Random Numbers

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***Abstract*—This solution manual provides solutions and link to codes used for generation of random numbers.**

1 Uniform Random Variables

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution:
Download the following C code and run it to generate samples of U.

```
wget https://github.com/TYCN129/AI1110−Assignments/blob/main/Manual%201/1.1/1.1.c
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: Code used to plot empirical CDF of U

```
wget https://github.com/TYCN129/AI1110−Assignments/blob/main/Manual%201/1.2/1.2.py
```

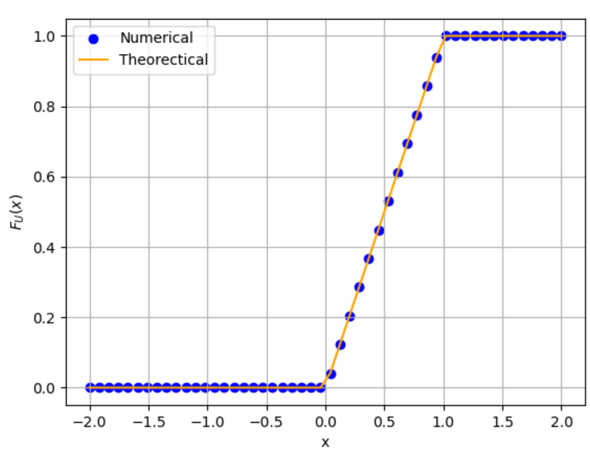

Fig. 1.2: CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution:

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \tag{1.2}$$

```
wget https://github.com/TYCN129/AI1110−Assignments/blob/main/Manual%201/1.3/1.3.py
```

1.4 Find mean and variance of U.
Solution:
Mean of Random Variable U is given by,

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.3}$$

and its Variance is given by,

$$E\left[[U - E[U]]^2\right] = E\left[U^2\right] - [E[U]]^2 \tag{1.4}$$

Using the above two formulas, we get,

$$\text{Mean of } U = 0.500031 \tag{1.5}$$
$$\text{Variance of } U = 0.083247 \tag{1.6}$$

Download and run the C code for mean and variance

```
wget https://github.com/TYCN129/AI1110−Assignments/blob/main/Manual%201/1.4/1.4.c
```

1.5 Verify your result theoretically that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.7}$$

Solution:

For $k = 1$,

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.8}$$

$$E[U] = \int_{0}^{1} x d(x) \tag{1.9}$$

$$E[U] = 0.5 \tag{1.10}$$

Similarly, for $k = 2$,

$$E\left[U^2\right] = 0.3333 \tag{1.11}$$

$$\text{Variance} = 0.0833 \tag{1.12}$$

Thus the simulated and theoretical values of mean and variance of U are approximately equal.

2 Central Limit Theorem

2.1 **Solution:**
Download the following C code and run it to generate samples of X.

```
wget https://github.com/TYCN129/AI1110−
    Assignments/blob/main/Manual
    %201/2.1/2.1.c
```

2.2 **Solution:**
The plot was generated by running the following Python code

```
wget https://github.com/TYCN129/AI1110−
    Assignments/blob/main/Manual
    %201/2.2./2.2.py
```

The CDF is a non-decreasing function with its range between 0 and 1. It will be continuous if PDF is finite.

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.1}$$

What properties does the PDF have?
Solution:
The empirical PDF of X is plotted by the Python code

```
wget https://github.com/TYCN129/AI1110−
    Assignments/blob/main/Manual
    %201/2.3/2.3.py
```

The PDF takes non-negative values and area under its curve is 1.

2.4 Find the mean and variance of X by writing a C program.
Solution:
Run the following C file

