Math 217 Notes

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Contents

1 Multivariable Calculus

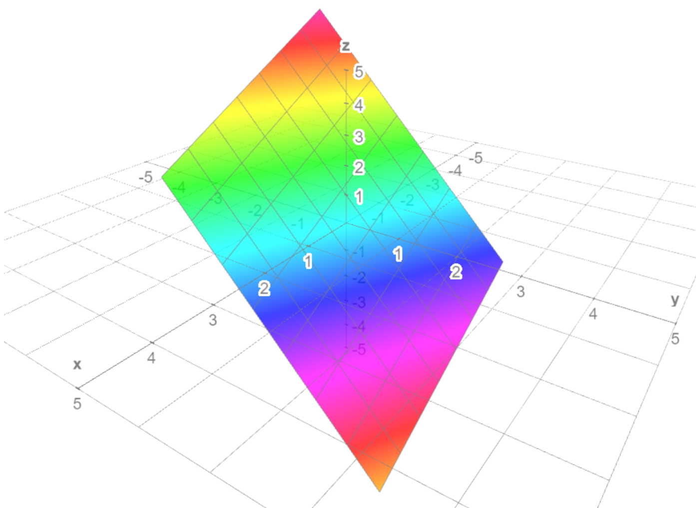
1.1 Functions of Multiple Variables

1.1.1 Classification of 3D Surfaces

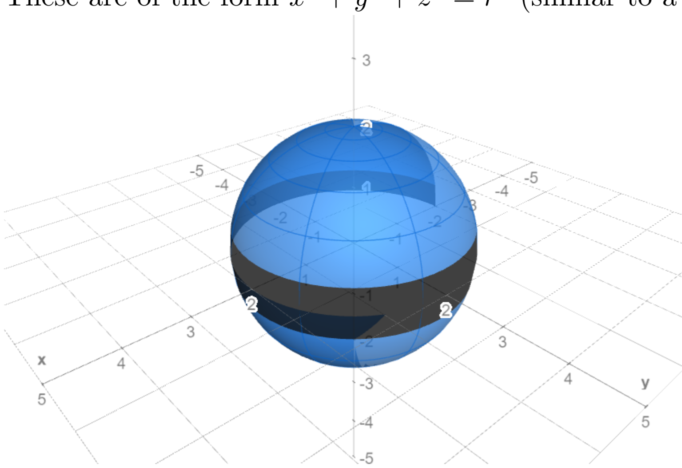
So far, we have only been working in the 2D plane with functions of 2 variables: $y = f(x)$. We can extend the same ideas to higher dimensions: A function $z = f(x, y)$ will form a surface in 3 dimensions and a function $w = f(x, y, z)$ will span 4 dimensions. While we are still able to analyze the behaviors of higher dimensional systems, we will often stick to functions of 3 variables for the sake of visualization.

As with 2D, there are a few common functions that you should recognize and be able to plot.

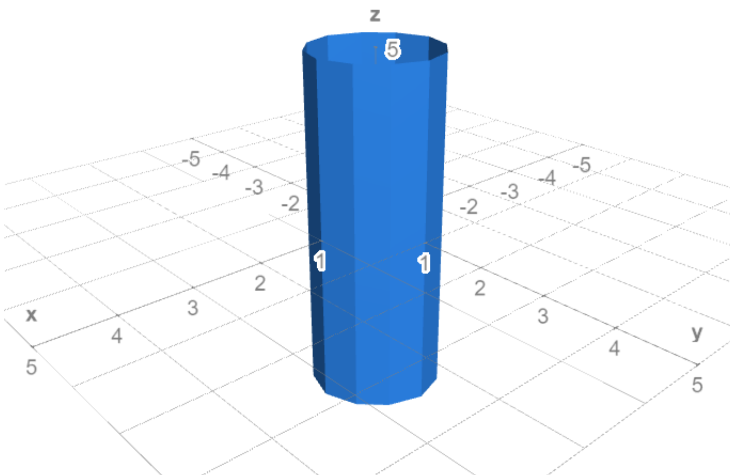
1. Planes
   These are of the form $z = ax + by + c$ (as seen in a previous section)



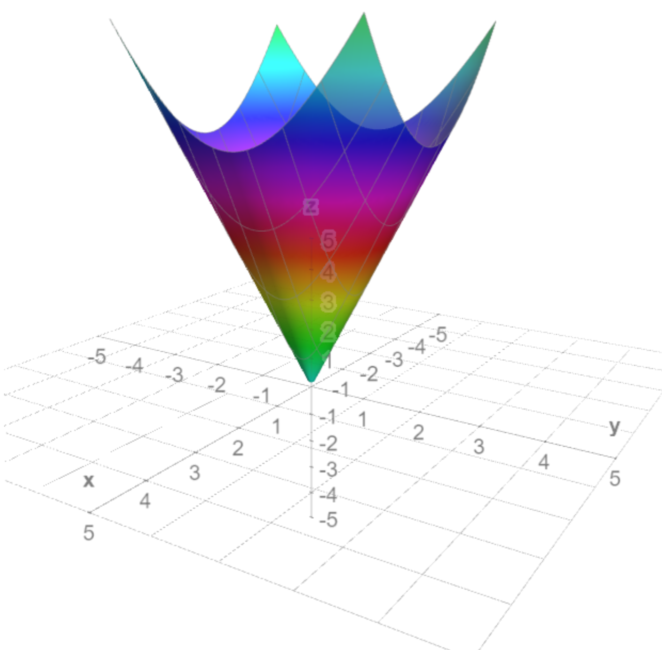
2. Spheres
   These are of the form $x^2 + y^2 + z^2 = r^2$ (similar to a circle in 2D)



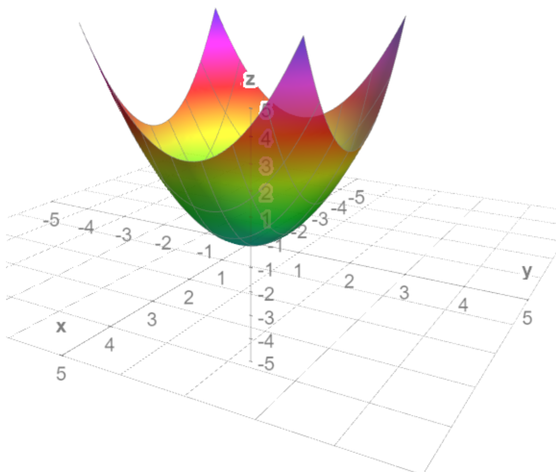
3. Cylinders
   These are of the form $x^2 + y^2 = 1$. Note that this is a function defined in terms of only x and y so it will be the same for all z.



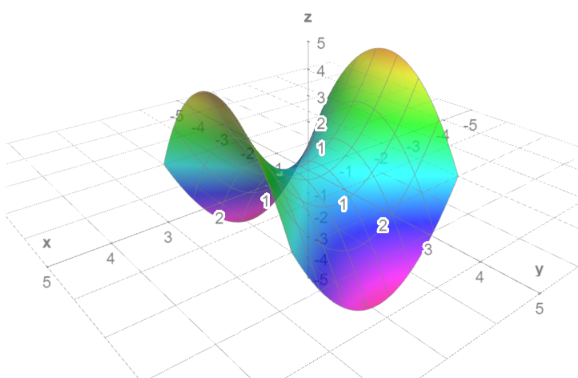
4. Cones
   These are of the form $z = \sqrt{x^2 + y^2}$ for a one-sided cone and $z^2 = x^2 + y^2$ for a two-sided cone.



5. Paraboloids
   These are of the form $z = x^2 + y^2$



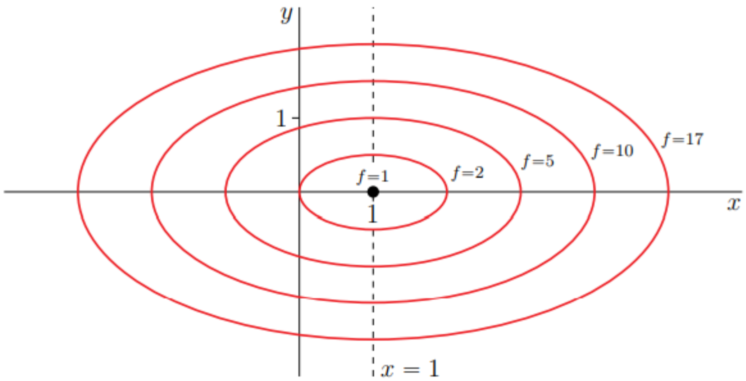
6. Saddles
   These are of the form $z = x^2 - y^2$



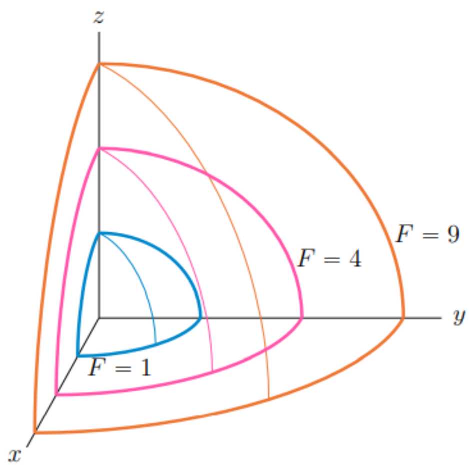
There are many more functions not mentioned such as ellipsoids, hyperboloids, and parabolic cylinders but this should be enough to begin to get an idea of what 3D functions look like.

1.1.2 Sketching Surfaces

Sketching surfaces in 3D is often trickier than in 2D but there are a few useful tricks to get an idea what the surface looks like.
One of the most useful ones is level curves. This is where if we have a function $z = f(x, y)$, we set z to be a constant and draw the curve on the xy-plane for varying z (in 2D these are called contour plots).
In short, a contour plot is an overlay of lines of $f(x, y) = k$ plotted on the xy-plane. The rate of greatest change (the gradient) is always perpundicular to the contour lines.
Similarly, a level surface is the overlay of surfaces of $f(x, y, z) = k$ plotted in 3-space (or higher).
Ex: $z = x^2 + 4y^2 - 2x + 2$



This becomes increasingly useful for visualizing functions in 4 dimensions as we can define their level curves as 3D functions.
Ex: $F = x^2 + y^2 + z^2$



Sometimes it can be useful to sketch contour plots in multiple planes as it may be easier or give different perspectives.
Other useful tricks in sketching curves that will be seen later may be to take the gradient (as it gives a vector field normal to the level curves at all points) or to find the critical points of the function.

