Math 255 Notes

Tyler Wilson

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Contents

1 Ordinary Differential Equations

1.1 First Order Differential Equations

These are differential equations of the form $F(x, y, y') = 0$

1.1.1 Integrals as Solutions

These are differential equations of the form $y' = f(x)$ or $y' = f(y)$

Ex: $y' = \cos x$
$$y = \int \cos x dx = \sin x + C$$

Ex2: $y' = e^{2y}$
$$y' = \frac{dy}{dx} = \frac{1}{dx/dy}$$
$$\frac{1}{y'} = \frac{1}{dy/dx} = x' = e{-}2y$$
$$x(y) = \int e^{-2y} = -2x + C$$
$$y = -\frac{1}{2}\ln(-2x + C)$$

1.1.2 Slope Fields and Unique Existence

A slope field is a vector field of equation $\langle 1, y' \rangle$.
A direction field is essentially the same thing but normalized: $\dfrac{\langle 1, y' \rangle}{\sqrt{1 + (y')^2}}$ If function is in terms of x only then plot along lines on the x-axis.
If the function is in terms of y only then plot along lines on the y-axis.

Existence and Uniqueness
If $f(x, y)$ and $\dfrac{\partial f}{\partial y}$ exists and is continuous near a point (x_0, y_0) then the ODE $y' = f(x, y)$, $y(x_0) = y_0$ has a solution locally and is unique.
A solution is considered unique if it has initial conditions such that it has only one solution.

Ex: For what range of values does $(x - 2)y'' + y' + (x - 2)\tan(x)y = 0$ have a unique solution?
$$(x - 2)y'' + y' + (x - 2)\tan(x)y = 0$$
$$y'' + \frac{1}{x - 2}y' + \tan(x)y = 0 \text{ such that } x \neq 2 \wedge x \neq \frac{\pi}{2} + n\pi,\ n \in \mathbb{Z}$$
$$y(3) = 0,\ y'(3) = 1$$
$\therefore$ the ODE will only have a unique solution in the continuous subdomain containing $x = 3$

$\therefore$ the longest interval for a unique solution is $x \in \left(2, \frac{3\pi}{2}\right)$

1.1.3 Separable Equations

These are of the form $y' = f(x)g(y)$ and have solution

$$\int \frac{dy}{g(y)} = \int f(x)dx$$

$$\begin{aligned}
&\text{Ex: } \frac{dy}{dx} = \frac{4x}{1-y},\ y(3) = 0\\
&\quad \int (1-y)dy = \int 4xdx \Rightarrow y - \frac{y^2}{2} = 2x^2 + C\\
&\quad y(3) = 0 \Rightarrow 0 = 18 + C \Rightarrow C = -18\\
&\quad y^2 - 2y = 36 - 4x^2\\
&\quad y^2 - 2y + 1 - 1 = 36 - 4x^2\\
&\quad (y-1)^2 = 37 - 4x^2\\
&\quad y - 1 = \pm\sqrt{37 - 4x^2}\\
&\quad y = 1 \pm \sqrt{37 - 4x^2}\\
&\quad y(3) = 0 \therefore\ y = 1 - \sqrt{37 - 4x^2}
\end{aligned}$$

Ex2:

$$\begin{aligned}
&x\frac{dy}{dx} = y\ln y\\
&\int \frac{dy}{y\ln y} = \int \frac{dx}{x}\\
&u = \ln y \Rightarrow du = \frac{dy}{y}\\
&\int \frac{dy}{y\ln y} = \int \frac{du}{u} = \ln|u| = \ln|\ln|y||\\
&\ln|\ln|y|| = \ln|x| + C\\
&\ln|y| = Cx\\
&y = e^{Cx}
\end{aligned}$$

1.1.4 Linear Differential Equations

These are of the form $y' + p(x)y = g(x)$
For where $p(x) = 0$, we get $y' = g(x)$ which we can solve. We can get to this point by introducing an integrating factor $r(x)$ such that

$$\begin{aligned}
&r(x)(y' + p(x)y) = (r(x)y)'\\
&ry' + rpy = ry' + r'y
\end{aligned}$$

$$rp = r'$$

We can solve to find

$$r(x) = e^{\int^x p(t)dt}$$

Ex: $ty' + 5y = 24t^3,\ y(1) = 2$

$$y' = \frac{5}{t}y = 24t^2$$

$$p(t) = \frac{5}{t},\ g(t) = 24t^2$$

$$r(t) = e^{\int^t \frac{5ds}{s}} = e^{5\ln|t|} = t^5$$

$$(t^5y)' = t^5\left(y'(t) + \frac{5}{t}y\right),\ y' + \frac{5}{t} = 24t^2$$

$$(t^5y)' = t^5(24t^2) = 24t^7$$

$$t^5y = \int^t 24s^7ds = 3t^8 + C$$

$$y(1) = 2,\ 2 = 3 + C \Rightarrow C = -1$$

$$y(t) = 3t^3 - \frac{1}{t^5}$$

Ex2:

$$(1 + x^2)y' + 2xy = \cot x$$

$$y' + \frac{2x}{1 + x^2}y = \frac{\cot x}{1 + x^2}$$

$$r = e^{\int pdx} = e^{\int\left(\frac{2x}{1+x^2}\right)dx} = e^{\ln|1+x^2|} = 1 + x^2$$

$$(ry)' = rg$$

$$y = r^{-1}\int rgdx$$

$$y = \frac{1}{1 + x^2}\int(1 + x^2)\frac{\cot x}{1 + x^2}dx = \frac{1}{1 + x^2}\int \cot xdx$$

$$y = \frac{1}{1 + x^2}(\ln|\sin x| + C)$$

1.1.5 Exact Differential Equations

An equation in the form $M(x,y) + N(x,y)y' = 0$ is exact if $M_y = N_x$.
We can solve an exact equation by taking the integral of M or N and then solving for the constants.

i.e. $\int Mdx = \phi(x,y) + h(y)$

$$\frac{\partial}{\partial y}(\phi(x,y) + h(y)) = N$$

solve for $h(y)$ and then $\phi(x,y) + h(y)$ will be the solution

Ex: $2x + y^2 + 2xyy' = 0$

$M = 2x + y^2,\ N = 2xy$

$M_y = 2y,\ N_x = 2y$

$M_y = N_x \therefore$ exact

$\int (2x + y^2)dx = x^2 + xy^2 + h(y)$

$\frac{\partial}{\partial y}\left(x^2 + xy^2 + h(y)\right) = 2xy + h'(y) \Rightarrow h'(y) = 0$

$\Rightarrow h(y) = C$

$\therefore x^2 + xy^2 = C$

If an equation is not in exact form, we can sometimes multiply the equation by an integrating factor, u where $u = u(x)$ or $u = u(y)$, to put it in the form of an exact equation.
$uM + uNy' = 0$ such that $(uM)_y = (uN)_x$

Ex: $(x + 2)\sin y + (x \cos y)y' = 0,\ y(1) = \frac{\pi}{2}$

$M = (x + 2)\sin y,\ N = (x \cos y)$

$M_y = (x + 2)\cos y,\ N_x = \cos y$

$M_y \neq N_x \therefore$ not exact

$(uM)_y = (uN)_x$

let $u = u(x)$

$uN_x + Nu' = uM_y$

$u \cos y + (x \cos y)u' = u(x + 2)\cos y$

$u + xu' = u(x + 2)$

$xu' = u(x + 1)$

$\int \frac{du}{u} = \int \left(1 + \frac{1}{x}\right) dx$

$\ln|u| = x + \ln|x|$

$|u| = e^{x+\ln|x|} = |x|e^x$

Integrating factor: $u = xe^x$

set up new DE $(x + 2)xe^x \sin y + (x^2 e^x \cos y)y' = 0,\ y(1) = \frac{\pi}{2}$

$M = (x + 2)xe^x \sin y,\ N = (x^2 e^x \cos y)$

$M_y = (x^2 e^x + 2xe^x)\cos y = N_x \therefore$ exact

$\int (x^2 e^x \cos y)dy = x^2 e^x \sin y + h(x)$

$\frac{\partial}{\partial x}(x^2 e^x \sin y + h(x)) = x^2 e^x \sin y + 2xe^x \sin y + h'(x) = (x^2 e^x \sin y + 2xe^x \sin y) \Rightarrow h'(x) = 0$

$\Rightarrow h(x) = C$

$x^2 e^x \sin y = C$

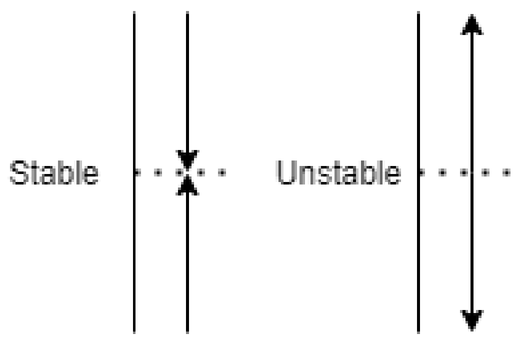
$$y(1) = \frac{\pi}{2} \Rightarrow e = C$$
$$\therefore x^2 e^x \sin y = e$$

1.1.6 Autonomous Differential Equations

This involves finding the properties of solutions of DEs without actually solving them.
If $\lim_{x\to\infty} y' = 0$ then the function will attain a constant value. For the case where $y' = y'(y)$ then we can analyze how specific values of $y(0)$ will behave as the function approaches infinity.
Definitions:

- If the value of a function at infinity does not change, it is a *fixed point*.
- If the function converges to a fixed point from both sides, it is said to be *asymptotically stable*.
- If the function diverges from a fixed point on either side, it is said to be *unstable*.
- If a function converges to a fixed point from only one side, it is said to be *metastable* or *semistable*.

Phase line diagrams:



1.1.7 Numerical Methods

These are methods used to approximate solutions.
One such method is Euler's Method which states that we can express the line $y = y(t)$ as a collection of tiny line segments.
We can express $y' = f(t, y)$ and $y(t_0) = y_0$ on the range from t_0 to T. In this range, we will have N intervals of width $h = \Delta t = \frac{T-t_0}{N}$ where $t_k = t_0 + kh$. This gives us

$$y(t) = \begin{cases} y_1 = y_0 + f(t_0, y_0),\ t \geq t_0 \\ y_2 = y_1 + f(t_1, y_1),\ t_0 \leq t < t_1 \\ y_3 = y_2 + f(t_2, y_2),\ t_1 \leq t < t_2 \\ \vdots \\ y_k = y_{k-1} + f(t_{k-1}, y_{k-1}),\ t_{k-1} \leq t < t_k \end{cases}$$

The error is approximately $E \approx Ch$ for some h.

Improved Euler's Method:

$$\begin{aligned}
k_1 &= f(t_n, y_n)\\
k_2 &= f(t_n + h, y_n + hk_1)\\
y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2)
\end{aligned}$$

Error is approximately $E \approx Ch^2$

Range-Kutta Method:

$$\begin{aligned}
k_1 &= f(t_n, y_n)\\
k_2 &= f\left(t_n + \frac{h}{2}, y_n + h\frac{k_1}{2}\right)\\
k_3 &= f\left(t_n + \frac{h}{2}, y_n + h\frac{k_2}{2}\right)\\
k_4 &= f(t_n + h, y_n + hk_3)\\
y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}$$

1.2 Second Order Linear Differential Equations

These are integrals in the form of $F(x, y, y', y'') = 0$
An equation is homogeneous if it is of the form $y'' + p(x)y' + q(x)y = 0$ (essentially if the equation equals 0)
If an equation is homogeneous is then then the general solution is the superposition of all solutions. $y(x) = y_1(x) + y_2(x)$

1.2.1 Reduction of Order

If we know one solution of a homogeneous solution $y_1(x)$ then the second solution can be found to be $y_2(x) = v(x)y_1(x)$

Ex: solve for $(1 - t^2)y'' + 2ty' - 2y' = 0$ given $y_1(t) = t$

find $y_2(t) = vy_1(t)$

$$\begin{aligned}
&y_2 = tv\\
&y_2' = tv' + v\\
&y_2'' = tv'' + 2v'\\
&(1 - t^2)(tv'' + 2v') + 2t(tv' + v) - 2tv = 0\\
&(1 - t^2)tv'' + (2(1 - t^2) + 2t^2)v' + 2tv - 2tv = 0\\
&tv'' + \left(2 + \frac{2t^2}{1 - t^2}\right)v' = 0\\
&w = v'\\
&w' + \left(\frac{2}{t} + \frac{2t}{1 - t^2}\right)w = 0
\end{aligned}$$

$$
\begin{aligned}
&\left(\frac{2}{t} + \frac{2t}{1-t^2}\right) w = -\frac{dw}{dt} \\
&-\int \frac{dw}{w} = \int \frac{2}{t} dt + \int \frac{2t}{1-t^2} dt \\
&-\ln|w| = 2\ln|t| - \ln|1-t^2| + C \\
&\text{take } C = 0 \\
&\ln|w| = \ln|1-t^2| - \ln|t^2| = \ln\left|\frac{1-t^2}{t^2}\right| \\
&\text{take } \left|\frac{1-t^2}{t^2}\right| = \frac{t^2-1}{t^2} \\
&w = \frac{t^2-1}{t^2} \\
&v = \int w dt = \int dt + \int \frac{1}{t^2} dt = t + \frac{1}{t} + C \\
&\text{take } C = 0 \\
&v = t + \frac{1}{t} \\
&y_2 = vy_1 = tv = t^2 + 1 \\
&y(t) = C_1 y_1(t) + C_2 y_2(t) \\
&y(t) = C_1 t + C_2(t^2+1)
\end{aligned}
$$

1.2.2 Constant Coefficients

For a homogeneous equation with constant coefficients, we can solve for the general solution using the characteristic equation.

$$
\begin{aligned}
&ay'' + by' + cy = 0 \\
&\text{guess } y = e^{rx} \\
&y' = re^{rx},\ y'' = r^2 e^{rx} \\
&ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0 \\
&ar^2 + br + c = 0 \\
&r = \frac{-b \pm \sqrt{b^2-4ac}}{2a} \\
&r = \frac{-b + \sqrt{b^2-4ac}}{2a},\ \frac{-b - \sqrt{b^2-4ac}}{2a} \\
&y = C_1 e^{\frac{-b+\sqrt{b^2-4ac}}{2a}} + C_2 e^{\frac{-b-\sqrt{b^2-4ac}}{2a}}
\end{aligned}
$$

We can have 3 cases:
Real roots: $b^2 - 4ac > 0$

$$
r = \frac{-b + \sqrt{b^2-4ac}}{2a},\ \frac{-b - \sqrt{b^2-4ac}}{2a}
$$

$$y = C_1 e^{\frac{-b+\sqrt{b^2-4ac}}{2a}} + C_2 e^{\frac{-b-\sqrt{b^2-4ac}}{2a}}$$

Ex: $y'' + 4y' + 3y = 0,\ y(0) = 2,\ y'(0) = -1$

$$r^2 + 4r + 3 = (r+3)(r+1) = 0$$
$$r = \{-3, -1\}$$
$$y = C_1 e^{-3x} + C_2 e^{-x}$$
$$y' = -3C_1 e^{-3x} - C_2 e^{-x}$$
$$y(0) = C_1 + C_2 = 2$$
$$y'(0) = -3C_1 - C_2 = -1$$
$$\Rightarrow -2C_1 = 1 \Rightarrow C_1 = -\frac{1}{2} \Rightarrow C_2 = \frac{5}{2}$$
$$y = -\frac{1}{2}e^{-3x} + \frac{5}{2}e^{-x}$$

Repeated Roots: $b^2 - 4ac = 0$

$$r = -\frac{b}{2a}$$
$$y_1 = e^{rt}$$

by reduction of order $y_2 = vy_1 = ve^{rt}$

$$y_2' = v'e^{rt} + vre^{rt}$$
$$y_2'' = v''e^{rt} + 2v're^{rt} + vr^2e^{rt}$$
$$ay'' + by' + cy = 0$$
$$a(v''e^{rt} + 2v're^{rt} + vr^2e^{rt}) + b(v'e^{rt} + vre^{rt}) + cve^{rt} = 0$$
$$av'' + (2ar + b)v' + (ar^2 + br + c)v = 0$$
$$r = -\frac{b}{2a},\ b^2 - 4ac$$
$$av'' + (-b + b)v' + \left(\frac{b^2}{4a} - \frac{b^2}{2a} + c\right)v = 0$$
$$av'' + \left(-\frac{b^2}{4a} + c\right)v = av'' - \frac{1}{4a}(b^2 - 4ac)v = 0$$
$$av'' = 0$$

