

Random Numbers

Vedant Bhandare (CS21BTECH11007)

Abstract—This solution manual provides solutions and link to codes used for generation of random numbers.

1 UNIFORM RANDOM VARIABLES

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution:

Download the following C code and run it to generate samples of U .

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/1.1/1.1.c
```

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: Code used to plot empirical CDF of U

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/1.2/1.2.py
```

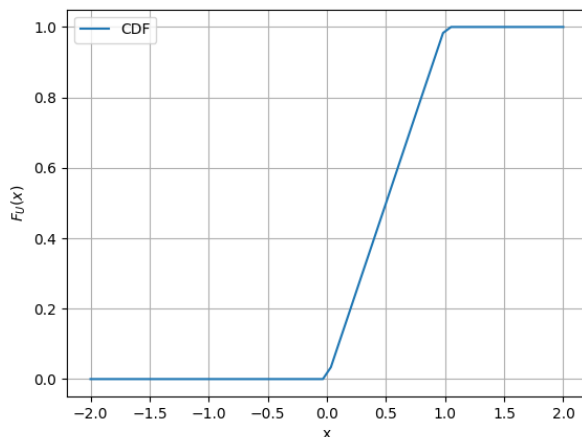


Fig. 1.2: CDF of U

- 1.3 Find a theoretical expression for $F_U(x)$.

Solution:

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \quad (1.2)$$

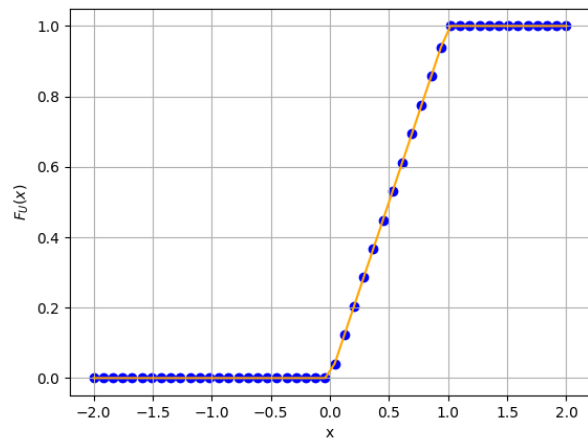


Fig. 1.3: CDF of U

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/1.3/1.3.py
```

- 1.4 Find mean and variance of U .

Solution:

Mean of Random Variable U is given by,

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.3)$$

and its Variance is given by,

$$E[U^2] - [E[U]]^2 \quad (1.4)$$

Using the above two formulas, we get,

$$\text{Mean of } U = 0.500031 \quad (1.5)$$

$$\text{Variance of } U = 0.083247 \quad (1.6)$$

Download and run the C code for mean and variance

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/1.4/1.4.c
```

1.5 Verify your result theoretically that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.7)$$

Solution:

For $k = 1$,

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.8)$$

$$E[U] = \int_0^1 x d(x) \quad (1.9)$$

$$E[U] = 0.5 \quad (1.10)$$

Similarly, for $k = 2$,

$$E[U^2] = 0.3333 \quad (1.11)$$

$$\text{Variance} = 0.0833 \quad (1.12)$$

Thus the simulated and theoretical values of mean and variance of U are approximately equal.

2 CENTRAL LIMIT THEOREM

2.1 **Solution:**

Download the following C code and run it to generate samples of X .

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/2.1/2.1.c
```

2.2 **Solution:**

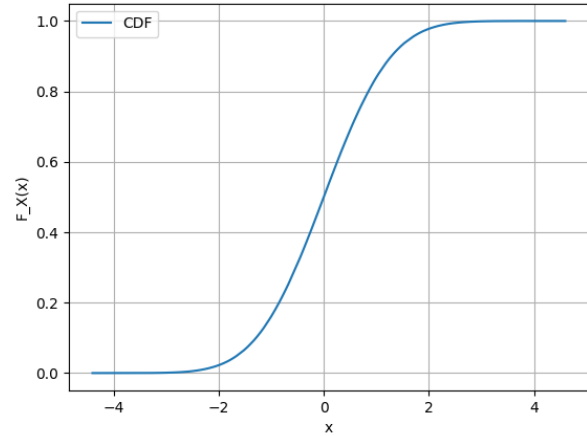


Fig. 2.2: CDF of X

The plot was generated by running the following Python code

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/2.2./2.2.py
```

The CDF is a non-decreasing function with its range between 0 and 1. It will be continuous if PDF is finite.

