4-MINUTE PIECE
FROM
TWO SIMPLE MUSICAL IDEAS

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ABSTRACT. These are notes combining fragments of my journal entries while composing *Voice Leading & Beatmatching Study* with fragments of notes I've been writing as I put together a personal understanding of tonal harmony and musical poetics. These notes are very incomplete and cobbled together in places.

CONTENTS

1. TWO SIMPLE MUSICAL PROTO-IDEAS.

1.1. **Melodies and chords as "motion shapes".** Working on the previous *interference_trial* études and the more recent piece $\mathbb{Z}/180\mathbb{Z}$-*Atonal*, $\mathbb{S}^1$-*Tonal Composition 001*, I found that $\mathbb{Z}/\ell\mathbb{Z}$-tonal melodies and chords can sometimes sound like a sequence of shifts in physical movement. This stands in contrast to $\mathbb{S}^1$-tonal melodies and chords, i.e., melodies and chords in standard tonal harmony, which sound more like sequences of changes in emotion.

Musical proto-idea 1. Build a simple but compelling $\mathbb{Z}/\ell\mathbb{Z}$-tonal melody or chord for a small untuned percussion ensemble. Use $\mathbb{Z}/\ell\mathbb{Z}$-tonal transformations, i.e., $\mathbb{Z}/\ell\mathbb{Z}$-analogues of movement up octaves, perfect fifths, major thirds, etc., to compose with it, roughly in the manner of the simpler and more successful parts of $\mathbb{Z}/180\mathbb{Z}$-*Atonal*, $\mathbb{S}^1$-*Tonal Composition 001*.

See §4 below for the development and realization of this first proto-idea into the first simple musical idea for the piece.

1.2. **Simultaneous $\mathbb{Z}/\ell\mathbb{Z}$- and $\mathbb{S}^1$-voice leading.** I shared some of the *interference_trial* études with two Tallahassee friends who have strong and somewhat exotic musical tastes/interests. They both expressed special interest in *interference_trial_014*. I agree that there's something particularly compelling about 014. I think that part of what makes it compelling is the way it briefly exhibits simultaneous $\mathbb{S}^1$-tonal voice leading — where signal interference happens at the standard time scale — and $\mathbb{Z}/\ell\mathbb{Z}$-tonal voice leading — where signal interference happens at the level of individual bits or individual percussive strikes.

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Musical proto-idea 2. Build a simple but compelling $\mathbb{Z}/\ell\mathbb{Z}$-tonal chord for a small ensemble of mutually interfering $\pm$-pairs of tuned instruments. Use $\mathbb{Z}/\ell\mathbb{Z}$-tonal voice leading to compose with it, roughly in the manner of *interference_trial_014*.

See §5 below for the development and realization of this first proto-idea into the first simple musical idea for the piece.

Remark 1.2.1. Simplicity. This text is long and it includes a bunch of mathematics; two common markers of "complexity."

I want to stress that the ideas here are actually quite simple.

The length of the text comes from the fact that tonal harmony via representation theory entails a kind of "1st-principle" way of thinking about tonal harmony. The mathematics is a language I know really well, kind of like a classically-trained musician with the language of tonal harmony. I think quickly and intuitively in the language of mathematics, and I experience the mathematical ideas in this text as simple.

2. My access-points to tonal theory

2.1. **Tonal harmony and Pontryagin duality.** I want to clarify that I don't conceptualize what I'm doing here as some kind of musical illustration of mathematics. I'm not trying to highlight the novelty of a language I know. I'm trying to use a language I know as a tool for composing music. I believe that the overlapping branches of mathematics *abstract harmonic analysis* and *representation theory of topological groups* can function together as a partial avatar of tonal harmony, one that clarifies possibilities for extending tonal harmony in ways that are difficult to discuss in the existing language of tonal harmony.

In abstract harmonic analysis, the relationship between the horizontal and vertical musical dimensions is an instance of general phenomenon called *Pontryagin duality* [Pon39] [HR63]. In the representation theory of topological groups, Pontryagin duality is generalized in the *Peter-Weyl Theorem* [Kna02] [Vog87] [Pro07]. I want to briefly explain three instances of Pontryagin duality in a musical setting:

2.1.1. The 2 dimensions in standard tonal harmony. There is a hidden assumption underlying standard tonal harmony, which gives it much of its structure. The assumption is that the musical audio signals of interest are periodic, and that time passes continuously over the course of each cycle of any periodic musical audio signal. This assumption lets us represent individual tones, and even chords, as the real parts of functions

$$f : \mathbb{S}^1 \longrightarrow \mathbb{C}, \tag{2.1.1.1}$$

where $\mathbb{S}^1$ is the circle, and where $\mathbb{C}$ is the set of complex numbers. In this situation, Pontryagin duality and the Peter-Weyl Theorem say that all reasonable functions[1] decompose as sums of pure tones. The pure tones that appear in these sums constitute the overtone series. In the setting of abstract harmonic analysis, this amounts to the observation that the Pontryagin dual of $\mathbb{S}^1$ (the circle, domain of any periodic signal) is $\mathbb{Z}$ (the integers, parametrizing the overtone series). A large component of standard tonal harmony derives from this structure (see§**??** below).

The horizontal dimension of staff paper is time. Periodic time is the domain of any function (2.1.1.1). The vertical dimension of staff paper is a discrete tone set. This tone set can be derived directly from the the beginning of overtone series.

[1] called L^2-*functions*.

Abstract harmonic analysis suggests that any theory built to guide a musical practice that uses periodic audio signals and a continuous conception of time will include overlapping fragments of what we recognize as standard tonal harmony.

2.1.2. The 2 dimensions in "spectral musical" strategies. It is interesting that the compositional strategies now collected under the name *spectral music* tend to break from standard tonality in two ways simultaneously.

2.1.2.i. Spectral music rejects the sort of heavy reliance on periodic cells in musical composition that are so common to earlier Western music composition. Spectral music tends to focus, instead, on non-periodic sounds that occupy time intervals of unbound duration [Mur05].

2.1.2.ii. Spectral music rejects heavy reliance fixed tone scales that consist of a discrete set of fixed pitches. Instead, spectral music tends to focus on the possibility of using entire, continuous bands of frequencies to create "clouds" or "continuous distributions" of tones [Mur05].