$a \neq 0$ for 2nd order equations

$$\therefore v'' = 0$$
$$v' = C$$

take $C = 1$

$$v = t + C$$

take $C = 0$

$$\Rightarrow v = t$$
$$y_2 = te^{rt}$$

$$y(t) = C_1 e^{-\frac{b}{2a}t} + C_2 t e^{-\frac{b}{2a}t}$$

Complex Roots: $b^2 - 4ac < 0$

$$r = -\frac{b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a} i$$
$$r = \alpha \pm \mu i$$
$$y = C_1 e^{\alpha t + \mu i t} + C_2 e^{\alpha t - \mu i t}$$
$$y = e^{\alpha t}(C_1 e^{\mu i t} + C_2 e^{-\mu i t})$$
$$y = e^{\alpha t}(C_1(\cos(\mu t) + i\sin(\mu t)) + C_2(\cos(-\mu t) + i\sin(-\mu t)))$$
$$y = e^{\alpha t}((C_1 + C_2)\cos(\mu t) + (C_1 - C_2) i \sin(\mu t))$$
$$\text{let } C_1 = (C_1 + C_2) \text{ and } C_2 = (C_1 - C_2)i$$
$$y = e^{\alpha t}(C_1 \cos(\mu t) + C_2 \sin(\mu t))$$

Ex:

$$y'' - 5y' + 4y = 0$$
$$r^2 - 5r + 4 = 0$$
$$(r-4)(r-1) = 0 \Rightarrow r = 1, 4$$
$$y = C_1 e^x + C_2 e^{4x}$$

Ex2:

$$4y'' + 4y' + 2y = 0, \ y(0) = 2, \ y'(0) = 3$$
$$4r^2 + 4r + 2 = 0$$
$$r = \frac{-4 \pm \sqrt{16 - 32}}{8} = -\frac{1}{2} \pm \frac{1}{2} i$$
$$y = e^{-\frac{x}{2}}\left(C_1 \cos\left(\frac{x}{2}\right) + C_2 \sin\left(\frac{x}{2}\right)\right)$$
$$y(0) = C_1 = 2$$
$$y' = e^{-\frac{x}{2}}\left(-\frac{C_1}{2}\cos\left(\frac{x}{2}\right) - \frac{C_2}{2}\sin\left(\frac{x}{2}\right) - \frac{C_1}{2}\sin\left(\frac{x}{2}\right) + \frac{C_2}{2}\cos\left(\frac{x}{2}\right)\right)$$
$$y'(0) = -\frac{C_1}{2} + \frac{C_2}{2} = 3$$
$$-1 + \frac{C_2}{2} = 3 \Rightarrow C_2 = 8$$
$$y = e^{-\frac{x}{2}}\left(2\cos\left(\frac{x}{2}\right) + 8\sin\left(\frac{x}{2}\right)\right)$$

1.2.3 Nonhomogeneous Equations

These are equations of the form $ay'' + by' + cy = f(x)$
These do not follow the principle of superposition. The general solution will be the particular solution plus the solution to the homogeneous equation (the complementary solution).

Reduction of Order:
Reduction of order also works for nonhomogeneous equations. Given y_c you can set $y_p = vy_c$

Ex: $ty'' - 2ty' + 2y = 5t^2$

$$ty_c'' - 2ty_c' + 2y_c = 0,\ y_1 = t$$
$$\text{let } y_2 = vy_1 = vt$$
$$y_2' = tv' + v$$
$$y_2'' = tv'' + 2v'$$
$$t^2v'' + 2tv' - 2t^2v' - 2tv + 2vt = 5t^2$$
$$v''t^2 + 2v't - 2v't^2 - 2tv + 2tv = 5t^2$$
$$v''t^2 + v'(2t - 2t^2) = 5t^2$$
$$v' = w$$
$$w't^2 + w(2t - 2t^2) = 5t^2$$
$$w' + w\left(\frac{2}{t} - 2\right) = 5$$
$$r = e^{\int(\frac{2}{t}-2)dt} = e^{2\ln t - 2t} = e^{\ln t^2}e^{-2t} = t^2e^{-2t}$$
$$(rw)' = 5r \Rightarrow w = r^{-1}\int 5rdt$$
$$w = 5\frac{e^{2t}}{t^2}\int t^2e^{-2t}dt$$
$$I_1 = \int t^2e^{-2t}dt$$
$$u = t^2 \Rightarrow du = 2tdt$$
$$dv = e^{-2t}dt \Rightarrow v = -\frac{e^{-2t}}{2}$$
$$I_1 = -\frac{t^2e^{-2t}}{2} + \int te^{-2t}dt$$
$$I_2 = \int te^{-2t}dt$$
$$u = t \Rightarrow du = dt$$
$$dv = e^{-2t}dt \Rightarrow v = -\frac{e^{-2t}}{2}$$
$$I_2 = -\frac{te^{-2t}}{2} + \int\frac{e^{-2t}}{2} = -\frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}$$
$$w = 5\frac{e^{2t}}{t^2}\left(-\frac{t^2e^{-2t}}{2} - \frac{te^{-2t}}{2} - \frac{e^{-2t}}{4}\right) = -\frac{5}{4}\left(2 + \frac{2}{t} + \frac{1}{t^2}\right)$$
$$v = \int vdt = -\frac{5}{4}\int\left(2 + \frac{2}{t} + \frac{1}{t^2}\right) = \frac{5}{4}\left(\frac{1}{t} - 2\ln t - 2t\right)$$
$$y_p = vy_c = tv = \frac{5}{4}(1 - 2t\ln t - 2t^2)$$

Variation of Parameters

This is a more general case of reduction of order for when there are more than one complementary solution:

$y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$

We can impose the condition $u_1'y_1 + u_2'y_2 = 0$ and derive another condition from the differential equation:

$$
\begin{aligned}
&y'' + py' + qy = f(t)\\
&y_p' = y_1'u_1 + y_2'u_2 + y_1u_1' + y_2u_2' = y_1'u_1 + y_2'u_2\\
&y_p'' = y_1'u_1' + y_2'u_2' + y_1''u_1 + y_2''u_2\\
&y_1'u_1' + y_2'u_2' + y_1''u_1 + y_2''u_2 + py_1'u_1 + py_2'u_2 + qy_1u_1 + qy_2u_2 = f(t)\\
&y_i'' + py_i' + qy_i = 0 \Rightarrow y_i'' = -qy_i' - p_i\\
&y_1'u_1' + y_2'u_2' = f(t)
\end{aligned}
$$

This gives us the system of equations

$$
\begin{cases} u_1'y_1 + u_2'y_2 = 0\\ u_1'y_1' + u_2'y_2' = f(t)\end{cases}
$$

Solving this gives the following general solution

$$
\begin{aligned}
&\begin{bmatrix} y_1 & y_2\\ y_1' & y_2'\end{bmatrix}\begin{bmatrix} u_1'\\ u_2'\end{bmatrix} = \begin{bmatrix} 0\\ f(t)\end{bmatrix}\\
&\begin{bmatrix} u_1'\\ u_2'\end{bmatrix} = \begin{bmatrix} y_1 & y_2\\ y_1' & y_2'\end{bmatrix}^{-1}\begin{bmatrix} 0\\ f(t)\end{bmatrix}\\
&\text{define } W = \begin{vmatrix} y_1 & y_2\\ y_1' & y_2'\end{vmatrix}\\
&\begin{bmatrix} u_1'\\ u_2'\end{bmatrix} = \frac{1}{W}\begin{bmatrix} y_2' & -y_2\\ -y_1' & y_2\end{bmatrix}\begin{bmatrix} 0\\ f(t)\end{bmatrix} = \frac{1}{W}\begin{bmatrix} -y_2f(t)\\ y_1f(t)\end{bmatrix}\\
&y_p = y_1\int^t \frac{y_2f(\tau)}{y_2y_1' - y_1y_2'}d\tau + y_2\int^t \frac{y_1f(\tau)}{y_1y_2' - y_2y_1'}d\tau
\end{aligned}
$$

Undetermined Coefficients:

This method is basically strategic guessing.

$f(x)$	guess
$e^{\alpha x}$	$ae^{\alpha x}$
$\sin(\omega x)$	$a\cos(\omega x) + b\sin(\omega x)$
$\cos(\omega x)$	$a\cos(\omega x) + b\sin(\omega x)$
t^n	$a_0 + a_1t + a_2t^2 + \cdots + a_nt^n$
$g(t) + h(t)$	$\text{guess}(g(t)) + \text{guess}(h(t))$
$g(t)h(t)$	$(\text{guess}(g(t)))\,(\text{guess}(h(t)))$

If there is any overlap with the complementary solution then you multiply your guess by x.

Ex: $y'' - 2y' + y = e^x$

$y_c:\ r^2 - 2r + r = 0$

$(r-1)^2 = 0 \Rightarrow r = 1$

$y_c = C_1e^x + C_2xe^x$

1st guess: $y_p = ae^x$

ae^x overlaps with C_1e^x

2nd guess: $y_p = ax^2e^x$

no overlap $\therefore$ guess is valid

Ex2: $y'' - y' - 6y = e^{2x}$

Characteristic equation:

$r^2 - r - 6 = 0$

$(r-3)(r+2)$

$r = -2, 3$

$y_c = C_1e^{-2x} + C_2e^{3x}$

guess $y_p = ae^{2x}$

no overlap

$\Rightarrow y_p = ae^{2x}$

$y_p' = 2ae^{2x}$

$y_p'' = 4ae^{2x}$

$4ae^{2x} - 2ae^{2x} - 6ae^{2x} = e^{2x}$

$-4a = 1 \Rightarrow a = -\frac{1}{4}$

$y_p = -\frac{e^{2x}}{4}$

$y(x) = C_1e^{-2x} + C_2e^{3x} - \frac{e^{2x}}{4}$

Ex3: $y'' + 2y' + 5y = 3e^{-t}\cos 2t$

$y_c:\ y_c'' + 2y_c' + 5y_c = 0$

$r^2 + 2r + 5 = 0$

$r = \frac{-2 \pm \sqrt{4 - 4(5)}}{2} = -1 \pm 2i$

$y_c = e^{-t}(C_1\cos 2t + C_2\sin 2t)$

guess $y_p = te^{-t}(a\cos 2t + b\sin 2t)$

$y_p' = -\mathrm{e}^{-t}\left(\left(\left(b+2a\right)t - b\right)\sin\left(2t\right) + \left(\left(a-2b\right)t - a\right)\cos\left(2t\right)\right)$

$y_p'' = -\mathrm{e}^{-t}\left(\left(\left(3b-4a\right)t + 2b + 4a\right)\sin\left(2t\right) + \left(\left(4b+3a\right)t - 4b + 2a\right)\cos\left(2t\right)\right)$

$$\begin{cases} -3b+4a-2b-4a+5b=0 \\ -2b-4a+2b=0 \\ -4b-3a-2a+4b+5a=0 \\ 4b-2a+2a=3 \end{cases} = \begin{cases} 0=0 \\ a=0 \\ 0=0 \\ 4b=3 \end{cases} \Rightarrow b=\frac{3}{4}$$

$$y_p = \frac{3}{4}te^{-t}\sin 2t$$

$$y(t) = e^{-t}\left(C_1\cos 2t + C_2\sin 2t + \frac{3}{4}t\sin 2t\right)$$

1.2.4 Mechanical Vibrations

One of the most common examples of second order constant coefficient equations is in mechanical vibrations such as a mass on a spring. It's system is composed of the force of the spring and the dampening force.

$$F = ma = -cv - kx$$

This can be written as a 2nd order system as

$$mx'' + cx' + kx = 0$$

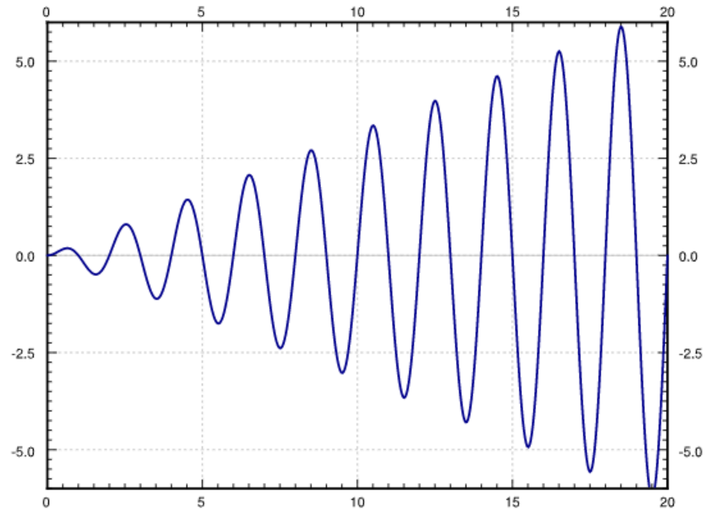
This system can be solved in the same way as we saw earlier and depending on the values, we will have one of 4 types of solutions.

- If $c^2 - 4mk > 0$, the system is *overdamped* and will take the form $x = Ae^{r_1 t} + Be^{r_2 t}$ where $r_1, r_2 \leq 0$
- If $c^2 - 4mk = 0$, the system is *critically damped* and will take the form $x = Ae^{rt} + Bte^{rt}$ where $r \leq 0$
- If $c^2 - 4mk < 0$, the system will be *underdamped* and take the form $x = e^{-pt}(A\cos(\omega t) + B\sin(\omega t))$ which can be rewritten as $x = Re^{-pt}\cos(\omega t + \phi)$ where $R = \sqrt{A^2 + B^2}$ and $\tan\phi = \frac{B}{A}$ where $p = \frac{c}{2m}$ and $\omega = \sqrt{\frac{k}{m} - p^2}$
- If $c = 0$, the system has no dampening and will take the form $x = A\cos(\omega t) + B\sin(\omega_0 t)$ which can also be rewritten as $x = R\cos(\omega t + \phi)$ where $\omega_0 = \sqrt{\frac{k}{m}}$.

Forced Vibrations:
These are equations in the form of $mx'' + cx' + kx = F_0\cos(\omega t)$ or $mx'' + cx' + kx = F_0\sin(\omega t)$
Where x_p is of the form $x_p = a\cos(\omega t) + b\sin(\omega t)$.
Note, x_c will not overlap with the particular solution unless $c = 0$ and $\omega_1 = \omega$. This gives us three distinct cases: resonance, beats, and steady periodic.

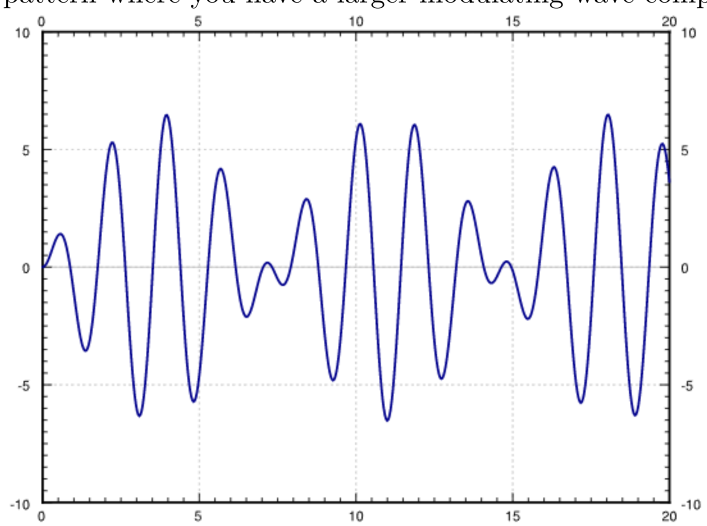
Resonance
Both resonance and beats occur where there is no dampening so the differential equation will take the form $mx'' + kx = F_0\cos(\omega t)$ and will have a complementary solution of $x_c = A\cos(\omega_0 t) + B\sin(\omega_0 t)$. We call the frequency ω_0 the natural frequency, because it is what the system naturally wants to attune, and we call ω the driving frequency.
Resonance occurs when the natural frequency equals the driving frequency $\omega = \omega_0$. When this

occurs, the particular solution will take the form of $at\cos(\omega t)$ and the system will experience very large vibrations (as was the case in the Tacoma Narrows bridge collapse).



