Math 152
Linear Systems

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Contents

1 Vectors and Geometry

1.1 Vectors

Vectors arise from describing things that have both magnitude and direction (such as force or velocity). The coordinates of a vector $\vec{a}$ can be defined as $\langle a_1, a_2, a_3 \rangle$. This is the expression for a vector in three dimensions but a vector can be also be defined in an arbitrary number of dimensions.

The length of a vector can be determined using Pythagorean's theorem and is denoted by $\|\vec{a}\|$ for where $\vec{a}$ is a vector. The length is equal to,

$$\|\vec{a}\| = \sqrt{a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2}$$

Ex: The length of $\langle 1, 2, 3\rangle$ is $\sqrt{1+4+9} = \sqrt{14}$

A vector can be multiplied by a scalar as such:

$$k\vec{a} = \langle ka_1, ka_2, ka_3, \ldots, ka_n\rangle$$

This stretches and contracts the vector. If you multiply by a negative, the vector will flip directions.
Ex: $2\langle 1, 1\rangle = \langle 2, 2\rangle$
Ex2: $-1\langle 2, 1\rangle = \langle -2, -1\rangle$

A unit vector is a vector whose magnitude is 1 and is considered to only have a directional component (sometimes called a direction vector)

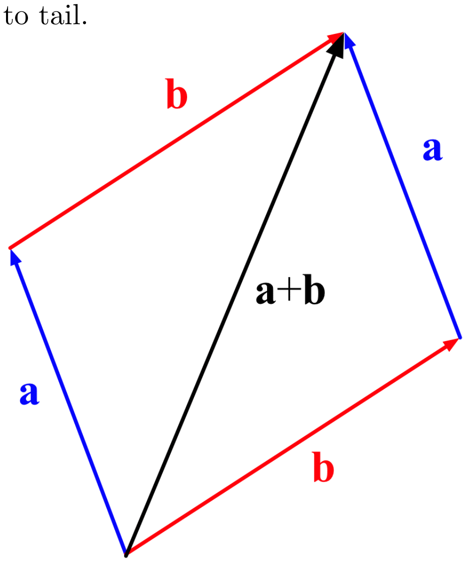
$$\hat{a} = \operatorname{dir}\vec{a} = \frac{\vec{a}}{\|\vec{a}\|}$$

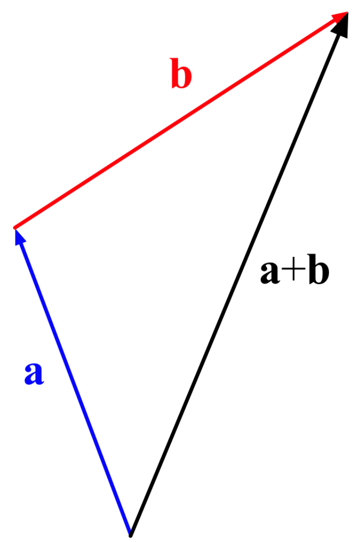
$$\text{Ex: } \operatorname{dir}\langle 2, 7\rangle = \frac{\langle 2, 7\rangle}{\sqrt{4+49}} = \left\langle \frac{2}{\sqrt{53}}, \frac{7}{\sqrt{53}} \right\rangle$$

Two vectors can be added by adding each element.

$$\vec{a} + \vec{b} = \langle a_1 + b_1, a_2 + b_2\rangle$$

Ex: $\langle 2, 0\rangle + \langle 1, 2\rangle = \langle 3, 2\rangle$ Geometrically, vector addition is the same as adding the two vectors tip to tail.

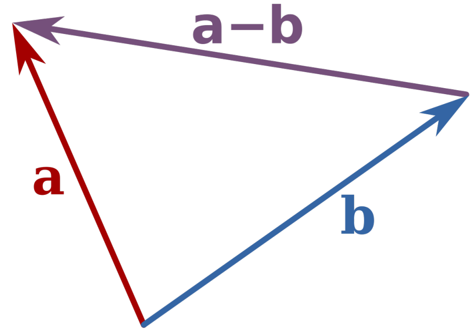




Similarly, vector subtraction is represented as

$$\vec{a} - \vec{b} = \langle a_1 - b_1,\, a_2 - b_2\rangle$$

and is shown geometrically as the vector from $\vec{b}$ to $\vec{a}$



The dot product is one form of multiplication of vectors. It is defined as

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2$$

*Notice that the output of a dot product is a scalar.
Ex: $\langle 2, 3 \rangle \cdot \langle 2, 1 \rangle = 2 \cdot 2 + 3 \cdot 1 = 4 + 3 = 7$ One useful identity of the dot product is

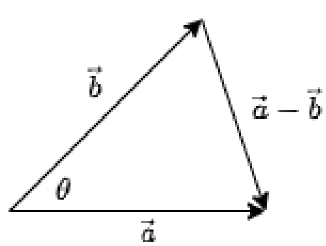
$$\vec{a} \cdot \vec{a} = \|\vec{a}\|^2$$

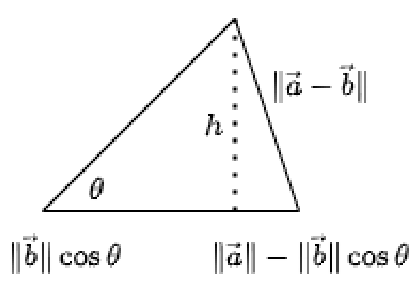
Another identity is

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos \theta$$

where θ is the angle between $\vec{a}$ and $\vec{b}$ Proof:

$$\|\vec{a} - \vec{b}\|^2 = (\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b}) = \vec{a} \cdot \vec{a} - \vec{a} \cdot \vec{b} - \vec{b} \cdot \vec{a} + \vec{b} \cdot \vec{b} = \|\vec{a}\|^2 - 2\vec{a} \cdot \vec{b} + \|\vec{b}\|^2$$





$$\|\vec{a} - \vec{b}\|^2 = h^2 + (\|\vec{a}\| - \|\vec{b}\| \cos \theta)^2$$
$$h = \|\vec{b}\| \sin \theta$$

$$\|\vec{a}-\vec{b}\|^2 = \|\vec{b}\|^2\sin^2\theta + \|\vec{a}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta + \|\vec{b}\|^2\cos^2\theta$$
$$= \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$$
$$\Rightarrow \|\vec{a}\|^2 - 2\vec{a}\cdot\vec{b} + \|\vec{b}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$$
$$\Rightarrow \vec{a}\cdot\vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$$

Ex: Find the angle between $\langle 1,2\rangle$ and $\langle 1,1\rangle$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{\|\vec{a}\|\|\vec{b}\|} = \frac{\langle 1,2\rangle\cdot\langle 1,1\rangle}{\sqrt{5}\sqrt{2}} = \frac{3}{\sqrt{10}}$$
$$\theta = \arccos\left(\frac{3}{\sqrt{10}}\right)$$

Note that if $\vec{a}\cdot\vec{b}=0$ it implies that the angle between them is 90° and that $\vec{a}\perp\vec{b}$. In general, a vector orthogonal to $\vec{a}$ can be defined as $\vec{a}^\perp = \pm\langle a_2, -a_1\rangle$

Proof:

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 = 0$$
$$a_1b_1 = -a_2b_2$$
$$\Rightarrow b_1 = a_2,\ b_2 = -a_1$$
$$\Rightarrow \vec{b} = \vec{a}^\perp = \pm\langle a_2, -a_1\rangle$$

Projections:

The projection of a vector $\vec{a}$ in the direction of $\vec{b}$ is denoted by $\text{proj}_{\vec{b}}\,\vec{a}$.