1.2 Partial Derivatives

1.2.1 Partial Derivatives and the Gradient

Notation: $\dfrac{\partial f}{\partial x} = f_x$
Definition:

$$\text{for } z = f(x, y)$$
$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$
$$\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

Partial derivatives are effectively the same but where you hold the rest of the variables to be constants.

$$\text{Ex: } f(x,y) = x^3y + y^2$$
$$\frac{\partial f}{\partial x} = 3x^2y$$
$$\frac{\partial f}{\partial y} = x^3 + 2y$$

Note that the order of differentiation for higher order derivatives does not matter. $f_{xy} = f_{yx}$
Ex2: Find an expression for $\frac{\partial^2 z}{\partial x \partial y}$ for the function $2yz + x^2 + y^2 + z^4 = 18$

$$\frac{\partial}{\partial x}\left(2yz + x^2 + y^2 + z^4 = 18\right)$$
$$2yz_x + 2x + 4z^3z_x = 0 \Rightarrow (2y + 4z^3)z_x = -2x$$
$$z_x = \frac{-x}{y + 2z^3}$$
$$\frac{\partial}{\partial y}\left(2yz + x^2 + y^2 + z^4 = 18\right)$$
$$2z + 2yz_y + 2y + 4z^3z_y = 0$$
$$(y + 2z^3)z_y = -y - z$$
$$z_y = \frac{-y - z}{y + 2z^3}$$
$$z_{xy} = \frac{\partial}{\partial y}\left(\frac{-x}{y + 2z^3}\right) = \frac{x}{(y + 2z^3)^2}(1 + 6z^2z_y) = \frac{x}{(y + 2z^3)^2}\left(1 + 6z^2\left(\frac{-y - z}{y + 2z^3}\right)\right)$$
$$z_{xy} = \frac{xy - 6xyz^2 - 4xz^3}{(y + 2z^3)^3}$$

Chain rule: $z = (x(t), y(t))$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Chain rule for functions of multiple variables: $z(u(s,t), v(s,t))$

$$\begin{bmatrix} \frac{\partial z}{\partial s} \\ \frac{\partial z}{\partial t} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial s} & \frac{\partial y}{\partial s} \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} \end{bmatrix} \begin{bmatrix} \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \\ \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} \end{bmatrix}$$

The above matrix is an example of the Jacobian which essentially maps (x, y) to (s, t).

Directional Derivative:
The directional derivative is when the take the rate of change of f is some arbitrary direction $\hat{u}$.
$x(t) = x_0 + ta,\ y(t) = y_0 + tb$ where $\hat{u} = \langle a, b \rangle$
$(D_{\hat{u}}f)(x_0, y_0) = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = f_x(x_0, y_0)a + f_y(x_0, y_0)b$

$$(D_{\hat{u}}f)(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0) \rangle$$

This can be simplified by defining the gradient.

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Using this, we can simplify the directional derivative to be.

$$(D_{\hat{u}} f)(x_0, y_0) = (\nabla f)(x_0, y_0) \cdot \hat{u}$$

Special note about the gradient: the gradient will always be normal to the surface of the level curve (think a balloon expanding)

1.2.2 Linear Approximation and Tangent Planes

Recall with linear approximations of one variable, they estimate the function at a point using a tangent line. For a 2 variable function, we use the tangent plane as an approximation.
A trick to finding tangent planes is to treat the function as a function of 3 variables as the gradient can be expressed as the normal vector. So we get,

$$f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0) + f_z(x_0, y_0, z_0)(z - z_0) = 0$$

for a function of 3 variables.
For functions of 2 variables, we can write it as a functions of 3 variables as $f(x, y, z) = g(x, y) - z = 0$ so $f_z = -1$.
This gives the the equation,

$$f(x, y) \approx f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f(x_0, y_0)$$

Ex: Given the cosine law, $c^2 = a^2 + b^2 - 2ab\cos\theta$, find the linear approximation of θ at the point $(a_0, b_0, c_0) = (1/2, 1, \sqrt{3}/2)$

$$\frac{3}{4} = \frac{1}{4} + 1 - \cos\theta$$
$$\frac{1}{2} = \cos\theta \Rightarrow \theta = \frac{\pi}{3}$$
$$\frac{\partial}{\partial a}: \ 0 = 2a - 2b\cos\theta + 2ab\sin\theta \cdot \theta_a$$
$$\Rightarrow 1 - 1 + \frac{\sqrt{3}}{2}\theta_a \Rightarrow 0 = \frac{\sqrt{3}}{2}\theta_a \Rightarrow \theta_a = 0$$
$$\frac{\partial}{\partial b}: \ 0 = 2b - 2a\cos\theta + 2ab\sin\theta \cdot \theta_b$$
$$0 = \frac{3}{2} + \frac{\sqrt{3}}{2}\theta_b \Rightarrow \theta_b = -\sqrt{3}$$
$$\frac{\partial}{\partial c}: \ 2c = 2ab\sin\theta \cdot \theta_c$$
$$\sqrt{3} = \frac{\sqrt{3}}{2}\theta_c \Rightarrow \theta_c = 2$$
$$\theta \approx \frac{\pi}{3} - \sqrt{3}(b - 1) + 2\left(c - \frac{\sqrt{3}}{2}\right)$$

1.2.3 Optimization

A critical point of a multivariable function is defined to be where all the partial derivatives equal 0, or more generally, $\nabla f = 0$

When we take the critical point of a 3D function, there are 3 different types of points it could have:

1) A local minimum

2) A local maximum

3) A saddle point (neither max or min)

We can also have a degenerate critical point which is when we have a line that acts as a critical point.

To find out which of these three types it is, we can use the 2nd derivative test.

If (x_0, y_0) is a critical point of f, we can define a discriminant to be,

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$$

- If $D(x_0, y_0) < 0$ and $f_{xx} > 0$, then it's a local maximum
- If $D(x_0, y_0) > 0$ and $f_{xx} < 0$, then it's a local minimum.
- If $D(x_0, y_0) < 0$ then it's a saddle point
- If $D(x_0, y_0) = 0$ then we can't conclude (degenerate critical point)

Ex: Find and classify the critical points of $f(x, y) = (x^2 + y^2 - 5)(y - 1)$

$$\nabla f = \begin{bmatrix} 2x(y-1) \\ 2y(y-1) + x^2 + y^2 - 5 \end{bmatrix} = \vec{0}$$

$x = 0: \ 2y(y-1) + y^2 - 5 = 0$

$3y^2 - 2y - 5 = 0 \Rightarrow (3y-5)(y+1) = 0 \Rightarrow y = \frac{5}{3}, \ -1$

$\left(0, \frac{5}{3}\right), \ (0, -1)$

$y = 1: \ x^2 + 1 - 5 = 0 \Rightarrow x = \pm 2$

$(-2, 1), \ (2, 1)$

$f_{xx} = 2(y-1)$

$f_{yy} = 6y - 2$

$f_{xy} = 2x$

$D = f_{xx}f_{yy} - f_{xy}^2$

$D(2, 1) < 0 \rightarrow \text{saddle}$

$D(-2, 1) < 0 \rightarrow \text{saddle}$

$D\left(0, \frac{5}{3}\right) > 0, \ f_{xx}\left(0, \frac{5}{3}\right) > 0 \rightarrow \text{local min}$

$D\,(0, -1) > 0, \ f_{xx}(0, -1) < 0 \rightarrow \text{local max}$

1.2.4 Lagrange Multipliers

In cases where we want to find the max/min of a function over a closed domain, we can use Lagrange multipliers.
Given some constraint equation $g(x,y,z)=0$, the maximum/minimum value along the boundary will occur where $\nabla f // \nabla g$ or more formally, our critical points will occur where

$$\begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$$

note that λ is some scaling constant.
The case where $\lambda = 0$ corresponds the the max/min along the boundary also being a critical point of the function ($\nabla f = 0$). So if we also compute the critical points of the function, we can ignore the $\lambda = 0$ case.

Ex: The plane $x+y+2z=2$ intersects the paraboloid $z=x^2+y^2$ in an ellipse. Find the points on the ellipse nearest and farthest from the origin.

$$D = \sqrt{x^2+y^2+z^2}$$
let $D = D^2 = x^2+y^2+z^2$
along region $\{x+y+2z=2\} \cap \left\{z = x^2+y^2\right\}$
$$\Rightarrow g(x,y) = x+y+2(x^2+y^2)-2 = 0$$
$$f(x,y) = x^2+y^2+(x^2+y^2)$$
$$\begin{cases} f_x = \lambda g_x \\ f_y = \lambda g_y \\ g = 0 \end{cases} \Rightarrow \begin{cases} 2x+4x(x^2+y^2) = \lambda(1+4x) \\ 2y+4y(x^2+y^2) = \lambda(1+4y) \\ x+y+2(x^2+y^2) = 2 \end{cases} \Rightarrow \begin{cases} 2xy+4xy(x^2+y^2) = \lambda y + 4\lambda xy \\ 2xy+4xy(x^2+y^2) = \lambda x + 4\lambda xy \\ x+y+2(x^2+y^2) = 2 \end{cases}$$
$$\Rightarrow \lambda x = \lambda y$$
$\lambda = 0$ or $x = y$
$\lambda = 0$ case:
$$\begin{cases} 2x+4x(x^2+y^2) = 0 \\ 2y+4y(x^2+y^2) = 0 \end{cases} \Rightarrow \begin{cases} x = 0 \\ y = 0 \end{cases}$$
if $x=y=0,\ g = x+y+2(x^2+y^2)-2 = -2 \neq 0 \therefore x = y \neq 0$
$x = y$ case:
$$2y+2(2y^2) = 2 \Rightarrow 2y^2+y-1 = 0 \Rightarrow (2y-1)(y+1) = 0$$
$$x = y = \left\{-1, \frac{1}{2}\right\},\ z = x^2+y^2$$
gives points $(-1,-1,2)$, $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

Ex2: Let (a,b) be a point on the ellipse $x^2+3y^2=3$ and $(c,3-c)$ be a point on the line $x+y=3$. Find the coordinates of the pair of points which are closest to each other.

