# Math 257 Notes

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## Contents



## 1 Ordinary Differential Equations

### 1.1 Analytical Solutions

#### 1.1.1 Cauchy-Euler Equation

The Cauchy-Euler equation is a 2nd order ODE of the form

$$ax^2y'' + bxy' + cy = 0$$

To solve this equation, we will guess that the solution will be of the form $y = x^r$.
This gives $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$
So if we plug these into the original equation we get

$$ar(r-1)x^r + brx^r + cx^r = 0$$

$$x^r(ar^2 + (b-a)r + c) = 0$$
$$ar^2 + (b-a)r + c = 0$$

giving the characteristic equation.
We can solve for $r$ and get 3 cases:
Real roots: $(b-a)^2 - 4ac > 0$

$$r = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a}$$
$$y(x) = C_1 x^{r_1} + C_2 x^{r_2}$$

Imaginary roots: $(b-a)^2 - 4ac < 0$

$$r = -\frac{(b-a)}{2a} \pm i\frac{\sqrt{4ac - (b-a)^2}}{2a} = \lambda \pm i\mu$$
$$y(x) = C_1 x^{(\lambda + i\mu)x} + C_2 x^{(\lambda - i\mu)x}$$
$$y = x^\lambda \left(C_1 x^{i\mu} + C_2 x^{-i\mu}\right)$$
$$y = x^\lambda \left(C_1 e^{i\mu \ln x} + C_2 e^{-i\mu \ln x}\right)$$
$$y = x^\lambda \left(C_1 \cos(\mu \ln x) + C_2 \sin(\mu \ln x)\right)$$

Repeated roots: $(b-a)^2 - 4ac = 0$

$$r = \frac{a-b}{2a}$$
$$y_1 = x^r$$
$$y_2 = u(x)y_1 = u(x)x^r$$
$$y_2' = urx^{r-1} + u'x^r$$
$$y_2'' = x^r u'' + 2u'rx^{r-1} + ur(r-1)x^{r-2}$$
$$ax^{r+2}u'' + 2arx^{r+1}u' + ar(r-1)x^r u + bx^{r+1}u' + brx^r u + cx^r u = 0$$

Simplifying we can get that all the $u$ terms cancel out

$$\frac{a-b}{2}\left(\frac{a-b}{2a} - 1\right) + \frac{b(a-b)}{2a} + c$$
$$\frac{a^2 - 2ab + b^2}{4a} + \frac{-2a^2 + 2ab}{4a} + \frac{2ab - 2b^2}{4a} + \frac{4ac}{4a}$$
$$\frac{-a^2 + 2ab - b^2 + 4ac}{4a} = -\frac{(a-b)^2 - 4ac}{4a} = 0$$

So our remaining terms are

$$ax^{r+2}u'' + 2arx^{r+1}u' + bx^{r+1}u' = 0$$
$$\text{let } v = u' \Rightarrow v' = u''$$
$$ax^{r+2}v' + 2arx^{r+1}v + bx^{r+1}v = 0$$

$$
\begin{aligned}
&v' + \frac{2ar+b}{a}\frac{v}{x} = 0 \\
&v' + \frac{a-b+b}{a}\frac{v}{x} = 0 \\
&v' + \frac{v}{x} = 0 \\
&\mu(x) = e^{\int \frac{dx}{x}} \\
&\mu v' + \mu\frac{v}{x} = xv' + v = \frac{d}{dx}vx = 0 \\
&\int \left(\frac{d}{dx}vx\right) dx = \int 0 dx = C \\
&vx = u'x = C \\
&\int \frac{du}{u} = \int \frac{dx}{x} \\
&u(x) = \ln|x| \\
&y_2 = x^r \ln|x| \\
&y(x) = C_1 x^r + C_2 x^r \ln|x|
\end{aligned}
$$

Note that we can also write a Cauchy-Euler Equation in the form of

$$a(x-\alpha)^2 y'' + b(x-\alpha)y' + cy = 0$$

and our fundamental guess would be of the form $y = (x-\alpha)^r$.
Ex:

$$
\begin{aligned}
&x^2y'' - xy' + y = 0 \\
&y = x^r,\ y' = rx^{r-1},\ y'' = r(r-1)x^{r-2} \\
&r(r-1) - r + 1 = 0 \\
&r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 \Rightarrow r = 1 \\
&y = C_1 x + C_2 x \ln|x|
\end{aligned}
$$

Ex2:

$$
\begin{aligned}
&x^2y'' - xy' + 5y = 0 \\
&y = x^r,\ y' = rx^{r-1},\ y'' = r(r-1)x^{r-2} \\
&r(r-1) - r + 5 = 0 \Rightarrow r^2 - 2r + 5 = 0 \\
&r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i \\
&y = x(C_1\cos(2\ln|x|) + C_2\sin(2\ln|x|))
\end{aligned}
$$

Ex3:

$$
\begin{aligned}
&2x^2y'' - xy' + y = x \\
&y_c = x^r,\ y_c' = rx^r,\ y_c = r(r-1)x^r
\end{aligned}
$$

$$
\begin{aligned}
&2r(r-1)-r+1=0 \Rightarrow 2r^2-3r+1=0 \Rightarrow (2r-1)(r-1)=0,\ r=\frac{1}{2},1\\
&y_c = C_1\sqrt{x}+C_2x\\
&y_p = ax\ln x\\
&y_p' = a\ln x + a\\
&y_p'' = \frac{a}{x}\\
&2ax - ax\ln x - ax + ax\ln x = x\\
&\Rightarrow a = 1\\
&y = C_1\sqrt{x}+C_2x + x\ln x
\end{aligned}
$$

## 1.2 Series Solutions

### 1.2.1 Power Series Solutions

We can express the solution to many ODEs in the form of a power series. For some differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

we can express the general solution about some point $x_0$ as

$$y = \sum_{n=0}^{\infty} a_n(x-x_0)^n$$

This method is best shown through examples.
Ex:

$$
\begin{aligned}
&y' - y - 2xy = 0\\
&y = \sum_{n=0}^{\infty} a_n x^n\\
&y' = \sum_{n=1}^{\infty} a_n n x^{n-1}\\
&\sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0
\end{aligned}
$$

Now we want to combine the three sums into one summation. We can do this by changing the indexes so that they all have matching $x^m$ terms and then peel off the lower terms in the sum so they all start at the same point.

$$
\begin{aligned}
&\underbrace{\sum_{n=1}^{\infty} a_n n x^{n-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{m=n} - 2\underbrace{\sum_{n=0}^{\infty} a_n x^{n+1}}_{m=n+1} = 0\\
&\sum_{m=0}^{\infty} a_{m+1}(m+1)x^m + \sum_{m=0}^{\infty} a_m x^m - 2\sum_{m=1}^{\infty} a_{m-1}x^m = 0
\end{aligned}
$$

$$a_0 + a_1 + \sum_{m=1}^{\infty} \left( a_{m+1}(m+1) + a_m - 2a_{m-1} \right) x^m = 0$$

Because each $x^2,\ x^3,\ x^4, \ldots$ term is linearly independent of one another we will have each of these terms sum to 0

$$x^0 \text{ terms: } a_0 + a_1 = 0 \Rightarrow a_1 = -a_0$$

$$x^1 \text{ terms: } a_2(2) + a_1 - 2a_0 = 0 \Rightarrow a_2 = \frac{2a_0 - a_1}{2}$$

$$x^m \text{ terms: } a_{m+1}(m+1) + a_m - 2a_{m-1} = 0 \Rightarrow a_{m+1} = \frac{2a_{m-1} - a_m}{m+1}$$

This gives a recursive formula we can use to solve for each $a_m$ term.

$$m = 1:\ a_2 = \frac{2a_0 - a_1}{2} = \frac{2a_0 + a_0}{2} = \frac{3}{2}a_0$$

$$m = 2:\ a_3 = \frac{2a_1 - a_2}{3} = \frac{-2a_0 - \frac{3}{2}a_0}{3} = -\frac{7}{6}a_0$$

$$\vdots$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$y(x) = a_0 \left( 1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3 + \cdots \right)$$

Note that the soluyion is in terms of an arbitrary constant $a_0$. For 2nd order ODEs the solution will be in terms of two arbitrary constants (usually $a_0$ and $a_1$).
Ex2:

$$(1 + x^3)y'' + 12xy = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n,\ y' = \sum_{n=1}^{\infty} a_n n x^{n-1},\ y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_n n(n-1)x^{n+1} + 12\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m + \sum_{m=3}^{\infty} a_{m-1}(m-1)(m-2)x^m + 12\sum_{m=1}^{\infty} a_{m-1}x^m = 0$$

$$2a_2 + 6a_3 x + 12a_4 x^2 + 12a_0 x + 12a_1 x^2 + \sum_{m=3}^{\infty} \left( a_{m+2}(m+2)(m+1) + a_{m-1}(m-1)(m-2) + 12a_{m-1} \right) x^m$$

$$2a_2 = 0$$

$$6a_3 + 12a_0 = 0 \Rightarrow a_3 = -2a_0$$

$$12a_4 + 12a_1 \Rightarrow a_4 = -a_1$$

$$a_{m+2}(m+2)(m+1) + a_{m-1}(m-1)(m-2) + 12a_{m-1} = 0$$

$$a_{m+2}(m+2)(m+1) = -a_{m-1}(m^2 - 3m + 14)$$

$$a_{m+2} = -a_{m-1}\frac{m^2 - 3m + 14}{(m+2)(m+1)}$$

$$
\begin{aligned}
&a_5 = -\frac{14a_2}{20} = 0\\
&a_6 = -a_3\left(\frac{18}{30}\right) = \frac{36a_0}{30} = \frac{6a_0}{5}\\
&a_7 = -a_4\left(\frac{24}{42}\right) = \frac{4a_1}{7}\\
&a_8 = 0\\
&a_9 = -a_6\left(\frac{42}{72}\right) = -\frac{7a_0}{10}\\
&a_{10} = -a_7\frac{54}{90} = -\frac{12a_1}{35}\\
&a_{12} = -a_9\frac{84}{132} = \frac{49a_0}{110}\\
&a_{13} = -a_{10}\frac{102}{156} = \frac{102a_1}{455}\\
&y = a_0\left(1 - 2x^3 + \frac{6}{5}x^6 - \frac{7}{10}x^9 + \cdots\right) + a_1\left(x - x^4 + \frac{4}{7}x^7 - \frac{12}{35}x^{10} + \cdots\right)
\end{aligned}
$$

Something to note when we are using power series is the radius of convergence. If we rewrite the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$ as

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$$

then it will be nonsensical for where $P(x) = 0$. Points where this happens are called *singular points*. So in our previous example we had $(1 + x^3)y'' + 12xy = 0$ so our singular points would be

$$x = -1,\ x = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

More rigorously, singular points are defined to be where the function is not analytic. This is the case where the function or the derivative of the function is divided by 0.
For example, $y'' + \sqrt{x}y' - y = 0$ would have a singular point at $x = 0$.
Our power series solution is only valid so long as we don't pass any singular points. This will mean that there is some radius of convergence within where the solution is valid. This radius of convergence can be found as the distance (in real and imaginary space) between the expansion point, $x_0$, and the nearest singular point.
So in our previous example, our expansion point was $x_0 = 0$ and our singular points are all of a distance $R = 1$ away so we find that our radius of convergence is $R = 1$. This means that our solution is only valid for $|x| < 1$ or more generally,

$$|x - x_0| < R$$

Another way to show this is using the ratio test for the recursive $a_m$

$$
\begin{aligned}
&\lim_{m\to\infty}\left|\frac{a_{m+2}}{a_{m-1}}x\right| = \lim_{m\to\infty}\left|\frac{-a_{m-1}\frac{m^2-3m+3}{(m+2)(m+1)}}{a_{m-1}}x\right| = \lim_{m\to\infty}\left|\frac{m^2-3m+3}{(m+2)(m+1)}x\right| = |x| < 1\\
&\therefore R = 1
\end{aligned}
$$

### 1.2.2 Frobenius Solutions

When we have a singular point we are not able to expand about that point using power series. One way to get around this is to use Frobenius series.
Because the Cauchy-Euler equation gives a solution for equations with singular points we can try to mimic it with the Frobenius series. We can take an arbitrary ODE and manipulate it to get it into the form of a Cauchy-Euler equation.

$$\begin{aligned}
&P(x)y'' + Q(x)y' + R(x)y = 0 \\
&y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0 \\
&(x-x_0)^2y'' + (x-x_0)^2\frac{Q(x)}{P(x)}y' + (x-x_0)^2\frac{R(x)}{P(x)}y = 0 \\
&\text{let } \alpha(x) = (x-x_0)\frac{Q(x)}{P(x)},\ \beta(x) = (x-x_0)\frac{R(x)}{P(x)} \\
&(x-x_0)^2y'' + (x-x_0)\alpha(x)y' + (x-x_0)^2\beta(x)y = 0
\end{aligned}$$

So long as $\lim_{x\to x_0}\alpha(x)$ and $\lim_{x\to x_0}\beta(x)$ are both finite then the point we are expanding about is considered to be a *regular singular point* and we are able to apply the Frobenius series.
The Frobenius series looks similar to the power series and is of the form

$$y = (x-x_0)^r\sum_{n=0}^{\infty}(x-x_0)^n = \sum_{n=0} a_n(x-x_0)^{n+r}$$

Finding the solution using Frobenius series works out very similar to using power series but with a few slight differences.
Ex:

$$6x^2(1+x)y'' + 5xy' - y = 0 \text{ about } x_0 = 0$$

We must first see if the point we are expanding about is a regular singular point

$$\begin{aligned}
&\lim_{x\to 0}\frac{5x^2}{6x^2(1+x)} = \lim_{x\to 0}\frac{5}{6(x+1)} = \frac{5}{6} \\
&\lim_{x\to 0}\frac{-x^2}{6x^2(x+1)} = \lim_{x\to 0}\frac{-1}{6(x+1)} = -\frac{1}{6} \\
&\therefore\ x_0 = 0 \text{ is a regular singular point}
\end{aligned}$$

These limits that we found give the terms $\alpha_0$ and $\beta_0$. We can use these terms to craft what we call the indicial equation which allow us to solve for the two $r$ values that will show up in our solution (similar to how we solved for $r$ in Cauchy-Euler).

$$\begin{aligned}
&x^2y_c'' + \alpha_0xy_c' + \beta_0y_c = 0 \\
&x^2y_c'' + \frac{5}{6}xy_c' - \frac{1}{6}y_c = 0 \\
&y_c = x^r,\ y_c' = rx^{r-1},\ y_c'' = r(r-1)x^{r-2}
\end{aligned}$$

$$r(r-1)+\frac{5}{6}r-\frac{1}{6}=0$$
$$6r^2-r-1=0$$
$$r=\frac{1\pm\sqrt{1+24}}{12}=\frac{1}{2},-\frac{1}{3}$$
$$y_1=\sum_{n=0}^{\infty}a_nx^{\frac{1}{2}+n}$$
$$y_2=\sum_{n=0}^{\infty}a_nx^{n-\frac{1}{3}}$$

