

1 L01: Derivatives, Limits, Continuity

1.1 Intuitive Notion of Limits

$\lim_{x \rightarrow c} f(x) = L$ means $f(x)$ approaches L as x approaches c (from both sides).

Note: The value $f(c)$ itself does not matter — only the behavior near c .

1.2 One-Sided Limits

Left-hand limit: $\lim_{x \rightarrow c^-} f(x) = L$

Right-hand limit: $\lim_{x \rightarrow c^+} f(x) = L$

$\lim_{x \rightarrow c} f(x) = L$ exists $\Leftrightarrow \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L$

1.3 Limit Laws

Suppose $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$.

Then:

$$\lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm M$$

$$\lim_{x \rightarrow c} [f(x) \cdot g(x)] = L \cdot M$$

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M} \text{ (provided } M \neq 0\text{)}$$

$$\lim_{x \rightarrow c} [f(x)]^n = L^n$$

$$\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L} \text{ (if } L > 0 \text{ or } n \text{ odd)}$$

$$\lim_{x \rightarrow c} k \cdot f(x) = k \cdot L$$

1.4 Important Standard Limits

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$$

$$\lim_{x \rightarrow 0} (1+x)^{1/x} = e$$

$$\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = e$$

$$\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$$

1.5 Squeeze Theorem

If $g(x) \leq f(x) \leq h(x)$ near c (except possibly at c), and $\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$, then

$$\lim_{x \rightarrow c} f(x) = L.$$

Classic example: $\lim_{x \rightarrow 0} x \sin(\frac{1}{x}) = 0$ since $-|x| \leq x \sin(\frac{1}{x}) \leq |x|$.

1.6 Limits at Infinity

$$\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0 \text{ for } n > 0$$

For rational functions: divide numerator and denominator by highest power of x in denominator.

Dominant terms: $e^x \gg x^n \gg \ln x$ as $x \rightarrow \infty$

1.7 Continuity

Definition: f is continuous at $x = c$ if:

(i) $f(c)$ is defined, (ii) $\lim_{x \rightarrow c} f(x)$ exists, (iii) $\lim_{x \rightarrow c} f(x) = f(c)$

Types of discontinuity:

• Removable: Limit exists but $f(c)$ undefined or $f(c) \neq \lim f(x)$

• Jump: Left and right limits exist but differ

• Infinite: At least one side goes to $\pm\infty$

Intermediate Value Theorem (IVT): If f is continuous on $[a, b]$ and N is between $f(a)$ and $f(b)$, then $\exists c \in (a, b)$ with $f(c) = N$.

1.8 Definition of the Derivative

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x-a}$$

Geometric meaning: Slope of tangent line at $(a, f(a))$

Tangent line: $y - f(a) = f'(a)(x - a)$

Differentiable implies continuous (but NOT converse).

f is NOT differentiable at: corners, cusps, vertical tangents, discontinuities.

1.9 Basic Derivatives

$$\frac{d}{dx}(c) = 0 \quad \frac{d}{dx}(x^n) = nx^{n-1}$$
 (Power Rule)

$$\frac{d}{dx}(\sin x) = \cos x \quad \frac{d}{dx}(\cos x) = -\sin x$$

$$\frac{d}{dx}(\tan x) = \sec^2 x \quad \frac{d}{dx}(\cot x) = -\operatorname{csc}^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \quad \frac{d}{dx}(\csc x) = -\operatorname{csc} x \cot x$$

2 L02: Differentiation Rules

2.1 Sum, Product, Quotient Rules

Sum/Difference: $(f \pm g)' = f' \pm g'$

Constant Multiple: $(cf)' = cf'$

Product Rule: $(fg)' = f'g + fg'$

Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

2.2 Chain Rule

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)$$

Leibniz notation: $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Examples:

$$\frac{d}{dx} \sin(x^2) = \cos(x^2) \cdot 2x$$

$$\frac{d}{dx} e^{3x+1} = 3e^{3x+1}$$

2.3 Higher-Order Derivatives

$$f''(x) = \frac{d}{dx} f'(x), \quad f^{(n)}(x) = \frac{d}{dx} f^{(n-1)}(x)$$

Example: $f(x) = x^4 \rightarrow f'(x) = 4x^3 \rightarrow f''(x) =$

$$12x^2 \rightarrow f'''(x) = 24x$$

2.4 Implicit Differentiation

When y is defined implicitly by $F(x, y) = 0$:

Differentiate both sides w.r.t. x , treat y as $y(x)$, then solve for $\frac{dy}{dx}$.