2.3 Load `gau.dat` in python and plot the empirical PDF of X using the samples in `gau.dat`. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.1)$$

What properties does the PDF have?

Solution:

The empirical PDF of X is plotted by the Python code

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/2.3/2.3.py
```

The PDF takes non-negative values and area under its curve is 1.

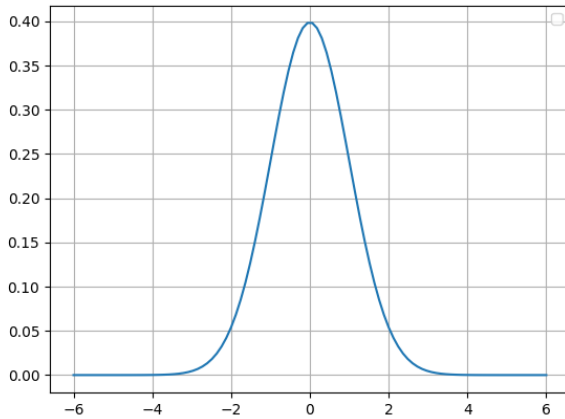


Fig. 2.3: The empirical PDF of X

2.4 Find the mean and variance of X by writing a C program.

Solution:

Run the following C file

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/2.4/2.4.c
```

$$E[X] = 0.000630 \quad (2.2)$$

$$\text{var}[X] = 1.000149 \quad (2.3)$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.4)$$

repeat the above exercise theoretically.

Solution:

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.5)$$

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.6)$$

Taking $\frac{x^2}{2} = t$,

$$E[X] = - \int_{\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp(-t) dt \quad (2.7)$$

$$E[X] = 0 \quad (2.8)$$

To calculate variance,

$$\text{var}[X] = E[(X - E[X])^2] \quad (2.9)$$

$$\text{var}[X] = E[X^2] \quad (2.10)$$

$$\text{var}[X] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.11)$$

$$\text{var}[X] = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.12)$$

We know that,

$$\int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2}\right) = \sqrt{2\pi} \quad (2.13)$$