From the perspective of abstract harmonic analysis, it's natural that 2.1.2.i and 2.1.2.ii would occur together. This is because the Pontryagin dual of $\mathbb{R}$ (all time past and present, i.e., an unbound, continuous time dimension) can be thought of as another copy of $\mathbb{R}$ (now the continuous spectrum of possible frequencies). Tones and chords become functions

$$f : \mathbb{R} \longrightarrow \mathbb{C}, \tag{2.1.2.1}$$

often supported on some compact domain in $\mathbb{R}$. In this setting, it becomes less natural to consider sums over discrete collections of tones, and instead to consider integrals over continuous spaces of tones, i.e., tone clouds

2.1.3. The 2 dimensions for discrete audio signals. Any theory built to guide a musical practice that uses discrete audio signals, i.e., audio signals where time passes in fixed, discrete increments, will end up with a tonal theory that uses some equivalent of power operations $\omega \mapsto \omega^m$ on groups of ℓ^{th} roots of 1. This is because the Pontryagin dual of $\mathbb{Z}/\ell\mathbb{Z}$ (a measure passing in ℓ discrete steps) is $\boldsymbol{\mu}_\ell$ (the group of ℓ^{th} roots of 1, parametrizing the "pure tones" for a measure passing in ℓ discrete steps). Tones become functions

$$f : \mathbb{Z}/\ell\mathbb{Z} \longrightarrow \mathbb{C}. \tag{2.1.3.1}$$

2.2. **A bit of $\mathbb{S}^1$-tonal harmony, i.e., standard tonal harmony as I understand it.** The core features of tonality, within the base-$2^{1/12}$ equal tempered scale, come from two primary sources:

2.2.i. The system of cadential, or "tension-resolution" relationships, ranked in part by levels of relative, perceptual consonance or dissonance, comes from specific rational relationships between sonic frequencies. A directed, cadential relationship occurs from a frequency λ_1 to another frequency λ_2 if

$$\lambda_1/\lambda_2 \;=\; 2^i \cdot 3^j \cdot 5^k$$

for integers i, j, and k of relatively small absolute value.

2.2.ii. Several structures, some of a cyclical nature, emerge from near coincidences between products of integer powers of 3, 5, and $2^{1/12}$. The circle of fifths, for example, arises from the approximate identity $2^{-19} \cdot 3^{12} \sim 1$. The fact that a major and minor triad differ by a half step arises from the approximate identity $2^{35/12} \cdot 3 \cdot 5^{-2} \sim 1$.

We can collect the tonal principals that generate the basic melodic and harmonic patterns, and thus generate larger musical forms, into tonnetz diagrams that exhibit natural movement between idealized pure tones. We call these diagrams "cadential diagrams," and we construct them as follows:

We can understand various "cadential" or "tension-release" relationships between tones or chords as relationships between the characters $\chi_n(t)$, and we can depict systems of these relationships by drawing diagrams consisting of arrows between the functions. Every continuous character $\chi : \mathbb{S}^1 \longrightarrow \mathbb{C}$ is of the form $\chi(t) = e^{\sqrt{\text{-}1}\, nt}$ for some $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$, we define

$$\begin{aligned} \chi_n(t) &:= e^{\sqrt{\text{-}1}\, nt} \\ &= \cos(nt) + \sqrt{\text{-}1}\, \sin(nt). \end{aligned} \tag{2.2.0.1}$$

With this indexing convention, we have the identity

$$\chi_n(mt) = \chi_{mn}(t) \quad \text{for all } m \in \mathbb{Z}. \tag{2.2.0.2}$$

To give a concrete example, for any $n \in \mathbb{Z}_{>0}$, the periodic audio signal $\chi_n(3t) = \chi_{3n}(t)$ is the just intoned 3^{rd}-overtone of the audio signal $\chi_n(t)$. In pitch-class space, $\chi_n(3t)$ represents a perfect fifth relative to $\chi_n(t)$.

We now construct tonnetze, i.e., "tone diagrams." For each $n \in \mathbb{Z}_{>0}$, we give ourselves the option of including the symbol "χ_n" as a node in our diagram. For each integer m, we give ourselves the option of including the arrow

$$\chi_n \xrightarrow{\ m\ } \chi_{mn}$$

in our diagram, corresponding to the identity in (2.2.0.2). For example, if we start with χ_1 and connect to other χ_n's in this manner, but only along arrows labelled "3" and "5," we obtain the diagram at left in Figure 1. This infinite diagram is a version of the tonnetz of Leonhard Euler [Eul39] and Hugo Riemann [Coh98]. We refer to the symbol "χ_1" at bottom in this particular tonnetz as the *root* of the diagram.

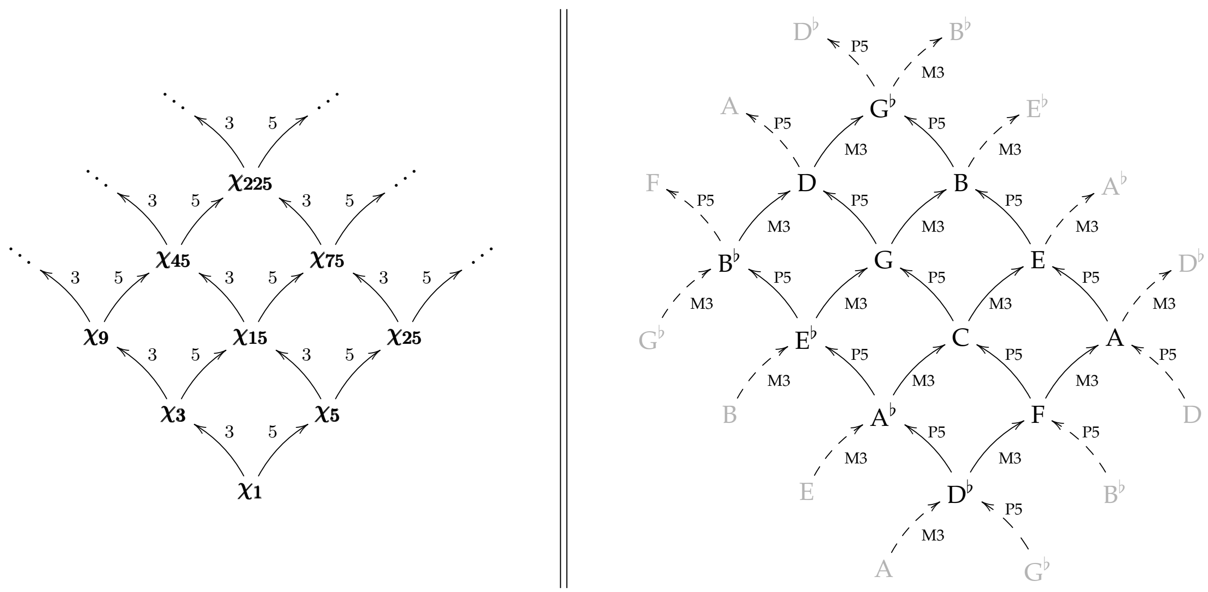


Figure 1. Cadential movement in tonal harmony. The diagram at left consists of all symbols "χ_n," for $n = 3^j 5^k$, $j, k \in \mathbb{Z}_{\geq 0}$, along with all arrows labelled 3 or 5. The approximate identities $2^{19/12} \cdot 3^{-1} \sim 1$ and $2^{28/12} \cdot 5^{-1} \sim 1$ turn the diagram at left into the pitch class diagram at right, called a *tonnetz*.

The diagram at right in Figure 1 shows how equal temperament introduces two cyclical structures into the diagram at left. The upper-right to lower-left cyclic structure comes from the approximate identity $2^{-7} \cdot 5^3 \sim 1$. The upper-left to lower right cyclic structure, which comes with a twist whereby each cycle shifts up by a major third, comes from the approximate identity $2^{-4} \cdot 3^4 \cdot 5^{-1} \sim 1$. This diagram at right in Figure 1 is the tonnetz of neo-Riemannianism [Coh12, Figure 2.9, p. 28] [Tym11, Figure C1, p. 413]. Observe that every one of the 12 equal tempered pitch classes appears in the diagram at right.