Remember: $\frac{d}{dx} y^n = ny^{n-1} \frac{dy}{dx}$

$$\text{Example: } x^2 + y^2 = 25$$

$$2x + 2y \frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = -\frac{x}{y}$$

$$\text{Example: } x^3 + y^3 = 6xy$$

$$3x^2 + 3y^2 \frac{dy}{dx} = 6y + 6xy' \rightarrow y' = \frac{6y - 3x^2}{3x^2 - 6y} = \frac{2y - x^2}{x^2 - 2y}$$

2.5 Inverse Functions

If f is one-to-one and differentiable, then f^{-1} is differentiable and:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))} \quad \text{or equivalently} \quad \frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

2.6 Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}} \quad \frac{d}{dx} \cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2} \quad \frac{d}{dx} \cot^{-1}(x) = -\frac{1}{1+x^2}$$

$$\frac{d}{dx} \sec^{-1}(x) = \frac{1}{|x|\sqrt{x^2-1}} \quad \frac{d}{dx} \csc^{-1}(x) = -\frac{1}{|x|\sqrt{x^2-1}}$$

2.7 Related Rates Strategy

1. Draw a diagram and identify variables
2. Write an equation relating the variables
3. Differentiate both sides with respect to t
4. Substitute known values and solve

2.8 Logarithmic Differentiation

For products/quotients/powers, take \ln of both sides first:

$$y = f(x) \rightarrow \ln y = \ln f(x) \rightarrow \frac{y'}{y} = (\ln f(x))' \rightarrow y' = f'(x) \cdot (\ln f(x))'$$

$$\text{Example: } y = x^x$$

$$\ln y = \ln x \ln x \rightarrow \frac{y'}{y} = \ln x + 1 \rightarrow y' = x^x (\ln x + 1)$$

General power: $\frac{d}{dx}[f(x)]^{g(x)}$ — use $e^{g(x) \ln f(x)}$ form.

3 L03: Exponential, Logarithmic, Optimization

3.1 Exponential Function

$$\frac{d}{dx} e^x = e^x \quad \frac{d}{dx} a^x = a^x \ln a$$

$$a = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n \approx 2.71828$$

Properties: $a^{b+c} = a^b \cdot a^c$, $a^{ab} = (a^b)^a$

3.2 Natural Logarithm

$$\frac{d}{dx} \ln x = \frac{1}{x} \quad \frac{d}{dx} \ln|x| = \frac{1}{x}$$

$$\frac{d}{dx} \log_a x = \frac{1}{x \ln a}$$

Properties:

$$\ln(ab) = \ln a + \ln b \quad \ln\left(\frac{a}{b}\right) = \ln a - \ln b$$

$$\ln(a^r) = r \ln a \quad \ln e = 1 \quad \ln 1 = 0$$

$$e^{\ln x} = x \text{ (for } x > 0\text{)} \quad \ln(e^x) = x \text{ (for all } x\text{)}$$

3.3 Exponential Growth & Decay

Growth: $k > 0$ (doubling time $t_d = \frac{\ln 2}{k}$)

Decay: $k < 0$ (half-life $t_{1/2} = \frac{-\ln 2}{k}$)

Continuously compounded: $A = Pe^{rt}$

3.4 Hyperbolic Functions (from MIT 18.01)

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad \tanh x = \frac{\sinh x}{\cosh x}$$

$$\sech^2 x = 1 - \tanh^2 x$$

$$\text{Identity: } \cosh^2 x - \sinh^2 x = 1$$

3.5 Critical Points & Extrema

Critical point: $f'(c) = 0$ or $f'(c)$ undefined (and c in domain)

Absolute extrema on $[a, b]$: Compare f at critical points and endpoints.

Fermat's Theorem: If f has a local extremum at c and $f'(c)$ exists, then $f'(c) = 0$.

