

# Random Numbers

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**Abstract**—This solution manual provides solutions and link to codes used for generation of random numbers.

## 1 UNIFORM RANDOM VARIABLES

- 1.1 Generate  $10^6$  samples of  $U$  using a C program and save into a file called uni.dat .

**Solution:**

Download the following C code and run it to generate samples of  $U$ .

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/1.1/1.1.c
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of  $U$  using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

**Solution:** Code used to plot empirical CDF of  $U$

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/1.2/1.2.py
```

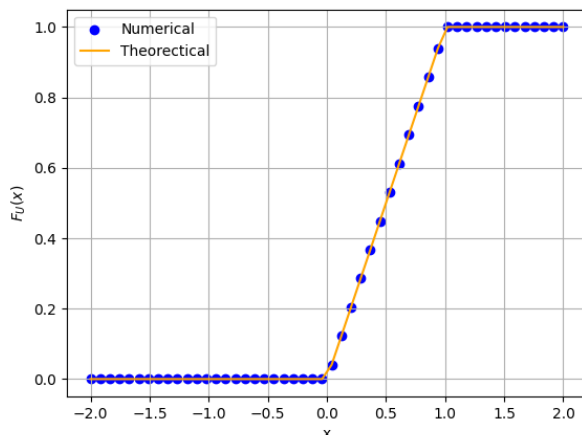


Fig. 1.2: CDF of  $U$

- 1.3 Find a theoretical expression for  $F_U(x)$ .

**Solution:**

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \quad (1.2)$$

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/1.3/1.3.py
```

- 1.4 Find mean and variance of  $U$ .

**Solution:**

Mean of Random Variable  $U$  is given by,

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.3)$$

and its Variance is given by,

$$E[U - E[U]]^2 = E[U^2] - [E[U]]^2 \quad (1.4)$$

Using the above two formulas, we get,

$$\text{Mean of } U = 0.500031 \quad (1.5)$$

$$\text{Variance of } U = 0.083247 \quad (1.6)$$

Download and run the C code for mean and variance

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/1.4/1.4.c
```

- 1.5 Verify your result theoretically that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.7)$$

**Solution:**

For  $k = 1$ ,

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.8)$$

$$E[U] = \int_0^1 x d(x) \quad (1.9)$$

$$E[U] = 0.5 \quad (1.10)$$

Similarly, for  $k = 2$ ,

$$E[U^2] = 0.3333 \quad (1.11)$$

$$\text{Variance} = 0.0833 \quad (1.12)$$

Thus the simulated and theoretical values of mean and variance of  $U$  are approximately equal.

## 2 CENTRAL LIMIT THEOREM

### 2.1 Solution:

Download the following C code and run it to generate samples of  $X$ .

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/2.1/2.1.c
```

### 2.2 Solution:

The plot was generated by running the following Python code

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/2.2./2.2.py
```

The CDF is a non-decreasing function with its range between 0 and 1. It will be continuous if PDF is finite.

2.3 Load gau.dat in python and plot the empirical PDF of  $X$  using the samples in gau.dat. The PDF of  $X$  is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.1)$$

What properties does the PDF have?

#### Solution:

The empirical PDF of  $X$  is plotted by the Python code

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/2.3/2.3.py
```

The PDF takes non-negative values and area under its curve is 1.

