

1 L01: Derivatives, Limits, Continuity
<p><b>1.1 Intuitive Notion of Limits</b></p> $\lim_{x \rightarrow c} f(x) = L$ means $f(x)$ approaches $L$ as $x$ approaches $c$ (from both sides). <p><b>Note:</b> The value <math>f(c)</math> itself does not matter — only the behavior <i>near</i> <math>c</math>.</p> <p><b>1.2 One-Sided Limits</b></p> <p><b>Left-hand limit:</b> <math>\lim_{x \rightarrow c^-} f(x) = L</math></p> <p><b>Right-hand limit:</b> <math>\lim_{x \rightarrow c^+} f(x) = L</math></p> <p><math>\lim_{x \rightarrow c} f(x) = L</math> exists <math>\leftrightarrow \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = L</math></p> <p><b>1.3 Limit Laws</b></p> <p>Suppose <math>\lim_{x \rightarrow c} f(x) = L</math> and <math>\lim_{x \rightarrow c} g(x) = M</math>. Then:</p> <p><math>\lim_{x \rightarrow c} [f(x) \pm g(x)] = L \pm M</math></p> <p><math>\lim_{x \rightarrow c} [f(x) \cdot g(x)] = L \cdot M</math></p> <p><math>\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}</math> (provided <math>M \neq 0</math>)</p> <p><math>\lim_{x \rightarrow c} [f(x)]^n = L^n</math></p> <p><math>\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L}</math> (if <math>L &gt; 0</math> or <math>n</math> odd)</p> <p><math>\lim_{x \rightarrow c} k \cdot f(x) = k \cdot L</math></p>

1.4 Important Standard Limits
$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$ $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$ $\lim_{x \rightarrow 0} (1 + x)^{1/x} = e$ $\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = e$ $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x} = 1$
1.5 Squeeze Theorem
<p>If <math>g(x) \leq f(x) \leq h(x)</math> near <math>c</math> (except possibly at <math>c</math>), and <math>\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L</math>, then <math>\lim_{x \rightarrow c} f(x) = L</math>.</p> <p><b>Classic example:</b> <math>\lim_{x \rightarrow 0} x \sin(\frac{1}{x}) = 0</math> since <math>- x  \leq x \sin(\frac{1}{x}) \leq  x </math>.</p>

1.6 Limits at Infinity
$\lim_{x \rightarrow \infty} \frac{1}{x^n} = 0$ for $n > 0$ For rational functions: divide numerator and denominator by highest power of $x$ in denominator. <b>Dominant terms:</b> $e^x \gg x^n \gg \ln x$ as $x \rightarrow \infty$
1.7 Continuity
<p><b>Definition:</b> <math>f</math> is continuous at <math>x = c</math> if:</p> <p>(i) <math>f(c)</math> is defined,   (ii) <math>\lim_{x \rightarrow c} f(x)</math> exists,   (iii) <math>\lim_{x \rightarrow c} f(x) = f(c)</math></p> <p><b>Types of discontinuity:</b></p> <ul style="list-style-type: none"> <li><b>Removable:</b> Limit exists but <math>f(c)</math> undefined or <math>f(c) \neq \text{limit}</math></li> <li><b>Jump:</b> Left and right limits exist but differ</li> <li><b>Infinite:</b> At least one side goes to <math>\pm\infty</math></li> </ul> <p><b>Intermediate Value Theorem (IVT):</b> If <math>f</math> is continuous on <math>[a, b]</math> and <math>N</math> is between <math>f(a)</math> and <math>f(b)</math>, then <math>\exists c \in (a, b)</math> with <math>f(c) = N</math>.</p>

1.8 Definition of the Derivative
$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$ <b>Geometric meaning:</b> Slope of tangent line at $(a, f(a))$ <b>Tangent line:</b> $y - f(a) = f'(a)(x - a)$ <b>Differentiable implies continuous</b> (but NOT converse). $f$ is NOT differentiable at: corners, cusps, vertical tangents, discontinuities.
1.9 Basic Derivatives
$\frac{d}{dx}(c) = 0$ $\frac{d}{dx}(x^n) = nx^{n-1}$ (Power Rule) $\frac{d}{dx}(\sin x) = \cos x$ $\frac{d}{dx}(\cos x) = -\sin x$ $\frac{d}{dx}(\tan x) = \sec^2 x$ $\frac{d}{dx}(\cot x) = -\csc^2 x$ $\frac{d}{dx}(\sec x) = \sec x \tan x$ $\frac{d}{dx}(\csc x) = -\csc x \cot x$