Many scales and chords that play a central role in standard tonal harmony appear as connected, 2- and 3-node paths in the diagrams in Figure 1. Figure 2 exhibits the examples of major, minor, and augmented triads, as well as major 6 and major 7 chords. The major scale also makes a natural appearance in Figure 2 as a connected, 7-node path of perfect fifths (a "well-formed scale," in the language of [CC89]). The diagram at right in Figure 1 also exhibits the tritone as the most "tonally remote" interval in our equal tempered system. The point I want to make here is that some of the simplest possible movement in the Euler-Riemannian and neo-Riemannian tonnetze leads to fundamental tonal materials.

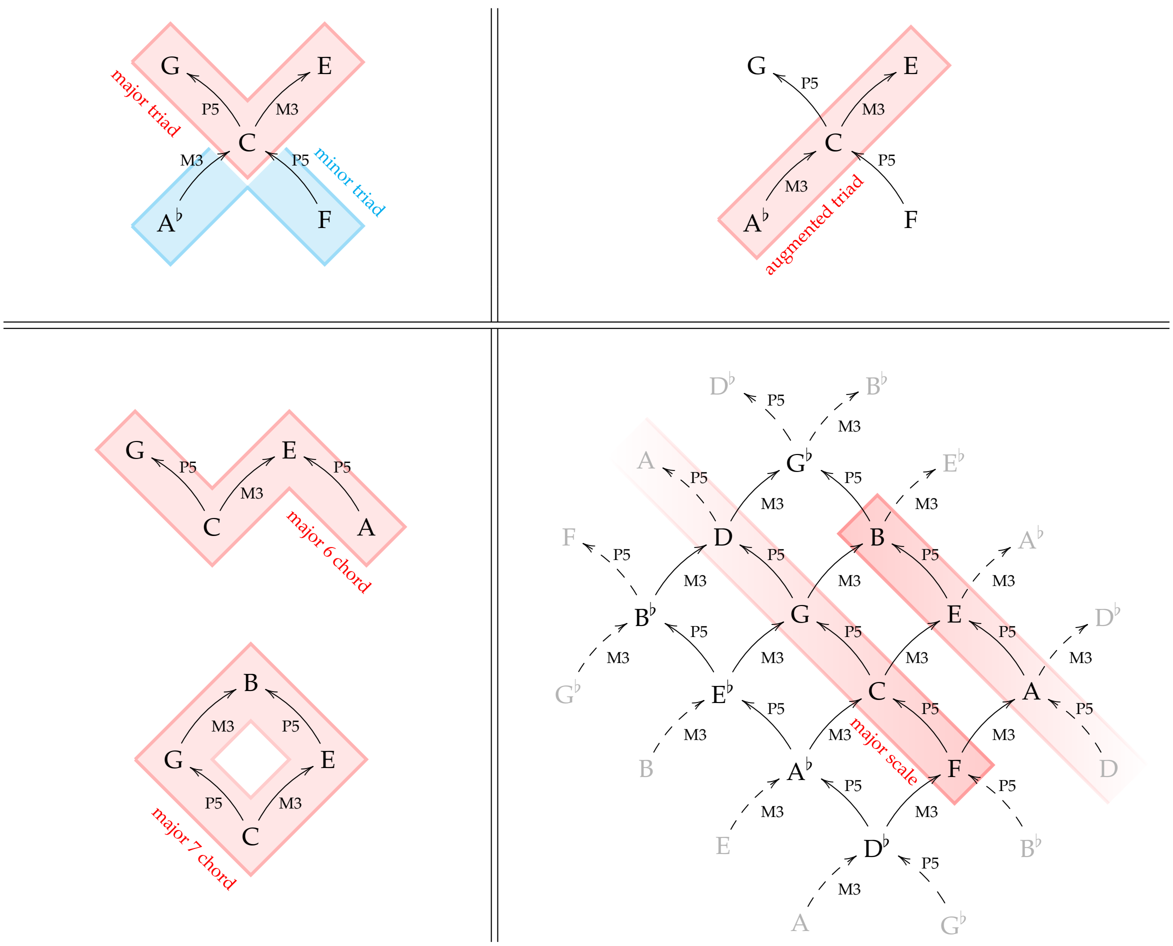


Figure 2. Modal materials. Basic modal materials of tonal harmony in the form of natural, relatively short connected paths within the diagrams in Figure 1.

For a tonnetz in $\mathbb{S}^1$-tonal harmony containing 27 tones, see Figure 19 in §6.

Modulation of key can also be explained in this context, but it is a bit more involved, and not really relevant to the present piece.

3. A bit of $\mathbb{Z}/\ell\mathbb{Z}$-tonal harmony.

3.1. **Differences in the behavior of continuous and discrete audio signals.** If we take some recording of a regular rhythm and speed it up by a factor of at least $2^5 = 32$, it sounds like a musical tone. If we take some recording of a musical tone and slow it down by a factor of less than $2^{-5} = \frac{1}{32}$, it sounds like a regular rhythm. These complementary observations suggest that we can interpret a single beat of a regular rhythm as constituting a single cycle of a musical tone, and vice versa. One concrete mathematical manifestation of this analogy is the frequency-to-tempo conversion

$$\lambda \,\mathrm{Hz} \;=\; 60\lambda \,\mathrm{bpm}. \tag{3.1.0.1}$$

Playing with tempo in this way leads to the following folk observation:

Observation 3.1.1. A continuous drop in tempo results in a continuous transition from the frequency regime of tone — roughly 20 Hz to 6000 Hz — down to the frequency regime of rhythm — below about 10 Hz $= 600$ bpm.

For some composers, musicians, and musicologists, an observation of this sort leads to the following conjecture:

Conjecture 3.1.2. The theory of tonal harmony extends down into the frequency regime of rhythm, where it becomes a theory of rhythm that operates tonaly.

Attempts at such an extension of tonal harmony, though, can leave one with a sense that the emotive power of tonal harmony fails or fades somewhere between the frequency regimes of tone and rhythm. One conclusion we might draw is that this failure reflects some fundamental reality of rhythm, that the project of constructing a tonal theory of rhythm with anything near the functional power of our theory of musical tone is doomed from the start. But there is a hidden assumption in all of this. The very claim that we can drop continuously from the frequency regime of tone down to the frequency regime of rhythm depends on an assumption that audio signals arise from a continuous sampling of time, often coupled with an assumption that the parameter of time is unbound. This hidden assumption can cause would-be rhythmic tonal theorists to overlook two critical facts, thus hindering the effectiveness of a tonal theory of rhythm:

Fact 3.1.3. The arithmetic behavior of frequency, in the form of transformations $f(t) \mapsto f(mt)$, for an audio signal $f(t)$ and a positive integer m, is not the same for discrete and continuous audio signals.

Because the arithmetic behavior of frequency is central to the structure of the "tonal force field" [KK10], failure to address Fact 3.1.4 weakens the performative strength of a tonal theory of rhythm.

Fact 3.1.4. The time-scale at which continuous audio signals interfere with one another does not match the time-scale at which discrete audio signals interfere with one another.

Because interference patterns created by collections of musical tones create the phenomena of consonance and dissonance, which are central to the structure of the tonal force field, failure to address Fact 3.1.4 weakens the performative strength of a tonal theory of rhythm.