The magnitude of the projection is $\|\vec{a}\|\cos\theta$ so we can define the projection vector as

$$\text{proj}_{\vec{b}}\,\vec{a} = (\vec{a}\cdot\hat{b})\hat{b} = \frac{\vec{a}\cdot\vec{b}}{\|\vec{b}\|^2}\vec{b}$$

Ex: Find the projection of $\langle 2,3\rangle$ onto $\langle 2,1\rangle$

$$\vec{a} = \langle 2,3\rangle,\ \vec{b} = \langle 2,1\rangle$$
$$\text{proj}_{\vec{b}}\,\vec{a} = \frac{\langle 2,3\rangle\cdot\langle 2,1\rangle}{\|\langle 2,1\rangle\|^2}\langle 2,1\rangle = \frac{7}{5}\langle 2,1\rangle$$
$$= \left\langle \frac{14}{5}, \frac{7}{5}\right\rangle$$

1.2 Determinants and Cross Products

The 2x2 determinant is defined as

$$\det\begin{bmatrix} a_1 & a_2 \\ b_1 & b_2\end{bmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2\end{vmatrix} = a_1b_2 - a_2b_1$$

Geometrically, it represents the signed area of the parallelogram spanned by vectors $\vec{a}$, $\vec{b}$.

$$A_{parallelogram} = \left|\det\begin{bmatrix} -\vec{a}- \\ -\vec{b}-\end{bmatrix}\right|$$

If the determinant is equal to zero it implies the area is zero and it means that $\vec{a}$ and $\vec{b}$ are along the same line and are considered *colinear*.

Determinants in $\mathbb{R}^3$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix} = a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1$$

$$\begin{aligned} \text{Ex: } & \det \begin{bmatrix} 1 & 2 & 0 \\ 2 & 7 & 1 \\ 5 & 3 & 3 \end{bmatrix} \\ & = 1 \begin{vmatrix} 7 & 1 \\ 3 & 3 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 5 & 3 \end{vmatrix} + 0 \begin{vmatrix} 2 & 7 \\ 5 & 3 \end{vmatrix} \\ & = (21 - 3) - 2(6 - 5) \\ & = 16 \end{aligned}$$

Geometrically, the 3x3 determinant represents the area of the parellelapiped spanned by $\vec{a}$, $\vec{b}$, $\vec{c}$

$$A_{parallelapiped} = \det \begin{bmatrix} -\vec{a}- \\ -\vec{b}- \\ -\vec{c}- \end{bmatrix} = \vec{a} \cdot (\vec{b} \times \vec{c})$$

Cross Product:
The cross product is defined as

$$\vec{a} \times \vec{b} = \langle a_2b_3 - a_3b_2,\, a_3b_1 - a_1b_3,\, a_1b_2 - a_2b_1 \rangle$$

It can be more easily interpreted as

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

where $\hat{i} = \langle 1, 0, 0 \rangle$, $\hat{j} = \langle 0, 1, 0 \rangle$, and $\hat{k} = \langle 0, 0, 1 \rangle$
The result of $\vec{a} \times \vec{b}$ will be a vector orthogonal to both $\vec{a}$ and $\vec{b}$.
*Note that $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$$\begin{aligned} \text{Ex: } & \langle 1, 1, 1 \rangle \times \langle 1, 2, 3 \rangle \\ & = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{vmatrix} \\ & = \hat{i} \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} - \hat{j} \begin{vmatrix} 1 & 1 \\ 1 & 3 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} \\ & = \langle 1, -2, 1 \rangle \end{aligned}$$

A useful identity is that $\|\vec{a} \times \vec{b}\| = \|\vec{a}\| \|\vec{b}\| \sin\theta$. This also happens to be the area of a parallelogram in $\mathbb{R}^3$.

1.3 Lines and Planes

Parametric Representation of a line:
The parametric form of a plane can be thought of as a scaled direction vector plus a vector from the origin to some point on the line. If we let $\vec{x}$ represent our line then we have

$$L = \{\vec{x} : \vec{x} = s\vec{a} + \vec{p},\ s \in \mathbb{R}\}$$

where $\vec{a}$ is the direction vector, s is a scaling variable, and $\vec{p}$ is some point on the line.
Ex: Find the parametric equation of a line that passes through $(1, 2)$ and $(3, 3)$

$$\begin{aligned}
&\vec{a} = (3, 3) - (1, 2) = \langle 2, 1 \rangle \\
&\vec{p} = \langle 1, 2 \rangle \\
&\Rightarrow \vec{x} = \langle 2, 1 \rangle\, s + \langle 1, 2 \rangle
\end{aligned}$$

Parametric Form of a Plane:
A plane is a 2D object, meaning that we will require 2 parameters (s and t) to describe it. The parametric form for a plane is very similar to that of a line in $\mathbb{R}^2$, just with the addition of another scaling variable, t, and another direction vector $\vec{b}$.

$$P = \left\{\vec{x} : \vec{x} = s\vec{a} + t\vec{b} + \vec{p},\ s, t \in \mathbb{R}\right\}$$

Ex: The points $(1, 1, 1)$, $(2, 3, 7)$, and $(0, 2, 0)$ lie on a plane. Write the parametric form of the plane.

$$\begin{aligned}
&\vec{a} = (2, 3, 7) - (1, 1, 1) = \langle 1, 2, 6 \rangle \\
&\vec{b} = (0, 2, 0) - (1, 1, 1) = \langle -1, 1, -1 \rangle \\
&\vec{p} = \langle 1, 1, 1 \rangle \\
&\Rightarrow \vec{x} = s\,\langle 1, 2, 6 \rangle + t\,\langle -1, 1, -1 \rangle + \langle 1, 1, 1 \rangle
\end{aligned}$$

Equation Form of a line in R^2:
Another way to express the equation of a line is

$$n_1 x_1 + n_2 x_2 = d$$

where n_1, n_2 are components of the normal vector and d is some constant that is found by plugging in a point on the line.
This equation is derived the following way:

$$\begin{aligned}
&\text{let } \vec{n} = \vec{a}^{\perp} \text{ and } \vec{n} \perp \vec{a} \\
&\vec{a} \cdot \vec{n} = 0 \\
&\vec{x} = \vec{a} + \vec{p} \Rightarrow \vec{a} = \vec{x} - \vec{p} \\
&(\vec{x} - \vec{p}) \cdot \vec{n} = 0 \\
&\vec{x} \cdot \vec{n} - \vec{p} \cdot \vec{n} = 0 \\
&\vec{x} \cdot \vec{n} = \vec{p} \cdot \vec{n} \\
&\text{let } \vec{p} \cdot \vec{n} = d
\end{aligned}$$

$n_1x_1 + n_2x_2 = d$

*Note: a handy trick for calculating $\vec{n}$ in $\mathbb{R}^2$ is to use a similar method to the cross product:

$$\vec{n} = \pm\begin{vmatrix} \hat{i} & \hat{j} \\ a_1 & a_2 \end{vmatrix}$$

Ex: Write the equation of the line passing through $(1,2)$ and $(3,3)$.

$\vec{a} = (3,3) - (1,2) = \langle 2,1\rangle$
$\vec{n} = \langle -1,2\rangle$
$\vec{x}\cdot\vec{n} = d$
$-x_1 + 2x_2 = d$
plug in point $(1,2):\ -1+4 = d = 3$
$-x_1 + 2x_2 = 3$

Ex2: Determine if the point $(5,5)$ is on the line in the example above.

$-5 + 2(5) = 5 \neq 3 \therefore (5,5)$ is not on the line.

Equation Form of a Plane in $\mathbb{R}^3$:
The equation form of a plane follows the exact same pattern but contains one more term.

$$n_1x_1 + n_2x_2 + n_3x_3 = d$$

The only part that is slightly different is finding the normal vector $\vec{n}$. In this case, you must take the cross product of two direction vectors to find the normal (as it will be orthogonal to both and, therefore, the plane). $\vec{n} = \pm\vec{a}\times\vec{b}$
Ex: Find the equation form of the plane that contains $(1,1,1)$, $(2,3,7)$, and $(0,2,0)$

$\vec{a} = (2,3,7) - (1,1,1) = \langle 1,2,6\rangle$
$\vec{b} = (0,2,0) - (1,1,1) = \langle -1,1,-1\rangle$
$\vec{n}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 6 \\ -1 & 1 & -1 \end{vmatrix} = \langle -8,-5,3\rangle$
$-8x_1 - 5x_2 + 3x_3 = d$
plug in $(0,2,0):\ -5(2) = d = -10$
$-8x_1 - 5x_2 + 3x_3 = -10$

Equation Form of a Line in $\mathbb{R}^3$:
This consists of two equations for x. for a point to be on the line, it must satisfy both equations.