The rest follows very similarly to finding a power series solution

$$y=\sum_{n=0}^{\infty}a_nx^{n+r}$$
$$y'=\sum_{n=0}^{\infty}a_n(n+r)x^{n+r-1}$$
$$y''=\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)x^{n+r-2}$$
$$6\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)x^{n+r}+6\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)x^{n+r+1}$$
$$+5\sum_{n=0}^{\infty}a_n(n+r)x^{n+r}-\sum_{n=0}^{\infty}a_nx^{n+r}$$
$$6\sum_{m=0}^{\infty}a_m(m+r)(m+r-1)x^{m+r}+6\sum_{m=1}^{\infty}a_{m-1}(m+r-1)(m+r-2)x^{m+r}$$
$$+5\sum_{m=0}^{\infty}a_m(m+r)x^{m+r}-\sum_{m=0}^{\infty}a_mx^{m+r}$$
$$6a_0(r)(r-1)x^r+5a_0(r)x^r-a_0x^r$$
$$+\sum_{m=1}^{\infty}\left(6a_m(m+r)(m+r-1)+6a_{m-1}(m+r-1)(m+r)+5a_m(m+r)-a_m\right)x^{m+r}$$
$$\Rightarrow r=\frac{1}{2},-\frac{1}{3}$$
$$a_m(6(m+r)(m+r-1)+5(m+r)-1)=-6a_{m-1}(m+r-1)(m+r-2)$$
$$a_m=\frac{-6a_{m-1}(m+r-1)(m+r-2)}{6(m+r)(m+r-1)+5(m+r)-1}$$
$$r=\frac{1}{2}:\ a_m=-a_{m-1}\frac{3(2m-1)(2m-3)}{2m(6m+5)}$$
$$a_1=\frac{3}{22}a_0$$
$$a_2=-\frac{9}{68}a_1=-\frac{27}{1496}a_0$$

$$r = -\frac{1}{3}: \;\; a_m = \frac{2(3m-4)(3m-7)}{3m(6m-5)}$$
$$a_1 = -\frac{8}{3}a_0$$
$$a_2 = a_1\frac{2}{21} = -\frac{16}{63}a_0$$
$$y = C_1 x^{1/2}\left(1 + \frac{3}{22}x - \frac{27}{1496}x^2 + \cdots\right) + C_2 x^{-\frac{1}{3}}\left(1 - \frac{8}{3}x - \frac{16}{63}x^2 + \cdots\right)$$

Ex2: Special case of the Bessel equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \;\; \nu = \frac{1}{2}$$
SP: $x = 0$
$$\alpha = 1$$
$$\alpha' = 0$$
$$\beta = x^2 - \nu^2$$
$$\beta' = 2x$$
$$\lim_{x\to 0} 1 = 1$$
$$\lim_{x\to 0} x^2 - \nu^2 = -\nu^2 = -\frac{1}{4}$$

So $x = 0$ is a regular singular point

$$r(r-1) + r - \frac{1}{4} = 0$$
$$r^2 - \frac{1}{4} = 0$$
$$\left(r - \frac{1}{2}\right)\left(r + \frac{1}{2}\right) \Rightarrow r = \pm\frac{1}{2}$$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$
$$y' = \sum_{n=0}^{\infty} a_n(n+r)x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r-1}$$
$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)x^{n+r} + \sum_{n=0}^{\infty} a_n(n+r)x^{n+r} + \sum_{n=0}^{\infty} a_n x^{n+r+2} - \nu^2\sum_{n=0}^{\infty} a_n x^{n+r} = 0$$
$$\sum_{m=0}^{\infty} a_m(m+r)(m+r-1)x^{m+r} + \sum_{m=0}^{\infty} a_m(m+r)x^{m+r} + \sum_{m=2}^{\infty} a_{m-2}x^{m+r} - \nu^2\sum_{m=0}^{\infty} a_m x^{m+r} = 0$$
$$a_0(r)(r-1)x^r + a_0 r x^r - \nu^2 a_0 x^r + a_1(1+r)(r)x^{1+r} + a_1(1+r)x^{1+r} - \nu^2 a_1 x^{1+r}$$

$$+\sum_{m=2}^{\infty}\left(a_m(m+r)(m+r-1)+a_m(m+r)+a_{m-2}-\nu^2 a_m\right)x^{m+r}=0$$

$$a_0 r(r-1)+a_0 r-\nu^2 a_0=0 \Rightarrow a_0(r^2-\nu^2)=0 \Rightarrow r^2=\nu^2=\pm\frac{1}{2}$$

$$a_1(1+r)r+a_1(1+r)-a_1\nu^2=0 \Rightarrow a_1\left(r^2+2r+1-\nu^2\right)=a_1(2r+1)$$

$$\Rightarrow a_1=0 \text{ or } r=-\frac{1}{2}$$

$$a_m=\frac{-a_{m-2}}{(m+r)^2-\nu^2}$$

$$r_1=\frac{1}{2}:\ a_1=0$$

$$a_m=\frac{-a_{m-2}}{m(m+1)}$$

$$m=2:\ a_2=-\frac{a_0}{3\cdot 2}=-\frac{a_0}{3!}$$

$$m=3:\ a_3=-\frac{a_1}{3\cdot 4}=0$$

$$m=4:\ a_4=-\frac{a_2}{5\cdot 4}=\frac{a_0}{5!}$$

$$y_1=a_0x^{1/2}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)$$

$$y_1=a_0x^{-1/2}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)$$

$$y_1=a_0x^{-1/2}\sin x$$

$$r=-\frac{1}{2}$$

$$a_m=\frac{-a_{m-2}}{m(m-1)}$$

$$m=2:\ a_2=-\frac{a_0}{2\cdot 1}=-\frac{a_0}{2!}$$

$$m=3:\ a_3=-\frac{a_1}{3\cdot 2}=-\frac{a_1}{3!}$$

$$m=4:\ a_4=-\frac{a_2}{4\cdot 3}=\frac{a_0}{4!}$$

$$m=5:\ a_5=-\frac{a_3}{5\cdot 4}=\frac{a_1}{5!}$$

$$y_2=a_0x^{-1/2}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)+a_1x^{-1/2}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)$$

$$y_2=a_0x^{-1/2}\cos x+a_1x^{-1/2}\sin x$$

$$y(x)=y_1+y_2=x^{-1/2}\left(A\sin x+B\cos x\right)$$

Ex3:

$$(1-x^2)y''-xy'+\alpha^2y=0 \text{ about } x=1$$

$$\lim_{x\to 1}(x-1)\frac{-x}{1-x^2}=\lim_{x\to 1}\frac{x}{1+x}=\frac{1}{2}$$

$$\lim_{x\to 1}(x-1)^2\frac{\alpha^2}{1-x^2}=\lim_{x\to 1}(x-1)\frac{\alpha^2}{1+x}=0$$

So $x=1$ is a regular singular point

At $x=1$

$$r(r-1)+\frac{1}{2}r=0$$

$$r^2+\frac{1}{2}r=0\Rightarrow r=0,\ \frac{1}{2}$$

$$\text{let } t=x-1,\ dt=dx\Rightarrow\frac{dy}{dx}=\frac{dy}{dt}$$

$$(1-x)(1+x)y''-xy'+\alpha^2y=0$$

$$-t(t+2)y''-(t+1)y'+\alpha^2y=0$$

$$y=\sum_{n=0}^{\infty}a_nt^{n+r}$$

$$y'=\sum_{n=0}^{\infty}a_n(n+r)t^{n+r-1}$$

$$y''=\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)t^{n+r-1}$$

$$-\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)t^{n+r}-2\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)t^{n+r-1}-\sum_{n=0}^{\infty}a_n(n+r)t^{n+r}$$

$$-\sum_{n=0}^{\infty}a_n(n+r)t^{n+r-1}+\alpha^2\sum_{n=0}^{\infty}a_nt^{n+r}=0$$

$$n=m+1\Rightarrow m=n-1$$

$$-\sum_{m=0}^{\infty}a_m(m+r)(m+r-1)t^{m+r}-2\sum_{m=-1}^{\infty}a_{m+1}(m+r+1)(m+r)t^{m+r}-\sum_{m=0}^{\infty}a_m(m+r)t^{m+r}$$

$$-\sum_{m=-1}^{\infty}a_{m+1}(m+r+1)t^{m+r}+\alpha^2\sum_{m=0}^{\infty}a_mt^{m+r}=0$$

$$-2a_0(r)(r-1)t^{r-1}-a_0(r)t^{r-1}$$

$$+\sum_{m=0}^{\infty}\left(-a_m(m+r)(m+r-1)-2a_{m+1}(m+r+1)(m+r)-a_m(m+r)-a_{m+1}(m+r+1)+\alpha^2a_m\right)t^{m+r}$$

$$-2r(r-1)-r=0\Rightarrow -2r^2+r=0\Rightarrow r(1-2r)=0\Rightarrow r=0,\ \frac{1}{2}$$

$$a_m\left(-(m+r)(m+r-1)-(m+r)+\alpha^2\right)=a_{m+1}\left(2(m+r+1)(m+r)+(m+r+1)\right)$$

$$a_{m+1}=a_m\frac{\alpha^2-(m+r)(m+r-1)-(m+r)}{2(m+r+1)(m+r)+(m+r+1)}$$

$$r=0:\ \ a_{m+1}=a_m\frac{\alpha^2-m(m-1)-m}{2(m+1)(m)+(m+1)}=a_m\frac{\alpha^2-m^2}{2m^2+3m+1}$$

$$m = 0: \ a_1 = \alpha^2 a_0$$
$$m = 1: \ a_2 = \frac{\alpha^2 - 1}{6} a_1 = \frac{(\alpha^2 - 1)\alpha^2}{6} a_0$$
$$m = 2: \ a_3 = \frac{\alpha^2 - 4}{15} a_2 = \frac{(\alpha^2 - 4)(\alpha^2 - 1)\alpha^2}{90} a_0$$
$$y_1(t) = a_0 \left(1 + \alpha^2 t + \frac{(\alpha^2 - 1)\alpha^2}{6} t^2 + \frac{(\alpha^2 - 4)(\alpha^2 - 1)\alpha^2}{90} t^3 + \cdots\right)$$
$$y_1(x) = a_0 \left(1 + \alpha^2 (x-1) + \frac{(\alpha^2 - 1)\alpha^2}{6} (x-1)^2 + \frac{(\alpha^2 - 4)(\alpha^2 - 1)\alpha^2}{90} (x-1)^3 + \cdots\right)$$
$$r = \frac{1}{2}: \ a_{m+1} = a_m \frac{\alpha^2 - (m+\frac{1}{2})(m-\frac{1}{2}) - (m+\frac{1}{2})}{2(m+\frac{3}{2})(m+\frac{1}{2}) + (m+\frac{3}{2})} = a_m \frac{4\alpha^2 - (2m+1)(2m-1) - 2(2m+1)}{2(2m+3)(2m+1) + 2(2m+3)}$$
$$m = 0: \ a_1 = \frac{\alpha^2 + 1 - 2}{6 + 6} a_0 = \frac{4\alpha^2 - 1}{12} a_0$$
$$m = 1: \ a_2 = \frac{4\alpha^2 - (3)(1) - 2(3)}{2(5)(3) + 2(5)} a_1 = \frac{4\alpha^2 - 9}{40} a_1 = \frac{(4\alpha^2 - 9)(4\alpha^2 - 1)}{480} a_0$$
$$m = 2: \ a_3 = \frac{\alpha^2 - (5)(3) - 2(5)}{2(7)(5) + 2(7)} a_2 = \frac{\alpha^2 - 25}{84} a_2 = \frac{(\alpha^2 - 9)(\alpha^2 - 4)(\alpha^2 - 1)}{40320}$$
$$y_2(t) = a_0 \left(t^{1/2} + \frac{4\alpha^2 - 1}{12} t^{3/2} + \frac{(4\alpha^2 - 9)(4\alpha^2 - 1)}{480} t^{5/2} + \frac{(\alpha^2 - 9)(\alpha^2 - 4)(\alpha^2 - 1)}{40320} t^{7/2} + \cdots\right)$$
$$y_2(x) = a_0 \left((x-1)^{1/2} + \frac{4\alpha^2 - 1}{12} (x-1)^{3/2} + \frac{(4\alpha^2 - 9)(4\alpha^2 - 1)}{480} (x-1)^{5/2} + \cdots\right)$$
$$y = A + B(x-1)^{1/2} + \sum_{n=1}^{\infty} \left( A(x-1)^n \prod_{m=0}^{n-1} \left( \frac{\alpha^2 - m^2}{2m^2 + 3m + 1} \right) + B(x-1)^{n+1/2} \prod_{m=0}^{n-1} \left( \frac{4\alpha^2 - (2m+1)^2}{4(2m^2 + 5m + 3)} \right) \right)$$

# 2 Partial Differential Equations

Partial differential equations are a differential equations involving functions of multiple variables. Some common ones are

- Laplace equation
$$\nabla^2 u = 0$$
- Heat equation
$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
- Wave Equation
$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 u}{\partial t^2}$$

## 2.1 Heat Equation

### 2.1.1 Finite Difference Approximations

We can approximate derivatives using Taylor approximations as follows:

$$f(x_0 + \Delta x) = f(x_0) + f'(x_0)\Delta x + \mathcal{O}(\Delta x^2)$$
$$f'(x_0) = \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} + \mathcal{O}(\Delta x)$$

$$f(x_0 + \Delta x) = f(x_0) + \Delta x f'(x_0) + \frac{\Delta x^2}{2} f''(x_0) + \mathcal{O}(\Delta x^3)$$
$$f(x_0 - \Delta x) = f(x_0) - \Delta x f'(x_0) + \frac{\Delta x^2}{2} f''(x_0)$$
$$f(x_0 + \Delta x) + f(x_0 - \Delta x) = 2f(x_0) + \Delta x^2 f''(x_0) + \mathcal{O}(\Delta x^4)$$
$$f''(x_0) = \frac{f(x_0 + \Delta x) - 2f(x_0) + f(x_0 - \Delta x)}{\Delta x^2} + \mathcal{O}(\Delta x^2)$$

This gives us approximate solutions of $f'(x_0)$ and $f''(x_0)$ which we can make use of in our PDE.

$$u_t = \alpha^2 u_{xx}$$
$$u_t = \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t}$$
$$u_{xx} = \frac{u(x, t + \Delta t) - 2u(x, t) + u(x - \Delta x, t)}{\Delta x^2}$$
$$\Rightarrow \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t} = \alpha^2 \frac{u(x, t + \Delta t) - 2u(x, t) + u(x - \Delta x, t)}{\Delta x^2}$$

Writing $u(x + \Delta x, t)$ and similar expressions over and over can get quite long and tedious so we can make use of index notation to clean this up.
If we write $x = i\Delta x$ and $t = k\Delta t$ where $i, k \in \mathbb{Z}$ then we can express each $u$ term as $u_i^k$ where $i$ is the node number and $k$ is the time step number.
So our expression above turns into

$$\frac{u_i^{k+1} - u_i^k}{\Delta t} = \alpha^2 \frac{u_{i+1}^k - 2u_i^k + u_{i-1}^k}{\Delta x^2}$$
$$u_i^{k+1} = \alpha^2 \frac{\Delta t}{\Delta x^2}(u_{i+1}^k - 2u_i^k + u_{i-1}^k) + u_i^k$$