2 L02: Differentiation Rules
<p><b>2.1 Sum, Product, Quotient Rules</b></p> <p><b>Sum/Difference:</b> <math>(f \pm g)' = f' \pm g'</math></p> <p><b>Constant Multiple:</b> <math>(cf)' = c f'</math></p> <p><b>Product Rule:</b> <math>(fg)' = f'g + fg'</math></p> <p><b>Quotient Rule:</b> <math>(\frac{f}{g})' = \frac{f'g - fg'}{g^2}</math></p> <p><b>2.2 Chain Rule</b></p> <p><math>\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)</math></p> <p><b>Leibniz notation:</b> <math>\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}</math></p> <p><b>Examples:</b></p> <p><math>\frac{d}{dx} \sin(x^2) = \cos(x^2) \cdot 2x</math></p> <p><math>\frac{d}{dx} e^{3x+1} = 3e^{3x+1}</math></p> <p><math>\frac{d}{dx} (2x + 1)^5 = 5(2x + 1)^4 \cdot 2 = 10(2x + 1)^4</math></p> <p><math>\frac{d}{dx} \ln(\sin x) = \frac{\cos x}{\sin x} = \cot x</math></p> <p><b>2.3 Higher-Order Derivatives</b></p> <p><math>f''(x) = \frac{d}{dx} f'(x)</math>,   <math>f^{(n)}(x) = \frac{d}{dx} f^{(n-1)}(x)</math></p> <p><b>Example:</b> <math>f(x) = x^4 \rightarrow f'(x) = 4x^3 \rightarrow f''(x) = 12x^2 \rightarrow f'''(x) = 24x</math></p> <p><b>2.4 Implicit Differentiation</b></p> <p>When <math>y</math> is defined implicitly by <math>F(x, y) = 0</math>: Differentiate both sides w.r.t. <math>x</math>, treat <math>y</math> as <math>y(x)</math>, then solve for <math>\frac{dy}{dx}</math>.</p> <p>Remember: <math>\frac{d}{dx} y^n = ny^{n-1} \frac{dy}{dx}</math></p> <p><b>Example:</b> <math>x^2 + y^2 = 25</math></p> <p><math>2x + 2y \frac{dy}{dx} = 0 \rightarrow \frac{dy}{dx} = -\frac{x}{y}</math></p> <p><b>Example:</b> <math>x^3 + y^3 = 6xy</math></p> <p><math>3x^2 + 3y^2 y' = 6y + 6xy' \rightarrow y' = \frac{6y - 3x^2}{3y^2 - 6x} = \frac{2y - x^2}{y^2 - 2x}</math></p> <p><b>2.5 Inverse Functions</b></p> <p>If <math>f</math> is one-to-one and differentiable, then <math>f^{-1}</math> is differentiable and:</p> <p><math>(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}</math>   or equivalently   <math>\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}</math></p> <p><b>2.6 Derivatives of Inverse Trigonometric Functions</b></p> <p><math>\frac{d}{dx} \sin^{-1}(x) = \frac{1}{\sqrt{1-x^2}}</math>   <math>\frac{d}{dx} \cos^{-1}(x) = -\frac{1}{\sqrt{1-x^2}}</math></p> <p><math>\frac{d}{dx} \tan^{-1}(x) = \frac{1}{1+x^2}</math>   <math>\frac{d}{dx} \cot^{-1}(x) = -\frac{1}{1+x^2}</math></p> <p><math>\frac{d}{dx} \sec^{-1}(x) = \frac{1}{ x \sqrt{x^2-1}}</math>   <math>\frac{d}{dx} \csc^{-1}(x) = -\frac{1}{ x \sqrt{x^2-1}}</math></p> <p><b>2.7 Related Rates Strategy</b></p> <ol style="list-style-type: none"> <li>Draw a diagram and identify variables</li> <li>Write an equation relating the variables</li> <li>Differentiate both sides with respect to <math>t</math></li> <li>Substitute known values and solve</li> </ol>

2.8 Logarithmic Differentiation
<p>For products/quotients/powers, take <math>\ln</math> of both sides first:</p> <p><math>y = f(x) \rightarrow \ln y = \ln f(x) \rightarrow \frac{y'}{y} = (\ln f(x))' \rightarrow y' = f(x) \cdot (\ln f(x))'</math></p> <p><b>Example:</b> <math>y = x^x</math></p> <p><math>\ln y = x \ln x \rightarrow \frac{y'}{y} = \ln x + 1 \rightarrow y' = x^x (\ln x + 1)</math></p> <p><b>General power:</b> <math>\frac{d}{dx} [f(x)]^{g(x)} = \frac{f(x)}{f(x)} [f(x)]^{g(x)} \text{ — use } e^{g(x) \ln f(x)} \text{ form.}</math></p>
3 L03: Exponential, Logarithm, Optimization
<p><b>3.1 Exponential Function</b></p> <p><math>\frac{d}{dx} e^x = e^x</math>   <math>\frac{d}{dx} a^x = a^x \ln a</math></p> <p><math>e = \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n \approx 2.71828</math></p> <p><b>Properties:</b> <math>e^{a+b} = e^a e^b</math>,   <math>e^{-a-b} = \frac{e^a}{e^b}</math>,   <math>(e^a)^b = e^{ab}</math></p> <p><b>3.2 Natural Logarithm</b></p> <p><math>\frac{d}{dx} \ln x = \frac{1}{x}</math>   <math>\frac{d}{dx} \ln x  = \frac{1}{x}</math>   <math>\frac{d}{dx} \log_a x = \frac{1}{x \ln a}</math></p> <p><b>Properties:</b></p> <p><math>\ln(ab) = \ln a + \ln b</math>   <math>\ln(\frac{a}{b}) = \ln a - \ln b</math></p> <p><math>\ln(a^r) = r \ln a</math>   <math>\ln e = 1</math>   <math>\ln 1 = 0</math></p> <p><math>e^{\ln x} = x</math> (for <math>x &gt; 0</math>)   <math>\ln(e^x) = x</math> (for all <math>x</math>)</p>

3.3 Exponential Growth & Decay
<p><math>\frac{dy}{dt} = ky \rightarrow y(t) = y_0 e^{kt}</math></p> <p><b>Growth:</b> <math>k &gt; 0</math> (doubling time <math>t_d = \frac{\ln 2}{k}</math>)</p> <p><b>Decay:</b> <math>k &lt; 0</math> (half-life <math>t_{1/2} = \frac{\ln 2}{ k }</math>)</p> <p><b>Continuously compounded:</b> <math>A = Pe^{rt}</math></p> <p><b>3.4 Hyperbolic Functions (from MIT 18.01)</b></p> <p><math>\sinh x = \frac{e^x - e^{-x}}{2}</math>   <math>\cosh x = \frac{e^x + e^{-x}}{2}</math>   <math>\tanh x = \frac{\sinh x}{\cosh x}</math></p> <p><math>\frac{d}{dx} \sinh x = \cosh x</math>   <math>\frac{d}{dx} \cosh x = \sinh x</math>   <math>\frac{d}{dx} \tanh x = \text{sech}^2 x</math></p> <p><b>Identity:</b> <math>\cosh^2 x - \sinh^2 x = 1</math></p> <p><b>3.5 Critical Points &amp; Extrema</b></p> <p><b>Critical point:</b> <math>f'(c) = 0</math> or <math>f'(c)</math> undefined (and <math>c</math> in domain)</p> <p><b>Absolute extrema</b> on <math>[a, b]</math>: Compare <math>f</math> at critical points and endpoints.</p> <p><b>Fermat's Theorem:</b> If <math>f</math> has a local extremum at <math>c</math> and <math>f'(c)</math> exists, then <math>f'(c) = 0</math>.</p> <p><b>3.6 First Derivative Test</b></p> <p>At a critical point <math>c</math>:</p> <ul style="list-style-type: none"> <li><math>f'</math> changes from <math>+</math> to <math>-</math> → <b>local maximum</b></li> <li><math>f'</math> changes from <math>-</math> to <math>+</math> → <b>local minimum</b></li> <li><math>f'</math> does not change sign → <b>neither</b></li> </ul> <p><b>3.7 Second Derivative Test</b></p> <p>If <math>f'(c) = 0</math>:</p> <ul style="list-style-type: none"> <li><math>f''(c) &gt; 0</math> → <b>local minimum</b> (concave up)</li> <li><math>f''(c) &lt; 0</math> → <b>local maximum</b> (concave down)</li> <li><math>f''(c) = 0</math> → <b>inconclusive</b> (use First Derivative Test)</li> </ul>

