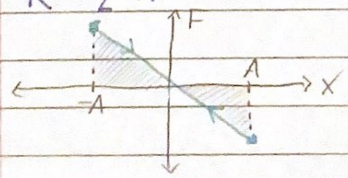
An oscillating object with mass m in simple harmonic motion moving with speed v has kinetic energy $\frac{1}{2}mv^2$



$x = -A$, $v = 0$, $K = 0$

$x = 0$, $v = v_{max}$, $K = \frac{1}{2}mv_{max}^2$

$K = \frac{1}{2}mv^2$



$W = \int F dx$

No matter where on the graph, work will always be $W = \frac{1}{2}kx^2$

This is same as elastic potential energy: $U_{el} = \frac{1}{2}kx^2$

Energy in SHM is always constant

$x=-A,\ K=0,\ U=\frac{1}{2}kA^2$

$x=0,\ K=\frac{1}{2}mv_{max}^2,\ U=0$

$x=A,\ K=0,\ U=\frac{1}{2}kA^2$

Ex: A 0.5kg mass is attached to a spring ($k=200\,N/m$). If the spring is compressed 0.1m and released, what is the speed at equilibrium?

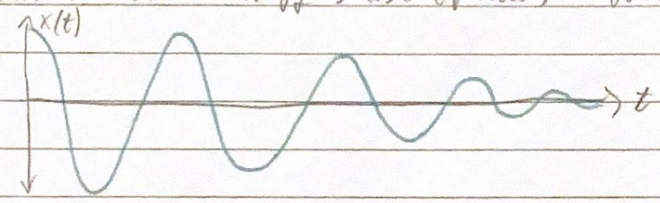
Initially, $v=0$, $x=-A=-0.1$

$E=U=\frac{1}{2}kA^2$

$E=K=\frac{1}{2}mv^2 \Rightarrow v=\sqrt{\frac{kA^2}{m}}=2\,m/s$ *For SHM, $E \propto A^2$

2.2 - Damped and Driven Oscillators

Real Oscillators: energy is lost (friction, air/fluid drag, heating of system)



the energy at any point of the curve is $E=\frac{1}{2}kx^2$

Common Situation: Amplitude decreases by same fraction each full oscillation

$t=0 \rightarrow A=A_0$

$t=T \rightarrow A=A_0 r$ $\quad r$ = fraction of decrease per period

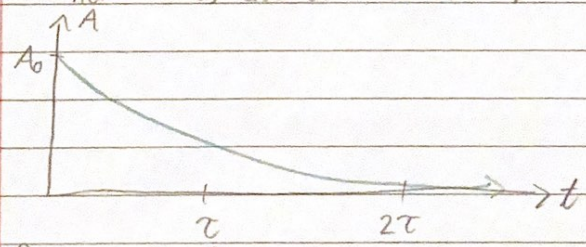
$t=2T \rightarrow A=A_0 r^2$

$t=3T \rightarrow A=A_0 r^3$

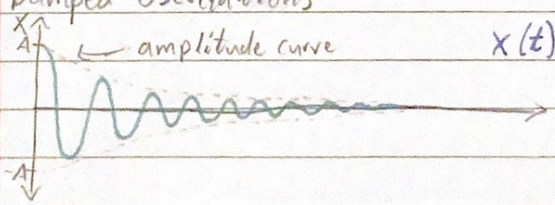
in general, $t \rightarrow A=A_0 r^{t/T}$ → number of periods

We can rewrite this as $A(t)=A_0 e^{-t/\tau}$ → gives $A_0 = e^{\ln r \cdot t/T} = A_0 r^{t/T}$

where τ is the time constant, $\tau = -\frac{T}{\ln(r)}$



Damped Oscillations



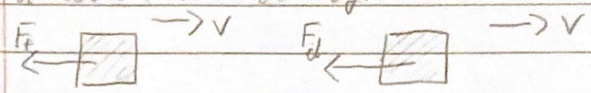
$x(t)=A_0 e^{-t/\tau}\cos(\omega t+\phi)$

Ex: If $A_0 = 8\,cm$ and at $t=2$, $A = 6\,cm$, what's the time constant?

$$A = A_0 e^{-t/\tau}$$

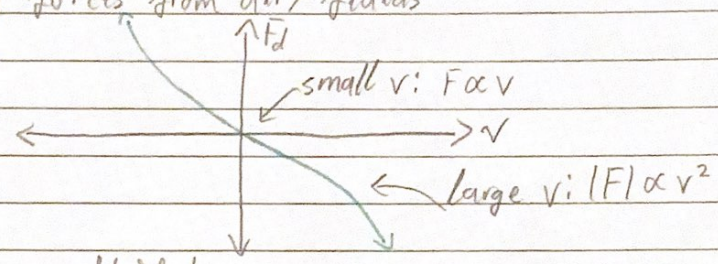
$$\frac{A}{A_0} = e^{-t/\tau} \Rightarrow \frac{-t}{\tau} = \ln\left(\frac{A}{A_0}\right) \Rightarrow \frac{-t}{\ln\left(\frac{A}{A_0}\right)} = \tau = \frac{-2}{\ln(3/4)} \approx 7.0\,s$$

Forces that lead to damping are velocity dependent and opposite in direction to velocity.

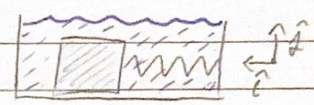


friction drag

Drag forces from air/fluids



Viscous fluid drag



$F_d = -bv$, b = damping constant

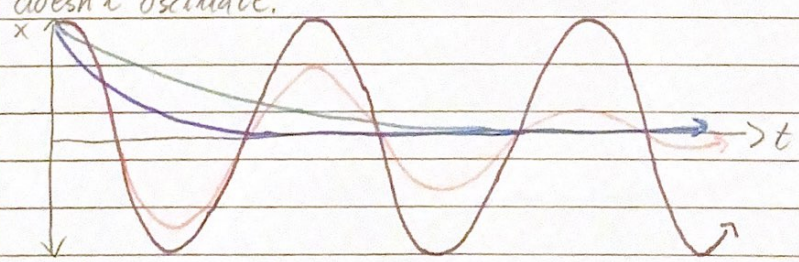
$$\sum F = -kx - bv$$

$$a = \frac{dv}{dt} = -\frac{k}{m}x - \frac{b}{m}v$$

Solution is: $x(t) = A_0 e^{-t/\tau}\cos(\omega t + \phi)$

$$\tau = \frac{2m}{b} \qquad \omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$ valid for $b < 2\sqrt{km}$

if $b > 2\sqrt{km}$ then ω is a complex number, implying that the system doesn't oscillate.



- no damping $b = 0$
- some damping $0 < b < 2\sqrt{km}$
- critical damping $b = 2\sqrt{km} \Rightarrow \omega = 0$, pure decay, no oscillations
- overdamping $b > 2\sqrt{km}$, slightly slower than critical damping

Ex: An object, mass 2kg, oscillates on a spring. $A_0 = 2$, $A = 1.55$ at $t = 2$

Find b. $A = A_0 e^{-t/\tau} \Rightarrow \frac{-t}{\tau} = \ln\left(\frac{A}{A_0}\right) \Rightarrow \tau = \frac{-t}{\ln\left(\frac{A}{A_0}\right)}$

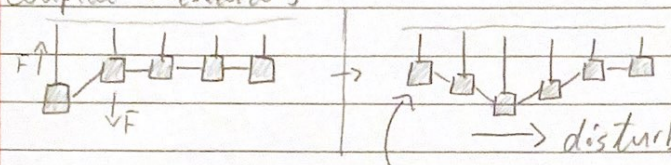
$\tau = \frac{2m}{b}$ $\quad b = \frac{2m\ln\left(\frac{A}{A_0}\right)}{-t} = \boxed{0.51\,kg/s}$

Find k. $\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} = \frac{2\pi}{T} \Rightarrow \frac{4\pi^2}{T^2} = \frac{k}{m} - \frac{b^2}{4m^2}$

$m\left(\frac{4\pi^2}{T^2} + \frac{b^2}{4m^2}\right) = k = 19.2\ N/m$

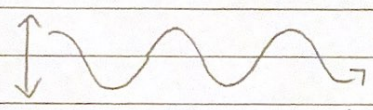
note: It's typically accurate to use $\omega = \sqrt{\frac{k}{m}}$ unless highly damped.

Coupled Oscillators



→ disturbance propogates along the chain

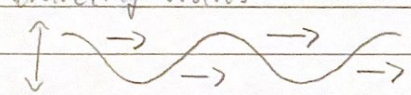
previous ones settle down to eq as energy is transferred



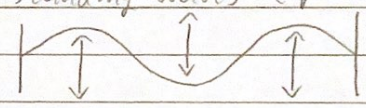
- If we drive the leftmost oscillator, we get a traveling wave (continuously adding energy to system)

Transverse wave: oscillations perpendicular to direction wave travels

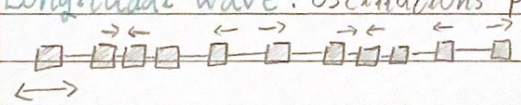
traveling waves



standing waves (system with boundaries)



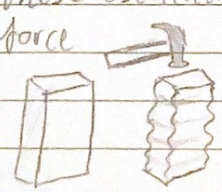
Longitudal wave: oscillations parallel to direction wave travels



Solids can be thought of as a large amount of coupled oscillators

When heat is added, each atom has small random oscillations about its equilibrium position

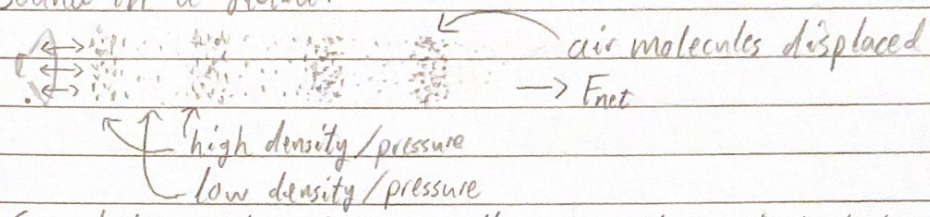
These oscillations can be coordinated through a macroscopic external force



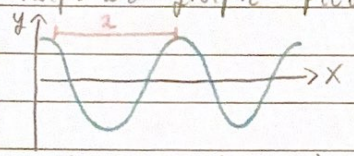
← longitudinal waves (sound waves in solid)

You can also have transverse waves in solids, such as in an earthquake.
- longitudal waves travel faster and are felt first
- transverse waves are slower and are felt after

Sound in a fluid:



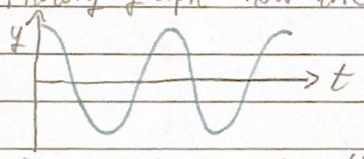
Snapshot graph: picture of the wave at an instant in time



λ = wavelength

$y(x) = A\cos(kx + \phi)$ $\quad$ k = wave number: $k = \frac{2\pi}{\lambda}$

History graph: how the wave moves with respect to time



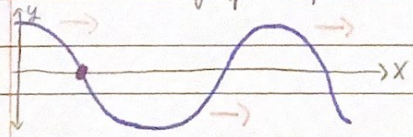
In equation $y = A\cos(kx + \phi)$, we can treat ϕ as a variable that changes with time (ωt).

$y(x,t) = A\cos(kx - \omega t + \phi_0)$ is the general equation for a right traveling wave.

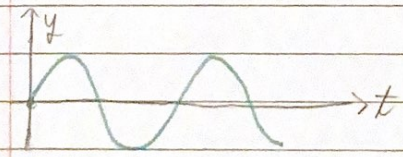
A left traveling wave has $A\cos(kx + \omega t + \phi_0)$ $\quad$ *recall $\omega = \frac{2\pi}{T}$

The amount a wave travels in one period is the same as one wavelength. This gives the wave speed as $v = \frac{\lambda}{T}$ or $v = \lambda \cdot f$ or $v = \frac{\omega}{k}$

Ex: What graph represents the motion of the dot on the wave?



* as the wave moves forward, the dot moves up the wave so the graph would look like:



* The history graph of a point will typically be the reflection of the snapshot graph in the vertical axis.

Ex2: If a wave has $v = 0.5\,m/s$, $\lambda = 3m$, $A = 1m$, what is the maximum vertical velocity?

$y = A\cos(kx - \omega t) = A\cos(\omega t - kx)$ $\qquad T = \frac{\lambda}{v} = 6s$

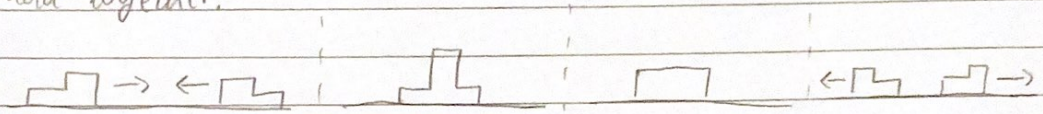
$\frac{\partial y}{\partial t} = v_y = -A\omega\sin(\omega t - kx)$ $\qquad \omega = \frac{2\pi}{T} = \frac{\pi}{3}$

$-A\omega \le v_y \le A\omega \Rightarrow v_{max} = A\omega = 1 \cdot \frac{\pi}{3} = \frac{\pi}{3}\,m/s$

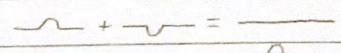
2.3 - Mechanical Wave Interference

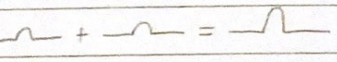
If $y_1(x,t)$ and $y_2(x,t)$ are allowed waves then $y_1(x,t) + y_2(x,t)$ is an allowed wave

This is called wave interference - when two waves overlap, their displacements add together.



The motion of each wave doesn't change.

Destructive interference is when two waves cancel: 

Constructive interference is when two waves add: 

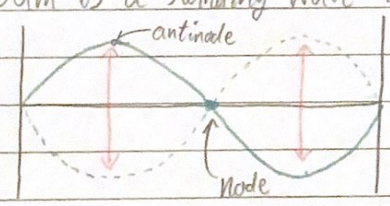
Standing waves: waves traveling opposite directions on a taut string.

*The waves will appear to bounce up and down.

right moving wave: $A\cos(kx - \omega t) = A\cos(kx)\cos(\omega t) + A\sin(kx)\sin(\omega t)$

left moving wave: $A\cos(kx + \omega t) = A\cos(kx)\cos(\omega t) - A\sin(kx)\sin(\omega t)$

sum is a standing wave: $2A\cos(kx)\cos(\omega t)$



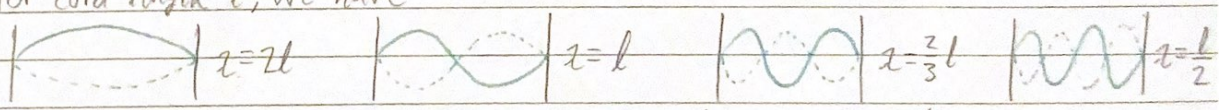
$y(x,t) = A\cos(kx)\cos(\omega t)$

node = points displacement is always zero

antinode = places of maximum displacement

For standing waves, there are only certain wavelengths that fit the constraints

For cord length l, we have



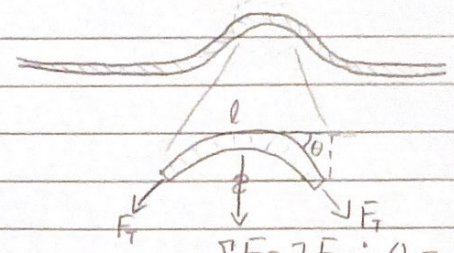
For a string fixed at both ends, the possible wavelengths are:

$\lambda_n = \frac{2l}{n}$ and possible frequencies are: $f_n = nf_1 = \frac{v}{\lambda_n} = \frac{nv}{2l}$ where $n \in \mathbb{N}$

f_1 is the fundamental frequency, f_2 is the second harmonic, f_3 is third harmonic.

These are also called normal modes

The Wave Equation:



- the tension on the rope pulls it down

corresponds to $2\sin\theta$ as a small l means small θ

$$\Sigma F = 2F_T\sin\theta = F_T l \frac{d^2y}{dx^2}$$ – also corresponds to θ. Gives the concavity.

$$a = \frac{\Sigma F}{m} = \frac{d^2y}{dt^2} = \frac{F_T l}{m}\frac{d^2y}{dx^2}$$ let μ be linear mass density, where $\mu = \frac{m_c}{l}$

$$\frac{\partial^2 y}{\partial t^2} = \frac{F_T}{\mu}\frac{\partial^2 y}{\partial x^2}$$ – gives general solution which is the sum of $F_1(x-vt)$ and $F_2(x+vt)$ with $v = \sqrt{\frac{F_T}{\mu}}$

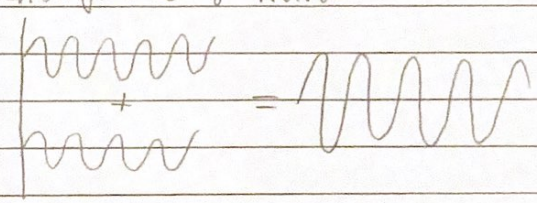
This means v only depends on the properties of the string (not λ or f)

Ex: A crane lifts a 2500kg mass using a 30m long cable that weighs 19.5kg. What is the speed of the transverse waves?

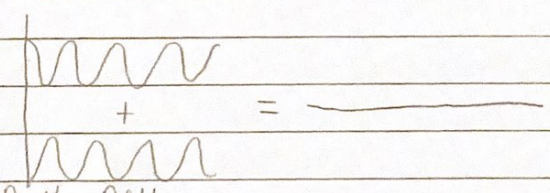
$F_T = F_g = m_m g$ $\quad \mu = \frac{m_c}{l} = \frac{19.5}{30} = 0.65\ kg/m$

$$v = \sqrt{\frac{F_T}{\mu}} = \sqrt{\frac{2500}{0.65}} = \underline{194\ m/s}$$

Interference of waves



- constructive interference
- 0° phase difference
- in phase

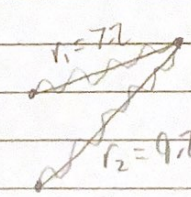


- destructive interference
- 180° phase difference
- out of phase

Path Difference

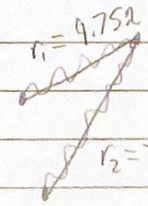
- Waves interfere constructively if their path lengths differ by an integer number of wavelengths. Half integer gives destructive.

$r_2 - r_1 = m\lambda$



$r_2 - r_1 = 2\lambda$

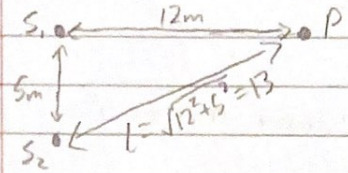
∴ constructive



$r_2 - r_1 = -2.5\lambda$

∴ destructive

constructive: $r_2 - r_1 = m\lambda$, $m \in \mathbb{Z}$
destructive: $r_2 - r_1 = m\lambda + \frac{\lambda}{2}$, $m \in \mathbb{Z}$
Ex: The speed of sound is ≈ 340 m/s in air, what is the lowest frequency for constructive interference at P? (assume start in phase)



$\Delta r = 13 - 12 = 1m = n\lambda \Rightarrow \lambda = \frac{1}{n}$

$f = \frac{v}{\lambda} = 340n$

lowest frequency (f_1) corresponds to $n = 1$ $\therefore f_1 = 340$ Hz

Wave Intensity: average power per unit area
- For a plane wave, the intensity stays the same as the wave travels
However, if the waves spread out uniformly in all directions, the intensity falls off like $1/r^2$, so $I \propto 1/r^2$

- imagine a ripple in a pond.