let D be the distance squared

$$D = (a-c)^2 + (b-3+c)^2$$
$$g = a^2 + 3b^2 - 3 = 0$$
$$\nabla D = \langle 2(a-c), 2(b-3+c), -2(a-c)\rangle$$
$$\nabla g = \langle 2a, 6b, 0\rangle$$
$$\begin{cases} a-c = \lambda a \\ b-3+c = 3\lambda b \\ a-c = b-3+c \\ a^2+3b^2 = 3 \end{cases}$$
$$a - c = 3\lambda b$$
$$3\lambda b = \lambda a \Rightarrow \lambda = 0 \text{ or } 3b = a$$
$$\text{if } \lambda = 0, a-c = 0 \Rightarrow a = c$$
$$b-3+c = 0 \Rightarrow b = 3-c$$
$$c^2 + 3(3-c)^2 = 4c^2 - 18c + 27 = 3 \Rightarrow 2c^2 - 9c + 12 = 0$$
$$c = \frac{9 \pm \sqrt{81-96}}{4} \rightarrow \text{no real solutions}$$
$$\therefore a = 3b$$
$$9b^2 + 3b^2 = 3 \Rightarrow b^2 = \frac{1}{4} \Rightarrow b = \pm\frac{1}{2} \Rightarrow a = \pm\frac{3}{2}$$
$$3b - c = b - 3 + c \Rightarrow 2b + 3 = 2c \Rightarrow c = b + \frac{3}{2} = \pm 2$$
$$D\left(\frac{3}{2}, \frac{1}{2}, 2\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$
$$D\left(-\frac{3}{2}, -\frac{1}{2}, -2\right) = \frac{1}{4} + \frac{169}{4} = \frac{85}{2}$$
$$\therefore \text{ the closest points are } \left(\frac{3}{2}, \frac{1}{2}\right) \text{ and } (2,1)$$

1.2.5 Least Squares Interpolation

Given experimental data, (x_1, y_1), (x_2, y_2), $\ldots (x_i, y_i)$, find the best fit line.
Minimize $D = \sum_{i=1}^{n} (y_i - (ax_i + b))^2$
We can find the minimum by finding the critical points.

$$\frac{\partial D}{\partial a} = 0 \Rightarrow \frac{\partial D}{\partial a} = \sum_{i=1}^{n} 2(y_i - (ax_i + b))(-x_i) = 0$$
$$\frac{\partial D}{\partial b} = 0 \Rightarrow \sum_{i=1}^{n} 2(y_i - (ax_i + b))(-1) = 0$$
$$\sum_{i=1}^{n} (x_i^2 a + x_i b - x_i y_i) = 0$$

$$\sum_{i=1}^{n}(x_i a + b - y_i) = 0$$

$$\begin{cases} \left(\sum_{i=1}^{n} x_i^2\right) a + \left(\sum_{i=1}^{n} x_i\right) b = \sum_{i=1}^{n} x_i y_i \\ \left(\sum_{i=1}^{n} x_i\right) a + nb = \sum_{i=1}^{n} y_i \end{cases}$$

$\rightarrow$ gives a 2x2 linear system

$\rightarrow$ solve for a and b

For interpolating non-linear plots, we can linearize the data.
Ex: $y = ce^{ax} \Rightarrow \ln y = \ln c + ax$
For a polynomial function, we can expand the number of coefficients we have.
Ex: for $y = ax^2 + bx + c$, we get:
$D(a, b, c) = \sum_{i=1}^{n}(y_i - (ax_i^2 + bx_i + c))^2$ which gives a 3x3 linear system.

1.3 Multiple Integrals

1.3.1 Double Integrals

By definition, we get

$$\iint_R f(x, y)dA = \lim_{M,N\to\infty} \sum_{i=1}^{M}\sum_{j=1}^{N} f(x_{ij}^*, y_{ij}^*)\Delta x \Delta y$$

where $dA = dydx$
The double integral can be interpreted as volume under the curve $f(x, y)$.
We compute the double integral by first integrating with respect to one variable and then the other.
A double integral will be computed in either of the following forms:

$$\iint_R f(x, y) = \int_{x=a}^{x=b}\int_{y=y_1(x)}^{y=y_2(x)} f(x, y)dydx$$

$$\iint_R f(x, y) = \int_{y=a}^{y=b}\int_{x=x_1(y)}^{x=x_2(y)} f(x, y)dxdy$$

Ex: Find $\iint_R ydA$ over the region bounded by $x = y$ and $x = 2 - y^2$

intersections:

$y = 2 - y^2 \Rightarrow y^2 + y - 2 = 0 \Rightarrow (y + 2)(y - 1) = 0 \Rightarrow y = 1,\ -2$

$$\iint ydA = \int_{y=-2}^{1}\int_{x=y}^{x=2-y^2} ydydx = \int_{-2}^{1} y\,[x]_y^{2-y^2}\, dy$$

$$= \int_{-2}^{1} y(2 - y^2 - y)dy = \int_{-2}^{1}(2y - y^3 - y^2)dy$$

$$= \left[y^2 - \frac{y^4}{4} - \frac{y^3}{3}\right]_{-2}^{1} = 1 - \frac{1}{4} - \frac{1}{3} - \left(4 - 4 + \frac{8}{3}\right) = -\frac{9}{4}$$

Note that the rules for even/odd functions are the same as with single integrals. If you have an odd function over a symmetric region, the entire integral will be zero. If you have an even function over a symmetric region, you can simplify the region and double the integral.
Applications of double integrals:
Area of a region:

$$\text{Area}(R) = \iint_R dA$$

The average value of a function can be computed as:

$$\overline{f(x,y)} = \frac{1}{\text{Area}(R)} \iint_R f(x,y)dA$$

Average height of a region:

$$\overline{y} = \frac{1}{\text{Area}(R)} \iint_R ydA$$

Surface Area:

$$S = \iint_R \sqrt{1 + f_x^2 + f_y^2}dA$$

Center of mass:

$$x_{CM} = \frac{1}{\text{Mass}(R)} \iint_R x\rho(x,y)dA$$

Moment of inertia about an axis a where $D(x,y)$ is the distance from the axis.

$$I_a = \iint_R (D(x,y))^2 \rho(x,y)dA$$

1.3.2 Polar Coordinates

Polar coordinates uses the variables r and θ to describe functions.
r is the distance from the origin
θ is the angle from the x-axis to the line formed by r.
We can convert between polar coordinates and rectangular coordinates using the following conversions:

$$r = \sqrt{x^2 + y^2}$$
$$x = r\cos\theta$$
$$y = r\sin\theta$$

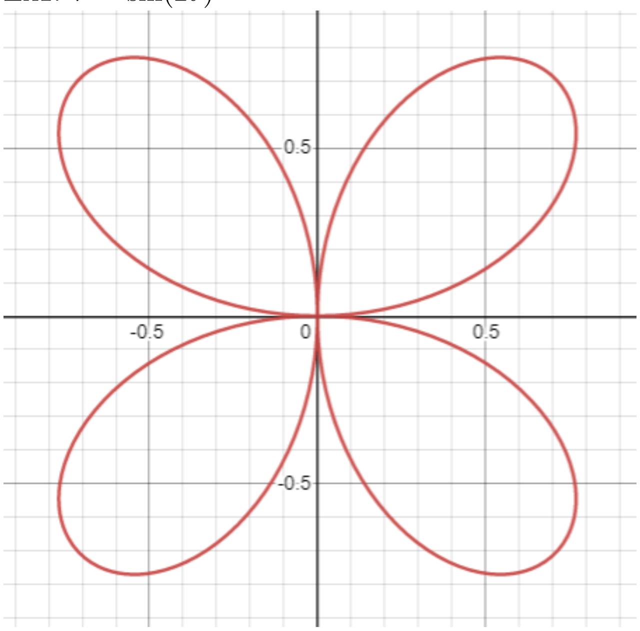
Some common expressions in polar coordinates are:

circle: $r = a$
ray: $\theta = a$

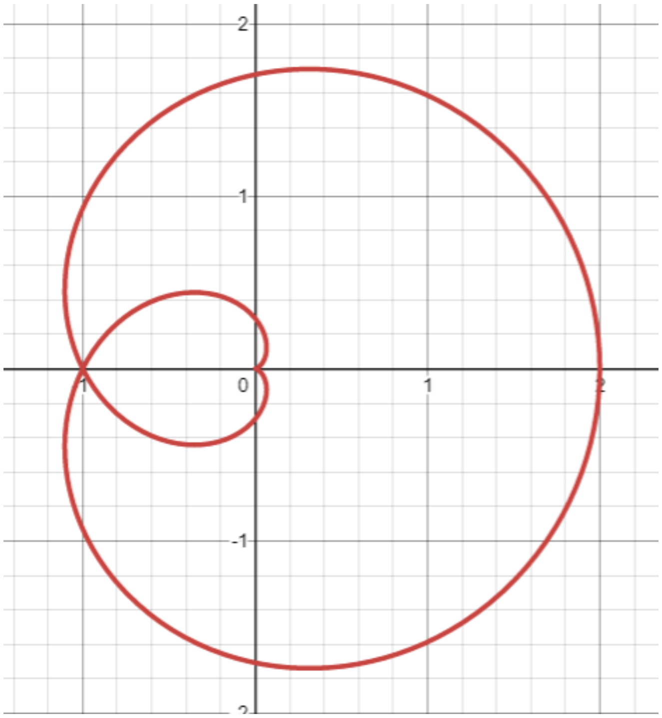
Ex: Find the equation of an off-center circle in polar coordinates

$$(x-a)^2 + y^2 = a^2$$
$$x^2 - 2ax + a^2 + y^2 = a^2$$
recall $r^2 = x^2 + y^2$
$$\Rightarrow r^2 - 2ax = 0$$
$$r^2 - 2ar\cos\theta = 0$$
$$r^2 = 2ar\cos\theta$$
$$r = 2a\cos\theta$$

Graphing in Polar Coordinates:
We can create some interesting graphs using polar coordinates. Here are a few examples:
Ex1: $r = \sin(2\theta)$



Ex2: $r = 1 + \cos\left(\frac{\theta}{2}\right)$



For regions with circular symmetry, it is often easier to integrate in polar coordinates. The differential area is given by:

$$dA = r dr d\theta$$

Note: This distortion factor of r in $dA = r dr d\theta$ comes from the arc length.
Ex: Area of a circle of radius R

$$\int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=R} r dr d\theta$$

$$\int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^R d\theta = \int_0^{2\pi} \frac{R^2}{2} d\theta = \pi R^2$$

Ex2: $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx$

let $I = \displaystyle\int_{-\infty}^{\infty} e^{-x^2} dx$, $I^2 = \displaystyle\int_{-\infty}^{\infty} e^{-x_1^2} dx_1 \int_{-\infty}^{\infty} e^{-x_2^2} dx_2 = \int_{x_1=-\infty}^{\infty} \int_{x_2=-\infty}^{\infty} e^{-x_1^2} e^{-x_2^2} dx_2 dx_1$

$$I^2 = \iint_{\mathbb{R}^2} e^{-x_1^2 - x_2^2} dA$$

$$r^2 = x_1^2 + x_2^2$$

$$I^2 = \int_{\theta=0}^{2\pi} \int_{r=0}^{\infty} e^{-r^2} r dr d\theta = 2\pi \int_0^{\infty} r e^{-r^2} dr = 2\pi \left[-\frac{e^{-r^2}}{2}\right]_0^{\infty} = 2\pi \left(\frac{1}{2}\right)$$

$$I^2 = \pi$$

$$I = \sqrt{\pi}$$

1.3.3 Triple Integrals

By definition, we get

$$\iiint f(x,y,z)dV = \lim_{\Delta x,\Delta y,\Delta z\to 0} \sum_{i=1}^{L}\sum_{j=1}^{M}\sum_{k=1}^{N} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\Delta x\Delta y\Delta z$$

These are computed in the same manner as double integrals, just with an additional step.

