Random Numbers

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***Abstract*—This solution manual provides solutions and link to codes used for generation of random numbers.**

1 Uniform Random Variables

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution:
Download the following C code and run it to generate samples of U.

```
wget https://github.com/TYCN129/AI1110−Assignments/blob/main/Manual%201/1.1/1.1.c
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: Code used to plot empirical CDF of U

```
wget https://github.com/TYCN129/AI1110−Assignments/blob/main/Manual%201/1.2/1.2.py
```

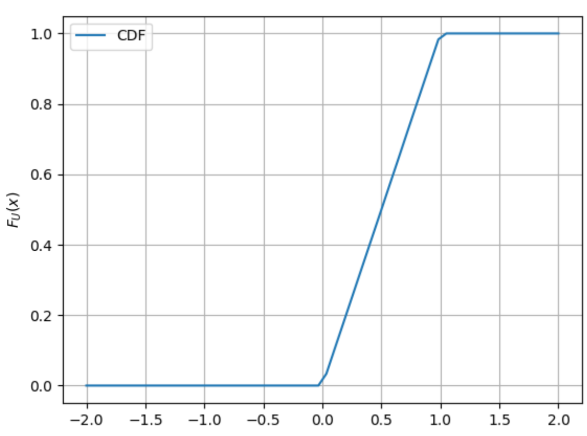

Fig. 1.2: CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution:

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{1.2}$$

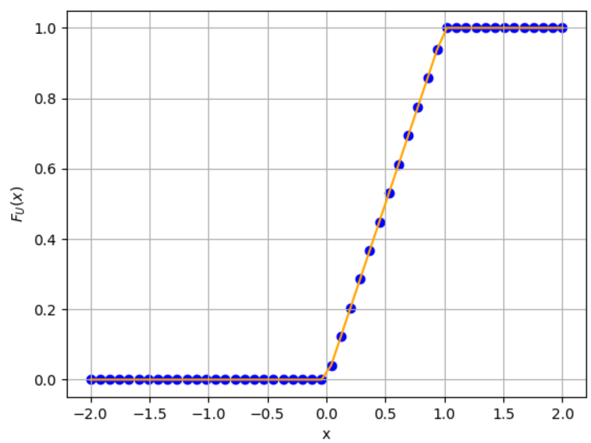

Fig. 1.3: CDF of U

```
wget https://github.com/TYCN129/AI1110−Assignments/blob/main/Manual%201/1.3/1.3.py
```

1.4 Find mean and variance of U.
Solution:
Mean of Random Variable U is given by,

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.3}$$

and its Variance is given by,

$$E\left[[U - E[U]]^2\right] = E\left[U^2\right] - [E[U]]^2 \tag{1.4}$$

Using the above two formulas, we get,

$$\text{Mean of } U = 0.500031 \quad (1.5)$$
$$\text{Variance of } U = 0.083247 \quad (1.6)$$

Download and run the C code for mean and variance

```
wget https://github.com/TYCN129/AI1110-Assignments/blob/main/Manual%201/1.4/1.4.c
```

1.5 Verify your result theoretically that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.7)$$

Solution:
For $k = 1$,

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.8)$$
$$E[U] = \int_{0}^{1} x d(x) \quad (1.9)$$
$$E[U] = 0.5 \quad (1.10)$$

Similarly, for $k = 2$,

$$E\left[U^2\right] = 0.3333 \quad (1.11)$$
$$\text{Variance} = 0.0833 \quad (1.12)$$

Thus the simulated and theoretical values of mean and variance of U are approximately equal.

2 Central Limit Theorem

2.1 **Solution:**
Download the following C code and run it to generate samples of X.

```
wget https://github.com/TYCN129/AI1110-Assignments/blob/main/Manual%201/2.1/2.1.c
```

2.2 **Solution:**

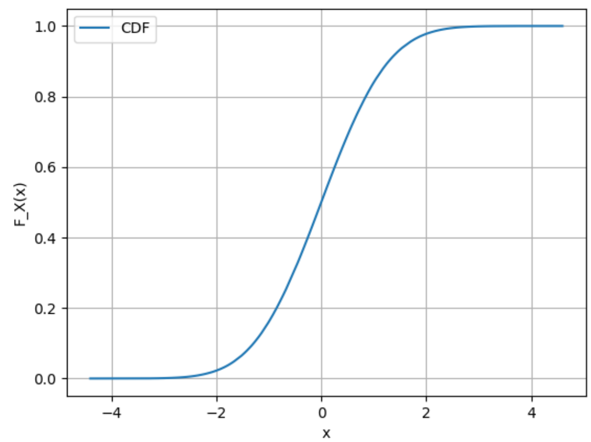

Fig. 2.2: CDF of X

The plot was generated by running the following Python code