```
wget https://github.com/TYCN129/AI1110−
    Assignments/blob/main/Manual
    %201/2.4/2.4.c
```

$$E[X] = 0.000630 \tag{2.2}$$

$$\text{var}[X] = 1.000149 \tag{2.3}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.4}$$

repeat the above exercise theoretically.
Solution:

$$E[X] = \int_{-\infty}^{\infty} x p_X(x)\, dx \tag{2.5}$$

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.6}$$

Taking $\frac{x^2}{2} = t$,

$$E[X] = -\int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-t)\, dt \tag{2.7}$$

$$E[X] = 0 \tag{2.8}$$

To calculate variance,

$$var[X] = E\left[(X - E[X])^2\right] \tag{2.9}$$

$$var[X] = E\left[X^2\right] \tag{2.10}$$

$$var[X] = \int_{-\infty}^{\infty} x^2 p_X(x)\, dx \tag{2.11}$$

$$var[X] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.12}$$

We know that,

$$\int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2}\right) = \sqrt{2\pi} \tag{2.13}$$

$$var[X] = 1 \tag{2.14}$$

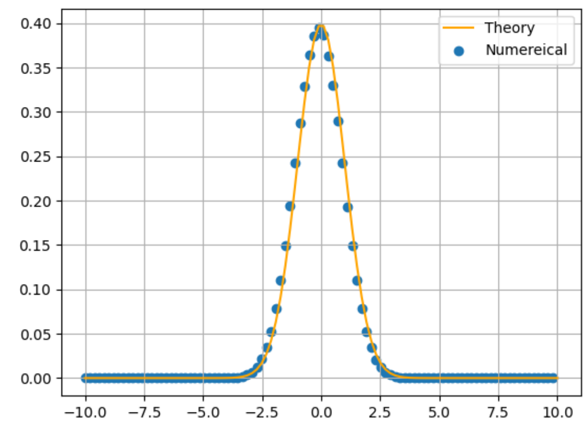

Fig. 2.5: PDF of X

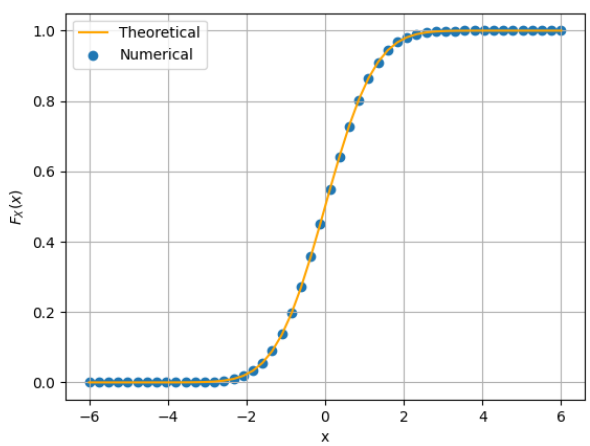

Fig. 2.5: CDF of X

Theoretical expression for X,

$$F_X(x) = 1 - Q(\frac{x - E[X]}{var[X]}) \tag{2.15}$$

$$F_X(x) = 1 - Q(x) \tag{2.16}$$

The Python codes plot the CDF and PDF of X

```
wget https://github.com/TYCN129/AI1110–
    Assignments/blob/main/Manual
    %201/2.5/2.5.py

wget https://github.com/TYCN129/AI1110–
    Assignments/blob/main/Manual
    %201/2.5/2.5_2.py
```

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF.
Solution:
Download and run the following code to generate samples of V

```
wget https://github.com/TYCN129/AI1110–
    Assignments/blob/main/Manual
    %201/3.1/3.1.c
```

The following Python code plots CDF of V

```
wget https://github.com/TYCN129/AI1110–
    Assignments/blob/main/Manual
    %201/3.1/3.1.py
```

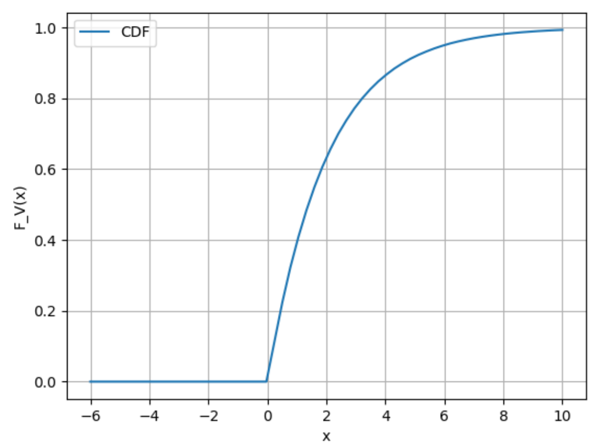

Fig. 3.1: The empirical CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$F_V(x) = Pr(V \le x) \tag{3.2}$$

$$F_V(x) = Pr(-2\ln(1 - U) \le x) \tag{3.3}$$

$$F_V(x) = Pr\left(\ln(1 - U) \ge -\frac{x}{2}\right) \tag{3.4}$$

$$F_V(x) = Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$

$$F_V(x) = Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.6}$$

$$F_V(x) = F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.7}$$

$$F_V(x) = \begin{cases} 0 & 1 - \exp\left(-\frac{x}{2}\right) < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & 0 \le 1 - \exp\left(-\frac{x}{2}\right) \le 1 \\ 1 & 1 - \exp\left(-\frac{x}{2}\right) > 1 \end{cases} \tag{3.8}$$

This simplifies to

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x \ge 0 \end{cases} \tag{3.9}$$

The following python code plots the theoritical CDF