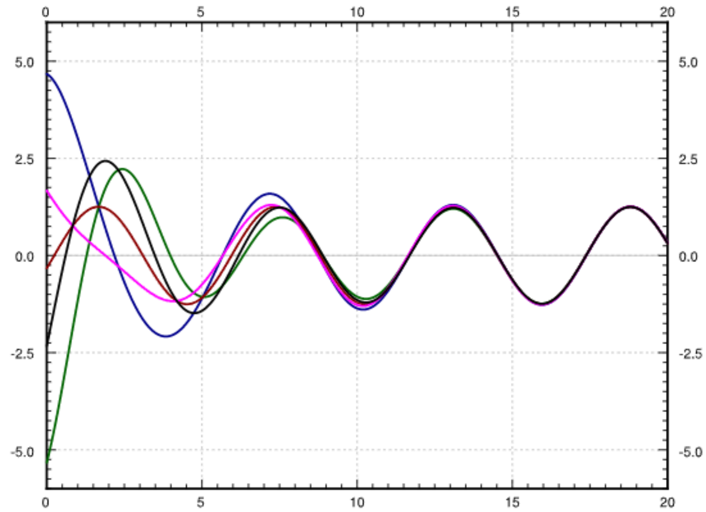
Beats

This also occurs where there is no dampening force but where the natural frequency and the driving frequency are different. The particular solution will take the form $x_p = a\cos(\omega t) + b\sin(\omega t)$. This will result the general solution being a sum of waves with different frequencies. This creates a pattern where you have a larger modulating wave comprised of smaller oscillations.



Steady State Response

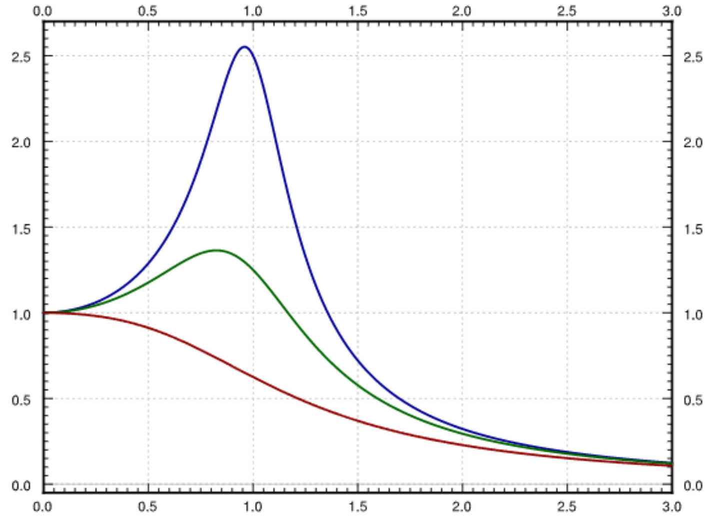
This is the case when there is a dampening force. This means that the complementary solution will experience exponential decay, going away quickly while the particular solution will take the form $x_p = a\cos(\omega t) + b\sin(\omega t)$. In this case, we call the complementary solution the *transient* solution as it goes away fast, leaving the particular solution as what is seen after a long time which we call the *steady state response*. $\lim_{t\to\infty} x(t) = x_p$



Derivation of general form:

$$mx'' + cx' + kx = F_0\cos(\omega t)$$
$$x_p = a\cos(\omega t) + b\sin(\omega t)$$
$$x_p' = -a\omega\sin(\omega t) + b\omega\cos(\omega t)$$
$$x_p'' = -a\omega^2\cos(\omega t) - b\omega^2\sin(\omega t)$$
$$\begin{cases} -ma\omega^2 + bc\omega + ak = F_0 \\ -mb\omega^2 - ac\omega + bk = 0 \end{cases} \Rightarrow \begin{bmatrix} k - m\omega^2 & c\omega \\ -c\omega & k - m\omega^2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} k - m\omega^2 & c\omega \\ -c\omega & k - m\omega^2 \end{bmatrix}^{-1} = \frac{1}{(k - m\omega^2)^2 + (c\omega)^2} \begin{bmatrix} k - m\omega^2 & -c\omega \\ c\omega & k - m\omega^2 \end{bmatrix}$$
$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{(k - m\omega^2)^2 + (c\omega)^2} \begin{bmatrix} k - m\omega^2 & -c\omega \\ c\omega & k - m\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix} = \frac{F_0}{(k - m\omega^2)^2 + (c\omega)^2} \begin{bmatrix} k - m\omega^2 \\ c\omega \end{bmatrix}$$
$$x_p = \frac{F_0}{(k - m\omega^2)^2 + (c\omega)^2}((k - m\omega^2)\cos(\omega t) + c\omega\sin(\omega t)) = R\cos(\omega t + \phi)$$
$$R = \sqrt{a^2 + b^2} = \frac{F_0}{(k - m\omega^2)^2 + (c\omega)^2}\sqrt{(k - m\omega^2)^2 + (c\omega)^2} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$$
$$\phi = \arctan\left(\frac{b}{a}\right) = \arctan\left(\frac{c\omega}{k - m\omega^2}\right)$$
$$x_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}\cos\left(\omega t + \phi\right)$$

While this solution cannot achieve pure resonance as in an undamped solution, it can attune a resonance frequency such that the amplitude will approach a maximum. If we consider the amplitude as a function of frequency, we can get the following graph for various values of m, k, and c.



A frequency of 0 will give $R = \frac{F_0}{k}$ and a very large frequency will give an amplitude of $R = 0$. To find the maximum frequency, we can take the derivative.

$$R'(\omega) = \frac{-F_0(2(k - m\omega^2)(-2m\omega) + 2c^2\omega)}{2\left((k - m\omega^2)^2 + (c\omega)^2\right)^{3/2}} = 0$$
$$\omega(-4m(k - m\omega^2)) + 2c^2) = 0$$
$$\omega = 0 \text{ or,}$$
$$k - m\omega^2 = \frac{2c^2}{4m} \Rightarrow k - \frac{2c^2}{4m} = m\omega^2$$
$$\omega = \sqrt{\frac{k}{m} - \frac{c^2}{2m^2}}$$

This gives the resonance frequency. If there are no nonzero solutions, we say that the system has no resonance frequency.

1.2.5 Cauchy-Euler Equation

The Cauchy-Euler equation is a 2nd order ODE of the form

$$ax^2y'' + bxy' + cy = 0$$

To solve this equation, we will guess that the solution will be of the form $y = x^r$.
This gives $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$
So if we plug these into the original equation we get

$$ar(r-1)x^r + brx^r + cx^r = 0$$
$$x^r(ar^2 + (b-a)r + c) = 0$$
$$ar^2 + (b-a)r + c = 0$$

giving the characteristic equation.
We can solve for r and get 3 cases:

Real roots: $(b-a)^2 - 4ac > 0$

$$r = \frac{-(b-a) \pm \sqrt{(b-a)^2 - 4ac}}{2a}$$
$$y(x) = C_1 x^{r_1} + C_2 x^{r_2}$$

Imaginary roots: $(b-a)^2 - 4ac < 0$

$$r = -\frac{(b-a)}{2a} \pm i\frac{\sqrt{4ac-(b-a)^2}}{2a} = \lambda \pm i\mu$$
$$y(x) = C_1 x^{(\lambda+i\mu)x} + C_2 x^{(\lambda-i\mu)x}$$
$$y = x^\lambda \left(C_1 x^{i\mu} + C_2 x^{-i\mu}\right)$$
$$y = x^\lambda \left(C_1 e^{i\mu \ln x} + C_2 e^{-i\mu \ln x}\right)$$
$$y = x^\lambda \left(C_1 \cos(\mu \ln x) + C_2 \sin(\mu \ln x)\right)$$

Repeated roots: $(b-a)^2 - 4ac = 0$

$$r = \frac{a-b}{2a}$$
$$y_1 = x^r$$
$$y_2 = u(x)y_1 = u(x)x^r$$
$$y_2' = urx^{r-1} + u'x^r$$
$$y_2'' = x^r u'' + 2u'rx^{r-1} + ur(r-1)x^{r-2}$$
$$ax^{r+2}u'' + 2arx^{r+1}u' + ar(r-1)x^r u + bx^{r+1}u' + brx^r u + cx^r u = 0$$

Simplifying we can get that all the u terms cancel out

$$\frac{a-b}{2}\left(\frac{a-b}{2a} - 1\right) + \frac{b(a-b)}{2a} + c$$
$$\frac{a^2-2ab+b^2}{4a} + \frac{-2a^2+2ab}{4a} + \frac{2ab-2b^2}{4a} + \frac{4ac}{4a}$$
$$\frac{-a^2+2ab-b^2+4ac}{4a} = -\frac{(a-b)^2-4ac}{4a} = 0$$

So our remaining terms are

$$ax^{r+2}u'' + 2arx^{r+1}u' + bx^{r+1}u' = 0$$
$$\text{let } v = u' \Rightarrow v' = u''$$
$$ax^{r+2}v' + 2arx^{r+1}v + bx^{r+1}v = 0$$
$$v' + \frac{2ar+b}{a}\frac{v}{x} = 0$$
$$v' + \frac{a-b+b}{a}\frac{v}{x} = 0$$
$$v' + \frac{v}{x} = 0$$

$$\mu(x) = e^{\int \frac{dx}{x}}$$
$$\mu v' + \mu \frac{v}{x} = xv' + v = \frac{d}{dx} vx = 0$$
$$\int \left(\frac{d}{dx} vx \right) dx = \int 0 dx = C$$
$$vx = u'x = C$$
$$\int \frac{du}{u} = \int \frac{dx}{x}$$
$$u(x) = \ln |x|$$
$$y_2 = x^r \ln |x|$$
$$y(x) = C_1 x^r + C_2 x^r \ln |x|$$

Note that we can also write a Cauchy-Euler Equation in the form of

$$a(x-\alpha)^2 y'' + b(x-\alpha)y' + cy = 0$$

and our fundamental guess would be of the form $y = (x-\alpha)^r$.
Ex:

$$x^2 y'' - xy' + y = 0$$
$$y = x^r,\ y' = rx^{r-1},\ y'' = r(r-1)x^{r-2}$$
$$r(r-1) - r + 1 = 0$$
$$r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 \Rightarrow r = 1$$
$$y = C_1 x + C_2 x \ln |x|$$

Ex2:

$$x^2 y'' - xy' + 5y = 0$$
$$y = x^r,\ y' = rx^{r-1},\ y'' = r(r-1)x^{r-2}$$
$$r(r-1) - r + 5 = 0 \Rightarrow r^2 - 2r + 5 = 0$$
$$r = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$$
$$y = x(C_1 \cos(2\ln|x|) + C_2 \sin(2\ln|x|))$$

Ex3:

$$2x^2 y'' - xy' + y = x$$
$$y_c = x^r,\ y_c' = rx^r,\ y_c = r(r-1)x^r$$
$$2r(r-1) - r + 1 = 0 \Rightarrow 2r^2 - 3r + 1 = 0 \Rightarrow (2r-1)(r-1) = 0,\ r = \frac{1}{2}, 1$$
$$y_c = C_1 \sqrt{x} + C_2 x$$
$$y_p = ax \ln x$$
$$y_p' = a \ln x + a$$

$$y_p'' = \frac{a}{x}$$
$$2ax - ax\ln x - ax + ax\ln x = x$$
$$\Rightarrow a = 1$$
$$y = C_1\sqrt{x} + C_2 x + x\ln x$$

1.3 Series Solutions

1.3.1 Power Series Solutions

We can express the solution to many ODEs in the form of a power series. For some differential equation

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

we can express the general solution about some point x_0 as

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

This method is best shown through examples.
Ex:

$$y' - y - 2xy = 0$$
$$y = \sum_{n=0}^{\infty} a_n x^n$$
$$y' = \sum_{n=1}^{\infty} a_n n x^{n-1}$$
$$\sum_{n=1}^{\infty} a_n n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n - 2\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Now we want to combine the three sums into one summation. We can do this by changing the indexes so that they all have matching x^m terms and then peel off the lower terms in the sum so they all start at the same point.

$$\underbrace{\sum_{n=1}^{\infty} a_n n x^{n-1}}_{m=n-1} + \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{m=n} - 2\underbrace{\sum_{n=0}^{\infty} a_n x^{n+1}}_{m=n+1} = 0$$
$$\sum_{m=0}^{\infty} a_{m+1}(m+1)x^m + \sum_{m=0}^{\infty} a_m x^m - 2\sum_{m=1}^{\infty} a_{m-1} x^m = 0$$
$$a_0 + a_1 + \sum_{m=1}^{\infty} \left(a_{m+1}(m+1) + a_m - 2a_{m-1}\right) x^m = 0$$

Because each $x^2,\ x^3,\ x^4, \ldots$ term is linearly independent of one another we will have each of these terms sum to 0

x^0 terms: $a_0 + a_1 = 0 \Rightarrow a_1 = -a_0$

$$x^1 \text{ terms: } a_2(2) + a_1 - 2a_0 = 0 \Rightarrow a_2 = \frac{2a_0 - a_1}{2}$$

$$x^m \text{ terms: } a_{m+1}(m+1) + a_m - 2a_{m-1} = 0 \Rightarrow a_{m+1} = \frac{2a_{m-1} - a_m}{m+1}$$

This gives a recursive formula we can use to solve for each a_m term.

$$m = 1: \; a_2 = \frac{2a_0 - a_1}{2} = \frac{2a_0 + a_0}{2} = \frac{3}{2}a_0$$

$$m = 2: \; a_3 = \frac{2a_1 - a_2}{3} = \frac{-2a_0 - \frac{3}{2}a_0}{3} = -\frac{7}{6}a_0$$

$$\vdots$$

$$y(x) = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$y(x) = a_0 \left(1 - x + \frac{3}{2}x^2 - \frac{7}{6}x^3 + \cdots\right)$$

Note that the soluyion is in terms of an arbitrary constant a_0. For 2nd order ODEs the solution will be in terms of two arbitrary constants (usually a_0 and a_1).
Ex2:

$$(1 + x^3)y'' + 12xy = 0$$

$$y = \sum_{n=0}^{\infty} a_n x^n, \; y' = \sum_{n=1}^{\infty} a_n n x^{n-1}, \; y'' = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2}$$

$$\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_n n(n-1)x^{n+1} + 12\sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m + \sum_{m=3}^{\infty} a_{m-1}(m-1)(m-2)x^m + 12\sum_{m=1}^{\infty} a_{m-1}x^m = 0$$

$$2a_2 + 6a_3 x + 12a_4 x^2 + 12a_0 x + 12a_1 x^2 + \sum_{m=3}^{\infty} \left(a_{m+2}(m+2)(m+1) + a_{m-1}(m-1)(m-2) + 12a_{m-1}\right) x^m$$

$$2a_2 = 0$$

$$6a_3 + 12a_0 = 0 \Rightarrow a_3 = -2a_0$$

$$12a_4 + 12a_1 \Rightarrow a_4 = -a_1$$

$$a_{m+2}(m+2)(m+1) + a_{m-1}(m-1)(m-2) + 12a_{m-1} = 0$$

$$a_{m+2}(m+2)(m+1) = -a_{m-1}(m^2 - 3m + 14)$$

$$a_{m+2} = -a_{m-1}\frac{m^2 - 3m + 14}{(m+2)(m+1)}$$

$$a_5 = -\frac{14a_2}{20} = 0$$

$$a_6 = -a_3\left(\frac{18}{30}\right) = \frac{36a_0}{30} = \frac{6a_0}{5}$$

$$a_7 = -a_4\left(\frac{24}{42}\right) = \frac{4a_1}{7}$$

$$
\begin{aligned}
&a_8 = 0 \\
&a_9 = -a_6\left(\frac{42}{72}\right) = -\frac{7a_0}{10} \\
&a_{10} = -a_7\frac{54}{90} = -\frac{12a_1}{35} \\
&a_{12} = -a_9\frac{84}{132} = \frac{49a_0}{110} \\
&a_{13} = -a_{10}\frac{102}{156} = \frac{102a_1}{455} \\
&y = a_0\left(1 - 2x^3 + \frac{6}{5}x^6 - \frac{7}{10}x^9 + \cdots\right) + a_1\left(x - x^4 + \frac{4}{7}x^7 - \frac{12}{35}x^{10} + \cdots\right)
\end{aligned}
$$

Something to note when we are using power series is the radius of convergence. If we rewrite the ODE $P(x)y'' + Q(x)y' + R(x)y = 0$ as

$$y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$$

then it will be nonsensical for where $P(x) = 0$. Points where this happens are called *singular points*. So in our previous example we had $(1 + x^3)y'' + 12xy = 0$ so our singular points would be

$$x = -1,\ x = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

More rigorously, singular points are defined to be where the function is not analytic. This is the case where the function or the derivative of the function is divided by 0.
For example, $y'' + \sqrt{x}y' - y = 0$ would have a singular point at $x = 0$.
Our power series solution is only valid so long as we don't pass any singular points. This will mean that there is some radius of convergence within where the solution is valid. This radius of convergence can be found as the distance (in real and imaginary space) between the expansion point, x_0, and the nearest singular point.
So in our previous example, our expansion point was $x_0 = 0$ and our singular points are all of a distance $R = 1$ away so we find that our radius of convergence is $R = 1$. This means that our solution is only valid for $|x| < 1$ or more generally,

$$|x - x_0| < R$$

Another way to show this is using the ratio test for the recursive a_m

$$
\begin{aligned}
&\lim_{m\to\infty}\left|\frac{a_{m+2}}{a_{m-1}}x\right| = \lim_{m\to\infty}\left|\frac{-a_{m-1}\frac{m^2-3m+3}{(m+2)(m+1)}}{a_{m-1}}x\right| = \lim_{m\to\infty}\left|\frac{m^2-3m+3}{(m+2)(m+1)}x\right| = |x| < 1 \\
&\therefore R = 1
\end{aligned}
$$

1.3.2 Frobenius Solutions

When we have a singular point we are not able to expand about that point using power series. One way to get around this is to use Frobenius series.