This now gives us an equation for time $k + 1$ that is only dependent on time $k$ so as long as we have the initial positions at time $k$, we can calculate every position of time $k + 1$. Ex: Calculate $u(x, 0.2)$ with $\Delta t = 0.01$ and $\Delta x = 0.2$ with boundary conditions $u(0, t) = u_x(1, t) = 0$ and initial condition

$$u(x, 0) = \begin{cases} 0 & 0 \le x < 0.4 \\ 1 & 0.4 \le x \le 0.8 \\ 0 & 0.8 < x \le 1 \end{cases}$$

$$\frac{u_n^{k+1} - u_n^k}{\Delta t} = \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{\Delta x^2}$$

$$u_n^{k+1} = \frac{\Delta t}{\Delta x^2}\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right) + u_n^k$$
$$u_0^0 = 0,\ u_1^k = 0,\ u_2^0 = 1,\ u_3^0 = 1,\ u_4^0 = 1,\ u_5^0 = 0,\ \frac{\Delta t}{\Delta x^2} = \frac{1}{4}$$
$$u_k^0 = 0\ \forall k$$
$$\frac{\partial}{\partial x}(1,t) = 0 = \frac{u_6^k - u_4^k}{2\Delta x} \Rightarrow u_6^k = u_4^k$$
$$\Rightarrow u_5^{k+1} = \frac{\Delta t}{\Delta x^2}\left(2u_4^k - 2u_5^k\right) + u_5^k$$
$$u_1^1 = \frac{1}{4}(1 - 0 + 0) + 0 = \frac{1}{4}$$
$$u_2^1 = \frac{1}{4}(1 - 2(1) + 0) + 1 = \frac{3}{4}$$
$$u_3^1 = \frac{1}{4}(1 - 2(1) + 1) + 1 = 1$$
$$u_4^1 = \frac{1}{4}(0 - 2(1) + 1) + 1 = \frac{3}{4}$$
$$u_5^1 = \frac{1}{4}(2(1) - 0) + 0 = \frac{1}{2}$$
$$u_1^2 = \frac{1}{4}\left(0 - 2\frac{1}{4} + \frac{3}{4}\right) + \frac{1}{4} = \frac{5}{16}$$
$$u_2^2 = \frac{1}{4}\left(\frac{1}{4} - 2\frac{3}{4} + 1\right) + \frac{3}{4} = \frac{11}{16}$$
$$u_3^2 = \frac{1}{4}\left(\frac{3}{4} - 2(1) + \frac{3}{4}\right) + 1 = \frac{14}{16}$$
$$u_4^2 = \frac{1}{4}\left(\frac{1}{2} - 2\frac{3}{4} + 1\right) + \frac{3}{4} = \frac{12}{16}$$
$$u_5^2 = \frac{2}{4}\left(\frac{3}{4} - \frac{1}{2}\right) + \frac{1}{2} = \frac{10}{16}$$

### 2.1.2 Separation of Variables

In order to solve the heat equation, we require one initial condition and two boundary conditions. The general form of the heat equation will be

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x}$$

where the domain is $0 \leq x \leq L$ and $t > 0$
We can solve this type of problem using separation of variables. This assumes that we will have a solution of the form $u(x,t) = X(x)T(t)$

$$u(x,t) = X(x)T(t)$$
$$u_t = \alpha^2 u_{xx}$$
$$XT' = \alpha^2 X''T$$

If we divide both sides by $XT$ then we get $X$ and $T$ on separate sides of the equation

$$\frac{T'}{T} = \alpha^2 \frac{X''}{X} \Rightarrow \frac{1}{\alpha^2}\frac{T'}{T} = \frac{X''}{X}$$

This implies that the $x$ and $t$ are independent of one another which is why separation of variables is able to work. Because they are separate, we can also set this equation equal to a constant, $-\lambda$. (The negative sign is cleverly put in place to make the math a little nicer later)

$$\frac{1}{\alpha^2}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$

We can use this to write out an ODE for both $X$ and $T$.
The equation for $T$ works out to be a first order ODE and will always have the following result:

$$\frac{1}{\alpha^2}\frac{T'}{T} = -\lambda \Rightarrow \frac{dT}{dt} = -\alpha^2\lambda T \Rightarrow \int \frac{dT}{T} = -\int \alpha^2 \lambda dt$$
$$T(t) = Ce^{-\lambda\alpha^2 t}$$

The equation for $X$ will be 2nd order and is subject to the boundary conditions of the problem.

$$\frac{X''}{X} = -\lambda X \Rightarrow X'' + \lambda X = 0$$

It is easy to see that $X(x) = 0$ is a solution to this ODE, however, we are interested in finding the general solution to this ODE. This means finding values of $\lambda$ that give non-trivial (nonzero) solutions for $X(x)$ subject to the given boundary conditions.
There are 5 standard sets of boundary conditions for the initial boundary value problem (IBVP).

1. Dirichelt Problem
   $u(0,t) = u(L,t) = 0$
   $$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$
   $$X_n = \sin\left(\frac{n\pi x}{L}\right),\ n \geq 1$$
2. Neumann Problem
   $u_x(0,t) = u_x(L,t) = 0$
   $$\lambda_n = 0,\ \left(\frac{n\pi}{L}\right)^2$$
   $$X_n = 1,\ \cos\left(\frac{n\pi x}{L}\right),\ n \geq 1$$
3. Periodic Boundary Conditions
   $u(0,t) = u(L,t)$ and $u_x(0,t) = u_x(L,t)$
   $$\lambda_n = 0,\ \left(\frac{n\pi}{L}\right)^2$$
   $$X_n = 1,\ \sin\left(\frac{n\pi x}{L}\right),\ \cos\left(\frac{n\pi x}{L}\right),\ n \geq 1$$
4. Mixed Type 1
   $u(0,t) = u_x(L,t) = 0$
   $$\lambda_n = \left(\frac{2n-1}{2L}\pi\right)^2$$
   $$X_n = \sin\left(\frac{2n-1}{2L}\pi x\right),\ n \geq 1$$
5. Mixed Type 2
   $u_x(0,t) = u(L,t)$
   $$\lambda_n = \left(\frac{2n-1}{2L}\pi\right)^2$$
   $$X_n = \cos\left(\frac{2n-1}{2L}\pi x\right),\ n \geq 1$$

To go into how these values were determined, let's look at the Dirichlet problem.
We will start with

$$X'' + \lambda X = 0,$$
$$X(0) = X(L) = 0$$

Recall that for constant coefficient ODES, depending on the value of the eigenvalue, $\lambda$, we can have 3 different forms of solutions and so we must look at each case.

1. $\lambda < 0 \Rightarrow \lambda = -\mu^2$

   $$X'' - \mu^2 X = 0 \Rightarrow r^2 - \mu^2 = 0 \Rightarrow r = \pm\mu$$
   $$X(x) = C_1 e^{-\mu x} + C_2 e^{\mu x}$$

   Alternatively, we can use the relationships $\sinh x = \frac{e^x - e^{-x}}{2}$ and $\cosh x = \frac{e^x + e^{-x}}{2}$ to rewrite the solution as

   $$X(x) = A\sinh(\mu x) + B\cosh(\mu x)$$

   Now, applying the boundary conditions,

   $$X(0) = 0 \Rightarrow B = 0$$
   $$X(L) = 0 \Rightarrow A\sinh(\mu L) = 0 \Rightarrow A = 0$$
   $$\therefore X(x) = 0$$

   This makes the solution trivial so $\lambda \not< 0$.
   (In general, for the heat equation we will not have $\lambda < 0$ so we can usually skip this case.)

2. $\lambda = 0$

   $$X'' = 0 \Rightarrow X(x) = Ax + B$$

   Applying the BCs;

   $$X(0) = 0 \Rightarrow B = 0$$
   $$X(L) = 0 \Rightarrow A = 0$$
   $$\therefore X(x) = 0$$

   So the solution is trivial and $\lambda \neq 0$

3. $\lambda > 0 \Rightarrow \lambda = \mu^2$

   $$X'' + \mu^2 X = 0 \Rightarrow r = \pm i\mu$$
   $$X(x) = A\sin(\mu x) + B\cos(\mu x)$$

   Applying the BCs;

   $$X(0) = 0 \Rightarrow B = 0$$

$$X(L) = 0 \Rightarrow A\sin(\mu L) = 0$$

This means that either $A = 0$ or $\sin(\mu L) = 0$. In order to have a non-trivial solution, we require that $\sin(\mu L) = 0$. This gives

$$\mu L = n\pi,\ n \in \mathbb{N}$$
$$\mu = \frac{n\pi}{L},\ n \in \mathbb{N}$$

Note that we have $n \in \mathbb{N}$ and $n \geq 1$ because we require that $\mu \neq 0$ and having negative $n$ would not give any new linearly independent solutions.
And so our eigenvalues are

$$\lambda = \left(\frac{n\pi}{L}\right)^2,\ n \in \mathbb{N}$$

and our eigenfunctions are

$$X_n(x) = A_n \sin\left(\frac{n\pi x}{L}\right)$$

This same process is done for each of the other 4 standard boundary conditions.

Keeping with the example of Dirichlet boundary conditions, we will have the general solution of

$$u_n(x,t) = X_n T_n$$
$$u_n(x,t) = A_n \sin\left(\frac{n\pi x}{L}\right) e^{\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$

This gives solutions for each value of $n$. We know that the specific solution is going to be some linear combination of each individual solution though so we can write the general solution as

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$

### 2.1.3 Fourier Series

To find the particular solution of a PDE, we require the use of Fourier series. At this point, we have the general solution expressed in terms of a sum with an unknown coefficient, $A_n$. We can use the initial condition to determine this coefficient. However, the initial condition will almost never be expressed as a sum so we need to make it into a sum. This is what Fourier series allows us to do.
The Fourier series expresses a function in terms of sines and cosines of different frequencies.

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

This may seem a little messy and it turns out that there is a nicer way to express the Fourier series using complex numbers:
Right now we have

$$f(\alpha) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\alpha) + \sum_{n=1}^{\infty} \sin(n\alpha)$$

We can make use of the identities

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2},\ \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2}$$

So we have

$$f(\alpha) = a_0 + \sum_{n=1}^{\infty}\frac{a_n}{2}\left(e^{in\alpha}+e^{-in\alpha}\right) + \sum_{n=1}^{\infty}\frac{b_n}{2}\left(e^{in\alpha}-e^{-in\alpha}\right)$$

$$f(\alpha) = \sum_{n=1}^{\infty}\left(\frac{a_n - ib_n}{2}\right)e^{in\alpha} + \sum_{n=1}^{\infty}\left(\frac{a_n + ib_n}{2}\right)e^{-in\alpha}$$

$$f(\alpha) = \underbrace{a_0}_{c_0} + \sum_{n=1}^{\infty}\underbrace{\left(\frac{a_n - ib_n}{2}\right)}_{c_n}e^{in\alpha} + \sum_{n=-\infty}^{-1}\underbrace{\left(\frac{a_{-n} + ib_{-n}}{2}\right)}_{c_n}e^{in\alpha}$$

$$f(\alpha) = \sum_{n=-\infty}^{\infty}c_n e^{in\alpha},\ c_n = \frac{1}{2L}\int_{-L}^{L}f(\alpha)e^{-in\alpha}d\alpha$$

To match functions, we can use the concept of orthogonality.
$f(x)$ and $g(x)$ are orthogonal if $\displaystyle\int_{-L}^{L}f(x)g(x)dx = 0$
Trig functions have some particularly useful orthogonality properties.
For where $m, n \in \mathbb{N}$

$$\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \begin{cases}0 & \text{if } m \neq n\\ L & \text{if } m = n\end{cases}$$

$$\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = \begin{cases}0 & \text{if } m \neq n\\ L & \text{if } m = n \neq 0\\ 2L & \text{if } m = n = 0\end{cases}$$

$$\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0$$

Derivation:

$$I = \int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

case $m \neq n$ :

$$I = 2\int_{0}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\sin A\sin B = \frac{1}{2}\left(\cos(A-B) - \cos(A+B)\right)$$

$$I = \frac{2}{2}\int_{0}^{L}\left(\cos\left(\frac{n-m}{L}\pi x\right) - \cos\left(\frac{n+m}{L}\pi x\right)\right)dx$$

$$I = \left[\frac{L}{(n-m)\pi}\sin\left(\frac{n-m}{L}\pi x\right)\right]_0^L - \left[\frac{L}{(n+m)\pi}\sin\left(\frac{n+m}{L}\pi x\right)\right]_0^L = 0$$

case $m = n$ :

$$I = 2\int_0^L \sin^2\left(\frac{n\pi x}{L}\right)dx$$

$$\cos 2A = 1 - \sin^2 A \Rightarrow \sin^2 A = \frac{1-\cos 2A}{2}$$

$$I = \frac{2}{2}\int_0^L \left(1 - \cos\left(\frac{2n\pi x}{L}\right)\right)dx$$

$$I = x|_0^L - \frac{L}{2n\pi}\sin\left(\frac{2n\pi x}{L}\right)\bigg|_0^L = L$$

Note that this result makes use of the fact that $\sin(kx) = 0$ for where $k \in \mathbb{Z}$, hence why some terms appear to drop out. This also means that these specific orthogonality calculations only hold for integer values of $m$ and $n$. This is fine for use in Fourier series but it is something to be aware of.

So far, we have introduced what Fourier series is; now let's get into how to use it.
When using Fourier series, we are trying to solve for the unknown coefficients, $a_0, a_m, b_m$ in the series.
We can make use of the trig orthogonality rules we determined earlier to determine these coefficients.
To solve for $b_m$, we can multiply by $\sin\left(\frac{m\pi x}{L}\right)$ and integrate from $-L$ to $L$.

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)$$

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = \sum_{n=0}^{\infty} a_n \underbrace{\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{=0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{=L \text{ if } m=n}$$

$$\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx = b_m L$$

$$b_m = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

To find $a_m$, we do the same thing but multiply by $\cos\left(\frac{m\pi x}{L}\right)$

$$f(x) = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)$$

$$\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx = \sum_{n=0}^{\infty} a_n \underbrace{\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx}_{=\begin{cases}2L & \text{if } m=0\\ L & \text{if } m=n\end{cases}} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx}_{=0}$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_m = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$

If $f(x)$ is odd then

$$a_0 = a_m = 0$$
$$b_m = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{m\pi x}{L}\right)dx$$

If $f(x)$ is even then

$$b_m = 0$$
$$a_0 = \frac{1}{L}\int_{0}^{L} f(x)dx$$
$$a_m = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{m\pi x}{L}\right)dx$$

If you are trying to match a function that is only defined between 0 and $L$ then you must make an odd or even extension.
Ex: for the function $1-x$ defined on $0 \leq x \leq 1$, the even extension would be

$$f^e(x) = \begin{cases} 1-x & x \geq 0 \\ x+1 & x < 0 \end{cases}$$

The odd extension would be

$$f^e(x) = \begin{cases} 1-x & x > 0 \\ 0 & x = 0 \\ -1-x & x < 0 \end{cases}$$

Note that when there is a discontinuity, the function will take the average value of the limit from both sides, hence why $f^e(x) = 0$ for where $x = 0$ as we have

$$f^e(0) = \frac{1}{2}\left(\lim_{x\to 0^-} f^e(x) + \lim_{x\to 0^+} f^e(x)\right) = \frac{1}{2}(-1+1) = 0$$

When dealing with a function that is neither even or odd, we will have to use the full Fourier series with both sines and cosines.