2.4 Find the mean and variance of  $X$  by writing a C program.

#### Solution:

Run the following C file

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/2.4/2.4.c
```

$$E[X] = 0.000630 \quad (2.2)$$

$$\text{var}[X] = 1.000149 \quad (2.3)$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.4)$$

repeat the above exercise theoretically.

#### Solution:

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.5)$$

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.6)$$

Taking  $\frac{x^2}{2} = t$ ,

$$E[X] = - \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-t) dt \quad (2.7)$$

$$E[X] = 0 \quad (2.8)$$

To calculate variance,

$$\text{var}[X] = E[(X - E[X])^2] \quad (2.9)$$

$$\text{var}[X] = E[X^2] \quad (2.10)$$

$$\text{var}[X] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.11)$$

$$\text{var}[X] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.12)$$

We know that,

$$\int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2}\right) = \sqrt{2\pi} \quad (2.13)$$

$$\text{var}[X] = 1 \quad (2.14)$$

### 3 FROM UNIFORM TO OTHER

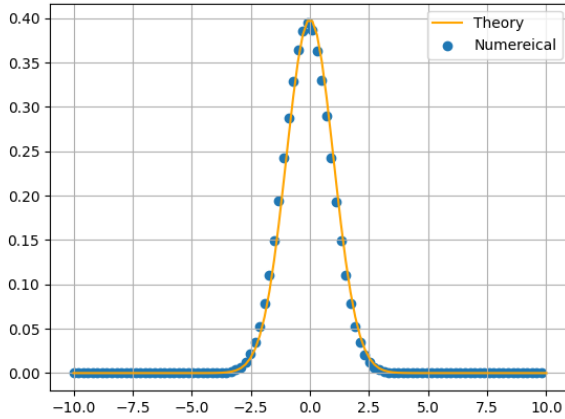


Fig. 2.5: PDF of  $X$

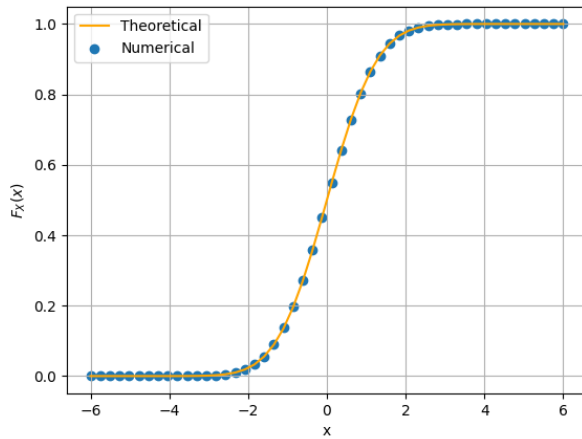


Fig. 2.5: CDF of  $X$

Theoretical expression for  $X$ ,

$$F_X(x) = 1 - Q\left(\frac{x - E[X]}{\text{var}[X]}\right) \quad (2.15)$$

$$F_X(x) = 1 - Q(x) \quad (2.16)$$

The Python codes plot the CDF and PDF of  $X$

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/2.5/2.5.py
```

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/2.5/2.5_2.py
```

#### 3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

**Solution:**

Download and run the following code to generate samples of  $V$

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/3.1/3.1.c
```

The following Python code plots CDF of  $V$

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/3.1/3.1.py
```

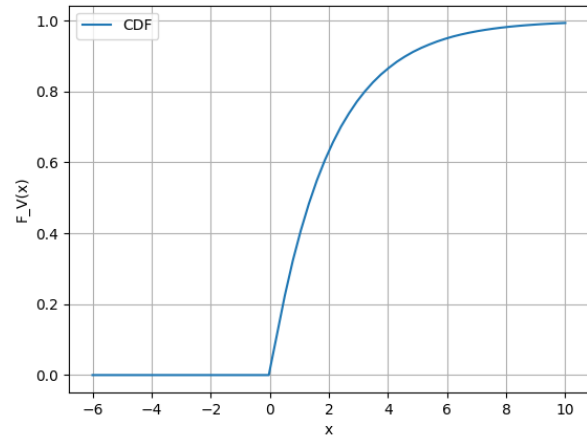


Fig. 3.1: The empirical CDF of  $V$

#### 3.2 Find a theoretical expression for $F_V(x)$ .

**Solution:**

$$F_V(x) = \Pr(V \leq x) \quad (3.2)$$

$$F_V(x) = \Pr(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$F_V(x) = \Pr\left(\ln(1 - U) \geq -\frac{x}{2}\right) \quad (3.4)$$

$$F_V(x) = \Pr\left(1 - U \geq \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

$$F_V(x) = \Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.6)$$

$$F_V(x) = F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.7)$$

$$F_V(x) = \begin{cases} 0 & 1 - \exp\left(-\frac{x}{2}\right) < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & 0 \leq 1 - \exp\left(-\frac{x}{2}\right) \leq 1 \\ 1 & 1 - \exp\left(-\frac{x}{2}\right) > 1 \end{cases} \quad (3.8)$$

This simplifies to

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \end{cases} \quad (3.9)$$

The following python code plots the theoretical CDF

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/3.2/3.2.py
```