```
wget https://github.com/TYCN129/AI1110-
    Assignments/blob/main/Manual
    %201/3.2/3.2.py
```

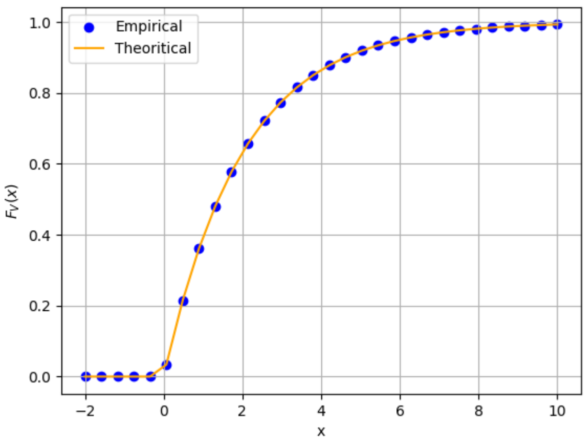

Fig. 3.2: CDF of V

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution:
Download and run the following C code to generate tri.dat file.

```
wget https://github.com/TYCN129/AI1110-
    Assignments/blob/main/Manual
    %201/4.1/4.1.c
```

4.2 Find the CDF of T.
Solution:
The following code plots the CDF of T

```
wget https://github.com/TYCN129/AI1110-
    Assignments/blob/main/Manual
    %201/4.2/4.2.py
```

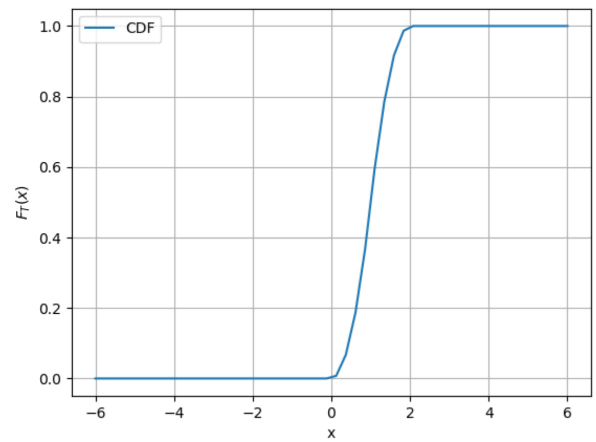

Fig. 4.2: CDF of T

4.3 Find the CDF of T.
Solution:

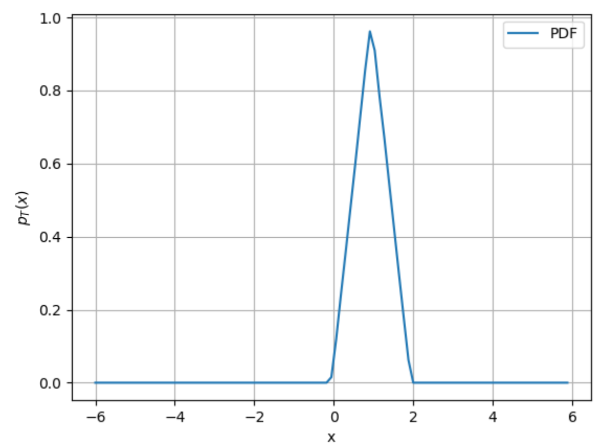

Fig. 4.3: PDF of T

The following code plots the PDF of T