### 2.1.4 Homogeneous Heat Equation

Now we are able to put it all together and solve the heat equation.
Any of the following forms of the heat equation are able to be solved using the separation of variables method (provided they have homogeneous BCs).

$$u_t = \alpha^2 u_{xx}$$
$$u_t = \alpha^2 u_{xx} - \gamma u$$
$$u_t = \alpha^2 u_{xx} - \gamma(t)u$$

$$u_t = \alpha^2 u_{xx} - \gamma(t)$$
$$u_t + \eta(t)u$$

Ex:

$$\begin{cases} u_t = \alpha^2 u_{xx},\ 0 \le x \le L,\ t \ge 0 \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x) = x(L-x),\ 0 \le x \le L \end{cases}$$
$$u(x,t) = X(x)T(t)$$
$$XT' = \alpha^2 X'' T \Rightarrow \frac{1}{\alpha^2}\frac{T'}{T} = \frac{X''}{X} = -\lambda$$
$$u(0,t) = 0 \Rightarrow X(0) = 0$$
$$u(L,t) = 0 \Rightarrow X(L) = 0$$
$$X'' + \lambda X = 0$$

These are Dirichlet boundary conditions so we can skip some steps and get the eigenvalues and eigenfunctions to be

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$
$$X_n = \sin\left(\frac{n\pi x}{L}\right),\ n \ge 1$$
$$T_n = e^{-\alpha^2\lambda_n t} = e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$
$$u_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$
$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$

Now we can use the initial condition to solve for $c_n$.

$$u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi}{L}\right)$$

Because we are working with a Fourier sine series, we will want the odd extension of our initial condition function.

$$f(x) = x(L-x)$$
$$f^e(x) = \begin{cases} x(L-x) & x \ge 0 \\ x(L+x) & x < 0 \end{cases}$$
$$b_n = \frac{1}{L}\int_{-L}^{L} f^e(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Recall that an odd function times an odd function gives an even function. Because both $f^e$ and sin are odd functions in this case, their product gives an even and so we are integrating an even function over a symmetric domain and thus, can simplify

$$b_n = \frac{2}{L}\int_0^L f^e(x)\sin\left(\frac{n\pi x}{L}\right) dx = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{2}{L}\int_0^L x(L-x)\sin\left(\frac{n\pi x}{L}\right)dx$$

This integral works out to be

$$\frac{4L^2}{(n\pi)^3}\left((-1)^{n+1}+1\right)$$

Note that the identity $\cos(n\pi) = (-1)^n$ was used in simplifying this integral. There will be a few occasions where we make use of integer or half-integer multiples of pi in the arguments of trig functions to simplify.

Putting everything together now, we get

$$u(x,t) = \sum_{n=1}^{\infty}\frac{4L^2}{(n\pi)^3}\left((-1)^{n+1}+1\right)\sin\left(\frac{n\pi x}{L}\right)e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$

Simple, right? (it gets much worse later)

Ex2:

$$\begin{cases} u_t = u_{xx},\ -2\pi \le x \le 2\pi,\ t \ge 0 \\ u(-2\pi,t) = u(2\pi,t) \\ u_x(-2\pi,t) = u_x(2\pi,t) \\ u(x,0) = f(x) = \cos\left(\frac{x}{2}\right) + \sin x \end{cases}$$

$$\begin{aligned}
&u(x,t) = X(x)T(t) \\
&\Rightarrow \frac{T'}{T} = \frac{X''}{X} = -\lambda \\
&X'' + \lambda X = 0
\end{aligned}$$

We have periodic boundary conditions so the eigenvalues and eigenfunctions will be

$$\begin{aligned}
&\lambda_0 = 0,\ \lambda_n = \left(\frac{n\pi}{L}\right)^2 = \left(\frac{n}{2}\right)^2 \\
&X_0 = 1,\ X_{n1} = \sin\left(\frac{nx}{2}\right),\ X_{n2} = \cos\left(\frac{nx}{2}\right) \\
&T_0 = 1,\ T_n = e^{-\left(\frac{n}{2}\right)^2 t} \\
&u_0(x,t) = 1 \\
&u_{n1}(x,t) = \sin\left(\frac{nx}{2}\right)e^{-\left(\frac{n}{2}\right)^2 t} \\
&u_{n2} = \cos\left(\frac{nx}{2}\right)e^{-\left(\frac{n}{2}\right)^2 t} \\
&u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{nx}{2}\right)e^{-\left(\frac{n}{2}\right)^2 t} + \sum_{n=1}^{\infty} b_n\sin\left(\frac{nx}{2}\right)e^{-\left(\frac{n}{2}\right)^2 t} \\
&u(x,0) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{nx}{2}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{nx}{2}\right) = f(x) = \cos\left(\frac{x}{2}\right) + \sin x
\end{aligned}$$

Now, using Fourier series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n\sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx = \frac{1}{4\pi}\int_{-2\pi}^{2\pi}\left(\cos\left(\frac{x}{2}\right) + \sin x\right)dx = 0$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2\pi}\int_{-2\pi}^{2\pi}\left(\cos\left(\frac{x}{2}\right) + \sin x\right)\cos\left(\frac{nx}{2}\right)dx$$

$$a_n = \underbrace{\frac{1}{2\pi}\int_{-2\pi}^{2\pi}\sin x\cos\left(\frac{nx}{2}\right)dx}_{=0} + \underbrace{\frac{1}{2\pi}\int_{-2\pi}^{2\pi}\cos\left(\frac{x}{2}\right)\cos\left(\frac{nx}{2}\right)dx}_{=2\pi \text{ if } n=1}$$

$$a_n = \begin{cases} 1 & n = 1 \\ 0 & n \neq 1 \end{cases}$$

This follows the definition of the Kronecker delta function,

$$\delta_{m,n} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

(not to be confused with the Dirac delta function which takes on an impulse of infinity, not 1) This allows us to rewrite $a_n$ in a more compact manner.

$$a_n = \delta_{n,1}$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{2\pi}\int_{-2\pi}^{2\pi}\left(\cos\left(\frac{x}{2}\right) + \sin x\right)\sin\left(\frac{nx}{2}\right)dx$$

$$b_n = \underbrace{\frac{1}{2\pi}\int_{-2\pi}^{2\pi}\cos\left(\frac{x}{2}\right)\sin\left(\frac{nx}{2}\right)dx}_{=0} + \underbrace{\frac{1}{2\pi}\int_{-2\pi}^{2\pi}\sin x\sin\left(\frac{nx}{2}\right)dx}_{=1 \text{ if } n=2}$$

$$b_n = \begin{cases} 1 & n = 2 \\ 0 & n \neq 2 \end{cases} = \delta_{n,2}$$

$$u(x,t) = \sum_{n=1}^{\infty}\delta_{1,n}\cos\left(\frac{nx}{2}\right)e^{-\left(\frac{n}{2}\right)^2 t} + \sum_{n=1}^{\infty}\delta_{2,n}\sin\left(\frac{nx}{2}\right)e^{-\left(\frac{n}{2}\right)^2 t}$$

From here, we can simplify using the delta functions, making the solution look a lot nicer

$$u(x,t) = \cos\left(\frac{x}{2}\right)e^{-\frac{t}{4}} + \sin(x)e^{-t}$$

You may notice that we didn't actually need to solve the integrals but could rather find the delta functions from the initial conditions by inspection. This is a nice benefit of having the initial condition consist of trig functions.

### 2.1.5 Inhomogeneous Heat Equation

The inhomogeneous heat equation is of the general form

$$\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2} + S(x,t)$$

with $u(0,t) = A(t)$ and/or $u(L,t) = B(t)$ For the inhomogeneous heat equation, we cannot use separation of variables. There are a few different techniques we can use to solve them. With the exception of eigenfunction expansion, they all involve decomposing the solution into two functions: One "guess" function to remove the inhomogeneous boundary conditions and one resulting function which can be solved using the homogeneous case. This best seen through some examples:

Ex: Regular inhomogeneous time-independent boundary conditions

$$
\begin{aligned}
&u_t = \alpha^2 u_{xx} \\
&u(0,t) = u_0 \neq 0 \\
&u(L,t) = u_1 \neq 0 \\
&u(L,t) = B(t)
\end{aligned}
$$

Because the boundary conditions do not depend on time, we can assume that there will be a steady-state response in the solution so we can decompose the solution into a a time-dependent and a time-independent part:

$$u(x,t) = w(x) + v(x,t)$$

$w(x)$ is called the steady-state solution and $v(x,t)$ is called the transient solution.
Because a superposition of solutions is also a solution, we can solve each part separately.

$$
\begin{aligned}
&w_t = \alpha^2 w_{xx} \\
&w_t = 0 \Rightarrow w_{xx} = 0 \\
&\Rightarrow w(x) = Ax + B \\
&w(0) = u_0 \Rightarrow B = u_0 \\
&w(L) = u_1 \Rightarrow u_1 = AL + B \Rightarrow A = \frac{u_1 - u_0}{L} \\
&w(x) = \frac{u_1 - u_0}{L} x + u_0
\end{aligned}
$$

Now that we've solved for $w(x)$, we can use it to find a homogeneous PDE we can use to solve for $v(x,t)$.

$$
\begin{aligned}
&u_t = \alpha^2 u_{xx} \\
&w_t + v_t = \alpha^2 w_{xx} + \alpha^2 v_{xx} \\
&v_t = \alpha^2 v_{xx} \\
&u(0,t) = u_0 \Rightarrow w(0) + v(0,t) = u_0 \Rightarrow v(0,t) = 0 \\
&u(L,0) = u_1 \Rightarrow w(L) + v(L,t) = u_1 \Rightarrow v(L,0) = 0 \\
&u(x,0) = f(x) \Rightarrow w(x) + v(x,0) = f(x) \Rightarrow v(x,0) = f(x) - w(x) = f(x) - \left(\frac{u_1 - u_0}{L} x + u_0\right) = g(x)
\end{aligned}
$$

This now gives a homogeneous PDE for $v(x.t)$

$$
\begin{aligned}
&v_t = \alpha^2 v_{xx} \\
&v(0,t) = v(L,t) = 0 \\
&v(x,0) = f(x) - w(x) = g(x)
\end{aligned}
$$

Solving for $v(x,t)$ gives

$$v(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t},$$

$$b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right) dx$$

Putting it all together we have the final solution of

$$u(x,t) = w(x) + v(x,t) = \frac{u_1 - u_0}{L}x + u_0 + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$$

Ex2: Regular inhomogeneous time-independent Neumann boundary conditions

$$u_t = \alpha^2 u_{xx}$$
$$u_x(0,t) = q_0$$
$$u_x(L,t) = q_1$$
$$u(x,0) = f(x)$$

When we have Neumann boundary conditions, we are not able to do the exact same thing we did in the previous example. If we have the same guess $w(x) = Ax + B$ we see

$$w_x(0) = q_0 \Rightarrow A = q_0$$
$$w_x(L) = q_1 \Rightarrow A = q_1$$

So unless $q_0 = q_1$, there is no steady state solution. We've seen in ODE's that multiplying our guess by $x$ seems to work for constant coefficient equations and fortunately that works in this case as well with the addition of a time dependent term

$$w(x,t) = Ax^2 + Bx + Ct$$
$$w_x(0,t) = q_0 = B$$
$$w_x(L,t) = q_1 = 2AL + B \Rightarrow A = \frac{q_1 - q_0}{2L}$$
$$w_t = \alpha^2 w_{xx}$$
$$C = \alpha^2(2A) \Rightarrow C = \alpha^2\frac{q_1 - q_0}{L}$$
$$w(x,t) = \frac{q_1 - q_0}{2L}x^2 + q_0 x + \alpha^2\frac{q_1 - q_0}{L}t$$

From here, we follow the same steps to get an inhomogeneous PDE for $v(x,t)$

$$w_t + v_t = \alpha^2(w_{xx} + v_{xx}) \Rightarrow v_t = \alpha^2 v_{xx}$$
$$w_x(0,t) + v_x(0,t) = q_0 \Rightarrow v_x(0,t) = 0$$
$$w_x(L,t) + v_x(L,t) = q_1 \Rightarrow v_x(L,t) = 0$$
$$w(x,0) + v(x,0) = f(x) \Rightarrow v(x,0) = f(x) - w(x,0) = g(x)$$

Solving for $v(x,t)$ and putting it all together, we get

$$u(x,t) = w(x,t) + v(x,t) = \frac{q_1 - q_0}{2L}x^2 + q_0 x + \alpha^2 \frac{q_1 - q_0}{L} t + a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) e^{-\alpha^2 \left(\frac{n\pi}{L}\right)^2 t},$$
$$a_0 = \frac{1}{L}\int_0^L g(x)dx,\ a_n = \frac{2}{L}\int_0^L g(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

Ex3: Forcing Function/Irregular Equation

$$u_t = u_{xx} - u + x,\ 0 < x < 1,\ t > 0$$
$$u_x(0,t) = 1,\ u_x(1,t) = 2$$
$$u(x,0) = x$$

This case is similar to the first example but a bit more general, extending to different forms of the heat equation as well as those with a forcing function, $S(x)$.
We start by splitting up $u(x,t)$ and then solve for $w(x)$. This is often done by solving an ODE.

$$u(x,t) = w(x) + v(x,t)$$
$$w_t = w_{xx} - w + x = 0 \Rightarrow w_{xx} - w = -x$$
$$r^2 - 1 = 0 \Rightarrow r = \pm 1$$
$$w_c(x) = A\cosh(x) + B\sinh(x)$$
$$w_p = Ax + B,\ w_p' = A,\ w_p'' = 0$$
$$-(Ax + B) = -x \Rightarrow A = 1,\ B = 0$$
$$w_p(x) = x$$
$$w(x) = A\cosh(x) + B\sinh(x) + x$$

Now that we've found $w(x)$, we can find the equation for $v(x,t)$ and solve the rest of the PDE.

$$w_x = A\sinh(x) + B\cosh(x) + 1$$
$$w_x(0) = 1 = B + 1 \Rightarrow B = 0$$
$$w_x(1) = 2 = A\sinh(1) + 1 \Rightarrow A = \frac{1}{\sinh(1)}$$
$$w(x) = \frac{\cosh(x)}{\sinh(1)} + x$$
$$u_t = u_{xx} - u + x$$
$$v_t + w_t = v_{xx} + w_{xx} - v - w + x \Rightarrow v_t = v_{xx} - v$$
$$u_x(0,t) = 1 \Rightarrow v_x(0,t) + w_x(0) = 1 \Rightarrow v_x(0,t) = 0$$
$$u_x(1,t) = 2 \Rightarrow v_x(1,t) + w_x(1) = 2 \Rightarrow v_x(1,t) = 0$$
$$u(x,0) = v(x,0) + w(x) = x \Rightarrow v(x,0) = -\frac{\cosh(x)}{\sinh(1)} \equiv g(x)$$
$$v(x,t) = X(x)T(t)$$
$$XT' = X''T - XT \Rightarrow \frac{T'}{T} + 1 = \frac{X''}{X} = -\lambda$$