Because the Cauchy-Euler equation gives a solution for equations with singular points we can try to mimic it with the Frobenius series. We can take an arbitrary ODE and manipulate it to get it into the form of a Cauchy-Euler equation.

$$
\begin{aligned}
&P(x)y'' + Q(x)y' + R(x)y = 0 \\
&y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0 \\
&(x-x_0)^2y'' + (x-x_0)^2\frac{Q(x)}{P(x)}y' + (x-x_0)^2\frac{R(x)}{P(x)}y = 0 \\
&\text{let } \alpha(x) = (x-x_0)\frac{Q(x)}{P(x)},\ \beta(x) = (x-x_0)^2\frac{R(x)}{P(x)} \\
&(x-x_0)^2y'' + (x-x_0)\alpha(x)y' + (x-x_0)^2\beta(x)y = 0
\end{aligned}
$$

So long as $\lim\limits_{x\to x_0}\alpha(x)$ and $\lim\limits_{x\to x_0}\beta(x)$ are both finite then the point we are expanding about is considered to be a *regular singular point* and we are able to apply the Frobenius series.
The Frobenius series looks similar to the power series and is of the form

$$
y = (x-x_0)^r\sum_{n=0}^{\infty}(x-x_0)^n = \sum_{n=0} a_n(x-x_0)^{n+r}
$$

Finding the solution using Frobenius series works out very similar to using power series but with a few slight differences.
Ex:

$$
6x^2(1+x)y'' + 5xy' - y = 0 \text{ about } x_0 = 0
$$

We must first see if the point we are expanding about is a regular singular point

$$
\begin{aligned}
&\lim_{x\to 0}\frac{5x^2}{6x^2(1+x)} = \lim_{x\to 0}\frac{5}{6(x+1)} = \frac{5}{6} \\
&\lim_{x\to 0}\frac{-x^2}{6x^2(x+1)} = \lim_{x\to 0}\frac{-1}{6(x+1)} = -\frac{1}{6} \\
&\therefore\ x_0 = 0 \text{ is a regular singular point}
\end{aligned}
$$

These limits that we found give the terms α_0 and β_0. We can use these terms to craft what we call the indicial equation which allow us to solve for the two r values that will show up in our solution (similar to how we solved for r in Cauchy-Euler).

$$
\begin{aligned}
&x^2y_c'' + \alpha_0xy_c' + \beta_0y_c = 0 \\
&x^2y_c'' + \frac{5}{6}xy_c' - \frac{1}{6}y_c = 0 \\
&y_c = x^r,\ y_c' = rx^{r-1},\ y_c'' = r(r-1)x^{r-2} \\
&r(r-1) + \frac{5}{6}r - \frac{1}{6} = 0 \\
&6r^2 - r - 1 = 0
\end{aligned}
$$

$$r = \frac{1 \pm \sqrt{1+24}}{12} = \frac{1}{2}, -\frac{1}{3}$$
$$y_1 = \sum_{n=0}^{\infty} a_n x^{\frac{1}{2}+n}$$
$$y_2 = \sum_{n=0}^{\infty} a_n x^{n-\frac{1}{3}}$$

The rest follows very similarly to finding a power series solution

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$
$$y' = \sum_{n=0}^{\infty} a_n (n+r) x^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r-2}$$
$$6\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r} + 6\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) x^{n+r+1}$$
$$+ 5\sum_{n=0}^{\infty} a_n (n+r) x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r}$$
$$6\sum_{m=0}^{\infty} a_m (m+r)(m+r-1) x^{m+r} + 6\sum_{m=1}^{\infty} a_{m-1} (m+r-1)(m+r-2) x^{m+r}$$
$$+ 5\sum_{m=0}^{\infty} a_m (m+r) x^{m+r} - \sum_{m=0}^{\infty} a_m x^{m+r}$$
$$6a_0(r)(r-1)x^r + 5a_0(r)x^r - a_0 x^r$$
$$+ \sum_{m=1}^{\infty} \left(6a_m(m+r)(m+r-1) + 6a_{m-1}(m+r-1)(m+r) + 5a_m(m+r) - a_m\right) x^{m+r}$$
$$\Rightarrow r = \frac{1}{2}, -\frac{1}{3}$$
$$a_m(6(m+r)(m+r-1) + 5(m+r) - 1) = -6a_{m-1}(m+r-1)(m+r-2)$$
$$a_m = \frac{-6a_{m-1}(m+r-1)(m+r-2)}{6(m+r)(m+r-1) + 5(m+r) - 1}$$
$$r = \frac{1}{2}:\ a_m = -a_{m-1}\frac{3(2m-1)(2m-3)}{2m(6m+5)}$$
$$a_1 = \frac{3}{22}a_0$$
$$a_2 = -\frac{9}{68}a_1 = -\frac{27}{1496}a_0$$
$$r = -\frac{1}{3}:\ a_m = \frac{2(3m-4)(3m-7)}{3m(6m-5)}$$

$$
\begin{aligned}
a_1 &= -\frac{8}{3}a_0 \\
a_2 &= a_1\frac{2}{21} = -\frac{16}{63}a_0 \\
y &= C_1x^{1/2}\left(1+\frac{3}{22}x-\frac{27}{1496}x^2+\cdots\right)+C_2x^{-\frac{1}{3}}\left(1-\frac{8}{3}x-\frac{16}{63}x^2+\cdots\right)
\end{aligned}
$$

Ex2: Special case of the Bessel equation

$$x^2y''+xy'+(x^2-\nu^2)y=0,\ \nu=\frac{1}{2}$$

SP: $x=0$

$$
\begin{aligned}
&\alpha = 1 \\
&\alpha' = 0 \\
&\beta = x^2-\nu^2 \\
&\beta' = 2x \\
&\lim_{x\to 0} 1 = 1 \\
&\lim_{x\to 0} x^2-\nu^2 = -\nu^2 = -\frac{1}{4}
\end{aligned}
$$

So $x=0$ is a regular singular point

$$
\begin{aligned}
&r(r-1)+r-\frac{1}{4}=0 \\
&r^2-\frac{1}{4}=0 \\
&\left(r-\frac{1}{2}\right)\left(r+\frac{1}{2}\right)\Rightarrow r=\pm\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
&y=\sum_{n=0}^{\infty}a_nx^{n+r} \\
&y'=\sum_{n=0}^{\infty}a_n(n+r)x^{n+r-1} \\
&y''=\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)x^{n+r-1} \\
&\sum_{n=0}^{\infty}a_n(n+r)(n+r-1)x^{n+r}+\sum_{n=0}^{\infty}a_n(n+r)x^{n+r}+\sum_{n=0}^{\infty}a_nx^{n+r+2}-\nu^2\sum_{n=0}^{\infty}a_nx^{n+r}=0 \\
&\sum_{m=0}^{\infty}a_m(m+r)(m+r-1)x^{m+r}+\sum_{m=0}^{\infty}a_m(m+r)x^{m+r}+\sum_{m=2}^{\infty}a_{m-2}x^{m+r}-\nu^2\sum_{m=0}^{\infty}a_mx^{m+r}=0 \\
&a_0(r)(r-1)x^r+a_0rx^r-\nu^2a_0x^r+a_1(1+r)(r)x^{1+r}+a_1(1+r)x^{1+r}-\nu^2a_1x^{1+r} \\
&+\sum_{m=2}^{\infty}\left(a_m(m+r)(m+r-1)+a_m(m+r)+a_{m-2}-\nu^2a_m\right)x^{m+r}=0
\end{aligned}
$$

$$a_0r(r-1)+a_0r-\nu^2a_0=0 \Rightarrow a_0(r^2-\nu^2)=0 \Rightarrow r^2=\nu^2=\pm\frac{1}{2}$$
$$a_1(1+r)r+a_1(1+r)-a_1\nu^2=0 \Rightarrow a_1\left(r^2+2r+1-\nu^2\right)=a_1(2r+1)$$
$$\Rightarrow a_1=0 \text{ or } r=-\frac{1}{2}$$
$$a_m=\frac{-a_{m-2}}{(m+r)^2-\nu^2}$$
$$r_1=\frac{1}{2}:\ a_1=0$$
$$a_m=\frac{-a_{m-2}}{m(m+1)}$$
$$m=2:\ a_2=-\frac{a_0}{3\cdot 2}=-\frac{a_0}{3!}$$
$$m=3:\ a_3=-\frac{a_1}{3\cdot 4}=0$$
$$m=4:\ a_4=-\frac{a_2}{5\cdot 4}=\frac{a_0}{5!}$$
$$y_1=a_0x^{1/2}\left(1-\frac{x^2}{3!}+\frac{x^4}{5!}+\cdots\right)$$
$$y_1=a_0x^{-1/2}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)$$
$$y_1=a_0x^{-1/2}\sin x$$
$$r=-\frac{1}{2}$$
$$a_m=\frac{-a_{m-2}}{m(m-1)}$$
$$m=2:\ a_2=-\frac{a_0}{2\cdot 1}=-\frac{a_0}{2!}$$
$$m=3:\ a_3=-\frac{a_1}{3\cdot 2}=-\frac{a_1}{3!}$$
$$m=4:\ a_4=-\frac{a_2}{4\cdot 3}=\frac{a_0}{4!}$$
$$m=5:\ a_5=-\frac{a_3}{5\cdot 4}=\frac{a_1}{5!}$$
$$y_2=a_0x^{-1/2}\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots\right)+a_1x^{-1/2}\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\right)$$
$$y_2=a_0x^{-1/2}\cos x+a_1x^{-1/2}\sin x$$
$$y(x)=y_1+y_2=x^{-1/2}\left(A\sin x+B\cos x\right)$$

Ex3:

$$(1-x^2)y''-xy'+\alpha^2y=0 \text{ about } x=1$$
$$\lim_{x\to 1}(x-1)\frac{-x}{1-x^2}=\lim_{x\to 1}\frac{x}{1+x}=\frac{1}{2}$$
$$\lim_{x\to 1}(x-1)^2\frac{\alpha^2}{1-x^2}=\lim_{x\to 1}(x-1)\frac{\alpha^2}{1+x}=0$$

So $x = 1$ is a regular singular point

$$\text{At } x = 1$$
$$r(r-1) + \frac{1}{2}r = 0$$
$$r^2 + \frac{1}{2}r = 0 \Rightarrow r = 0,\ \frac{1}{2}$$

$$\text{let } t = x - 1,\ dt = dx \Rightarrow \frac{dy}{dx} = \frac{dy}{dt}$$
$$(1-x)(1+x)y'' - xy' + \alpha^2 y = 0$$
$$-t(t+2)y'' - (t+1)y' + \alpha^2 y = 0$$
$$y = \sum_{n=0}^{\infty} a_n t^{n+r}$$
$$y' = \sum_{n=0}^{\infty} a_n (n+r) t^{n+r-1}$$
$$y'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r-1}$$
$$-\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r} - 2\sum_{n=0}^{\infty} a_n (n+r)(n+r-1) t^{n+r-1} - \sum_{n=0}^{\infty} a_n (n+r) t^{n+r}$$
$$-\sum_{n=0}^{\infty} a_n (n+r) t^{n+r-1} + \alpha^2 \sum_{n=0}^{\infty} a_n t^{n+r} = 0$$
$$n = m + 1 \Rightarrow m = n - 1$$
$$-\sum_{m=0}^{\infty} a_m (m+r)(m+r-1) t^{m+r} - 2\sum_{m=-1}^{\infty} a_{m+1} (m+r+1)(m+r) t^{m+r} - \sum_{m=0}^{\infty} a_m (m+r) t^{m+r}$$
$$-\sum_{m=-1}^{\infty} a_{m+1} (m+r+1) t^{m+r} + \alpha^2 \sum_{m=0}^{\infty} a_m t^{m+r} = 0$$
$$-2a_0(r)(r-1)t^{r-1} - a_0(r)t^{r-1}$$
$$+\sum_{m=0}^{\infty} \left(-a_m(m+r)(m+r-1) - 2a_{m+1}(m+r+1)(m+r) - a_m(m+r) - a_{m+1}(m+r+1) + \alpha^2 a_m\right) t^{m+r}$$
$$-2r(r-1) - r = 0 \Rightarrow -2r^2 + r = 0 \Rightarrow r(1-2r) = 0 \Rightarrow r = 0,\ \frac{1}{2}$$
$$a_m\left(-(m+r)(m+r-1) - (m+r) + \alpha^2\right) = a_{m+1}\left(2(m+r+1)(m+r) + (m+r+1)\right)$$
$$a_{m+1} = a_m \frac{\alpha^2 - (m+r)(m+r-1) - (m+r)}{2(m+r+1)(m+r) + (m+r+1)}$$
$$r = 0:\ \ a_{m+1} = a_m \frac{\alpha^2 - m(m-1) - m}{2(m+1)(m) + (m+1)} = a_m \frac{\alpha^2 - m^2}{2m^2 + 3m + 1}$$
$$m = 0:\ \ a_1 = \alpha^2 a_0$$

$$m=1:\ a_2=\frac{\alpha^2-1}{6}a_1=\frac{(\alpha^2-1)\alpha^2}{6}a_0$$
$$m=2:\ a_3=\frac{\alpha^2-4}{15}a_2=\frac{(\alpha^2-4)(\alpha^2-1)\alpha^2}{90}a_0$$
$$y_1(t)=a_0\left(1+\alpha^2t+\frac{(\alpha^2-1)\alpha^2}{6}t^2+\frac{(\alpha^2-4)(\alpha^2-1)\alpha^2}{90}t^3+\cdots\right)$$
$$y_1(x)=a_0\left(1+\alpha^2(x-1)+\frac{(\alpha^2-1)\alpha^2}{6}(x-1)^2+\frac{(\alpha^2-4)(\alpha^2-1)\alpha^2}{90}(x-1)^3+\cdots\right)$$
$$r=\frac{1}{2}:\ a_{m+1}=a_m\frac{\alpha^2-(m+\frac{1}{2})(m-\frac{1}{2})-(m+\frac{1}{2})}{2(m+\frac{3}{2})(m+\frac{1}{2})+(m+\frac{3}{2})}=a_m\frac{4\alpha^2-(2m+1)(2m-1)-2(2m+1)}{2(2m+3)(2m+1)+2(2m+3)}$$
$$m=0:\ a_1=\frac{\alpha^2+1-2}{6+6}a_0=\frac{4\alpha^2-1}{12}a_0$$
$$m=1:\ a_2=\frac{4\alpha^2-(3)(1)-2(3)}{2(5)(3)+2(5)}a_1=\frac{4\alpha^2-9}{40}a_1=\frac{(4\alpha^2-9)(4\alpha^2-1)}{480}a_0$$
$$m=2:\ a_3=\frac{\alpha^2-(5)(3)-2(5)}{2(7)(5)+2(7)}a_2=\frac{\alpha^2-25}{84}a_2=\frac{(\alpha^2-9)(\alpha^2-4)(\alpha^2-1)}{40320}$$
$$y_2(t)=a_0\left(t^{1/2}+\frac{4\alpha^2-1}{12}t^{3/2}+\frac{(4\alpha^2-9)(4\alpha^2-1)}{480}t^{5/2}+\frac{(\alpha^2-9)(\alpha^2-4)(\alpha^2-1)}{40320}t^{7/2}+\cdots\right)$$
$$y_2(x)=a_0\left((x-1)^{1/2}+\frac{4\alpha^2-1}{12}(x-1)^{3/2}+\frac{(4\alpha^2-9)(4\alpha^2-1)}{480}(x-1)^{5/2}+\cdots\right)$$
$$y=A+B(x-1)^{1/2}+\sum_{n=1}^{\infty}\left(A(x-1)^n\prod_{m=0}^{n-1}\left(\frac{\alpha^2-m^2}{2m^2+3m+1}\right)+B(x-1)^{n+1/2}\prod_{m=0}^{n-1}\left(\frac{4\alpha^2-(2m+1)^2}{4(2m^2+5m+3)}\right)\right)$$

