1 [...]

1.1 [...]

In any setting where we have a monoid M acting on a set S, and a subset $\mathrm{X} \subset M$, we can form the graph $\Gamma_{\bullet}^{S/\mathrm{X}}$ with

$$\Gamma_0^{S/\mathrm{X}} := \ S \qquad \text{and} \qquad \Gamma_1^{S/\mathrm{X}} := \ \mathrm{X} \times S,$$

where the *source* and *target* maps

$$\partial_0^1, \partial_1^1 : \ \Gamma_1^{S/\mathrm{X}} \longrightarrow \Gamma_0^{S/\mathrm{X}}$$

that interpret each pair $(\chi,\ s) \in \mathrm{X} \times S = \Gamma_1^{S/\mathrm{X}}$ as the arrow

$$s \xrightarrow{\ \chi\ } \chi s.$$

In general, it's hard to "*zoom out from* $\Gamma_{\bullet}^{S/\mathrm{X}}$" to say anything substantuve about the large-scale structure of this graph. That said, there are many special casses where we can say a lot about the strucutre of this graph. In the special case that - $S = \mathbb{Z}/\ell\mathbb{Z}$, the set of integers modulo some positive integer ℓ, - $M = \mathbb{Z}/\ell\mathbb{Z}$ equipped with its mutliplicative structure,

we can say quite a bit about the strucutre of $\Gamma_{\bullet}^{S/\mathrm{X}}$. The positive integer ℓ has unique prime factorization

$$\ell \ = \ p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$$

The graph $\Gamma_{\bullet}^{S/\mathrm{X}}$ contains a subgraph $\mathrm{Div}(\ell)_{\bullet}$ with

$$\mathrm{Div}(\ell)_0 := \{d \in \mathbb{Z}_{>0} : d \text{ divides } \ell\}$$

and

$$\mathrm{Div}(\ell)_1 := \ \{d \xrightarrow{\ p\ } pd : \ pd \text{ divides } \ell,\ p \text{ prime}\}$$