$$L : \begin{cases} n_1x_1 + n_2x_2 = d_1 \\ m_1x_1 + m_2x_2 = d_2 \end{cases}$$

where $\vec{n}$ and $\vec{m}$ are both 2D normal vectors. A handy trick is to set $\vec{n}$ and $\vec{m}$ to be

$$\vec{n} = \det\begin{bmatrix} \hat{i} & \hat{j} \\ a_1 & a_2 \end{bmatrix},\ \vec{m} = \det\begin{bmatrix} \hat{j} & \hat{k} \\ a_2 & a_3 \end{bmatrix}$$

Ex: Find the equation form of the line passing through $(0, 1, 5)$ and $(2, 2, 2)$

$$\vec{a} = (2, 2, 2) - (0, 1, 5) = \langle 2, 1, -3\rangle$$
$$\vec{n} = \det \begin{bmatrix} \hat{i} & \hat{j} \\ 2 & 1 \end{bmatrix} = \langle 1, -2, 0\rangle$$
$$\vec{m} = \det \begin{bmatrix} \hat{j} & \hat{k} \\ 1 & -3 \end{bmatrix} = \langle 0, -3, -1\rangle$$
$$L : \begin{cases} x_1 - 2x_2 = d_1 \\ -3x_2 - x_3 = d_2 \end{cases}$$
plug in $(0, 1, 5)$
$$\begin{cases} -2(1) = d_1 = -2 \\ -3(1) - 5 = d_2 = -8 \end{cases}$$
$$L : \begin{cases} x_1 - 2x_2 = -2 \\ -3x_2 - x_3 = -8 \end{cases}$$

1.4 Distances in Space

Note: This section uses some math in it that is not covered until later sections
Intersection of Lines in $\mathbb{R}^2$:
To solve, put both lines in equation form and solve the augmented matrix

$$\left[\begin{array}{cc|c} n_1x_1 & n_2x_2 & d_! \\ m_1x_1 & m_2x_2 & d_2 \end{array}\right]$$

Alternatively, this can also be solved using Pre-Calculus methods. The calculations work out similarly in either case.

Intersection of Planes in $\mathbb{R}^3$:
The solution of two intersecting planes will generally be a line with direction vector $\vec{a} = \pm\vec{n} \times \vec{m}$
This line can be found by solving the following matrix in terms of a free variable

$$\left[\begin{array}{ccc|c} n_1x_1 & n_2x_2 & n_3x_3 & d_1 \\ m_1x_1 & m_2x_2 & m_3x_3 & d_2 \end{array}\right]$$

Ex: Find the intersection of the planes $x_1 + x_2 + x_3 = 2$ and $x_1 + 2x_2 + 3x_3 = -1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 1 & 2 & 3 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & -3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & 2 & -3 \end{array}\right]$$
$$x_3 = t \Rightarrow x_1 = 5 + t,\ x_2 = -3 - 2t$$
$$\vec{x} = \langle 1, -2, 1\rangle\, t + \langle 5, -3, 0\rangle$$

Similarly, we could take $\vec{a} = \vec{n} \times \vec{m} = \langle 1, 1, 1\rangle \times 1, \vec{2}, 3 = \langle 1, -2, 1\rangle$ to get the direction vector and then find some point that exists on both planes, giving the overall equation.

Intersection between a line and plane:

1. Set up with $L : \vec{x} = t\vec{a} + \vec{p}$ and $P : n_1x_1 + n_2x_2 + n_3x_3 = d$
2. Plug the line equation into the plane equation in place of $\vec{x}$
3. Solve for t
4. Plug t back into the line equation to calculate $\vec{x}$

*This same method works for finding the intersection of 2 lines.
Ex: Find the intersection of $x_1 + y_1 + 2x_3 = 5$ and the line passing through $(1, 0, 0)$ and $(1, 2, 3)$

$$\vec{a} = \langle 0, 2, 3\rangle\,,\ \vec{p} = \langle 1, 0, 0\rangle$$
$$\vec{x} = \langle 0, 2, 3\rangle\, t + \langle 1, 0, 0\rangle = \langle 1, 2t, 3t\rangle$$
$$(1) + (2t) + 2(3t) = 5$$
$$8t = 4 \Rightarrow t = \frac{1}{2}$$
$$\vec{x} = \left\langle 1, 2\left(\frac{1}{2}\right), 3\left(\frac{1}{2}\right)\right\rangle = (1, 1, 1.5)$$

Distance of an object from a hyperplane:
Def: A hyperplane is defined as an object in Euclidean space that separates it in two halves. For example, a hyperplane in $\mathbb{R}^2$ is a line and a hyperplane in $\mathbb{R}^3$ is a plane.

1. Let $\vec{p}$ be some point that lies on the hyperplane and let $\vec{q}$ be some point on the object you are trying to find the distance from.
2. Compute a vector $\vec{pq}$ (order doesn't matter)
3. Find the normal vector to the hyperplane, $\vec{n}$
4. The distance will be $|\operatorname{proj}_{\vec{n}}(\vec{pq})|$

General Method of Finding Distances:

1. Set up both objects in parametric form
2. Set the distance, d to be $d = \|\vec{x}_2 - \vec{x}_1\|$
3. Set $\nabla d = 0$ and solve the system of equations for t_1, t_2 and so on. (note that the denominator of the square root derivative can be ignored)
4. Plug in the values for $t_1, t_2, \ldots$ to get a value for $\vec{x}_1$ and $\vec{x}_2$ to calculate d.

2 Linear Systems

2.1 Linear Combinations

A combination of vectors $s_1\vec{a}_1$ and $s_2\vec{a}_2$, s_1, $s_2 \in \mathbb{R}$ is called a *linear combination* of $\{\vec{a}_1, \vec{a}_2\}$
The set of all linear combinations of a set is called the *span* of the set of vectors.

Ex: span $\{\langle 1,0,0\rangle, \langle 0,1,0\rangle, \langle 0,0,1\rangle\} = \mathbb{R}^3$
Ex2: span $\{\langle 1,0,0\rangle, \langle 0,1,0\rangle\} = \mathbb{R}^2$
Ex3: If $\vec{a} = \langle 2,3\rangle$ and $\vec{b} = \langle 1,2\rangle$, is $(1,1)$ in the span of $\left\{\vec{a}, \vec{b}\right\}$? If so, find the linear combination.

Require s and t such that $s\vec{a} + t\vec{b} = \langle 1,1\rangle$

$s\langle 2,3\rangle + t\langle 1,2\rangle = \langle 1,1\rangle$

$\langle 2s+t,\, 3s+2t\rangle = \langle 1,1\rangle$

$$\begin{cases} 2s+t=1 \\ 3s+2t=1 \end{cases} \Rightarrow s=1,\, t=-1$$

$\rightarrow$ any point, (c_1, c_2) can be written as a linear combination of $\vec{a}$ and $\vec{b}$

$\therefore$ span $\left\{\vec{a}, \vec{b}\right\} = \mathbb{R}^2$

A collection of vectors is called *linearly dependent* if some nontrivial (not all 0) combination of equal zero. Otherwise, it is called *linearly independent*
$\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly dependent if

$$\det \begin{bmatrix} -\vec{a}- \\ -\vec{b}- \\ -\vec{c}- \end{bmatrix} = 0$$

or if

$$\begin{bmatrix} | & | & | \\ \vec{a} & \vec{b} & \vec{c} \\ | & | & | \end{bmatrix} \begin{bmatrix} | \\ \vec{0} \\ | \end{bmatrix}$$

has rank $< n$. i.e. no unique solution
Ex: Is $\langle 1,0,1\rangle, \langle 1,1,0\rangle, \langle 0,1,1\rangle$ linearly independent?

$$\det \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 - 0 + 1 = 2 \neq 0$$

$\therefore$ linearly independent

A collection of n linearly dependent vectors in $\mathbb{R}^n$ dimensional space is called a *basis*. So if $\vec{a}$, $\vec{b}$, $\vec{c}$ are linearly independent they form a basis and any vector, $\vec{x}$, can be formed from a linear combination of $\vec{a}$, $\vec{b}$, $\vec{c}$.

$$\begin{bmatrix} | & | & | \\ \vec{a} & \vec{b} & \vec{c} \\ | & | & | \end{bmatrix} \begin{bmatrix} | \\ \vec{k} \\ | \end{bmatrix} = \begin{bmatrix} | \\ \vec{x} \\ | \end{bmatrix}$$