1.4 Laplace Transform

1.4.1 Definition of the Laplace Transform

The Laplace transform maps a function to another function. It effectively maps the frequency and exponential components of a function, converting it from the time domain to the frequency domain. It has some properties that become particularly useful in solving differential equations.
The Laplace transform is defined by

$$\mathscr{L}\{f(t)\}=\int_0^\infty f(t)e^{-st}dt$$

The domain of $\mathscr{L}\{f(t)\}$ is the set of s where $\mathscr{L}\{f(t)\}$ converges.
Note that because of the integral, the Laplace operator is linear.

$$\mathscr{L}\{c_1f(t)+c_2g(t)\}=c_1\mathscr{L}\{f\}+c_2\mathscr{L}\{g\}$$

However,

$$\mathscr{L}\{fg\}\neq\mathscr{L}\{f\}\cdot\mathscr{L}\{g\}$$

Ex: Find the Laplace transform of 1.

$$\mathscr{L}\{1\}=\lim_{A\to\infty}\int_0^A e^{-st}dt=\lim_{A\to\infty}\left[\frac{e^{-st}}{-s}\right]_0^A=\lim_{A\to\infty}-\frac{1}{s}\left(e^{-sA}-1\right)=\frac{1}{s},\ s>0$$

Ex2: Find the Laplace transform of e^{at}

$$\begin{aligned}
&\mathscr{L}\left\{e^{at}\right\} = \lim_{A\to\infty}\int_0^A e^{at}\cdot e^{-st}dt \lim_{A\to\infty}\int_0^A e^{(a-s)t}dt \\
&= \lim_{A\to\infty}\frac{1}{a-s}\left(e^{(a-s)A}-1\right) = \frac{1}{s-a},\ s>-a
\end{aligned}$$

We can use the result from this example to also compute the Laplace transforms of $\sin(at)$ and $\cos(at)$.

$$\begin{aligned}
&\mathscr{L}\left\{e^{iat}\right\} = \frac{1}{s-ia} = \frac{s+ia}{s^2+a^2} \\
&\mathscr{L}\left\{\sin(at)\right\} = \Re\left\{\frac{s+ia}{s^2+a^2}\right\} = \frac{s}{s^2+a^2},\ s>0 \\
&\mathscr{L}\left\{\sin(at)\right\} = \Im\left\{\frac{s+ia}{s^2+a^2}\right\} = \frac{a}{s^2+a^2},\ s>0
\end{aligned}$$

Ex3: Find the Laplace transform of t^n

$$\begin{aligned}
&\mathscr{L}\left\{t^n\right\} = \int_0^\infty t^n e^{-st}dt \\
&u = t^n \Rightarrow du = nt^{n-1}dt \\
&dv = e^{-st}dt \Rightarrow v = -\frac{e^{-st}}{s} \\
&\mathscr{L}\left\{t^n\right\} = \lim_{A\to\infty}\left(\left[-\frac{t^n}{s}e^{-st}\right]_0^A - \int_0^A -\frac{nt^{n-1}}{s}e^{-st}dt\right) = \lim_{A\to\infty}\frac{n}{s}\int_0^A t^{n-1}e^{-st}dt \\
&\vdots \\
&\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^n}\int_0^\infty e^{-st}dt = \frac{n!}{s^n}\mathscr{L}\left\{1\right\} = \frac{n!}{s^{n+1}},\ s>0
\end{aligned}$$

Ex4: Find the Laplace transform of $\int_0^t f(\tau)d\tau$

$$\begin{aligned}
&\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \int_0^\infty\int_0^t f(\tau)d\tau e^{-st}dt \\
&u = \int_0^t f(\tau)d\tau \Rightarrow du = f(t)dt \\
&dv = e^{-st}dt \Rightarrow v = -se^{-st} \\
&\mathscr{L}\left\{\int_0^t f(\tau)d\tau\right\} = \left[\int_0^t f(\tau)d\tau e^{-st}\right]_0^\infty + \frac{1}{s}\int_0^\infty f(t)e^{-st}dt = \frac{\mathscr{L}\left\{f(t)\right\}}{s}
\end{aligned}$$

Ex5: Find the Laplace transform of $f'(t)$

$$\begin{aligned}
&\mathscr{L}\left\{f'(t)\right\} = \int_0^\infty f'(t)e^{-st}dt \\
&u = e^{-st} \Rightarrow du = -\frac{e^{-st}}{s}dt
\end{aligned}$$

$$dv = f'(t)dt \Rightarrow v = f(t)$$

$$\mathscr{L}\left\{f'(t)\right\} = \left[f(t)e^{-st}\right]_0^\infty + s\int_0^\infty f(t)e^{-st}dt = s\,\mathscr{L}\left\{f(t)\right\} - f(0)$$

This can be defined recursively for multiple derivatives to be

$$\mathscr{L}\left\{f^{(n)}(t)\right\} = s^n\,\mathscr{L}\left\{f(t)\right\} - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$$

Ex6: Find the derivative of the Laplace transform

$$\frac{d}{ds}\mathscr{L}\left\{f(t)\right\} = \frac{d}{ds}\int_0^\infty f(t)e^{-st}dt = -t\int_0^\infty f(t)e^{-st}dt = -t\,\mathscr{L}\left\{f(t)\right\}$$

This can also be defined recursively to give the identity

$$\mathscr{L}\left\{(-t)^n f(t)\right\} = \frac{d^n}{ds^n}\mathscr{L}\left\{f(t)\right\}$$

Using the Laplace transforms we already have, we can combine them to find the transforms of other non-elementary functions.
Ex7: Find the Laplace tranform of

$$\mathscr{L}\left\{t^2\sin(at)\right\}$$

use $\mathscr{L}\left\{(-t)^n f(t)\right\} = \dfrac{d^n}{ds^n}\mathscr{L}\left\{f(t)\right\}$

$$(-t)^2 = t^2$$

$$\mathscr{L}\left\{t^2\sin(at)\right\} = \frac{d^2}{ds^2}\mathscr{L}\sin(at) = \frac{d^2}{ds^2}\frac{a}{s^2+a^2},\ s>0$$

$$= \frac{d}{ds}\frac{-2as}{(s^2+a^2)^2} = \frac{(s^2+a^2)^2(-2a) - (-2as)(2)(s^2+a^2)(2s)}{(s^2+a^2)^4} = \frac{-2a(s^2+a^2)+8as^2}{(s^2+a^2)^3}$$

$$= \frac{6as^2-2a^3}{(s^2+a^2)^3},\ s>0$$

Ex8: Given that $\mathscr{L}\left\{\sqrt{t}\right\} = \frac{\sqrt{\pi}}{2s^{3/2}}$ find $\mathscr{L}\left\{\frac{1}{\sqrt{t}}\right\}$

$$\frac{d}{dt}\sqrt{t} = \frac{1}{2\sqrt{t}} \Rightarrow \frac{d}{dt}2\sqrt{t} = \frac{1}{\sqrt{t}}$$

$$\mathscr{L}\left\{f'(t)\right\} = s\,\mathscr{L}\left\{f(t)\right\} - f(0) \Rightarrow \mathscr{L}\left\{\frac{1}{\sqrt{t}}\right\} = 2s\,\mathscr{L}\left\{\sqrt{t}\right\} - 2\sqrt{0} = \frac{2s\sqrt{\pi}}{2s^{3/2}} = \sqrt{\frac{\pi}{s}}$$

alternative solution:

$$\mathscr{L}\left\{\sqrt{t}\right\} = \mathscr{L}\left\{\frac{t}{\sqrt{t}}\right\} = -\frac{d}{ds}\mathscr{L}\left\{\frac{1}{\sqrt{t}}\right\}$$

$$\mathscr{L}\left\{\frac{1}{\sqrt{t}}\right\} = -\int \mathscr{L}\left\{\sqrt{t}\right\}ds = -\int \frac{\sqrt{\pi}}{2s^{3/2}}ds = \sqrt{\frac{\pi}{s}}$$

1.4.2 Inverse Laplace Transform

Using the table of Laplace transforms becomes particularly useful in finding inverse Laplace transforms. In addition to the functions found, there is also a shifting law such that

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s+a)$$

where $F(s)$ is the Laplace transform of $f(t)$. Some common occurrences of this shifting law are

$$\begin{aligned}
e^{-at}\sin(bt) &= \frac{b}{(s+a)^2+b^2}\ s > -a \\
e^{-at}\cos(bt) &= \frac{s+a}{(s+a)^2+b^2},\ s > -a \\
e^{-at}t^n &= \frac{n!}{(s+a)^{n+1}},\ s > -a
\end{aligned}$$

Ex: $\mathscr{L}^{-1}\left\{\dfrac{8s^2-4s+12}{s(s^2+4)}\right\}$

$$\begin{aligned}
&\frac{8s^2-4s+12}{s(s^2+4)} = \frac{A}{s} + \frac{Bs+C}{s^2+4} \\
&As^2+4A+Bs^2+Cs = 8s^2-4s+12 \Rightarrow A = 3 \\
&3s^2+12+Bs^2+Cs = 8s^2-4s+12 \\
&(3+B)s^2+Cs = 8s^2-4s \Rightarrow C = -4 \Rightarrow B = 5 \\
&\frac{8s^2-4s+12}{s(s^2+4)} = \frac{3}{s} + \frac{5s-4}{s^2+4} = \frac{3}{s} + \frac{5s}{s^2+4} - \frac{4}{s^2+4} \\
&\mathscr{L}^{-1}\left\{\frac{8s^2-4s+12}{s(s^2+4)}\right\} = 3 + 5\cos(2t) - 2\sin(2t)
\end{aligned}$$

Ex2: $\mathscr{L}^{-1}\left\{\dfrac{s^3-1}{(s^2+1)(s+2)^2}\right\}$

$$\begin{aligned}
&\frac{s^3-1}{(s^2+1)(s+2)^2} = \frac{As+B}{s^2+1} + \frac{C}{s+2} + \frac{D}{(s+2)^2} \\
&(As+B)(s^2+4s+4) + C(s^2+1)(s+2) + Ds^2 + D = s^3-1 \\
&As^3+4As^2+4As+Bs^2+4Bs+4B+Cs^3+2Cs^2+Cs+2C+Ds^2+D = s^3-1
\end{aligned}$$

$$\begin{cases} A+C=1 \\ 4A+B+2C+D=0 \\ 4A+4B+C=0 \\ 4B+2C+D=-1 \end{cases} \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 4 & 1 & 2 & 1 & 0 \\ 4 & 4 & 1 & 0 & 0 \\ 0 & 4 & 2 & 1 & -1 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & -4 \\ 0 & 4 & -3 & 0 & -4 \\ 0 & 4 & 2 & 1 & -1 \end{array}\right]$$

$$\to \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & -4 \\ 0 & 0 & 5 & -4 & 12 \\ 0 & 0 & 10 & -3 & 15 \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{1}{25} \\ 0 & 1 & 0 & 0 & -\frac{7}{25} \\ 0 & 0 & 1 & 0 & \frac{24}{25} \\ 0 & 0 & 0 & 1 & -\frac{9}{5} \end{array}\right] \Rightarrow \begin{cases} A = \frac{1}{25} \\ B = -\frac{7}{25} \\ C = \frac{24}{25} \\ D = -\frac{9}{5} \end{cases}$$

$$
\begin{aligned}
&\frac{s^3-1}{(s^2+1)(s+2)^2} = \frac{1}{25}\frac{s-7}{s^2+1} + \frac{24/25}{s+2} - \frac{9/5}{(s+2)^2} = \frac{s/25}{s^2+1} - \frac{7/25}{s^2+1} + \frac{24/25}{s+2} - \frac{9/5}{(s+2)^2} \\
&\mathscr{L}^{-1}\left\{\frac{s^3-1}{(s^2+1)(s+2)^2}\right\} = \frac{1}{25}\cos(t) - \frac{7}{25}\sin(t) + \frac{24}{25}e^{-2t} - \frac{9}{5}\mathscr{L}\left\{\frac{1}{(s+2)^2}\right\} \\
&\text{use } \mathscr{L}\left\{t^n e^{-at}\right\} = \frac{n!}{(s+a)^{n+1}},\ n=1 \\
&\mathscr{L}\left\{\frac{1}{(s+2)^2}\right\} = te^{-2t} \\
&\Rightarrow \mathscr{L}^{-1}\left\{\frac{s^3-1}{(s^2+1)(s+2)^2}\right\} = \frac{1}{25}\cos(t) - \frac{7}{25}\sin(t) + \frac{24}{25}e^{-2t} - \frac{9}{5}te^{-2t}
\end{aligned}
$$

Sometimes the inverse Laplace can be difficult to find using conventional methods or even be impossible to express. In this case, we can define the convolution

$$
\mathscr{L}\left\{f(t) * g(t)\right\} = \mathscr{L}\left\{f(t)\right\} \cdot \mathscr{L}\left\{g(t)\right\}
$$

$$
\begin{aligned}
&\mathscr{L}\left\{f * g\right\} = FG = \left(\int_0^\infty f(\tau)e^{-s\tau}d\tau\right)\left(\int_0^\infty g(\nu)e^{-s\nu}d\nu\right) = \int_0^\infty \int_0^\infty f(\tau)g(\nu)e^{-s\nu}e^{-s\tau}d\nu d\tau \\
&\text{let } \nu = t-\tau \\
&FG = \int_0^\infty \int_\tau^\infty f(\tau)g(t-\tau)e^{-st}dt d\tau = \int_0^\infty \left(\int_0^t f(\tau)g(t-\tau)d\tau\right)e^{-st}dt \\
&\Rightarrow f * g = \int_0^t f(\tau)g(t-\tau)d\tau
\end{aligned}
$$

1.4.3 Solving Differential Equations

We can use the derivative properties of the Laplace transform to help solve differential equations. The expression $y'' + ay' + by = f(t)$ can be solved using the following method:

$$
\begin{aligned}
&\mathscr{L}\left\{y'' + ay' + by\right\} = \mathscr{L}\left\{f(t)\right\} \\
&\text{let } Y = \mathscr{L}\left\{y(t)\right\} \\
&s^2Y - sy(0) - y'(0) + saY - ay(0) + bY = \mathscr{L}\left\{f(t)\right\} \\
&Y = \frac{\mathscr{L}\left\{f(t)\right\} + sy(0) + y(0) + y'(0)}{s^2+as+b} \\
&y(t) = \mathscr{L}^{-1}\left\{Y\right\} = \mathscr{L}^{-1}\left\{\frac{\mathscr{L}\left\{f(t)\right\} + sy(0) + y(0) + y'(0)}{s^2+as+b}\right\}
\end{aligned}
$$