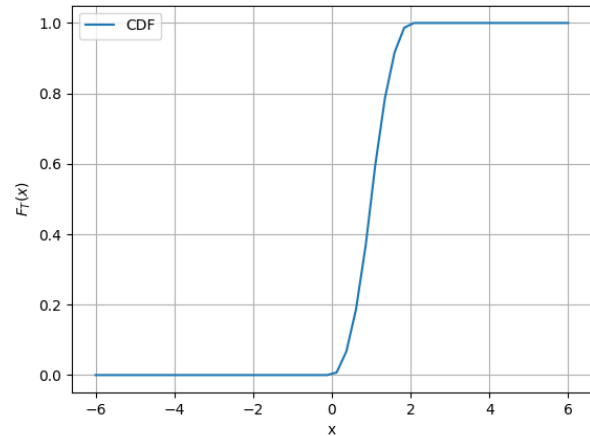


Fig. 4.2: CDF of  $T$

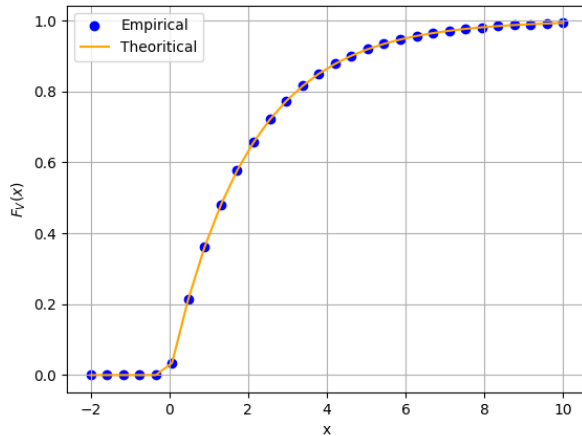


Fig. 3.2: CDF of  $V$

#### 4 TRIANGULAR DISTRIBUTION

##### 4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

##### Solution:

Download and run the following C code to generate tri.dat file.

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/4.1/4.1.c
```

##### 4.2 Find the CDF of $T$ .

##### Solution:

The following code plots the CDF of  $T$

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/4.2/4.2.py
```

##### 4.3 Find the CDF of $T$ .

##### Solution:

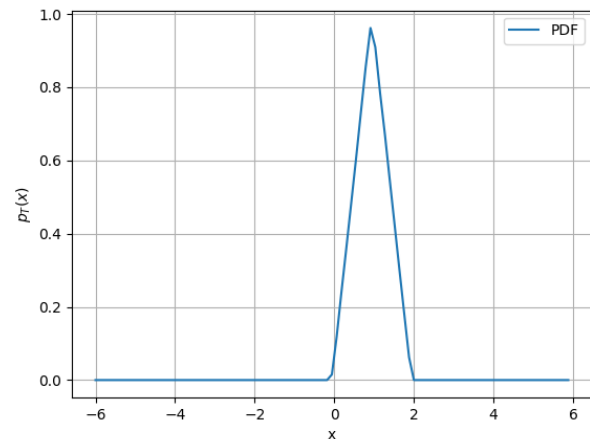


Fig. 4.3: PDF of  $T$

The following code plots the PDF of  $T$

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/4.3/4.3.py
```

##### 4.4 Find the theoretical expressions for PDF and CDF of $T$ .

##### Solution:

$$p_T(x) = p_{U_1+U_2}(x) = p_{U_1}(x) * p_{U_2}(x) \quad (4.2)$$

$$p_T(x) = \int_{-\infty}^{\infty} p_{U_1}(\tau) p_{U_2}(x - \tau) d\tau \quad (4.3)$$

$$p_T(x) = \int_0^1 p_{U_2}(x - \tau) d\tau \quad (4.4)$$

$$p_T(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x 1 d\tau & 0 < x < 1 \\ \int_{x-1}^1 1 d\tau & 1 \leq x < 2 \\ 0 & x > 2 \end{cases} \quad (4.5)$$

$$p_T(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & x > 2 \end{cases} \quad (4.6)$$

Expression for CDF can be obtained by integrating  $p_T(x)$  w.r.t.  $x$

$$F_T(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{2} & 0 < x < 1 \\ -\frac{x^2}{2} + 2x - 1 & 1 \leq x < 2 \\ 1 & x > 2 \end{cases} \quad (4.7)$$

4.5 Verify the results through a plot.

**Solution:**

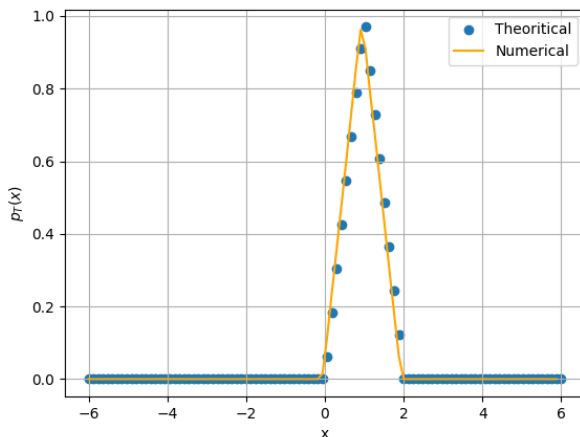


Fig. 4.5: Theoretical PDF of  $T$

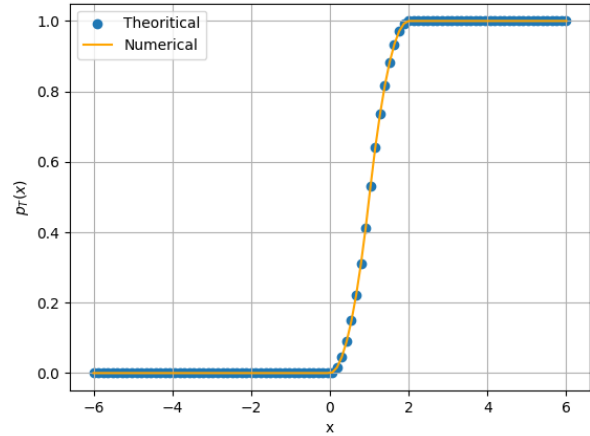


Fig. 4.5: The CDF of  $T$

PDF and CDF plotted by the Python codes

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/4.5/4.5_1.py
```