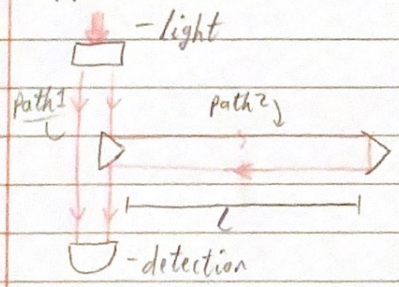
power $= P = \frac{1}{2}\mu\omega^2 A^2 v$

$I = \frac{\mu\omega^2 A^2 v}{8\pi r^2}$ - for spherical area

- We won't need these equations for this course

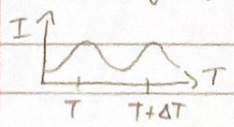
Ex: How does the amplitude of sound waves 20m away from the source compare with those 10m away?
$I \propto 1/r^2$ so $I_2 = \frac{1}{4} I_1$
$I \propto A^2$ so $A_2^2 = \frac{1}{4} A_1^2 \Rightarrow A_2 = \frac{1}{2} A_1$

Application of interfering waves: interferometer



- can detect very small changes in waves such as light waves and IR radiation

Ex: with $l = 1m$, you are measuring an aluminium block. The light has $\lambda = 650$ nm. How much does the block need to be heated in order for the detected intensity to go from one max to the next? ($\alpha_{al} = 2.4 \times 10^{-5}\ K^{-1}$)
The change from one max to the next is $1\lambda = 650$nm



$\Delta l = \alpha l_0 \Delta T$ $\quad 2\Delta l = \lambda$ (2 because of round trip)

$\Delta T = \frac{\Delta l}{\alpha l} = \frac{325 \times 10^{-9}}{(2.4 \times 10^{-5})(1)} = 0.0135K$

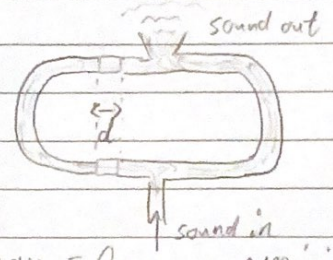
Inferometers are incredibly sensitive. Large enough ones were used to measure gravity waves ($\Delta l = 4 \times 10^{-18} m$) or 0.001 the size of a proton!

Physics 158 Notes

Tyler Wilson

*class picks up from pg. 31 (superposition, path difference, intensity)

Ex: Cancellation trombone



- by moving the side, and changing the length d, (such as a trombone), it changes the path length to give the following measurements. Use these to estimate the speed of sound.
- use $f = 3kHz$

$max_0 = 0cm$
$min_0 = 3.4cm$
$max_1 = 5.4cm$
$min_1 = 9.3cm$

• minimums are easier to experimentally determine, so we will use those.

$\Delta l = 2d$, Δl = change in path length

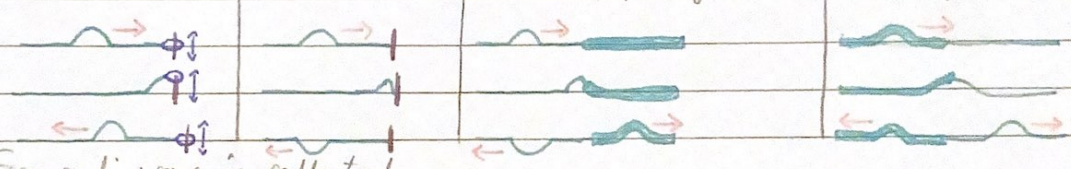
$\Delta min = 5.9cm = d$

$\Delta l = 2(5.9cm) = 10.8cm = \lambda$

$v_s = \lambda f = 354 Hz$ note this is an experimental value.

Transverse wave reflections

- Free End	Fixed End	low density → high density	high density → low density



Free end: wave is reflected

Fixed end: wave is reflected and inverted

low ρ → high ρ: part of transmitted wave is reflected and inverted

high ρ → low ρ: part of transmitted wave is reflected

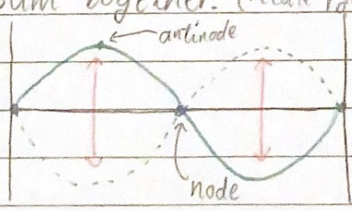
- the amplitudes for waves with different densities can be described by:

$$A_r = \frac{Z_1 - Z_2}{Z_1 + Z_2} A_0 \quad \text{and} \quad A_t = \frac{2Z_1}{Z_1 + Z_2} A_0$$

where A_0 is the original amplitude, A_r is the reflected amplitude, and A_t is the transmitted wave amplitude

Z = the characteristic impedance of the material, where $Z = \rho v$

a standing wave can be formed when waves with fixed end boundaries sum together. (recall pg. 31)



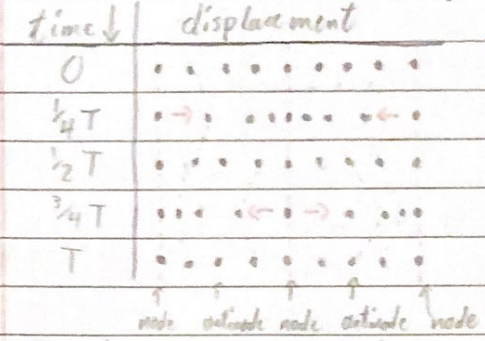
standing wave = transmitted wave + reflected wave

$SW = y_1(x,t) + y_2(x,t) = A\cos(kx - \omega t) - A\cos(kx + \omega t)$
$= 2A\sin(kx)\sin(\omega t)$

so, for standing waves, $y(x,t) = A_{sw}\sin(kx)\sin(\omega t)$

Longitudinal Standing Waves

- a similar idea but with gases and fluids



note that displacement nodes = pressure antinodes.
This makes sense becease if we have an open tube, the pressure at the end of the tube must equal the atmospheric pressure

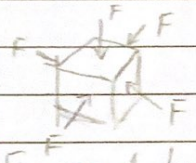
for longitudal waves we have the equation $D(x,t) = A\cos(kx - \omega t)$ where D and x are colinear.

to get the equation for pressure, we can express volume as $V = A\Delta D$

$$V = A(D_1(x_1,t) - D_2(x_2,t)) = A(D(x+\Delta x,t) - D(x,t))$$

$$\frac{dV}{V} = \lim_{\Delta x \to 0} \frac{A(D(x+\Delta x,t) - D(x,t))}{A\Delta x} = \frac{\partial D(x,t)}{\partial x}$$

Bulk stress and strain:



- applied pressure causes the box to shrink

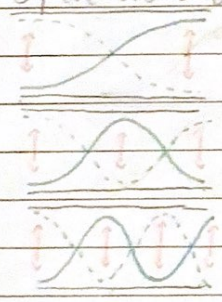
Expressed by $\Delta P = -B\frac{\Delta V}{V_0}$, where B is the bulk modulus.

we can use this to get, $P(x,t) = -B\frac{dV}{V} = \frac{-B\,\partial D(x,t)}{\partial x}$

so, $P(x,t) = BkA\sin(kx - \omega t)$

Standing Waves in tubes

Open at both ends

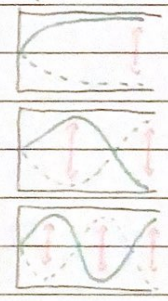


1st harmonic $f_1 = \frac{v}{2l}$

2nd harmonic (first overtone) $f_2 = \frac{2v}{2l} = 2f_1$

3rd harmonic (2nd overtone) $f_3 = \frac{3v}{2l} = 3f_1$

Open at one end



1st harmonic $f_1 = \frac{v}{4l}$

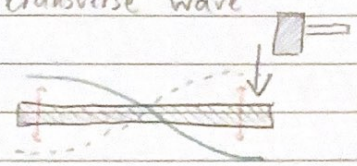
3rd harmonic (first overtone) $f_3 = \frac{3v}{4l} = 3f_1$

5th harmonic (2nd overtone) $f_5 = \frac{5v}{4l} = 5f_1$

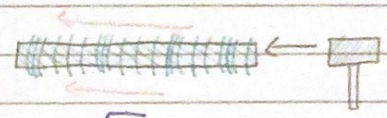
Waves through a solid

- depending on how we hit the rod, it can have either transverse, or longitudinal waves, or even both.

transverse wave

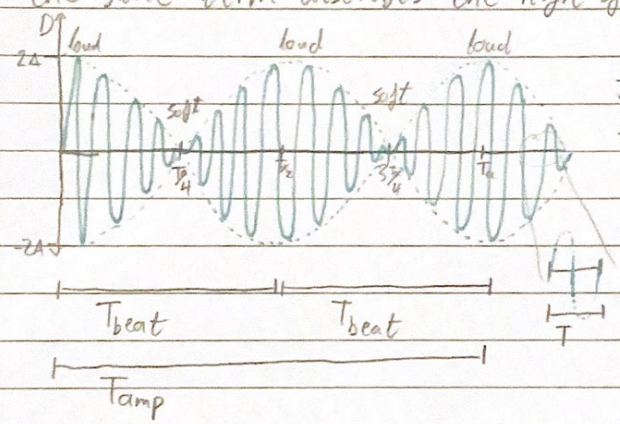


longitudinal wave



$v_T = \sqrt{\frac{S}{\rho}}$ $\qquad$ $v_L = \sqrt{\frac{Y}{\rho}}$

where S = shear modulus and Y = young's modulus
note this is different for solids vs. strings. $v_{string} = \sqrt{\frac{F_T}{\mu}}$
Waves with slightly mismatched frequencies cause beats
The wave resulting from this interference can be expressed as
$D = D_1 + D_2 = A(\sin(\omega_1 t) + \sin(\omega_2 t))$ for any point x.
$D(t) = 2A\cos(\frac{1}{2}(\omega_1 - \omega_2)t)\sin(\frac{1}{2}(\omega_1 + \omega_2)t)$
the cosine term describes the amplitude curve
the sine term describes the high-frequency change between waves.



the total wave has an oscillation frequency of $\omega = \frac{1}{2}(\omega_1 + \omega_2)$
the amplitude changes with an oscillation frequency of $\omega_{amp} = \frac{1}{2}(\omega_1 - \omega_2)$

Your ears hear intensity which is amplitude squared so we hear maximums twice as often. $f_{beat} = 2f_{amp}$, $T_{beat} = \frac{1}{2}T_{amp}$
Also what our ears hear is the average frequency modulated by the beat frequency: $f_{av} = \frac{\frac{1}{2}(\omega_1 + \omega_2)}{2\pi} \Rightarrow f_{avg} = \frac{1}{2}(f_1 + f_2)$

$f_{beat,av} = 2f_{amp,av} = 2(\frac{1}{2})\frac{(\omega_1 - \omega_2)}{2\pi} = \frac{\omega_1 - \omega_2}{2\pi} \Rightarrow f_{beat} = f_1 - f_2$

Ex: You hear a sound with $f = 256$ Hz. It takes 2 seconds to go from loud to soft to loud. If the wave is the sum of two waves, what are the frequencies?

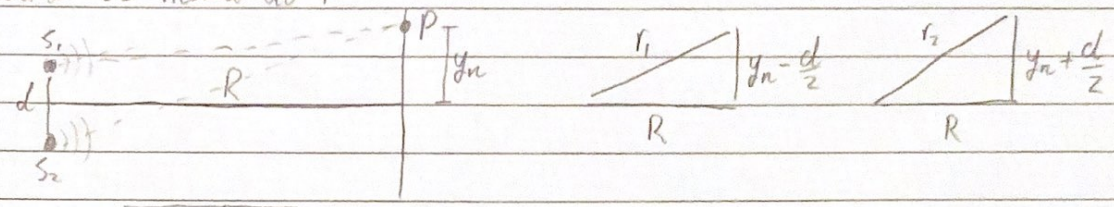
$f_{avg} = \frac{f_1 + f_2}{2} = 256 \Rightarrow f_1 + f_2 = 512$

$f_{beat} = f_1 - f_2 = \frac{1}{T_{beat}} = \frac{1}{2} \Rightarrow f_1 - f_2 = 0.5$

adding together gives, $2f_1 = 512.5 \Rightarrow \underline{f_1 = 256.25Hz} \therefore \underline{f_2 = 255.75Hz}$

Interference

Ex: Two speakes emit similar soundwaves. For what values, y_n, will no sound be heard at P

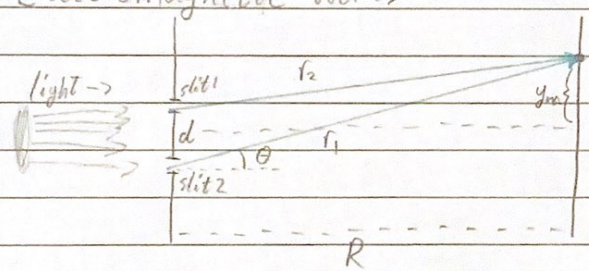


$r_1 = \sqrt{\left(y_n - \frac{d}{2}\right)^2 + R^2} \quad r_2 = \sqrt{\left(y_n + \frac{d}{2}\right)^2 + R^2}$

destructive interference happens for $\Delta r = \left(1 + \frac{n}{2}\right)\lambda$

$\Delta r = \sqrt{\left(y_n + \frac{d}{2}\right)^2 + R^2} - \sqrt{\left(y_n - \frac{d}{2}\right)^2 + R^2} = \left(1 + \frac{n}{2}\right)\lambda$

2.4 - Electromagnetic Waves



← Double Slit Experiment

in reality, $R \gg d$

- bright bands will appear for constructive interference and dark spots for destructive

some useful equations can be derived, relating these quantities

Method 1:

from before, we have $r_2 = \sqrt{\left(y_m + \frac{d}{2}\right)^2 + R^2}$, $r_1 = \sqrt{\left(y_m - \frac{d}{2}\right)^2 + R^2}$

$r_2 = \sqrt{y_m^2 + R^2 + \frac{d^2}{4} + dy_m}$, $r_1 = \sqrt{y_m^2 + R^2 + \frac{d^2}{4} - dy_m}$

let $\frac{d^2}{4} \pm dy_m = \varepsilon$ where $|\varepsilon| \ll 1$

$r_2 = \sqrt{y_m^2 + R^2 + \varepsilon}$, $r_1 = \sqrt{y_m^2 + R^2 + \varepsilon}$

we can use a 2nd order taylor polynomial to approximate r with $\varepsilon = 0$

$r_2 \approx \sqrt{y_m^2 + R^2} + \frac{1}{2\sqrt{y_m^2 + R^2}}(\varepsilon - 0) \quad r_1 \approx \sqrt{y_m^2 + R^2} + \frac{1}{2\sqrt{y_m^2 + R^2}}(\varepsilon - 0)$

$\Delta r \approx \frac{\frac{d^2}{4} + dy_m}{2\sqrt{y_m^2 + R^2}} - \frac{\frac{d^2}{4} - dy_m}{2\sqrt{y_m^2 + R^2}} = \frac{dy_m}{\sqrt{y_m^2 + R^2}}$

In the case of constructive interference: $\Delta r = m\lambda$

$$\frac{d\,y_m}{\sqrt{y_m^2+R^2}} \approx m\lambda = \Delta r$$

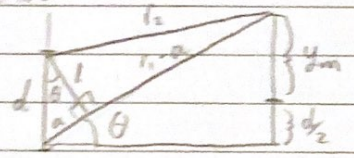
$$\Rightarrow d \approx \frac{m\lambda\sqrt{y_m^2+R^2}}{y_m} = m\lambda\sqrt{1+\frac{R^2}{y_m^2}}$$

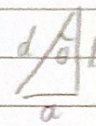
for $R \gg y_m$, $y_m \approx \frac{Rm\lambda}{d}$

$$d \approx m\lambda\sqrt{1+\left(\frac{R}{y_m}\right)^2} \quad \text{for } R \gg d$$

we also get from the same equation: $y_m \approx \frac{Rm\lambda}{\sqrt{d^2-m^2\lambda^2}}$

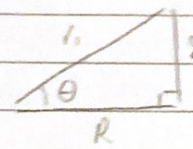
Method 2:



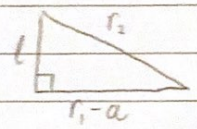


$a = d\sin\theta$

$d^2 = a^2 + l^2$



$r_1^2 = R^2 + (y_m + \frac{d}{2})^2$



$r_2^2 = l^2 + (r_1-a)^2$

$$r_2^2 = l^2 + (r_1+a)^2 = (d^2-a^2) + (r_1^2 + 2ar_1 + a^2) = d^2 + r_1^2 + 2ar_1$$

$$r_1^2 - r_2^2 = 2ar_1 - d^2 = (r_1-r_2)(r_1+r_2)$$

$$\Delta r = r_1 - r_2 = \frac{2ar_1}{r_1+r_2} - \frac{d^2}{r_1+r_2}$$

assuming $r_1 \approx r_2$, $r_1 + r_2 \approx 2r_1$

$$\Delta r \approx \frac{2ar_1}{2r_1} - \frac{d^2}{2r_1} = a - \frac{d^2}{2r_1}$$

*when d gets smaller, the distance between bands, Δy increases

assuming $\frac{d^2}{r_1} \approx 0$

$\Delta r \approx a = d\sin\theta \Rightarrow \Delta r \approx d\sin\theta$

note: these equations are only valid for $R \gg d$

- Light in a Medium

n is the index of refraction

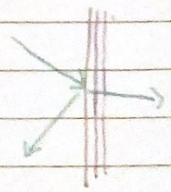
where n is defined as $n = \frac{c}{v_{med}}$, c = speed of light in vacuum, v_{med} = speed of light in medium

$c = 3.0\times10^8\,m/s$

this means that $n \geq 1$ ($n_{air} \approx 1$)

- f is the same for all mediums $f = \frac{v}{\lambda}$

as a result, the velocity and wavelength changes



$$f = \frac{v_{air}}{\lambda_{air}} = \frac{v_{med}}{\lambda_{med}} \Rightarrow \lambda_{med} = \frac{v_{med}}{v_{air}}\lambda_{air} \approx \frac{v_{med}}{c}\lambda_{air}$$

$$n = \frac{\lambda}{\lambda_{med}}$$

$\lambda_{med} \approx \frac{\lambda_{air}}{n_{med}}$

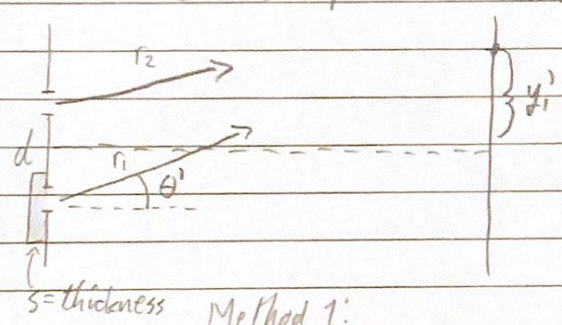
Ex: Two lasers travel through two different glass blocks of 2cm thickness. with $n_1 = 1.25$ and $n_2 = 1.5$. Which wave is faster and what is the time difference?