3.8 Concavity & Inflection Points
<p><b>Concave up:</b> <math>f''(x) &gt; 0</math> (graph curves upward, tangent below)</p> <p><b>Concave down:</b> <math>f''(x) &lt; 0</math> (graph curves downward, tangent above)</p> <p><b>Inflection point:</b> Where concavity changes (<math>f''(x) = 0</math> or undefined <i>and</i> sign change)</p> <p><b>3.9 Optimization Strategy</b></p> <ol style="list-style-type: none"> <li>Read problem, identify quantity to optimize</li> <li>Express quantity as function of one variable</li> <li>Find domain (usually a closed interval)</li> <li>Find critical points: <math>f'(x) = 0</math> or undefined</li> <li>Evaluate <math>f</math> at critical points and endpoints</li> <li>Compare values to find absolute max/min</li> </ol> <p><b>Closed interval:</b> Must check endpoints too!</p> <p><b>Open interval:</b> If only one critical point and Second Derivative Test confirms, that's the answer.</p>

4 L04: Newton's Method, MVT, Antiderivatives
<p><b>4.1 Newton's Method</b></p> <p>Iterative root-finding: <math>x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}</math></p> <p><b>Geometric idea:</b> Follow tangent line to <math>x</math>-axis, repeat.</p> <p><b>Convergence:</b> Quadratic near simple roots (digits roughly double each step).</p> <p><b>Failure cases:</b> <math>f'(x_n) = 0</math>, cycling, divergence, wrong root.</p> <p><b>Example:</b> Find <math>\sqrt{2}</math> using <math>f(x) = x^2 - 2</math></p> <p><math>x_{n+1} = x_n - \frac{x_n^2 - 2}{2x_n} = \frac{x_n + \frac{2}{x_n}}{2}</math></p> <p><b>4.2 Mean Value Theorem (MVT)</b></p> <p>If <math>f</math> is <b>continuous</b> on <math>[a, b]</math> and <b>differentiable</b> on <math>(a, b)</math>, then <math>\exists c \in (a, b)</math>:</p> <p><math>f'(c) = \frac{f(b) - f(a)}{b - a}</math></p> <p><b>Geometric meaning:</b> There's a point where the tangent is parallel to the secant line.</p> <p><b>Consequences:</b></p> <ul style="list-style-type: none"> <li><math>f'(x) \geq 0</math> for all <math>x \in (a, b) \rightarrow f</math> is constant on <math>[a, b]</math></li> <li><math>f'(x) &gt; 0</math> for all <math>x \in (a, b) \rightarrow f</math> is increasing on <math>[a, b]</math></li> <li><math>f'(x) &lt; 0</math> for all <math>x \in (a, b) \rightarrow f</math> is decreasing on <math>[a, b]</math></li> </ul> <p><math>\frac{d}{dx} [g \circ f](x) = g'(f(x)) \cdot f'(x)</math> for all <math>x \rightarrow f(x) = g(x) + C</math></p>

4.3 Rolle's Theorem (Special Case of MVT)
<p>If <math>f</math> is continuous on <math>[a, b]</math>, differentiable on <math>(a, b)</math>, and <math>f(a) = f(b)</math>, then <math>\exists c \in (a, b)</math>: <math>f'(c) = 0</math>.</p> <p><b>4.4 Differentials &amp; Linear Approximation</b></p> <p><b>Differential:</b> <math>dy = f'(x) dx</math></p> <p><b>Linear approximation:</b> <math>f(x) \approx f(a) + f'(a)(x - a)</math></p> <p>near <math>x = a</math></p> <p><b>Example:</b> Approximate <math>\sqrt{4.1}</math>.</p> <p><math>f(x) = \sqrt{x}</math>, <math>a = 4</math>: <math>\sqrt{4.1} \approx 2 + \frac{1}{2}(0.1) = 2.025</math></p> <p><b>Error bound:</b> <math> f(x) - L(x)  \leq \frac{M}{2}  x - a ^2</math> where <math>M = \max f'' </math> near <math>a</math></p> <p><b>4.5 Antiderivatives (Indefinite Integrals)</b></p> <p><math>F(x)</math> is an antiderivative of <math>f(x)</math> if <math>F'(x) = f(x)</math>.</p> <p>General antiderivative: <math>F(x) + C</math> (arbitrary constant).</p> <p><b>4.5.1 Basic Antiderivative Formulas</b></p> <p><math>\int x^n dx = \frac{x^{n+1}}{n+1} + C</math> (<math>n \neq -1</math>)</p> <p><math>\int \frac{1}{x} dx = \ln x  + C</math></p> <p><math>\int e^x dx = e^x + C</math>   <math>\int a^x dx = \frac{a^x}{\ln a} + C</math></p> <p><math>\int \sin x dx = -\cos x + C</math>   <math>\int \cos x dx = \sin x + C</math></p> <p><math>\int \sec^2 x dx = \tan x + C</math>   <math>\int \csc^2 x dx = -\cot x + C</math></p> <p><math>\int \sec x \tan x dx = \sec x + C</math>   <math>\int \csc x \cot x dx = -\csc x + C</math></p> <p><math>\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + C</math>   <math>\int \frac{1}{1+x^2} dx = \tan^{-1} x + C</math></p> <p><b>4.5.2 Antiderivative Rules</b></p> <p><math>\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx</math></p> <p><math>\int k f(x) dx = k \int f(x) dx</math></p> <p><b>4.6 Initial Value Problems</b></p> <p>Given <math>f'(x) = g(x)</math> and <math>f(a) = b</math>:</p> <ol style="list-style-type: none"> <li>Find general antiderivative <math>F(x) + C</math></li> <li>Use initial condition to solve for <math>C</math>: <math>F(a) + C = b</math></li> </ol> <p><b>Example:</b> <math>f'(x) = 2x</math>, <math>f(1) = 3 \rightarrow f(x) = x^2 + C</math>, <math>1 + C = 3</math>, so <math>f(x) = x^2 + 2</math></p>