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/4.5/4.5_1.py
```

## 5 MAXIMUM LIKELIHOOD

5.1 Generate equiprobable  $X \in \{1, -1\}$ .

**Solution:**

Download and run the following codes to generate  $X$ .

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/5.1/5.1_X.c
```

5.2 Generate

$$Y = AX + N, \quad (5.1)$$

where  $A = 5$  dB, and  $N \sim \mathcal{N}(0, 1)$ .

**Solution:**

Download and run the following code to generate  $Y$ .

```
wget https://github.com/TYCN129/AI1110-
Assignments/tree/main/Manual
%201/5.1/5.1_Y.c
```

### 5.3 Plot $Y$

**Solution:**

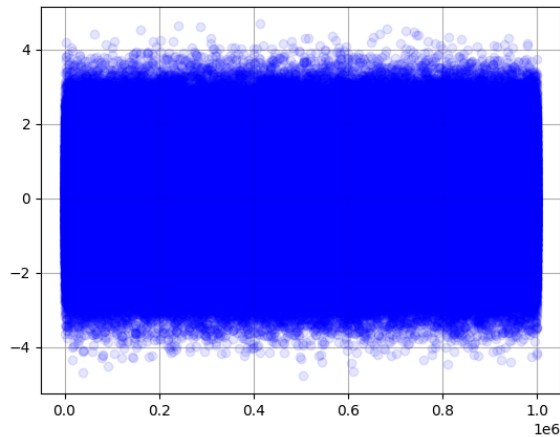


Fig. 5.3: Plot of  $Y$

The Python code plots  $Y$ .