$v_1 = \frac{c}{n_1}$, $v_2 = \frac{c}{n_2}$ $\therefore$ v_1 is faster

$v = \frac{\Delta x}{t} \Rightarrow t = \frac{\Delta x}{v} = \frac{\Delta x n}{c} \Rightarrow t_1 = \frac{x n_1}{c}$, $t_2 = \frac{x n_2}{c}$

$\Delta t = \frac{x}{c}(n_2 - n_1) = \frac{2cm}{3\times10^8 m}(0.25) = 1.7\times10^{-11} s$

- Suppose now we place a thin piece of glass ($n = 1.50$) over one of the slits in the double slit experiment. What is the new position of y_1. (y_1')



assume $s \ll R$ and $s \sim d$

Method 1:

there will be an effective path difference because r_1 moves slower through the glass.

recall $\Delta r \approx d\sin\theta$

$\Delta r' \approx d\sin\theta' +$ extra effective path difference, also $\Delta r = m\lambda_{air}$

the effective path difference will be the time delay

$v = \frac{c}{n} = \frac{s}{t} \Rightarrow t = \frac{sn}{c} \Rightarrow \Delta t = \frac{s}{c}(n_g - n_{air})$

effective path difference $= \Delta x = v_{air}\Delta t = v_{air}\frac{s}{c}(n_g - n_{air}) = \frac{c}{n_{air}} \cdot \frac{s}{c}(n_g - n_{air})$

$\Rightarrow \Delta x = \frac{s}{n_{air}}(n_g - n_{air}) \Rightarrow \Delta r \approx d\sin\theta + \frac{s}{n_{air}}(n_g - n_{air}) \Rightarrow n_{air} d\sin\theta' \approx \Delta r' n_{air} - s(n_g - n_{air})$

Method 2:

For two waves of same frequency interfering,

$u_1 = A\sin(k_1 x - \omega t + \phi_1)$ and $u_2 = A\sin(k_2 x - \omega t + \phi_2)$

we can express the interference as the difference in the arguments.

$(k_2 x - \omega t + \phi_2) - (k_1 x - \omega t + \phi_1) = k_2 x - k_1 x + \phi_2 - \phi_1$

This will equal $2\pi m$ for constructive interference and $(2m-1)\pi$ for destructive.

* this is the same thing as multiplying Δr by $k = \frac{2\pi}{\lambda}$

$m\lambda \cdot \frac{2\pi}{\lambda} = 2\pi m$ and $(m-\frac{1}{2})\lambda \cdot \frac{2\pi}{\lambda} = (2m-1)\pi$

$k_g s - k_{air} s + k_{air}(r_1 - r_2) + \phi_g - \phi_{air} = 2\pi m$ for maximum

note: $\phi_g = \phi_{air}$ because they start at the same source

note: the $k_g s - k_{air} s$ part comes from the difference in the glass and the $k_{air}(r_1 - r_2)$ part is where both waves are travelling in air.

$$\frac{2\pi s}{\lambda_g} - \frac{2\pi s}{\lambda_{air}} + \frac{2\pi}{\lambda_{air}}\Delta r' = \frac{2\pi s n_g}{\lambda} - \frac{2\pi s n_{air}}{\lambda} + \frac{2\pi n_{air}}{\lambda} d\sin\theta' = 2\pi m$$

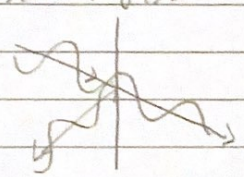
$\Rightarrow n_{air} d\sin\theta \approx m\lambda - s(n_g - n_{air})$, $\lambda = \lambda_{air} n_{air}$

$n_{air} d\sin\theta \approx \Delta r_{air} n_{air} - s(n_g - n_{air})$ which is the same as method 1.

note that in $\Delta r = m\lambda$, the λ corresponds to the medium it travels through, so in most cases, λ_{air}

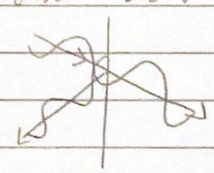
Phase Shifts During Reflection

$n_1 > n_2$ slow → fast



no phase change

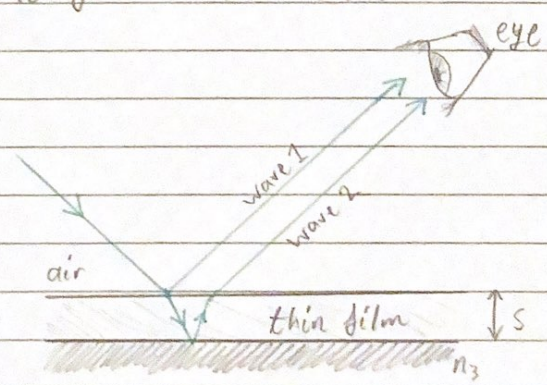
$n_1 < n_2$ fast → slow



180° phase change

* recall boundary conditions for mechanical waves

Interference in Thin Films



Wave 1: $\arg_1 = k_1 x - \omega t + \Delta\phi_1$

$n_{air} < n_{film} \therefore \Delta\phi_1 = \pi$

Wave 2: $\arg_2 = k_2(2s) + k_1 x - \omega t + \Delta\phi_2$

extra path length $\approx k_2(2s)$

$n_{film} < n_3 \therefore \Delta\phi_2 = \pi$

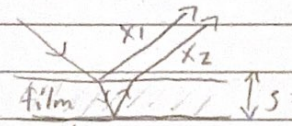
note $k_i = \frac{2\pi}{\lambda_i} = \frac{2\pi n_i}{\lambda}$

if $\Delta\arg = 2\pi m \rightarrow$ constructive

if $\Delta\arg = (2m-1)\pi \rightarrow$ destructive

Ex: Nonreflecting Film

$n_{film} = 1.38$, $n_{glass} = 1.55$, $s = \frac{1}{4}\lambda/n_{film}$

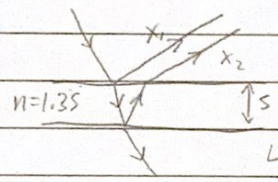


$$arg_1 = k_{air}x_1 - \omega t + \pi$$
$$arg_2 = k_{air}x_2 - \omega t + \pi + k_{film}(2s) \quad \text{note } x_1 \approx x_2$$
$$\approx k_{air}x_1 - \omega t + \pi + k_{film}(2s)$$
$$\Delta arg \approx k_{film}(2s) = (2m-1)\pi \text{ for destructive}$$

$$\frac{2\pi}{\lambda_{film}} \cdot 2s = \frac{4\pi s}{\lambda_{film}} = (2m-1)\pi$$

$$s = \frac{(2m-1)\lambda_{film}}{4} \Rightarrow s = \frac{\lambda_{film}}{4} = \boxed{\frac{\lambda_{air}}{4n_{film}}}$$

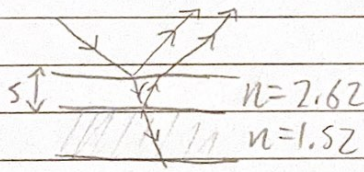
Ex2: a thin soap bubble ($n = 1.35$) appears green ($\lambda = 550nm$). what's the minimum thickness?



$$arg_1 = k_{air}x_1 - \omega t + \pi$$
$$arg_2 = k_{air}x_2 + k_b(2s) - \omega t$$
$$\Delta arg = 2sk_b - \pi = 2\pi m$$
$$\frac{4\pi s}{\lambda_b} - \pi = 2\pi m$$

$$\frac{4sn_b}{\lambda} = 2m+1 \Rightarrow s = \frac{(2m+1)\lambda}{4n_b} \Rightarrow \frac{\lambda}{4n_b} = \frac{550}{4(1.35)} = \boxed{102nm}$$

Ex3: A thin film of TiO_2 ($n_f = 2.62$) is deposited onto a glass ($n_g = 1.52$) To provide maximum transmission of red light what is the minimum coating thickness we can use? ($\lambda_{red} = 690nm$)



$$arg_1 = k_{air}x - \omega t + \pi$$
$$arg_2 = k_{air}x + 2sk_f - \omega t$$
$$\Delta arg = 2sk_f - \pi = (2m-1)\pi$$
$$\frac{4\pi s}{\lambda_f} = 2m\pi$$

note: it's destructive interference because we want the light to go through and not get reflected

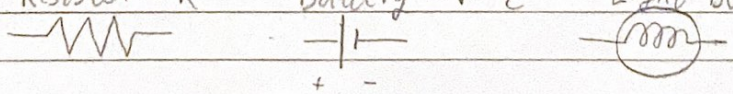
$$\frac{2sn_a}{\lambda} = 1 \Rightarrow s = \frac{\lambda}{2n_a} = \frac{690}{2(2.62)} = \boxed{132nm}$$

Unit 3 - Circuits

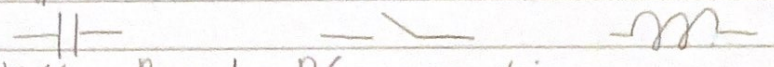
3.1 - Time-Independent DC Circuits

Basic DC components

Resistor = R Battery = V = $\mathcal{E}$ Light bulb = R



Capacitor = C Switch = S Inductor = L



*inductors resist the changing in current. When the current is constant, it acts as a wire.

Voltage Drop for DC components:

Resistor: $V_R = IR$

Battery: $V = \mathcal{E}$

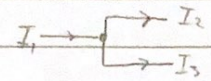
Capacitor: $V_C = \frac{Q}{C} = \frac{1}{C}\int I(t)dt$

Inductor: $\mathcal{E}_L = -L\frac{dI}{dt}$ - voltage acts opposite to the change in current

Kirchoff's Laws

K1 - Junction Law

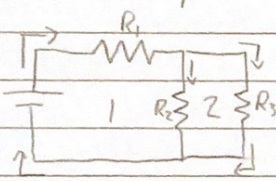
$\sum I_{in} = \sum I_{out}$



$\Rightarrow I_1 = I_2 + I_3$

K2 - Loop Voltage Law

$\sum V_{closed\ loop} = 0$



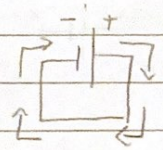
$\sum V_{loop\,1} = 0$

$\sum V_{loop\,2} = 0$

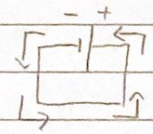
$\sum V_{loop\,1+2} = 0$

so, $V_{R_2} = V_{R_3}$ and $V_{R_1} + V_{R_2} = V_{battery}$

note about current flow



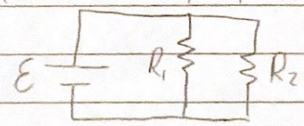
positive current flow



positive electron flow

*in many cases, you can guess the direction of current flow and if you have a negative current, you guessed wrong.

Ex: $\mathcal{E} = 12V$, $R_1 = 2\Omega$, $R_2 = 3\Omega$. Find the current through the battery.

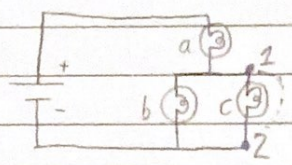


$I_1 = \frac{V}{R_1} = \frac{12}{2} = 6A$, $I_2 = \frac{V}{R_2} = \frac{12}{3} = 4A$

$\sum I = \boxed{10A}$

Ex2: What happens to the bulbs as a wire connects 1 to 2?

Note, the brightness is proportional to the power. $P = IV$

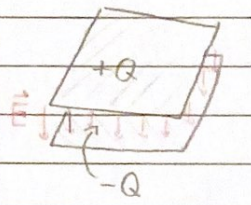


the voltage from 1->2 is 0 ∴ the voltage for b and c is also 0.
This means that the voltage for a goes up to match the battery voltage.

∴ a gets brighter and b & c go out.

- Capacitors

A capacitor is a device that stores a charge and therefore energy



} potential difference (ΔV)

because of the difference in charges, there is an electric field and a potential difference

C is the capacitance (unit F)

$C = \frac{|Q|}{\Delta V}$ where Q is the magnitude of the charge stored on each conductor and ΔV is the potential difference between the two

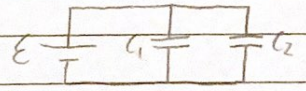
Another formula derived from the electric field is:

$C = \varepsilon_0 \frac{A}{d}$ where A is the area of each plate, d is the distance between plates, and ε_0 is the electric constant.

Batteries	Capacitors
- constant ΔV	- changing ΔV (decreases as it is discharged)
- lower voltages (can get higher voltage with batteries in series)	- very high voltages possible
- Internal resistance leads to a maximum current possible.	- no internal resistance -> fast discharge

Capacitors in Parallel

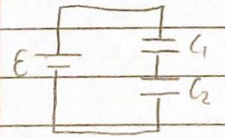
$\Delta V_{bat} = \Delta V_1 = \Delta V_2$ note $Q = It \Rightarrow Q_{bat} = Q_1 + Q_2$, $Q = \Delta V C$



$Q_{bat} = \Delta V C_1 + \Delta V C_2 = \Delta V C_{eq}$

$\Rightarrow C_{eq} = C_1 + C_2 + \cdots$

Capacitors in Series



note $Q = It \Rightarrow Q_{bat} = Q_1 = Q_2$, $\Delta V = \frac{Q}{C}$

$\Delta V_{bat} = \Delta V_1 + \Delta V_2 = \frac{Q_1}{C_1} + \frac{Q_2}{C_2} = \frac{Q_{bat}}{C_{eq}}$

$Q\left(\frac{1}{C_1} + \frac{1}{C_2}\right) = \frac{Q}{C_{eq}} \Rightarrow \frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots$

Energy Stored in a Capacitor

from the electric field equation, $\vec{F} = \frac{kq_1q_2}{r^2}$,

we can get $U = \int_r^0 \frac{kq_1q_2}{r^2} = \frac{kq_1q_2}{r} = qV$

$\Rightarrow \Delta U = q\Delta V$

$dW = dU = dq\Delta V = \frac{q}{C}dq$

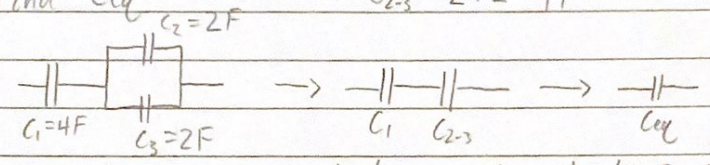
$W = \int_0^Q \frac{q}{C}dq = \frac{Q^2}{2C} = \frac{1}{2}CV^2$ note Q is a steady charge and dq is a changing charge

$U_c = \frac{1}{2}CV^2$

Ex: Find C_{eq}

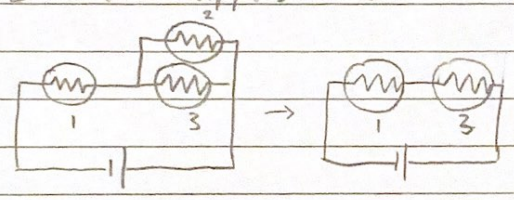
$C_{2\text{-}3} = 2+2 = 4F$

$\frac{1}{C_{eq}} = \frac{1}{4} + \frac{1}{4} = \frac{2}{4}$



$\Rightarrow C_{eq} = 2F$

Ex2: What happens to bulb 3 when bulb 2 burns out?



$\frac{1}{R_{2\text{-}3}} = \frac{1}{R_2} + \frac{1}{R_3} = \frac{R_3 + R_2}{R_2R_3}$ $R_{eq} = R_1 + R_3$

$R_{eq_0} = R_1 + \frac{R_2R_3}{R_2+R_3}$

R_{eq_0} vs. R_{eq}

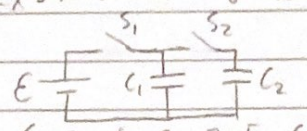
$R_1 + \frac{R_2R_3}{R_2+R_3}$	$R_1 + R_3$
$\frac{R_2R_3}{R_2+R_3}$	R_3
$\frac{R_3}{R_2+R_3}$	1

$\frac{R_3}{R_2+R_3} < 1 \therefore R_{eq} > R_{eq_0}$

this means that the total current goes down $\therefore$ bulb 1 gets dimmer

We also observe the current through R_3 goes up $\therefore$ bulb 3 gets brighter

Ex 3: Find the final charges.



$\mathcal{E} = 24V$, $C_1 = 2\mu F$, $C_2 = 1\mu F$

1. Close S_1 for long time
2. Open S_1
3. Close S_2

Once S_1 is closed, the battery charges C_1

$$V_{bat} = V_{C_1} = \frac{Q_{1i}}{C_1} \Rightarrow Q_{1i} = 48\mu C$$

When S_1 is open and S_2 is closed, C_1 charges C_2.
Note, charge is conserved.

$$Q_{1i} + Q_{2i} = Q_{1f} + Q_{2f} = 48\mu C$$

$$Q_{1i} = C_1 V_f + C_2 V_f$$

$$\frac{Q_{1i}}{C_1 + C_2} = V_f = \frac{48\mu}{2\mu + 1\mu} = 16V$$

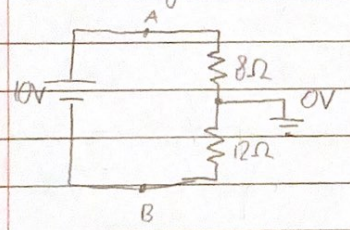
$\Rightarrow Q_{1f} = C_1 V_f = 32\mu C$ and $Q_{2f} = C_2 V_f = 16\mu C$

note: $U_{initial} = \frac{1}{2} C_1 V_i^2 = 576\mu J$

$U_{final} = \frac{1}{2}(C_1 + C_2) V_f^2 = 384\mu J$

$\therefore$ in this case, energy is lost to the surroundings.

Grounding:



note: in some schematics, the ground represents the negative terminal of a battery. (this would also have $V_{ground} = 0V$)

grounding does not change the current, it only allows for you to find a point at which the potential is 0V.

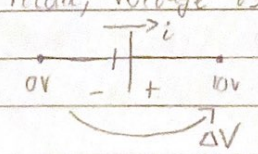
$R_{eq} = 20\Omega \Rightarrow I = \frac{V}{R_{eq}} = \frac{10}{20} = \frac{1}{2}A$

$\Delta V_1 = R_1 I = 4V$ $\quad \Delta V_2 = R_2 I = 6V$

$\Rightarrow$ the potential at A is 4V, and the potential at B is -6V.
note that A→B still gives 10V.

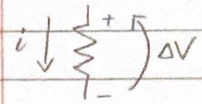
Sign Convention for Passive Components:

recall, voltage is measured as $V_{high} - V_{low}$



this gives a positive value for voltage ($\Delta V > 0$)

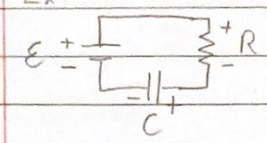
In the case of passive components such as resistors, the voltage drop will oppose the direction of the current.



for calculation purposes, we say this has a negative voltage ($\Delta V < 0$)

* passive components include capacitors and inductors as well.

Ex:



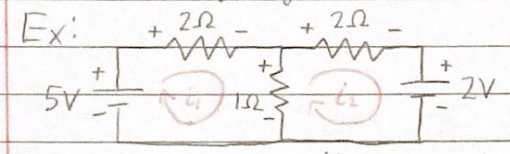
using the loop voltage law, we get

$$\mathcal{E} - V_R - V_C = 0$$

Mesh Current Method of Solving Circuits

- this is useful for when we don't know the direction of current.

Ex:



* a mesh is each small loop. so, this circuit has 2 meshes.

we can define the current to go clockwise in each mesh and write the polarities accordingly.

note: the current must travel the same way in each mesh for the setup.