5 L05: Definite Integrals & FTC
<p><b>5.1 Riemann Sums</b></p> <p>Partition <math>[a, b]</math> into <math>n</math> subintervals of width <math>\Delta x = \frac{b-a}{n}</math>.</p> <p><b>Left sum:</b> <math>L_n = \sum_{i=1}^n f(x_i) \Delta x</math></p> <p><b>Right sum:</b> <math>R_n = \sum_{i=1}^n f(x_i) \Delta x</math></p> <p><b>Midpoint sum:</b> <math>M_n = \sum_{i=1}^n f(x_{i-1}^* \Delta x</math></p> <p><math>f(x) dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*) \Delta x</math></p> <p><b>5.2 Properties of Definite Integrals</b></p> <p><math>\int_a^a f(x) dx = 0</math>   <math>\int_a^b f(x) dx = -\int_b^a f(x) dx</math></p> <p><math>\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx</math></p> <p><math>\int_a^b k f(x) dx = k \int_a^b f(x) dx</math></p> <p><math>\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx</math> (additivity)</p> <p>If <math>f(x) \geq 0</math> on <math>[a, b]</math>, then <math>\int_a^b f(x) dx \geq 0</math></p> <p>If <math>f(x) \geq g(x)</math> on <math>[a, b]</math>, then <math>\int_a^b f(x) dx \geq \int_a^b g(x) dx</math></p> <p><b>5.3 Fundamental Theorem of Calculus (FTC)</b></p> <p><b>5.3.1 FTC Part 1 (Derivative of Integral)</b></p> <p>If <math>f</math> is continuous on <math>[a, b]</math>, then <math>F(x) = \int_a^x f(t) dt</math> is differentiable and:</p> <p><math>F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)</math></p> <p><b>With chain rule:</b> <math>\frac{d}{dx} \int_{g(x)}^x f(t) dt = f(g(x)) \cdot g'(x)</math></p> <p><b>Variable in both limits:</b></p> <p><math>\frac{d}{dx} \int_{h(x)}^{g(x)} f(t) dt = f(g(x)) \cdot g'(x) - f(h(x)) \cdot h'(x)</math></p> <p><b>5.3.2 FTC Part 2 (Evaluation Theorem)</b></p> <p>If <math>f</math> is continuous on <math>[a, b]</math> and <math>F</math> is any antiderivative of <math>f</math>:</p> <p><math>\int_a^b f(x) dx = F(b) - F(a) = [F(x)]_a^b</math></p>