$v(x,t)$ has Neumann boundary conditions so it will have the following eigenvalues and eigenfunctions

$$\lambda_n = 0,\ (n\pi)^2$$
$$X_n = 1,\ \cos(n\pi x)$$
$$T' = -T(\lambda+1)$$
$$T_n = e^{-((n\pi)^2+1)t}$$
$$v(x,t) = a_0e^{-t} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)e^{-((n\pi)^2+1)t}$$
$$a_0 = \frac{1}{2}\int_{-1}^{1} g(x)dx = -\int_0^1 \frac{\cosh(x)}{\sinh(1)}dx = -\left.\frac{\sinh(x)}{\sinh(1)}\right|_0^1 = -1$$
$$a_n = \frac{1}{1}\int_{-1}^{1} g(x)\cos(n\pi x)dx = -2\int_0^1 \frac{\cosh(x)}{\sinh(1)}\cos(n\pi x)dx$$
$$\rightsquigarrow a_n = -\frac{2}{\sinh(1)}\left[\frac{n\pi\cosh(x)\sin(n\pi x)+\sinh(x)\cos(n\pi x)}{(n\pi)^2+1}\right]_0^1 = -\frac{2}{\sinh(1)}\left(\frac{\sinh(1)(-1)^n}{(n\pi)^2+1}\right)$$
$$a_n = \frac{2(-1)^{n+1}}{(n\pi)^2+1}$$
$$v(x,t) = -e^{-t} + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{(n\pi)^2+1}\cos(n\pi x)e^{-((n\pi)^2+1)t}$$
$$u(x,t) = w(x) + v(x,t)$$
$$u(x,t) = \frac{\cosh(x)}{\sinh(1)} - e^{-t} + x + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{(n\pi)^2+1}\cos(n\pi x)e^{-((n\pi)^2+1)t}$$

Ex4: Time dependent boundary conditions

$$u_t = u_{xx} - u + 2x$$
$$u(0,t) = e^{-t}$$
$$u(1,t) = 1$$
$$u(x,0) = \sin(6x) - \frac{\sinh(x)}{\sinh(1)} + x + 1$$

When we have time dependent boundary conditions we will always want to remove them with a guess solution before proceeding to the rest of the problem

$$u(x,t) = w(x,t) + v(x,t)$$

We will always make the guess

$$w(x,t) = A(t)x + B(t)$$
$$u(0,t) = B(t) = e^{-t}$$
$$u(1,t) = A(t) + e^{-t} = 1 \Rightarrow A(t) = 1 - e^{-t}$$
$$w(x,t) = (1-e^{-t})x + e^{-t}$$

Now that we solved for $w(x,t)$ we should be able to get a PDE for $v(x,t)$ with time-independent boundary conditions.

$$
\begin{aligned}
&w_t + v_t = w_{xx} + v_{xx} - w - v + 2x \\
&e^{-t}x - e^{-t} + v_t = v_{xx} - x + e^{-t} - e^{-t} - v + 2x \\
&v_t = v_{xx} - v + x \\
&u(x,0) = w(x,0) + v(x,0) = \sin(6x) - \frac{\sinh(x)}{\sinh(1)} + x + 1 \\
&1 + v(x,0) = \sin(6x) - \frac{\sinh(x)}{\sinh(1)} + x + 1 \Rightarrow v(x,0) = \sin(6x) - \frac{\sinh(x)}{\sinh(1)} + x \\
&v(0,t) = v(1,t) = 0
\end{aligned}
$$

Now we have a PDE for $v(x,t)$, which we are able to solve. The rest of the steps follow from previous examples. Because $v(x,t)$ has a forcing function, we'll split it up as follows

$$
\begin{aligned}
&v(x,t) = p(x) + q(x,t) \\
&p_t = p_{xx} - p + x \Rightarrow 0 = p'' - p + x \\
&\rightsquigarrow p_c = A\cosh(x) + B\cosh(x) \\
&p_p = x \Rightarrow p(x) = A\cosh(x) + B\sinh(x) + x \\
&p(0) = A = 0 \\
&p(1) = B\sinh(1) + 1 = 0 \Rightarrow B = -\frac{1}{\sinh(1)} \\
&p(x) = -\frac{\sinh(x)}{\sinh(1)} + x
\end{aligned}
$$

Now we can find a PDE for $q(x,t)$

$$
\begin{aligned}
&p_t + q_t = p_{xx} + q_{xx} - p - q + x \\
&q_t = -\frac{\sinh(x)}{\sinh(1)} + q_{xx} + \frac{\sinh(x)}{\sinh(1)} - x - q + x \\
&q_t = q_{xx} - q \\
&v(0,t) = p(0) + q(0,t) = 0 \Rightarrow q(0,t) = 0 \\
&v(1,t) = p(1) + q(1,t) = 0 \Rightarrow q(1,t) = 0 \\
&v(x,0) = p(x) + q(x,0) = -\frac{\sinh(x)}{\sinh(1)} + x + q(x,0) = \sin(6x)
\end{aligned}
$$

Now we can solve for $q(x,t)$

$$
\begin{aligned}
&q(x,t) = X(x)T(t) \\
&XT' = X''T - XT \Rightarrow \frac{T'}{T} = \frac{X''}{X} - 1 \\
&\frac{T'}{T} + 1 = \frac{X''}{X} = -\lambda \\
&\rightsquigarrow \lambda = (n\pi)^2
\end{aligned}
$$

$$
\begin{aligned}
&X_n = \sin(n\pi x)\\
&T_n = e^{-((n\pi)^2+1)t}\\
&q(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-((n\pi)^2+1)t}\\
&q(x,0) = \sin(6x) \Rightarrow b_n = \delta_{6,n}\\
&q(x,t) = \sin(6\pi x) e^{-(36\pi^2+1)t}
\end{aligned}
$$

Putting it all together we have

$$
\begin{aligned}
&v(x,t) = p(x) + q(x,t) = -\frac{\sinh(x)}{\sinh(1)} + x + \sin(6\pi x) e^{-(36\pi^2+1)t}\\
&u(x,t) = w(x,t) + v(x,t) = (1-e^{-t})x + e^{-t} - \frac{\sinh(x)}{\sinh(1)} + x + \sin(6\pi x) e^{-(36\pi^2+1)t}\\
&\ = 2x + (1-x)e^{-t} - \frac{\sinh(x)}{\sinh(1)} + \sin(6\pi x) e^{-(36\pi^2+1)t}
\end{aligned}
$$

Ex5: Eigenfunction expansion

$$
\begin{aligned}
&u_t = u_{xx} + e^{-t}\cos\left(\frac{3\pi}{2}x\right) + 1,\ 0 < x < 1,\ t > 0\\
&u_x(0,t) = 1,\ u(1,t) = t\\
&u(x,0) = 1
\end{aligned}
$$

When we have a time dependent source/sink term in the PDE, we will have to use the method of eigenfunction expansion. Eigenfunction expansion is a general approach to solving any of these more complicated PDEs.
We'll first start by removing the time-dependent boundary conditions and getting a homogeneous equation.

$$
\begin{aligned}
&w(x,t) = A(t)x + B(t)\\
&w_x(x,t) = A(t)\\
&w_x(0,t) = 1 = A(t)\\
&w(1,t) = t = 1 + B(t) \Rightarrow B(t) = t-1\\
&w(x,t) = x + t - 1\\
&w_t + v_t = w_{xx} + v_{xx} + e^{-t}\cos\left(\frac{3\pi}{2}x\right) + 1\\
&1 + v_t = v_{xx} + e^{-t}\cos\left(\frac{3\pi}{2}x\right) + 1\\
&v_t = v_{xx} + e^{-t}\cos\left(\frac{3\pi}{2}x\right)\\
&u_x(0,t) = w_x(0,t) + v_x(0,t) = 1 \Rightarrow v_x(0,t) = 0\\
&u(1,t) = w(1,t) + v(1,t) = t \Rightarrow v(1,t) = 0\\
&u(x,0) = w(x,0) + v(x,0) = x - 1 + v(x,0) = 1 \Rightarrow v(x,0) = 2 - x
\end{aligned}
$$

Now that we have a homogeneous equation for $v(x,t)$, we are able to apply the method of eigenfunction expansion.
The first step is to solve the for the eigenvalues of the PDE without the source term.

$$\begin{aligned}
&v_t = v_{xx}\\
&v_x(0,t) = v(1,t) = 0\\
&v(x,0) = 2 - x
\end{aligned}$$

Mixed type 2 so the eigenvalues and eigenfunctions are:

$$\begin{aligned}
&\lambda_n = \left(\frac{2n-1}{2}\pi\right)^2\\
&X_n = \cos\left(\frac{2n-1}{2}\pi x\right)\\
&v(x,t) = \sum_{n=1}^{\infty} V_n(t)\cos\left(\frac{2n-1}{2}\pi x\right)
\end{aligned}$$

Note that we did not put in the time-dependent eigenfunction into $v(x,t)$ but rather express it as some unknown function $V_n(t)$. We will get to solving this in a bit.
The next step is to expand the source term so that we are able to express it as a sum.

$$\begin{aligned}
&S(x,t) = e^{-t}\cos\left(\frac{3\pi}{2}x\right) = \sum_{n=1}^{\infty} S_n(t)\cos\left(\frac{2n-1}{2}\pi x\right)\\
&S_n(t) = \delta_{2,n}e^{-t}\\
&S(x,t) = \sum_{n=1}^{\infty} \delta_{2,n}e^{-t}\cos\left(\frac{2n-1}{2}\pi x\right)
\end{aligned}$$

Now that we have the source term expressed as a sum, we can write out $v_t$ and $v_{xx}$ and combine all of the summations.

$$\begin{aligned}
&v_t = \sum_{n=1}^{\infty} V_n'(t)\cos\left(\frac{2n-1}{2}\pi x\right)\\
&v_{xx} = \sum_{n=1}^{\infty} -V_n(t)\cos\left(\frac{2n-1}{2}\pi x\right)\left(\frac{2n-1}{2}\pi\right)^2\\
&v_t - v_{xx} - S(x,t) = 0\\
&\sum_{n=1}^{\infty}\left(V_n'(t) + V_n(t)\left(\frac{2n-1}{2}\pi\right)^2 - e^{-t}\delta_2(n)\right)\cos\left(\frac{2n-1}{2}\pi x\right) = 0
\end{aligned}$$

This gives us an ODE for $V_n$ that we can solve.

$$V_n'(t) + V_n(t)\left(\frac{2n-1}{2}\pi\right)^2 - e^{-t}\delta_{2,n} = 0$$

$$\mu = e^{\int \left(\frac{2n-1}{2}\pi\right)^2 dt} = e^{\left(\frac{2n-1}{2}\pi\right)^2 t}$$
$$(V_n\mu)' = \mu e^{-t}\delta_{2,n}$$
$$V_n = \frac{1}{e^{\left(\frac{2n-1}{2}\pi\right)^2 t}} \int e^{\left(\left(\frac{2n-1}{2}\pi\right)^2 - 1\right)t}\delta_{2,n} dt$$
$$V_n = \frac{1}{e^{\left(\frac{2n-1}{2}\pi\right)^2 t}} \left( \frac{1}{\left(\frac{2n-1}{2}\pi\right)^2 - 1} e^{\left(\left(\frac{2n-1}{2}\pi\right)^2 - 1\right)t}\delta_{2,n} + C_n \right)$$
$$V_n = \frac{\delta_{2,n}}{\left(\frac{2n-1}{2}\pi\right)^2 - 1} e^{-t} + C_n e^{-\left(\frac{2n-1}{2}\pi\right)^2 t}$$

Now that we have solved for $V_n$, we can write a more complete form of $v(x,t)$ and then use Fourier series to solve for the last remaining constant. We will introduce a new constant, $D_n$, just to make computation slightly simpler.

$$v(x,t) = \sum_{n=1}^{\infty} \left( \frac{\delta_{2,n}}{\left(\frac{2n-1}{2}\pi\right)^2 - 1} e^{-t} + C_n e^{-\left(\frac{2n-1}{2}\pi\right)^2 t} \right) \cos\left(\frac{2n-1}{2}\pi x\right)$$
$$v(x,0) = 2 - x = \sum_{n=1}^{\infty} \underbrace{\left( \frac{\delta_{2,n}}{\left(\frac{2n-1}{2}\pi\right)^2 - 1} + C_n \right)}_{D_n} \cos\left(\frac{2n-1}{2}\pi x\right)$$

Fourier series:

$$D_n = 2\int_0^1 (2-x)\cos\left(\frac{2n-1}{2}\pi x\right) dx$$
$$D_n = \frac{4\left((2\pi n - \pi)\sin\left(\frac{2\pi n - \pi}{2}\right) - 2\cos\left(\frac{2\pi n - \pi}{2}\right) + 2\right)}{\pi^2 \cdot (4n^2 - 4n + 1)}$$
$$D_n = \frac{4((2n-1)\pi(-1)^{n+1} + 2)}{(2n-1)^2\pi^2} = \frac{(2n-1)(-1)^{n+1}\pi + 2}{\left(\frac{2n-1}{2}\pi\right)^2}$$
$$\Rightarrow C_n = \frac{(2n-1)(-1)^{n+1}\pi + 2}{\left(\frac{2n-1}{2}\pi\right)^2} - \frac{\delta_2(n)}{\left(\frac{2n-1}{2}\pi\right)^2 - 1}$$

$$v(x,t) = \sum_{n=1}^{\infty} \left( \frac{\delta_2(n)}{\left(\frac{2n-1}{2}\pi\right)^2 - 1} e^{-t} + \left( \frac{(2n-1)(-1)^{n+1}\pi + 2}{\left(\frac{2n-1}{2}\pi\right)^2} - \frac{\delta_2(n)}{\left(\frac{2n-1}{2}\pi\right)^2 - 1} \right) e^{-\left(\frac{2n-1}{2}\pi\right)^2 t} \right) \cos\left(\frac{2n-1}{2}\pi x\right)$$
$$u(x,t) = x + t - 1 + \sum_{n=1}^{\infty} \left( \frac{\delta_2(n)}{\left(\frac{2n-1}{2}\pi\right)^2 - 1} e^{-t} + \left( \frac{(2n-1)(-1)^{n+1}\pi + 2}{\left(\frac{2n-1}{2}\pi\right)^2} - \frac{\delta_2(n)}{\left(\frac{2n-1}{2}\pi\right)^2 - 1} \right) e^{-\left(\frac{2n-1}{2}\pi\right)^2 t} \right) \cos\left(\frac{2n-1}{2}\pi x\right)$$

These examples covered a lot of information so here is an algorithmic summary of what the steps are for each method:

1. Regular Inhomogeneous Boundary Conditions
   Form: $u_t = \alpha^2 u_{xx}$ with $u(0,t) = u_0$ and $u(L,t) = u_1$ (or mixed BCs)
   Guess $w(x) = Ax + B$

(a) Use the BCs to solve for $A$ and $B$

(b) Plug $u = w + v$ into the PDE to find the homogeneous problem for $v(x,t)$ and solve for $v$

2. Regular Inhomogeneous Boundary Conditions (Neumann)
Form: $u_t = \alpha^2 u_{xx}$ with $u_x(0,t) = q_0$ and $u_x(L,t) = q_1$ Guess $w(x,t) = Ax^2 + Bx + Ct$

   (a) Use the BCs and plug into the PDE to get $C = 2\alpha^2 A$ and solve for $A, B, C$

   (b) Plug $u = w + v$ into the PDE to find the homogeneous problem for $v(x,t)$ and solve for $v$

3. Forcing Function/Irregular Equation
Form: $u_t = \alpha^2 u_{xx} - \gamma u + g(x)$ with constant boundary conditions
Guess will be the $w(x)$ that solves the resulting ODE: $\alpha^2 w_{xx} - \gamma w + g(x) = 0$

   (a) Use the BCs to solve for the unknown coefficients coming from the complementary solution of $w(x)$

   (b) Plug $u = w + v$ into the PDE to find the homogeneous problem for $v(x,t)$ and solve for $v$

4. Time Dependent Boundary Conditions:
Form: when the 2 boundary conditions are functions of time, such as $u(0,t) = p(t)$ and $u(L,t) = q(t)$
Guess $w(x,t) = A(t)x + B(t)$

   (a) Use the BCs to find the functions $A(t)$ and $B(t)$

   (b) Plug $u = w + v$ into the PDE to find a new PDE for $v(x,t)$ with time independent BCs. Note that $v(x,t)$ may still be inhomogeneous, requiring you to then solve for $v$ using one of the other methods

5. Eigenfunction Expansion/Time Dependent Source/Sink (General solution of the heat equation)
Form: $u_t = \alpha^2 u_{xx} + S(x,t)$
This is the most general solution and will always work

   (a) First use one of the other methods to remove any inhomogeneous boundary conditions. This will provide a new PDE of the form $v_t = \alpha^2 v_{xx} + S_2(x,t)$ with homogeneous boundary conditions.