```
wget https://github.com/TYCN129/AI1110-Assignments/blob/main/Manual%201/2.2./2.2.py
```

The CDF is a non-decreasing function with its range between 0 and 1. It will be continuous if PDF is finite.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.1)$$

What properties does the PDF have?
Solution:
The empirical PDF of X is plotted by the Python code

```
wget https://github.com/TYCN129/AI1110-Assignments/blob/main/Manual%201/2.3/2.3.py
```

The PDF takes non-negative values and area under its curve is 1.

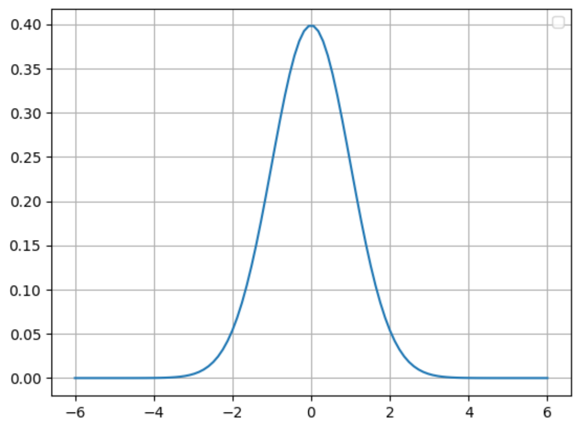

Fig. 2.3: The empirical PDF of X

2.4 Find the mean and variance of X by writing a C program.
Solution:
Run the following C file

```
wget https://github.com/TYCN129/AI1110−
    Assignments/blob/main/Manual
    %201/2.4/2.4.c
```

$$E[X] = 0.000630 \tag{2.2}$$
$$\text{var}[X] = 1.000149 \tag{2.3}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.4}$$

repeat the above exercise theoretically.
Solution:

$$E[X] = \int_{-\infty}^{\infty} x p_X(x)\, dx \tag{2.5}$$
$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.6}$$

Taking $\frac{x^2}{2} = t$,

$$E[X] = -\int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-t)\, dt \tag{2.7}$$
$$E[X] = 0 \tag{2.8}$$

To calculate variance,

$$var[X] = E\left[(X - E[X])^2\right] \tag{2.9}$$
$$var[X] = E\left[X^2\right] \tag{2.10}$$
$$var[X] = \int_{-\infty}^{\infty} x^2 p_X(x)\, dx \tag{2.11}$$
$$var[X] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.12}$$

We know that,

$$\int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2}\right) = \sqrt{2\pi} \tag{2.13}$$
$$var[X] = 1 \tag{2.14}$$

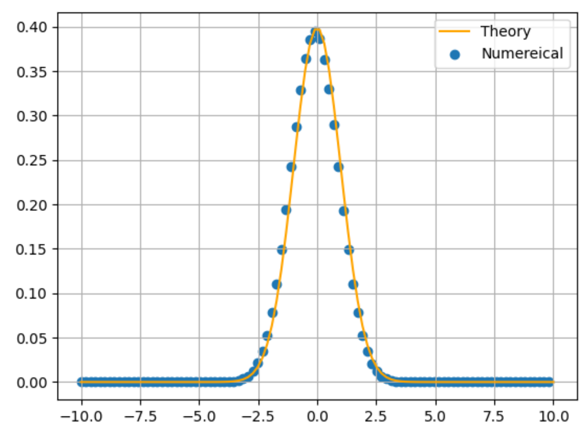

Fig. 2.5: PDF of X

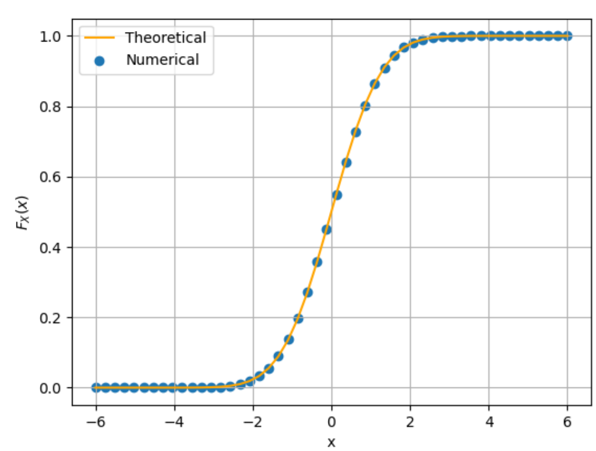

Fig. 2.5: CDF of X

Theoretical expression for X,

$$F_X(x) = 1 - Q(\frac{x - E[X]}{var[X]}) \quad (2.15)$$

$$F_X(x) = 1 - Q(x) \quad (2.16)$$

The Python codes plot the CDF and PDF of X