5.4 Summation Formulas (for Riemann Sums)
$\sum_{i=1}^n 1 = n$ $\sum_{i=1}^n i = \frac{n(n+1)}{2}$ $\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$ $\sum_{i=1}^n i^3 = \left(\frac{n(n+1)}{2}\right)^2$
5.5 u-Substitution (Change of Variable)
<p><math>\int_a^b f(g(x))g'(x) dx = \int_{g(a)}^{g(b)} f(u) du</math> where <math>u = g(x)</math></p> <p><b>Steps:</b></p> <ol style="list-style-type: none"> <li>Choose <math>u = g(x)</math>, compute <math>du = g'(x) dx</math></li> <li>Change limits: <math>x = a \rightarrow u = g(a)</math>, <math>x = b \rightarrow u = g(b)</math></li> <li>Substitute everything to <math>u</math> and integrate</li> <li>(For indefinite: substitute back to <math>x</math> at end)</li> </ol>
5.6 Symmetry Properties
<p><b>Even function</b> (<math>f(-x) = f(x)</math>): <math>\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx</math></p> <p><b>Odd function</b> (<math>f(-x) = -f(x)</math>): <math>\int_{-a}^a f(x) dx = 0</math></p> <p><b>Examples:</b> <math>x^2, \cos x</math> are even; <math>x^3, \sin x</math> are odd</p>
6 L06: Applications, Volume, Averages
<p><b>6.1 Area Between Curves</b></p> <p><b>Vertical slices</b> (<math>x</math>-axis): <math>A = \int_a^b  f(x) - g(x)  dx</math></p> <p>If <math>f(x) \geq g(x)</math> on <math>[a, b]</math>: <math>A = \int_a^b [f(x) - g(x)] dx</math></p> <p><b>Horizontal slices</b> (<math>y</math>-axis): <math>A = \int_c^d  f(y) - g(y)  dy</math></p> <p><b>Strategy:</b> If curves cross, split into subintervals where one function dominates.</p> <p><b>Finding bounds:</b> Solve <math>f(x) = g(x)</math> for intersection points.</p> <p><b>6.2 Volume by Cross-Sections</b></p> <p><math>V = \int_a^b A(x) dx</math> where <math>A(x)</math> is the cross-sectional area at position <math>x</math>.</p> <p><b>Common cross-sections:</b></p> <ul style="list-style-type: none"> <li>Square: <math>A = s^2</math></li> <li>Semicircle: <math>A = \frac{\pi}{2} r^2</math></li> <li>Equilateral triangle: <math>A = \frac{\sqrt{3}}{4} s^2</math></li> </ul> <p><b>6.3 Volume of Revolution: Disk Method</b></p> <p>Revolving <math>y = f(x)</math> around the <math>x</math>-axis (<math>f(x) \geq 0</math>): <math>V = \pi \int_a^b [f(x)]^2 dx</math></p> <p>Revolving <math>x = g(y)</math> around the <math>y</math>-axis: <math>V = \pi \int^d [g(y)]^2 dy</math></p> <p><b>6.4 Volume of Revolution: Washer Method</b></p> <p>Revolving region between <math>f(x)</math> and <math>g(x)</math> around the <math>x</math>-axis (<math>f(x) \geq g(x) \geq 0</math>): <math>V = \pi \int_a^b ([f(x)]^2 - [g(x)]^2) dx</math></p> <p><math>R(x)</math>: outer radius (farther from axis), <math>r(x)</math>: inner radius (closer to axis)</p> <p><b>6.5 Volume of Revolution: Shell Method</b></p> <p>Revolving around the <b><math>y</math>-axis</b> using vertical slices: <math>V = 2\pi \int_a^b x  f(x)  dx</math></p> <p>Revolving around the <b><math>x</math>-axis</b> using horizontal slices: <math>V = 2\pi \int_c^d y  g(y)  dy</math></p> <p><b>6.6 Choosing Disk/Washer vs Shell</b></p> <p><b>Disk/Washer:</b> Slice <math>\perp</math> to axis of rotation. Need to express radius as function of the slicing variable.</p> <p><b>Shell:</b> Slice <math>\parallel</math> to axis of rotation. Need to express height and radius.</p> <p><b>Rule of thumb:</b></p> <ul style="list-style-type: none"> <li>Revol around <math>x</math>-axis, integrate w.r.t. <math>x \rightarrow</math> Disk/Washer</li> <li>Revol around <math>y</math>-axis, integrate w.r.t. <math>x \rightarrow</math> Shell</li> <li>Revol around <math>x</math>-axis, integrate w.r.t. <math>y \rightarrow</math> Shell</li> <li>Revol around <math>y</math>-axis, integrate w.r.t. <math>y \rightarrow</math> Disk/Washer</li> </ul> <p><b>6.7 Revolution Around Non-Standard Axes</b></p> <p><b>Around <math>y = k</math>:</b> Adjust radii: <math>R =  f(x) - k </math></p> <p><b>Around <math>x = k</math>:</b> Adjust shell radius: <math>r =  x - k </math></p> <p><b>6.8 Average Value of a Function</b></p> <p><math>f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx</math></p>

**Mean Value Theorem for Integrals:** If  $f$  is continuous on  $[a, b]$ , then  $\exists c \in [a, b]$ :  
 $f(c) = f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) dx$

7 L07: Integration Techniques

7.1 Substitution (Review)

$\int f(g(x))g'(x) dx = \int f(u) du$  where  $u = g(x)$   
**Key patterns:**

$\int f(ax + b) dx = \frac{1}{a} F(ax + b) + C$   
 $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C$

7.2 Integration by Parts (IBP)

$\int u dv = uv - \int v du$   
**Definite form:**  $\int_a^b u dv = [uv]_a^b - \int_a^b v du$   
**LIATE rule** for choosing  $u$  (first):

**Logarithmic**  $\rightarrow$  **Inverse trig**  $\rightarrow$  **Algebraic**  $\rightarrow$  **Trig**  $\rightarrow$

**Exponential**  
**Common IBP integrals:**

$\int xe^x dx = xe^x - e^x + C$   
 $\int x \sin x dx = -x \cos x + \sin x + C$   
 $\int x \cos x dx = x \sin x + \cos x + C$   
 $\int \ln x dx = x \ln x - x + C$   
 $\int x^n \ln x dx = \frac{x^{n+1}}{n+1} \ln x - \frac{x^{n+1}}{(n+1)^2} + C$   
 $\int \sin^{-1} x dx = x \sin^{-1} x + \sqrt{1-x^2} + C$   
 $\int \tan^{-1} x dx = x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$

**Tabular method:** For  $\int x^n \cdot (\text{trig/exp}) dx$ , repeatedly differentiate  $u$  and integrate  $dv$  in columns with alternating signs.  
**Cyclic IBP:**  $\int e^x \sin x dx \rightarrow$  IBP twice, then solve for the integral algebraically.  
 $\int e^x \sin x dx = \frac{e^x (\sin x - \cos x)}{2} + C$   
 $\int e^x \cos x dx = \frac{e^x (\sin x + \cos x)}{2} + C$

7.3 Trigonometric Integrals

7.3.1 Powers of sin and cos

$\int \sin^m x \cos^n x dx$ :  
• **m odd:** Save one  $\sin x$ , convert rest using  $\sin^2 x = 1 - \cos^2 x$ ,  $u = \cos x$   
• **n odd:** Save one  $\cos x$ , convert rest using  $\cos^2 x = 1 - \sin^2 x$ ,  $u = \sin x$   
• **Both even:** Use half-angle identities:  
 $\sin^2 x = \frac{1 - \cos 2x}{2}$ ,  $\cos^2 x = \frac{1 + \cos 2x}{2}$   
 $\sin x \cos x = \frac{\sin 2x}{2}$

7.3.2 Powers of tan and sec

$\int \tan^m x \sec^n x dx$ :  
• **n even:** Save  $\sec^2 x$ , convert rest using  $\sec^2 x = 1 + \tan^2 x$ ,  $u = \tan x$   
• **m odd:** Save  $\sec x \tan x$ , convert rest using  $\tan^2 x = \sec^2 x - 1$ ,  $u = \sec x$

**Key integrals:**  
 $\int \tan x dx = \ln|\sec x| + C = -\ln|\cos x| + C$   
 $\int \sec x dx = \ln|\sec x + \tan x| + C$   
 $\int \sec^3 x dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$