   (b) Find the general eigenfunctions of $X_n$ for the homogeneous heat equation, $v_t = \alpha^2 v_{xx}$ omitting the source term: $v_t = \alpha^2 v_{xx}$

   (c) Expand the source term in terms of $X_n$ and write the source term as:

$$S_2(x,t) = \sum_{n=1}^{\infty} S_n(t) X_n(x)$$

   Use the Fourier analysis $S_n(t) = \frac{2}{L}\int_0^L s(x,t)X_n(x)$ or intuition to solve for $S_n(t)$ given that you know $S_2(x,t)$

(d) Express the solution for $v$ as

$$v(x,t) = \sum_{n=1}^{\infty} V_n(t) X_n(x)$$

where $V_n$ is an undetermined function. Express $v_t$ and $v_{xx}$ as an infinite series as well.

(e) Substitute all values that are now in terms of $X_n$ back into the original PDE.

$$\sum_{n=1}^{\infty} V_n' X_n = \sum_{n=1}^{\infty} V_n X_n'' + \sum_{n=1}^{\infty} S_n X_n$$

rearrange to get an expression of the form

$$\sum_{n=1}^{\infty} [V_n' + \lambda_n V_n - S_n] X_n = 0$$

(f) Solve the ODE for $V_n$ inside the series: $V_n'(t) + \lambda_n V_n(t) - S_n(t) = 0$ (Integrating factor technique will likely work well)

(g) The ODE will yield a general solution with an undetermined coefficient for the homogeneous portion of the ODE: $V_n(t) = f(n,t) + C_n g(n,t)$

(h) Use Fourier series and the initial condition $v(x,0) = h(x)$ to solve for the unknown constant that comes from the solution of $V_n$:

$$v(x,t) = \sum_{n=1}^{\infty} [f(n,t) + C_n g(n,t)] X_n$$

$$v(x,0) = h(x) = \sum_{n=1}^{\infty} \underbrace{[f(n,0) + C_n g(n,0)]}_{D_n} X_n$$

Then $D_n = \frac{2}{L}\int_0^L h(x) X_n(x) dx$, solve for $C_n$ from $D_n$, then write out full solution.

## 2.2 Wave Equation

### 2.2.1 Separation of Variables

The general form of the wave equation is

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

In order to find a solution, we will again require two boundary conditions as well as two initial conditions: $u(x,0)$ and $u_t(x,0)$ normally.
We can use many of the same techniques to solve the wave equation as we did to solve the heat equation.
Ex:

$$u_{tt} = u_{xx},\ 0 \leq x \leq \pi,\ t \geq 0$$

$$u_x(0,t) = u_x(\pi,t) = 0$$
$$u(x,0) = f(x) = 5\cos x$$
$$u_t(x,0) = g(x) = 1 - 2\cos(2x)$$

We will start by using separation of variables and finding the eigenvalues and eigenfunctions, the same as was done with the heat equation.

$$u(x,t) = X(x)T(t)$$
$$XT'' = X''T \Rightarrow \frac{T''}{T} = \frac{X''}{X} = -\lambda$$
$$X'' + \lambda X = 0,\; X'(0) = X'(\pi) = 0$$
$$\rightsquigarrow \lambda_0 = 0,\; \lambda_n = \left(\frac{n\pi}{L}\right)^2 = n^2$$
$$X_0 = 1,\; X_n = \cos(nx)$$
$$T_n'' + \lambda_n T_n = 0$$
$$\lambda_0 = 0:\; T_0'' = 0 \Rightarrow T_0 = A_0 + B_0 t$$
$$\lambda_n = n^2:\; T_n'' + n^2 T_n = 0 \Rightarrow T_n = A_n\cos(nt) + B_n\sin(nt)$$
$$u(x,t) = A_0 + B_0 t + \sum_{n=1}^{\infty} \cos(nx)\left(A_n\cos(nt) + B_n\sin(nt)\right)$$
$$u_t(x,t) = B_0 + \sum_{n=1}^{\infty} \cos(nx)(n)\left(-A_n\sin(nt) + B_n\cos(nt)\right)$$

Now we can use our two initial conditions to solve for $A_0, A_n, B_0, B_n$

$$u(x,0) = 5\cos x = A_0 + \sum_{n=1}^{\infty} A_n\cos(nx)$$
$$\Rightarrow A_0 = 0,\; A_n = 5\delta_{1,n}$$
$$u_t(x,0) = 1 - 2\cos(2x) = B_0 + \sum_{n=1}^{\infty}(B_n n)\cos(nx)$$
$$\Rightarrow B_0 = 1,\; B_n = -\delta_{n,2}$$
$$u(x,t) = t + \sum_{n=1}^{\infty}\left(5\delta_{1,n}\cos(nt) - \delta_{2,n}\sin(nt)\right)\cos(nx)$$
$$u(x,t) = t + 5\cos t\cos x - \sin(2t)\cos(2x)$$

The separation of variables method will work for any of the following forms:

$$u_{tt} = c^2 u_{xx}$$
$$u_{tt} = c^2 u_{xx} - \gamma u$$
$$u_{tt} = c^2 u_{xx} - \gamma(t) u$$
$$u_{tt} = c^2 u_{xx} - \gamma u_t + \eta(t) u$$

Ex2:

$$
\begin{aligned}
&u_{tt} = u_{xx} - \gamma u,\ 0 < x < 1,\ t > 0 \\
&u(0,t) = u(1,t) = 0 \\
&u(x,0) = 0,\ u_t(x,0) = g(x) \\
&u(x,t) = X(x)T(t) \\
&\frac{T''}{T} = \frac{X''}{X} - \gamma \Rightarrow \frac{T''}{T} + \gamma = \frac{X''}{X} = -\lambda
\end{aligned}
$$

Dirichlet boundary conditions

$$
\begin{aligned}
&\lambda_n = (n\pi)^2 \\
&X_n = \sin(n\pi x) \\
&T_n'' + T_n(\gamma + \lambda) = T_n'' + T_n(\gamma + (n\pi)^2) = 0 \\
&\text{assuming } \gamma \geq 0,\ \mu_n = \sqrt{\gamma + (n\pi)^2} \\
&T_n = A_n \cos(\mu_n t) + B_n \sin(\mu_n t) \\
&u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(A_n \cos(\mu_n t) + B_n \sin(\mu_n t)\right) \\
&u_t(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(-A_n \mu_n \sin(\mu_n t) + B_n \mu_n \cos(\mu_n t)\right) \\
&u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = 0 \Rightarrow A_n = 0 \\
&u_t(x,0) = \sum_{n=1}^{\infty} B_n \mu_n \sin(n\pi x) = g(x) \\
&B_n \mu_n = 2\int_0^1 g(x)\sin(n\pi x)dx \Rightarrow B_n = \frac{2}{\mu_n}\int_0^1 g(x)\sin(n\pi x)dx \\
&u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\sin(t\sqrt{\gamma + (n\pi)^2}) = \sum_{n=1}^{\infty} \frac{2\sin(n\pi x)\sin(t\sqrt{\gamma + (n\pi)^2})}{\sqrt{\gamma + (n\pi)^2}} \int_0^1 g(x)\sin(n\pi x)dx
\end{aligned}
$$

We can also use the same methods of dealing with inhomogeneous equations for the wave equation as we did with the heat equation.
Ex3:

$$
\begin{aligned}
&u_{tt} = u_{xx},\ t > 0,\ 0 \leq x \leq 1 \\
&u(0,t) = 0,\ u(1,t) = 1 \\
&u(x,0) = \sin(\pi x) + x,\ u_t(x,0) = \sin(3\pi x)
\end{aligned}
$$

Steady state solution:

$$
\begin{aligned}
&u(x,t) = w(x) + v(x,t) \\
&w_{tt} = 0 = w_{xx} \Rightarrow w(x) = Ax + B
\end{aligned}
$$

$$w(0) = 0 \Rightarrow B = 0$$
$$w(1) = 1 = A$$
$$w(x) = x$$
$$w_{tt} + v_{tt} = w_{xx} + v_{xx} \Rightarrow v_{tt} = v_{xx}$$
$$w(0) + v(0) = 0 \Rightarrow v(0) = 0$$
$$w(1) + v(1) = 1 \Rightarrow v(1) = 0$$
$$w(x) + v(x,0) = x + v(x,0) = \sin(\pi x) + x \Rightarrow v(x,0) = \sin(\pi x)$$
$$w_t(x) + v_t(x,0) = \sin(3\pi x) \Rightarrow v_t(x,0) = \sin(3\pi x)$$

Separation of variables:
Dirichlet BCs

$$\lambda_n = (n\pi)^2$$
$$X_n = \sin(n\pi x)$$
$$T_n'' + \lambda_n T_n = T_n'' + (n\pi)^2 T_n' = 0$$
$$\Rightarrow T_n = A_n \cos(n\pi t) + B_n \sin(n\pi t)$$
$$v(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(A_n \cos(n\pi t) + B_n \sin(n\pi t)\right)$$
$$v_t(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(-A_n n\pi \sin(n\pi t) + B_n n\pi \cos(n\pi t)\right)$$
$$v(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = \sin(\pi x)$$
$$A_n = \delta_{1,n}$$
$$v_t(x,0) = \sum_{n=1} B_n n\pi \sin(n\pi x) = \sin(3\pi x)$$
$$B_n n\pi = \delta_{3,n} \Rightarrow B_n = \frac{\delta_{3,n}}{n\pi}$$
$$v(x,t) = \sin(\pi x)\cos(\pi t) + \frac{1}{3\pi}\sin(3\pi x)\sin(3\pi t)$$
$$u(x,t) = x + \sin(\pi x)\cos(\pi t) + \frac{1}{3\pi}\sin(3\pi x)\sin(3\pi t)$$

Ex4:

$$u_{tt} = c^2 u_{xx} + e^{-t}\cos(3x),\ 0 < x < \frac{\pi}{2},\ t > 0$$
$$u_x(0,t) = 0,\ u\left(\frac{\pi}{2}, t\right) = t$$
$$u(x,0) = \cos x,\ u_t(x,0) = 1 + \cos(5x)$$

Steady state solution:

$$u(x,t) = w(x,t) + v(x,t)$$

$$
\begin{aligned}
&w(x,t) = A(t)x + B(t) \\
&w_x(x,t) = A(t) \Rightarrow w_x(0,t) = A(t) = 0 \\
&w\left(\frac{\pi}{2}, t\right) = B(t) = t \\
&w(x,t) = t \\
&w_{tt} + v_{tt} = c^2 w_{xx} + c^2 v_{xx} + e^{-t}\cos(3x) \Rightarrow v_{tt} = c^2 v_{xx} + e^{-t}\cos(3x) \\
&w_x(0,t) + v_x(0,t) = 0 \Rightarrow v_x(0,t) = 0 \\
&w\left(\frac{\pi}{2}, t\right) + v\left(\frac{\pi}{2}, t\right) = t \Rightarrow v\left(\frac{\pi}{2}, t\right) = 0 \\
&w(x,0) + v(x,0) = v(x,0) = \cos x \\
&w_t(x,0) + v_t(x,0) = 1 + v_t(x,0) = 1 + \cos(5x) \Rightarrow v_t(x,0) = \cos(5x)
\end{aligned}
$$

Eigenfunction expansion:
Mixed BCs:

$$
\begin{aligned}
&\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2 = (2n-1)^2 \\
&X_n = \cos(\mu_n x),\ \ \mu_n = 2n-1,\ \ n \geq 1 \\
&v(x,t) = \sum_{n=1}^{\infty} V_n(t)\cos(\mu_n x) \\
&S(x,t) = e^{-t}\cos(3x) = \sum_{n=1}^{\infty} S_n(t)\cos(\mu_n x) \\
&\mu_n = 2n-1 = 3 \Rightarrow n = 2 \\
&S_n = e^{-t}\delta_{2,n} \\
&v_{tt} = \sum_{n=1}^{\infty} V_n''(t)\cos(\mu_n x) \\
&v_{xx} = \sum_{n=1}^{\infty} -V_n(t)\mu_n^2\cos(\mu_n x) \\
&v_{tt} - c^2 v_{xx} - S(x,t) = 0 \\
&\sum_{n=1}^{\infty} \left(V_n'' + c^2\mu_n^2 V_n - e^{-t}\delta_{2,n}\right)\cos(\mu_n x) = 0
\end{aligned}
$$

2nd Order ODE:

$$
\begin{aligned}
&V_n'' + c^2\mu_n^2 V_n = \delta_{2,n}e^{-t} \\
&(V_n)_c = A_n\cos(c\mu_n t) + B_n\sin(c\mu_n t) \\
&(V_n)_p = C_n e^{-t} = (V_n)_p'' \\
&C_n e^{-t}(1 + c^2\mu_n^2) = \delta_{2,n}e^{-t} \\
&C_n = \frac{\delta_{2,n}}{1 + c^2\mu_n^2}
\end{aligned}
$$

$$V_n(t) = A_n \cos(c\mu_n t) + B_n \sin(c\mu_n t) + \frac{\delta_{2,n} e^{-t}}{1 + c^2\mu_n^2}$$

Impose initial conditions:

$$\begin{aligned}
v(x,t) &= \sum_{n=1}^{\infty} \left( A_n \cos(c\mu_n t) + B_n \sin(c\mu_n t) + \frac{\delta_{2,n} e^{-t}}{1 + c^2\mu_n^2} \right) \cos(\mu_n x) \\
v_t(x,t) &= \sum_{n=1}^{\infty} \left( -A_n c\mu_n \sin(c\mu_n t) + B_n c\mu_n \cos(c\mu_n t) - \frac{\delta_{2,n} e^{-t}}{1 + c^2\mu_n^2} \right) \cos(\mu_n x) \\
v(x,0) &= \cos x = \sum_{n=1}^{\infty} \left( A_n + \frac{\delta_{2,n}}{1 + c^2\mu_n^2} \right) \cos(\mu_n x) \\
\mu_n &= 1 = 2n - 1 \Rightarrow n = 1 \\
A_n &+ \frac{\delta_{2,n}}{1 + c^2\mu_n^2} = \delta_{1,n} \Rightarrow A_n = \delta_{1,n} - \frac{\delta_{2,n}}{1 + c^2\mu_n^2} \\
v_t(x,0) &= \cos(5x) = \sum_{n=1}^{\infty} \left( B_n c\mu_n - \frac{\delta_{2,n}}{1 + c^2\mu_n^2} \right) \cos(\mu_n x) \\
\mu_n &= 5 = 2n - 1 \Rightarrow n = 3 \\
B_n c\mu_n &- \frac{\delta_{2,n}}{1 + c^2\mu_n^2} = \delta_{3,n} \Rightarrow B_n = \frac{\delta_{3,n}}{c\mu_n} + \frac{\delta_{2,n}}{c\mu_n(1 + c^2\mu_n^2)} \\
v(x,t) &= \cos(ct)\cos(x) + \left( \frac{e^{-t} - \cos(3ct)}{1 + 9c^2} + \frac{\sin(3ct)}{3c(1 + 9c^2)} \right) \cos(3x) + \frac{\sin(5ct)}{5c} \cos(5x) \\
u(x,t) &= t + \cos(ct)\cos(x) + \left( \frac{3ce^{-t} - 3c\cos(3ct) + \sin(3ct)}{3c(1 + 9c^2)} \right) \cos(3x) + \frac{1}{5c} \sin(5ct)\cos(5x)
\end{aligned}$$