```
wget https://github.com/TYCN129/AI1110-
    Assignments/blob/main/Manual
    %201/4.3/4.3.py
```

4.4 Find the theoritical expressions for PDF and CDF of T.
Solution:

$$p_T(x) = p_{U_1+U_2}(x) = p_{U_1}(x) * p_{U_2}(x) \quad (4.2)$$

$$p_T(x) = \int_{-\infty}^{\infty} p_{U_1}(\tau) p_{U_2}(x-\tau) d\tau \quad (4.3)$$

$$p_T(x) = \int_0^1 p_{U_2}(x-\tau) d\tau \quad (4.4)$$

$$p_T(x) = \begin{cases} 0 & x \le 0 \\ \int_0^x 1 d\tau & 0 < x < 1 \\ \int_{x-1}^1 1 d\tau & 1 \le x < 2 \\ 0 & x > 2 \end{cases} \quad (4.5)$$

$$p_T(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x < 1 \\ 2 - x & 1 \le x < 2 \\ 0 & x > 2 \end{cases} \quad (4.6)$$

Expression for CDF can be obtained by integrating $p_T(x)$ w.r.t. X

$$F_T(x) = \begin{cases} 0 & x \le 0 \\ \frac{x^2}{2} & 0 < x < 1 \\ -\frac{x^2}{2} + 2x - 1 & 1 \le x < 2 \\ 1 & x > 2 \end{cases} \quad (4.7)$$

4.5 Verify the results through a plot.
Solution:

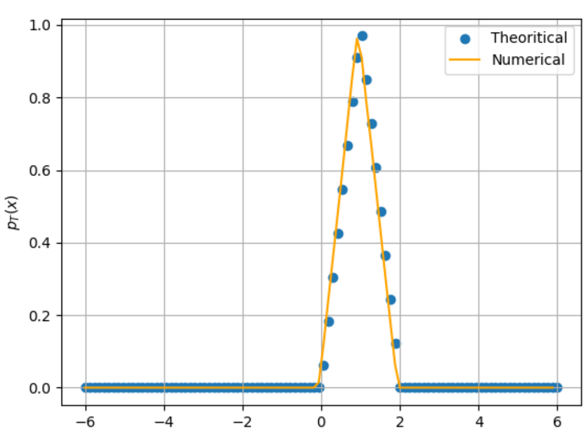

Fig. 4.5: Theoretical PDF of T

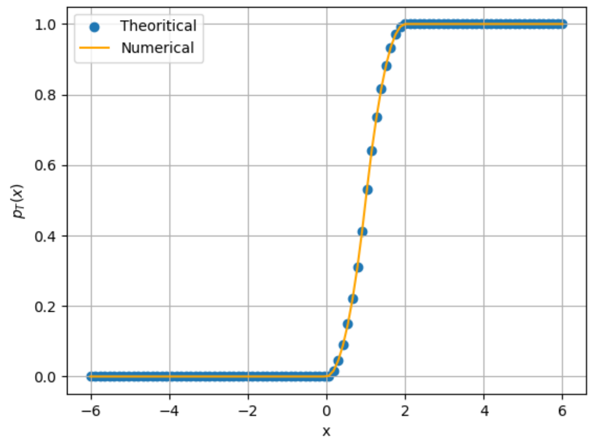

Fig. 4.5: The CDF of T

PDF and CDF plotted by the Python codes

```
wget https://github.com/TYCN129/AI1110−
    Assignments/blob/main/Manual
    %201/4.5/4.5_1.py

wget https://github.com/TYCN129/AI1110−
    Assignments/blob/main/Manual
    %201/4.5/4.5_1.py
```

5 Maximul Likelihood

5.1 Generate equiprobable $X \in \{1, -1\}$.
Solution:

Download and run the following codes to generate X.

```
wget https://github.com/TYCN129/AI1110−
    Assignments/blob/main/Manual
    %201/5.1/5.1_X.c
```

5.2 Generate

$$Y = AX + N, \quad (5.1)$$

where $A = 5$ dB, and $N \sim 01$.
Solution:
Download and run the following code to generate Y.