Ex: $y'' - 2y' + 2y = e^{-t},\ y(0) = 0,\ y'(0) = 1$

$$
\begin{aligned}
&\mathscr{L}\left\{y'' - 2y' + 2y\right\} = \mathscr{L}\left\{e^{-t}\right\} \\
&s^2Y - 1 - 2sY + 2Y = \frac{1}{s+1} \\
&Y(s^2 - 2s + 2) = \frac{1}{s+1} + 1 = \frac{1}{s+1} + \frac{s+1}{s+1} = \frac{s+2}{s+1}
\end{aligned}
$$

$$Y = \frac{s+2}{(s+1)(s^2-2s+2)}$$

$$y = \mathscr{L}^{-1}\{y\} = \mathscr{L}^{-1}\left\{\frac{s+2}{(s+1)(s^2-2s+2)}\right\}$$

$$\frac{s+2}{(s+1)(s^2-2s+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-2s+2}$$

$$As^2 - 2sA + 2A + Bs^2 + Cs + Bs + C = s+2$$

$$\begin{cases} A+B=0 \\ -2A+C+B=1 \\ 2A+C=2 \end{cases} \Rightarrow A=-B \Rightarrow \begin{cases} 3B+C=1 \\ -2B=2-C \end{cases} \Rightarrow B=-\frac{1}{5} \Rightarrow A=\frac{1}{5} \Rightarrow C=\frac{8}{5}$$

$$\frac{s+2}{(s+1)(s^2-2s+2)} = \frac{1/5}{s+1} + \frac{-s/5+8/5}{(s^2-2s+2)}$$

$$-\frac{1}{5}\frac{s-8}{s^2-2s+1+1} = -\frac{1}{5}\frac{s-2}{(s-1)^2+1} = -\frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{7}{5}\frac{1}{(s-1)^2+1}$$

$$y = \mathscr{L}^{-1}\left\{\frac{1/5}{s+1} - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{7/5}{(s-1)^2+1}\right\}$$

$$y = \frac{1}{5}e^{-t} - \frac{1}{5}e^t\cos(t) + \frac{7}{5}e^t\sin(t)$$

$$y(t) = \frac{1}{5}(e^{-t} - e^t\cos(t) + 7e^t\sin(t))$$

Ex2: $x''' + x = 0,\ x(0)=1,\ x'(0)=0,\ x''(0)=0$

$$\mathscr{L}\left\{x''' + x\right\} = 0$$

$$s^3X - s^2 + X = 0 \Rightarrow X(s^3+1) = s^2$$

$$X = \frac{s^2}{s^3+1} = \frac{s^2}{(s+1)(s^2-s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2-s+1}$$

$$As^2 - As + A + Bs^2 + Cs + Bs + C = s^2$$

$$\begin{cases} A+B=1 \\ -A+C+B=0 \\ A+C=0 \end{cases} \Rightarrow A=-C \Rightarrow \begin{cases} -C+B=1 \\ 2C=-B \end{cases} \Rightarrow C=-\frac{1}{3} \Rightarrow B=\frac{2}{3} \Rightarrow A=\frac{1}{3}$$

$$X = \frac{1/3}{s+1} + \frac{1}{3}\frac{2s-1}{s^2-s+1}$$

$$\frac{2s-1}{s^2-s+1} = \frac{2s-1}{\left(s^2-s+\frac{1}{4}\right)+\frac{3}{4}} = \frac{2(s-\frac{1}{2})}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$X = \frac{1/3}{s+1} + \frac{2}{3}\frac{\left(s-\frac{1}{2}\right)}{\left(s-\frac{1}{2}\right)^2+\frac{3}{4}}$$

$$x = \mathscr{L}\{X\} = \frac{1}{3}e^{-t} + \frac{2}{3}e^{\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right)$$

1.4.4 Heaviside and Dirac Delta Functions

We can combine piecewise functions into one function by introducing the Heaviside (or unit step) function $u(t)$. This function is defined to be 0 up until the time $t = 0$ when it instantaneously jumps to 1.

$$u(t) = \begin{cases} 0 \text{ if } t < 0 \\ 1 \text{ if } t > 0 \end{cases}$$

If we shift the function, we get $u(t-a)$ or $u_a(t)$ which means that the function jumps to the value 1 at time a instead of time 0.
The Dirac delta function $\delta(x)$ can be thought of as the derivative of the Heaviside function. It is defined to be zero everywhere except for at $x = 0$ where it is an impulse jump of infinity.

$$\delta(t) = \begin{cases} 0 \text{ if } t \neq 0 \\ \infty \text{ if } t = 0 \end{cases}$$

It also has the property that the area of this impulse is 1.

$$\int_{-\epsilon}^{\epsilon} \delta(x)dx = 1$$

The Laplace transform of the delta function can be computed as

$$\mathscr{L}\{\delta(t-a)\} = \int_0^\infty \delta(t)e^{-st}dt = \lim_{\epsilon\to 0}\int_{a-\epsilon}^{a+\epsilon} \delta(t)e^{-st}dt = e^{-at}$$

The Laplace transform of the Heaviside function gives a second shifting law

$$\begin{aligned}
&\mathscr{L}\{u(t-a)f(t-a)\} = \int_0^\infty u(t-a)f(t-a)e^{-st}dt = \int_a^\infty u(t-a)f(t-a)e^{-st}dt \\
&\text{let } \tau = t-a \Rightarrow d\tau = dt \\
&t = \tau + a,\ \tau(a) = 0,\ \lim_{t\to\infty} \tau = \lim_{A\to\infty}(A-a) = \infty \\
&\mathscr{L}\{u(t-a)f(t-a)\} = \int_0^\infty u(\tau)f(\tau)e^{-(\tau+a)s}d\tau = e^{-as}\int_0^\infty f(\tau)d\tau = e^{-as}\mathscr{L}\{f(\tau)\} \\
&= e^{-as}\mathscr{L}\{f(t)\}
\end{aligned}$$

Ex: $\mathscr{L}\{f(t)\},\ f(t) = \begin{cases} t & 0 \le t < 1 \\ 2-t & 1 \le t < 2 \\ 0 & 2 \le t \end{cases}$

$$\begin{aligned}
&f(t) = t(u(t) - u(t-1)) + (2-t)(u(t-1) - u(t-2)) \\
&f(t) = tu(t) - 2tu(t-1) + 2u(t-1) - 2u(t-2) + tu(t-2) \\
&\text{use } \mathscr{L}\{u(t-a)f(t-a)\} = e^{-as}\mathscr{L}(f(t)) \\
&f(t) = tu(t) - 2(t-1)u(t-1) + (t-2)u(t-2) \\
&\mathscr{L}\{t-a\} = \frac{1}{s^2},\ s > 0 \\
&\mathscr{L}\{f(t)\} = \frac{1}{s^2} - 2e^{-s}\left(\frac{1}{s^2}\right) + e^{-2s}\left(\frac{1}{s^2}\right),\ s > 0
\end{aligned}$$

Ex2: $\mathscr{L}^{-1}\left\{\frac{e^{-s}+e^{-2s}-e^{-3s}-e^{-4s}}{s}\right\} = \mathscr{L}^{-1}\left\{\frac{e^{-s}}{s}\right\} + \mathscr{L}^{-1}\left\{\frac{e^{-2s}}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\} - \mathscr{L}^{-1}\left\{\frac{e^{-4s}}{s}\right\}$

$= u(t-1) + u(t-2) - u(t-3) - u(t-4)$

Ex3: $x'' + 4x' + 3x = 2\delta(t-\pi),\ x(0) = 2,\ x'(0) = 0$

$$\mathscr{L}\left\{x'' + 4x' + 3x\right\} = 2\,\mathscr{L}\left\{\delta(t-\pi)\right\}$$

$$s^2X - 2s + 4sX - 8 + 3X = 2e^{-\pi s}$$

$$X = \frac{2e^{-\pi s} + 2s + 8}{s^2 + 4s + 3}$$

$$X = 2\left(\frac{e^{-\pi s}}{(s+3)(s+1)} + \frac{s}{(s+3)(s+1)} + \frac{4}{(s+3)(s+1)}\right)$$

$$\frac{s}{(s+3)(s+1)} = \frac{A}{s+3} + \frac{B}{s+1}$$

$$sA + A + sB + 3B = s$$

$$\begin{cases} A + B = 1 \\ A + 3B = 0 \end{cases} \Rightarrow -3B + B = 1 \Rightarrow B = -\frac{1}{2} \Rightarrow A = \frac{3}{2}$$

$$\frac{1}{(s+3)(s+1)} = \frac{A}{s+3} + \frac{B}{s+1}$$

$$sA + A + sB + 3B = 1$$

$$\begin{cases} A + B = 0 \\ A + 3B = 1 \end{cases} \Rightarrow -B + 3B = 1 \Rightarrow B = \frac{1}{2} \Rightarrow A = -\frac{1}{2}$$

$$X = -\frac{e^{-\pi s}}{s+3} + \frac{e^{-\pi s}}{s+1} + \frac{3}{s+3} - \frac{1}{s+1} - \frac{4}{s+3} + \frac{4}{s+1}$$

$$x = -u(t-\pi)e^{-3(t-\pi)} + u(t-\pi)e^{-(t-\pi)} + 3e^{-3t} - e^{-t} - 4e^{-3t} + 4e^{-t}$$

Ex4: $y'' + y' + \frac{5}{4}y = \begin{cases} \sin(t), & 0 \le t \le \pi \\ 0, & \pi \le t \end{cases},\ y(0) = 0,\ y'(0) = 0$

$$g(t) = \sin(t)(u(t) - u(t-\pi)) = u(t)\sin(t) + u(t-\pi)\sin(t-\pi)$$

$$\mathscr{L}\left\{y'' + y' + \frac{5}{4}y\right\} = \mathscr{L}\left\{u(t)\sin(t) + u(t-\pi)\sin(t-\pi)\right\}$$

$$s^2Y + sY + \frac{5}{4}Y = \frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}$$

$$Y = \left(\frac{1}{s^2 + s + \frac{5}{4}}\right)\left(\frac{1}{s^2+1} + \frac{e^{-\pi s}}{s^2+1}\right)$$

$$Y = \frac{1}{(s^2 + s + \frac{5}{4})(s^2+1)} + \frac{e^{-\pi s}}{(s^2 + s + \frac{5}{4})(s^2+1)}$$

$$y = \mathscr{L}^{-1}\left\{\frac{1}{(s^2 + s + \frac{5}{4})(s^2+1)}\right\} + u(t-\pi)\,\mathscr{L}^{-1}\left\{\frac{1}{(s^2 + s + \frac{5}{4})(s^2+1)}\right\}$$

$$\frac{1}{(s^2 + s + \frac{5}{4})(s^2+1)} = \frac{As + B}{s^2 + s + \frac{5}{4}} + \frac{Cs + D}{s^2+1}$$

$$\rightsquigarrow \frac{4}{17}\frac{1-4s}{s^2+1} + \frac{4}{17}\frac{4s+3}{s^2+s+\frac{5}{4}}$$

$$\frac{1}{s^2+s+\frac{5}{4}} = \frac{1}{\left(s+\frac{1}{2}\right)^2+1}$$

$$\frac{4}{17}\frac{1-4s}{s^2+1} + \frac{4}{17}\frac{4s+3}{s^2+s+\frac{5}{4}} = \frac{4}{17}\left(\frac{1}{s^2+1} - \frac{4s}{s^2+1} + \frac{4s+2}{(s^2+\frac{1}{2})^2+1} + \frac{1}{(s+\frac{1}{2})^2+1}\right)$$

$$\frac{4}{17}\mathscr{L}^{-1}\left\{\frac{1}{(s^2+s+\frac{5}{4})(s^2+1)}\right\} = \frac{4}{17}\left(\sin(t) - 4\cos(t) + 4e^{-\frac{t}{2}}\cos(t) + e^{-\frac{t}{2}}\sin(t)\right)$$

$$y = \frac{4}{17}\left(\sin(t) - 4\cos(t) + 4e^{-\frac{t}{2}}\cos(t) + e^{-\frac{t}{2}}\sin(t)\right)$$
$$+ \frac{4}{17}u(t-\pi)\left(\sin(t-\pi) - 4\cos(t-\pi) + 4e^{-\frac{(t-\pi)}{2}}\cos(t-\pi) + e^{-\frac{(t-\pi)}{2}}\sin(t-\pi)\right)$$

1.5 First Order Systems of Differential Equations

1.5.1 Converting to First Order

A system of differential equations is where we are dealing with multiple dependent variables, $x_1(t), x_2(t), \ldots, x_n(t)$.
In some cases, each system can be solved independently, making them easy to solve.
Ex: solve $\begin{cases} x_1' = x_2 - x_1 + t \\ x_2' = x_2 \end{cases}$

$$x_2 = C_2e^t$$
$$x_1' = C_2e^t - x_1 + t$$
$$x_1' + x_1 = C_2e^t + t$$
$$r = e^t$$
$$x_1 = r^{-1}\int rg\,dt$$
$$x_1 = e^{-t}\int e^t(C_2e^t + t)dt$$
$$\int te^t dt$$
$$u = t \Rightarrow du = dt$$
$$dv = e^t dt \Rightarrow v = e^t$$
$$\int te^t dt = te^t - \int e^t dt = te^t - e^t + C$$
$$x_1 = e^{-t}\left(\frac{C_2}{2}e^{2t} + te^t - e^t + C\right)$$
$$x_1 = \frac{C_2}{2}e^t + t - 1 + C_1e^{-t}$$

$$\vec{x} = \begin{bmatrix} \frac{C_2}{2}e^t + t - 1 + C_1 e^{-t} \\ C_2 e^t \end{bmatrix}$$

This is quite uncommon though. In most cases we will be required to express the system as a matrix and solve using linear algebra methods.
A general first order system of differential equations is given by

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}(t)$$

Many systems of equations can be represented as first order systems.
Ex: express $y'' - 4y' + 3y = 0$ as a first order system.

$$\begin{aligned}
&x_1 = y, \ x_2 = y' \\
&x_1' = x_2 \\
&x_2' - 4x_2 + 3x_1 = 0 \Rightarrow x_2' = 4x_2 - 3x_1 \\
&\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&\vec{x}' = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} \vec{x}
\end{aligned}$$

Ex2: express the system $\begin{cases} m_1 x_1'' + cx_1' = -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 x_2'' + cx_2' = -k_2(x_2 - x_1) - k_3 x_2 \end{cases}$ as a first order system.

$$\text{let } u_1 = x_1, \ u_2 = x_1', \ u_3 = x_2, \ u_4 = x_2'$$

$$\begin{cases} m_1 u_2' + cu_2 = -ku_1 + k_2 u_3 - k_2 u_1 \\ m_2 u_4' + cu_4 = -k_2 u_3 + k_2 u_1 - k_3 u_3 \\ u_1' = u_2 \\ u_3' = u_4 \end{cases} \Rightarrow \begin{cases} u_2' = \frac{1}{m_1}\left(k_2 u_3 - (k_1 + k_2)u_1 - cu_2\right) \\ u_4' = \frac{1}{m_2}\left(k_2 u_1 - (k_2 + k_3)u_3 - cu_4\right) \\ u_1' = u_2 \\ u_3' = u_4 \end{cases}$$

$$\frac{d\vec{u}}{dt} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{k_1+k_2}{m_1} & -\frac{c}{m_1} & \frac{k_2}{m_1} & 0 \\ 0 & 0 & 0 & 1 \\ \frac{k_2}{m_2} & 0 & -\frac{k_2+k_3}{m_2} & -\frac{c}{m_2} \end{bmatrix} \vec{u}$$

1.5.2 Linear Systems of Differential Equations

These are differential equations in the form of $\frac{d\vec{x}}{dt} = A\vec{x}$ and can be solved with the following method:

1. Set $\frac{d\vec{x}}{dt} = A\vec{x}$ and find the matrix A
2. Find eigenvalues and eigenvectors of A
3. Write the general solution
4. Express the general solution as $\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots$
5. If imaginary, express conjugate pairs as $c_1\vec{P}(t) + c_2\vec{Q}(t)$ where $\vec{P}(t) = \Re\left(e^{\alpha t}\vec{v}_1(\cos(\beta t) + i\sin(\beta t))\right)$ and $\vec{Q}(t) = \Im\left(e^{\alpha t}\vec{v}_1(\cos(\beta t) + i\sin(\beta t))\right)$