Ex: $\iiint_E 2xdV$, where E is the region in the first octant bounded by $2x+3y+z=6$

$$I = \int_{x=0}^{3}\int_{y=0}^{y=-\frac{2}{3}x+2}\int_{z=0}^{z=6-2x-3y} 2xdzdydx$$

$$I = \int_{x=0}^{3}\int_{y=0}^{y=-\frac{2}{3}x+2} 2x(6-2x-3y)dydx$$

$$I = \int_{0}^{3}\left[12xy - 4x^2y - 3xy^2\right]_{y=0}^{y=-\frac{2}{3}x+2} dx$$

$$I = \int_{0}^{3}\left(\frac{4}{3}x^3 - 8x^2 + 12x\right)dx$$

$$I = 9$$

Ex2: Find the volume of intersection between the cylinders $x^2+y^2=R^2$ and $x^2+z^2=R^2$

$$V = \iiint dV = \int_{x=-R}^{R}\int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{z=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dzdydx$$

$$V = \int_{x=-R}^{R}\int_{y=-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} 2\sqrt{R^2-x^2}dydx$$

$$V = \int_{x=-R}^{R} 4(R^2-x^2)dx$$

$$V = 4\left[R^2x - \frac{x^3}{3}\right]_{-R}^{R} = 8\left(R^3 - \frac{R^3}{3}\right) = \frac{16R^3}{3}$$

1.3.4 Change of Coordinate Systems

Two common change of coordinates in triple integrals are cylindrical coordinates and spherical coordinates.
Cylindrical coordinates is an extention of polar coordinates where the conversions are as follows:

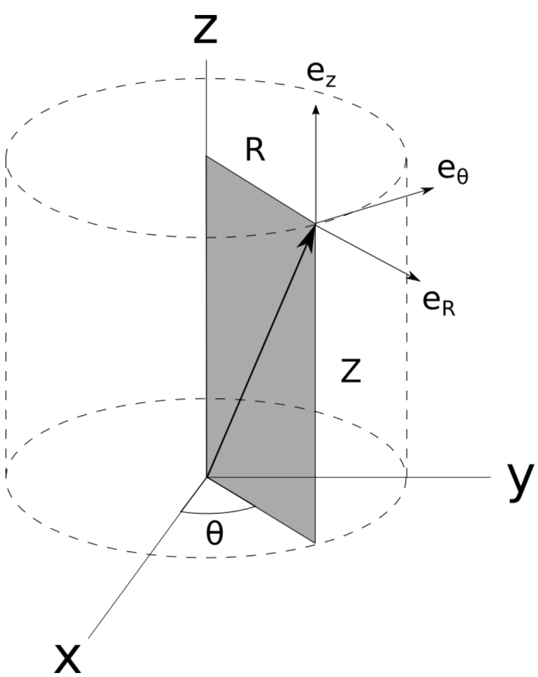
$$r = \sqrt{x^2+y^2}$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

$$dV = rdzdrd\theta$$

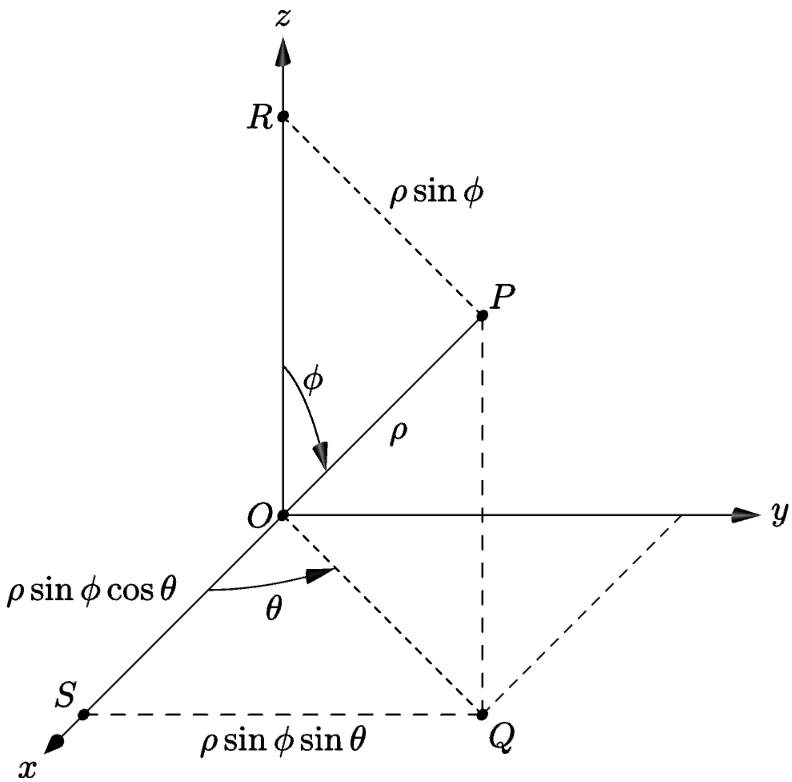


Ex: Find the volume of the region below the paraboloid $z = 5 - x^2 - y^2$ and above the plane $z = 1$

$$\text{Projected area is } \{z = 5 - x^2 - y^2\} \cap \{z = 1\} \Rightarrow x^2 + y^2 = 4$$
$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z=1}^{5-r^2} r dz dr d\theta = 2\pi \int_0^2 (4r - r^3) dr = 2\pi \left[2r^2 - \frac{r^4}{4}\right]_0^2$$
$$V = 2\pi (8 - 4) = 8\pi$$

Spherical coordinates is given by

$$\rho^2 = x^2 + y^2 + z^2$$
$$x = \rho \cos\theta \sin\phi$$
$$y = \rho \sin\theta \sin\phi$$
$$z = \cos\phi$$
$$dV = \rho^2 \sin\phi d\rho d\phi d\theta$$



Ex: Find the volume below the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$

$$V = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{2} \rho^2 \sin\phi d\rho d\phi d\theta = 2\pi \int_0^{\pi/4} \frac{8}{3} \sin\phi d\phi = \frac{16\pi}{3} \left[-\cos\phi\right]_0^{\pi/4}$$

$$V = \frac{16\pi}{3} \left(1 - \frac{1}{\sqrt{2}}\right) = \frac{8\pi}{3}(2 - \sqrt{2})$$

We can define an arbitrary change of coordinates in the following way:
If we have the equations

$$\begin{cases} x = x(u, v, w) \\ y = y(u, v, w) \\ z = z(u, v, w) \end{cases}$$

we can represent the distortion factor as the determinant of the Jacobian.

$$\det J = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix}$$

Ex: Find the volume of the ellipsoid enclosed by the surface $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$

$$\text{let } \begin{cases} x = au \\ y = bv \\ z = cw \end{cases}$$

$$\det J = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

$$V = \iiint_{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\le 1} dV = \iiint_{u^2+v^2+w^2\le 1} abc\,du\,dv\,dw$$
$$V = abc\text{Volume(unit sphere)} = \frac{4\pi abc}{3}$$

2 Vector Calculus

2.1 Parameterizations of Curves and Surfaces

2.1.1 Parametric Equations of Curves

We have often seen equations in the form of $y = f(x)$. We can express both x and y in terms of a common variable t in order to create a vector form of the function and show how the function changes over time.
A parametric equation will be in the form of $\vec{r}(t) = \langle \vec{x}(t), \vec{y}(t), \vec{z}(t) \rangle$
Ex: What curve is represented by $x = \cos t$ and $y = \sin t$ for $t \in [0, 2\pi]$

$$\sin^2 t + \cos^2 t = 1$$
$$y^2 + x^2 = 1$$

This is the equation of the unit circle. By plugging in points, we can see that it starts at $(1, 0)$ and rotates counterclockwise.
Ex2: What curve is represented by $x = t$ and $y = t^2$
$y = x^2$
The function is an upward opening parabola.
We can also define slightly more complicated functions through parametrics such as a cycloid:
$\vec{r}(\theta) = \langle a\theta - a\sin\theta, a - a\cos\theta \rangle$ where a is the radius.
Ex3: Find a parameterization of the curve given by the intersection of $\{z = \sqrt{x^2 + y^2}\} \cap \{z = 1 + y\}$

$$\text{try } x = t$$
$$z = \sqrt{t^2 + y^2},\ z = 1 + y$$
$$1 + y = \sqrt{t^2 + y^2}$$
$$(1 + y)^2 = t^2 + y^2 \Rightarrow 1 + 2y + y^2 = t^2 + y^2$$
$$y = \frac{1}{2}(t^2 - 1)$$
$$z = 1 + \frac{1}{2}(t^2 - 1) = \frac{1}{2}(t^2 + 1)$$
$$\vec{r}(t) = \left\langle t, \frac{1}{2}(t^2 - 1), \frac{1}{2}(t^2 + 1) \right\rangle$$

Derivatives of Parametric Curves
When taking the derivative of a parametric equation, we can either analyze the time derivatives of each component or we can use the chain rule to compare components to each other as we would normally do.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