```
wget https://github.com/TYCN129/AI1110-
    Assignments/blob/main/Manual
    %201/2.5/2.5.py

wget https://github.com/TYCN129/AI1110-
    Assignments/blob/main/Manual
    %201/2.5/2.5_2.py
```

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \quad (3.1)$$

and plot its CDF.
Solution:
Download and run the following code to generate samples of V

```
wget https://github.com/TYCN129/AI1110-
    Assignments/blob/main/Manual
    %201/3.1/3.1.c
```

The following Python code plots CDF of V

```
wget https://github.com/TYCN129/AI1110-
    Assignments/blob/main/Manual
    %201/3.1/3.1.py
```

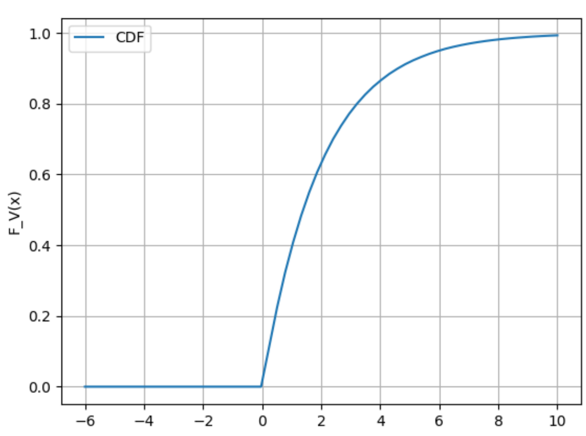

Fig. 3.1: The empirical CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$F_V(x) = Pr(V \le x) \quad (3.2)$$

$$F_V(x) = Pr(-2\ln(1 - U) \le x) \quad (3.3)$$

$$F_V(x) = Pr\left(\ln(1 - U) \ge -\frac{x}{2}\right) \quad (3.4)$$

$$F_V(x) = Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

$$F_V(x) = Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.6)$$

$$F_V(x) = F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.7)$$

$$F_V(x) = \begin{cases} 0 & 1 - \exp\left(-\frac{x}{2}\right) < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & 0 \le 1 - \exp\left(-\frac{x}{2}\right) \le 1 \\ 1 & 1 - \exp\left(-\frac{x}{2}\right) > 1 \end{cases} \quad (3.8)$$

This simplifies to

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x \ge 0 \end{cases} \quad (3.9)$$

The following python code plots the theoritical CDF

```
wget https://github.com/TYCN129/AI1110-
    Assignments/blob/main/Manual
    %201/3.2/3.2.py
```

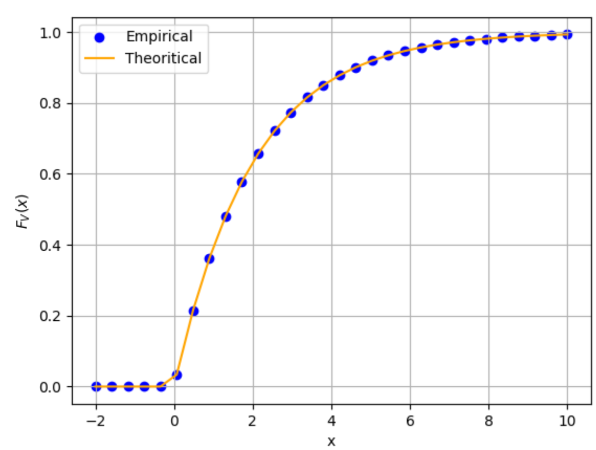

Fig. 3.2: CDF of V

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution:
Download and run the following C code to generate tri.dat file.

```
wget https://github.com/TYCN129/AI1110-
    Assignments/blob/main/Manual
    %201/4.1/4.1.c
```

4.2 Find the CDF of T.
Solution:

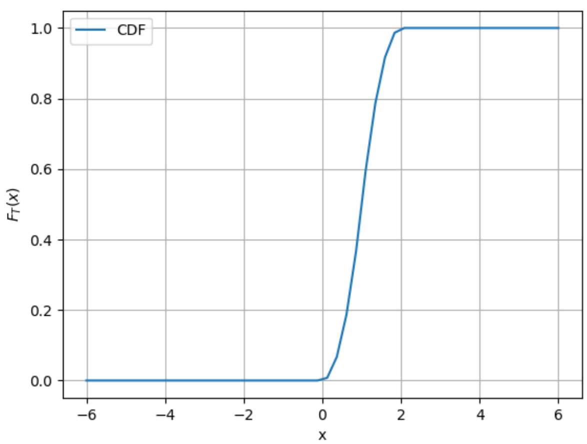

Fig. 4.2: CDF of T

The following code plots the CDF of T