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual-
%201/5.2/5.2.py
```

### 5.4 Guess how to estimate $X$ from $Y$

**Solution:**

Estimate of  $X = \hat{X}$  is found out from  $Y$  by,

$$\hat{X} = \begin{cases} -1 & Y < 0 \\ 1 & Y \geq 0 \end{cases} \quad (5.2)$$

### 5.5 Find

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.3)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) \quad (5.4)$$

**Solution:**

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.5)$$

$$= 0.4998032 \quad (5.6)$$

and

$$P_{e|1} = \Pr(\hat{X} = 1|X = -1) \quad (5.7)$$

$$= 0.4995446 \quad (5.8)$$

### 5.6 Find $P_e$ assuming $X$ has equiprobable symbols.

**Solution:**

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.9)$$

$$= \Pr(AX + N < 0|X = 1) \quad (5.10)$$

$$= \Pr(N < -A) \quad (5.11)$$

$$= \int_{-\infty}^{-A} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (5.12)$$

$$= \int_A^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = Q_N(A) \quad (5.13)$$

where,  $Q_N(A)$  is the Q-function of Normal distribution. Similarly,

$$P_{e|1} = Q_N(A) \quad (5.14)$$

Thus,

$$P_e = P_{e|0} \times \Pr(X = 1) + P_{e|1} \times \Pr(X = -1) \quad (5.15)$$

$$P_e = \frac{1}{2} \times P_{e|0} + \frac{1}{2} \times P_{e|1} \quad (5.16)$$

$$P_e = \frac{P_{e|0} + P_{e|1}}{2} \quad (5.17)$$

$$P_e = Q_N(A) \quad (5.18)$$

### 5.7 Verify by plotting theoretical $P_e$ by varying $A$ from 0 to 10 dB

**Solution:**

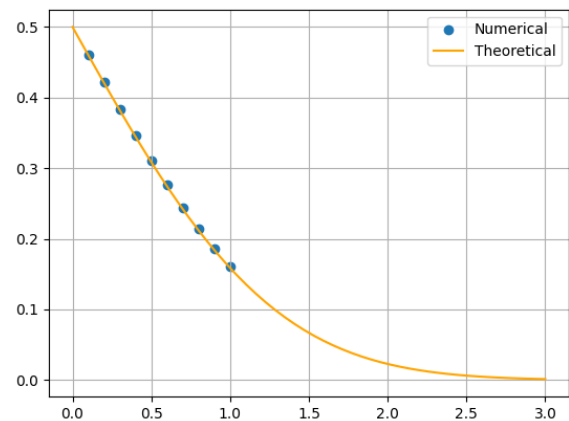


Fig. 5.7: Plot of  $P_e$  with varying  $A$