```
wget https://github.com/TYCN129/AI1110−
    Assignments/tree/main/Manual
    %201/5.1/5.1_Y.c
```

5.3 Plot Y
Solution:

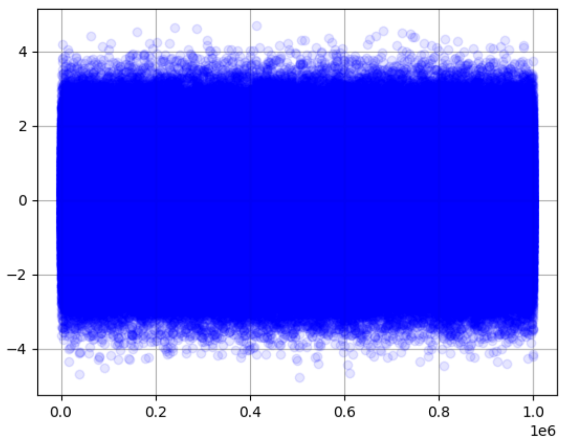

Fig. 5.3: Plot of Y

The Python code plots Y.

```
wget https://github.com/TYCN129/AI1110-
    Assignments/blob/main/Manual
    %201/5.2/5.2.py
```

5.4 Guess how to estimate X from Y
Solution:
Estimate of $X = \hat{X}$ is found out from Y by,

$$\hat{X} = \begin{cases} -1 & Y < 0 \\ 1 & Y \geq 0 \end{cases} \tag{5.2}$$

5.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.3}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.4}$$

Solution:

$$P_{e|0} = \Pr\left(\hat{X} = -1|X = 1\right) \tag{5.5}$$
$$= 0.4998032 \tag{5.6}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1|X = -1\right) \tag{5.7}$$
$$= 0.4995446 \tag{5.8}$$

5.6 Find P_e assuming X has equiprobable symbols.
Solution:

$$P_{e|0} = Pr(\hat{X} = -1|X = 1) \tag{5.9}$$
$$= Pr(AX + N < 0|X = 1) \tag{5.10}$$
$$= Pr(N < -A) \tag{5.11}$$
$$= \int_{-\infty}^{-A} \frac{1}{\sqrt{2\pi}} \exp^{\frac{-x^2}{2}} \tag{5.12}$$
$$= \int_{A}^{\infty} \frac{1}{\sqrt{2\pi}} \exp^{\frac{-x^2}{2}} \quad = Q_N(A) \tag{5.13}$$

where, $Q_N(A)$ is the Q-function of Normal distribution. Similarly,

$$P_{e|1} = Q_N(A) \tag{5.14}$$

Thus,

$$P_e = P_{e|0} \times Pr(X = 1) + P_{e|1} \times Pr(X = -1) \tag{5.15}$$

$$P_e = \frac{1}{2} \times P_{e|0} + \frac{1}{2} \times P_e e|1 \tag{5.16}$$
$$P_e = \frac{P_{e|0} + P_{e|1}}{2} \tag{5.17}$$
$$P_e = Q_N(A) \tag{5.18}$$

5.7 Verify by plotting theoretical P_e by varying A from 0 to 10 dB
Solution:

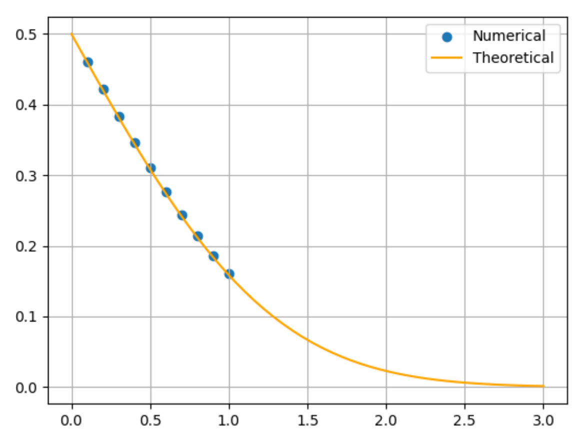

Fig. 5.7: Plot of P_e with varying A