2.2 Gaussian Elimination

Say we have a system of 3 equations and 3 unknowns. We can use an augmented matrix to solve this system.

$$\text{Ex: } \begin{cases} x_2 + x_3 = 1 \\ x_1 + x_2 = 2 \\ x_1 + x_2 + x_3 = 3 \end{cases} = \left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right]$$

Once in augmented matrix form on the right, we can perform the following elementary row operations:

- Multiply a row by a non-zero number
- Add a multiple of one row to another row
- Interchange two rows

Using these operations we can perform Gaussian elimination to put the matrix into echelon form and come to our solution.
Gaussian Elimination Method:

1. kill off first column so that there is only one nonzero number remaining
2. Use a row with a 0 first entry to kill off the second column
3. Continue the process until you are able to solve for a variable
4. Use the variable you solved for to work backwards and solve for the rest

Ex: $\left[\begin{array}{ccc|c} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$

$$x_3 = 1$$
$$x_2 = 1 - x_3 = 1 - 1 = 0$$
$$x_1 = 2 - x_2 = 2$$
$$\vec{x} = \langle 2, 0, 1 \rangle$$

2.3 Solution Spaces

Not all systems will have a unique solution. Some may have no solutions and some may have infinite solutions

Ex2: $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$

if you look at the last row, it states that $0 = 1$. This implies that there is no solution to the system

Ex3: $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -2 & -2 & 1 \end{array}\right]$

In this example, we have no equation to determine x_3 so we call it a free variable and assign it a variable value t instead of a number.

$$x_3 = t$$
$$-x_2 - 2x_3 = 1 \Rightarrow x_2 = -1 - 2x_3 = -1 - 2t$$
$$x_1 + 2x_2 + 3x_3 = 0 \Rightarrow x_1 = -2x_2 - 3x_3 = -2(-1 - 2t) - 3t = 2 + t$$

$\vec{x} = \langle 2 + t, -1 - 2t, t \rangle$

Notice, the solution space is also the equation of a line.
A handy way to imagine solution spaces is through lines and planes. In a 3x3 matrix, each row represents a plane and the solution is the intersection of the three planes. Similarly, if two rows are linearly dependent, we will effectively only have two planes and the solution space will be the intersection of these two planes which is a line.
One way to determine the number of solutions a system has is using rank.
The *rank* of a matrix is the number of nonzero rows of the matrix in echelon form.
r represents the rank of the matrix
r_A represents the rank of the augmented matrix (inclusive of the rightmost column)
If we consider a matrix to be of size $m \times n$ then m is the number of rows and n is the number of columns. (n is also the number of variables).

- If $r_A > r$, there are no solutions (irrational argument)
- If $r_A = r = n$, there will be a unique solution
- If $r_A = r < n$, there will be infinite solutions

A special type of augmented matrix comes from homogeneous systems. These have 0 for all the constant terms and the matrix will either have one solution ($\vec{x} = 0$) or infinite solutions.

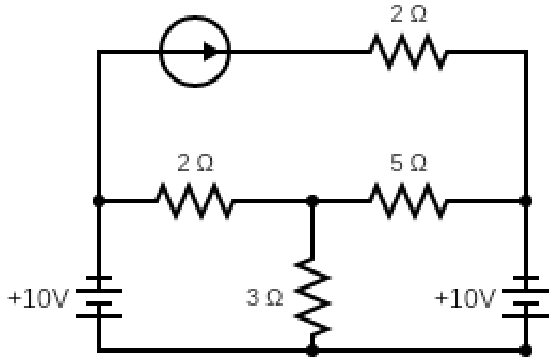
$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & 0 \\ a_{21} & a_{22} & a_{23} & 0 \\ a_{31} & a_{32} & a_{33} & 0 \end{array}\right]$$

2.4 Resistor Networks

Mesh Current Method:

1. Determine variables: the current in every loop (I_1, I_2, $\ldots I_n$) and the voltage drop across the current sources (E_1, E_2 $\ldots E_n$)
2. Set up equations so that the sum of the voltage in each loop is zero and equations that relate the specific currents of the current source
3. Set up in a matrix and solve for the unknowns

*The current must be travelling in the same direction in each loop (i.e. arbitrarily set all clockwise)
*We consider the current specific to the loop we are considering to be positive
Ex: If the current source is $4A$, find the voltage drop across all resistors and the current source



Loop 1: $-E = 2I_1 = 5(I_1 - I_3) + 2(I_1 - I_2) = 0$
Loop 2: $10 + 2(I_2 - I_1) + 3(I_2 - I_3) = 0$
Loop 3: $5(I_3 - I_1) - 10 + 3(I_3 - I_2) = 0$
Current Source: $I_1 = 4$

$$\begin{cases} 2I_2 + 5I_3 + E = 36 \\ 5I_2 - 3I_3 = -2 \\ -3I_2 + 8I_3 = 30 \end{cases}$$

$$\left[\begin{array}{ccc|c} 2 & 5 & 1 & 36 \\ 5 & -3 & 0 & -2 \\ -3 & 8 & 0 & 30 \end{array}\right] \rightsquigarrow \begin{cases} I_1 = 4A \\ I_2 \approx 2.3871A \\ I_3 \approx 4.6452A \\ E = 8V \end{cases}$$

$V_{R_2} = 2|I_1 - I_2| \approx 3.2V$
$V_{R_5} = 5|I_1 - I_3| \approx 3.2V$
$V_{R_3} = 3|I_2 - I_3| \approx 6.8V$

3 Matrices

3.1 Matrix Operations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \\ a_{m1} & a_{m2} & & a_{mn} \end{bmatrix} = \left[a_{ij}\right]_{m\times n}$$

*m represents the number of rows and n represents the number of columns. ij represents the current subscripts.
For a matrix A to multiply a matrix B, the number of columns of A must match the number of rows of B. Ex: $A_{4\times 2}B_{2\times 3}$ is allowed because the bottom middle numbers are both 2.

The method for matrix multiplication, AB, is to dot the rows of A with the columns of B

$$\begin{aligned} \text{Ex: } & \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 5 \\ 2 & 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2+2 & 0+3 & 2\cdot 5+1 \\ 1+2\cdot 2 & 0+2\cdot 3 & 5+2 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 3 & 11 \\ 5 & 6 & 7 \end{bmatrix} \end{aligned}$$

*note that in general, $AB \neq BA$
A matrix of particular interest is the identity matrix. Multiplying by this matrix is the same as multiplying by 1.

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & 1 \end{bmatrix}$$

Transpose of a Matrix:
The transpose is an operation that makes the rows become columns and the columns become rows and is expressed as A^T.

$$\text{Ex: if } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \text{ then } A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$$

If $A = A^T$ then we consider A to be a symmetric matrix.
Properties of the transpose:

$$\begin{aligned} &\left(A^T\right)^T = a \\ &(A+B)^T = A^T + B^T \\ &(sA)^T = sA^T \\ &(AB)^T = B^T A^T \end{aligned}$$

3.2 Linear Transformations

We can express vectors as matrices: A $m \times 1$ matrix is a column vector and a $1 \times n$ matrix is a row vector.
In the case of column vectors, we can multiply them by a matrix and get out another vector.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} a_{11}x_1 + a_{12}x_2 \\ a_{21}x_1 + a_{22}x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$A\vec{x} = \vec{b}$$

We can think of this matrix A as some function applied to $\vec{x}$. $\vec{f}(\vec{x}) = A\vec{x} = \vec{b}$
This is represented as

$$T(\vec{x}) = A\vec{x} = \vec{b}$$

and has the properties that

$$T(\vec{x}+\vec{y}) = T(\vec{x}) + T(\vec{y}) \text{ and } T(s\vec{x}) = sT(\vec{x})$$

We can take advantage of these identities and express the transformation matrix in the form

$$A = \begin{bmatrix} | & | & & | \\ T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \\ | & | & & | \end{bmatrix}$$

where $\vec{e}_1 = \hat{i}$, $\vec{e}_2 = \hat{j}$, ...