```
wget https://github.com/TYCN129/AI1110-
Assignments/tree/main/Manual%201/5.7
```

5.8 Now, consider a threshold  $\delta$  while estimating  $X$  from  $Y$ . Find the value of  $\delta$  that maximizes the theoretical  $P_e$ .

**Solution:**

Defining estimate of  $X$  as,

$$\hat{X} = \begin{cases} 1 & Y > \delta \\ -1 & Y < \delta \end{cases} \quad (5.19)$$

$$P_{e|0} = \Pr(\hat{X} = -1|X = 1) \quad (5.20)$$

$$= \Pr(Y < \delta|X = 1) \quad (5.21)$$

$$= \Pr(AX + N < \delta|X = 1) \quad (5.22)$$

$$= \Pr(N < \delta - A) \quad (5.23)$$

$$= \int_{-\infty}^{\delta-A} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (5.24)$$

$$= \int_{A-\delta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \quad (5.25)$$

$$= Q_N(A - \delta) \quad (5.26)$$

Similarly,

$$(5.27)$$

$$P_{e|1} = Q_N(A + \delta) \quad (5.28)$$

Therefore,

$$P_e = P(X = 1) \times P_{e|0} + P(X = -1) \times P_{e|1} \quad (5.29)$$

$$= \frac{Q(A - \delta) + Q_N(A + \delta)}{2} \quad (5.30)$$

To maximize theoretical  $P_e$ , we differentiate equation (5.30) w.r.t  $\delta$ .

$$\frac{dP_e}{d\delta} = \frac{1}{2} \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta-A)^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}} \right) \quad (5.31)$$

$$0 = \frac{1}{2} \left( \frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta-A)^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}} \right) \quad (5.32)$$

Thus,

$$\delta = 0 \quad (5.33)$$

5.9 Repeat the above exercise when

$$p_X(0) = p \quad (5.34)$$

**Solution:**

$$P_e = P(X = 1) \times P_{e|0} + P(X = -1) \times P_{e|1} \quad (5.35)$$

$$P_e = pP_{e|0} + (1 - p)P_{e|1} \quad (5.36)$$

On differentiating both sides,

$$0 = \frac{1}{2} \left( \frac{p}{\sqrt{2\pi}} e^{-\frac{(\delta-A)^2}{2}} - \frac{(1-p)}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}} \right) \quad (5.37)$$

Thus,

$$\delta = \frac{1}{2A} \ln \left( \frac{(1-p)}{p} \right) \quad (5.38)$$

5.10 Repeat the above exercise using the MAP criterion.

**Solution:**

$$\begin{aligned} p_Y(y) &= p_{Y|X=1}(y|1) \times P(X = 1) + \\ &\quad p_{Y|X=-1}(y|-1) \times P(X = -1) \\ &= p_{N+A}(y) \times p + p_{N-A}(y) \times (1 - p) \end{aligned} \quad (5.39)$$

Here,  $p_Y(y)$  is the PDF of  $Y$ ,  $p_{N+A}$  and  $p_{N-A}$  are the PDF of shifted normal distribution.

$$p_Y(y) = \frac{p}{\sqrt{2\pi}} e^{-\frac{(y+A)^2}{2}} + \frac{(1-p)}{\sqrt{2\pi}} e^{-\frac{(y-A)^2}{2}} \quad (5.40)$$

Now, we need to find  $p_{X|Y}(x|y)$  using the formula,

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x) \times p_X(x)}{p_Y(y)} \quad (5.41)$$

For  $X = 1$ ,

$$p_{X|Y}(1|y) = \frac{p_{Y|X}(y|1) \times p}{p_Y(y)} \quad (5.42)$$

$$= \frac{\frac{p}{\sqrt{2\pi}} e^{-\frac{(y+A)^2}{2}}}{\frac{p}{\sqrt{2\pi}} e^{-\frac{(y+A)^2}{2}} + \frac{(1-p)}{\sqrt{2\pi}} e^{-\frac{(y-A)^2}{2}}} \quad (5.43)$$

$$= \frac{pe^{2Ay}}{pe^{2Ay} + (1-p)} \quad (5.44)$$

Similarly, for  $X = -1$ ,

$$p_{X|Y}(-1|y) = \frac{p_{Y|X}(y) - 1 \times (1-p)}{p_Y(y)} \quad (5.45)$$

$$= \frac{\frac{(1-p)}{\sqrt{2\pi}} e^{-\frac{(y-A)^2}{2}}}{\frac{p}{\sqrt{2\pi}} e^{-\frac{(y+A)^2}{2}} + \frac{(1-p)}{\sqrt{2\pi}} e^{-\frac{(y-A)^2}{2}}} \quad (5.46)$$

$$= \frac{(1-p)}{pe^{2Ay} + (1-p)} \quad (5.47)$$

- Case 1)  $p_{X|Y}(1|y) > p_{X|Y}(-1|y)$

$$\frac{pe^{2Ay}e^{2Ay}}{pe^{2Ay} + (1-p)} > \frac{(1-p)}{pe^{2Ay} + (1-p)} \quad (5.48)$$

$$e^{-2Ay} < \frac{p}{1-p} \quad (5.49)$$

$$y > \frac{1}{2A} \ln\left(\frac{p}{1-p}\right) \quad (5.50)$$

Equation (5.50) holds good when  $X = 1$ .

- Case 2)  $p_{X|Y}(1|y) < p_{X|Y}(-1|y)$

$$\frac{pe^{2Ay}}{pe^{2Ay} + (1-p)} < \frac{(1-p)}{pe^{2Ay} + (1-p)} \quad (5.51)$$

$$e^{-2Ay} > \frac{p}{1-p} \quad (5.52)$$

$$y < \frac{1}{2A} \ln\left(\frac{p}{1-p}\right) \quad (5.53)$$

Equation (5.53) holds good when  $X = -1$ .

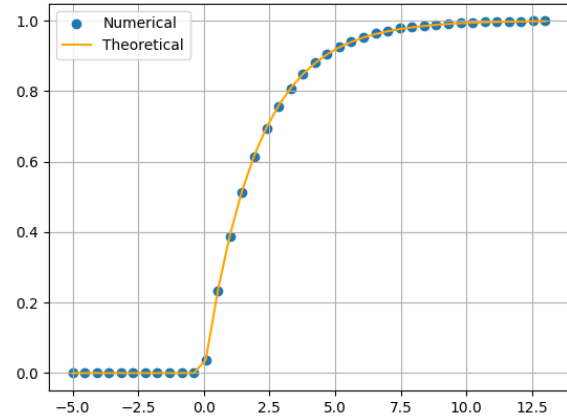


Fig. 6.1: CDF of  $V$

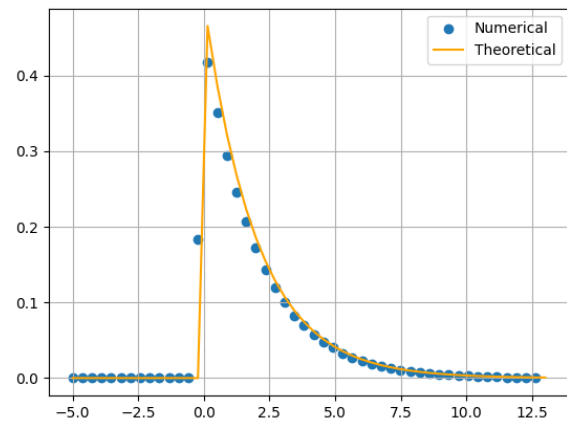


Fig. 6.1: PDF of  $V$

## 6 GAUSSIAN TO OTHER

6.1 Let  $X_1 \sim 01$  and  $X_2 \sim 01$ . Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \quad (6.1)$$

**Solution:**

Download and run the C code to generate  $V$ .

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual%201/6.1/
%201/6.1/6.1.c
```