For the example of the cycloid, we have $\frac{d\vec{r}}{d\theta} = \langle a - a\cos\theta, a\sin\theta\rangle$.
We can also get $\frac{dy}{dx} = \frac{a\sin\theta}{a - a\cos\theta}$
Ex: If $x = 2t^2 + 3$ and $y = t^4$, find $\frac{dy}{dx}$

$$\frac{dy}{dt} = 4t^3$$
$$\frac{dx}{dt} = 4t$$
$$\frac{dy}{dx} = \frac{4t^3}{4t} = t^2$$
$$x = 2t^2 + 3 \Rightarrow t^2 = \frac{x-3}{2}$$
$$\frac{dy}{dx} = \frac{x-3}{2}$$

We can also define higher order derivatives using the same method:

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

The derivatives for linear operations follow naturally from the product rule:

$$\frac{d}{dt}(f(t)\vec{r}(t)) = f'(t)\vec{r}(t) + f(t)\vec{r}'(t)$$

$$\frac{d}{dt}(\vec{u}(t)\cdot\vec{v}(t)) = \vec{u}'\cdot\vec{v} + \vec{u}\cdot\vec{v}'$$
$$\frac{d}{dt}(\vec{u}(t)\times\vec{v}(t)) = \vec{u}'\times\vec{v} + \vec{u}\times\vec{v}'$$

Motion:
$\vec{r}$ is the position vector, s is the arc length (distance travelled along trajectory), $\vec{v}$ is the velocity, $\|\vec{v}\|$ is the speed, and $\vec{a}$ is the acceleration.
Note that $\vec{v}$ is always the direction tangent to the path of motion.

$$\vec{v} = \frac{d\vec{r}}{dt}$$
$$\vec{a} = \frac{d\vec{v}}{dt}$$
$$\|\vec{v}\| = \frac{ds}{dt}$$
$$\vec{v} = \hat{T}\|\vec{v}\| = \hat{T}\frac{ds}{dt}$$
$$s = \int_{t_0}^{t_1} \|\vec{r}'(t)\| dt$$

Ex: Find the speed of an object travelling around the unit circle.

$$\vec{r} = \langle \cos t, \sin t\rangle$$

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2$$
$$\left(\frac{\Delta s}{\Delta t}\right)^2 = \left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2$$
$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$
$$\|\vec{v}\| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

Ex2: Find the circumference of a circle of radius a

$$\vec{r} = \langle a\cos t, a\sin t\rangle$$
$$ds = \sqrt{a^2\sin^2 t + a^2\cos^2 t}dt = adt$$
$$s = \int_0^{2\pi} adt = at|_0^{2\pi} = 2\pi a$$

Kepler's Law:
states that for a planet in an elliptical orbit, the area swept out over time is always constant.
$\|\vec{r} \times \vec{v}\| = \text{constant}$
With this we can prove that $\vec{a}_{\|}\vec{r}$

$$\frac{d}{dt}(\vec{r} \times \vec{v}) = 0$$
$$\frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt} = 0$$
$$\vec{v} \times \vec{v} + \vec{r} \times \vec{a} = 0$$
$$\vec{r} \times \vec{a} = 0$$
$$\therefore \vec{a}_{\|}\vec{r}$$

2.1.2 Curvature

Curvature is defined to be how "curvy" a curve is. This is done by using a tangent circle to approximate the curve (similar to how a tangent line works).
The circle which best approximates the curve near a point is called the *circle of curvature*.
The radius of this circle is called the *radius of curvature* (represented by ρ) and the center of the circle is called the *center of curvature*.
The value of curvature itself is defined to be

$$\kappa = \frac{1}{\rho}$$

Note as $\kappa \to \infty$ the curve is *very* curvy and as $\kappa \to 0$ the curve is linear.
The equation for curvature is given by

$$\kappa = \left|\frac{\vec{v}(t) \times \vec{a}(t)}{\left(\frac{ds}{dt}\right)^3}\right|$$

which, expressed in cartesian coordinates can simplify to

$$\kappa = \frac{\left|\frac{d^2y}{dx^2}\right|}{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{3/2}}$$

Additionally, the radius of curvature is given by

$$\rho(t) = \frac{1}{\kappa(t)}$$

and the center of curvature is given by

$$\vec{r}(t) + \rho(t)\hat{N}(t)$$

2.1.3 Parametric Equations of Surfaces

Similar to lines, we can represent a surface with a parameterization $\vec{r}(u, v)$.
There are often many different ways to parameterize a surface but some will result in simpler forms than others.
If your surface has spherical symmetry, try an analog to spherical coordinates where one variable is taken to be a constant (usually ρ).
If your surface has circular symmetry about an axis, try cylindrical coordinates where one variable is taken to be a constant (usually r).
Otherwise, a good option may be to use Cartesian where you set your two variables to x and y and set the z-position to be a function of x and y.
Ex: Parameterize the hemisphere of radius 5 that lies above the xy-plane using 3 different parameterizations

$$S = \left\{x^2 + y^2 + z^2 = 25,\ z \geq 0\right\}$$
$$\vec{r}(x, y) = \left\langle x, y, \sqrt{25 - x^2 - y^2}\right\rangle$$
$$\vec{r}(r, \theta) = \left\langle r\cos\theta, r\sin\theta, \sqrt{25 - r^2}\right\rangle$$
$$\vec{r}(\theta, \phi) = \langle 5\cos\theta\sin\phi, 5\sin\theta\sin\phi, 5\cos\phi\rangle$$

Ex2: Find a parameterization of a torus with larger radius A (from the origin) smaller radius a (internal radius)

circular symmetry so try polar

let θ be the revolution around the central z-axis

let φ be the revolution about the inside of the torus

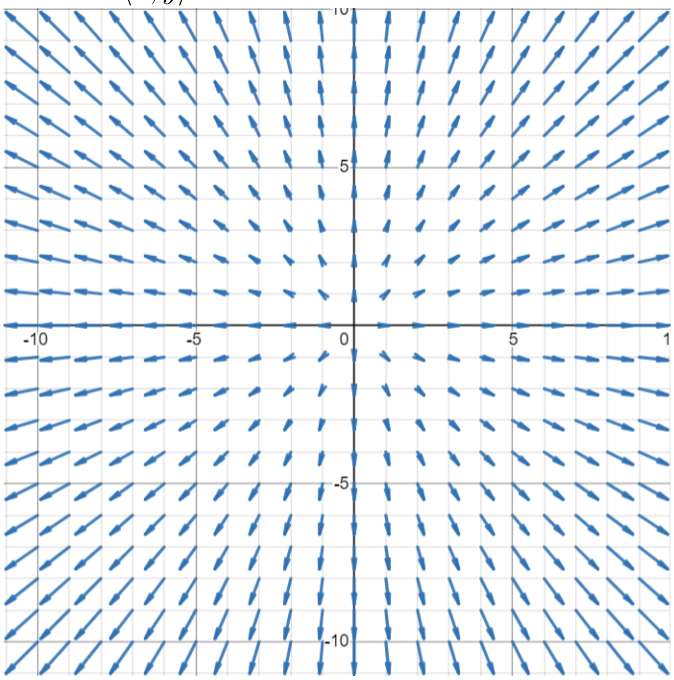
$$(r(\varphi), z(\varphi)) = (A + a\sin\varphi, a\cos\varphi)$$
$$x(\theta, \varphi) = r(\varphi)\cos\theta$$
$$y(\theta, \varphi) = r(\varphi)\sin\theta$$
$$z(\theta, \varphi) = z(\varphi)$$
$$\vec{r}(\theta, \varphi) = \langle (A + a\sin\varphi)\cos\theta, (A + a\sin\varphi)\sin\theta, a\cos\varphi\rangle$$

2.2 Integrals in the Plane

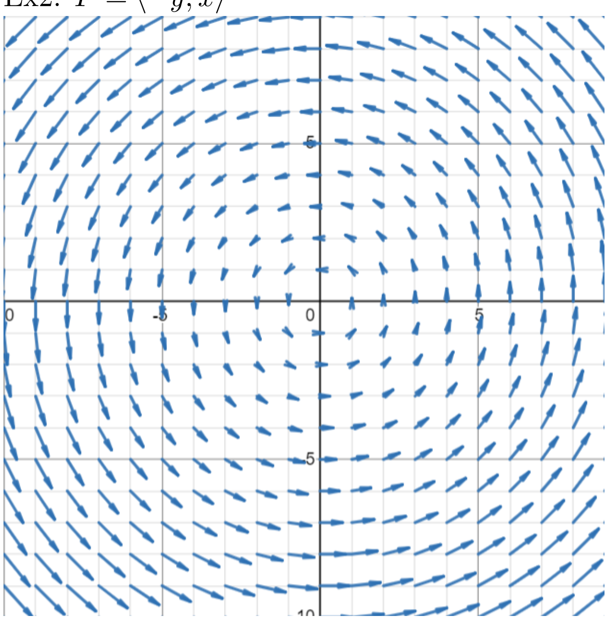
2.2.1 Vector Fields

A vector field can be defined as $\vec{F} = \langle P(x,y), Q(x,y) \rangle$ where every point (x,y) maps to a corresponding vector.

Ex: $\vec{F} = \langle x, y \rangle$



Ex2: $\vec{F} = \langle -y, x \rangle$



A vector field is defined to be conservative if it can be represented as the gradient of a function

$$\vec{F} = \nabla f$$

Ex: Conservation of energy:

$$\vec{F} = m\vec{a}$$
$$\nabla f = m\frac{d\vec{v}}{dt}$$
$$m\vec{v} \cdot \frac{d\vec{v}}{dt} = \vec{v} \cdot \nabla f$$
$$\left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle \cdot \langle f_x, f_y, f_z \rangle = \frac{1}{2}m\frac{d}{dt}(\vec{v} \cdot \vec{v})$$
$$\frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\|v\|^2\right)$$
$$\frac{d}{dt}\left(\frac{1}{2}mv^2 - f(x, y, z)\right) = 0$$

A vector field is conservative if the curl is zero and the domain is simply connected (meaning it is continuous and defined everywhere in question).

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Note that just a curl of $\vec{0}$ is not enough to determine if a vector field is conservative (we will see an example of this later).

Divergence and Curl

There are two new operations that can be performed on a vector field. These are divergence and curl.