Ex: $T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$ and $T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix}$. Find $T\left(\begin{bmatrix}3\\2\end{bmatrix}\right)$

$T(\vec{e}_1) = T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\3\end{bmatrix}$

$T(\vec{e}_2) = T\left(\begin{bmatrix}1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\1\end{bmatrix} - \begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}0\\-2\end{bmatrix}$

$\Rightarrow A = \begin{bmatrix}2 & 0\\3 & -2\end{bmatrix}$

$T\left(\begin{bmatrix}3\\2\end{bmatrix}\right) = \begin{bmatrix}2 & 0\\3 & -2\end{bmatrix}\begin{bmatrix}3\\2\end{bmatrix} = \begin{bmatrix}6\\5\end{bmatrix}$

If we have a composition of transformations, $S(\vec{x}) = B\vec{x}$ and $T(\vec{x}) = A\vec{x}$, then $S(T(\vec{x})) = BA\vec{x}$.
2D Rotation Matrix:

$$\text{Rot}_\theta(\vec{x}) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

Ex: $\text{Rot}_{\pi/2} = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} \rightarrow \text{Rot}_{\pi/2}\begin{bmatrix}a_1\\a_2\end{bmatrix} = \begin{bmatrix}-a_2\\a_1\end{bmatrix} = \vec{a}^\perp$

2D Projection Matrix:

$$\text{proj}_{\hat{a}}(\vec{x}) = \begin{bmatrix}a_1^2 & a_1a_2\\a_1a_2 & a_2^2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

where $\hat{a}$ is the unit vector of the line that the vector $\vec{x}$ is projected onto. Proof:

$$\text{proj}_{\hat{a}}(\vec{x}) = (\vec{x}\cdot\hat{a})\hat{a} = (x_1a_1 + x_2a_2)\begin{bmatrix}a_1\\a_2\end{bmatrix} = \begin{bmatrix}(x_1a_1+x_2a_2)a_1\\(x_1a_1+x_2a_2)a_2\end{bmatrix} = \begin{bmatrix}x_1a_1^2 + x_2a_1a_2\\x_1a_1a_2 + x_2a_2^2\end{bmatrix}$$
$$= \begin{bmatrix}a_1^2 & a_1a_2\\a_1a_2 & a_2^2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

We can also express the line of projection as the angle of a line.
If $\hat{a} = \langle\cos\theta, \sin\theta\rangle$, $\theta \in [0, 2\pi)$ we can then denote $\text{proj}_{\hat{a}} = \text{proj}_\theta$ where

$$\text{proj}_\theta = \begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \cos\theta\sin\theta & \sin^2\theta\end{bmatrix}$$

Ex: Find the matrix that represents the projection onto the line $y = x$.

$$\vec{a} = \langle 1, 1 \rangle \Rightarrow \|\vec{a}\| = \sqrt{2}$$
$$\hat{a} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$
$$\text{proj}_{\hat{a}} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

2D Reflection Matrix:
Ref_θ is the reflection across the line making an angle θ with the x-axis.

$$\text{Ref}_\theta(\vec{x}) = 2\,\text{proj}_\theta(\vec{x}) - \vec{x} = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

3.3 Inverse of a Matrix

Consider a linear system $ax = b$. Its solution will be $x = a^{-1}b$. The same sort of thing can be done with matrices and we can take the inverse of a matrix.
One way to do this is using a super-augmented matrix.

$$[A|I] \to [I|A]$$

Also, it is important to note that A^{-1} only exists if $\det A \neq 0$.

Ex: Find the inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 3 & 8 & -1 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & 2 & -3 & 1 \end{array}\right]$$
$$\to \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & -\frac{3}{2} & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 0 & \frac{3}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & -3 & 4 & -1 \\ 0 & 0 & 1 & 1 & -\frac{3}{2} & \frac{1}{2} \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 1 & 0 & -3 & 4 & 1 \\ 0 & 0 & 1 & 1 & -\frac{3}{2} & \frac{1}{2} \end{array}\right]$$
$$A^{-1} = \begin{bmatrix} 3 & -\frac{5}{2} & \frac{1}{2} \\ -3 & 4 & 1 \\ 1 & -\frac{3}{2} & \frac{1}{2} \end{bmatrix}$$

Another way to find the inverse is to use the method

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)}$$

This method is more complex but can be quicker in some cases. The adjoint of a matrix is calculated by finding the cofactors of each entry and then taking the transpose of the cofactor matrix.

$$\text{adj}(A) = C^T = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix}$$

where

$$C_{ij} = (-1)^{i+j} M_{ij}$$

M_{ij} is the minor which is defined to be the determinant of the entries not in the row i and column j.

Ex: The minor of cell 32 in a 3x3 matrix, $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is det $\begin{bmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{bmatrix}$

The cofactors are calculated similarly to how determinants are calculated.
This method leads to a handy formula for 2x2 matrices

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$C = \begin{bmatrix} a_{22} & -a_{21} \\ -a_{12} & a_{11} \end{bmatrix}$$

$$\text{adj}(A) = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$A^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

Larger matrices don't have a nice formula but still use the same method.

Ex: $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 3 & 1 \\ 4 & 2 & 0 \end{bmatrix}$

$$C_{11} = \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} = 2$$

$$C_{12} = -\begin{vmatrix} 0 & 1 \\ 4 & 0 \end{vmatrix} = 4$$

$$C_{13} = \begin{vmatrix} 0 & 3 \\ 4 & 2 \end{vmatrix} = -12$$

$$C_{21} = -\begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} = 2$$

$$C_{22} = \begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} = -4$$

$$C_{23} = -\begin{vmatrix} 1 & 3 \\ 4 & 2 \end{vmatrix} = 10$$

$$C_{31} = \begin{vmatrix} 3 & 1 \\ 3 & 1 \end{vmatrix} = 0$$

$$C_{32} = -\begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1$$

$$C_{33} = \begin{vmatrix} 1 & 3 \\ 0 & 3 \end{vmatrix} = 3$$

$$C = \begin{bmatrix} -2 & 4 & -12 \\ 2 & -4 & 10 \\ 0 & -1 & 3 \end{bmatrix}$$

$$\operatorname{adj}(A) = C^T = \begin{bmatrix} -2 & 2 & 0 \\ 4 & -4 & -1 \\ -12 & 10 & 3 \end{bmatrix}$$

$$\det(A) = \begin{vmatrix} 3 & 1 \\ 2 & 0 \end{vmatrix} - 3\begin{vmatrix} 0 & 1 \\ 4 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 3 \\ 4 & 2 \end{vmatrix} = -2 + 12 - 12 = -2$$

$$A^{-1} = \frac{\operatorname{adj}(A)}{\det(A)} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & \frac{1}{2} \\ 6 & -5 & -\frac{3}{2} \end{bmatrix}$$

One use for matrix inverses is to give an alternative way to solve systems of equations.

Ex: $\begin{cases} x + 3y = 5 \\ 2x + 4y = 6 \end{cases}$

$$AX = B$$

$$A^{-1}AX = A^{-1}B$$

$$X = A^{-1}B$$

$$A^{-1} = -\frac{1}{2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

$$X = -\frac{1}{2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

Inverse Properties:

$$AA^{-1} = A^{-1}A = I$$

$$\left(A^{-1}\right)^{-1} = A$$

$$(kA)^{-1} = \frac{1}{k}A^{-1}$$

$$(AB)^{-1} = B^{-1}A^{-1}$$

$$\left(A^{-1}\right)^T = \left(A^T\right)^{-1}$$

Ex: Simplify $(AB^TB)^{-1}(BB^TBA^T)^T(A^TB^{-1})^T$

$$(B^TB)^{-1}A^{-1}(BA^T)^T(BB^T)^T(B^{-1})^TA$$

$$B^{-1}(B^T)^{-1}A^{-1}AB^TBB^T(B^{-1})^TA$$

$$B^{-1}(B^T)^{-1}B^TBA$$

$$B^{-1}BA$$

$$A$$

3.4 Determinants

While we know how to calculate determinants already, there are some tricks that make calculating them easier, especially larger determinants.

If A is an upper or lower triangular matrix then $\det(A)$ is the product of the diagonals.
Also, $\det(A) = \det(A^T)$
We can also perform row operations on determinants to simplify them as much as possible.