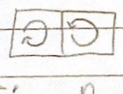
mesh 1: $\mathcal{E}_1 - V_{R_1} - V_{R_3} = 0 \Rightarrow 5 - 2i_1 - 1(i_1 - i_2) = 0$

mesh 2: $-V_{R_2} - \mathcal{E}_2 + V_{R_3} = 0 \Rightarrow 1(i_1 - i_2) - 2 - 2i_2 = 0$

$$\Rightarrow \begin{aligned} -3i_1 + i_2 &= -5 \\ i_1 - 3i_2 &= 2 \end{aligned} \Rightarrow \left[\begin{array}{cc|c} -3 & 1 & -5 \\ 1 & -3 & 2 \end{array}\right] \xrightarrow{(2)=(2)+\frac{1}{3}(1)} \left[\begin{array}{cc|c} -3 & 1 & -5 \\ 0 & -\frac{8}{3} & \frac{1}{3} \end{array}\right] \Rightarrow \begin{aligned} -\tfrac{8}{3} i_2 &= \tfrac{1}{3} \\ i_2 &= -\tfrac{1}{8} \end{aligned}$$

$i_1 = 2 + 3i_2 = 1.625\,A$, $i_2 = -0.125\,A$

The negative sign in i_2 indicates it's going in the opposite direction.



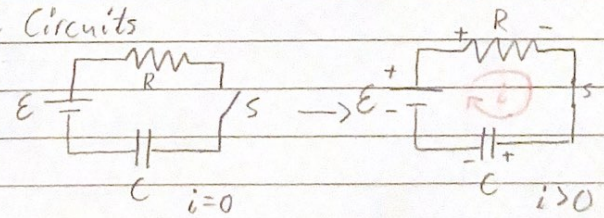
from here we can solve the circuit:

$V_{R_1} = 2(1.625) = 3.25\,V$, $V_{R_2} = 2(-0.125) = -0.5\,V$, $V_{R_3} = 1.625 - (-0.125) = 1.75\,V$

3.2 - Time-Dependent DC Circuits

notation: i and q mean changing with time while I and Q are constants.

I - RC Circuits



once the switch is closed, the battery begins to charge the capacitor

recall $V_c = \frac{q}{C}$ when the capacitor has no charge, $V_c = 0$ and the capacitor acts as a wire.

From the loop-voltage law, we get

$\mathcal{E} - iR - \frac{q}{C} = 0$

→ take the derivative with respect to time:

$-R\frac{di}{dt} - \frac{1}{C}\frac{dq}{dt} = 0$ recall, $i = \frac{dq}{dt}$

$\Rightarrow -R\frac{di}{dt} - \frac{i}{C} = 0 \Rightarrow \int \frac{di}{i} = -\int \frac{dt}{RC} \Rightarrow \ln|i| = \frac{-t}{RC} + \text{const}$

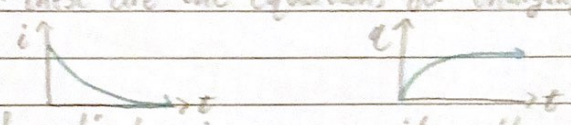
$\Rightarrow i(t) = i(0)\, e^{-t/RC}$

at $t=0$, $\frac{q}{C} = 0 \Rightarrow \mathcal{E} - i_0 R = 0 \Rightarrow i_0 = \frac{\mathcal{E}}{R}$

so, $i(t) = \frac{\mathcal{E}}{R} e^{-t/RC}$

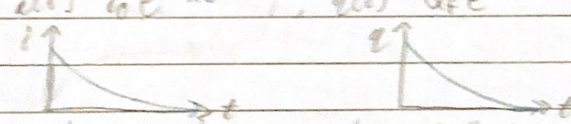
similarly, using $i = \frac{dq}{dt}$, we get $q(t) = Q_f\left(1 - e^{-t/RC}\right)$

→ these are the equations for charging a capacitor



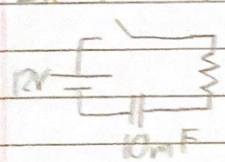
for discharging a capacitor, it would be the opposite:

$i(t) = i_0 e^{-t/RC}$, $q(t) = Q_f e^{-t/RC}$



- As the capacitor comes to a full charge, the voltage of the capacitor directly opposes the voltage of the battery, and so, the current tends to 0.

Ex: Find $i(t)$ and compute $q(0.2s)$



$i(t) = i_0 e^{-t/RC}$, $i_0 = \frac{\mathcal{E}}{R}$

$q(t) = Q_f\left(1 - e^{-t/RC}\right)$, $Q_f = \mathcal{E}C$

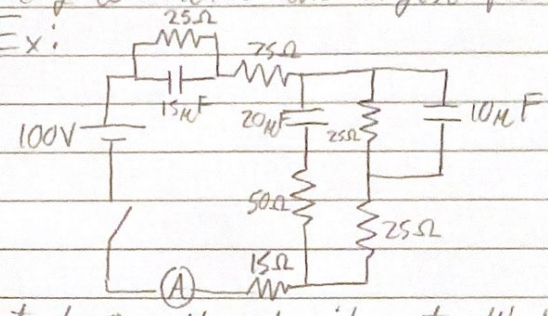
$q(0.2s) = \underline{0.076C}$

- Capacitors and Time Dependence:

→ at $t=0$, the capacitor will act as a wire (short) because the voltage drop is 0. ($V_c = \frac{q}{C} = \frac{0}{C} = 0$)

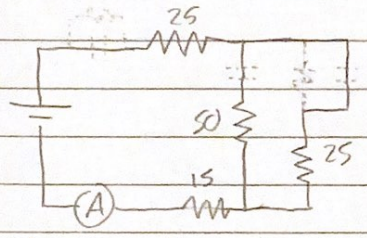
→ at equilibrium ($t \to \infty$) the capacitor will have no current flowing through it (acts as a break). This is because it will try to attain the largest possible voltage.

Ex:



Find $I_A(0+)$
Find $I_A(\infty)$

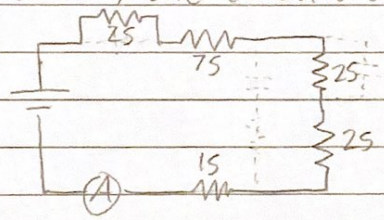
at $t=0$, the circuit acts like:



$R_{eq} = 75 + \left(\frac{1}{50} + \frac{1}{25}\right)^{-1} + 15 = 106.67$

$I = \frac{\mathcal{E}}{R_{eq}} = \boxed{0.94A}$

at $t \to \infty$, the circuit acts like:



$R_{eq} = 25 + 75 + 25 + 25 + 15 = 165\Omega$

$I = \frac{\mathcal{E}}{R_{eq}} = \boxed{0.61A}$

II — Inductors

- When the current changes, the inductor generates a voltage to try and oppose this change. (comes from magnetic field)

$$V_L = -L\frac{di}{dt}$$

Energy in an inductor

recalling $P = IV$

$P = iV_L = iL\frac{di}{dt}$ (ignore the negative for simplicity)

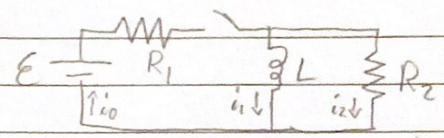
$$p = \frac{dU}{dt} = iL\frac{di}{dt}$$

$$U = \int_0^{I_f} iL\,di = \frac{1}{2}LI_f^2 \Rightarrow U = \frac{1}{2}LI_f^2$$

- Inductors and Time Dependence

- At $t=0$, the inductor opposes any change in current so the current will be 0.
- At equilibrium ($t \to \infty$) the current is no longer changing so the voltage across the inductor is 0, so the inductor acts as a wire.

Ex: the switch is closed at $t=0$.
Find the currents at $t=0$ and $t=\infty$



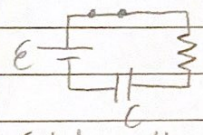
at $t=0$, $i_1=0$ and $i_2=\frac{\mathcal{E}}{R_1+R_2}$ because no current goes through inductor
at $t\to\infty$, $i_1=\mathcal{E}/R_1$ and $i_2=0$ because inductor acts as a short.

Ex2: at $t=t_1$, the switch is opened again.
Find the currents i_0 and i_2

- the inductor resists change so it will retain its current of $\mathcal{E}/R_1$. From $L\frac{di}{dt}=i_2R_2$, we see $\frac{di}{dt}<0 \therefore i_2<0$ and $i_2=-\mathcal{E}/R_1$.

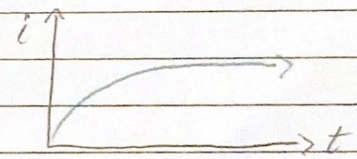
↑ This means i_2 switches directions

Consider the circuit:

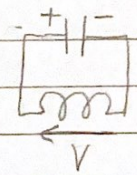


we get $\mathcal{E}=iR+L\frac{di}{dt}$

Solving the ODE gives $i(t)=I_f\left(1-e^{-\frac{Rt}{L}}\right)$
- this is known as an RL circuit.

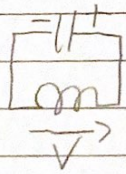


III – RLC Circuits



Capacitor starts at full charge
Inductor opposes changing current, $i(0)=0$

Current slowly changes to deliver the charge
Once the capacitor is balanced, the inductor continues to force the current, switching the polarity of the capacitor



Process repeats indefinetly.

Derivation: note $i = -\frac{dq}{dt}$ in this case

so, $\frac{q}{C} + L\frac{di}{dt} = 0 \Rightarrow L\frac{d^2q}{dt^2} + \frac{q}{C} = 0$

let $q(t)$ be in form of $e^{rt} \Rightarrow \frac{dq}{dt} = re^{rt} \Rightarrow \frac{d^2q}{dt^2} = r^2e^{rt}$

$\rightarrow Lr^2e^{rt} + \frac{e^{rt}}{C} = 0 = e^{rt}(Lr^2 + \frac{1}{C})$

solve for r: $Lr^2 + \frac{1}{C} \Rightarrow r^2 = \frac{-1}{LC} \Rightarrow r = \sqrt{\frac{-1}{LC}} = \pm\frac{i}{\sqrt{LC}}$

let $\omega = \frac{1}{\sqrt{LC}} \Rightarrow r = \pm i\omega$

$q(t) = Z_1e^{i\omega t} + Z_2e^{-i\omega t}$, Z is some constant

$q(t) = Z_1(\cos(\omega t) + i\sin(\omega t)) + Z_2(\cos(-\omega t) + i\sin(-\omega t))$

$q(t) = (Z_1 + Z_2)\cos(\omega t) + i(Z_1 - Z_2)\sin(\omega t)$

$q(t) = Z_1'\cos(\omega t) + Z_2'\sin(\omega t)$

$q(0) = Q \Rightarrow Q = Z_1' \therefore Z_2' = 0$

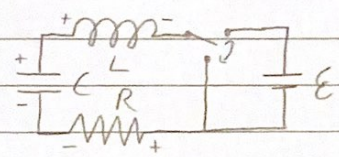
$\rightarrow q(t) = Q\cos(\omega t + \phi)$ where $\omega = \sqrt{\frac{1}{LC}}$

energy

$U_{circuit} = \frac{q^2}{2C} + \frac{1}{2}Li^2$

$U = \frac{Q_{max}}{2C} = \frac{1}{2}LI_{max}^2$

- RLC Circuit



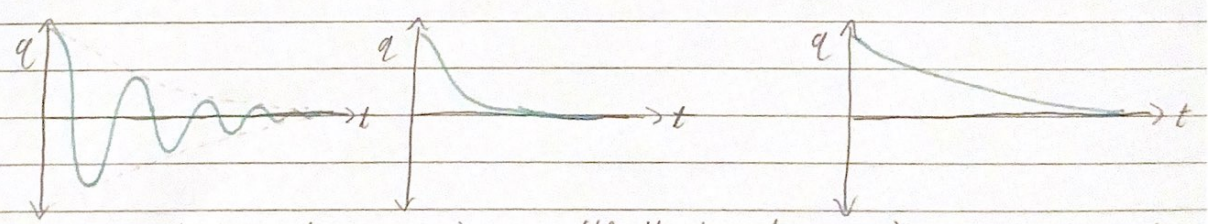
$\frac{q}{C} - L\frac{di}{dt} - Ri = 0$

$\Rightarrow L\frac{d^2q}{dt} + R\frac{dq}{dt} + \frac{q}{C} = 0$

- solving this ODE gives:

$q(t) = Ae^{-t/\tau}\cos(\omega' t + \phi)$ where $\tau = \frac{2L}{R}$ and $\omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

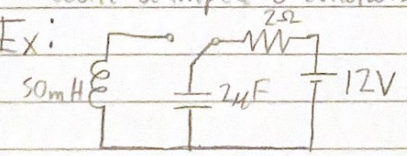
$i(t) = \frac{-dq(t)}{dt} = \omega Q_0 \sin(\omega t + \phi)$ for $\omega' = \omega$



$\omega' > 0$ - underdamped $\quad$ $\omega' = 0$ - critically damped $\quad$ $\omega' < 0$ - overdamped

$\rightarrow$ recall damped oscillators from Phys 157.

Ex:



The capacitor is charged and the switch is moved. Find I_{max} through the inductor and the first time of the max

at $t = 0+$, $V_c = 12V$

$Q_0 = CV_0 = (2\mu)(12) = 24\mu C$

$U_{cap} = \frac{Q_0^2}{2C} = U_L = \frac{1}{2}LI_{max}^2 \Rightarrow I_{max}^2 = \frac{Q_0^2}{LC} = \frac{(24\mu)^2}{(50m)(2\mu)} \Rightarrow I_{max} = \underline{75.9mA}$

$q(t) = Q_0 \cos(\omega t)$

$i(t) = Q_0 \omega \sin(\omega t) = I_{max} \sin(\omega t)$

$\sin(\frac{\pi}{2}) = 1 \Rightarrow \omega t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{2\omega} = \frac{\pi\sqrt{LC}}{2} = \underline{0.497ms}$

Ex2: You build an LC circuit with $f = 10kHz$ where the max current must be $< 0.1A$ and the max capacitor energy must be $< 10^{-5}J$. What inductance and capacitance must you use?

$\omega = 2\pi f = \frac{1}{\sqrt{LC}} \Rightarrow LC = \frac{1}{4\pi^2 f^2}$

$U = \frac{Q^2}{2C} = \frac{1}{2}LI_T^2 \Rightarrow Q^2 = LCI_T^2 = \frac{I^2}{4\pi^2 f^2}$

$C = \frac{Q^2}{2U_c} = \frac{I^2}{8\pi^2 f^2 U_c} = \frac{(0.1)^2}{8\pi^2(10k)^2(10^{-5})} = \boxed{0.123\mu F}$

$L = \frac{1}{4\pi^2 f^2 C} = \boxed{2mH}$

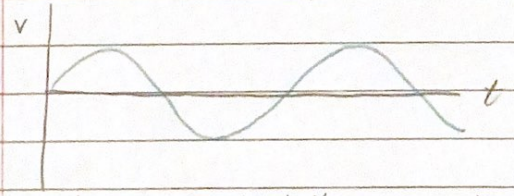
Note on inductors in series vs. parallel:

Series	Parallel
$L_{eq} = \Sigma L$	$\frac{1}{L_{eq}} = \Sigma \frac{1}{L}$

3.3 - AC Circuits

AC circuits have an emf that fluctuates with time.

$v(t) = V_{max} \sin(\omega t)$



* the current follows a similar trend, though it may be off by a phase shift. $i(t) = I_{max} \sin(\omega t + \phi)$

- The time average of a function is the average over one period, denoted by $\langle f(t) \rangle$.

$$\langle f(t)\rangle = \frac{1}{T}\int_0^T f(t)\,dt$$

note the average of sine over a period is 0 so instead, we can calculate the root mean squared average instead.

$$x_{RMS} = \sqrt{\sum_{i=1}^{n} \frac{x_i^2}{n}}$$

so, the RMS voltage is

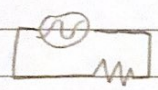
$$V_{RMS} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt} = \sqrt{\langle v^2(t)\rangle}$$

$$\langle v^2(t)\rangle = \frac{1}{T}\int_0^T V_{max}^2 \sin^2(\omega t)\,dt = \frac{V_{max}^2}{T}\int_0^T \frac{1-\cos(2\omega t)}{2}\,dt = \frac{V_{max}^2}{2}$$

$$\Rightarrow V_{RMS} = \frac{V_{max}}{\sqrt{2}}$$

similarly, $I_{RMS} = \frac{I_{max}}{\sqrt{2}}$

- Resistors in AC

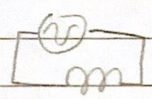


$$v - iR = 0, \quad v = V\sin(\omega t)$$

$$i = \frac{v}{R} = \frac{V_m}{R}\sin(\omega t)$$

$\therefore$ the current is in phase with the voltage in this circuit.

- Inductors in AC.

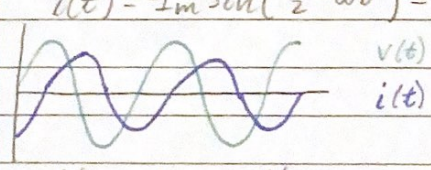


$$v - L\frac{di}{dt} = 0 = V_m\sin(\omega t) - L\frac{di}{dt} = 0$$

$$\frac{di}{dt} = \frac{V_m}{L}\sin(\omega t)$$

$$\rightarrow i(t) = \frac{-V_m}{\omega L}\cos(\omega t) = \frac{-V_m}{X_L}\cos(\omega t) = -I_m\cos(\omega t)$$

$$i(t) = -I_m\sin\left(\frac{\pi}{2} - \omega t\right) = I_m\sin\left(\omega t - \frac{\pi}{2}\right)$$

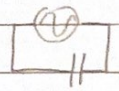


notice here the voltage 'leads' the current

note V_{max} and I_{max} do not occur at the same time.

We also defined the term X_L.
X is defined to be the reactance and is measured in Ohms.
$X_L = \omega L$

- Capacitors in AC



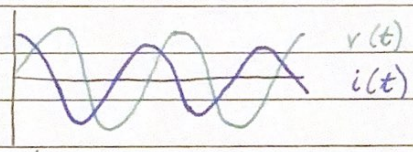
$$v - \frac{q}{C} = 0 = V_m \sin(\omega t) - \frac{q}{C}$$

$$q(t) = C v(t) = C V_m \sin(\omega t)$$

$$i(t) = \frac{dq}{dt} = \omega C V_m \cos(\omega t)$$

we can define $X_C = \frac{1}{\omega C}$

$$i(t) = \frac{V_m}{X_C} \cos(\omega t) = I_m \sin(\omega t + \tfrac{\pi}{2})$$

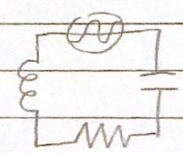


notice here the current leads the voltage.