If we think of a vector filed as a fluid flow, divergence determines how compressible a vector field is. If there is more fluid being created then destroyed in a region, the divergence in that region will be positive (think of a source of a sink). Similarly, if there is more fluid being destroyed then the divergence will be negative (think of a drain of a sink). A divergence of 0 implies that the fluid is incompressible. The restult of the divergence will be a scalar and it is analogous to the dot product.

$$\text{div}(\vec{F}) = \nabla \cdot \vec{F} = P_x + Q_y + R_z$$

Curl is a measure of how non-conservative a vector field is. A curl of zero implies that that a vector field is irrotational.

Some notes on divergence and curl.

- The image of the gradient are conservative vector fields
- The kernal (null space) of curl is the irrotational vector field
- If the domain is simply connected then the kernel of curl is equal to the image of the gradient.
- When a vector filed is divergence free, every surface is a boundary of a solid.

There are many different product rules that relate to divergence and curl. Here is an example:

$$\begin{aligned}
\nabla \cdot (f\vec{F}) &= \nabla \cdot \langle fP, fQ, fR \rangle \\
&= (fP)_x + (fQ)_y + (fR)_z \\
&= f_xP + f_yQ + f_zR + fP_x + fQ_y + fR_z \\
&= \langle f_x, f_y, f_z \rangle \cdot \langle P, Q, R \rangle + f\,\langle P_x, Q_y, R_z \rangle \\
\nabla \cdot (f\vec{F}) &= (\nabla f) \cdot \vec{F} + f(\nabla \cdot \vec{F})
\end{aligned}$$

Ex: Compute $\nabla \cdot (r^k\vec{r})$ for where $\vec{r} = \langle x, y, z \rangle$ and $r = \|\vec{r}\| = \sqrt{x^2+y^2+z^2}$

$$\nabla \cdot (r^k\vec{r}) = (\nabla r^k) \cdot \vec{r} + r^k \nabla \cdot \vec{r}$$

$$\nabla \cdot \vec{r} = \nabla \cdot \langle x, y, z \rangle = 1 + 1 + 1 = 3$$

$$\nabla r^k = kr^{k-1}\nabla r,\ \nabla r = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle = \frac{\vec{r}}{r}$$

$$\nabla r^k = kr^{k-2}\vec{r}$$

$$\nabla \cdot (r^k\vec{r}) = kr^{k-2}\vec{r} \cdot \vec{r} + 3r^k = kr^{k-2}\|\vec{r}\|^2 + 3r^k$$

$$= (3+k)r^k$$

2.2.2 Line Integrals

Line integrals over some path C can be computed the same way as with arc length. We first find a parameterization for C and then compute using the following formula,

$$\int_C f ds = \int_{t=a}^{b} f(\vec{r}(t))\|\vec{r}'(t)\|dt$$

Ex: Integrate $f(x, y) = xy^4$ along the right half of the circle $x^2 + y^2 = 16$

$$C:\ \vec{r}(t) = \langle 4\cos t, 4\sin t \rangle\,,\ -\frac{\pi}{2} \le t \le \frac{\pi}{2}$$

$$\vec{r}'(t) = \langle -4\sin t, 4\cos t \rangle$$

$$\|\vec{r}'(t)\| = \sqrt{16\sin^2 t + 16\cos^2 t} = 4$$

$$\int_C f ds = \int_{-\pi/2}^{\pi/2} (4\cos t)(4\sin t)^4(4)dt = 4^6\left[\frac{\sin^5 t}{5}\right]_{-\pi/2}^{\pi/2} = \frac{8192}{5}$$

2.2.3 Work Integrals

We can define a path integral inside a force field using the concept of work. Work is the force of the vector field times the distance travelled in the same direction. So we can define an infinitesimal amount of work to be $dW = \vec{F} \cdot d\vec{r}$ which gives the integral formula

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_C (\vec{F} \cdot \hat{T})ds = \int_{t=a}^{b} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt$$

Note that the shorthand notation $d\vec{r} = \|\vec{r}'(t)\|dt$.
Ex: Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \langle x, x \rangle$ and C is the curve $3x = y^2$ starting at $(0,0)$ and ending at $(3,3)$.

$$\vec{r}(t) = \left\langle \frac{t^2}{3}, t \right\rangle$$

$$\vec{r}'(t) = \left\langle \frac{2}{3}t, 1 \right\rangle$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^3 \left\langle \frac{t^2}{3}, \frac{t^2}{3} \right\rangle \cdot \left\langle \frac{2}{3}t, 1 \right\rangle dt = \int_0^3 \left(\frac{2}{9}t^3 + \frac{t^2}{3} \right) dt = \left[\frac{t^4}{18} + \frac{t^3}{9} \right]_0^3 = \frac{15}{2}$$

Fundamental Theorem of Line Integrals:
This states that if $\vec{F}$ is conservative ($\vec{F} = \nabla f$) then

$$\int_C \vec{F} \cdot d\vec{r} = f(P_2) - f(P_1)$$

meaning that the work integral is path independent.
Note that this formula implies that the integral of any closed loop over a conservative vector field is always 0.

$$\oint_C \vec{F} \cdot d\vec{r} = 0$$

$\vec{F}$ is conservative when the curl is zero and the domain is simply connected.
Ex2: Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \left\langle e^{-x^2} + 3y^3 - 3x^2y, \arctan(y^3) + 6xy^2 - x^3 \right\rangle$ and C is the closed loop formed by the unit circle in the clockwise direction.

$$\nabla \times \vec{F} = \left(6y^2 - 3x^2 - 9y^2 + 6x^2\right)\hat{k} = -3y^2\hat{k}$$

we can break this up into two different focre fields $\vec{F}_1$ and $\vec{F}_2$

$$\vec{F}_1 = \left\langle e^{-x^2} + 2y^3 - 3x^2y, \arctan(y^3) + 6xy^2 - x^3 \right\rangle, \ \nabla \times \vec{F}_1 = \vec{0}$$

$$\vec{F}_2 = \left\langle y^3, 0 \right\rangle$$

$\vec{F}_1$ is conservative so $\displaystyle\int_C \vec{F}_1 \cdot d\vec{r} = 0$

$\displaystyle\int_C \vec{F}_2 \cdot d\vec{r}$ will require a parameterization

$$\vec{r}(t) = \langle -\cos t, \sin t \rangle$$

$$\vec{r}'(t) = \langle \sin t, \cos t \rangle$$

$$\int_C \vec{F}_2 \cdot \ d\vec{r} = \int_0^{2\pi} \left\langle \sin^3 t, 0 \right\rangle \cdot \langle \sin t, \cos t \rangle \, dt = \int_0^{2\pi} \sin^4 t dt = \frac{3\pi}{4}$$

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F}_1 \cdot d\vec{r} + \int_C \vec{F}_2 \cdot d\vec{r} = \frac{3\pi}{4}$$

Ex3: Find $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$ and C is the counterclockwise circle of radius R around the origin.

$$\nabla \times \vec{F} = \vec{0}$$

$\vec{F}$ is undefined at $(0,0)$, meaning it is not simply connected

This means we cannot apply the fundamental theorem of line integrals

$\vec{r}(t) = \langle R\cos t, \mathbb{R}\sin t\rangle$

$\vec{r}'(t) = \langle -R\sin t, R\cos t\rangle$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left\langle \frac{-R\sin t}{R^2}, \frac{R\cos t}{R^2}\right\rangle \cdot \langle -R\sin t, R\cos t\rangle\, dt = \int_0^{2\pi} (\sin^2 t + \cos^2 t)dt = 2\pi$$

2.2.4 Green's Theorem

Green's Theorem is an extention of the fundamental theorem of line integrals for closed path integrals. Essentially, it transforms a line integral into a double integral over the enclosed region. For a vector field $\vec{F} = \langle P, Q\rangle$

$$\oint_{\partial R} \vec{F}\cdot d\vec{r} = \iint_R (Q_x - P_y)dA$$

Some caveats to this is that the vector field must be defined over the entire region and that the loop travels counterclockwise (it can go counterclockwise but with a change of sign) so that the region will always be to the left of the direction of travel along the loop.

We can see this using some previous examples:

Ex: Find $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \left\langle e^{-x^2} + 3y^3 - 3x^2y, \arctan(y^3) + 6xy^2 - x^3\right\rangle$ and C is the closed loop formed by the unit circle in the clockwise direction.

$$Q_x - P_y = \left(6y^2 - 3x^2 - 9y^2 + 6x^2\right) = -3y^2$$

$$\oint_C \vec{F}\cdot d\vec{r} = -\iint_{x^2+y^2=1} -3y^2 dA = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} 3r^3\sin^2\theta\, dr d\theta$$

$$= \frac{3\pi}{4}$$

If a loop is not closed or has a hole in it, we can add another line integral to close the loop so that we can compute with Green's Theorem.

Ex2: Find $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right\rangle$ and C is the counterclockwise circle of radius R around the origin.

Notice with this vector field the center is undefined so we can create another infinitesimally small loop around the origin so that the vector field is now defined everywhere.

$$C' = \left\{x^2 + y^2 = \epsilon^2,\ |\epsilon| \ll 1\right\} \text{ travelling clockwise}$$

$$\oint_{\partial R} \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_{C'} \vec{F}\cdot d\vec{r}$$

$$Q_x - P_y = 0$$

$$\oint_{\partial R} \vec{F}\cdot d\vec{r} = \iint 0 d\vec{r} = 0$$

$$\vec{r} = \langle -\epsilon\cos t, \epsilon\sin t\rangle$$

$$\vec{r}' = \langle \epsilon\sin t, \epsilon\cos t\rangle$$

$$\int_{C'} \vec{F}\cdot d\vec{r} = \lim_{\epsilon\to 0}\int_0^{2\pi} \left\langle \frac{-\epsilon\sin t}{\epsilon^2}, \frac{\epsilon\cos t}{\epsilon^2}\right\rangle \cdot \langle -\epsilon\sin t, \epsilon\cos t\rangle = 2\pi$$

$$\Rightarrow \int_C \vec{F} \cdot d\vec{r} = \oint_{\partial R} \vec{F} \cdot d\vec{r} - \int_{C'} \vec{F} \cdot d\vec{r} = 2\pi$$

Ex3: Find the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\begin{aligned}
&\text{need } Q_x - P_y = 1 \\
&\vec{F} = \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle \\
&\vec{r} = \langle a\cos t, b\sin t \rangle \\
&\vec{r}' = \langle -a\sin t, b\cos t \rangle \\
&\oint \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \frac{1}{2} \langle -b\sin t, a\cos t \rangle \cdot \langle -a\sin t, b\cos t \rangle \, dt = \frac{1}{2} \int_0^{2\pi} (ab\sin^2 t + ab\cos^2 t) dt \\
&= \frac{ab}{2} \int_0^{2\pi} dt = \pi ab
\end{aligned}$$