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual%201/6.1/
CDF.py
```

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual%201/6.1/
PDF.py
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \quad (6.2)$$

find  $\alpha$ .

**Solution:**

Let  $X_1 = R \sin \theta$  and  $X_2 = R \cos \theta$



Jacobian matrix is given as follows

$$J = \begin{pmatrix} \frac{\delta x_1}{\delta r} & \frac{\delta x_1}{\delta \theta} \\ \frac{\delta x_2}{\delta r} & \frac{\delta x_2}{\delta \theta} \end{pmatrix} \quad (6.3)$$

$$J = \begin{pmatrix} \cos \theta & -R \sin \theta \\ \sin \theta & R \cos \theta \end{pmatrix} \quad (6.4)$$

$$|J| = R \quad (6.5)$$

Also,

$$f_{r,\theta}(r, \theta) = |J|f_{X_1, X_2}(x_1, x_2) \quad (6.6)$$

Since,  $X_1$  and  $X_2$  are independent,

$$f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1)f_{X_2}(x_2) \quad (6.7)$$

$$= \frac{1}{2\pi} e^{-\frac{(x_1^2 + x_2^2)}{2}} \quad (6.8)$$

$$= \frac{1}{2\pi} e^{-\frac{r^2}{2}} \quad (6.9)$$

$$= \frac{1}{2\pi} e^{-\frac{r^2}{2}} \quad (6.10)$$

Put in Equation 6.6, we get,

$$f_{r,\theta}(r, \theta) = \frac{r}{2\pi} e^{-\frac{r^2}{2}} \quad (6.11)$$

Now,

$$f_R(r) = \int_0^{2\pi} f_{r,\theta}(r, \theta) d\theta \quad (6.12)$$

$$= \int_0^{2\pi} \frac{r}{2\pi} e^{-\frac{r^2}{2}} d\theta \quad (6.13)$$

$$= r e^{-\frac{r^2}{2}} \quad (6.14)$$

CDF is given by,

$$F_R(r) = \int_0^r r e^{-\frac{r^2}{2}} dr \quad (6.15)$$

$$= 1 - e^{-\frac{r^2}{2}} \quad (6.16)$$

And,

$$V = X_1^2 + X_2^2 \quad (6.17)$$

$$= R^2 \quad (6.18)$$

Now,

$$F_V(x) = \Pr(V \leq x) \quad (6.19)$$

$$= \Pr(R^2 \leq x) \quad (6.20)$$

$$= \Pr(R \leq \sqrt{x}) \quad (6.21)$$

$$F_V(x) = F_R(\sqrt{x}) \quad (6.22)$$

$$= \begin{cases} 0, & x < 0 \\ 1 - e^{-\frac{x}{2}} & x \geq 0 \end{cases} \quad (6.23)$$

$$\alpha = \frac{1}{2} \quad (6.24)$$

6.3 Plot the CDF and PDF of A.

$$A = \sqrt{V} \quad (6.25)$$

**Solution:**

Download the C code to generate  $A = \sqrt{V}$

wget <https://github.com/TYCN129/AI1110-Assignments/blob/main/Manual%201/6.3/A.c>

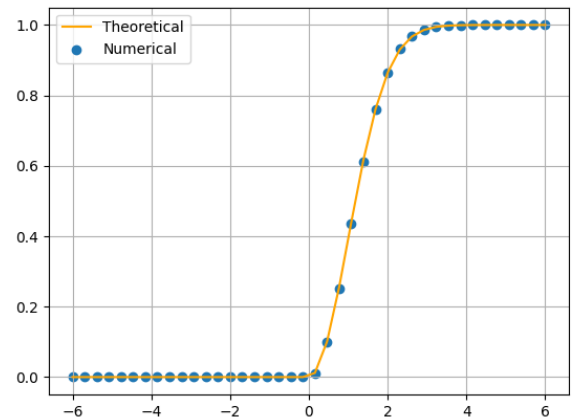


Fig. 6.3: CDF of A

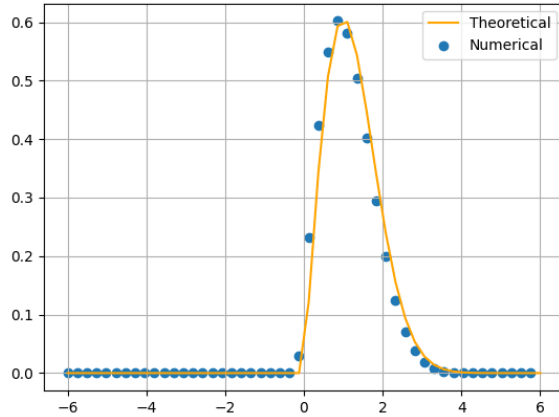


Fig. 6.3: PDF of A

## 7 CONDITIONAL PROBABILITY

### 7.1 Plot

$$P_e = \Pr(\hat{X} = -1|X = 1) \quad (7.1)$$

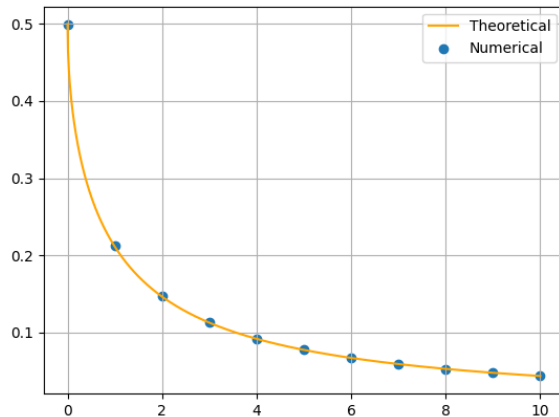
for

$$Y = AX + N, \quad (7.2)$$

where A is Rayleigh with  $E[A^2] = \gamma$ ,  $N \sim \mathcal{N}(0, 1)$ ,  $X \in (-1, 1)$  for  $0 \leq \gamma \leq 10$  dB.

**Solution:**

Download and run the following C code to

Fig. 7.1:  $P_e$  as a function for  $\gamma$ 

generate Rayleigh distribution.

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual%201/7.1/
ral_gen.c
```