```
wget https://github.com/TYCN129/AI1110-
    Assignments/tree/main/Manual%201/5.7
```

5.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

Solution:

Defining estimate of X as,

$$\hat{X} = \begin{cases} 1 & Y > \delta \\ -1 & Y < \delta \end{cases} \tag{5.19}$$

$$P_{e|0} = Pr(\hat{X} = -1|X = 1) \tag{5.20}$$
$$= Pr(Y < \delta|X = 1) \tag{5.21}$$
$$= Pr(AX + N < \delta|X = 1) \tag{5.22}$$
$$= Pr(N < \delta - A) \tag{5.23}$$
$$= \int_{-\infty}^{\delta-A} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \tag{5.24}$$
$$= \int_{A-\delta}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} \tag{5.25}$$
$$= Q_N(A - \delta) \tag{5.26}$$

Similarly,

$$\tag{5.27}$$

$$P_{e|1} = Q_N(A + \delta) \tag{5.28}$$

Therefore,

$$P_e = P(X = 1) \times P_{e|0} + P(X = -1) \times P_{e|1} \tag{5.29}$$

$$= \frac{Q_(A - \delta) + Q_N(A + \delta)}{2} \tag{5.30}$$

To maximize theoretical P_e, we differentiate equation (5.30) w.r.t δ.

$$\frac{dP_e}{d\delta} = \frac{1}{2}(\frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta-A)^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}}) \tag{5.31}$$
$$0 = \frac{1}{2}(\frac{1}{\sqrt{2\pi}} e^{-\frac{(\delta-A)^2}{2}} - \frac{1}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}}) \tag{5.32}$$

Thus,

$$\delta = 0 \tag{5.33}$$

5.9 Repeat the above exercise when

$$p_X(0) = p \tag{5.34}$$

Solution:

$$P_e = P(X = 1) \times P_{e|0} + P(X = -1) \times P_{e|1} \tag{5.35}$$

$$P_e = pP_{e|0} + (1 - p)P_{e|1} \tag{5.36}$$

On differentiating both sides,

$$0 = \frac{1}{2}(\frac{p}{\sqrt{2\pi}} e^{-\frac{(\delta-A)^2}{2}} - \frac{(1-p)}{\sqrt{2\pi}} e^{-\frac{(A+\delta)^2}{2}}) \tag{5.37}$$

Thus,

$$\delta = \frac{1}{2A} ln\left(\frac{(1-p)}{p}\right) \tag{5.38}$$

5.10 Repeat the above exercise using the MAP criterion.

Solution:

$$p_Y(y) = p_{Y|X=1}(y|1) \times P(X = 1) + p_{Y|X=-1}(y|-1) \times P(X = -1)$$
$$= p_{N+A}(y) \times p + p_{N-A}(y) \times (1 - p) \tag{5.39}$$

Here, $p_Y(y)$ is the PDF of Y, p_{N+A} and p_{N-A} are the PDF of shifted normal distribution.

$$p_Y(y) = \frac{p}{\sqrt{2\pi}} e^{-\frac{(y+A)^2}{2}} + \frac{(1-p)}{\sqrt{2\pi}} e^{-\frac{(y-A)^2}{2}} \tag{5.40}$$

Now, we need to find $p_{X|Y}(x|y)$ using the formula,

$$p_{X|Y}(x|y) = \frac{p_{Y|X}(y|x) \times p_X(x)}{p_Y(y)} \tag{5.41}$$

For $X = 1$,

$$p_{X|Y}(1|y) = \frac{p_{Y|X}(y|1) \times p}{p_Y(y)} \tag{5.42}$$
$$= \frac{\frac{p}{\sqrt{2\pi}} e^{-\frac{(y+A)^2}{2}}}{\frac{p}{\sqrt{2\pi}} e^{-\frac{(y+A)^2}{2}} + \frac{(1-p)}{\sqrt{2\pi}} e^{-\frac{(y-A)^2}{2}}} \tag{5.43}$$
$$= \frac{pe^{2Ay}}{pe^{2Ay} + (1-p)} \tag{5.44}$$

Similarly, for $X = -1$,

$$p_{X|Y}(-1|y) = \frac{p_{Y|X}(y|-1) \times (1-p)}{p_Y(y)} \tag{5.45}$$

$$= \frac{\frac{(1-p)}{\sqrt{2\pi}} e^{-\frac{(y-A)^2}{2}}}{\frac{p}{\sqrt{2\pi}} e^{-\frac{(y+A)^2}{2}} + \frac{(1-p)}{\sqrt{2\pi}} e^{-\frac{(y-A)^2}{2}}} \tag{5.46}$$

$$= \frac{(1-p)}{pe^{2Ay} + (1-p)} \tag{5.47}$$

- Case 1) $p_{X|Y}(1|y) > p_{X|Y}(-1|y)$

$$\frac{pe^{2Ay}e^{2Ay}}{pe^{2Ay} + (1-p)} > \frac{(1-p)}{pe^{2Ay} + (1-p)} \tag{5.48}$$

$$e^{-2Ay} < \frac{p}{1-p} \tag{5.49}$$

$$y > \frac{1}{2A} ln\left(\frac{p}{1-p}\right) \tag{5.50}$$

Equation (5.50) holds good when $X = 1$.

- Case 2) $p_{X|Y}(1|y) < p_{X|Y}(-1|y)$

$$\frac{pe^{2Ay}}{pe^{2Ay} + (1-p)} < \frac{(1-p)}{pe^{2Ay} + (1-p)} \tag{5.51}$$

$$e^{-2Ay} > \frac{p}{1-p} \tag{5.52}$$

$$y < \frac{1}{2A} ln\left(\frac{p}{1-p}\right) \tag{5.53}$$

Equation (5.53) holds good when $X = -1$.

6 Gaussian to Other

6.1 Let $X_1 \sim 01$ and $X_2 \sim 01$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{6.1}$$

Solution:
Download and run the C code to generate V.