3.2. **Musical implications of audio signal implementations.** Consider a continuous, periodic audio signal in the form of a repeating loop of a single untuned drum playing a rhythm, with the amplitude of individual strikes changes from strike to strike. The image labelled (A) in Figure 6 depicts this signal. The excess of timbral information in the signal makes it difficult to isolate the idealized percussive information that a tonal theory of rhythm might address. We want to encode the percussive line in an audio signal that treats each strike as a rapid, isolated change in amplitude, i.e., as a *pulse*.

One common way of doing this is to approximate the original signal with a substantially smoother audio signal. The smoothed-out signal omits all timbral information from the original signal, and replaces each drum strike in the original signal with a single bump function. The image labelled (B) in Figure 6 depicts this sort of signal, replacing the original signal (A).

A second way to create an audio signal, where each strike is an isolated pulse, is to let δ be the *least common duration* between strikes. In other words, choose δ so that the duration between any two strikes in the percussion line is some integer multiple of δ. This puts distinct drum strikes in distinct time increments, and allows us to encode the percussion line as a discrete audio signal where each drum strike is encoded only as its amplitude, in the form of a real value associated to the time increment where the strike happens. The image labelled (C) in Figure 6 depicts an instance of this sort of signal replacing the original (A).

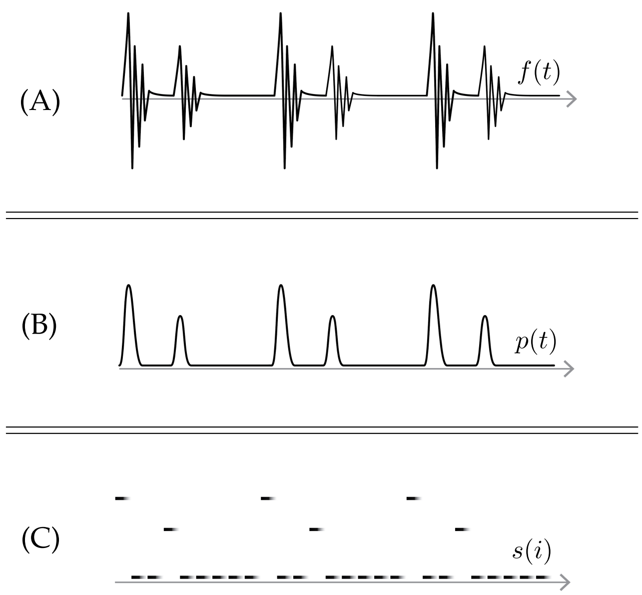


Figure 3. Distinct encodings of a percussive audio signal. Image (A) depicts the original percussive signal. Image (B) depicts its encoding as a continuous audio signal consisting of isolated pulses. Image (C) depicts its encoding as a discrete audio signal.

As musical objects, the three distinct encodings of percussive audio signals in Figure 6 function very differently.

3.2.1. Discrepancies in the behavior of frequency. If $f(t)$ is a continuous periodic signal, then we can change its frequency by changing its playback speed, i.e., by changing the rate at which its argument t increases. If λ is the frequency of $f(t)$, then for any $a \in \mathbb{R}_{>0}$, the frequency of $f(at)$ will be $a\lambda$. If T is the period of $f(t)$, then $\frac{1}{a}T$ will be the period of $f(at)$. Figure 4 depicts this operation of passing from $f(t)$ to $f(at)$ in the case $a = 4$.

If we start with a discrete audio signal

$$s : \mathbb{Z}/\ell\mathbb{Z} \longrightarrow \mathbb{C}$$

instead of a continuous audio signal $f : \mathbb{S}^1 \longrightarrow \mathbb{C}$, and we fix $m \in \mathbb{Z}_{>0}$, then running through the argument i of $s(i)$ at m times the original rate means counting increments by m. If ℓ and m are relatively prime, then counting the periodic set

$$\{0,\ 1,\ 2,\ \ldots,\ i,\ \ldots,\ \ell{-}1\}$$

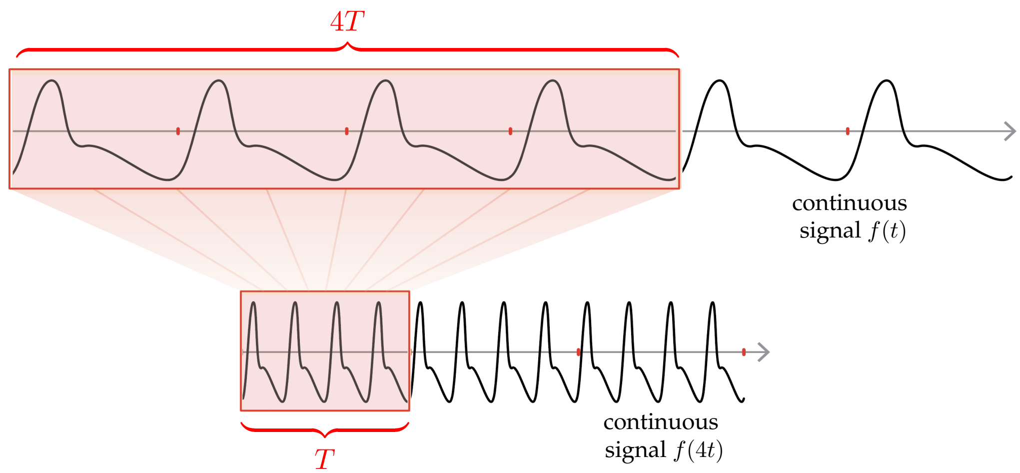


Figure 4. Multiplying the argument of a continuous signal by 4. If the signal $f(t)$ has period T and frequency λ, then the new signal $f(4t)$ has period $\frac{1}{4}T$ and frequency 4λ. The new function $f(4t)$ moves through the values of f at 4 times the rate of $f(t)$. In the span of one cycle of $f(t)$, the new signal $f(4t)$ moves through 4 full cycles.

of ℓ successive discrete increments by m can introduce a re-ordering of the increments, and thus a re-ordering of the values $s(i)$. Stated in more formal mathematical language, when m and n are relatively prime, multiplication by m induces an invertible homomorphism of Abelian groups

$$m : \mathbb{Z}/\ell\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/\ell\mathbb{Z}$$

that takes each $i \in \mathbb{Z}/\ell\mathbb{Z}$ to $mi \pmod{\ell}$. Figure 5 depicts one example of this phenomenon in the case that $n = 9$ and $m = 4$.