6. Use the initial conditions to solve for the coefficients, $\vec{c}$

Real Eigenvalues

This is the most general case where the number of eigenvalues matches the size of the matrix. In this case the general solution can be expressed as

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + \cdots$$

Ex: $\begin{cases} \frac{dx_1}{dt} = x_1 + x_3,\ x_1(0) = 1 \\ \frac{dx_2}{dt} = x_2,\ x_2(0) = 1 \\ \frac{dx_3}{dt} = x_1 + x_3,\ x_3(0) = 0 \end{cases}$

$$\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = A\vec{x} \Rightarrow A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix},\ \vec{x}(0) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Eigen-analysis gives

$$\begin{cases} \lambda_1 = 0,\ \vec{v}_1 = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}^T \\ \lambda_2 = 1,\ \vec{v}_2 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T \\ \lambda_3 = 2,\ \vec{v}_3 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T \end{cases}$$

$$\vec{x}(t) = c_1 e^{0t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$c_1\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \rightsquigarrow \vec{c} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

$$\vec{x}(t) = -\frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + e^{t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{2}e^{2t}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2}e^{2t} \\ e^{t} \\ -\frac{1}{2} + \frac{1}{2}e^{2t} \end{bmatrix}$$

Complex Eigenvalues

This case will have the same general solution but because of the imaginary components, it can be simplified using Euler's equation. We can express any conjugate pairs as $c_1\vec{P}(t) + c_2\vec{Q}(t)$ where

$$\vec{P}(t) = \Re\left\{e^{\alpha t}\vec{v}_1(\cos(\beta t) + i\sin(\beta t))\right\} \text{ and } \vec{Q}(t) = \Im\left\{e^{\alpha t}\vec{v}_1(\cos(\beta t) + i\sin(\beta t))\right\}$$

Ex: $\dfrac{d\vec{x}}{dt} = A\vec{x},\ A = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 3 \end{bmatrix}$

eigen-analysis gives

$$\begin{cases} \lambda_1 = 3,\ \vec{v}_1 = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}^T \\ \lambda_2 = i,\ \vec{v}_2 = \begin{bmatrix} -3i & -3+i & 1 \end{bmatrix}^T \\ \lambda_3 = -i,\ \vec{v}_3 = \begin{bmatrix} 3i & -3-i & 1 \end{bmatrix}^T \end{cases}$$

Real part: $e^{\lambda_1 t}\vec{v}_1 = e^{3t}\begin{bmatrix}1\\0\\1\end{bmatrix}$

Imaginary part: $e^{\lambda_2 t}\vec{v}_2 = e^{it}\begin{bmatrix}-3i\\-3+i\\1\end{bmatrix} = (\cos(t) + i\sin(t))\begin{bmatrix}-3i\\-3+i\\1\end{bmatrix}$

$= \begin{bmatrix}3\sin(t) - 3i\cos(t)\\-3\cos(t) + i\cos(t) - 3i\sin(t) - \sin(t)\\\cos(t) + i\sin(t)\end{bmatrix}$

$\vec{P}(t) = \Re\begin{bmatrix}3\sin(t) - 3i\cos(t)\\-3\cos(t) + i\cos(t) - 3i\sin(t) - \sin(t)\\\cos(t) + i\sin(t)\end{bmatrix} = \begin{bmatrix}3\sin(t)\\-3\cos(t) - \sin(t)\\\cos(t)\end{bmatrix}$

$\vec{Q}(t) = \Im\begin{bmatrix}3\sin(t) - 3i\cos(t)\\-3\cos(t) + i\cos(t) - 3i\sin(t) - \sin(t)\\\cos(t) + i\sin(t)\end{bmatrix} = \begin{bmatrix}-3\cos(t)\\\cos(t) - 3\sin(t)\\\sin(t)\end{bmatrix}$

$\vec{x}(t) = c_1 e^{3t}\begin{bmatrix}1\\0\\1\end{bmatrix} + c_2\begin{bmatrix}3\sin(t)\\-3\cos(t) - \sin(t)\\\cos(t)\end{bmatrix} + c_3\begin{bmatrix}-3\cos(t)\\\cos(t) - 3\sin(t)\\\sin(t)\end{bmatrix}$

Repeated Eigenvalues

With repeated eigenvalues, we may not have enough eigenvectors to capture the solution space. We consider the size of the matrix to be called the *algebraic multiplicity* of the system and the number of eigenvectors the *geometric multiplicity* of the system.

In the case where the geometric multiplicity is the same as the algebraic multiplicity, the solution will follow the from the regular, non-repeated, case.

Ex: $\vec{x}' = \begin{bmatrix}2 & 0\\0 & 2\end{bmatrix}\vec{x}$

$\lambda = 2$

$A - \lambda I = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} \to \vec{v}_1 = \begin{bmatrix}1\\0\end{bmatrix},\ \vec{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}$

$\vec{x} = c_1 e^{2t}\begin{bmatrix}1\\0\end{bmatrix} + c_2 e^{2t}\begin{bmatrix}0\\1\end{bmatrix}$

$\vec{x} = e^{2t}\begin{bmatrix}c_1\\c_2\end{bmatrix}$

When the algebraic and geometric multiplicities are not equal, we can introduce generalized eigenvectors.

$\vec{v}_1$ is an eigenvector if $(A - \lambda I)\vec{v}_1 = 0$.

Similarly, $\vec{v}_2$ is a generalized eigenvector of $\vec{v}_1$ if $(A - \lambda I)\vec{v}_2 = \vec{v}_1$

In the most general sense, we can determine if $\vec{v}$ is a generalized eigenvector if

$$(A - \lambda I)^k\vec{v} \neq \vec{0} \text{ and } (A - \lambda I)^{k+1}\vec{v} = \vec{0}$$

Note that the choice of generalized eigenvectors will always be somewhat arbitrary, as there are infinitely many choices.

The general solution for multiple eigenvalues will resemble the form

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_1 t}\left(\vec{v}_k + t\vec{v}_{k-1} + \frac{t}{2}\vec{v}_{k-2} + \cdots + \frac{t^{k-2}}{(k-2)!}\vec{v}_2 + \frac{t^{k-1}}{(k-1)!}\vec{v}_1\right)$$

Ex: $\vec{x}' = \begin{bmatrix} 5 & -3 \\ 3 & -1 \end{bmatrix}\vec{x}$

$\det A = 4,\ \operatorname{tr} A = 4 \Rightarrow \lambda = 2$

$\begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix} \to \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 3 & -3 \\ 3 & -3 \end{bmatrix}\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \vec{v}_2 = \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}$

$\vec{x} = c_1 e^{2t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{2t}\left(t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} \\ 0 \end{bmatrix}\right)$

Ex2: $\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\vec{x}$

$\lambda = 0,\ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \to \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$\vec{x} = c_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2\left(\frac{t^2}{2}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$

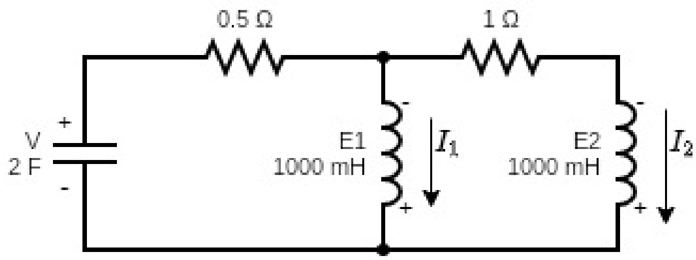
1.5.3 LCR Circuits

Recall, $V_C = \frac{q}{C} \Rightarrow \frac{dV_C}{dt} = \pm\frac{i}{C}$ and $\frac{dI}{dt} = \pm\frac{E}{L}$
where V is the voltage of the capacitor, i is the current through the capacitor, E is the voltage of the inductor, and I is the current throuh the inductor. C and L are the capacitance and inductance.

1. Write voltage and current equations
2. Express i and E in terms of I and V
3. Write differential equations of form $\frac{dV}{dt} = \pm\frac{i}{C}$ and $\frac{dI}{dt} = \pm\frac{E}{L}$
4. Express the system of differential equations in matrix form

5. Solve differential equation for various $V(t)$ and $I(t)$

Ex:



Voltage equations

$0.5i_1 - E_1 - V = 0$

$i_2 - E_2 + E_1 = 0$

current equations

$I_1 = i_1 - i_2$

$I_2 = i_2$

writing expressions in terms of V, I_1, I_2

$i_2 = I_2$

$i_1 = I_1 + i_2 = I_1 + I_2$

$E_1 = 0.5i_1 - V = 0.5I_1 + 0.5I_2 - V$

$E_2 = i_2 + E_1 = 0.5I_1 + 1.5I_2 - V$

Differential equations for V, I_1, I_2

$$\frac{dV}{dt} = -\frac{i_1}{C} = -\frac{i_1}{2} = -\frac{(I_1 + I_2)}{2}$$

$$\frac{dI_1}{dt} = -\frac{E_1}{L} = -\frac{E_1}{1} = -0.5I_1 + 0.5I_2 - V$$

$$\frac{dI_2}{dt} = -\frac{E_2}{L} = -\frac{E_2}{1} = -0.5I_1 + 1.5I_2 - V$$

$$\frac{d}{dt}\begin{bmatrix} V \\ I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -\frac{1}{2} & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} V \\ I_1 \\ I_2 \end{bmatrix}$$

eigenvalues and eigenvectors

$\lambda_1 = \dfrac{-1+i}{2}$, $\vec{v}_1 = \begin{bmatrix} 1 & -i & 1 \end{bmatrix}^T$

$\lambda_2 = \dfrac{-1-i}{2}$, $\vec{v}_2 = \begin{bmatrix} 1 & i & 1 \end{bmatrix}^T$

$\lambda_3 = -1$, $\vec{v}_3 = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}^T$

general solution

$$\begin{bmatrix} V(t) \\ I_1(t) \\ I_2(t) \end{bmatrix} = c_1 e^{\lambda_1 t}\vec{v}_1 + c_2 e^{\lambda_2 t}\vec{v}_2 + c_3 e^{\lambda_3 t}\vec{v}_3$$

$$\begin{bmatrix} V(t) \\ I_1(t) \\ I_2(t) \end{bmatrix} = c_1 e^{-t/2} \begin{bmatrix} \cos\left(\frac{t}{2}\right) \\ \sin\left(\frac{t}{2}\right) \\ \cos\left(\frac{t}{2}\right) \end{bmatrix} + c_2 e^{-t/2} \begin{bmatrix} \sin\left(\frac{t}{2}\right) \\ -\cos\left(\frac{t}{2}\right) \\ \sin\left(\frac{t}{2}\right) \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$\rightarrow$ $\vec{c}$ can be solved from initial conditions

This particular example would show damped oscillations in the circuit.

1.5.4 Phase Portraits

Phase portraits are a snapshot of the vector field and phase lines formed by the solution to a system of differential equations.

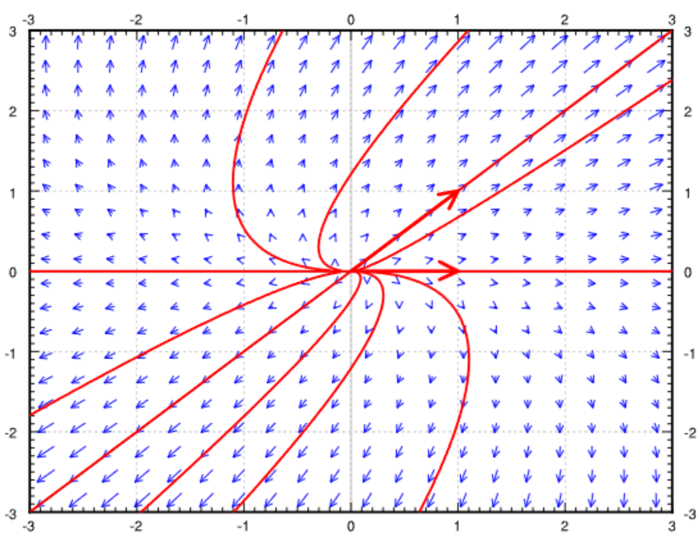
Ex: $\vec{x}' = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \vec{x}$

$$\lambda = 1, 2,\ \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix},\ \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

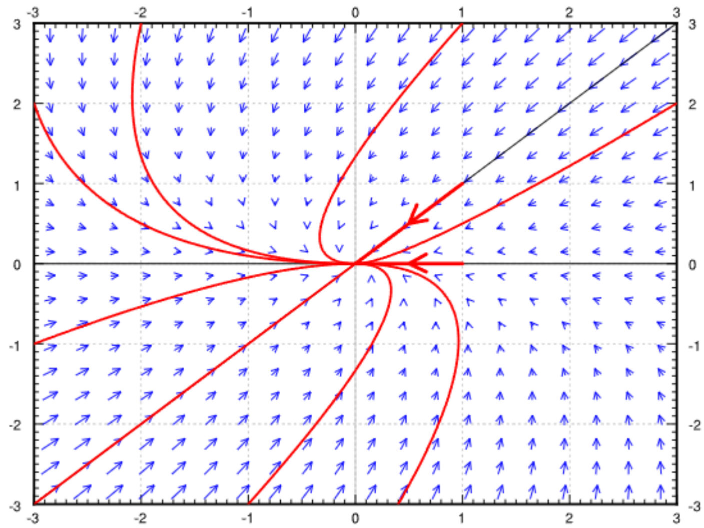
$$\vec{x} = c_1 e^{t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lim_{t\to\infty} \vec{x} = \infty \text{ parallel to } \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

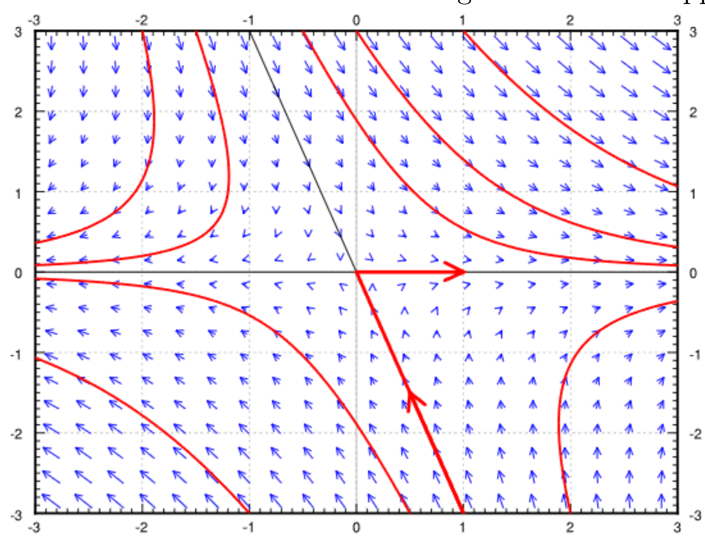
$$\lim_{t\to\infty} \vec{x} = 0 \text{ parallel to } \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



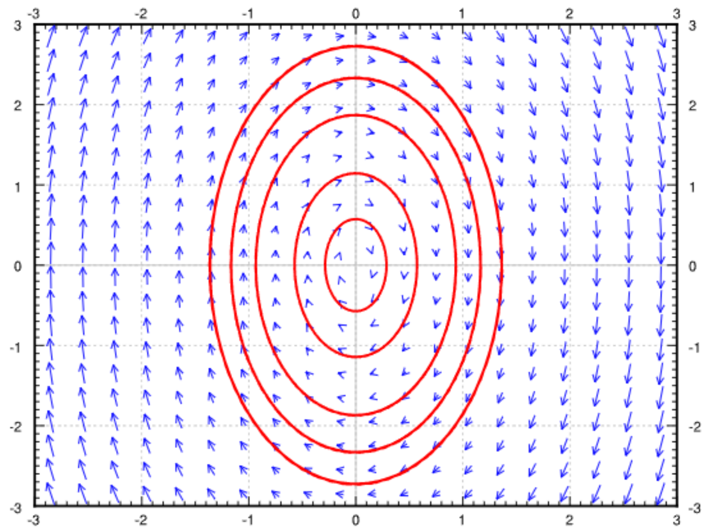
If we take the system $\vec{x}' = \begin{bmatrix} -1 & -1 \\ 0 & -2 \end{bmatrix}$, we get the same eigenvectors as before but with $\lambda = -1, -2$. This gives the plot



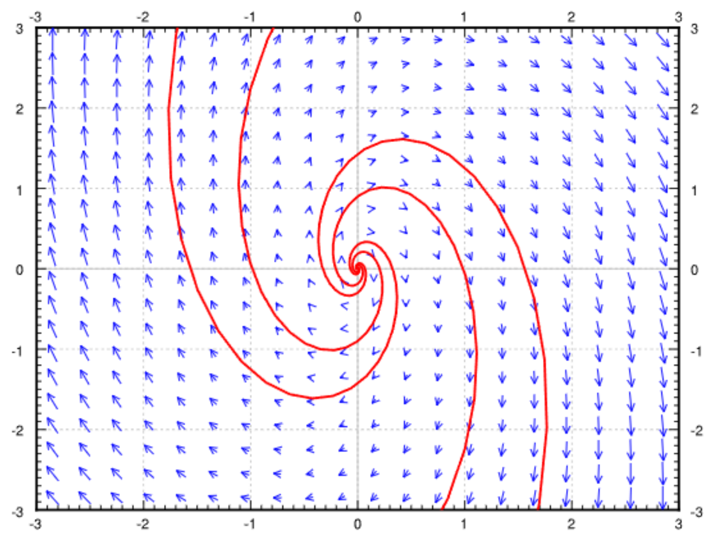
where the system tends toward the origin.
We can also have cases where the eigenvalues are of opposite signs which will form a saddle



The cases with repeated eigenvalues will react similarly to the first two examples but with the all the lines evenly diverging or converging to/from the origin.
In the case of complex eigenvalues, if we have no real component, the phase portrait will follow an elliptical orbit about the origin.