7.4 Trigonometric Substitution

Expression	Substitution	Identity	Range
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$1 - \sin^2 = \cos^2$	$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$1 + \tan^2 = \sec^2$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$\sec^2 - 1 = \tan^2$	$0 \leq \theta < \frac{\pi}{2}$

7.5 Partial Fractions

For  $\int \frac{P(x)}{Q(x)} dx$  where  $\deg P < \deg Q$  (do polynomial division first if not):  
**Linear factor**  $(ax + b)$ :  $\frac{A}{ax+b} \rightarrow$  integrates to  $\frac{A}{a} \ln|ax + b|$   
**Repeated linear**  $(ax + b)^k$ :  $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \dots +$

$\frac{A_k x + B_k}{(ax^2 + bx + c)^k}$   
**Irreducible quadratic**  $(ax^2 + bx + c)$ :  $\frac{Ax+B}{ax^2+bx+c} \rightarrow$  complete the square, split  
**Repeated quadratic**  $(ax^2 + bx + c)^k$ :  $\frac{A_1 x + B_1}{ax^2+bx+c} + \dots + \frac{A_k x + B_k}{(ax^2+bx+c)^k}$   
**Cover-up method:** For distinct linear factors, plug in root of each factor to find each constant directly.

7.6 Useful Reference Integrals

$\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}(\frac{x}{a}) + C$   
 $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \ln|\frac{x-a}{x+a}| + C$   
 $\int \frac{1}{\sqrt{x^2+a^2}} dx = \ln|x + \sqrt{x^2+a^2}| + C$   
 $\int \frac{1}{\sqrt{x^2-a^2}} dx = \ln|x + \sqrt{x^2-a^2}| + C$

8 L08: Parametric Equations, Arc Length, Polar

8.1 Parametric Equations

$x = f(t)$ ,  $y = g(t)$ ,  $t \in [a, b]$   
**Eliminate parameter:** Solve for  $t$  from one equation, substitute into the other.

**Common parametrizations:**  
• Circle:  $x = r \cos t$ ,  $y = r \sin t$ ,  $t \in [0, 2\pi]$   
• Ellipse:  $x = a \cos t$ ,  $y = b \sin t$ ,  $t \in [0, 2\pi]$   
• Line:  $x = x_0 + at$ ,  $y = y_0 + bt$

8.2 Calculus with Parametric Curves

**First derivative:**  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{y'(t)}{x'(t)}$  (provided  $x'(t) \neq 0$ )  
**Second derivative:**  $\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(\frac{dy}{dx})}{dx/dt}$   
**Area under parametric curve:**  $A = \int_a^b g(t) f'(t) dt$   
**Tangent lines:**  
• Horizontal tangent:  $g'(t) = 0$  (and  $f'(t) \neq 0$ )  
• Vertical tangent:  $f'(t) = 0$  (and  $g'(t) \neq 0$ )

8.3 Arc Length

**Cartesian** ( $y = f(x)$ ):  $L = \int_a^b \sqrt{1 + (f'(x))^2} dx$   
**Parametric:**  $L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} dt$   
**Polar:**  $L = \int_a^b \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$

8.4 Surface Area of Revolution

**Around x-axis** (Cartesian):  $S = 2\pi \int_b^a |f(x)| \sqrt{1 + (f'(x))^2} dx$   
**Around x-axis** (Parametric):  $S = 2\pi \int_b^a |g(t)| \sqrt{(f'(t))^2 + (g'(t))^2} dt$   
**Around y-axis** (Cartesian):  $S = 2\pi \int_b^a |x| \sqrt{1 + (f'(x))^2} dx$   
**Key:** Multiply arc length element  $ds$  by circumference  $2\pi r$ , where  $r$  is the distance from the curve to the axis.

8.5 Polar Coordinates

**Conversion:**  
 $x = r \cos \theta$ ,  $y = r \sin \theta$   
 $r^2 = x^2 + y^2$ ,  $\tan \theta = \frac{y}{x}$   
**Common polar curves:**  
• Circle:  $r = a$  (centered at origin),  $r = 2a \cos \theta$  (centered at  $(a, 0)$ )  
• Cardioid:  $r = a(1 - \cos \theta)$  or  $r = a(1 + \sin \theta)$   
• Rose:  $r = a \cos(n\theta)$  or  $r = a \sin(n\theta)$  ( $n$  petals if  $n$  odd,  $2n$  if  $n$  even)  
• Limacon:  $r = g + b \cos \theta$  (inner loop if  $|b| > |a|$ )  
• Lemniscate:  $r^2 = a^2 \cos 2\theta$ ,  $r^2 = a^2 \sin 2\theta$   
• Spiral:  $r = a\theta$

8.6 Slope of Polar Curves

$\frac{dy}{dx} = \frac{r' \sin \theta + r \cos \theta}{r' \cos \theta - r \sin \theta}$  where  $r' = \frac{dr}{d\theta}$   
**Derivation:**  $x = r \cos \theta$ ,  $y = r \sin \theta \rightarrow$  use parametric derivative with parameter  $\theta$ .

8.7 Area in Polar Coordinates

**Area of sector:**  $A = \frac{1}{2} \int_a^b r^2 d\theta$   
**Area between two curves:**  $A = \frac{1}{2} \int_a^b (r_{\text{outer}}^2 - r_{\text{inner}}^2) d\theta$   
**Warning:** Find correct limits by checking where  $r = 0$  or where curves intersect ( $r_1(\theta) = r_2(\theta)$ ). Check

whether curves pass through the origin at different  $\theta$  values.