```
wget https://github.com/TYCN129/AI1110-Assignments/blob/main/Manual%201/6.1/6.1.c
```

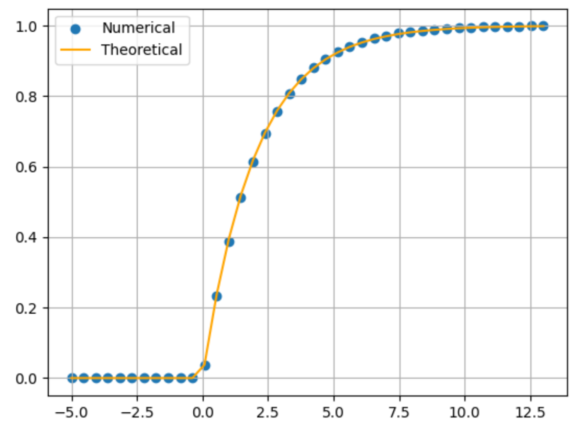

Fig. 6.1: CDF of V

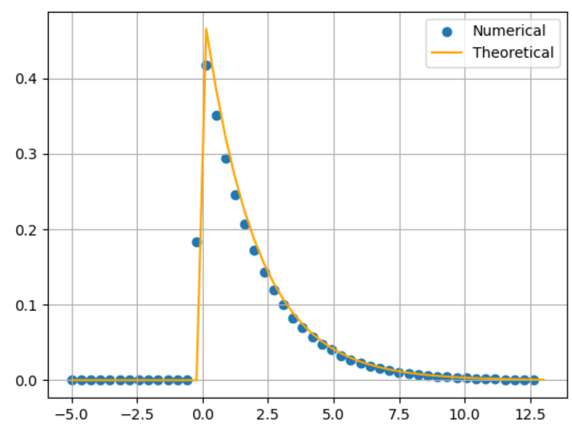

Fig. 6.1: PDF of V

```
wget https://github.com/TYCN129/AI1110-Assignments/blob/main/Manual%201/6.1/CDF.py