If the eigenvalues are complex and do have a real part, they will form a spiral. For a positive real part, it will spiral out toward infinity, and a negative real part will give a spiral toward the origin.



Based on the phase portraits, we can classify the stability of different solutions

Eigenvalues	Behavior
both real and positive	unstable source
both real and negative	stable sink
both real with opposite signs	unstable saddle
complex with no real part	center point/ellipses
complex with positive real part	spiral source
complex with negative real part	spiral sink

1.5.5 Nonhomogeneous Equations

These are systems in the form of

$$\vec{x}' = A\vec{x} + \vec{f}(t)$$

The general solution will be $\vec{x}(t) = \vec{x}_c + \vec{x}_p$.

Undetermined Coefficients

This is very similar to the method of undetermined coefficients seen earlier. Based on the functions in $\vec{f}(t)$, we make a guess that encompasses those functions. The only difference is if there is overlap, we don't multiply the entire guess by t but rather add in another term that's multiplied by t. This is because in a system, it is not always evident if there will be overlap even if two of the same terms appear.

Ex: Write the expression for $\vec{x}_p$ of the system $\vec{x}' = \begin{bmatrix} 3 & -2 \\ 2 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} e^{2t} \\ t \end{bmatrix}$

$\vec{x}_c = c_1 e^{-t} \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

initial guess: $\vec{x}_p = \vec{a} e^{2t} + \vec{b} t + \vec{c}$

there may be overlap with e^{2t}

revised guess: $\vec{x}_p = \vec{a} t e^{2t} + \vec{b} e^{2t} + \vec{c} t + \vec{d}$

Ex2: Solve $\vec{x}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ e^t \\ e^t \end{bmatrix}$

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 - \lambda - 6 = 0$$

$(\lambda - 3)(\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda = 3, 2, -1$

$$\begin{bmatrix} 0 & -1 & -1 \\ -2 & 0 & 2 \\ 4 & -1 & -5 \end{bmatrix} \to \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ -2 & 1 & 2 \\ 4 & -1 & -4 \end{bmatrix} \to \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & -1 \\ -2 & 4 & 2 \\ 4 & -1 & -1 \end{bmatrix} \to \vec{v}_3 = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$$

$$\vec{x}_c = c_1 e^{3t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$$

guess $\vec{x}_p = \vec{a} e^t + \vec{b}$

$\vec{x}_p' = \vec{a} e^t$

$\vec{f} = \vec{a} e^t - A\vec{a} e^t - A\vec{b}$

$$\begin{cases} \begin{bmatrix}0\\1\\1\end{bmatrix} = (I-A)\vec{a} \\ \quad\begin{bmatrix}1\\0\\0\end{bmatrix} = A\vec{b} \end{cases}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 2 & -2 & -2 & 1 \\ -4 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & -2 & 0 & 2 \end{array}\right] \rightsquigarrow \vec{a} = \begin{bmatrix}-\frac{1}{2}\\-1\\0\end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & -1 & 1 \\ -2 & 3 & 2 & 0 \\ 4 & -1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & -1 & -1 & 1 \\ 4 & 1 & 0 & 2 \\ -2 & 1 & 0 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & -1 & -1 & 1 \\ 6 & 0 & 0 & 4 \\ -2 & 1 & 0 & -2 \end{array}\right] \rightsquigarrow \vec{b} = \begin{bmatrix}\frac{2}{3}\\-\frac{2}{3}\\\frac{5}{3}\end{bmatrix}$$

$$\vec{x}_p = e^t \begin{bmatrix}-\frac{1}{2}\\-1\\0\end{bmatrix} + \frac{1}{3}\begin{bmatrix}2\\-2\\5\end{bmatrix}$$

Variation of Parameters

We can define the fundamental matrix X to be comprised of the complementary solution $\left[x_1|x_2|\cdots|x_n\right]$. Using this, we can express the particular solution as

$$\vec{x}_p = X\int^t (X(\tau)^{-1})\vec{f}(\tau)d\tau$$

Ex: Solve $\vec{x}' = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\vec{x} + \begin{bmatrix}1\\t\end{bmatrix}$

$$\det A = 1,\ \operatorname{tr} A = 0 \Rightarrow \lambda = \pm i$$

$$\begin{bmatrix}-i & 1\\-1 & -i\end{bmatrix} \to \vec{v} = \begin{bmatrix}1\\i\end{bmatrix}$$

$$\begin{bmatrix}\cos t + i\sin t\\ i\cos t - \sin t\end{bmatrix} \to \begin{bmatrix}\cos t\\-\sin t\end{bmatrix} + \begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$$

$$\vec{x}_c = c_1\begin{bmatrix}\cos t\\-\sin t\end{bmatrix} + c_2\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}$$

$$X = \begin{bmatrix}\cos t & \sin t\\-\sin t & \cos t\end{bmatrix}$$

$$X^{-1} = \begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}$$

$$\vec{u} = \int^t X(\tau)^{-1}\vec{f}(\tau)d\tau$$

$$\vec{u} = \int^t \begin{bmatrix}\cos\tau - \tau\sin\tau\\ \sin\tau + \tau\cos\tau\end{bmatrix}d\tau = \begin{bmatrix}\sin t + t\cos t - \sin t\\ -\cos t + t\sin t + \cos t\end{bmatrix} = \begin{bmatrix}t\cos t\\ t\sin t\end{bmatrix}$$

$$\vec{x}_p = X\vec{u} = \begin{bmatrix}t\cos^2 t + t\sin^2 t\\ -t\sin t\cos t + t\cos t\sin t\end{bmatrix} = \begin{bmatrix}t\\0\end{bmatrix}$$

$$\vec{x}(t) = c_1 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + c_2 \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Ex2: Solve $\vec{x}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ e^t \\ e^t \end{bmatrix}$

$$\begin{vmatrix} 3-\lambda & -1 & -1 \\ -2 & 3-\lambda & 2 \\ 4 & -1 & -2-\lambda \end{vmatrix} = -\lambda^3 + 4\lambda^2 - \lambda - 6 = 0$$

$(\lambda - 3)(\lambda - 2)(\lambda + 1) = 0 \Rightarrow \lambda = 3, 2, -1$

$$\begin{bmatrix} 0 & -1 & -1 \\ -2 & 0 & 2 \\ 4 & -1 & -5 \end{bmatrix} \to \vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ -2 & 1 & 2 \\ 4 & -1 & -4 \end{bmatrix} \to \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -1 & -1 \\ -2 & 4 & 2 \\ 4 & -1 & -1 \end{bmatrix} \to \vec{v}_3 = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$$

$$\vec{x}_c = c_1 e^{3t} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-t} \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix}$$

guess $\vec{x}_p = \vec{a}e^t + \vec{b}$

$\vec{x}_p' = \vec{a}e^t$

$\vec{f} = \vec{a}e^t - A\vec{a}e^t - A\vec{b}$

$$\begin{cases} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = (I - A)\vec{a} \\ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = A\vec{b} \end{cases}$$

$$\left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 2 & -2 & -2 & 1 \\ -4 & 1 & 3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & -1 & 1 & 1 \end{array}\right] \to \left[\begin{array}{ccc|c} -2 & 1 & 1 & 0 \\ 0 & -1 & -1 & 1 \\ 0 & -2 & 0 & 2 \end{array}\right] \rightsquigarrow \vec{a} = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 3 & -1 & -1 & 1 \\ -2 & 3 & 2 & 0 \\ 4 & -1 & -2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & -1 & -1 & 1 \\ 4 & 1 & 0 & 2 \\ -2 & 1 & 0 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 3 & -1 & -1 & 1 \\ 6 & 0 & 0 & 4 \\ -2 & 1 & 0 & -2 \end{array}\right] \rightsquigarrow \vec{b} = \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{5}{3} \end{bmatrix}$$

$$\vec{x}_p = e^t \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 0 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 2 \\ -2 \\ 5 \end{bmatrix}$$

Ex3: Solve $\vec{x}' = \frac{1}{t}\begin{bmatrix} 1 & t \\ -t & 1 \end{bmatrix}\vec{x} + t\begin{bmatrix} \cos t \\ \sin t \end{bmatrix}$ where $X = t\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$

$$X^{-1} = \frac{1}{t}\begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix}$$
$$\vec{u} = \int^t \begin{bmatrix} \cos^2\tau - \sin^2\tau \\ \sin\tau\cos\tau + \sin\tau\cos\tau \end{bmatrix} d\tau = \int^t \begin{bmatrix} \cos 2\tau \\ \sin 2\tau \end{bmatrix} d\tau = \frac{1}{2}\begin{bmatrix} \sin 2t \\ -\cos 2t \end{bmatrix}$$
$$\vec{x}_p = X\vec{u} = \frac{t}{2}\begin{bmatrix} \cos t \sin 2t - \sin t \cos 2t \\ -\sin t \sin 2t - \cos t \cos 2t \end{bmatrix}$$

1.5.6 Nonlinear Systems

These can often be very complex. To make it easier, we can analyze the case of autonomous systems. Autonomous systems are where $\vec{F}$ does not depend on t.

$$\vec{F} = \begin{cases} x' = f(x,y) \\ y' = g(x,y) \end{cases}$$

We will not be solving these systems but rather analyzing their behavior through phase portraits. We can find critical points in the system where the system converges or diverges from. These critical points can be linearized and then treated the same as with linear phase portraits. Then, in combining the phase portraits, we can get an idea of what the system looks like.

Ex: Find the critical points of $\begin{cases} x' = xy + x \\ y' = x^2 - x \end{cases}$

$$x(y+1) = 0 \Rightarrow x = 0,\ y = -1$$
$$x(x-1) = 0 \Rightarrow x = 0,\ x = 1$$
gives CPs $(1,-1), (0,y)$

This would mean that the point $(1,-1)$ is a critical point along with the entire line $x = 0$. The point $(1,-1)$ is said to be an isolated critical point while the line $x = 0$ is a non-isolated critical point.

For any isolated critical point, we can linearize it by computing the Jacobian about that point. Recall, the Jacobian is equal to

$$J = \begin{bmatrix} f_x & f_y \\ g_x & g_y \end{bmatrix}$$

with the linearization about a point P becoming

$$\frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f_x(P) & f_y(P) \\ g_x(p) & g_y(P) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

We can do this for where P is an isolated critical point and $J(P)$ is invertible.

From here, we can draw the phase portraits and sketch our solution.

Ex: Sketch the solution of $\begin{cases} x' = y \\ y' = x^2 - x \end{cases}$

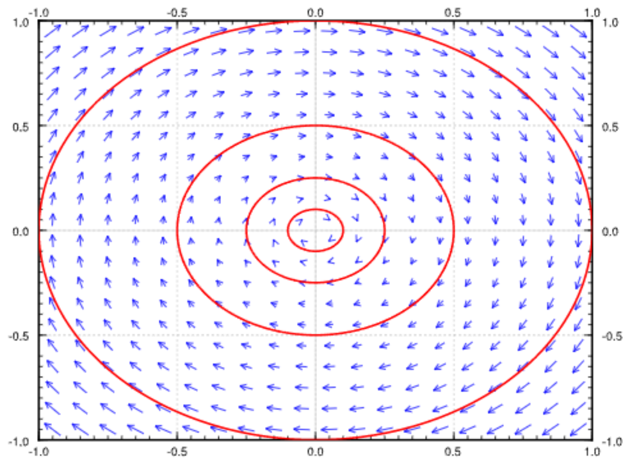
$$y = 0$$

$x(x-1) = 0 \Rightarrow x = 0,\ x = 1$

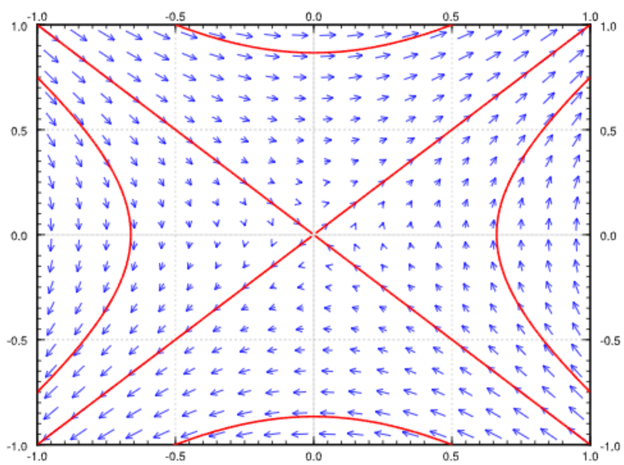
CPs: $(0,0), (1,0)$

$J = \begin{bmatrix} 0 & 1 \\ 2x-1 & 0 \end{bmatrix}$

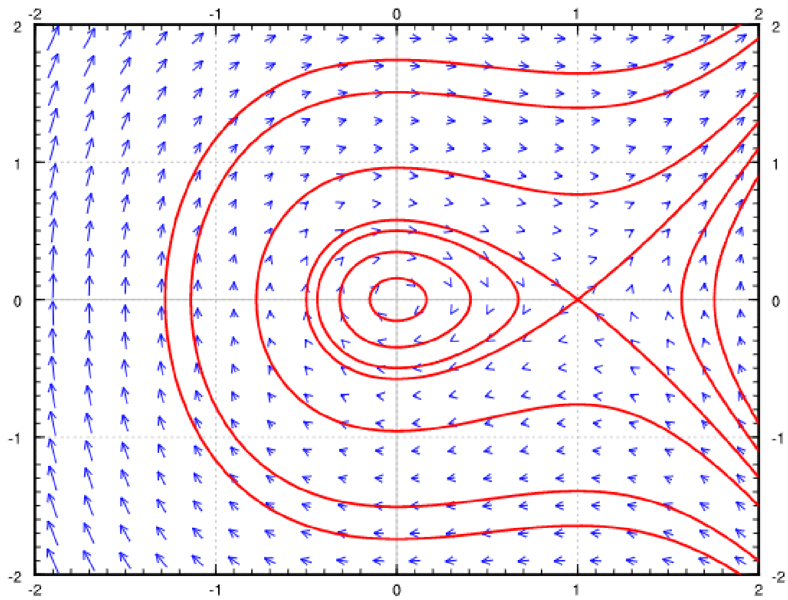
$J(0,0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \lambda = \pm i$



$J(1,0) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda = \pm 1$



Combining the phase portraits gives a total system that looks something like


-2
-1
0
1
2
2
1
0
-1
-2