8.8 Key Exam Reminders

- **Parametric:** Always check direction of traversal (orientation)
- **Arc length:** Never forget the square root and the sum under it
- **Polar area:** Use  $\frac{1}{2}r^2$ , NOT  $\pi r^2$ . Factor of  $\frac{1}{2}$  is crucial
- **Polar intersections:** Curves may intersect at origin even if  $r = 0$  at different  $\theta$  values — graph the curves to be safe

9 L09: L'Hôpital's Rule & Improper Integrals

9.1 Indeterminate Forms

**Basic types:**  $\frac{0}{0}$ ,  $\frac{\infty}{\infty}$  — L'Hôpital applies directly  
**Other types** (must be converted):  
 $0 \cdot \infty$ : Rewrite as  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$  (e.g.,  $f \cdot g = \frac{f}{1/g}$ )  
 $\infty - \infty$ : Combine into single fraction  
 $0^0$ ,  $1^\infty$ ,  $\infty^0$ : Take  $\ln$  first, then exponentiate result

9.2 L'Hôpital's Rule

If  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$  gives  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ , and  $g'(x) \neq 0$  near  $c$ , then:  
 $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$  (if right side exists or is  $\pm \infty$ )

**Also works for**  $x \rightarrow \pm\infty$ ,  $x \rightarrow c^\pm$ .  
**Can apply repeatedly** if result is still indeterminate.  
**Common pitfall:** Verify the form IS indeterminate before applying! If not  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$ , L'Hôpital does NOT apply.

**Examples:**  
 $\lim_{x \rightarrow 0} \frac{\sin x}{x} = \lim_{x \rightarrow 0} \frac{\cos x}{1} = 1$   
 $\lim_{x \rightarrow \infty} \frac{x}{e^x} = \lim_{x \rightarrow \infty} \frac{1}{e^x} = 0$   
 $\lim_{x \rightarrow 0^+} x \ln x = \lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} = \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} =$   
 $\lim_{x \rightarrow 0^+} (-x) = 0$

9.3 Exponential Indeterminate Forms

For  $\lim_{x \rightarrow c} f(x)^{g(x)}$  with form  $0^0$ ,  $1^\infty$ , or  $\infty^0$ :  
1. Let  $L = \lim_{x \rightarrow c} g(x) \ln f(x)$  (this is often  $0 \cdot \infty$  or  $\frac{\infty}{\infty}$ )  
2. Apply L'Hôpital to find  $L$   
3. Answer is  $e^L$   
**Classic:**  $\lim_{x \rightarrow 0^+} x^x = e^{\lim_{x \rightarrow 0^+} x \ln x} = e^0 = 1$   
**Classic:**  $\lim_{x \rightarrow \infty} (1 + \frac{1}{x})^x = e^{\lim_{x \rightarrow \infty} x \ln(1+1/x)} = e^1 = e$

9.4 Improper Integrals: Type 1 (Infinite Limits)

$\int_a^\infty f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx$   
 $\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx$   
 $\int_{-\infty}^\infty f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^\infty f(x) dx$  (both must converge)

9.5 Improper Integrals: Type 2 (Discontinuities)

If  $f$  has a discontinuity at  $b$ :  $\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$   
If  $f$  has a discontinuity at  $a$ :  $\int_a^b f(x) dx = \lim_{t \rightarrow a^+} \int_t^b f(x) dx$   
If  $f$  has a discontinuity at  $c \in (a, b)$ : Split into two improper integrals at  $c$  (both must converge)

9.6 p-Integral Test (Key Reference)

$\int_1^\infty \frac{1}{x^p} dx$ : **Converges** if  $p > 1$ , **Diverges** if  $p \leq 1$   
 $\int_0^1 \frac{1}{x^p} dx$ : **Converges** if  $p < 1$ , **Diverges** if  $p \geq 1$

9.7 Comparison Tests for Improper Integrals

9.7.1 Direct Comparison Test (DCT)

If  $0 \leq f(x) \leq g(x)$  for  $x \geq a$ :  
•  $\int_a^\infty g(x) dx$  converges  $\rightarrow \int_a^\infty f(x) dx$  converges  
•  $\int_a^\infty f(x) dx$  diverges  $\rightarrow \int_a^\infty g(x) dx$  diverges

9.7.2 Limit Comparison Test (LCT)

If  $f(x), g(x) > 0$  and  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$  with  $0 < L < \infty$ :  
Both integrals converge or both diverge.  
**Strategy:** Compare with  $\frac{1}{x^p}$  integrals.  
**Example:**  $\int_1^\infty \frac{1}{x^{2+1}} dx$  — compare with  $\frac{1}{x^2}$  (converges,  $p = 2 > 1$ )

9.8 Quick Reference: Common Results

$\int_0^\infty e^{-x} dx = 1$ ,  $\int_0^\infty e^{-ax} dx = \frac{1}{a}$  ( $a > 0$ )  
 $\int_1^\infty \frac{1}{x} dx = \infty$  (diverges),  $\int_1^\infty \frac{1}{x^2} dx = 1$   
 $\int_0^\infty x e^{-x} dx = 1$  (by IBP)  
 $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$  (Gaussian integral)

10 L10: Infinite Series & Taylor Series

10.1 Sequences

A sequence  $\{a_n\}$  converges if  $\lim_{n \rightarrow \infty} a_n = L$  exists and is finite.  
**Monotone Convergence Theorem:** A bounded monotone sequence converges.  
**Common limits:**

$\lim_{n \rightarrow \infty} \frac{1}{n^p} = 0$  ( $p > 0$ ),  $\lim_{n \rightarrow \infty} r^n = 0$  ( $|r| < 1$ )  
 $\lim_{n \rightarrow \infty} n^{1/p} = 1$ ,  $\lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$   
 $\lim_{n \rightarrow \infty} \frac{x^n}{n!} = 0$  (for any fixed  $x$ ),  $\lim_{n \rightarrow \infty} \frac{\ln n}{n} = 0$

10.2 Infinite Series

$\sum_{n=1}^\infty a_n = \lim_{N \rightarrow \infty} S_N$  where  $S_N = \sum_{n=1}^N a_n$  (partial sums)  
**Converges** if  $\lim_{N \rightarrow \infty} S_N$  exists and is finite; otherwise **diverges**.