wget https://github.com/TYCN129/AI1110-Assignments/blob/main/Manual%201/6.1/PDF.py
```

6.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{6.2}$$

find α.

Solution:
Let $X_1 = R \sin\theta$ and $X_2 = R \cos\theta$

Jacobian matrix is given as follows

$$J = \begin{pmatrix} \frac{\delta x_1}{\delta r} & \frac{\delta x_1}{\delta \theta} \\ \frac{\delta x_2}{\delta r} & \frac{\delta x_2}{\delta \theta} \end{pmatrix} \tag{6.3}$$

$$J = \begin{pmatrix} \cos\theta & -R\sin\theta \\ \sin\theta & R\cos\theta \end{pmatrix} \tag{6.4}$$

$$|J| = R \tag{6.5}$$

Also,

$$f_{r,\theta}(r,\theta) = |J| f_{X_1,X_2}(x_1, x_2) \tag{6.6}$$

Since, X_1 and X_2 are independent,

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) \tag{6.7}$$

$$\tag{6.8}$$

$$= \frac{1}{2\pi} e^{-\frac{(x_1^2+x_2^2)}{2}} \tag{6.9}$$

$$= \frac{1}{2\pi} e^{-\frac{r^2}{2}} \tag{6.10}$$

Put in Equation 6.6, we get,

$$f_{R,\theta}(r,\theta) = \frac{r}{2\pi} e^{-\frac{r^2}{2}} \tag{6.11}$$

Now,

$$f_R(r) = \int_0^{2\pi} f_{R,\theta}(r,\theta) \tag{6.12}$$

$$= \int_0^{2\pi} \frac{r}{2\pi} e^{-\frac{r^2}{2}} d\theta \tag{6.13}$$

$$= re^{-\frac{r^2}{2}} \tag{6.14}$$

CDF is given by,

$$F_R(r) = \int_0^r re^{-\frac{r^2}{2}} \tag{6.15}$$

$$= 1 - e^{\frac{-r^2}{2}} \tag{6.16}$$

And,

$$V = X_1^2 + X_2^2 \tag{6.17}$$

$$= R^2 \tag{6.18}$$

Now,

$$F_V(x) = \Pr(V \le x) \tag{6.19}$$

$$= \Pr\left(R^2 \le x\right) \tag{6.20}$$

$$= \Pr\left(R \le \sqrt{x}\right) \tag{6.21}$$

$$F_V(x) = F_R(\sqrt{x}) \tag{6.22}$$

$$= \begin{cases} 0, & x < 0 \\ 1 - e^{-\frac{x}{2}} & x \ge 0 \end{cases} \tag{6.23}$$

$$\alpha = \frac{1}{2} \tag{6.24}$$

6.3 Plot the CDF and PDF of A.

$$A = \sqrt{V} \tag{6.25}$$

Solution:
Download the C code to generate $A = \sqrt{V}$

```
wget https://github.com/TYCN129/AI1110-
    Assignments/blob/main/Manual%201/6.3/
    A.c
```

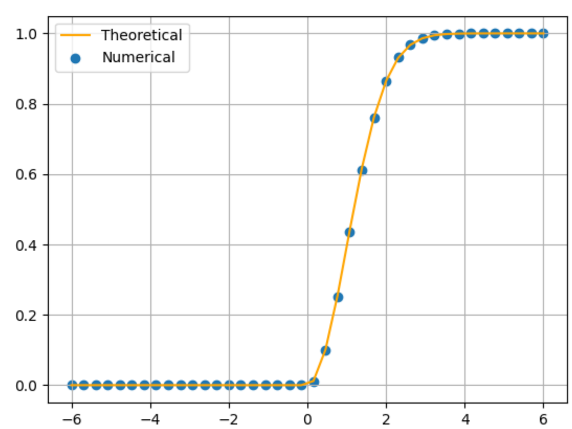

Fig. 6.3: CDF of A

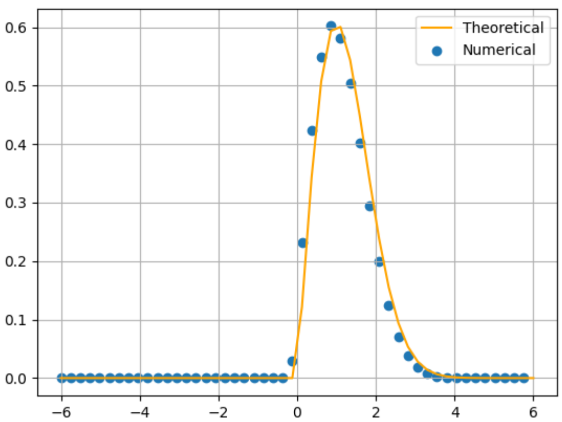

Fig. 6.3: PDF of A