- Swap Rows: $\det(B) = -\det(A)$
- Multiply a Row by a Constant, k: $\det(B) = k\det(A)$
- Add a Multiple of One Row to Another: $\det(B) = \det(A)$

Ex: $\det \begin{bmatrix} 1 & 2 & -2 & -7 \\ 1 & 2 & -1 & -5 \\ 0 & 3 & 0 & -3 \\ -1 & 4 & 1 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1 & 2 & -2 & -7 \\ 1 & 2 & -1 & -5 \\ 0 & 3 & 0 & -3 \\ -1 & 4 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -2 & -7 \\ 0 & 0 & 1 & 2 \\ 0 & 3 & 0 & -3 \\ -1 & 4 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -2 & -7 \\ 0 & 0 & 1 & 2 \\ 0 & 3 & 0 & -3 \\ 0 & 6 & -1 & -6 \end{vmatrix} = \begin{vmatrix} 1 & 2 & -2 & -7 \\ 0 & 0 & 1 & 2 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & -1 & 0 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 2 & -2 & -7 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & -2 & -7 \\ 0 & 3 & 0 & -3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 2 \end{vmatrix} = -(1)(3)(1)(2) = -6$$

Some properties of determinants are as follows:

$$\det(A)\det(A^T)$$
$$\det(A^{-1}) = \frac{1}{\det(A)}$$
$$\det(A^x) = (\det(A))^x$$
$$\det(AB) = \det(A)\det(B)$$
$$\det(kA) = k^n \det(A) \text{ where } n \text{ is the matrix size}$$

4 Eigen-Analysis

4.1 Complex Numbers

Complex numbers arise from the roots of polynomials.
Ex: $x^2 + 1 = 0 \Rightarrow x^2 = -1 \Rightarrow x = \pm\sqrt{-1}$. This polynomial has no real roots, however, we can introduce an imaginary number i such that $i^2 = -1$. Then we will have the solution $x = \pm i$
We can introduce *complex numbers* which are numbers in the form $z = x + iy$, where x is the real part of z, $\Re(z)$, and y is the imaginary part of z, $\Im(z)$. These numbers can also be expressed in vector notation along the complex plane.
Complex Arithmetic:
Addition, subtraction, and multiplication works the same, just with the addition of the fact $i^2 = -1$.
For division, we require what is called the conjugate.

The conjugate of a complex number is the same number, just with the sign of the imaginary component flipped.

$$\overline{z} = x - yi$$

where $\overline{z}$ is the conjugate of z.
Similarly to vectors, we can also define the modulus (length) of a complex number

$$|z|^2 = x^2 + y^2 = z \cdot \overline{z}$$

Using this, we can define division of a complex number and also define the real and imaginary components of a complex number.

$$\frac{u}{z} = \frac{s+it}{x+iy} = \frac{(s+it)(x-iy)}{(x+iy)(x-iy)} = \frac{u\overline{z}}{x^2+y^2} = \frac{u\overline{z}}{|z|^2}$$

Another way to represent complex numbers is through polar form. To use this we must first introduce Euler's identity:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

This then helps us with the equation for polar form

$$z = a + ib = |z|(\cos\theta + i\sin\theta) = |z|e^{i\theta}$$

Ex: Represent $1 + i$ in polar coordinates

$$|z| = \sqrt{2}$$
$$\theta = \arctan\frac{b}{a} = \arctan 1 = \frac{\pi}{4}$$
$$z = \sqrt{2}e^{\frac{\pi i}{4}+2n},\ n \in \mathbb{R}$$

Ex2: Find the roots of $x^3 = 2$

$$x = |x|e^{i\theta}$$
$$|x| = \sqrt[3]{2}$$
$$x^3 = 2 = 2e^{3i\theta} \Rightarrow 1 = e^{3i\theta} = \cos(3\theta) + i\sin(3\theta)$$
$$\Rightarrow \cos(3\theta) = 1$$
$$3\theta = 2\pi n,\ n \in \mathbb{R}$$
$$\theta = \frac{3\pi}{2}n,\ n \in \mathbb{R}$$
$$x = \begin{cases} \sqrt[3]{2} \\ \sqrt[3]{2}e^{i\frac{2\pi}{3}} \\ \sqrt[3]{2}e^{i\frac{4\pi}{3}} \end{cases}$$

4.2 Eigenvalues and Eigenvectors

If we think of a matrix A as a linear transformation, an eigenvector is a nonzero vector that's direction is unchanged by the transformation. This can be expressed as

$$A\vec{v} = \lambda\vec{v}$$

or more generally as,

$$A^n\vec{v} = \lambda^n\vec{v}$$

The value λ is called the *eigenvalue* that corresponds to the given eigenvector.
To solve for these eigenvalues, we can do the following,

$$\begin{aligned}
&A\vec{v} = \lambda\vec{v} \\
&A\vec{v} = \lambda I\vec{v} \\
&A\vec{v} - \lambda I\vec{v} = 0 \\
&(A - \lambda I)\vec{v} = 0 \\
&\Rightarrow \det(A - \lambda I) = 0
\end{aligned}$$

To then find the eigenvectors, we can solve the homogeneous system for each λ_i formed by $(A - \lambda_i I)\vec{v} = 0$
This will give a solution space of a line for $\vec{v}_i$. Because we only care about the direction of the eigenvectors, we can take the simplest eigenvector.

Ex: Compute eigenvectors for the matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (2-\lambda)(2-\lambda) - 1 = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$$

$\lambda_1 = 1,\ \lambda_2 = 3$

$$\lambda_1 :\ A - \lambda_1 I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow x_1 + x_2 = 0 \Rightarrow x_1 = -x_2 \Rightarrow \begin{cases} x_1 = -t \\ x_2 = t \end{cases}$$

$\vec{v}_1 = [1, -1]^T$

$$\lambda_2 :\ A - \lambda_2 I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow -x_1 + x_2 = 0 \Rightarrow x_1 = x_2 \Rightarrow \begin{cases} x_1 = t \\ x_2 = t \end{cases}$$

$\vec{v}_2 = [1, 1]^T$

The eigenvectors of a matrix also form a basis, meaning that we can express any vector in that space as a linear combination of eigenvectors. This makes eigen-analysis very helpful for computing large powers of matrices.

Using the matrix from the above example, compute $A^{10}\begin{bmatrix} 2 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ -1 & 1 & 3 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 2 & 5 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & \frac{5}{2} \end{array}\right] \to \left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{5}{2} \end{array}\right]$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix}2\\3\end{bmatrix} = -\frac{1}{2}\vec{v}_1 + \frac{5}{2}\vec{v}_2$$

$$A^{10}\begin{bmatrix}2\\3\end{bmatrix} = A^{10}\left(-\frac{1}{2}\vec{v}_1 + \frac{5}{2}\vec{v}_2\right) = A^{10}\left(-\frac{1}{2}\vec{v}_1\right) + A^{10}\left(\frac{5}{2}\vec{v}_2\right)$$

recall $A^n\vec{v} = \lambda^n\vec{v}$

$$A^{10}\begin{bmatrix}2\\3\end{bmatrix} = -\frac{1}{2}(1^{10}\vec{v}_1) + \frac{5}{2}(3^{10}\vec{v}_2) = \begin{bmatrix}-\frac{1}{2} + \frac{5}{2}\cdot 3^{10}\\ \frac{1}{2} + \frac{5}{2}\cdot 3^{10}\end{bmatrix} = \begin{bmatrix}147622\\147623\end{bmatrix}$$

A handy method for finding/checking eigenvalues is to use the trace and determinant of the matrix. (Trace is the sum of the diagonal elements)

$$\mathrm{tr}(A) = \sum_{i=1}^{n}\lambda_i$$

$$\det(A) = \prod_{i=1}^{n}\lambda_i$$

4.3 Complex Eigen-Analysis

If A is a real matrix with complex eigenvalues then the complex eigenvalues and eigenvectors will always come in conjugate pairs.
This will also lead to the coefficients being conjugate pairs and allows us to express these conjugate pairs as,

$$2\Re(c_i\lambda_i^n\vec{v}_i)$$

We can do this because the imaginary components in a real matrix will always cancel.