The Python code plots the plot of  $P_e$  vs  $\gamma$

wget https://github.com/TYCN129/AI1110-  
Assignments/blob/main/Manual  
%201/7.1/7.1.py

7.2 Assuming that  $N$  is a constant, find an expression for  $P_e$ . Call this  $P_e(N)$

**Solution:**

$$P_e = \Pr(\hat{X} = -1|X = 1) \quad (7.3)$$

$$= \Pr(A + N < 0) \quad (7.4)$$

$$= \Pr(A < -N) \quad (7.5)$$

$$= \int_{-\infty}^{-N} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad (7.6)$$

$$= \int_N^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \quad (7.7)$$

$$= Q_A(N) \quad (7.8)$$

$$= F_A(-N) \quad (7.9)$$

$$F_A(-N) = \begin{cases} 1 - e^{-\frac{N^2}{2}} & N \leq 0 \\ 0 & N > 0 \end{cases} \quad (7.10)$$

7.3 For a function  $g$ ,

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) p_X(x) dx \quad (7.11)$$

Find  $P_e = E[P_e(N)]$ .

**Solution:**

$$P_e = \Pr(\hat{X} = -1|X = 1) \quad (7.12)$$

$$= \Pr(A + N < 0) \quad (7.13)$$

$$= \Pr(A < -N) \quad (7.14)$$

$$= \int_{-\infty}^0 \Pr(A < -\tau) P_N(\tau) d\tau \quad (7.15)$$

$$= \int_{-\infty}^0 (1 - e^{-\frac{\tau^2}{\gamma}}) \frac{1}{\sqrt{2\pi}} e^{-\frac{\tau^2}{2}} d\tau \quad (7.16)$$

$$= \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} e^{-\frac{\tau^2}{2}} d\tau - \int_{-\infty}^0 \frac{1}{\sqrt{2\pi}} e^{-\frac{\tau^2}{2}(\frac{2}{\gamma}+1)} d\tau \quad (7.17)$$

$$= \frac{1}{2} - \frac{1}{2} \frac{\sqrt{\gamma}}{\sqrt{2+\gamma}} \quad (7.18)$$

7.4 Plot  $P_e$  in problems 7.1 and 7.3 on the same graph w.r.t  $\gamma$ . Comment.