3.6 First Derivative Test

At a critical point c :

- f' changes from $+ \rightarrow - \rightarrow$ local maximum
- f' changes from $- \rightarrow + \rightarrow$ local minimum
- f' does not change sign \rightarrow neither

3.7 Second Derivative Test

If $f'(c) = 0$:

- $f''(c) > 0 \rightarrow$ local minimum (concave up)
- $f''(c) < 0 \rightarrow$ local maximum (concave down)
- $f''(c) = 0 \rightarrow$ inconclusive (use First Derivative Test)

3.8 Concavity & Inflection Points

Concave up: $f''(x) > 0$ (graph curves upward, tangent below)

Concave down: $f''(x) < 0$ (graph curves downward, tangent above)

Inflection point: Where concavity changes ($f''(x) = 0$ or undefined and sign change)

3.9 Optimization Strategy

1. Read problem, identify quantity to optimize
2. Express quantity as function of one variable
3. Find domain (usually, a closed interval)
4. End critical points: $f'(x) = 0$ or undefined
5. Evaluate f at critical points and endpoints
6. Compare values to find absolute max/min

Closed interval: Must check endpoints too!

Open interval: If only one critical point and Second Derivative Test confirms, that's the answer.

4 L04: Newton's Method, MVT, Antiderivatives

4.1 Newton's Method

Iterative root-finding: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

Geometric idea: Follow tangent line to x -axis, repeat.

Convergence: Quadratic near simple roots (digits roughly double each step).

Failure cases: $f'(x_n) = 0$, cycling, divergence, wrong root.

Example: Find $\sqrt{2}$ using $f(x) = x^2 - 2$

$$x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n}$$

4.2 Mean Value Theorem (MVT)

If f is continuous on $[a, b]$ and differentiable on (a, b) , then $\exists c \in (a, b)$:

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Geometric meaning: There's a point where the tangent is parallel to the secant line.

Consequences:

- $f'(x) \geq 0$ for all $x \in (a, b) \rightarrow f$ is constant on $[a, b]$
- $f'(x) \leq 0$ for all $x \in (a, b) \rightarrow f$ is increasing on $[a, b]$
- $f'(x) < 0$ for all $x \in (a, b) \rightarrow f$ is decreasing on $[a, b]$

$$\frac{[g,b]}{[a,b]} = g'(x) \text{ for all } x \rightarrow f(x) = g(x) + C$$

4.3 Rolle's Theorem (Special Case of MVT)

If f is continuous on $[a, b]$, differentiable on (a, b) , and $f(a) = f(b)$, then $\exists c \in (a, b)$: $f'(c) = 0$.

4.4 Differentials & Linear Approximation

Differential: $dy = f'(x) dx$

Linear approximation: $f(x) \approx f(a) + f'(a)(x - a)$

near $x = a$

Example: Approximate $\sqrt{4.1}$.

$$f(x) = \sqrt{x}, a = 4: \sqrt{4.1} \approx 2 + \frac{1}{8}(0.1) = 2.025$$

Error bound: $|f(x) - L(x)| \leq \frac{M}{2} |x - a|^2$ where $M = \max|f''|$ near a

5.4 Summation Formulas (for Riemann Sums)

$$\sum_{i=1}^n 1 = n \quad \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

$$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

5.5 u-Substitution (Change of Variable)

$\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$ where $u = g(x)$

Steps:

1. Choose $u = g(x)$, compute $du = g'(x) dx$

2. Change limits: $x = a \rightarrow u = g(a)$, $x = b \rightarrow u = g(b)$

3. Substitute everything to u and integrate

4. (If indefinite: substitute back to x at end)

5.6 Symmetry Properties

Even function ($f(-x) = f(x)$): $\int_a^a f(x) dx = 0$

Odd function ($f(-x) = -f(x)$): $\int_a^a f(x) dx = 0$

Examples: $x^2, \cos x$ are even; $x^3, \sin x$ are odd

6 L06: Applications, Volume, Averages

6.1 Area Between Curves

Vertical slices (x-axis): $A = \int_a^b |f(x) - g(x)| dx$

If $f(x) \geq g(x)$ on $[a, b]$: $A = \int_a^b [f(x) - g(x)] dx$

Horizontal slices (y-axis): $A = \int_c^d [f(y) - g(y)] dy$

Strategy: If curves cross, split into subintervals where one function dominates.

Finding bounds: Solve $f(x) = g(x)$ for intersection points.

6.2 Volume by Cross-Sections

$V = \int_a^b A(x) dx$ where $A(x)$ is the cross-sectional area at position x .