Ex: for the matrix $A = \begin{bmatrix}0 & 0 & 3\\1 & 0 & -1\\0 & 1 & 3\end{bmatrix}$ compute $A^{13}\begin{bmatrix}1\\0\\0\end{bmatrix}$

$$\det(A-\lambda I) = \det\begin{bmatrix}-\lambda & 0 & 3\\1 & -\lambda & -1\\0 & 1 & 3-\lambda\end{bmatrix} = -\lambda\begin{vmatrix}-\lambda & -1\\1 & 3-\lambda\end{vmatrix} + 3\begin{vmatrix}1 & -\lambda\\0 & 1\end{vmatrix}$$

$$= -\lambda(-\lambda(3-\lambda)+1)+3 = -\lambda^3+3\lambda^2-\lambda+3 = 0$$

$$\Rightarrow (\lambda-3)(-\lambda^2-1) = 0 \Rightarrow \begin{cases}\lambda_1 = 3\\ \lambda_2 = i\\ \lambda_3 = -i\end{cases}$$

$$\lambda_1 : A-3I = \begin{bmatrix}-3 & 0 & 3\\1 & -3 & -1\\0 & 1 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1\\0 & -3 & 0\\0 & 1 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & -1\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}$$

$$\Rightarrow \vec{v}_1 = \begin{bmatrix}1 & 0 & 1\end{bmatrix}^T$$

$$\lambda_2 : A-iI = \begin{bmatrix}-i & 0 & 3\\1 & -i & -1\\0 & 1 & 3-i\end{bmatrix} \to \begin{bmatrix}1 & -i & -1\\0 & 1 & 3-i\\0 & 0 & 0\end{bmatrix}$$

$$\Rightarrow \vec{v}_2 = \begin{bmatrix} -3i & -3+i & 1 \end{bmatrix}^T$$
$$\therefore\ \vec{v}_3 = \begin{bmatrix} 3i & -3-i & 1 \end{bmatrix}^T$$
$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3 = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -3i \\ -3+i \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 3i \\ -3-i \\ 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & -3i & 3i \\ 0 & -3+i & -3-i \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$
$$\left[\begin{array}{ccc|c} 1 & -3i & 3i & 1 \\ 0 & -3+i & -3-i & 0 \\ 1 & 1 & 1 & 0 \end{array}\right] \rightsquigarrow \vec{c} = \begin{bmatrix} 1 \\ -\frac{1}{20} + \frac{3}{20}i \\ \frac{1}{20} - \frac{3}{20}i \end{bmatrix}$$
$$A^{13} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = c_1\lambda_1^{13}\vec{v}_1 + c_2\lambda_2^{13}\vec{v}_2 + c_3\lambda_3^{13}\vec{v}_3 = c_1\lambda_1^{13}\vec{v}_1 + 2\Re(c_2\lambda_2^{13}\vec{v}_2)$$
$$\Re(c_2\lambda_2^{13}\vec{v}_2) = \Re\left(\left(-\frac{1}{20} + \frac{3}{20}i\right) \begin{bmatrix} -3i \\ -3+i \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -\frac{3}{20} \\ \frac{1}{2} \\ -\frac{3}{20} \end{bmatrix}$$
$$A^{13} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 3^{13} + 2 \begin{bmatrix} -\frac{3}{20} \\ \frac{1}{2} \\ -\frac{3}{20} \end{bmatrix} = \begin{bmatrix} 3^{13} - \frac{3}{10} \\ 1 \\ 3^{13} - \frac{3}{10} \end{bmatrix}$$

*If A has repeated eigenvalues, it may or may not have a basis of eigenvectors. However, if A is symmetric ($A = A^T$), it always has a basis of eigenvectors, even if eigenvalues are repeated, and all eigenvalues are real.

4.4 Transition Matrices

Transition matrices, also known as random walks or probability matrices, are matrices that have every column sum to 1.
Take a 3x3 probability matrix. It will have 3 states, 1, 2, and 3 and there will be some probability that an object from one state moves to another state. The probability it goes from state i to state j is represented as P_{ij}. We can represent these probabilities in a matrix as shown

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}$$

We let the vector $\vec{x}^{(n)}$ be the probability that an object is in each state at the time n. So $x_1^{(0)}$ would be the probability it is in state 1 at time 0. Using $\vec{x}^{(0)}$ as the starting probability, we can find $\vec{x}^{(n)}$ through

$$\vec{x}^{(n)} = P^n\vec{x}^{(0)}$$

If $\lim_{n\to\infty} P^n\vec{x}_0 = \vec{p}$ for every $\vec{x}_0$, we say the transition matrix has an *equilibrium probability vector*, $\vec{p}$. A matrix will only contain $\vec{p}$ if it has an eigenvalue $\lambda = 1$. In this case, $\vec{p}$ can be found by scaling the eigenvector associated with $\lambda = 1$ such that the sum of its components is 1.

Ex: Find $\vec{p}$ for the random walk $P = \begin{bmatrix} \frac{1}{2} & \frac{2}{3} \\ \frac{1}{2} & \frac{1}{3} \end{bmatrix}$

$$\text{tr}(P) = \frac{5}{6} = \lambda_1 + \lambda_2$$

$$\det(P) = -\frac{1}{6} = \lambda_1 \lambda_2$$

$$\Rightarrow \lambda_1 = 1,\ \lambda_2 = -\frac{1}{6}$$

$$\lambda_1 = 1: \begin{bmatrix} -\frac{1}{2} & \frac{2}{3} \\ \frac{1}{2} & -\frac{2}{3} \end{bmatrix} \to \begin{bmatrix} -\frac{1}{2} & \frac{2}{3} \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} -3 & 4 \\ 0 & 0 \end{bmatrix} \to \vec{v}_1 = \begin{bmatrix} 4, 3 \end{bmatrix}^T$$

$$\sum \vec{v}_1 = 7 \Rightarrow \vec{p} = \frac{1}{7}\vec{v}_1$$

$$\therefore \vec{p} = \begin{bmatrix} \frac{4}{7} \\ \frac{3}{7} \end{bmatrix}$$

Trends of Transition Matrices:

- $|\lambda| \leq 1$ for all transition matrices
- Every transition matrix with all nonzero entries will contain $\lambda = 1$
- The term with $\lambda = 1$ will always tend to $\vec{p}$

4.5 Linear Systems of Differential Equations

These are differential equations in the form of $\frac{d\vec{x}}{dt} = A\vec{x}$ and can be solved with the following method:

1. Set $\frac{d\vec{x}}{dt} = A\vec{x}$ and find the matrix A
2. Find eigenvalues and eigenvectors of A
3. Write the general solution
4. Express the general solution as $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots$
5. If imaginary, express conjugate pairs as $c_1\vec{P}(t) + c_2\vec{Q}(t)$ where $\vec{P}(t) = \Re\left(e^{\alpha t}\vec{v}_1(\cos(\beta t) + i\sin(\beta t))\right)$ and $\vec{Q}(t) = \Im\left(e^{\alpha t}\vec{v}_1(\cos(\beta t) + i\sin(\beta t))\right)$
6. Use the initial conditions to solve for the coefficients, $\vec{c}$

Real Eigenvalues
This is the most general case where the number of eigenvalues matches the size of the matrix. In this case the general solution can be expressed as

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots$$

Ex: $\begin{cases} \frac{dx_1}{dt} = x_1 + x_3,\ x_1(0) = 1 \\ \frac{dx_2}{dt} = x_2,\ x_2(0) = 1 \\ \frac{dx_3}{dt} = x_1 + x_3,\ x_3(0) = 0 \end{cases}$

$$\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = A\vec{x} \Rightarrow A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},\ \vec{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Eigen-analysis gives

$$\begin{cases} \lambda_1 = 0,\ \vec{v}_1 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T \\ \lambda_2 = 1,\ \vec{v}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \\ \lambda_3 = 2,\ \vec{v}_3 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T \end{cases}$$

$$\vec{x}(t) = c_1 e^{0t} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \rightsquigarrow \vec{c} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

$$\vec{x}(t) = -\frac{1}{2} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + e^{t} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2} e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{2t} \\ e^{t} \\ -\frac{1}{2} + \frac{1}{2}e^{2t} \end{bmatrix}$$

Complex Eigenvalues

This case will have the same general solution but because of the imaginary components, it can be simplified using Euler's equation. We can express any conjugate pairs as $c_1\vec{P}(t) + c_2\vec{Q}(t)$ where

$$\vec{P}(t) = \Re\left\{e^{\alpha t}\vec{v}_1(\cos(\beta t) + i\sin(\beta t))\right\} \text{ and } \vec{Q}(t) = \Im\left\{e^{\alpha t}\vec{v}_1(\cos(\beta t) + i\sin(\beta t))\right\}$$