### 2.2.2 D'Alembert's Solution

We can think of the solution to the wave equation as two superimposed waves travelling in opposite directions. This is represented through d'Alembert's solution to the wave equation.
Derivation of formula:

$$\begin{aligned}
&u_{tt} = c^2 u_{xx} \\
&u_{tt} - c^2 u_{xx} = 0 \\
&(u_t - cu_x)(u_t + cu_x) = 0
\end{aligned}$$

We can show that any function of the form $F(x - ct)$ satisfies $u_t + cu_x = 0$.

$$\begin{aligned}
&u(x,t) = F(x - ct) \\
&\text{let } \xi = x - ct \\
&u_t = \frac{dF}{d\xi}\frac{d\xi}{dt} = F'(-c) \\
&u_x = \frac{dF}{d\xi}\frac{d\xi}{dx} = F'(1) \\
&u_t + cu_x = -cF' + cF' = 0
\end{aligned}$$

$\therefore F(x-ct)$ is a solution to $u_t + cu_x = 0$
Any function $G(x+ct)$ is a solution to $u_t - cu_x = 0$

$$\text{let } \eta = x + ct$$
$$u_t = \frac{dG}{d\eta}\frac{d\eta}{dt} = G'(c)$$
$$u_x = \frac{dG}{d\eta}\frac{d\eta}{dx} = G'(1)$$
$$\Rightarrow G'(c) - cG' = 0$$

The solution to $u_{tt} = c^2 u_{xx}$ is $u = F(x-ct) + G(x+ct)$.
Proving that it's a solution:

$$u_{tt} - c^2 u_{xx} = 0$$
$$u_t = \frac{dF}{d\xi}\frac{d\xi}{dt} + \frac{dG}{d\eta}\frac{d\eta}{dt}$$
$$u_{tt} = \frac{d^2F}{d\xi^2} \cdot c^2 + \frac{d^2G}{d\eta^2} \cdot c^2$$
$$u_x = \frac{dF}{d\xi}\frac{d\xi}{dx} + \frac{dF}{d\eta}\frac{d\eta}{dx}$$
$$u_{xx} = \frac{d^2F}{d\xi^2} + \frac{d^2G}{d\eta^2}$$
$$u_{tt} - c^2 u_{xx} = F''c^2 + G''c^2 - c^2(F'' + G'') = 0$$
$$u(x,t) = F(x-ct) + G(x+ct)$$
$$u(x,0) = f(x)$$
$$u_t(x,0) = g(x)$$
$$u(x,0) = f(x) = F(x) + G(x)$$
$$u_t(x,0) = g(x) = -cF'(x) + cG'(x)$$
$$-cF(x) + cG(x) = \int_0^x g(s)ds$$
$$2cG(x) = \int_0^x g(s)ds + cf(x)$$
$$G(x) = \frac{1}{2c}\int_0^x g(s)ds + \frac{1}{2}f(x)$$
$$2cF(x) = cf(x) - \int_0^x g(s)ds$$
$$\Rightarrow F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)ds$$
$$u(x,t) = F(x-ct) + G(x+ct)$$
$$= \frac{1}{2}f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)ds + \frac{1}{2c}\int_0^{x+ct} g(s)ds + \frac{1}{2}f(x+ct)$$
$$= \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)ds + \frac{1}{2c}\int_{x-ct}^0 g(s)ds$$

And so the final solution is of the form

$$u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$$

Ex:

$$\begin{aligned}
&u_{tt} = u_{xx},\ t > 0,\ 0 \leq x \leq 1 \\
&u(0,t) = 0,\ u(1,t) = 1 \\
&u(x,0) = \sin(\pi x) + x,\ u_t(x,0) = \sin(3\pi x)
\end{aligned}$$

D'Alembert's solution:

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds \\
u(x,t) &= \frac{1}{2}\left((x-t) + \sin(\pi(x-t)) + (x+t) + \sin(\pi(x+t)) + \int_{x-t}^{x+t} \sin(3\pi s)ds\right) \\
&= x + \frac{1}{2}\left(\sin(\pi x)\cos(\pi t) - \cos(\pi x)\sin(\pi t) + \sin(\pi x)\cos(\pi t) + \cos(\pi x)\sin(\pi t) - \frac{1}{3\pi}\left[\cos(3\pi s)\right]_{x-t}^{x+t}\right) \\
&= x + \frac{1}{2}\left(2\sin(\pi x)\cos(\pi t) - \frac{1}{3\pi}\left(\cos(3\pi(x+t)) - \cos(3\pi(x-t))\right)\right) \\
&= x + \frac{1}{2}\left(2\sin(\pi x)\cos(\pi t) - \frac{1}{3\pi}\left(\cos(3\pi x)\cos(3\pi t) - \sin(3\pi x)\sin(3\pi t) - (\cos(3\pi x)\cos(3\pi t) + \sin(3\pi x)\sin(3\pi t))\right)\right) \\
&= x + \frac{1}{2}\left(2\sin(\pi x)\cos(\pi t) + \frac{2}{3\pi}\sin(3\pi x)\sin(3\pi t)\right) \\
u(x,t) &= x + \sin(\pi x)\cos(\pi t) + \frac{1}{3\pi}\sin(3\pi x)\sin(3\pi t)
\end{aligned}$$

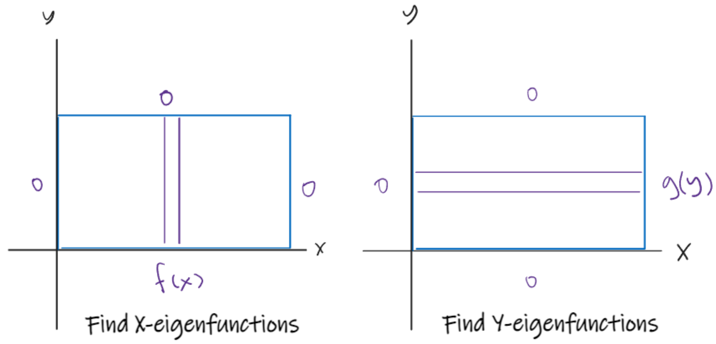
## 2.3 Laplace Equation

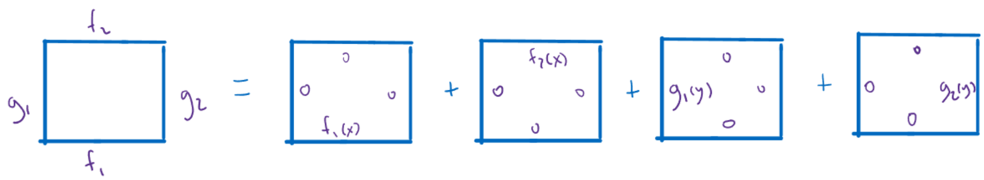
### 2.3.1 Rectangular Laplace Equation

The Laplace equation is of the form

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

and is solved over a 2D domain. It is commonly the steady state of the heat or wave equation. The solution method involves splitting the problem up into 4 subproblems with each subproblem having a solution similar to the wave equation.




1. Draw out the domain and BCs
2. Decompose into up to 4 subproblems
3. Assume $u(x, y) = X(x)Y(y)$ so that $\frac{Y''}{Y} = \frac{-X''}{X} = \pm\lambda$
4. Solve for eigenfunctions in the completely homogeneous axis first (see above). Here we cover the homogeneous $X$ case.
5. Apply homogeneous boundary conditions to $X$ to fix $X_n$, $\lambda_n$ identical to eigenfunctions of heat equation boundary conditions.
6. $Y_n$ will take the form of $Y_n(y) = A_n \cosh(\mu_n y) + B_n \sinh(\mu_n y)$
7. Use the remaining homogeneous boundary to eliminate a coefficient or reduce the expression to $A_n = C\, B_n$ for some constant $C$ and rewrite the equation in terms of a single constant $Q_n$. Using hyperbolic identities, the 8 common half-homogeneous boundary scenarios are listed below:

**Common Laplace Equation Boundary Solutions**

If $u(x,0) = 0$

$$u^{\text{top}} = \sum_{n=1}^{\infty} Q_n \sinh(\mu_n y) X_n$$

If $u(x,b) = 0$

$$u^{\text{bottom}} = \sum_{n=1}^{\infty} Q_n \sinh(\mu_n (y-b)) X_n$$

If $u(0,y) = 0$

$$u^{\text{right}} = \sum_{n=1}^{\infty} Q_n \sinh(\mu_n x) Y_n$$

If $u(a,y) = 0$

$$u^{\text{left}} = \sum_{n=1}^{\infty} Q_n \sinh(\mu_n (x-a)) Y_n$$

If $u_y(x,0) = 0$

$$u^{\text{top}} = \sum_{n=1}^{\infty} Q_n \cosh(\mu_n y) X_n$$

If $u_y(x,b) = 0$

$$u^{\text{bottom}} = \sum_{n=1}^{\infty} Q_n \cosh(\mu_n (y-b)) X_n$$

If $u_x(0,y) = 0$

$$u^{\text{right}} = \sum_{n=1}^{\infty} Q_n \cosh(\mu_n x) Y_n$$

If $u_x(a,y) = 0$

$$u^{\text{left}} = \sum_{n=1}^{\infty} Q_n \cosh(\mu_n (x-a)) Y_n$$

8. Use the final in-homogenous boundary condition to find a Fourier relation to determine the coefficients $Q_n$

9. Solve the remaining subproblems, the final solution is:

$$u(x,y) = u(x,y)^{\text{left}} + u(x,y)^{\text{right}} + u(x,y)^{\text{top}} + u(x,y)^{\text{bottom}}$$

* Some of the sub-problems have trivial solutions if there is no in-homogenous boundary condition for that side.

$u_{xx} + u_{yy} = 0,\ 0 \le x \le 3,\ 0 \le y \le 2$

$u_x(0,y) = 0.1,\ u_y(x,0) = 0$

$$u(x,2) = \begin{cases} x & 0 \le x < 1 \\ 2-x & 1 \le x < 3 \end{cases}$$

$$u(3,y) = \begin{cases} y & 0 \le y < 1 \\ 2-y & 1 \le y < 2 \end{cases}$$

$u = u^A + u^B + u^C$

$u^A :\ u_x(0,y) \neq 0$

$u^B :\ u(x,2) \neq 0$

$u^C :\ u(3,y) \neq 0$

Subproblem A:

$$u(x,y) = X(x)Y(y)$$
$$X''Y + XY'' = 0 \Rightarrow -\frac{X''}{X} = \frac{Y''}{Y} = -\lambda$$
$$u_y(x,0) = u(x,2) = 0$$
$$\rightsquigarrow \lambda_n = \left(\frac{2n-1}{2L}\pi\right)^2 = \left(\frac{2n-1}{4}\pi\right)^2$$
$$Y_n = \cos(\mu_n y),\ \mu_n = \frac{2n-1}{4}\pi$$
$$X_n = A_n \cosh(\mu_n x) + B_n \sinh(\mu_n x)$$
$$u(3,y) = 0 \Rightarrow X(3) = 0$$
$$A_n \cosh(3\mu_n) + B_n \sinh(3\mu_n) = 0 \Rightarrow A_n = -B_n \frac{\sinh(3\mu_n)}{\cosh(3\mu_n)}$$
$$X_n = -B_n \frac{\sinh(3\mu_n)}{\cosh(3\mu_n)} \cosh(\mu_n x) + B_n \sinh(\mu_n x)$$
$$X_n = \frac{B_n}{\cosh(3\mu_n)} \left(-\sinh(3\mu_n)\cosh(\mu_n x) + \sinh(\mu_n x)\cosh(3\mu_n)\right)$$
$$\sinh A \cosh B - \sinh B \cosh A = \sinh(A-B)$$
$$X_n = \frac{B_n}{\cosh(3\mu_n)} \sinh(\mu_n(x-3))$$
$$u^A = \sum_{n=1}^{\infty} Q_n \cos(\mu_n y) \sinh(\mu_n(x-3))$$
$$u_x^A = \sum_{n=1}^{\infty} Q_n \mu_n \cos(\mu_n y) \cosh(\mu_n(x-3))$$
$$u_x^A(0,y) = \sum_{n=1}^{\infty} \underbrace{Q_n \mu_n \cosh(-3\mu_n)}_{C_n} \sin(\mu_n y) = 0.1$$
$$C_n = \frac{2}{2}\int_0^2 0.1\cos(\mu_n y)dy = \left[0.1\frac{\sin \mu_n y}{\mu_n}\right]_0^2 = \frac{0.1\sin(2\mu_n)}{\mu_n}$$
$$\sin(2\mu_n) = (-1)^{n+1}$$
$$Q_n = \frac{0.1(-1)^{n+1}}{\mu_n^2 \cosh(-3\mu_n)}$$
$$u^A(x,y) = \sum_{n=1}^{\infty} \frac{0.1(-1)^{n+1}}{\mu_n^2 \cosh(-3\mu_n)} \cos(\mu_n y)\sinh(\mu_n(x-3)),\ \mu_n = \frac{2n-1}{4}\pi$$

Subproblem B:

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$$
$$u_x(0,y) = u(3,y) = 0$$

$$\rightsquigarrow \lambda_n = \left(\frac{2n-1}{2L}\pi\right)^2 = \left(\frac{2n-1}{6}\right)^2$$

$$X_n = \cos(\mu_n x),\ \mu_n = \frac{2n-1}{6}\pi$$

$$Y_n = A_n \cosh(\mu_n y) + B_n \sinh(\mu_n y)$$

$$u_y(x,0) = 0 \Rightarrow Y'(0) = 0$$

$$Y_n' = A_n\mu_n \sinh(\mu_n y) + B_n\mu_n \cosh(\mu_n y)$$

$$Y'(0) = B_n\mu_n = 0 \Rightarrow B_n = 0$$

$$Y_n = A_n \cosh(\mu_n y)$$

$$u^B = \sum_{n=1}^{\infty} Q_n \cos(\mu_n x)\cosh(\mu_n y)$$

$$u^B(x,2) = \sum_{n=1}^{\infty} \underbrace{Q_n \cosh(2\mu_n)}_{C_n} \cos(\mu_n x) = \begin{cases} x & 0 \le x < 1 \\ 2-x & 1 \le x < 3 \end{cases}$$

$$C_n = \frac{2}{3}\int_0^1 x\cos(\mu_n x)dx + \frac{2}{3}\int_1^3 (2-x)\cos(\mu_n x)dx$$

$$C_n = \frac{2\mu_n \sin(\mu_n) + 2\cos(\mu_n) - 2}{3\mu_n^2} + \frac{2\cos(\mu_n) - 2\mu_n \sin(3\mu_n) - 2\cos(3\mu_n) - 2\mu_n \sin(\mu_n)}{3\mu_n^2}$$

$$\sin(3\mu_n) = (-1)^{n+1},\ \cos(3\mu_n) = 0$$

$$C_n = \frac{4\cos(\mu_n) - 2\mu_n(-1)^{n+1} - 2}{3\mu_n^2}$$

$$Q_n = \frac{4\cos(\mu_n) - 2\mu_n(-1)^{n+1} - 2}{3\mu_n^2 \cosh(2\mu_n)}$$

$$u^B = \sum_{n=1}^{\infty} \frac{4\cos(\mu_n) - 2\mu_n(-1)^{n+1} - 2}{3\mu_n^2 \cosh(2\mu_n)} \cos(\mu_n x)\cosh(\mu_n y),\ \mu = \frac{2n-1}{6}\pi$$