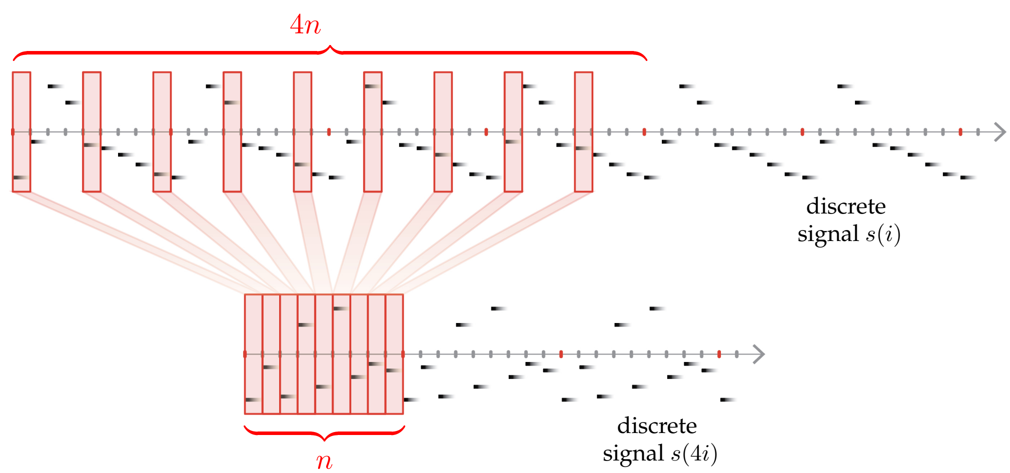


Figure 5. Multiplying the argument of a discrete signal by 4. Multiplying the argument of a discrete audio signal $s(i)$ by 4 means counting through the increments at 4 times the normal rate, i.e., counting by 4. In contrast to Figure 4, both the original signal $s(i)$ *and* the new signal $s(4i)$ complete 1 cycle every 9 increments. Thus $s(i)$ and $s(4i)$ have the same frequency. The effect of multiplying by 4 here is more like an arithmetic re-sequencing of the than a continuous re-scaling of the original signal.

Remark 3.2.2. Because the encoding in images (B) of Figure 6 produces a continuous signal, wheres the the encoding in image (C) of Figure 6 produces a discrete signal, the discrepancy between the behavior of frequency for continuous versus discrete signals is exacerbated if we do not clearly distinguish between these two encoding methods.

Figure 6 exhibits the radical mismatch in behavior that can emerge from applying the two different encoding methods to the same percussion line.

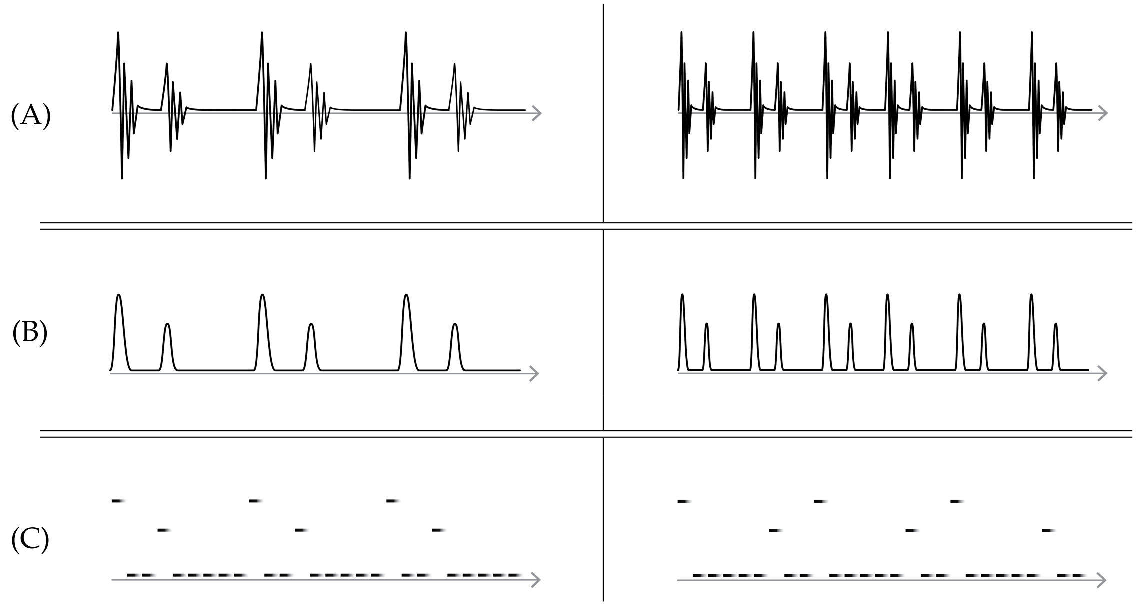


Figure 6. Radical mismatch in behavior for the two encoding methods. The three audio signals in the column at left come from Figure 6. Each corresponding signal in the column at right increases the rate of the passage of time by 2. For rows (A) and (B), the frequency doubles, but in row (C), the frequency remains the same, and the values of the signal are re-ordered with respect to the incremental passage of time.

3.2.3. Discrepancies in the time-scale of interference. The phenomena of consonance and dissonance emerge from a more basic sonic phenomenon. When an audio signal occupies a medium, for instance the air surrounding a listener, it makes the medium deviate back and forth from its equilibrium. If two different audio signals act on the same medium simultaneously, they interact with one another. The two signals partially cancel each other out when they push the medium in opposite directions, and the two signals add together when they push the medium in the same direction. The combined effect of this canceling and adding, for the listener, is called *interference* between the two signals.

When the two audio signals are periodic, the listener can often detect a strong pattern in their interference. This interference pattern can have two basic shapes. If λ_1 and λ_2 are the frequencies of the two periodic signals, then their interference pattern itself will repeat with a fixed period whenever the ratio λ_1/λ_2 is a rational number with relatively few, small prime factors. As the number and sizes of the prime factors of λ_1/λ_2 increase, it becomes more and more difficult to hear the regularities in the interference pattern. If λ_1/λ_2 is not a rational number, then the interference pattern will not repeat, but will instead gradually transform.

Our auditory experience of the regularity in these interference patterns is a central component of our experience of tonal consonance, and our auditory experience of the irregularity in these interference patterns is a central component of our experience of tonal dissonance. The theory of tonal harmony provides a system of guidelines for employing consonance and dissonance in musically effective ways by choosing sequences and sets of these ratios λ_1/λ_2 in particular ways.

Our continuous and discrete idealized audio signals in Figures 4 and 5 consist of a sequence of $\mathbb{R}$-valued deviations from some equilibrium 0, and so they are able to mimic interference phenomena. Given a pair of continuous audio signals $f_1(t)$ and $f_2(t)$, there exists a third audio signal $f_3(t)$ that a listener will perceive as the original pair of signals playing simultaneously, entangled through their interference pattern. This third signal is nothing but the sum of the original two signals:

$$f_3(t) \ := \ f_1(t) + f_2(t).$$

An m-note chord is a sum

$$f_1(t) + f_2(t) + \cdots + f_{m-1}(t) + f_m(t), \tag{3.2.3.1}$$

where each term is the periodic audio signal of one note on one instrument. A just intoned major triad is any sum

$$f(t) + f\left(\tfrac{5}{4}t\right) + f\left(\tfrac{3}{2}t\right),$$

where $f(t)$ is the periodic audio signal of one note on one instrument.

In close analogy with the case of continuous audio signals, given a pair of discrete audio signals $s_1(i)$ and $s_2(i)$, there exists a third audio signal $s_3(i)$ that consists of the original pair of signals playing simultaneously, entangled through a discrete interference pattern. This third signal is the sum of the original signals:

$$s_3(i) \ := \ s_1(i) + s_2(i).$$

The above statements about chords and sums carry over to discrete signals.

Problem 3.2.4. Interference scale in tonal theories of percussion. The *problem of interference scale in tonal theories of percussion* is my observation that for many tonal theories of percussion, the sum of audio signals does not accurately represent what we hear when we listen to percussion ensembles.