Ex: $\dfrac{d\vec{x}}{dt} = A\vec{x},\ A = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$

eigen-analysis gives

$$\begin{cases} \lambda_1 = 3,\ \vec{v}_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T \\ \lambda_2 = i,\ \vec{v}_2 = \begin{bmatrix} -3i & -3+i & 1 \end{bmatrix}^T \\ \lambda_3 = -i,\ \vec{v}_3 = \begin{bmatrix} 3i & -3-i & 1 \end{bmatrix}^T \end{cases}$$

Real part: $e^{\lambda_1 t}\vec{v}_1 = e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Imaginary part: $e^{\lambda_2 t}\vec{v}_2 = e^{it} \begin{bmatrix} -3i \\ -3+i \\ 1 \end{bmatrix} = (\cos(t) + i\sin(t)) \begin{bmatrix} -3i \\ -3+i \\ 1 \end{bmatrix}$

$$= \begin{bmatrix} 3\sin(t) - 3i\cos(t) \\ -3\cos(t) + i\cos(t) - 3i\sin(t) - \sin(t) \\ \cos(t) + i\sin(t) \end{bmatrix}$$

$$\vec{P}(t) = \Re \begin{bmatrix} 3\sin(t) - 3i\cos(t) \\ -3\cos(t) + i\cos(t) - 3i\sin(t) - \sin(t) \\ \cos(t) + i\sin(t) \end{bmatrix} = \begin{bmatrix} 3\sin(t) \\ -3\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix}$$

$$\vec{Q}(t) = \Im \begin{bmatrix} 3\sin(t) - 3i\cos(t) \\ -3\cos(t) + i\cos(t) - 3i\sin(t) - \sin(t) \\ \cos(t) + i\sin(t) \end{bmatrix} = \begin{bmatrix} -3\cos(t) \\ \cos(t) - 3\sin(t) \\ \sin(t) \end{bmatrix}$$

$$\vec{x}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3\sin(t) \\ -3\cos(t) - \sin(t) \\ \cos(t) \end{bmatrix} + c_3 \begin{bmatrix} -3\cos(t) \\ \cos(t) - 3\sin(t) \\ \sin(t) \end{bmatrix}$$

Repeated Eigenvalues

With repeated eigenvalues, we may not have enough eigenvectors to capture the solution space. We consider the size of the matrix to be called the *algebraic multiplicity* of the system and the number of eigenvectors the *geometric multiplicity* of the system.

In the case where the geometric multiplicity is the same as the algebraic multiplicity, the solution will follow the from the regular, non-repeated, case.

Ex: $\vec{x}' = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \vec{x}$

$$\lambda = 2$$

$$A - \lambda I = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\vec{x} = e^{2t} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

When the algebraic and geometric multiplicities are not equal, we can introduce generalized eigenvectors.

$\vec{v}_1$ is an eigenvector if $(A - \lambda I)\vec{v}_1 = 0$.

Similarly, $\vec{v}_2$ is a generalized eigenvector of $\vec{v}_1$ if $(A - \lambda I)\vec{v}_2 = \vec{v}_1$

In the most general sense, we can determine if $\vec{v}$ is a generalized eigenvector if

$$(A - \lambda I)^k \vec{v} \neq \vec{0} \text{ and } (A - \lambda I)^{k+1} \vec{v} = \vec{0}$$

Note that the choice of generalized eigenvectors will always be somewhat arbitrary, as there are infinitely many choices.

The general solution for multiple eigenvalues will resemble the form

$$\vec{x}(t) = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_1 t} \left(\vec{v}_k + t\vec{v}_{k-1} + \frac{t}{2}\vec{v}_{k-2} + \cdots + \frac{t^{k-2}}{(k-2)!}\vec{v}_2 + \frac{t^{k-1}}{(k-1)!}\vec{v}_1 \right)$$

Ex: $\vec{x}' = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix} \vec{x}$

$$\det A = 4,\ \operatorname{tr} A = 4 \Rightarrow \lambda = 2$$

$$\begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix} \rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}$$

$$\vec{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t} \left(t \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix} \right)$$

Ex2: $\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{x}$

$$\lambda = 0,\ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \rightarrow \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

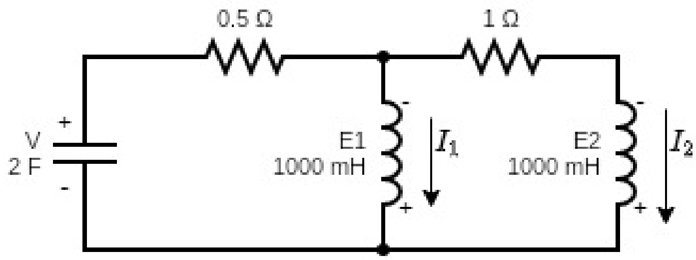
$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \left(\frac{t^2}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

4.6 LCR Circuits

Recall, $V_C = \frac{q}{C} \Rightarrow \frac{dV_C}{dt} = \pm\frac{i}{C}$ and $\frac{dI}{dt} = \pm\frac{E}{L}$
where V is the voltage of the capacitor, i is the current through the capacitor, E is the voltage of the inductor, and I is the current throuh the inductor. C and L are the capacitance and inductance.

1. Write voltage and current equations
2. Express i and E in terms of I and V
3. Write differential equations of form $\frac{dV}{dt} = \pm\frac{i}{C}$ and $\frac{dI}{dt} = \pm\frac{E}{L}$
4. Express the system of differential equations in matrix form
5. Solve differential equation for various $V(t)$ and $I(t)$

Ex:



Voltage equations

$0.5i_1 - E_1 - V = 0$

$i_2 - E_2 + E_1 = 0$

current equations

$I_1 = i_1 - i_2$

$I_2 = i_2$

writing expressions in terms of V, I_1, I_2

$i_2 = I_2$

$i_1 = I_1 + i_2 = I_1 + I_2$

$E_1 = 0.5i_1 - V = 0.5I_1 + 0.5I_2 - V$

$E_2 = i_2 + E_1 = 0.5I_1 + 1.5I_2 - V$

Differential equations for V, I_1, I_2

$$\frac{dV}{dt} = -\frac{i_1}{C} = -\frac{i_1}{2} = -\frac{(I_1 + I_2)}{2}$$

$$\frac{dI_1}{dt} = -\frac{E_1}{L} = -\frac{E_1}{1} = -0.5I_1 + 0.5I_2 - V$$

$$\frac{dI_2}{dt} = -\frac{E_2}{L} = -\frac{E_2}{1} = -0.5I_1 + 1.5I_2 - V$$

$$\frac{d}{dt}\begin{bmatrix} V \\ I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} V \\ I_1 \\ I_2 \end{bmatrix}$$

eigenvalues and eigenvectors

$\lambda_1 = \dfrac{-1+i}{2}$, $\vec{v}_1 = \begin{bmatrix} 1 & -i & 1 \end{bmatrix}^T$

$\lambda_2 = \dfrac{-1-i}{2}$, $\vec{v}_2 = \begin{bmatrix} 1 & i & 1 \end{bmatrix}^T$

$\lambda_3 = -1$, $\vec{v}_3 = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}^T$

general solution

$$\begin{bmatrix} V(t) \\ I_1(t) \\ I_2(t) \end{bmatrix} = c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2 + c_3 e^{\lambda_3 t} \vec{v}_3$$

$$\begin{bmatrix} V(t) \\ I_1(t) \\ I_2(t) \end{bmatrix} = c_1 e^{-t/2} \begin{bmatrix} \cos\left(\frac{t}{2}\right) \\ \sin\left(\frac{t}{2}\right) \\ \cos\left(\frac{t}{2}\right) \end{bmatrix} + c_2 e^{-t/2} \begin{bmatrix} \sin\left(\frac{t}{2}\right) \\ -\cos\left(\frac{t}{2}\right) \\ \sin\left(\frac{t}{2}\right) \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$\rightarrow$ $\vec{c}$ can be solved from initial conditions

This particular example would show damped oscillations in the circuit.