Common cross-sections:

• Square: $A = x^2$

• Semicircle: $A = \frac{\pi r^2}{2}$

• Equilateral triangle: $A = \frac{\sqrt{3}}{4}s^2$

6.3 Volume of Revolution: Disk Method

Revolving $y = f(x)$ around the x -axis ($f(x) \geq 0$): $V = \pi \int_a^b [f(x)]^2 dx$

Revolving $y = g(y)$ around the y -axis: $V = \pi \int_c^d [g(y)]^2 dy$

6.4 Volume of Revolution: Washer Method

Revolving region between $f(x)$ and $g(x)$ around the x -axis ($f(x) \geq g(x) \geq 0$

Mean Value Theorem for Integrals: If f is continuous on $[a, b]$, then $\exists c \in [a, b]$:
 $f(c) = \frac{1}{b-a} \int_a^b f(x) dx$

7 L07: Integration Techniques

7.1 Substitution (Review)

$\int f(g(x))g'(x) dx = \int f(u) du$ where $u = g(x)$

Key patterns:

- $\int f(ax+b) dx = \frac{1}{a} F(ax+b) + C$
- $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$

7.2 Integration by Parts (IBP)

$\int u dv = uv - \int v du$

Definite form: $\int_a^b u dv = [uv]_a^b - \int_a^b v du$

LIAE rule for choosing u (first match):
 Logarithmic \rightarrow Inverse trig \rightarrow Algebraic \rightarrow Trig \rightarrow Exponential

Common IBP integrals:

- $\int xe^x dx = xe^x - e^x + C$
- $\int x \sin x dx = -x \cos x + \sin x + C$
- $\int x \cos x dx = x \sin x + \cos x + C$
- $\int \ln x dx = x \ln x - x + C$
- $\int x^n \ln x dx = \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + C$
- $\int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1-x^2} + C$
- $\int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$

Tabular method: For $f x^n$ (trig/exp) dx , repeatedly differentiate u and integrate dv in columns with alternating signs.

Cyclic IBP: $\int e^x \sin x dx \rightarrow$ IBP twice, then solve for the integral algebraically.

- $\int e^x \sin x dx = \frac{e^x(\sin x - \cos x)}{2} + C$
- $\int e^x \cos x dx = \frac{e^x(\sin x + \cos x)}{2} + C$

7.3 Trigonometric Integrals

7.3.1 Powers of sin and cos

- m odd:** Save one $\sin x$, convert rest using $\sin^2 x = 1 - \cos^2 x$, $u = \cos x$
- n odd:** Save one $\cos x$, convert rest using $\cos^2 x = 1 - \sin^2 x$, $u = \sin x$
- Both even:** Use half-angle identities:
 $\sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$
 $\sin x \cos x = \frac{\sin 2x}{2}$

7.3.2 Powers of tan and sec

- n even:** Save $\sec^2 x$, convert rest using $\sec^2 x = 1 + \tan^2 x$, $u = \tan x$
- m odd:** Save $\sec x \tan x$, convert rest using $\tan^2 x = \sec^2 x - 1$, $u = \sec x$

Key integrals:

- $\int \tan x dx = \ln|\sec x| + C = -\ln|\cos x| + C$
- $\int \sec x dx = \ln|\sec x + \tan x| + C$
- $\int \sec^3 x dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$

7.4 Trigonometric Substitution

Expression	Substitution	Identity	Range
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$1 - \sin^2 = \frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}$	\cos^2
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$1 + \tan^2 = -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	\sec^2
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$\sec^2 - 1 = 0 \leq \theta < \frac{\pi}{2}$	\tan^2

7.5 Partial Fractions

For $\int \frac{P(x)}{Q(x)} dx$ where $\deg P < \deg Q$ (do polynomial division first if not):

Linear factor $(ax+b): \frac{A}{ax+b} \rightarrow$ integrates to $\frac{A}{a} \ln|ax+b|$

Repeated linear $(ax+b)^k: \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots +$

Irreducible quadratic $(ax^2 + bx + c): \frac{Ax+B}{ax^2+bx+c} \rightarrow$ complete the square, split

Repeated quadratic $(ax^2 + bx + c)^k: \frac{A_1x+B_1}{ax^2+bx+c} + \dots + \frac{A_kx+B_k}{(ax^2+bx+c)^k}$

Cover-up method: For distinct linear factors, plug in root of each factor to find each constant directly.

7.6 Useful Reference Integrals

- $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$
- $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C$
- $\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln|x + \sqrt{x^2+a^2}| + C$
- $\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln|x + \sqrt{x^2-a^2}| + C$

8 L08: Parametric Equations, Arc Length, Polar

8.1 Parametric Equations

$x = f(t), y = g(t), t \in [a, b]$

Eliminate parameter: Solve for t from one equation, substitute into the other.