In a tonal theory of percussion where a metronome keeping tempo is an instance of a "fundamental tone," normal interference of audio signals will not provide the kind of consonance/dissonance phenomena that will allow the tonal theory to perform at the same level as standard tonal harmony. If the tempo R of a drum playing a regular rhythm satisfies $60 \leq R \leq 180$ bpm, then the conversion (3.1.0.1) tells us that as a fundamental tone, the rhythm is on the order of $1 \leq 60R \leq 3$ Hz. Standard tonal information, that is, tonal information in the frequency range $27 \leq \lambda \leq 4200$ Hz sits in the range the 3rd overtone of the 3rd overtone of the 3rd overtone of the fundamental. We might say that standard tonal information has a purely timbral function within any tonal theory of percussion.

This means that if we compose with percussive audio signals using a tonal theory of percussion, the standard interference phenomena for audio signals will amount to timbral interference. Interference phenomena will not occur at a time scale that will allow the tonal theory to guide our movement through interference patterns of varying regularity.

3.3. **Tonnetze for discrete audio signals.** Let us now fix an integer $\ell \in \mathbb{Z}_{>0}$, can consider the group $\mathbb{Z}/\ell\mathbb{Z}$. Every character $\chi \in \mathrm{X}(\mathbb{Z}/\ell\mathbb{Z})$ is of the form $\chi(i) = e^{\sqrt{-1}\cdot 2\pi n i/\ell}$ for some $n \in \mathbb{Z}/\ell\mathbb{Z}$. For each $n \in \mathbb{Z}/\ell\mathbb{Z}$, we define

$$\chi_n(i) \ := \ e^{\sqrt{-1}\,\frac{2\pi}{\ell}\,ni}. \tag{3.3.0.1}$$

With this indexing convention, we have the identity

$$\chi_n(mi) \ = \ \chi_{mn}(i) \quad \text{for all } m \in \mathbb{Z}/\ell\mathbb{Z}. \tag{3.3.0.2}$$

The identities (3.3.0.2) constitute a system of overtone relationships for discrete periodic audio signals, analogous to the system of overtone relationships for continuous periodic audio signals. In much the same way we did in §2.2 for the topological group $\mathbb{S}^1$, we can depict this system of relationships as a diagram consisting of arrows between the characters $\chi \in \mathrm{X}(\mathbb{Z}/\ell\mathbb{Z})$, labelled by elements $m \in \mathbb{Z}/\ell\mathbb{Z}$. Nodes in our diagram are symbols "χ_n," for $n \in \mathbb{Z}/\ell\mathbb{Z}$. For any node χ_n and any $m \in \mathbb{Z}/\ell\mathbb{Z}$ of our choosing, if "χ_{mn}" also appears in our diagram, then we draw an arrow labelled "m," running from χ_n to χ_{mn}:

$$\chi_n \xrightarrow{\quad m \quad} \chi_{mn} \tag{3.3.0.3}$$

The arithmetic behavior of frequency for discrete signals, observed in §3.2.1, suggests that the system of overtone relationships for discrete audio signals will be more complicated to organize. Indeed, versions of the resulting cadential diagrams for $\ell = 8$ and 12 appear in Figures 7 and 8. Further examples, relevant to the two simple musical ideas underlying *Voice Leading & Beatmatching Study*, appear in Figure 11 in §4.2, and Figure 16 in §5.2.

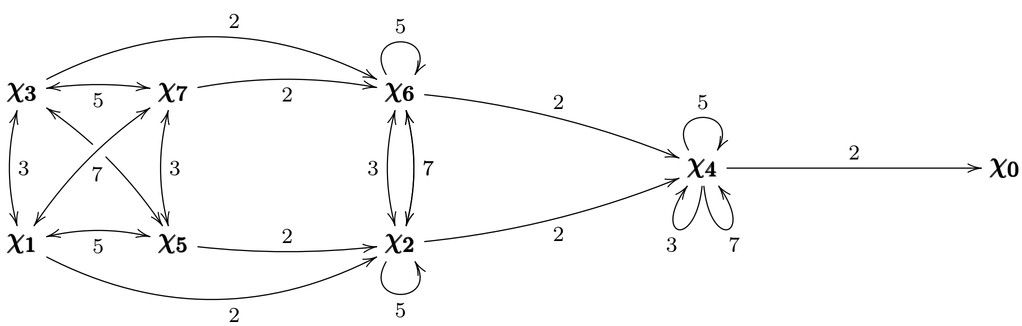


Figure 7. Cadential movement for order-8 discrete audio signals. This diagram is an analogue of the tonnetz that appears at left in Figure 1. The symbols "χ_n" are the characters of $\mathbb{Z}/8\mathbb{Z}$. We include every arrow $\chi_n \xrightarrow{m} \chi_{mn}$, for $m \in \mathbb{Z}/\ell\mathbb{Z}$, for $m = 2$, 3, 5, and 7, where the product mn is now taken in $\mathbb{Z}/8\mathbb{Z}$. Bi-directional arrows $\chi_i \xleftrightarrow{m} \chi_j$, for $m \in \mathbb{Z}/\ell\mathbb{Z}$ indicate that $mi = j$ and $mj = i$ in $\mathbb{Z}/8\mathbb{Z}$.

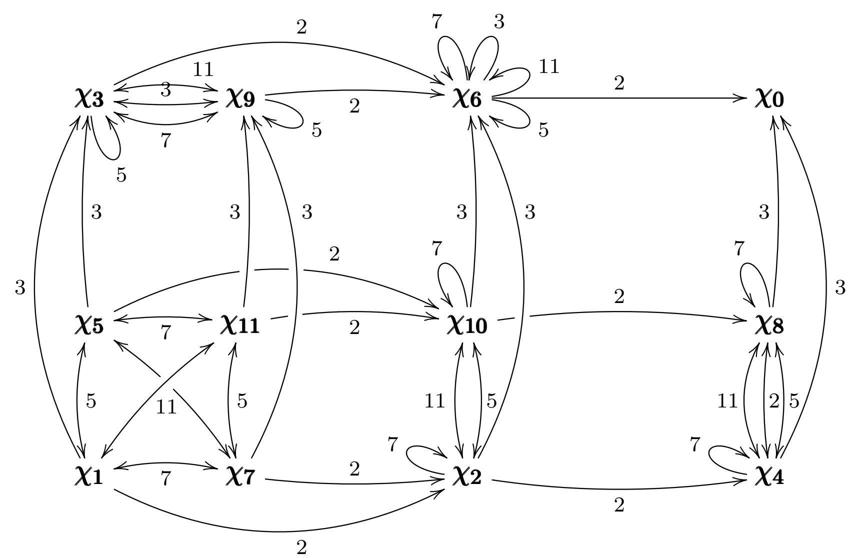


Figure 8. Cadential movement for order-12 discrete audio signals. The symbols "χ_0" through "χ_{11}" are the characters of $\mathbb{Z}/12\mathbb{Z}$. We include every arrow $\chi_n \xrightarrow{m} \chi_{mn}$, for $m \in \mathbb{Z}/\ell\mathbb{Z}$, for $m = 2$, 3, 5, 7, and 11, where the product "mn" is now taken in $\mathbb{Z}/12\mathbb{Z}$. Bi-directional arrows $\chi_i \xleftrightarrow{m} \chi_j$ indicate that $mi = j$ and $mj = i$ in $\mathbb{Z}/12\mathbb{Z}$.