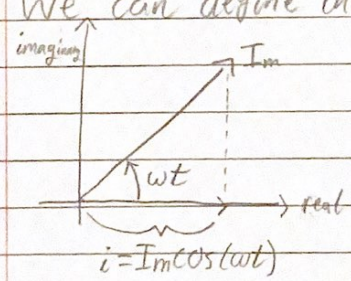
- RLC Circuits in AC.

in a series circuit, the current will be the same for all parts.
So, we can define a common current $i(t) = I_m \cos(\omega t)$
the voltages will be as follows:

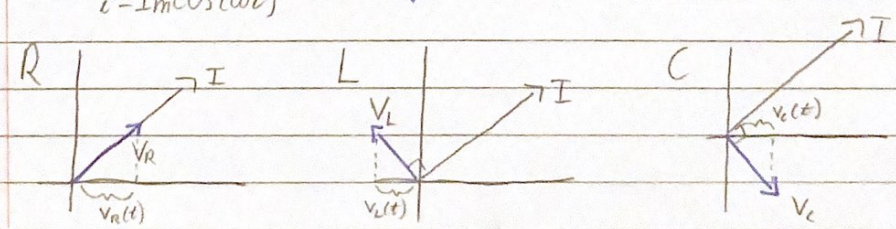
$v_R(t) = V_m \cos(\omega t)$
$v_L(t) = V_m \cos(\omega t + \pi/2) = -V_m \sin(\omega t)$
$v_C(t) = V_m \cos(\omega t - \pi/2) = V_m \sin(\omega t)$



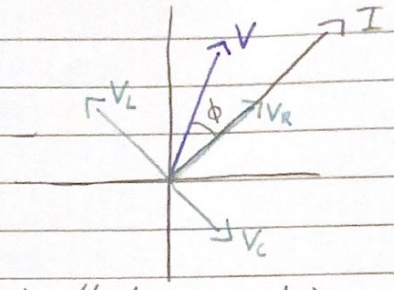
We can define these relationships through a phasor diagram.



the vector rotates around the orgin and the projected length along the real axis is the observed current.
$j = \sqrt{-1}$



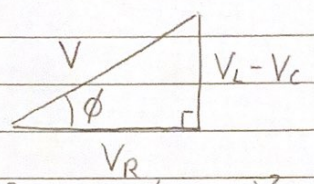
When we combine the phasor diagrams we get the following:



the total voltage, V, is the sum of all voltage vectors and will be observed in the circuit. ϕ is the phase difference between voltage and current.

note that V_L and V_c are colinear so they can be expressed as a single vector $(V_L - V_c)$

We can now form a triangle



where $V_R = IR$, $V_L = IX_L$, $V_c = IX_c$

$$V^2 = V_R^2 + (V_L - V_c)^2 = I^2R^2 + I^2(X_L - X_c)^2 = I^2\left(R^2 + (X_L - X_c)^2\right)$$

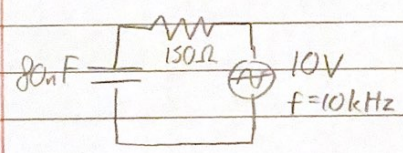
we can express the term $R^2 + (X_L - X_c)^2$ as impedance squared.

$|Z|^2 = R^2 + (X_L - X_c)^2$ so that $V_m = I_m Z$

the angle can be expressed as $\tan\phi = \frac{X_L - X_c}{R}$

note that the impedence can be more generally expressed through a vector sum and can thus be expressed as, $Z = R + j(X_L - X_c)$

Ex: Find the peak voltage across the capacitor



$V_c = I_m X_c$

- find I_m: $V_m = I_m Z$

$Z = \sqrt{R^2 + X_c^2}$

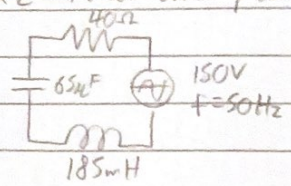
$X_c = \frac{1}{\omega C} = 199\Omega$

$\Rightarrow Z = 249\Omega$

$\Rightarrow I_m = \frac{V_m}{Z} = 0.04A$

$V_c = I_m X_c = \boxed{8V}$

Ex2: Find the peak voltage across the inductor and the phase shift



$X_L = \omega L = 58$, $X_C = \frac{1}{\omega C} = 49$

$Z = \sqrt{R^2 + (X_L - X_C)^2} = 41$

$I_m = V_m / Z = 3.66$

$V_L = I_m X_L = \boxed{212\ V}$

$\phi = \arctan\left(\frac{X_L - X_C}{R}\right) = \boxed{12.9°}$

- Reactance at high/low frequencies

Inductors	Capacitors
$X_L = \omega L$	$X_C = \frac{1}{\omega C}$
$\omega \to \infty$: $X_L \to \infty$	$\omega \to \infty$: $X_C \to 0$
$\omega \to 0$: $X_L \to 0$	$\omega \to 0$: $X_C \to \infty$

- for high frequencies, inductors act as a short and capacitors act as a wire
- for low frequencies, inductors act as a wire and capacitors act as a short.

- Power:

note that in an RLC circuit, power can only be dissapated in a resistor.

So, we can express the power dissapated in a circuit as $P = iV$ or $\langle P \rangle = \frac{1}{2} I_m V_m \cos\phi$ where $\cos\phi$ is called the 'power factor'

we also know $\langle P \rangle = I_{RMS} V_{RMS} \cos\phi$

and $\cos\phi = R/Z$

so, $\langle P \rangle = I_{RMS}(I_{RMS} Z) R/Z$

$\Rightarrow \langle P \rangle = I_{RMS}^2 R$

- Resonance

we can express current as a function of ω

$$I(\omega) = \frac{V_{source,m}}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

note as $\omega \to 0$, $I(\omega) = 0$
and as $\omega \to \infty$, $I(\omega) = 0$
This implies that there is some maximum current possible
This occurs when $X_L - X_C = 0$ and gives $I = \frac{V_{source}}{R}$

$$\omega L = \frac{1}{\omega C} \Rightarrow \omega = \frac{1}{\sqrt{LC}}$$

this is called the resonance frequency

Ex: a radio picks up a 1MHz signal with $V_{peak} = 5mV$. If $L = 60\mu H$, $R = 0.25\Omega$, what value should the capacitor be tuned to to listen?

$\omega = 2\pi \cdot 1MHz$

$$\omega = \frac{1}{\sqrt{LC}} \Rightarrow C = \frac{1}{L\omega^2} = \underline{4.22 \times 10^{-10} F}$$

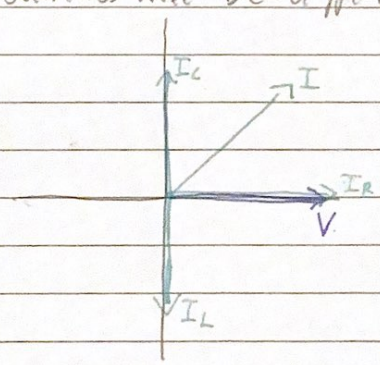
What is the peak current?

note $Z = R$

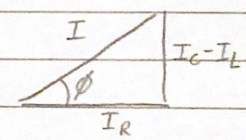
$$I_m = \frac{V_{source}}{R} = \underline{20mA}$$

AC Circuits in Parallel

in a true parallel circuit, the voltage will be common and the currents will be different.

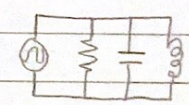


this gives



$$I = \frac{V}{Z} = \sqrt{I_R^2 + (I_C - I_L)^2}$$

$$\frac{1}{Z} = \sqrt{\frac{1}{R^2} + \left(\frac{1}{X_C} - \frac{1}{X_L}\right)^2}$$



$$\tan\phi = \frac{I_C - I_L}{I_R} = \frac{\frac{1}{X_C} - \frac{1}{X_L}}{\frac{1}{R}} \Rightarrow \tan\phi = R\left(\frac{1}{X_C} - \frac{1}{X_L}\right)$$

note: for more complex problems it is helpful to work it out in terms of Kirchoff's laws and phasor diagrams.
It may also be of use knowing that the vector equation of impedence is $Z = R + j(X_L - X_C)$ (for series) and we express it as $\|Z\| = \sqrt{R^2 + (X_L - X_C)^2}$

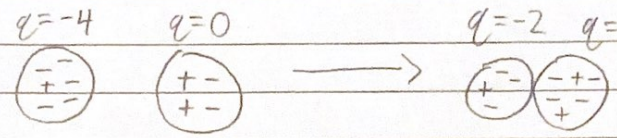
Unit 4 - Electricity and Magnetism

4.1 - Electric Charge and Electric Field

note that electric charge is always conserved

Insulators are objects that don't allow electrons to move freely. (wood, glass, plastic) - nothing really happens

Conductors allow electrons to move freely. (often metal)



the charges want to balance out so the electrons flow from one ball to the other.

note: this would not occur in an insulator

This occurs because of the attraction between charges.

Like charges repel: ⟵ ⊕⊕ ⟶

Opposite charges attract: ⊖ ⟶ ⟵ ⊕

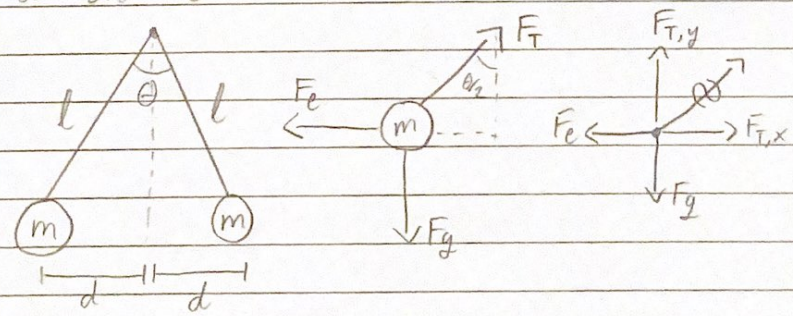
This is expressed through Coulomb's Law:

$\vec{F} = k_e \frac{|q_1||q_2|}{r^2} \hat{r}$ where q_1 and q_2 are two point charges and $\hat{r}$ is the direction unit vector that spans the line connecting the points.

$k_e = 8.99 \times 10^9 \, Nm^2/C^2$ and $\epsilon_0 = 8.854 \times 10^{-12} \, C^2/Nm^2$

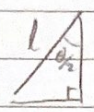
with $k_e = \frac{1}{4\pi\epsilon_0}$

Ex: find q for each ball such that they remain in equilibrium as shown



$F_{Ty} = F_T \cos(\frac{\theta}{2}) = mg$, $F_{Tx} = F_T \sin(\frac{\theta}{2}) = F_e$

$\frac{F_e}{mg} = \tan(\frac{\theta}{2}) \Rightarrow k \frac{|q_1||q_2|}{r^2} = mg \tan(\frac{\theta}{2})$



$\frac{r}{2} = l\sin(\frac{\theta}{2}) \Rightarrow r^2 = 4l^2\sin^2(\frac{\theta}{2})$

$|q_1||q_2| = \frac{4l^2\sin^2(\frac{\theta}{2})\, mg\tan(\frac{\theta}{2})}{k}$

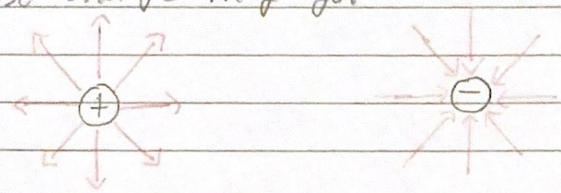
note: q_1 and q_2 must also have the same charge

Electric Fields

the electric field is how much pull a charge has at any point

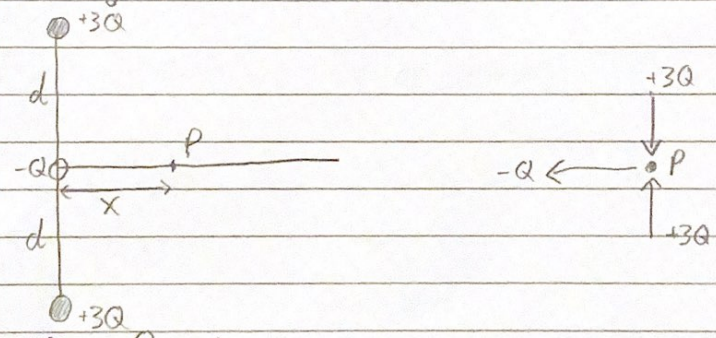
$\vec{E} = \frac{k_e q}{r^2}\hat{r}$

the electric field will always point in the direction a positive test charge may go.



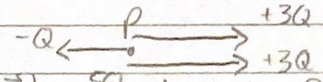
note that a positive test charge will move away from the positive charge and move towards the negative charge

Ex: if $d \gg x$ then what is the electric field strength at P?



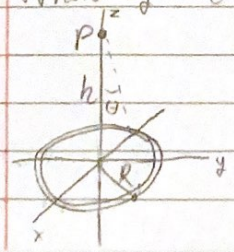
$|\vec{E}| = \frac{Q}{x^2}$ towards $-Q$

Ex2: What if $x \gg d$?



$|\vec{E}| = \frac{5Q}{x^2}$ towards P (away from $-Q$)

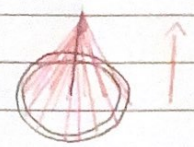
What if we aren't dealing with point charges?



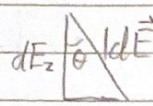
consider a ring with radius R and uniform total charge $+Q$.

What is the electric field at P?

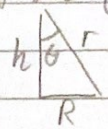
We know the electric field should look something like this:



we can think of this as a vector field consisting of an infinite number of vectors $d\vec{E}$.
notice because of the symmety, all x-y components cancel



$dE_z = |d\vec{E}|\cos\theta$



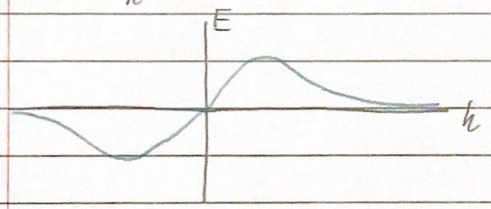
$\cos\theta = \frac{h}{r}$

$dE_z = |d\vec{E}|\frac{h}{r}$

$|d\vec{E}| = \left(\frac{k\,dQ}{r^2}\right) \Rightarrow dE_z = \left(\frac{k\,dQ}{r^2}\right)\left(\frac{h}{r}\right) = \left(\frac{k\,h\,dQ}{(R^2+h^2)^{3/2}}\right)$

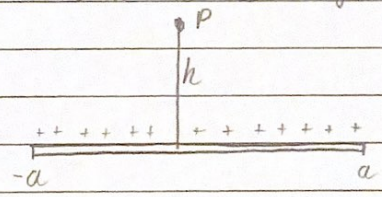
$E_z = \int dE_z = \int \frac{k\,h\,dQ}{(R^2+h^2)^{3/2}} = \frac{k\,h\,Q}{(R^2+h^2)^{3/2}}$ $\qquad \vec{E}_{ring} = \frac{k\,h\,Q}{(R^2+h^2)^{3/2}}\hat{n}$

note for $h \gg R$, the ring looks like a point charge and it tends to
$E = \frac{kQ}{h^2}$



note, these maximum values occur at
$h = \pm\frac{R}{\sqrt{2}}$

Ex2: Consider a uniformly charged line with charge +Q.



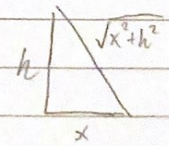
we can treat each point dx as a point charge of dQ with a field of $d\vec{E}$

$\sum dQ = Q$ and $\sum dx = 2a$

$\Rightarrow \frac{dQ}{Q} = \frac{dx}{2a} \Rightarrow dQ = \frac{Q\,dx}{2a}$

$\frac{Q}{2a}$ can be expressed as the linear charge density, λ

$dQ = \lambda\,dx$

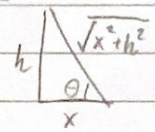


$d\vec{E} = \left(\frac{k\,dQ}{r^2}\right)\hat{r} = \left(\frac{k}{x^2+h^2}\right)\left(\frac{Q\,dx}{2a}\right)\hat{r}$

To avoid the vector notation of $d\vec{E}$, we can break it up into its x and y components

$dE_x = |dE|\cos\theta$

$dE_y = |dE|\sin\theta$



$\cos\theta = \frac{x}{\sqrt{x^2+h^2}}$

$\sin\theta = \frac{h}{\sqrt{x^2+h^2}}$

$$dE_x = \left(\frac{kQ}{2a}\right)\left(\frac{dx}{x^2+h^2}\right)\left(\frac{x}{\sqrt{x^2+h^2}}\right)$$

$$dE_y = \left(\frac{kQ}{2a}\right)\left(\frac{dx}{x^2+h^2}\right)\left(\frac{h}{\sqrt{x^2+h^2}}\right)$$

we can see that dE_x is an odd function so, by symmetry, the integral will work out to 0.

$$E_y = \frac{kQ}{2a}\int_{-a}^{a}\frac{h\,dx}{(x^2+h^2)^{3/2}} = \frac{Q}{4\pi\epsilon_0 h\sqrt{a^2+h^2}}$$

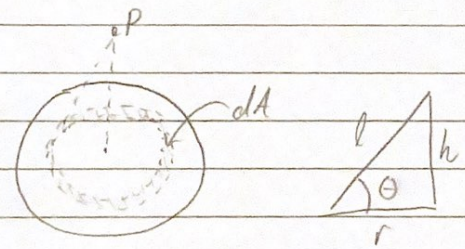
if $a \gg h$

$\sqrt{a^2+h^2} \approx a$

$$E_y \cong \frac{Q}{4\pi\epsilon_0 h a} = \frac{\lambda}{2\pi\epsilon_0 h} \qquad \vec{E}_{rod} = \frac{k\lambda l}{h\sqrt{h^2+\frac{l^2}{4}}}$$

Ex3: consider a charged disk

*note that the electric field <u>inside</u> any spherical closed loop is zero for all points.



$\sigma = \frac{Q}{A} \Rightarrow dQ = \sigma dA$

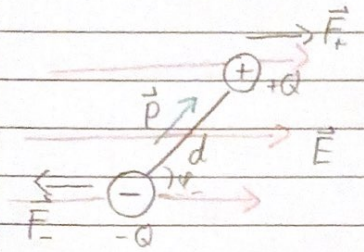
$$d\vec{E} = \frac{k\,dQ}{l^2} = \frac{k\sigma\,dA}{l^2} \qquad A = \pi r^2 \Rightarrow dA = 2\pi r\,dr$$

*only need to consider vertical components

$$dE_z = \frac{k\sigma\,dA}{l^2}\sin\theta = \frac{k\sigma h\,2\pi r\,dr}{(r^2+h^2)^{3/2}}$$

$$E_z = \int_0^R \frac{2\pi k\sigma h r\,dr}{(r^2+h^2)^{3/2}} = 2\pi k\sigma\left(\frac{1}{h} - \frac{1}{\sqrt{R^2+h^2}}\right) \qquad \vec{E}_{disk} = 2\pi k\sigma\left(\frac{1}{h} - \frac{1}{\sqrt{R^2+h^2}}\right)\hat{n}$$

Electric Dipole in a uniform electric field



$\vec{p}$ is an electric dipole moment
($\vec{p} = Qd$)

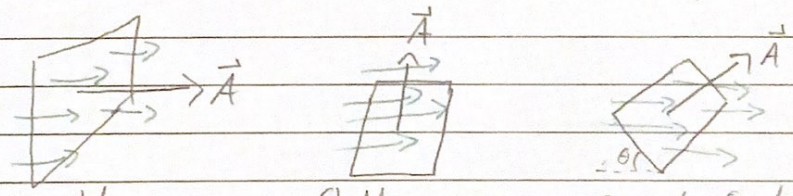
The electric forces cause a net torque on the dipole
$\vec{\tau} = \vec{F} \times \vec{d} = qE\,d\sin\varphi = pE\sin\varphi$
$\vec{\tau} = \vec{p} \times \vec{E}$
This torque does work on the dipole which is equal to a change in potential energy. $U_E = -\vec{p}\cdot\vec{E}$
→ stable equilibrium for $\varphi = 0$
→ unstable equilibrum for $\varphi = \pm\pi$

4.2- Gauss's Law

Gauss's law is a powerful tool for calculating electric fields

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} = \Phi_E$$

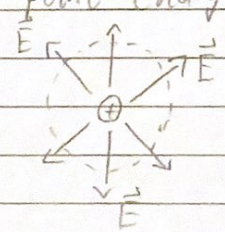
where Φ_E is the electric flux, Q_{enc} is the enclosed charge, and the direction of $d\vec{A}$ is normal to the surface
Flux is the amount of substance flowing through an area



$\Phi = EA\cos\theta$

max flux 0 flux compromised flux

If the field is passing out of a surface, it has positive flux
If the field is entering a surface, it has negative flux
Ex: point charge



$\vec{E}$ points radially out
$d\vec{A}$ also points radially out
$\therefore \vec{E} \parallel d\vec{A} \Rightarrow \vec{E}\cdot d\vec{A} = E\,dA$

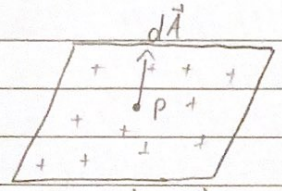
$$\oint \vec{E}\cdot d\vec{A} = \oint E\,dA = EA_{sphere} = \frac{Q}{\epsilon_0}$$

the surface area of a sphere is $4\pi r^2$

$$4\pi r^2 E = \frac{Q}{\epsilon_0} \Rightarrow E = \frac{Q}{4\pi\epsilon_0 r^2}$$

this is the familiar equation for a point charge.