Common parametrizations:

- Circle: $x = r \cos t, y = r \sin t, t \in [0, 2\pi]$
- Ellipse: $x = a \cos t, y = b \sin t, t \in [0, 2\pi]$
- Line: $x = x_0 + at, y = y_0 + bt$

8.2 Calculus with Parametric Curves

First derivative: $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{f'(t)}$ (provided $f'(t) \neq 0$)

Second derivative: $\frac{d^2y}{dx^2} = \frac{d}{dt} \left(\frac{dy}{dx} \right) = \frac{d^2y/dt^2}{(dx/dt)^2} = \frac{d^2y/dt^2}{(f'(t))^2}$

Area under parametric curve: $A = \int_a^b g(t)f'(t) dt$

Tangent lines:

- Horizontal tangent: $y'(t) = 0$ (and $f'(t) \neq 0$)
- Vertical tangent: $f'(t) = 0$ (and $g'(t) \neq 0$)

8.3 Arc Length

Cartesian ($y = f(x)$): $L = \int_a^b \sqrt{1+(f'(x))^2} dx$

Parametric: $L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2} dt$

Polar: $L = \int_{\alpha}^{\beta} \sqrt{r^2 + (dr/d\theta)^2} d\theta$

8.4 Surface Area of Revolution

Around x-axis (Cartesian): $S = 2\pi \int_a^b |f(x)| \sqrt{1+(f'(x))^2} dx$

Around x-axis (Parametric): $S = 2\pi \int_a^b |g(t)| \sqrt{(f'(t))^2 + (g'(t))^2} dt$

Around y-axis (Cartesian): $S = 2\pi \int_a^b |x| \sqrt{1+(f'(x))^2} dx$

Key: Multiply arc length element ds by circumference $2\pi r$, where r is the distance from the curve to the axis.

8.5 Polar Coordinates

Conversion:

- $x = r \cos \theta$
- $y = r \sin \theta$
- $r^2 = x^2 + y^2$
- $\tan \theta = \frac{y}{x}$

Common polar curves:

- Circle: $r = a$ (centered at origin), $r = 2a \cos \theta$ (centered at $(a, 0)$), $r = 2a \sin \theta$ (centered at $(0, a)$)
- Cardioid: $r = a(1 + \cos \theta)$ or $r = a(1 + \sin \theta)$
- Rose: $r = a \cos(n\theta)$ or $r = a \sin(n\theta)$ (n petals if n odd, $2n$ if n even)
- Lemniscate: $r^2 = a^2 \cos 2\theta$, $r = a \sin 2\theta$
- Spiral: $r = a\theta$

8.6 Slope of Polar Curves

$\frac{dy}{dx} = \frac{r \sin \theta + r \cos \theta}{r \cos \theta - r \sin \theta}$ where $r' = \frac{dr}{d\theta}$

Derivation: $x = r \cos \theta$, $y = r \sin \theta$ \rightarrow use parametric derivative with parameter θ .

8.7 Area in Polar Coordinates

Area of sector: $A = \frac{1}{2} \int_{\alpha}^{\beta} r^2 d\theta$

Area between two curves: $A = \frac{1}{2} \int_{\alpha}^{\beta} (r_{\text{outer}}^2 - r_{\text{inner}}^2) d\theta$

Warning: Find correct limits by checking where $r = 0$ or where curves intersect ($r_1(\theta) = r_2(\theta)$). Check

whether curves pass through the origin at different θ values.

8.8 Key Exam Reminders

- Parametric:** Always check direction of traversal (orientation)
- Arc length:** Never forget the square root and the sum under it
- Polar area:** Use $\frac{1}{2}r^2$, NOT πr^2 . Factor of $\frac{1}{2}$ is crucial
- Polar intersections:** Curves may intersect at origin even if $r = 0$ at different θ values — graph the curves to be safe

9 L09: L'Hôpital's Rule & Improper Integrals

9.1 Indeterminate Forms

Basic types: $\frac{0}{0}, \frac{\infty}{\infty} \rightarrow$ L'Hôpital applies directly
Other types (must be converted):
 $0 \cdot \infty$: Rewrite as $\frac{0}{0}$ or ∞ (e.g., $f \cdot g = \frac{f}{1/g}$)
 $\infty - \infty$: Combine into single fraction
 $0^0, 1^\infty, \infty^0$: Take ln first, then exponentiate result

9.2 L'Hôpital's Rule

If $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$ gives $\frac{0}{0}$ or $\frac{\infty}{\infty}$, and $g'(x) \neq 0$ near c , then: $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$ (if right side exists or is $\pm\infty$)

Also works for $x \rightarrow \pm\infty$, $x \rightarrow c^\pm$.