Subproblem C:

$$-\frac{X''}{X} = \frac{Y''}{Y} = -\lambda$$

$$u_y(x,0) = u(x,2) = 0$$

$$\rightsquigarrow \lambda_n = \left(\frac{2n-1}{2L}\pi\right)^2 = \left(\frac{2n-1}{4}\pi\right)^2$$

$$Y_n = \cos(\mu_n y),\ \mu_n = \frac{2n-1}{4}\pi$$

$$X_n = A_n \cosh(\mu_n x) + B_n \sinh(\mu_n x)$$

$$u_x(0,y) = 0 \Rightarrow X'(0) = 0$$

$$X' = A_n\mu_n \sinh(\mu_n x) + B_n\mu_n \cosh(\mu_n x)$$

$$X'(0) = B_n\mu_n = 0 \Rightarrow B_n = 0$$

$$X(x) = A_n \cosh(\mu_n x)$$

$$
\begin{aligned}
&u^C = \sum_{n=1}^{\infty} Q_n \cos(\mu_n y)\cosh(\mu_n x) \\
&u^C(3,y) = \sum_{n=1}^{\infty} \underbrace{Q_n \cosh(3\mu_n)}_{C_n} \cos(\mu_n y) = \begin{cases} y & 0 \le y < 1 \\ 2-y & 1 \le y < 2 \end{cases} \\
&C_n = \frac{2}{2}\int_0^1 y\cos(\mu_n y)dy + \frac{2}{2}\int_1^2 (2-y)\cos(\mu_n y)dy \\
&C_n = \frac{\mu_n \sin(\mu_n) + \cos(\mu_n) - 1}{\mu_n^2} + \frac{\cos(\mu_n) - \cos(2\mu_n) - \mu_n\sin(\mu_n)}{\mu_n^2} = \frac{2\cos(\mu_n) - \cos(2\mu_n) - 1}{\mu_n^2} \\
&\cos(2\mu_n) = 0 \\
&Q_n = \frac{2\cos(\mu_n) - 1}{\mu_n^2 \cosh(3\mu_n)} \\
&u^C = \sum_{n=1}^{\infty} \frac{2\cos(\mu_n) - 1}{\mu_n^2\cosh(3\mu_n)} \cos(\mu_n y)\cosh(\mu_n x), \ \ \mu_n = \frac{2n-1}{4}\pi
\end{aligned}
$$

Final solution:

$$
\begin{aligned}
u(x,y) = \sum_{n=1}^{\infty} \Bigg( &\left( \frac{1.6(-1)^{n+1}\sinh\left(\frac{2n-1}{4}\pi(x-3)\right)}{(2n-1)^2\pi^2\cosh(-3(\frac{2n-1}{4}\pi))} + \frac{16(2\cos(\frac{2n-1}{4}\pi)-1)\cosh(\frac{2n-1}{4}\pi x)}{(2n-1)^2\pi^2\cosh(3(\frac{2n-1}{4}\pi))} \right) \cos\left(\frac{2n-1}{4}\pi y\right) \\
&+ \frac{24(\cos(\frac{2n-1}{6}\pi) - \frac{(2n-1)\pi(-1)^{n+1}}{6} - 1)}{(2n-1)^2\pi^2\cosh(\frac{2n-1}{3}\pi)} \cos\left(\frac{2n-1}{6}\pi x\right)\cosh\left(\frac{2n-1}{6}\pi y\right) \Bigg)
\end{aligned}
$$

### 2.3.2 Circular Laplace Equation

We can solve the Laplace equation on slightly more complicated domains. One very common one is circular domains.
To do this we will want to solve the Laplace equation in polar coordinates.

$$
\begin{aligned}
&x = r\cos\theta, \ \ y = r\sin\theta \\
&\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}
\end{aligned}
$$

Using separation of variables we can get

$$
\begin{aligned}
&u(r,\theta) = R(r)\Theta(\theta) \\
&R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0
\end{aligned}
$$

Multiply by $r^2$ and divide by $R\Theta$

$$
-\left(r^2\frac{R''}{R} + r\frac{R'}{R}\right) = \frac{\Theta''}{\Theta} = -\lambda
$$

We will have 2 different types of solutions:

1. homogeneous boundary conditions in $\theta$

2. homogeneous boundary conditions in $r$

Along with the given boundary conditions, we also have the following implied boundary conditions:

- If the domain contains $r = 0$ then $\lim\limits_{r\to 0} u(r,\theta)$ must be finite
- If the domain contains $r = \infty$ then $\lim\limits_{r\to\infty} u(r,\theta)$ must be finite
- If the domain forms a full circle/ring then the boundary conditions in $\theta$ are periodic

It is also important to note that you cannot use superposition on a circular domain like you can on a rectangular domain.
The general solution method for boundary value problems in $\theta$ is outlined below:

1. The first thing to do is to make sure the boundary conditions in $\theta$ are homogeneous. If not then use the methods mentioned earlier to make them homogeneous.
2. Solve the eigenvalue problem in $\Theta$ to get $\lambda_n$ and $\Theta_n$
3. Plug in $\lambda_n$ and solve the ODE, $r^2R_n'' + rR_n' - \lambda_n R_n = 0$ to find $R_n$. The solution should be of the form $R_n = A_n r^{\mu_n} + B_n r^{-\mu_n}$
4. Write out $u(r,\theta) = a_0\Theta_0R_0 + \sum\limits_{n=1}^{\infty}\Theta_nR_n$ and impose the initial conditions in $r$ to solve for the unknown constants, $a_0, A_n, B_n$

Ex: full circular domain with periodic BCs

$$
\begin{aligned}
&u_{rr} + \frac{1}{r}u_\theta + \frac{1}{r^2}u_{\theta\theta}\\
&u(a,\theta) = f(\theta),\ 0 \le r \le a,\ -\pi \le \theta \le \pi
\end{aligned}
$$

This implies that the boundary conditions are

$$
\begin{aligned}
&u(0,\theta) \text{ is finite}\\
&u(a,\theta) = f(\theta)\\
&u(r,\pi) = u(r,-\pi)\\
&u_\theta(r,\pi) = u_\theta(r,-\pi)
\end{aligned}
$$

To solve, we will first solve the eigenvalue problem in $\theta$

$$
\begin{aligned}
&-\left(r^2\frac{R''}{R} + r\frac{R'}{R}\right) = \frac{\Theta''}{\Theta} = -\lambda\\
&\Theta'' + \lambda\Theta = 0\\
&\rightsquigarrow \lambda_0 = 0,\ \lambda_n = n^2,\ n \ge 1\\
&\Theta_0 = 1,\ \Theta_n = \begin{cases}\cos(n\theta)\\ \sin(n\theta)\end{cases}
\end{aligned}
$$

For each $\lambda_n$, we can solve for $R_n$

$$\lambda_0 :$$

$$r^2\frac{R_0''}{R_0}+r\frac{R_0'}{R_0}\Rightarrow r^2R_0''+rR_0=0$$
$$\beta(\beta-1)+\beta=0\Rightarrow\beta^2=0$$
$$R_0=C_0r^0+D_0\ln|r|r^0$$
$$R_0=C_0+D_0\ln r,\ 0<r\leq a$$
$$\lambda_n:$$
$$r^2\frac{R_n''}{R_n}+r\frac{R_n'}{R_n}=n^2\Rightarrow r^2R_n''+rR_n'-n^2R_n$$
$$\beta(\beta-1)+\beta-n^2=0\Rightarrow\beta^2=n^2\Rightarrow\beta=\pm n$$
$$R_n=C_nr^n+D_nr^{-n}$$
$$\lim_{r\to0}R_0(r)\neq\pm\infty\Rightarrow\lim_{r\to0}\left(C_0+D_0\ln r\right)\neq\pm\infty\Rightarrow D_0=0\Rightarrow R_0=1$$
$$\lim_{r\to0}R_n(r)\neq\pm\infty\Rightarrow\lim_{r\to0}\left(C_nr^n+D_nr^{-n}\right)\neq\pm\infty\Rightarrow D_n=0\Rightarrow R_n=r^n$$

Now that we have $R$ and $\Theta$ we can put them together to get

$$u(r,\theta)=C_0+\sum_{n=1}^{\infty}r^n\left(C_n\cos(n\theta)+D_n\sin(n\theta)\right)$$
$$u(a,\theta)=f(\theta)=C_0+\sum_{n=1}^{\infty}a^nC_n\cos(n\theta)+a^nD_n\sin(n\theta)$$
$$C_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta)d\theta$$
$$C_n=\frac{1}{\pi a^n}\int_{-\pi}^{\pi}f(\theta)\cos(n\theta)d\theta$$
$$D_n=\frac{1}{\pi a^n}\int_{-\pi}^{\pi}f(\theta)\sin(n\theta)d\theta$$

Ex2:

$$0<a\leq r\leq b,\ 0\leq\theta\leq\frac{\pi}{2}$$
$$u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}$$
$$u_\theta(r,0)=0,\ u\left(r,\frac{\pi}{2}\right)=2$$
$$u(a,\theta)=\cos(2\theta),\ u(b,\theta)=0$$
$$u^A:\ v\left(r,\frac{\pi}{2}\right)\neq0$$
$$u^B:\ v(a,\theta)\neq0$$

Steady State:

$$u(r,\theta)=w(\theta)+v(r,\theta)$$
$$\nabla^2w=0\Rightarrow w(\theta)=A\theta+B$$

$$w'(0) = 0 \Rightarrow A = 0$$
$$w\left(\frac{\pi}{2}\right) = 2 = B$$
$$w(\theta) = 2$$
$$\nabla^2 u = \nabla^2 v$$
$$u_\theta(r,0) = w_\theta(0) + v_\theta(r,0) = 0 \Rightarrow v_\theta(r,0) = 0$$
$$u\left(r, \frac{\pi}{2}\right) = w\left(\frac{\pi}{2}\right) + v\left(r, \frac{\pi}{2}\right) \Rightarrow v\left(r, \frac{\pi}{2}\right)$$
$$u(a,\theta) = w(\theta) + v(a,\theta) = 2 + v(a,\theta) = \cos(2\theta) \Rightarrow v(a,\theta) = \cos(2\theta) - 2$$
$$u(b,\theta) = w(\theta) + v(b,\theta) = 0 \Rightarrow v(b,\theta) = -2$$

Eigenvalue problem:

$$v_\theta(r,0) = 0,\ v\left(r, \frac{\pi}{2}\right) = 0$$
$$-\left(r^2 \frac{R''}{R} + r\frac{R'}{R}\right) = \frac{\Theta''}{\Theta} = -\lambda$$
$$\rightsquigarrow \lambda_n = \left(\frac{2n-1}{2L}\pi\right)^2 = (2n-1)^2$$
$$\Theta_n = \cos(\mu_n \theta),\ \mu_n = 2n-1$$
$$r^2 R_n'' + rR_n' - \mu_n^2 R_n = 0$$
$$\beta(\beta-1) + \beta - \mu_n^2 = \beta^2 - \mu_n^2 = 0 \Rightarrow \beta = \pm\mu_n$$
$$R_n = A_n r^{\mu_n} + B_n r^{-\mu_n}$$
$$v(r,\theta) = \sum_{n=1}^{\infty} (A_n r^{\mu_n} + B_n r^{-\mu_n}) \cos(\mu_n \theta)$$
$$v(b,\theta) = \sum_{n=1}^{\infty} \underbrace{(A_n b^{\mu_n} + B_n b^{-\mu_n})}_{C_n} \cos(\mu_n \theta) = -2$$
$$C_n = \frac{4}{\pi}\int_0^{\pi/2} -2\cos(\mu_n\theta)d\theta = \frac{-8}{\mu_n \pi}\ \sin(\mu_n\theta)\big|_0^{\pi/2} = \frac{8(-1)^n}{\mu_n \pi}$$
$$v(a,\theta) = \sum_{n=1}^{\infty} \underbrace{(A_n a^{\mu_n} + B_n a^{-\mu_n})}_{D_n} \cos(\mu_n \theta) = \cos(2\theta) - 2$$
$$D_n = \frac{4}{\pi}\int_0^{\pi/2} (\cos(2\theta) - 2)\cos(\mu_n\theta)d\theta = C_n + \frac{4}{\pi}\int_0^{\pi/2} \cos(2\theta)\cos(\mu_n\theta)d\theta$$
$$D_n = C_n + \frac{4}{\pi}\left(\frac{(\mu_n - 2)\sin(\frac{\mu_n+2}{2}\pi) + (\mu_n+2)\sin(\frac{\mu_n-2}{2}\pi))}{2(\mu_n^2-4)}\right) = C_n + \underbrace{\frac{4\mu_n(-1)^n}{(\mu_n^2-4)\pi}}_{K_n}$$
$$\begin{cases} A_n b^{\mu_n} + B_n b^{-\mu_n} = C_n \\ A_n a^{\mu_n} + B_n a^{-\mu_n} = C_n \end{cases} \Rightarrow \begin{bmatrix} b^{\mu_n} & b^{-\mu_n} \\ a^{\mu_n} & a^{-\mu_n} \end{bmatrix} \begin{bmatrix} A_n \\ B_n \end{bmatrix} = \begin{bmatrix} C_n \\ D_n \end{bmatrix}$$
$$\begin{bmatrix} A_n \\ B_n \end{bmatrix} = \frac{1}{(\frac{b}{a})^{\mu_n} - (\frac{a}{b})^{\mu_n}} \begin{bmatrix} a^{-\mu_n} & -b^{-\mu_n} \\ -a^{\mu_n} & b^{\mu_n} \end{bmatrix} \begin{bmatrix} C_n \\ D_n \end{bmatrix} = \frac{1}{(\frac{b}{a})^{\mu_n} - (\frac{a}{b})^{\mu_n}} \begin{bmatrix} (a^{-\mu_n} - b^{-\mu_n})C_n \\ (b^{\mu_n} - a^{\mu_n})D_n \end{bmatrix}$$

$$u(r,\theta) = 2 + \sum_{n=1}^{\infty} \left( \frac{8(-1)^n((a^{-(2n-1)} - b^{-(2n-1)})r^{(2n-1)} + (b^{(2n-1)} - a^{(2n-1)})r^{-(2n-1)})}{((\frac{b}{a})^{(2n-1)} - (\frac{a}{b})^{(2n-1)})(2n-1)\pi} \right.$$
$$\left. + \frac{\frac{4(2n-1)(-1)^n}{((2n-1)^2-4)\pi}(b^{(2n-1)} - a^{(2n-1)})r^{-(2n-1)}}{(\frac{b}{a})^{(2n-1)} - (\frac{a}{b})^{(2n-1)}} \right) \cos((2n-1)\theta)$$