2.3 Integrals in Space

2.3.1 Surface Integrals

Similar to line integrals, a surface integral is an integral evaluated over the surface of a region. It is computed by

$$\iint_S f dS = \iint_S f(\vec{r}(u,v)) \|\vec{r}_u \times \vec{r}_v\| dA$$

Note that dS decomposes into $dS = \|\vec{r}_u \times \vec{r}_v\| dA$

Ex: Find the surface area of a sphere or radius 2.

$$\begin{aligned}
&\vec{r}(\theta, \phi) = \langle 2\cos\theta\sin\phi, 2\sin\theta\sin\phi, 2\cos\phi \rangle \\
&\vec{r}_\theta = \langle -2\sin\theta\sin\phi, 2\cos\theta\sin\phi, 0 \rangle \\
&\vec{r}_\phi = \langle 2\cos\theta\cos\phi, 2\sin\theta\cos\phi, -2\sin\phi \rangle \\
&\vec{r}_\theta \times \vec{r}_\phi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin\theta\sin\phi & 2\cos\theta\sin\phi & 0 \\ 2\cos\theta\cos\phi & 2\sin\theta\cos\phi & -2\sin\phi \end{vmatrix} \\
&= \left\langle -4\cos\theta\sin^2\phi, -4\sin\theta\sin^2\phi, -4\sin^2\theta\sin\phi\cos\phi - 4\cos^2\theta\sin\phi\cos\phi \right\rangle \\
&= \left\langle -4\cos\theta\sin^2\phi, -4\sin\theta\sin^2\phi, -4\sin\phi\cos\phi \right\rangle \\
&= -4\sin\phi \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi \rangle \\
&\|\vec{r}_\theta \times \vec{r}_\phi\| = 4\sin\phi\sqrt{\cos^2\theta\sin^2\phi + \sin^2\theta\sin^2\phi + \cos^2\phi} \\
&= 4\sin\phi\sqrt{\sin^2\phi + \cos^2\phi} \\
&= 4\sin\phi \\
&\iint_S dS = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi} 4\sin\phi = 8\pi \left[-\cos\phi\right]_0^\pi = 16\pi
\end{aligned}$$

Ex2: Let S be the part of the one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$ which lies above the xy-plane and below the plane $z = A$ where A is a positive constant. Compute the integral $\iint_S z dS$.

$$\text{let } \vec{r}(\theta, z) = \left\langle \sqrt{1+z^2}\cos\theta, \sqrt{1+z^2}\sin\theta, z \right\rangle, \ 0 \le \theta \le 2\pi, \ 0 \le z \le A$$

$$\vec{r}_\theta = \left\langle -\sqrt{1+z^2}\sin\theta, \sqrt{1+z^2}\cos\theta, 0 \right\rangle$$

$$\vec{r}_z = \left\langle \frac{z}{\sqrt{1+z^2}}\cos\theta, \frac{z}{\sqrt{1+z^2}}\sin\theta, 1 \right\rangle$$

$$\vec{r}_\theta \times \vec{r}_z = \left\langle \sqrt{1+z^2}\cos\theta, \sqrt{1+z^2}\sin\theta, -z\sin^2\theta - z\cos^2\theta \right\rangle$$

$$= \left\langle \sqrt{1+z^2}\cos\theta, \sqrt{1+z^2}\sin\theta, -z \right\rangle$$

$$\|\vec{r}_\theta \times \vec{r}_z\| = \sqrt{(1+z^2)\cos^2\theta + (1+z^2)\sin^2\theta + z^2} = \sqrt{1+z^2+z^2} = \sqrt{1+2z^2}$$

$$\iint_S z dS = \int_{\theta=0}^{2\pi}\int_{z=0}^{A} z\sqrt{1+2z^2} dz d\theta = 2\pi \left[\frac{1}{4}\cdot\frac{2}{3}(1+2z^2)^{3/2}\right] = \frac{\pi}{3}\left((1+2A^2)^{3/2} - 1\right)$$

2.3.2 Flux Integrals

Integrating a vector field over a surface in $\mathbb{R}^3$ gives a flux integral. This can be thought of how much the vector field is flowing in or out of the surface.
Note that a flux integral also requires a choice of orientation. Similar to how a curve had a direction, an orientation states which side of the surface is positive.
A flux integral is formed as follows:

$$\iint_S \vec{F}\cdot d\vec{S} = \pm\iint_S \vec{F}\cdot\hat{n} dS = \pm\iint_S \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v) dA$$

Ex: Let S be the part of the surface $z = 4 - x^2 - y^2$ lying above the xy-plane, oriented upward. Let $\vec{F} = \langle x(x^2+y^2), y(x^2+y^2), z\rangle$. Compute $\iint_S \vec{F}\cdot d\vec{S}$.

$$\vec{r}(\theta, z) = \left\langle \sqrt{4-z}\cos\theta, \sqrt{4-z}\sin\theta, z \right\rangle, \ 0 \le \theta \le 2\pi, \ 0 \le z \le 4$$

$$\vec{F}(\vec{r}) = \left\langle \sqrt{4-z}\cos\theta(4-z), \sqrt{4-z}\sin\theta(4-z), z \right\rangle$$

$$\vec{r}_z = \left\langle \frac{-1}{2\sqrt{4-z}}\cos\theta, \frac{-1}{2\sqrt{4-z}}\sin\theta, 1 \right\rangle$$

$$\vec{r}_\theta = \left\langle -\sqrt{4-z}\sin\theta, \sqrt{4-z}\cos\theta, 0 \right\rangle$$

$$\vec{r}_z \times \vec{r}_\theta = \left\langle -\sqrt{4-z}\cos\theta, -\sqrt{4-z}\sin\theta, -\frac{1}{2}(\cos^2\theta + \sin^2\theta) \right\rangle$$

$$\iint_S \vec{F}\cdot d\vec{S} = -\int_{\theta=0}^{2\pi}\int_{z=0}^{4} \left(-(4-z)^2(\cos^2\theta+\sin^2\theta) - \frac{z}{2}\right) dz d\theta$$

$$= 2\pi\left[-\frac{1}{3}(4-z)^3 + \frac{z^2}{4}\right]_0^4$$

$$= 2\pi\left(4 + \frac{4^3}{3}\right) = \frac{152\pi}{3}$$

Ex2: Let S be the disk of radius 3, oriented upward on the xy-plane centered at $(15, 16, 0)$ and let $\vec{F} = \left\langle e^{\tan\sqrt{z}} + yz, \frac{x\ln(z+1)}{y^6}, 3 + 2\cos(z^2) \right\rangle$.

$$\text{note } z = 0 \text{ and } x, y > 0 \text{ for all points on } S$$

$$\vec{F} = \langle 1, 0, 5 \rangle$$

$$\hat{n} = \hat{k} = \langle 0, 0, 1 \rangle$$

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_S \langle 1, 0, 5 \rangle \cdot \langle 0, 0, 1 \rangle \, dS = \iint_S 5dS = 5(9\pi) = 45\pi$$

2.3.3 The Divergence Theorem

Divergence Theorem is given by

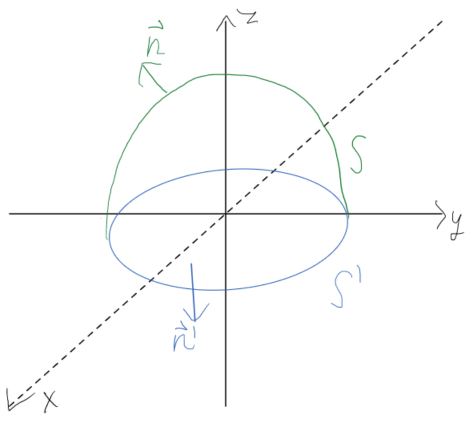
$$\oiint_{\partial E} \vec{F} \cdot d\vec{S} = \iiint_E (\nabla \cdot \vec{F}) dV$$

where E is a solid region in $\mathbb{R}^3$ and ∂E is the boundary of E oriented outward.
This formula can be thought of as the rate that fluid escapes E is equal to the total rate of fluid being produced.
Ex: Find the flux from the vector field $\vec{F} = \langle x, y, z \rangle$ out of the cube of side length 2 centered at the origin.

$$\nabla \cdot \vec{F} = 3$$

$$\iiint_E 3dV = 3 \cdot \text{Vol}(E) = 24$$

Ex2: Let S be the hemisphere of radius 1 located above the xy-plane, oriented outward and let $\vec{F} = \langle z^2 x, \frac{1}{3}y^3 + \tan\sqrt{z}, x^2 z + y^2 \rangle$. Find $\iint_S \vec{F} \cdot d\vec{S}$.



$$\nabla \cdot \vec{F} = z^2 + y^2 + x^2 = \rho^2$$

$$\partial E = S + S'$$

$$E = \left\{ x^2 + y^2 + z^2 \leq 1, \ z \geq 0 \right\}$$

$$S' = \left\{ x^2 + y^2 = 1 \ z = 0 \right\}$$

$$\iiint_E \rho^2 dV = \iint_S \vec{F}\cdot d\vec{S} + \iint_{S'} \vec{F}\cdot d\vec{S}$$

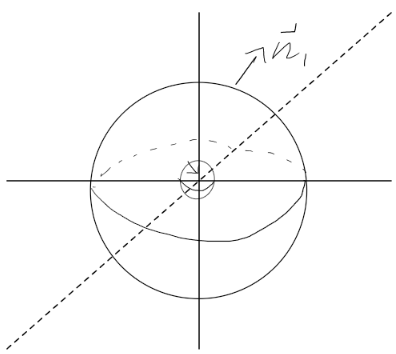
$$\iiint_E \rho^2 dV = \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=0}^{1} \rho^4 \sin\phi d\rho d\phi d\theta = 2\pi\left[-\cos\phi\right]_0^{\pi/2}\left[\frac{\rho^5}{5}\right]_0^1 = \frac{2\pi}{5}$$

$$\iint_{S'} \vec{F}\cdot d\vec{S} = \left\langle z^2x, \frac{1}{3}y^3 + \tan\sqrt{z}, x^2z + y^2\right\rangle \cdot \langle 0,0,-1\rangle \, dS = -\iint_{S'} x^2z + y^2 dS$$

$$\iint_{S'} \vec{F}\cdot d\vec{S} = -\iint_{S'} y^2 dS = -\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^3\sin^2\theta dr d\theta = -\pi\left[\frac{r^4}{4}\right]_0^1 = -\frac{\pi}{4}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \rho^2 dV - \iint_{S'} \vec{F}\cdot d\vec{S} = \frac{5\pi}{2} + \frac{\pi}{4} = \frac{13\pi}{20}$$