Can apply repeatedly if result is still indeterminate.

Common pitfall: Verify the form IS indeterminate before applying! If not $\frac{0}{0}$ or $\frac{\infty}{\infty}$, L'Hôpital does NOT apply.

Examples:

- $\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$
- $\lim_{x \rightarrow \infty} \frac{e^x}{x} = \lim_{x \rightarrow \infty} \frac{e^x}{1} = \infty$
- $\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (-x) = 0$

9.3 Exponential Indeterminate Forms

For $\lim_{x \rightarrow c} f(x)^{g(x)}$ with form $0^0, 1^\infty$, or ∞^0 :

- Let $L = \lim_{x \rightarrow c} g(x) \ln f(x)$ (this is often $0 \cdot \infty$ or $\infty \cdot 0$)
- Answer L'Hôpital to find L
- Classic: $\lim_{x \rightarrow 0^+} x^x = \lim_{x \rightarrow 0^+} e^{\ln x \ln x} = e^0 = 1$
- Classic: $\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = \lim_{x \rightarrow \infty} e^{\ln(1+1/x)} = e^1 = e$

9.4 Improper Integrals: Type 1 (Infinite Limits)

$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$

$\int_{-\infty}^c f(x) dx = \lim_{a \rightarrow -\infty} \int_a^c f(x) dx$

$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^0 f(x) dx + \int_0^{\infty} f(x) dx$ (both must converge)

9.5 Improper Integrals: Type 2 (Discontinuities)

If f has a discontinuity at $b: \int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$

If f has a discontinuity at $a: \int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$

If f has a discontinuity at $c \in (a, b)$: Split into two improper integrals at c (both must converge)

9.6 p-Integral Test (Key Reference)

$\int_1^{\infty} \frac{1}{x^p} dx: \text{Converges if } p > 1, \text{ Diverges if } p \leq 1$

$\int_0^{\infty} \frac{1}{x^p} dx: \text{Converges if } p < 1, \text{ Diverges if } p \geq 1$

9.7 Comparison Tests for Improper Integrals

9.7.1 Direct Comparison Test (DCT)

If $0 \leq f(x) \leq g(x)$ for $x \geq a$ and $\int_a^{\infty} g(x) dx$ converges, then $\int_a^{\infty} f(x) dx$ converges.

If $0 \leq g(x) \leq f(x)$ for $x \geq a$ and $\int_a^{\infty} f(x) dx$ diverges, then $\int_a^{\infty} g(x) dx$ diverges.

9.7.2 Limit Comparison Test (LCT)

If $f(x), g(x) > 0$ and $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$ with $0 < L < \infty$: Both integrals converge or both diverge.

Strategy: Compare with $\frac{1}{x^p}$ integrals.

Example: $\int_1^{\infty} \frac{1}{x^2+1} dx$ — compare with $\frac{1}{x^2}$ (converges, $p = 2 > 1$)

9.8 Quick Reference: Common Results

- $\int_0^{\infty} e^{-x} dx = 1$
- $\int_0^{\infty} e^{-ax} dx = \frac{1}{a} (a > 0)$
- $\int_1^{\infty} \frac{1}{x} dx = \infty$ (diverges)
- $\int_1^{\infty} \frac{1}{x^2} dx = 1$
- $\int_0^{\infty} x e^{-x} dx = 1$ (by IBP)
- $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ (Gaussian integral)

10 L10: Infinite Series & Taylor Series

10.1 Sequences

A sequence $\{a_n\}$ converges if $\lim_{n \rightarrow \infty} a_n = L$ exists and is finite.

Monotone Convergence Theorem: A bounded monotone sequence converges.

Common limits:

- $\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$ ($p > 0$)
- $\lim_{n \rightarrow \infty} n^p = \infty$
- $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$
- $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$ (for any fixed x)
- $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$

10.2 Infinite Series

$\sum_{n=1}^{\infty} a_n = \lim_{N \rightarrow \infty} S_N$ where $S_N = \sum_{n=1}^N a_n$ (partial sums)

Converges if $\lim_{N \rightarrow \infty} S_N$ exists and is finite; otherwise **diverges**.