Ex2: Find the electric field for a sheet with charge density σ



$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{enc}}{\epsilon_0} \qquad \sigma = \frac{Q}{A} \Rightarrow Q = \sigma A$$

note $\vec{E} \parallel d\vec{A}$ because of the symmetry

$$\therefore \oint \vec{E}\cdot d\vec{A} = EA$$

this gives the flux for one side. For both sides, $\Phi = 2EA = \frac{\sigma A}{\epsilon_0}$

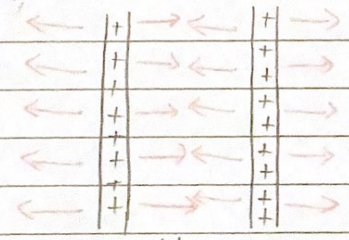
$$\Rightarrow \vec{E}_{plane} = \frac{\sigma}{2\epsilon_0}\hat{n}$$

note, we could also get this by looking at the limiting case of the electric field of a disk.

$$\vec{E}_{disk} = 2\pi k\sigma\left(\frac{1}{h} - \frac{1}{\sqrt{R^2+h^2}}\right) = 2\pi k\sigma\left(\frac{\sqrt{R^2+h^2}-h}{h\sqrt{R^2+h^2}}\right)$$

$$\lim_{R\to\infty} E = 2\pi k\sigma = \frac{\sigma}{2\epsilon_0}$$

Superposition:



If both sheets have the same magnitude, by superposition we get

$E_{center} = 0$

$|E_{left}| = |E_{right}| = 2\cdot\frac{\sigma}{2\epsilon_0} = \frac{\sigma}{\epsilon_0}$

Electric Fields inside an object:

Say we have a uniformly charged sphere with charge density ρ and charge Q

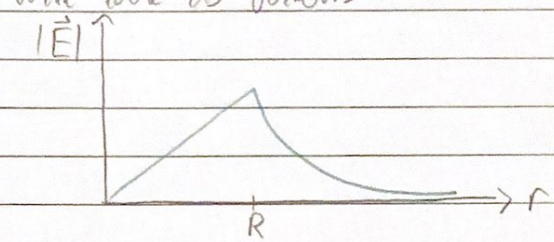
$$\Phi = \oint \vec{E}\cdot d\vec{A} = EA_{sphere} = E(4\pi r^2) = \frac{Q}{\epsilon_0} \Rightarrow E = \frac{Q}{4\pi\epsilon_0 r^2}$$

but note that this charge Q is the <u>total</u> charge and is not constant inside the sphere. ρ, however, is constant at all places.

$$\rho = \frac{Q}{V_{sphere}} \Rightarrow Q = \rho V = \rho \tfrac{4}{3}\pi r^3$$

$$E = \frac{(\rho \frac{4}{3}\pi r^3)}{4\pi r^2 \epsilon_0} = \frac{\rho r}{3\epsilon_0}$$

So, if we graph the electric field of a sphere of radius R, it will look as follows



Cylindrical Symmetry

If the cylinder is infinite then $\vec{E} \parallel d\vec{A}$

$$\oint \vec{E} \cdot d\vec{A} = EA = \frac{Q}{\epsilon_0} \qquad Q = \rho V = \rho \pi r^2 l \qquad A = 2\pi r l$$

$$E = \frac{Q}{A\epsilon_0} = \frac{\rho \pi r^2 l}{2\pi r l \epsilon_0} = \frac{\rho r}{2\epsilon_0}$$

note that over distances where $r \geq R$, the cylinder can be treated as a wire or line of charge and will have the equation $E = \frac{\lambda}{2\pi r \epsilon_0}$

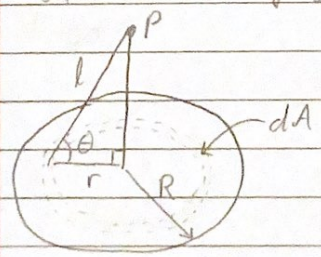
This is also true for spheres and slabs

-> if $r \geq R$, a sphere will emulate a point charge

-> if $s \geq S$, a slab will emulate a plane

Charged Disk Example

- this is one of the more complicated derivations we will do.



$$\sigma = \frac{Q}{A} \Rightarrow dQ = \sigma dA$$

$$A = \pi r^2 \Rightarrow dA = 2\pi r dr$$

$$dQ = \sigma (2\pi r dr)$$

note all horizontal vectors cancel.

$$d\vec{E} = \frac{k\,dQ}{l^2} = \frac{2k\sigma\pi r\,dr}{l^2}$$

$$dE_z = \frac{2k\sigma\pi r\,dr}{l^2}\sin\theta = \frac{2k\sigma\pi h r\,dr}{l^3}$$

$$E_z = \int_0^R \frac{2k\sigma\pi h r\,dr}{(r^2+h^2)^{3/2}} \qquad \text{let } u = r^2+h^2 \quad u(0) = h^2$$

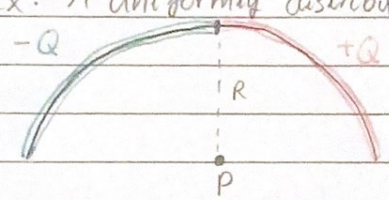
$$du = 2r\,dr \quad u(R) = R^2+h^2$$

$$E_z = \int_{h^2}^{R^2} \frac{k\sigma\pi h\,du}{u^{3/2}} = \left(\frac{-2k\sigma\pi h}{\sqrt{u}}\right)\Bigg|_{h^2}^{R^2} = 2k\pi\sigma\left(\frac{1}{h} - \frac{1}{\sqrt{R^2+h^2}}\right)$$

note if we evaluated this between $-\infty$ and ∞ we would get the electric field for a plane.

Sometimes it may make more sense to integrate with respect to θ

Ex: A uniformly distributed, charged semicircle is shown below. calculate the electric field.



note all vertical vectors cancel.

- will consider each quarter seperate

$$l = \frac{\pi}{2}R$$

$$\lambda = \frac{Q}{l} = \frac{2Q}{\pi R} \Rightarrow dQ = \lambda\,ds$$

$$s = R\theta \Rightarrow ds = R\,d\theta$$

$$d\vec{E} = \frac{k\,dQ}{R^2} = \frac{k\lambda\,ds}{R^2} = \frac{k\lambda R\,d\theta}{R^2} = \frac{k\lambda\,d\theta}{R}$$

$$dE_x = |d\vec{E}|\cos\theta = \frac{k\lambda}{R}\cos\theta\,d\theta$$

$$E_x = \int_0^{\pi/2} \frac{k\lambda}{R}\cos\theta\,d\theta = \frac{k\lambda}{R}\sin\theta\Big|_0^{\pi/2} = \frac{k\lambda}{R} = \frac{2kQ}{\pi R^2}$$

$$\Rightarrow E_{net} = 2E_x = \boxed{\frac{4kQ}{\pi R^2}}$$

Some Useful Derived Equations:

Rod	Ring	Disk
$\vec{E} = \frac{k\lambda l}{r\sqrt{r^2+\frac{l^2}{4}}}\hat{r}$	$\vec{E} = \frac{kQh}{(R^2+h^2)^{3/2}}\hat{n}$	$\vec{E} = 2\pi k\sigma\left(\frac{1}{h} - \frac{1}{\sqrt{R^2+h^2}}\right)\hat{n}$

Sphere: $\vec{E} = \frac{\rho r}{3\epsilon_0}\hat{r}$ for $r \in [0, R]$ $\xrightarrow[\text{for } r \geq R]{}$ Point: $\vec{E} = \frac{kQ}{r^2}\hat{r}$

Cylinder: (Infinite) $\vec{E} = \frac{\rho r}{2\epsilon_0}\hat{r}$ for $r \in [0, R]$ $\xrightarrow[\text{for } r \geq R]{}$ Wire: $\vec{E} = \frac{\lambda}{2\pi r \epsilon_0}\hat{r}$

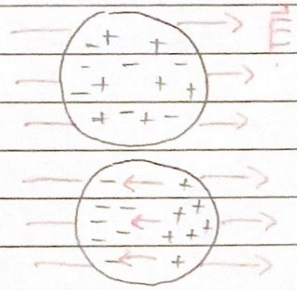
Slab: (Infinite) $\vec{E} = \frac{\rho s}{2\epsilon_0}\hat{n}$ for $s \in [0, S]$ $\xrightarrow[\text{for } s \geq S]{}$ Plane: $\vec{E} = \frac{\sigma}{2\epsilon_0}\hat{n}$

Conductors

A conductor is any material in which the charges are free to move.

Inside a conductor:

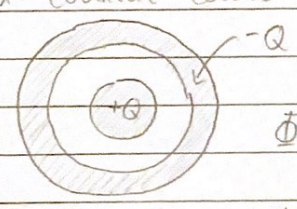
1) there is no net charge
2) there is no electric field



- as an electric field passes through, it induces a force on the electrons, causing them to move
- this creates a new electric field and, by law of superposition, there is no electric field inside and the system is at stable equilibrium

- notice all the charge is accumulated on the surfaces so the conductor will still emit an electric field outside the object

Ex: Coaxial Cable

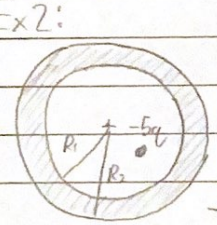


- for the outer shell, find Q on the inside wall and Q on the outside wall.

$\Phi_{cond} = 0 = \frac{Q_{enc}}{\epsilon_0}$ $\therefore$ $Q_{enc} = 0$ for a conductor

so, the inner wall will have $-Q$ and the outer wall must have $+Q$

Ex2:



- by the same logic as above, $Q_{inner} = +5q$ and $Q_{outer} = -5q$
- in this case, the inner charge will not be distributed even because the central point charge is off-center
- the outer charge, however, will be uniformly distributed because it has no connection to what is happening inside because there is no electric field.

- If you have a conductor with a hollow shell, the electric field

inside that hollow shell will also be zero. This is because all the inside electric field lines cancel out.

4.3- Electric Potential and Potential Energy

Relationship Between Gravitational and Electric fields

$$\vec{F}_g = G\frac{m_1 m_2}{r^2}\hat{r} \qquad \vec{F}_e = k\frac{q_1 q_2}{r^2}\hat{r}$$

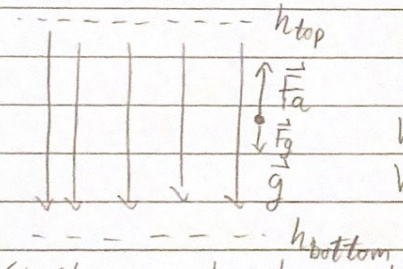
$$\vec{F}_g = m_1\vec{g}_2 \qquad \vec{F}_g = q_1\vec{E}_2$$

$$\vec{g} = \frac{Gm}{r^2}\hat{r} \qquad \vec{E} = \frac{kq}{r^2}\hat{r}$$

$$\Phi_g = \oint \vec{g}\cdot d\vec{A} = -4\pi G m_{enc} \qquad \Phi_e = \oint \vec{E}\cdot d\vec{A} = 4\pi k q_{enc}$$

Uniform Fields

Ex: gravitational fields



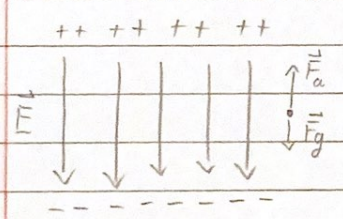
- if an object is moved from h_{bottom} to h_{top}, what is the work done by gravity?

$U = mgh = F_g h$

$W_{ext} = \Delta U = U_{top} - U_{bottom}$

$W_g = -W_{ext} = U_{bottom} - U_{top} = -\Delta U$

Similar can be done for electric fields:



$W_{ext} = F_a \Delta h = qE\Delta h$

$W_{ext} = U_{top} - U_{bottom} = \Delta U > 0$

$W_E = -\Delta U = U_{bottom} - U_{top}$

Recall the work energy theorem:

$$W = \int_a^b F\,dx = \int_a^b ma\,dx = \int_a^b ma\frac{dx}{dt}dt = \int_a^b mav\,dt = \int_a^b mv\frac{dv}{dt}dt = \int_a^b mv\,dv = \frac{1}{2}mv^2\Big|_a^b$$

$\therefore W = \Delta K$

If we consider our system to be comprised of only kinetic and potential energy, we get $K + U = const = E_{total}$

$\Rightarrow K_1 + U_1 = K_2 + U_2 \Rightarrow \Delta K = -\Delta U$

$\therefore W = -\Delta U$

Potential Energy of a point charge:

$$W = \int_a^b \vec{F} \cdot d\vec{r} = \int_a^b \frac{kq_1q_2}{r^2} \cdot d\vec{r} = -\frac{kq_1q_2}{r^2}\Big|_a^b = -\Delta U$$

$$\therefore U = \frac{kq_1q_2}{r}$$

- Electric Potential

note of notation: V = electric potential and ΔV = voltage. In circuits, V and ΔV are often used interchangably.

Voltage is defined to be the potential energy change per unit charge

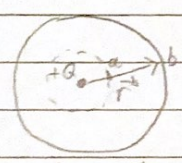
$$\Delta V = \frac{\Delta U}{q}$$

note that $\frac{U}{q}$ is analogous to $-\int \vec{E} \cdot d\vec{l}$

$$\begin{tikzcd} \vec{F} \arrow[r, "F/q"] \arrow[d, "\int F dx"] & \vec{E} \arrow[d, "\int E dx"] \\ U \arrow[r, "U/q"] & V \end{tikzcd}$$

* notation for ΔV
$V_{ab} = V_a - V_b$ and is denoted as the voltage of a with respect to b.
- This is why ΔV is opposite to ΔS and such.

Potential for a Point Charge



$$\Delta V = -\int_a^b \vec{E} \cdot d\vec{l} = -\int_a^b \frac{kQ}{r^2} dr = -\left(\frac{-kQ}{r}\right)\Big|_a^b = \frac{kQ}{b} - \frac{kQ}{a}$$

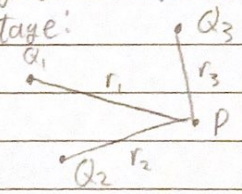
so V for a point charge is $V = \frac{kq}{r}$

note, this is for if you integrate outwards. If you integrate inwards, $d\vec{l} = -d\vec{r}$ and the negatives become messy.

In general, we state that $V(\infty) = 0$. This is to avoid having a constant in the equation of potential after integrating.

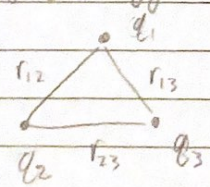
- Systems of Multiple Charges

Votage:



$$V_{system} = V_1 + V_2 + V_3 = k\left(\frac{Q_1}{r_1} + \frac{Q_2}{r_2} + \frac{Q_3}{r_3}\right)$$

Potential Energy: The potential energy of a system is equal to the amount of work required to create that system.



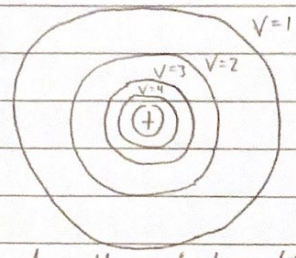
$$U_{system} = U_{12} + U_{13} + U_{23} = k\left(\frac{q_1q_2}{r_{12}} + \frac{q_1q_3}{r_{13}} + \frac{q_2q_3}{r_{23}}\right)$$

more generally, $U_{sys} = k \sum_{\substack{i=1 \\ j>i}}^{N} \frac{q_i q_j}{r_{ij}}$ for $i \neq j$

we could also use $U_{sys} = k \sum_{i=1, j=1}^{N} \frac{q_i q_j}{2 r_{ij}}$ for $i \neq j$ *note the 2 is to account for double counting in the notation

*can think of the summation as a double 'for loop'

- Equipotential Lines



- analogous to contour lines on a map
- all points on the same equipotential line have the same potential (hence the name) and ΔV on a line $= 0$
- the closer together the lines are, the faster the potential is changing.

- at all points, the equipotential lines are orthogonal to the electric field. ($\vec{E} \perp$ eq. PL)

*I refered to them as lines but they are actually a gaussian surface in 3D

- Electric Field and Potential

recall $dV = -\vec{E} \cdot d\vec{l}$

since V is usually easy to calculate, we can use this to find the electric field.

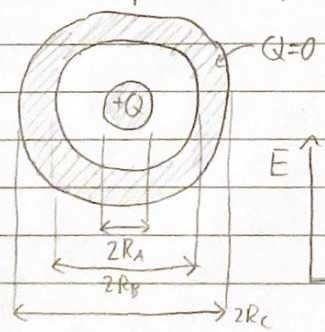
when traveling in a radial direction, $E_r = \frac{-dV(r)}{dr}$

more generally, $\vec{E} = -\vec{\nabla} V$

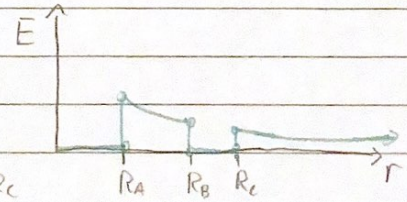
this the equivilant of saying $\vec{E} = -\frac{\partial V}{\partial x}\hat{i} - \frac{\partial V}{\partial y}\hat{j} - \frac{\partial V}{\partial z}\hat{k}$

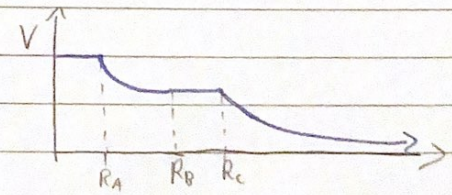
The electric field is the derivative of potential, meaning that when the potential is constant ($\Delta V = 0$) there is no electric field.

Ex: Graph the potential and electric field.