4. Building the first simple musical idea.

The present §4 begins with a short development of the musical proto-idea described in §1.1, followed by a detailed description of the fully developed, first simple musical idea, along with a , including a complete list of allowed transformations and compositional rules for the idea.

4.1. **Construction for first idea.** As noted in §1.1, some $\mathbb{Z}/\ell\mathbb{Z}$-tonal melodies and chords sound like complex physical movement, as opposed to $\mathbb{S}^1$-tonal melodies and chords, which sound more like sequences of changes in emotion. I think I experience this as more of a 2^{nd}-order than 1^{st}-order phenomenon, i.e., at the smallest level, it occurs for sequences (f_1, f_2, f_3) of three successive tones, rather tan sequences (f_1, f_2) of two successive tones.

Melody operates primarily in the horizontal notational dimension, i.e., along the parameter of time. For $\mathbb{S}^1$-tonal harmony, i.e., standard tonal harmony, the vertical notational dimension describes an acoustic version of the construction of (truncated) Fourier series, in the form of chords of musical tones. We can think of chords as being "tone-like musical signals that are more complex than individual tones." If musical tones evoke emotion, then chords evoke more complex emotions. This suggests that chords in $\mathbb{Z}/\ell\mathbb{Z}$-tonality might evoke "more complex movement."

It's easy to reverse-engineer this line of thinking, i.e., to start with an interesting example of complex repetitive movement, and then write down a $\mathbb{Z}/\ell\mathbb{Z}$-tonal chord that captures aspects of the complex movement. Figure 9 provides an example of repetitive movement that holds musical interest for me.

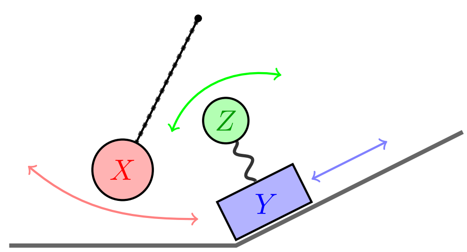


Figure 9. Movement & signal triad. Pendulum X swings back and forth. At its lowest point, pendulum X collides with block Y, propelling block Y up the short ramp. A third mass Z is attached to black Y by a spring, and Y's movement triggers a brief, decaying oscillation in Z. To clarrify, I'm not interested in the strict physics of this situation, which involves issues of friction, dampening, and and chaotic dynamics. Rather, I'm interested in the physics of this situation to the extent that I can, in my head, do a kind of "kinematic audiation" of what the movement in this situation might be like. See Figure 10 below.

I can easily imagine a triad of discrete audio signals that vaguely depict the movement in Figure 9. I imagine myself standing to the right of the configuration. I imagine each of the objects X and Z as generating its own fixed tone or pulse, with amplitude proportional to its speed, pitched shifted via a Doppler-like effect relative to my position. I imagine Y to generate sound via its friction with the surface under it, subject to the same sort of quasi-Doppler phenomenon. I fix $\ell = 36$, so that my resulting triad will consist of the real parts of audio signals of the form

$$f : \mathbb{Z}/36\mathbb{Z} \longrightarrow \mathbb{C}. \tag{4.1.0.1}$$

The resulting, *intuitively constructed* discrete audio signals appear in Figure 10 below.

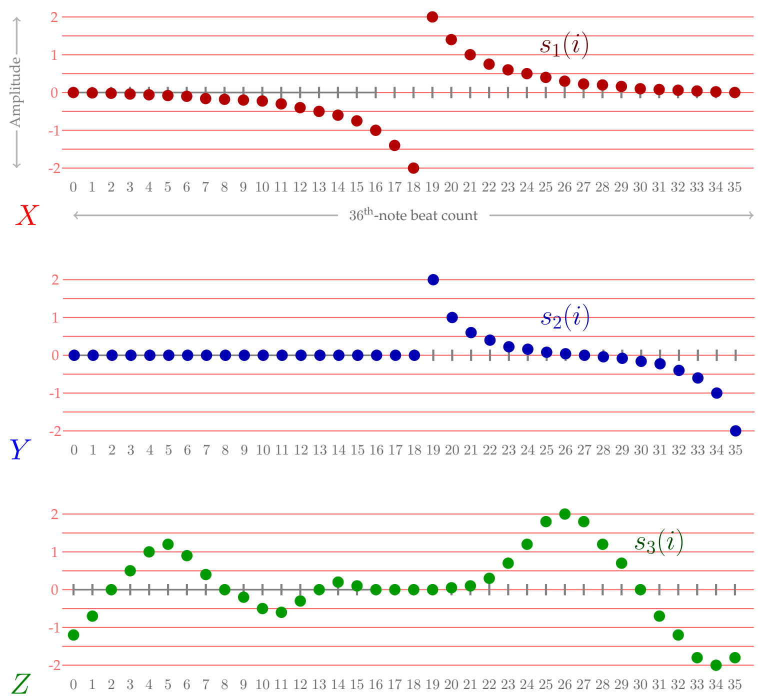


Figure 10. $\mathbb{Z}/36\mathbb{Z}$-tonal triad "approximating" Figure 9. There isn't any kind of strict physical reasoning happening in deriving these three $\mathbb{Z}/36\mathbb{Z}$-tones from Figure 9, but the loose idea is to imagine what it would sound like if each of the objects X, Y, and Z in Figure 9 generated an $\mathbb{S}^1$-tone that meets a listeners ear with Doppler-shifts.

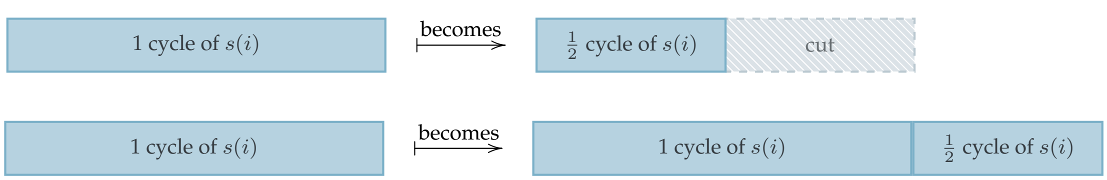
4.2. **Simple musical idea 1.** Use the $\mathbb{Z}/36\mathbb{Z}$-tones in the triad from Figure 10 as a set of basic musical objects, subject to the 4 transformation rules that appear in §4.2.1, and subject to the 2 further composition rules that appear 4.2.2.

A list of poetics, for composing with the simple musical idea in this and the next §5, appears in §7.

4.2.1. Allowed transformations & compositional moves for simple idea 1. Items 4.2.1.i through 4.2.1.iv constitute a complete, self-imposed list of allowed transformations and compositional moves for writing the piece.

Keep in mind that combinations of these transformations can produce quite complex results. Some of the restraint and management of this complexity comes from the further compositional rules appear in §4.2.2 below.

4.2.1.i. Changing the real time duration of any $\mathbb{Z}/36\mathbb{Z}$ signal by cutting or concatenating by increments in multiples of δ. This amounts to the basic cut-and-paste operations for building longer and shorter signal durations. We provide two simple examples immediately below.