```
wget https://github.com/TYCN129/AI1110-
    Assignments/blob/main/Manual
    %201/4.2/4.2.py
```

4.3 Find the CDF of T.
Solution:

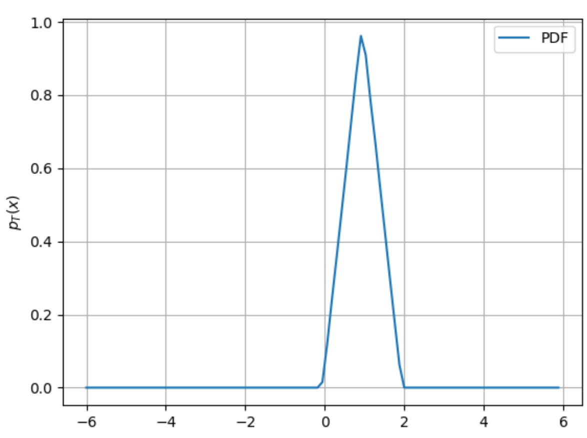

Fig. 4.3: PDF of T

The following code plots the PDF of T

```
wget https://github.com/TYCN129/AI1110-
    Assignments/blob/main/Manual
    %201/4.3/4.3.py
```

4.4 Find the theoritical expressions for PDF and CDF of T.
Solution:

$$p_T(x) = p_{U_1+U_2}(x) = p_{U_1}(x) * p_{U_2}(x) \tag{4.2}$$

$$p_T(x) = \int_{-\infty}^{\infty} p_{U_1}(\tau)p_{U_2}(x-\tau)d\tau \tag{4.3}$$

$$p_T(x) = \int_0^1 p_{U_2}(x-\tau)d\tau \tag{4.4}$$

$$p_T(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x 1d\tau & 0 < x < 1 \\ \int_{x-1}^1 1d\tau & 1 \leq x < 2 \\ 0 & x > 2 \end{cases} \tag{4.5}$$

$$p_T(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 2-x & 1 \leq x < 2 \\ 0 & x > 2 \end{cases} \tag{4.6}$$

Expression for CDF can be obtained by integrating $p_T(x)$ w.r.t. X

$$F_T(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{2} & 0 < x < 1 \\ -\frac{x^2}{2} + 2x - 1 & 1 \leq x < 2 \\ 1 & x > 2 \end{cases} \tag{4.7}$$

4.5 Verify the results through a plot.
Solution:

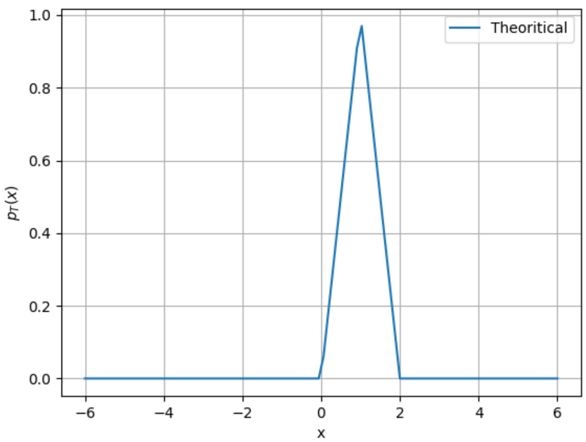

Fig. 4.5: Theoretical PDF of T

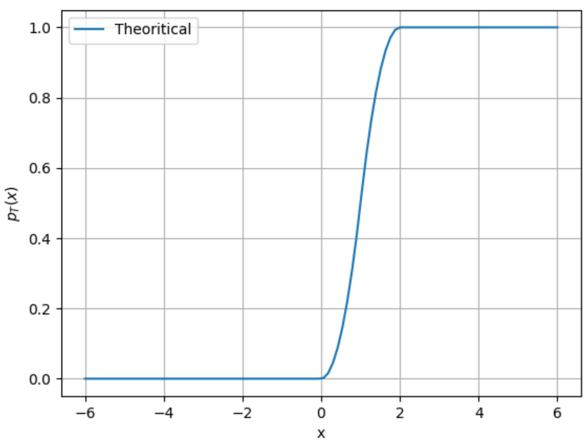

Fig. 4.5: The CDF of T

PDF and CDF plotted by the Python codes

```
wget https://github.com/TYCN129/AI1110−
    Assignments/blob/main/Manual
    %201/4.5/4.5_1.py

wget https://github.com/TYCN129/AI1110−
    Assignments/blob/main/Manual
    %201/4.5/4.5_1.py
```