1 [...]

1.1 [...]

In any setting where we have a monoid M acting on a set S, and a subset $\mathrm{X} \subset M$, we can form the graph $\Gamma_\bullet^{S/\mathrm{X}}$ with

$$\Gamma_0^{S/\mathrm{X}} := S \qquad \text{and} \qquad \Gamma_1^{S/\mathrm{X}} := \mathrm{X} \times S,$$

where the *source* and *target* maps

$$\partial_0^1, \partial_1^1 : \Gamma_1^{S/\mathrm{X}} \longrightarrow \Gamma_0^{S/\mathrm{X}}$$

that interpret each pair $(\chi,\ s) \in \mathrm{X} \times S = \Gamma_1^{S/\mathrm{X}}$ as the arrow

$$s \xrightarrow{\ \chi\ } \chi s.$$

In general, it's hard to "*zoom out from* $\Gamma_\bullet^{S/\mathrm{X}}$" to say anything substantuve about the large-scale structure of this graph. That said, there are many special casses where we can say a lot about the strucutre of this graph. In the special case that - $S = \mathbb{Z}/\ell\mathbb{Z}$, the set of integers modulo some positive integer ℓ, - $M = \mathbb{Z}/\ell\mathbb{Z}$ equipped with its mutliplicative structure,

we can say quite a bit about the strucutre of $\Gamma_\bullet^{S/\mathrm{X}}$. The positive integer ℓ has unique prime factorization

$$\ell \ = \ p_1^{e_1} p_2^{e_2} \cdots p_r^{e_r}$$

The graph $\Gamma_\bullet^{S/\mathrm{X}}$ contains a subgraph $\mathrm{Div}(\ell)_\bullet$ with

$$\mathrm{Div}(\ell)_0 := \{d \in \mathbb{Z}_{>0} : d \text{ divides } \ell\}$$

and

$$\mathrm{Div}(\ell)_1 := \ \{d \xrightarrow{\ p\ } pd : \ pd \text{ divides } \ell,\ p \text{ prime}\}$$

1.2 Continuous Fourier transform of discrete audio signals

I want to spend a moment trying to better understand continuous realizations of discrete audio signals.

1.2.1 Continuous realization of discrete audio signals

Consider the case where we realize a given discrete periodic audio signal $s : \mathbb{Z}/\ell\mathbb{Z} \longrightarrow \mathbb{C}$ as a continuous audio signal $f_s : \mathbb{S}^1 \longrightarrow \mathbb{C}$ by fixing a wavelet $\psi(t)$ and interpreting the values $s(n)$ as coefficients for translates of this wavelet:

$$f_s(t) \ := \sum_{n \in \mathbb{Z}/\ell\mathbb{Z}} s(n)\ \psi\big(t - \tfrac{2\pi}{\ell} n\big).$$

See [Mal09]

1.2.2 Fourier transform

In the special case that our wavelet $\psi(t)$ is the on/off pulse

$$\psi(t) \ = \ \begin{cases} 1 & \text{if } 0 \leq t < \frac{2\pi}{\ell}, \\ 0 & \text{otherwise,} \end{cases}$$

the Fourier transform of the translate $\psi_n(t) := \psi\big(t - \frac{2\pi}{\ell} n\big)$ is

$$\begin{aligned} \widehat{\psi_n}(\omega) \ &= \ e^{-i\frac{2\pi}{\ell} nt} \cdot \ \widehat{\psi}(\omega) \\ &= \ e^{-i\frac{2\pi}{\ell} nt} \cdot 2\pi \cdot \mathrm{sinc}\big(\tfrac{\omega}{\ell}\big) \end{aligned}$$

[...]

$$\widehat{f_s}(\omega) \quad = \sum_{n\in\mathbb{Z}/\ell\mathbb{Z}} s(n)\cdot e^{-i\frac{2\pi}{\ell}nt}\cdot \mathrm{sinc}\left(\frac{\omega}{\ell}\right)$$

[...]
[GR14] [Tym11] [Coh12]

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