$$\text{var}[X] = 1 \quad (2.14)$$

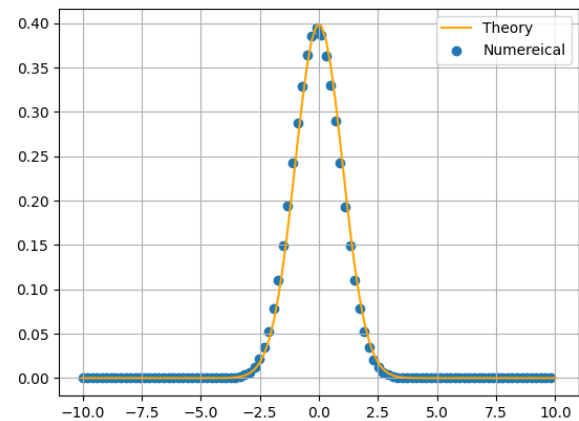


Fig. 2.5: Theoretical PDF of X

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

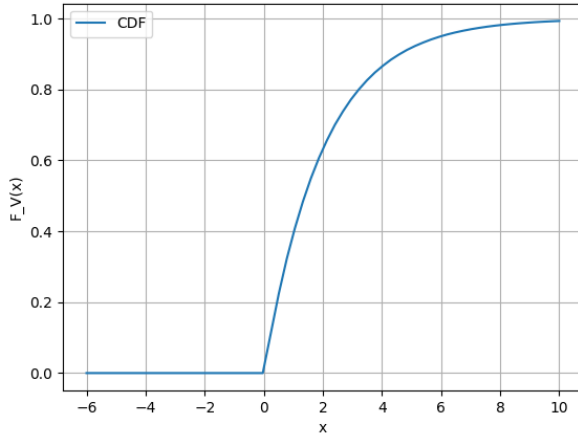
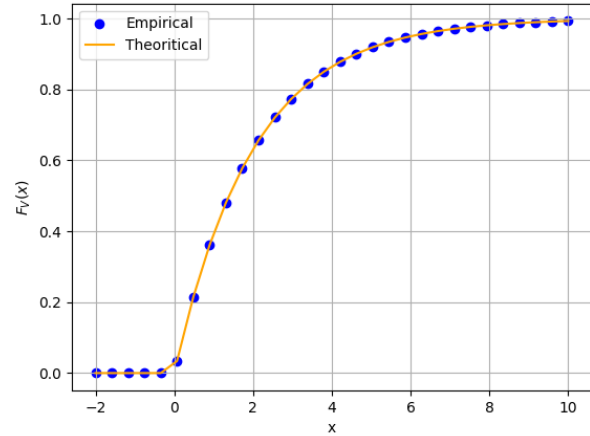
Solution:

Download and run the following code to generate samples of V

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/3.1/3.1.c
```

The following Python code plots CDF of V

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/3.1/3.1.py
```

Fig. 3.1: The empirical CDF of V Fig. 3.2: CDF of V

3.2 Find a theoretical expression for $F_V(x)$.

Solution:

$$F_V(x) = Pr(V \leq x) \quad (3.2)$$

$$F_V(x) = Pr(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$F_V(x) = Pr\left(\ln(1 - U) \geq -\frac{x}{2}\right) \quad (3.4)$$

$$F_V(x) = Pr\left(1 - U \geq \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

$$F_V(x) = Pr\left(U \leq 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.6)$$

$$F_V(x) = F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.7)$$

$$F_V(x) = \begin{cases} 0 & 1 - \exp\left(-\frac{x}{2}\right) < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & 0 \leq 1 - \exp\left(-\frac{x}{2}\right) \leq 1 \\ 1 & 1 - \exp\left(-\frac{x}{2}\right) > 1 \end{cases} \quad (3.8)$$

This simplifies to

$$F_V(x) = \begin{cases} 0 & x < 0 \\ 1 - \exp\left(-\frac{x}{2}\right) & x \geq 0 \end{cases} \quad (3.9)$$

The following python code plots the theoretical CDF

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/3.2/3.2.py
```

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

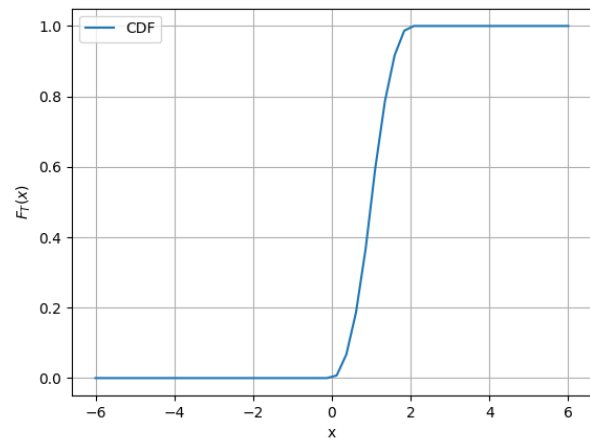
Solution:

Download and run the following C code to generate tri.dat file.

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/4.1/4.1.c
```

4.2 Find the CDF of T .

Solution:

Fig. 4.2: CDF of T

The following code plots the CDF of T

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/4.2/4.2.py
```

4.3 Find the CDF of T .

Solution:

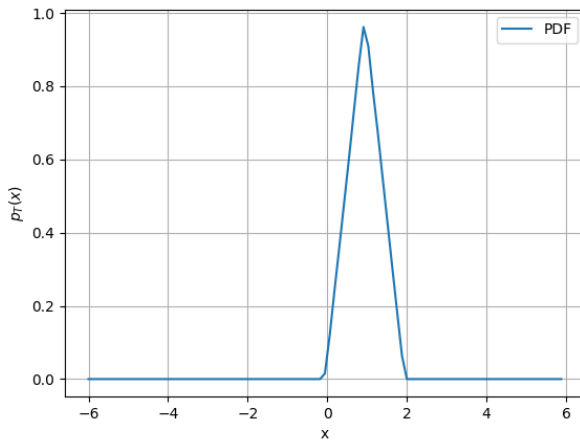


Fig. 4.3: PDF of T

The following code plots the PDF of T

```
wget https://github.com/TYCN129/AI1110-
Assignments/blob/main/Manual
%201/4.3/4.3.py
```

4.4 Find the theoretical expressions for PDF and CDF of T .

Solution:

$$p_T(x) = p_{U_1+U_2}(x) = p_{U_1}(x) * p_{U_2}(x) \quad (4.2)$$

$$p_T(x) = \int_{-\infty}^{\infty} p_{U_1}(\tau) p_{U_2}(x - \tau) \quad (4.3)$$

$$p_T(x) = \int_0^1 p_{U_2}(x - \tau) \quad (4.4)$$

$$p_T(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x 1 d\tau & 0 < x < 1 \\ \int_{x-1}^1 1 d\tau & 1 \leq x < 2 \\ 0 & x > 2 \end{cases} \quad (4.5)$$

$$p_T(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & x > 2 \end{cases} \quad (4.6)$$

Expression for CDF can be obtained by integrating $p_T(x)$ w.r.t. X

$$F_T(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{2} & 0 < x < 1 \\ -\frac{x^2}{2} + 2x - 1 & 1 \leq x < 2 \\ 1 & x > 2 \end{cases} \quad (4.7)$$

4.5 Verify the results through a plot.

Solution:

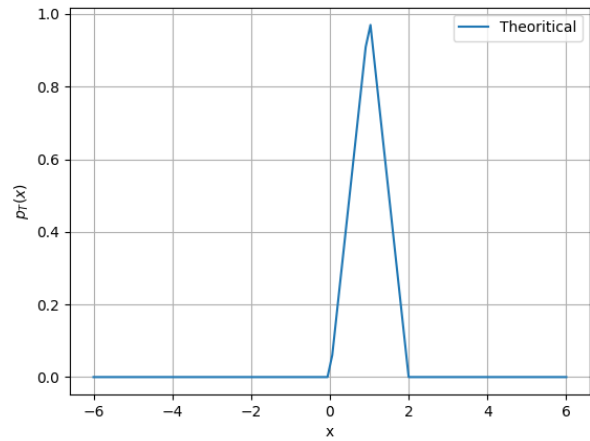


Fig. 4.5: Theoretical PDF of T

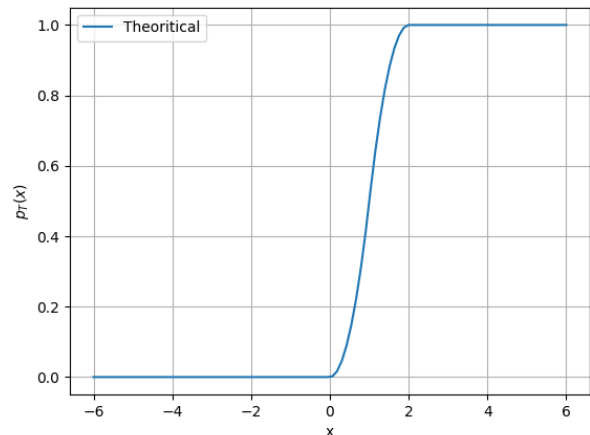


Fig. 4.5: The CDF of T

PDF and CDF plotted by the Python codes

```
wget https://github.com/TYCN129/AI1110-  
Assignments/blob/main/Manual  
%201/4.5/4.5_1.py
```

```
wget https://github.com/TYCN129/AI1110-  
Assignments/blob/main/Manual  
%201/4.5/4.5_1.py
```