10.3 Divergence Test (nth-Term Test)

If  $\lim_{n \rightarrow \infty} a_n \neq 0$  (or does not exist), then  $\sum a_n$  **diverges**.  
**WARNING:** If  $\lim_{n \rightarrow \infty} a_n = 0$ , the series may still diverge! (e.g., harmonic series)

10.4 Geometric Series

$\sum_{n=0}^\infty ar^n = \frac{a}{1-r}$  if  $|r| < 1$ ; diverges if  $|r| \geq 1$   
**Partial sum:**  $S_N = \frac{a(1-r^{N+1})}{1-r}$   
**Common trick:** Rewrite series to identify  $a$  and  $r$ .

10.5 Telescoping Series

$\sum_{n=1}^\infty (b_n - b_{n+1}) = b_1 - \lim_{n \rightarrow \infty} b_n$  (if limit exists)  
**Strategy:** Use partial fractions to express terms as differences.

10.6 p-Series

$\sum_{n=1}^\infty \frac{1}{n^p}$ : **Converges** if  $p > 1$ , **Diverges** if  $p \leq 1$   
**Harmonic series:**  $\sum \frac{1}{n}$  diverges ( $p = 1$ )

10.7 Convergence Tests

10.7.1 Integral Test

If  $f$  is positive, continuous, decreasing for  $x \geq 1$  and  $a_n = f(n)$ :  
 $\sum_{n=1}^\infty a_n$  and  $\int_1^\infty f(x) dx$  both converge or both diverge.

10.7.2 Direct Comparison Test

If  $0 \leq a_n \leq b_n$ :  
•  $\sum b_n$  converges  $\rightarrow \sum a_n$  converges  
•  $\sum a_n$  diverges  $\rightarrow \sum b_n$  diverges

10.7.3 Limit Comparison Test

If  $a_n, b_n > 0$  and  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = L$  with  $0 < L < \infty$ : both converge or both diverge.

10.7.4 Ratio Test

$L = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$ :  
•  $L < 1$ : **converges absolutely**  
•  $L < 1$  or  $L = 1$ : **inconclusive**  
•  $L > 1$ : **diverges**  
**Best for:** factorials  $n!$ , exponentials  $a^n$ , products involving  $n$ .

10.7.5 Root Test

$L = \lim_{n \rightarrow \infty} \sqrt[n]{|a_n|}$ : Same conclusions as Ratio Test.

10.8 Alternating Series

$\sum_{n=1}^\infty (-1)^{n+1} b_n$  converges if:  
(i)  $b_n > 0$ , (ii)  $b_{n+1} \leq b_n$  (decreasing), (iii)  $\lim_{n \rightarrow \infty} b_n = 0$   
**Error bound:**  $|S - S_N| \leq b_{N+1}$  (remainder  $\leq$  first omitted term)

10.9 Absolute vs Conditional Convergence

**Absolutely convergent:**  $\sum |a_n|$  converges  $\rightarrow \sum a_n$  converges  
**Conditionally convergent:**  $\sum a_n$  converges but  $\sum |a_n|$  diverges  
**Example:**  $\sum \frac{(-1)^{n+1}}{n}$  converges conditionally;  $\sum \frac{(-1)^{n+1}}{n^2}$  converges absolutely  
**10.10 Power Series**  
 $\sum_{n=0}^\infty c_n(x-a)^n$  with **center**  $a$  and **radius of convergence**  $R$   
**Finding  $R$ :**  $R = \lim_{n \rightarrow \infty} \left| \frac{c_n}{c_{n+1}} \right|$  or  $R = \frac{1}{\lim_{n \rightarrow \infty} \sqrt[n]{|c_n|}}$   
Converges absolutely for  $|x-a| < R$ , diverges for  $|x-a| > R$ .  
**Check endpoints separately** ( $x = a \pm R$ ).  
**Operations:** Can differentiate and integrate term-by-term within interval of convergence.

10.11 Taylor & Maclaurin Series

$f(x) = \sum_{n=0}^\infty \frac{f^{(n)}(a)}{n!} (x-a)^n$  (Taylor series centered at  $a$ )  
Maclaurin series:  $a = 0$   
**diverges.**  
**WARNING:** If  $\lim_{n \rightarrow \infty} a_n = 0$ , the series may still diverge! (e.g., harmonic series)

10.11.1 Essential Maclaurin Series

$e^x = \sum_{n=0}^\infty \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$ ,  $R = \infty$   
 $\sin x = \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$ ,  $R = \infty$   
 $\cos x = \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$ ,  $R = \infty$   
 $\frac{1}{1-x} = \sum_{n=0}^\infty x^n = 1 + x + x^2 + x^3 + \dots$ ,  $R = 1$   
 $\ln(1+x) = \sum_{n=1}^\infty \frac{(-1)^{n+1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$   
 $R = 1$   
 $\tan^{-1} x = \sum_{n=0}^\infty \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$ ,  $R = 1$   
 $(1+x)^\alpha = \sum_{n=0}^\infty \binom{\alpha}{n} x^n = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!} x^2 + \dots$   
 $R = 1$

10.12 Taylor's Theorem (Remainder)

$f(x) = T_n(x) + R_n(x)$  where  $T_n$  is the  $n$ th-degree Taylor polynomial.  
**Lagrange remainder:**  $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$  for some  $c$  between  $a$  and  $x$ .  
**Bound:**  $|R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1}$  where  $M = \max |f^{(n+1)}|$  on the interval.

10.13 Generating New Series from Known Ones

**Substitution:**  $e^{-x^2} = \sum_{n=0}^\infty \frac{(-1)^n x^{2n}}{n!}$  (replace  $x$  with  $-x^2$  in  $e^x$ )  
**Differentiation:**  $\frac{1}{(1-x)^2} = \sum_{n=1}^\infty n x^{n-1}$  (differentiate  $\frac{1}{1-x}$ )  
**Integration:**  $\ln(1+x) = \int_0^x \frac{1}{1+t} dt = \sum_{n=0}^\infty \frac{(-1)^n x^{n+1}}{n+1}$   
**Multiplication:** Multiply series term by term (Cauchy product)