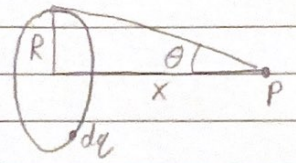


Note there is no electric field in the conductor so the potential will be constant





Ex2: Find the electric field of a ring at point P



$$dV = \frac{k\,dq}{r} = \frac{k\,dq}{\sqrt{x^2+R^2}}$$

note that x, and R are constants

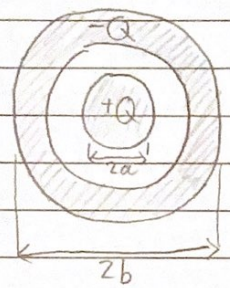
$$\therefore V = \frac{k}{\sqrt{x^2+R^2}}\int dq = \frac{kq}{\sqrt{x^2+R^2}}$$

* const = 0 because $V(\infty)=0$

$$E = -\vec{\nabla}V \Rightarrow -E_x = -\frac{\partial V}{\partial x} \Rightarrow E_x = \frac{kQx}{(x^2+R^2)^{3/2}}$$

4.4- Capacitance and Dielectrics

Ex: capacitance of a cylindrical capacitor (coaxial cable)



recall $C = \frac{Q}{\Delta V}$ where ΔV is the potential difference between the two conducting surfaces ($\Delta V = V_{ab}$)

Inner Surface: $\oint \vec{E}\,d\vec{A} = E_r A = \frac{Q}{\epsilon_0} \Rightarrow E_r = \frac{Q}{2\pi l \epsilon_0 r}$

Outer Surface: note Q_{enc} for the outer surface is $Q-Q=0 \therefore E=0$

$$V = -\int E_r\,dr = -\int \frac{2kQ}{lr}\,dr = -\frac{2kQ}{l}\ln|r| + C$$

we know $V(b)=0$

$$\therefore -\frac{2kQ}{l}\ln b + C = 0 \Rightarrow C = \frac{2kQ}{l}\ln b$$

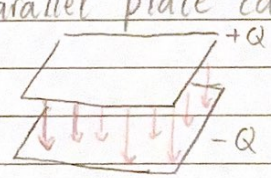
* we also could've integrated between a and b.

$$V_a = -\frac{2kQ}{l}\ln a + \frac{2kQ}{l}\ln b = \frac{2kQ}{l}\ln\left(\frac{b}{a}\right)$$

$$V_{ab} = V_a - V_b = \frac{2kQ}{l}\ln\left(\frac{b}{a}\right) - 0$$

$$C = \frac{Q}{V_{ab}} = \boxed{\frac{l}{2k\ln(b/a)}}$$

-Parallel plate capacitors



$$E_+ = \frac{\sigma}{2\epsilon_0}, \quad E_- = \frac{\sigma}{2\epsilon_0}$$

$$E_{total} = \frac{\sigma}{\epsilon_0}$$

$$|\Delta V| = \int_0^d E\,ds = \int_0^d \frac{\sigma}{\epsilon_0} = \frac{\sigma}{\epsilon_0} d$$

$$C = \frac{Q}{V} = \frac{\sigma A}{\sigma d}\epsilon_0 = \frac{A\epsilon_0}{d} \qquad C_{\parallel} = \frac{A\epsilon_0}{d}$$

Energy stored in capacitor

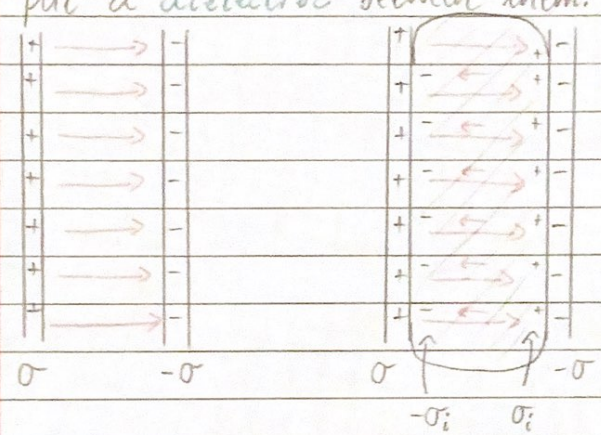
$$U = \frac{Q^2}{2C} = \frac{Q^2 d}{2A\epsilon_0}$$

If we want the attractive force between plates we can take

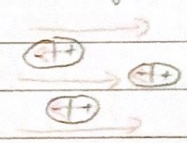
$$\vec{F} = -\vec{\nabla} U = -\frac{\partial}{\partial d}\frac{Q^2 d}{2A\epsilon_0} = \frac{-Q^2}{2A\epsilon_0}$$

- Dielectrics

to counteract this attractive force between capacitor plates we put a dielectric between them.



the electric field rotates the molecules in the dielectric, causing them to generate a small electric field in the other direction



The electric field with a dielectric is the original electric field minus the induced field ($E = E_0 - E_{ind}$)

We define the dielectric constant κ as $\kappa = \frac{E_0}{E} = \frac{\sigma}{\sigma - \sigma_i}$

We can also get,

$$E_0 = \kappa E \qquad \sigma_i = \sigma\left(1 - \frac{1}{\kappa}\right)$$

$$-\int E_0 ds = -\int \kappa E ds \Rightarrow V = \frac{V_0}{\kappa}$$

$$\frac{Q}{V} = \frac{Q\kappa}{V_0} \Rightarrow C = \kappa C_0$$

We can also adjust Gauss's law for dielectrics:

$$\Phi_E = \oint \kappa \vec{E} \cdot d\vec{A} = \frac{Q_{conductor}}{\epsilon_0}$$

note that we only consider the charge in the inductor, not the dielectric.

Ex: A parallel plate capacitor has $C_0 = 2pF$. A dielectric ($\kappa = 3.9$) is inserted. Find the new capacitance.

$C = \kappa C_0 = (3.9)(2p) = 7.8pF$

What happens to the energy?

$U = \frac{Q^2}{2C}$ - charge stays constant, capacitince goes up

$\therefore$ U goes down

What about if the capacitor is attached to a voltage source?

$U = \frac{Q^2}{2C} = \frac{1}{2}CV^2$ - V is constant, C goes up

$\therefore$ U goes up.

- Electric Field Energy Density

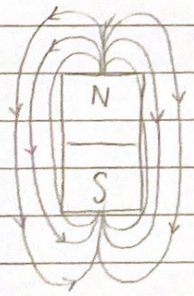
defined as $u = \frac{U}{V}$ and $U = \int u\,dV$

use $U = \frac{1}{2}CV^2 = \frac{1}{2}\left(\frac{A\epsilon_0}{d}\right)V^2$ for parallel plates

$V = Ad$

$u = \frac{U}{V} = \frac{1}{2}\left(\frac{A\epsilon_0}{Ad^2}\right)V^2 \Rightarrow u = \frac{1}{2}\epsilon_0 E^2$

4.5 - Magnetic Force, Torque, and Cyclotron Motion



- magnetic fields start from the north pole and travel to the south pole.

* The Earth's geographic north pole is actually a magnetic south pole.

- there are no magnetic monopoles (field is always continuous)

$\vec{F} = q\vec{v} \times \vec{B} = qvB\sin\theta$ where B is the magnetic field.

The direction of F can be determined using the right hand rule.

Magnetic field is measured in Teslas (T)

The work done by the magnetic force is zero:

$W = \int \vec{F} \cdot d\vec{r}$

$W = \int (q\vec{v} \times \vec{B}) \cdot d\vec{r}$, $d\vec{r} = \vec{v}\,dt$

$W = \int (q\vec{v} \times \vec{B}) \cdot \vec{v}\, dt$

note $(q\vec{v} \times \vec{B}) \perp \vec{v}$

$\therefore (q\vec{v} \times \vec{B}) \cdot \vec{v} = 0$

$\therefore W = 0$

• vector notations

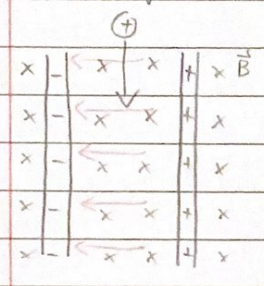
⊙ means pointing out of page

× means pointing into page

If an electric field is also present, the total force is given by Lorentz Law: $\vec{F} = q\vec{E} + (q\vec{v} \times \vec{B})$

Applications:

Velocity selector



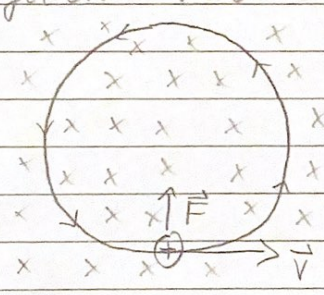
$\vec{F}_{tot} = (q\vec{v} \times \vec{B} - q\vec{E})\hat{i}$

only particles with a specific velocity satisfy $F = 0$, allowing them to travel through.

All other particles will be deflected.

A mass spectrometer is similar except after the particles exit, the amount they are deflected by the magnetic field will be different depending on their mass.

- Cyclotron Motion



A charge moving through a magnetic field will tend to have a circular path. This is because $\vec{F}$ always points radially.

$\vec{F} = q\vec{v} \times \vec{B} = qv_\perp B$

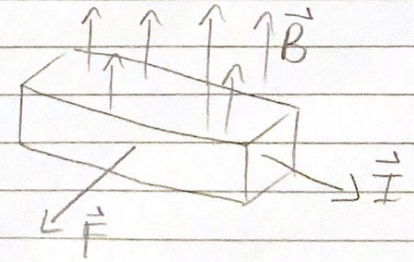
recall from centripetal motion, $F = \frac{mv_\perp^2}{R}$

$qv_\perp B = \frac{mv_\perp^2}{R} \Rightarrow R = \frac{mv_\perp}{qB}$

$v_\perp = \frac{2\pi R}{T} \Rightarrow \omega = \frac{v_\perp}{R}$

$\Rightarrow \omega = \frac{qB}{m}$

- Force on a current carrying wire.



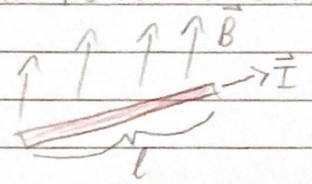
From, $\vec{F} = q\vec{v} \times \vec{B}$, we can break it into the sum of all infinitesimal point charges, dq flowing through the current.

$\vec{F} = \int d\vec{F} = \int dq\,\vec{v} \times \vec{B}, \quad \vec{v} = \frac{d\vec{l}}{dt}$

$d\vec{F} = \frac{dq}{dt} d\vec{l} \times \vec{B} = I\,d\vec{l} \times \vec{B}$

$\Rightarrow F = \int I\,d\vec{l} \times \vec{B}$ is the force on a current carrying wire in a magnetic field.

Ex: Force on a straight wire



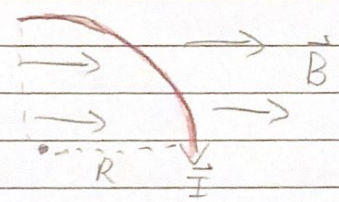
$F = \int I\,d\vec{l} \times \vec{B}$, note $d\vec{l} \times \vec{B} = dlB\sin\phi$

ϕ is the angle between $\vec{B}$ and $d\vec{l}$. Note that in this example, $\phi = 90°$ for all points.

$\Rightarrow d\vec{l} \times \vec{B} = dlB$

$F = \int I\,dlB = \underline{IlB}$

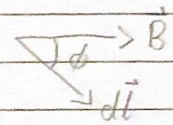
Ex2: Force on a curved wire



$F = \int I\,d\vec{l} \times \vec{B} = \int IdlB\sin\phi$

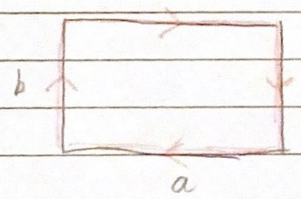
$l = R\phi \Rightarrow dl = Rd\phi$

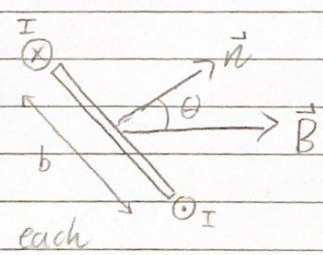
$F = \int_0^{\pi/2} IRBd\phi\sin\phi$



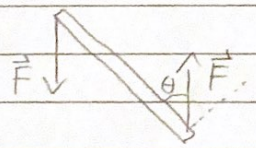
$F = IRB\cos\phi\Big|_0^{\pi/2} = IRB$

- Torque on a Current Carrying Loop





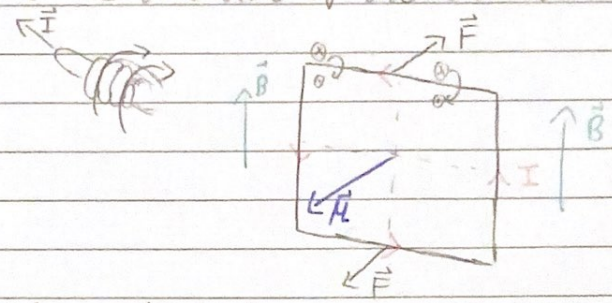
We can find the force on each wire. Note the side wires will have forces in opposite directions, resulting no torque from them.



$\vec{F} = IaB$

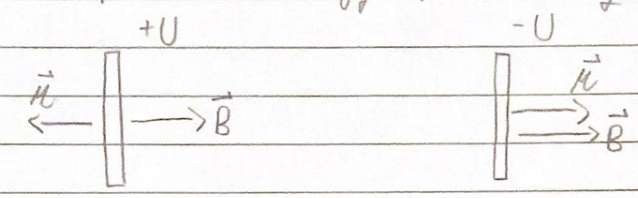
$\Rightarrow$ the net torque works out to $\vec{\tau} = IBab\sin\theta$ in the ccw direction

This expression for torque on a loop can be expressed more generally using the term $\vec{\mu}$ where $|\vec{\mu}| = NIA$ and $\vec{\tau} = \vec{\mu} \times \vec{B}$
$\vec{\mu}$ will point normal to the loop. The direction can be found with the right hand rule by curling your fingers with your thumb pointing in the direction of the current.

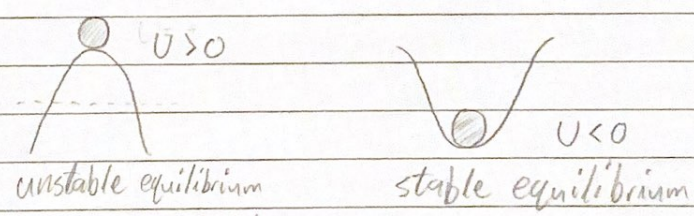


*N represents the number of coils. I most cases, we take N=1.

The potential energy of a rotating coil is given as $U = -\vec{\mu} \cdot \vec{B}$



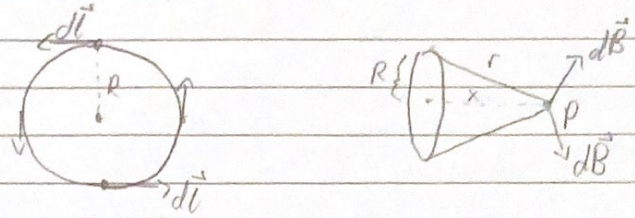
Of these two cases, the one on the left has potential energy +U, meaning that the system is unstable and wants to expel energy. The one on the right has potential energy of -U, meaning it is stable and requires you to input some energy to change it.
Analogy to gravity:



4.6 Magnetic Fields
note that unlike electric fields, magnetic fields are continuous and do not converge in any one place. (All magnetic field lines that enter a region must exit).
This is expressed through $\oiint \vec{B} \cdot d\vec{A} = 0$ and $\vec{\nabla} \cdot \vec{B} = 0$
Biot-Savart Law:
this is very useful for determining the magnetic field. (I won't go into the derivation).

$$d\vec{B} = \frac{\mu_0}{4\pi}\frac{I\,d\vec{l}\times\hat{r}}{r^2}$$

Ex: Field from a circular loop.



*notice the components cancel so only x-components are left

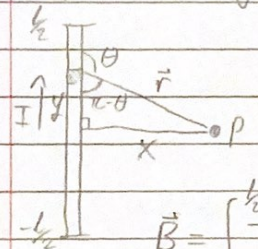
$$d\vec{B} = \frac{\mu_0 I\,d\vec{l}\times\hat{r}}{4\pi r^2} = \frac{\mu_0 I\,dl}{4\pi r^2}, \quad d\vec{B}_x = \frac{\mu_0 I\,dl}{4\pi r^2}\cdot\frac{R}{r}\,\hat{i}$$

$$\vec{B} = \oint\frac{\mu_0 I\,dl\,R\,\hat{i}}{4\pi(x^2+R^2)^{3/2}} = \frac{\mu_0 I R(2\pi R)}{4\pi(x^2+R^2)^{3/2}}\,\hat{i}$$

$$\vec{B} = \frac{\mu_0 I R^2}{2(h^2+R^2)^{3/2}}\,\hat{n}$$

as $x \to 0$, $\vec{B} = \frac{\mu_0 I}{2R}\,\hat{n}$

Ex 2: Field from a straight wire



$$d\vec{B} = \frac{\mu_0 I\,d\vec{l}\times\hat{r}}{4\pi r^2} = \frac{-\mu_0 I\,dy\sin\theta}{4\pi r^2}\,\hat{k}$$

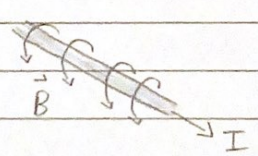
$$\sin\theta = \sin(\pi-\theta) = \frac{x}{r}$$

$$\vec{B} = \int_{-l/2}^{l/2}\frac{-\mu_0 I\,x\,dy}{4\pi(x^2+y^2)^{3/2}}\,\hat{k} = \frac{-\mu_0 I}{4\pi}\,\frac{y}{x\sqrt{x^2+y^2}}\,\hat{k}\Big|_{-l/2}^{l/2}$$

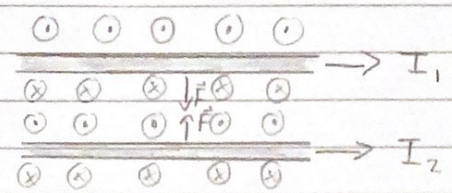
$$\vec{B} = \frac{-\mu_0 I\,l}{4\pi r\sqrt{r^2+\frac{l^2}{4}}}\,\hat{k}$$

as $l \gg r$, $\vec{B} = \frac{-\mu_0 I}{2\pi r}\,\hat{k}$

Note the direction of the magnetic field is such that it curls around the wire.

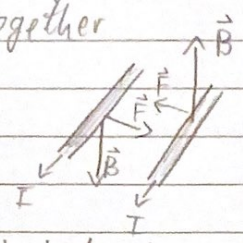


What if we have two wires with current in the same direction?



note in the center the magnetic fields interfere destructively.

The field from the opposite wire induces a force, pulling the wires together



Similarly, for wires with current in opposite directions, they would repel each other

assuming $l \gg d$, $B_1 = \frac{\mu_0 I_1}{2\pi d}$

$$\vec{F} = I_2 \vec{l} \times \vec{B}_1 = \frac{I_2 l \mu_0 I_1}{2\pi d}$$

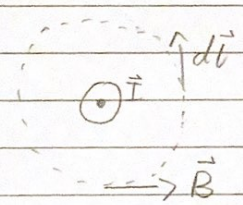
so, the force per unit length between wires is $\frac{F}{l} = \frac{\mu_0 I_1 I_2}{2\pi d}$

note the force on each wire is equal and opposite

- Ampère's Law

This is an easier alternative to Biot-Savart Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc}$$

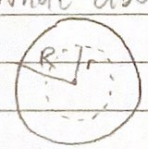


Similar to Gauss's law, we can create a path, $\vec{l}$ that is perpendicular to $\vec{B}$ at all points in order to easily evaluate the integral and find the magnetic field.

So, for a straight wire as shown, $\oint \vec{B} \cdot d\vec{l} = Bl$ where l is the circumference of a circle. $Bl = B 2\pi r = \mu_0 I_{enc}$

$\Rightarrow B = \frac{\mu_0 I_{enc}}{2\pi r}$ which is a familiar expression

What about inside a wire?