10.3 Divergence Test (nth-Term Test)

If $\lim_{n \rightarrow \infty} a_n \neq 0$ (or does not exist), then $\sum a_n$ **diverges**.

WARNING: If $\lim_{n \rightarrow \infty} a_n = 0$, the series may still diverge! (e.g., harmonic series)

10.4 Geometric Series

$\sum_{n=0}^{\infty} ar^n = \frac{1}{1-r}$ if $|r| < 1$; diverges if $|r| \geq 1$

Partial sum: $S_N = \frac{1-r^{N+1}}{1-r}$

Common trick: Rewrite series to identify a and r .

10.5 Telescoping Series

$\sum_{n=1}^{\infty} (b_n - b_{n+1}) = b_1 - \lim_{n \rightarrow \infty} b_n$ (if limit exists)

Strategy: Use partial fractions to express terms as differences.

10.6 p-Series

$\sum_{n=1}^{\infty} \frac{1}{n^p}$: **Converges** if $p > 1$, **Diverges** if $p \leq 1$

Harmonic series: $\sum \frac{1}{n}$ diverges ($p = 1$)

10.7 Convergence Tests

10.7.1 Integral Test

If f is positive, continuous, decreasing for $x \geq 1$ and $a_n = f(n)$:
 $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x) dx$ both converge or both diverge.

10.7.2 Direct Comparison Test

If $0 \leq a_n \leq b_n$:

- $\sum a_n$ converges $\rightarrow \sum b_n$ converges
- $\sum b_n$ diverges $\rightarrow \sum a_n$ diverges

10.7.3 Limit Comparison Test

If $a_n, b_n > 0$ and $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$ with $0 < L < \infty$: both converge or both diverge.

10.7.4 Ratio Test

$L = \lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|}$:

- $\frac{L}{2} < 1$: **Converges absolutely**
- $\frac{L}{2} > 1$ or $L = \infty$: **Diverges**
- $L = 1$: **Inconclusive**

Best for: factorials $n!$, exponentials a^n , products involving n .

10.8 Alternating Series

$\sum_{n=1}^{\infty} (-1)^{n+1} b_n$ converges if:
 $(i) b_n > 0$, $(ii) b_{n+1} \leq b_n$ (decreasing), $(iii) \lim_{n \rightarrow \infty} b_n = 0$

Error bound: $|S - S_N| \leq b_{N+1}$ (remainder \leq first omitted term)

10.9 Absolute vs Conditional Convergence

Absolutely convergent: $\sum |a_n|$ converges $\rightarrow \sum a_n$ converges

Conditionally convergent: $\sum a_n$ converges but $\sum |a_n|$ diverges

Example: $\sum \frac{(-1)^{n+1}}{n}$ converges conditionally;
 $\sum \frac{(-1)^{n+1}}{n^2}$ converges absolutely

10.10 Power Series

$\sum_{n=0}^{\infty} c_n (x-a)^n$ with center a and radius of convergence R

Finding R : $R = \lim_{n \rightarrow \infty} \frac{c_n}{c_{n+1}}$ or $R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|}}$

Converges absolutely for $|x-a| < R$, diverges for $|x-a| > R$.

Check endpoints separately ($x = a \pm R$).

Operations: Can differentiate and integrate term-by-term within interval of convergence.

10.11 Taylor & Maclaurin Series

$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$ (Taylor series centered at a)

Maclaurin series: $a = 0$

10.11.1 Essential Maclaurin Series

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots$

$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

10.12 Taylor's Theorem (Remainder)

$f(x) = T_n(x) + R_n(x)$ where T_n is the n th-degree Taylor polynomial.

Lagrange remainder: $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$ for some c between a and x .

Bound: $|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$ where $M = \max|f^{(n+1)}|$ on the interval.

10.13 Generating New Series from Known Ones

Substitution: $e^{-x^2} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$ (replace x with $-x^2$ in e^x)

Differentiation: $\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} nx^{n-1}$ (differentiate $\frac{1}{1-x}$)

Integration: $\ln(1+x) = \int_0^x \frac{1}{1+t} dt = \sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n+1}$

Multiplication: Multiply series term by term (Cauchy product)