It is also important to note that to apply the divergence theorem, the vector field must be defined within the region.
Ex3: Find the flux from the vector field $\vec{F} = \frac{\vec{r}}{r^3}$ out of the surface ∂E where E is the sphere of radius 2 centered about the origin.
Note that because the vector field is undefined at the origin, we need to compute the flux about the origin separately.
We can define the sphere S_ϵ to be the sphere of radius ϵ about the origin for where $\epsilon \to 0$ and subtract its flux from the total flux of the region.



$$\iiint_{\partial E} (\nabla\cdot\vec{F})dV = \iint_S \vec{F}\cdot d\vec{S} + \iint_{S_\epsilon} \vec{F}\cdot d\vec{S}$$

$$\vec{F} = \left\langle \frac{x}{(x^2+y^2+z^2)^{3/2}}, \frac{y}{(x^2+y^2+z^2)^{3/2}}, \frac{z}{(x^2+y^2+z^2)^{3/2}} \right\rangle$$

$$\nabla\cdot\vec{F} = 0$$

$$\hat{n}_2 = -\frac{\vec{r}}{r}$$

$$\iint_{S_\epsilon} \vec{F}\cdot d\vec{S} = \iint_{S_\epsilon} \frac{\vec{r}}{r^3}\cdot\frac{-\vec{r}}{r} dS = -\iint_{S_\epsilon} \frac{1}{r^2} dS = -\frac{1}{\epsilon^2}\iint_{S_\epsilon} dS = -\frac{1}{\epsilon^2}\cdot 4\pi\epsilon^2 = -4\pi$$

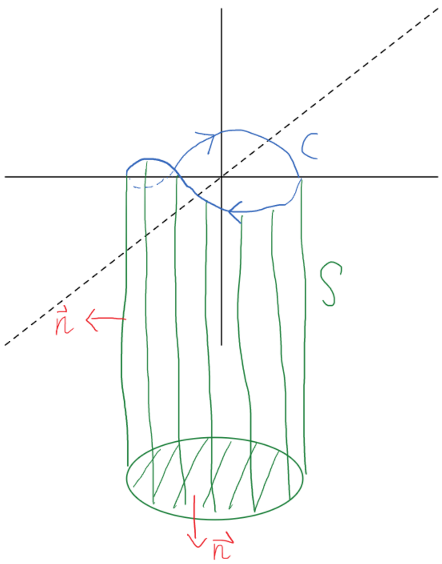
$$\iint_S \vec{F}\cdot d\vec{S} = -\iint_{S_\epsilon} \vec{F}\cdot d\vec{S} = 4\pi$$

2.3.4 Stokes' Theorem

Stokes' Theorem is a more general case of Green's Theorem and is given by

$$\oint_{\partial S} \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S}$$

Ex: Compute $\oint_C \vec{F} \cdot d\vec{r}$ where C is given by $\vec{r}(t) = \langle \sin t, \cos t, -\sin 2t \rangle$ and $\vec{F} = \left\langle \tan\sqrt{1+x^4}, zy + e^{y^3}, \frac{x^2}{2} + \sqrt[3]{\sin(z^2)} \right\rangle$



$$\nabla \times \vec{F} = \langle -y, -x, 0 \rangle$$
$$S_1 = \{x^2 + y^2 = 1\}$$
$$S_2 = \{z = -A\}$$
$$\oint \vec{F} \cdot d\vec{r} = \iint_{S_1} (\nabla \times \vec{F}) cdot d\vec{S} + \iint_{S_2} (\nabla \times \vec{F}) \cdot d\vec{S}$$
$$\iint_{S_2} \langle -y, -x, 0 \rangle \cdot \langle 0, 0, -1 \rangle \, dS = 0$$
$$S_1 : \ \vec{r}(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle, \ 0 \le \theta \le 2\pi, \ -A \le z \le -\sin 2\theta$$
$$\vec{r}_z = \langle 0, 0, 1 \rangle$$
$$\vec{r}_\theta = \langle -\sin\theta, \cos\theta, 0 \rangle$$
$$\vec{r}_z \times \vec{r}_\theta = \langle -\cos\theta, -\sin\theta, 0 \rangle$$
$$\iint_{S_1} (\nabla \times \vec{F}) \cdot d\vec{S} = -\int_{\theta=0}^{2\pi} \int_{z=-A}^{-\sin 2\theta} \langle -\sin\theta, -\cos\theta, 0 \rangle \cdot \langle -\cos\theta, -\sin\theta, 0 \rangle \, dz d\theta$$
$$= \int_0^{2\pi} \int_{-A}^{-\sin 2\theta} -2\sin\theta\cos\theta dz d\theta = \int_0^{2\pi} -2\sin\theta\cos\theta \, (-\sin 2\theta + A) \, d\theta$$
$$= \int_0^{2\pi} (\sin^2(2\theta) - A\sin(2\theta)) d\theta = \pi$$

$$\Rightarrow \oint \vec{F} \cdot d\vec{r} = \pi$$

2.3.5 Differential Forms

Differential forms are the unifying language of derivatives and integrals.
Let $x_1, \ldots, x_n$ be coordinates on $\mathbb{R}^n$. We can introduce the *wedge product* such that $dx_i \wedge dx_j = -dx_j \wedge dx_i$. Now let $U \subset \mathbb{R}^n$ be an open set. A differential k-form is a linear combination of k-fold products of dx's with functions on U as coefficients.
For example, a 1-form corresponds to $Pdx + Qdy + Rdz$ and a 2-form can be written as $Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy$
Notice that with wedge products there is also the rule $dx \wedge dx = 0$.
Zero forms are just functions.
A set of k-forms are written as $\Omega^k(U)$.

k-form	expression	geometric meaning
$\Omega^0(U)$	f	function
$\Omega^1(U)$	$F_1dx + F_2dy + F_3dz$	vector field
$\Omega^2(U)$	$F_1dy \wedge dz + F_2dz \wedge dx + F_3dx \wedge dy$	vector field
$\Omega^3(U)$	$fdx \wedge dy \wedge dz$	function

A wedge product is computed as

$$\Omega^k(U) \wedge \Omega^l(U) \to \Omega^{k+l}(U)$$

This allows us to express vector operations in terms of wedge products in a more compact notation.
Ex: The cross product is given by $\Omega^1(U) \wedge \Omega^1(U) \to \Omega^2(U)$

$$\begin{aligned}
&F, G \in \Omega^1(\mathbb{R}^3)\\
&F \wedge G = (F_1dx + F_2dy + F_3dz) \wedge (G_1dx + G_2dy + G_3dz)\\
&= F_1G_2dx \wedge dy + F_1G_3dx \wedge dz + F_2G_1dy \wedge dx + F_2G_3dy \wedge dz + F_3G_1dz \wedge dx + F_3G_2dz \wedge dy\\
&= (F_2G_3 - F_3G_2)dy \wedge dz + (F_3G_1 - F_1G_3)dz \wedge dx + (F_1G_2 - F_2G_1)dx \wedge dy\\
&= \vec{F} \times \vec{G}
\end{aligned}$$

Ex2: The dot product is given by $\Omega^1(U) \wedge \Omega^2(U) \to \Omega^3(U)$

$$\begin{aligned}
&F \in \Omega^1(\mathbb{R}^3)\\
&G \in \Omega^2(\mathbb{R}^3)\\
&F \wedge G = (F_1dx + F_2dy + F_3dz) \wedge (G_1dy \wedge dz + G_2dz \wedge dx + G_3dx \wedge dy)\\
&= F_1G_1dx \wedge dy \wedge dz + F_2G_2dy \wedge dz \wedge dx + F_3G_3dz \wedge dx \wedge dy\\
&= (F_1G_1 + F_2G_2 + F_3G_3)dx \wedge dy \wedge dz\\
&= \vec{F} \cdot \vec{G}
\end{aligned}$$

Differential forms also allows us to generalize the types of derivatives (gradient, curl, divergence) into one derivative.

$$d\Omega^k(U) \to \Omega^{k+1}(U)$$

$\Omega^k(U)$	derivative transformation	Type of derivative
$\Omega^0(U)$	scalar to vector field	gradient (∇f)
$\Omega^1(U)$	vector field to vector field	curl ($\nabla \times \vec{F}$)
$\Omega^2(U)$	vector field to scalar	divergence ($\nabla \cdot \vec{F}$)

In general, taking a derivative twice gives 0. $d(d\Omega^k(U)) = 0$
Differential forms also allows us to easily define product rules using the following formula.
If $\alpha \in \Omega^k(U)$ and $\beta \in \Omega^l(U)$ then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^k \alpha \wedge d\beta$$

Ex: $\nabla \times (f\vec{F})$

$f \in \Omega^0(U),\ \vec{F} \in \Omega^1(U)$

$\nabla \times (f\vec{F}) = \nabla f \times \vec{F} + f(\nabla \times \vec{F})$

Ex2: $\nabla \cdot (\vec{A} \times \vec{B})$

$\vec{A}, \vec{B} \in \Omega^1(U)$

$\nabla \cdot (\vec{A} \times \vec{B}) = (\nabla \times \vec{A}) \cdot \vec{B} - \vec{A} \cdot (\nabla \times \vec{B})$

Similar to derivatives, we can go in the opposite direction and generalize multiple integration using differential forms.
If M is a (k+1)-manifold, ∂M is a k-manifold, α is a k-form, and $d\alpha$ is a (k+1)-form then we can write the generalized Stokes' Theorem as

$$\int_{\partial M} \alpha = \int_M d\alpha$$

Differential forms also behave nicely under a change of variables.

Ex: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta = \cos\theta dr - r\sin\theta d\theta$

$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta = \sin\theta dr + r\cos\theta d\theta$

$dx \wedge dy = (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta)$

$= r\cos^2\theta dr \wedge d\theta + r\sin^2\theta dr \wedge d\theta = r dr \wedge d\theta$