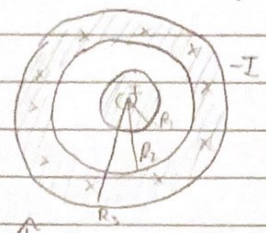
$$\oint \vec{B} \cdot d\vec{l} = B 2\pi r = \mu_0 I_{enc}$$

$$I_{enc} = \frac{\pi r^2}{\pi R^2} I = \frac{I r^2}{R^2}$$

$\Rightarrow B = \frac{\mu_0 I r}{2\pi R^2}$ for $r < R$

Ex: Coaxial Cable

The cable consists of a wire a the center of current I and an outer shell carrying an equal and opposite current.

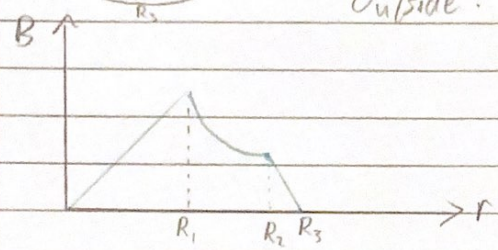


Inside: $B \propto r$

Between: $B \propto 1/r$

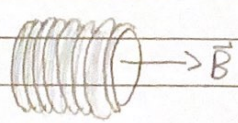
Outer Shell: $B \propto -r$

Outside: $B = 0$ (no I_{enc})

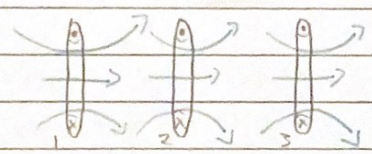


- Solenoids

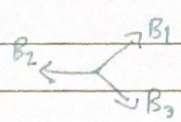
A solenoid is a long coiled piece of wire, similar to an inductor



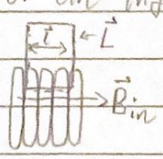
Cross Section



the field outside of the solenoid cancels while it is constant inside



For an infinite solenoid, we can use Ampere's Law:



*note we draw our box really high because at the top, the field will be 0.

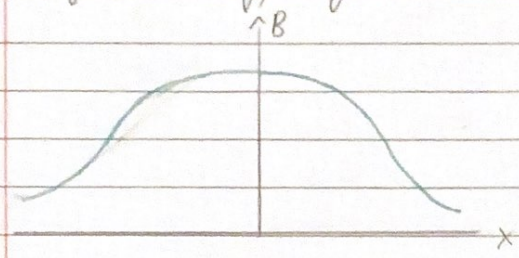
$\oint \vec{B} \cdot d\vec{L} = B_{in} l$ note we take l instead of L because $\vec{B} \cdot d\vec{L}$ is only non-zero for the stretch l.

$B_{in} l = \mu_0 I_{enc}$, $I_{enc} = IN$, where $N = \#$ of loops.

$B_{in} = \frac{\mu_0 I N}{l}$ we can express $n = \frac{N}{l}$ where n is the loop density.

so, $B = \mu_0 I n$ for an infinite solenoid

unfortunatley, engineers have to deal with finite length solenoids.



- finite solenoids do not behave with constant fields.

- Derivation of magnetic field for finite solenoid

recall the magnetic field of a circular loop as derived earlier:

$$B_x = \frac{\mu_0 I R^2}{2(x^2+R^2)^{3/2}}$$

We can think of the total magnetic field as being the sum of all of these loops.

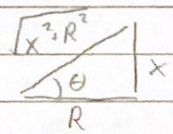
The current, I is analogous to I_{enc} and will be small for each loop, so

$$I_{enc} = \int dI = I dN$$

$$dB_x = \frac{\mu_0 dI R^2}{2(x^2+R^2)^{3/2}} = \frac{\mu_0 I dN R^2}{2(x^2+R^2)^{3/2}}$$

$$n = \frac{N}{L} = \frac{dN}{dx} \Rightarrow dN = n dx$$

$$dB_x = \frac{\mu_0 I n dx R^2}{2(x^2+R^2)^{3/2}}$$



$x = R\tan\theta$ $\quad \sqrt{x^2+R^2} = R\sec\theta$

$dx = R\sec^2\theta\, d\theta$

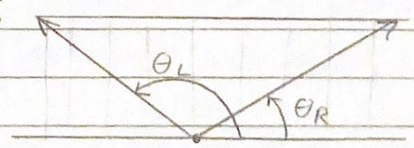
$$B = \int \frac{\mu_0 I n R^3 \sec^2\theta\, d\theta}{2R^3\sec^3\theta} = \int \frac{\mu_0 I n \cos\theta\, d\theta}{2} = \frac{\mu_0}{2} I n \sin\theta \Big|_{\theta_1}^{\theta_2}$$

$$\sin\theta = \frac{x}{\sqrt{x^2+R^2}}$$

$$B = \frac{\mu_0 I n x}{2\sqrt{x^2+R^2}} \Big|_{x_L}^{x_R}$$

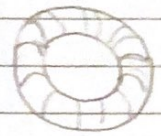
Similarly, we can get:

$$B = \frac{\mu_0 I n}{2} \cos\theta \Big|_{\theta_L}^{\theta_R}$$

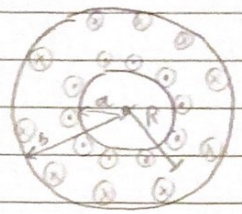


refering to the diagram

- Toroidial Solenoid



- like a donut wrapped in wire



Use Ampere's law:

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$$

$$I_{enc} = IN$$

$$\oint \vec{B}\cdot d\vec{l} = Bl = \mu_0 IN, \quad l = 2\pi r$$

$$\Rightarrow B = \frac{\mu_0 IN}{2\pi r} \quad \text{for } a<r<b$$

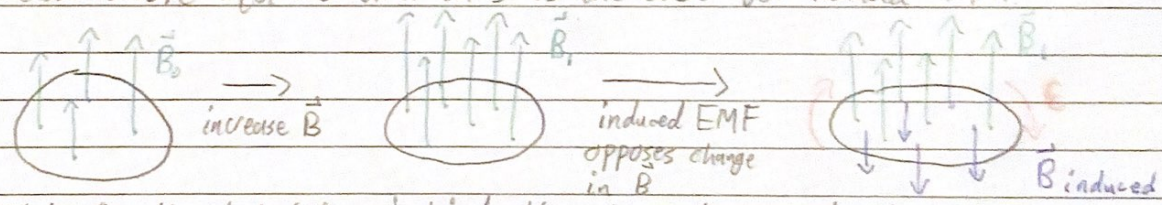
4.7- Electromagnetic Induction

Faraday's Law

$$\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt} = \mathcal{E}$$

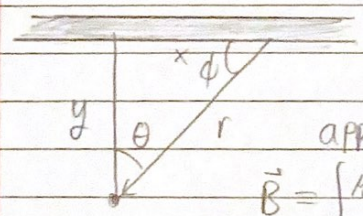
- A changing magnetic flux creates an induced electric field and, therefore, an induced EMF

Recall that when a system is changed, it will generally try to reestablish equilibrium. This is the case for induced EMF



This is the intuition behind the formula, and also where the negative comes from

- Revisit to magnetic field of a finite wire offcenter



$$d\vec{B} = \frac{\mu_0 I\, d\vec{x}\times\hat{r}}{4\pi r^2} = \frac{\mu_0 I\, dx}{4\pi r^2}\sin\phi$$

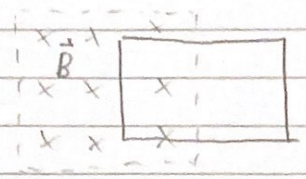
approximate $d\vec{x} = r\,d\theta$, also note $\sin\phi = \cos\theta$

$$\vec{B} = \int \frac{\mu_0 I \cos\theta\, d\theta}{4\pi r}$$

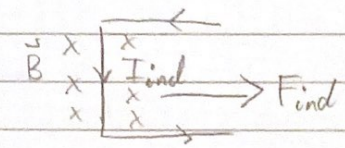
$$\Rightarrow B = \frac{\mu_0 I}{4\pi r}\sin\theta \Big|_{\theta_L}^{\theta_R} = \frac{\mu_0 I x}{4\pi r\sqrt{x^2+r^2}}\Big|_{x_L}^{x_R}$$

Ex: magnetic induction

A conducting loop is halfway inside a uniform B field. Suppose B is increased rapidly. What happens to the loop?

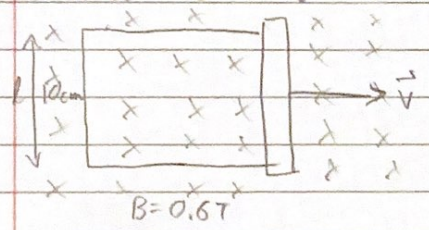


- induced B field is out of page
∴ current flows ccw.



⇒ the force on the left rod from the magnetic field pushes the loop right.

Ex2: You pull a metal rod to the right at a constant speed of 2.5m/s. The metal rod, which has an electrical resistance $R = 0.3\,\Omega$ is free to move on a U-shaped conductor. What is the magnitude and direction of the induced electrical current?



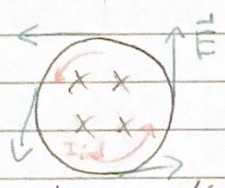
$\Phi_B = BA$, $A = lx$, B is const

$\Phi_B = Blx$

$\frac{d\Phi_B}{dt} = Bl\frac{dx}{dt} = Blv$

$\Rightarrow \mathcal{E} = -Blv = 0.5A$ ccw

Note that if there is a current flowing around this loop, there must also be an induced electric field. This is the other part of Faraday's law: $\oint \vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$



We know the magnetic field is related to the movement of charged particles and current. This is expressed in Ampere's law:

$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc}$.

However, current is also related to electric field so we can include

another term to get the Ampere-Maxwell Equation

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 i_{enc} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

the term $i_{disp} = \epsilon_0 \frac{d\Phi_E}{dt}$ is called the displacement current

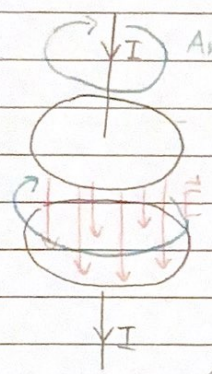
This gives us all four of Maxwell's equations:

$$\Phi_E = \oiint \vec{E}\cdot d\vec{A} = \frac{q_{enc}}{\epsilon_0} \qquad \oiint \vec{B}\cdot d\vec{A} = 0$$

$$\oint \vec{E}\cdot d\vec{s} = \frac{-d\Phi_B}{dt} \qquad \oint \vec{B}\cdot d\vec{s} = \mu_0 i_{enc} + \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

note that $\mu_0 \epsilon_0 = \frac{1}{c^2}$ are equivilant.

Example of displacement current: charging capacitor



loop around wire: $\frac{d\Phi_E}{dt} = 0$

$$\Rightarrow \oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} \Rightarrow B = \frac{\mu_0 I}{2\pi r}$$

loop around capacitor: $I_{enc} = 0$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

$$E = \frac{\sigma}{\epsilon_0} = \frac{q}{A\epsilon_0} \Rightarrow \Phi_E = EA = \frac{q}{\epsilon_0}$$

$$\frac{d\Phi_E}{dt} = \frac{1}{\epsilon_0}\frac{dq}{dt} = \frac{i}{\epsilon_0}$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 \epsilon_0 \frac{i}{\epsilon_0} = \mu_0 i \Rightarrow B = \frac{\mu_0 i}{2\pi r}$$

$\therefore$ the current is the same all the way through despite there not being any physical current through the capacitor

i.e. the current is conserved

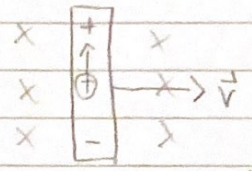
Ex: At what rate must the potential difference increase across a $1\mu F$ capacitor to create a $1A$ displacement current?

$$V = \frac{q}{C} \Rightarrow \frac{dV}{dt} = \frac{1}{C}\frac{dq}{dt} = \frac{i}{C} = 10^6 \, V/s$$

- Motional EMF

say we move a conducting bar through a magnetic field with some speed $\vec{v}$.

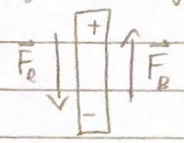
This motion induces a magnetic force on the charged particles in the bar, causing them to seperate



recall $\vec{F} = q\vec{v} \times \vec{B}$

so, positive and negative charges seperate

This seperation induces an electric field and electric force of equal magnitude



$F_e = F_B \Rightarrow qE = q\vec{v} \times \vec{B} \Rightarrow E = vB$

we can relate this to emf using $V = \int \vec{E} \cdot d\vec{l}$

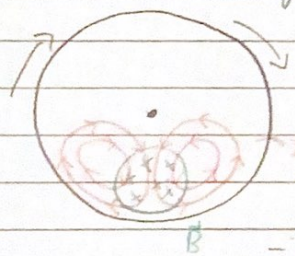
$V = \mathcal{E} = El = vBl$

$\rightarrow \mathcal{E}_{motional} = El = vBl$

In general, the equation for motional emf is $\mathcal{E} = \oint (\vec{v} \times \vec{B}) \cdot d\vec{l}$

- Eddy Currents

metal disk rotating through magnetic field

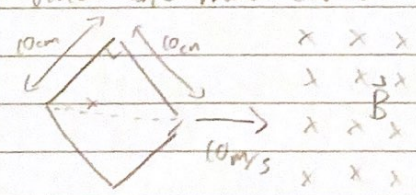


As the wheel passes through the magnetic field, the induced force on the electrons induces a current

- A common application is in metal detectors

- Examples of magnetic induction:

Ex1: A bar moves into a 0.8T magnetic field at 10m/s at $t=0$. If $R = 0.1\Omega$, find the max current



x for front half only

$A = 2\left(\frac{x^2}{2}\right) = x^2 \Rightarrow \frac{dA}{dt} = 2x\frac{dx}{dt} = 2xv$

$\Phi_B = BA$

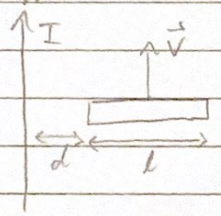
$|\mathcal{E}| = \left|\frac{d\Phi_B}{dt}\right| = B\frac{dA}{dt} = 2Bxv$

$\mathcal{E} = iR \Rightarrow i = \mathcal{E}/R$

$i = \frac{2Bxv}{R}, \quad x_{max} = \frac{10cm}{\sqrt{2}}$

$\Rightarrow i_{max} = \frac{2Bx_{max}v}{R} = \frac{2(0.8)(0.07)(10)}{0.1} = 11.3A$

Ex2: A metal bar moves next to a wire. find the potential difference in the bar.



$F_e = F_B \Rightarrow qE = q\vec{v}\times\vec{B}$

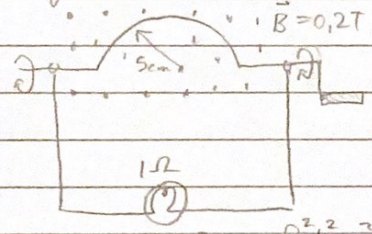
$E = vB$

$V = \int E\,dx = \int vB\,dx$

$\vec{B} = \frac{\mu_0 I}{2\pi r}$

$V = \int_d^{d+l} \frac{v\mu_0 I}{2\pi x}dx = \frac{v\mu_0 I}{2\pi}\ln\left(\frac{d+l}{d}\right)$

Ex3: A mechanical generator is shown at $t=0$. What frequency do you need to turn the crank to generate 4W of peak power?



$\mathcal{E} = -\frac{d\Phi_B}{dt} = -BA\frac{d}{dt}(\cos\theta)$

$|\mathcal{E}| = BA\sin\theta\frac{d\theta}{dt} = BA\omega\sin\theta$

$P = i^2R = \mathcal{E}^2/R = \frac{B^2A^2\omega^2\sin^2\theta}{R}$ for P_{max}, take $\sin\theta = 1$

$\frac{\sqrt{RP}}{BA} = \omega = \frac{\sqrt{RP}}{B\pi r^2} \Rightarrow f = \frac{\sqrt{RP}}{B\pi^2 r^2} = 405Hz$

4.8- Electromagnetic Waves

Maxwell understood light is an oscillation of the electromagnetic field.

In empty space, both charge and current are 0.

This makes Maxwell's equations assume the following form:

$\oint\vec{E}\cdot d\vec{A} = 0, \quad \oint\vec{E}\cdot d\vec{l} = -\frac{d\Phi_B}{dt}$

$\oint\vec{B}\cdot d\vec{A} = 0, \quad \oint\vec{B}\cdot d\vec{l} = \mu_0\epsilon_0\frac{d\Phi_E}{dt}$

$\Phi_B = \iint \vec{B} \cdot d\vec{A}$, $\Phi_E = \iint \vec{E} \cdot d\vec{A}$

$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt} = -\frac{d}{dt}\iint \vec{B} \cdot d\vec{A}$

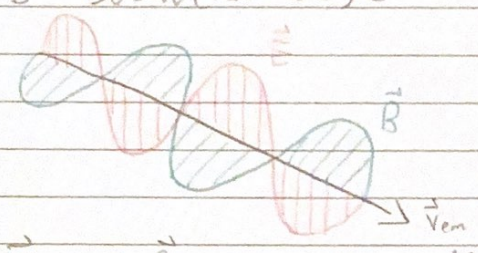
$\oint \vec{B} \cdot d\vec{l} = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt} = \mu_0 \epsilon_0 \frac{d}{dt}\iint \vec{E} \cdot d\vec{A}$

this shows that a time-varying electric field induces a time-varying magnetic field and vice versa

Solving for $\vec{B}$ and $\vec{E}$ gives

$\vec{E} = E_0 \sin(kx - \omega t)\,\hat{j}$

$\vec{B} = B_0 \sin(kx - \omega t)\,\hat{k}$



$\vec{E}$ and $\vec{B}$ are perpendicular to the direction the wave travels. ($\vec{E} \times \vec{B} \parallel \vec{v}_{em}$)

the speed this wave travels in a vacuum is $v_{em} = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$

this is defined by c.

$c = 3.00 \cdot 10^8 \, m/s$

and $\frac{1}{c^2} = \mu_0 \epsilon_0$

there is also the relationship $E = cB$ at any point of the wave.

The energy in an electromagnetic wave can be considered using energy densities.

recall $u = \frac{1}{2}\epsilon_0 E^2 + \frac{B^2}{2\mu_0}$

$u = \frac{1}{2}\epsilon_0 E^2 + \frac{c^2 E^2}{2\mu_0} = \frac{1}{2}\epsilon_0 E^2 + \frac{\mu_0 \epsilon_0 E^2}{2\mu_0}$

$u = \epsilon_0 E^2 = \frac{B^2}{\mu_0} = \frac{EB}{\mu_0 c}$

- Solar Wind

Although we don't normally think of photons and particle momentum at the same time, there is a pressure exerted by light.

S = rate of energy flow = $\frac{\text{power}}{\text{area}}$

$\vec{S} = \frac{\vec{E} \times \vec{B}}{\mu_0}$ and $S_{avg} = \frac{E_{max} B_{max}}{2\mu_0} = I$ = intensity of wave

This relates back to blackbody radiation in Phys 157

The radiation pressure is defined as

$$P_{rad} = \frac{I}{c} = \frac{S_{avg}}{c}$$

— End of Phys 158 —