Note that repeated application of this transformation can be used to create the effect of any discrete time shift $s_n(i) \mapsto s_n(i+j)$, for $j \in \mathbb{Z}/36\mathbb{Z}$.

4.2.1.ii. Argument "rescaling" $s_n(i) \mapsto s_n(mi)$, for $m \in \mathbb{Z}/36\mathbb{Z}$. This is the $\mathbb{Z}/36\mathbb{Z}$ analogue of moving up or down tonal intervals. The cadential diagram, or "tonnetz" for $\mathbb{Z}/36\mathbb{Z}$-tonality appears in Figure 11. Use this to guide choices of multiplying factor m. Figure 12 below provides one example of how the signals in Figure 10 change under this type of transformation.

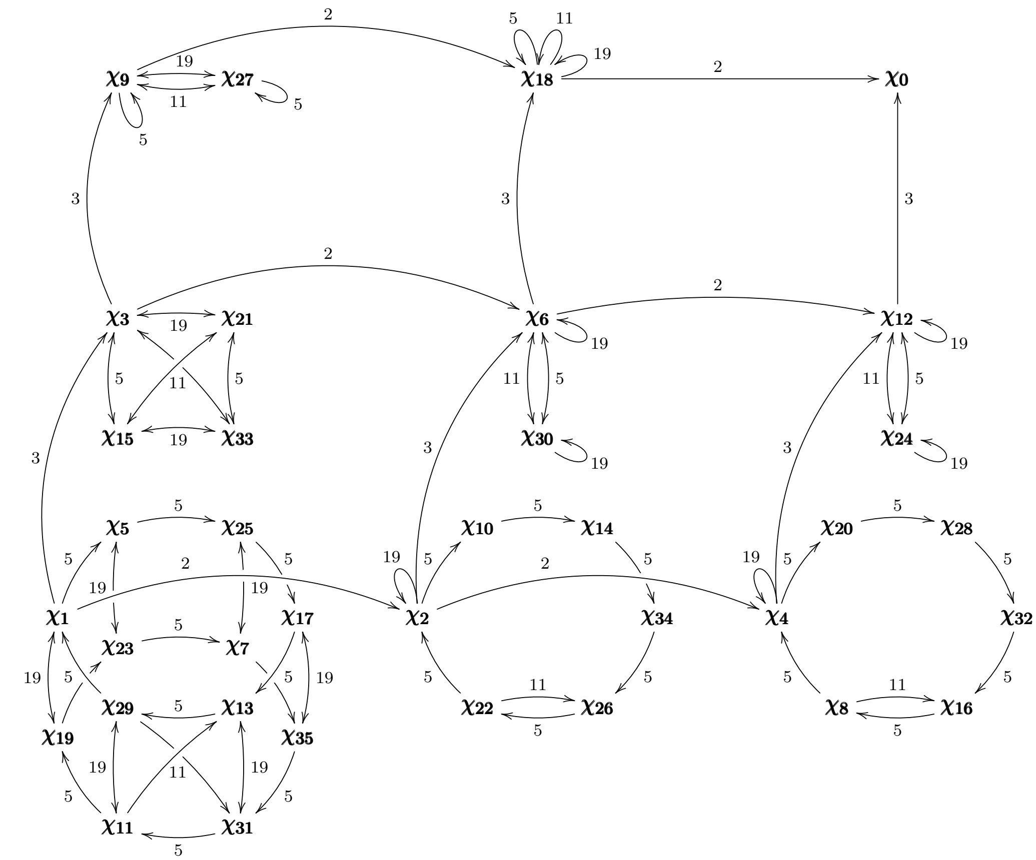


Figure 11. Cadential movement in order-36 tonality. Compare to Figures 7 and 8. This particular realization of the diagram includes very few of the possible "2"- and "3"-labelled arrows. While composing, I use this diagram as a tonal guide for which transformations $s_n(i) \mapsto s_n(mi)$ to carry out, and as a guide for how such transformations might interact in $\mathbb{Z}/36\mathbb{Z}$-melodies and -chords.

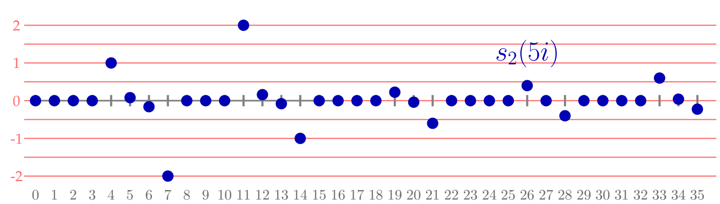
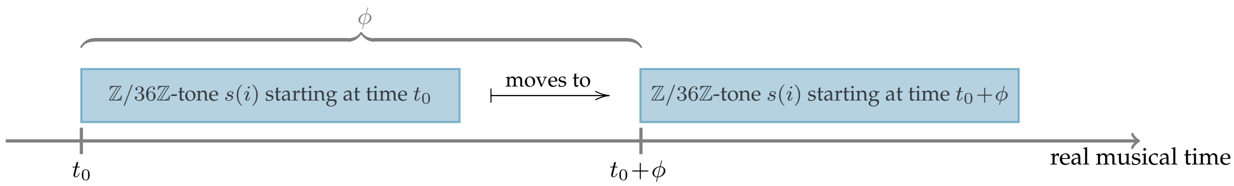


Figure 12. An example of argument rescaling for the "Y" signal $s_2(i)$ in Figure 10. The figure depicts the signal that results from Y via the transformation $s_2(i) \mapsto s_2(5i)$.

4.2.1.iii. Putting down $\mathbb{Z}/36\mathbb{Z}$-tones anywhere on the real musical timeline, along with any continuous time shift in the starting moment of any $\mathbb{Z}/36\mathbb{Z}$-tone.



4.2.1.iv. "Mild" changes in the specific percussive voice underlying one of the signals $s_n(i)$. For instance, changing from one hi-hat to a slightly different hi-hat, or changing the pitch of a bass drum.

4.2.2. Further composition rules for simple idea 1. The following additional rules guide use of the first simple musical idea as compositional material. They are intended, in part, to restrain and manage the potential explosion in complexity that can result from combinations of the transformations and moves that appear in §4.2.1.

4.2.2.i. Do not include any more variation than seems absolutely necessary!

4.2.2.ii. Some transformation $s_1(m_1 i) + s_2(m_2 i) + s_3(m_3 i)$ must occur in each of the 8 larger "cells" in the piece. See §6.

Note: I break this rule throughout the piece, already in the second cell. I mostly follow it.

Remark 4.2.3. Rapid hi-hats & rim-shots, with a tonal flavor, in trap & drill. In *Voice Leading & Beatmatching Study*, I voice the signal Y, i.e., $s_2(i)$ in Figure 10 with a hi-hat. The effect, as I experience it at least, is sort of like a cross between a tambourine and a mechanical drill.

Drill-like hi-hats and rim-shots are one of the defining features of several strands of contemporary hip hop, particularly in the closely related *trap* and *drill* sub-genres. These sub-genres focus heavily on aspects of drug dealing and gang warfare, delivered from the nihilistic and often brutal perspective of people living in that world. The term "drill," within these sub-genres, refers to both the hi-hat patterns in the music and to the act of committing a murder.

An example of a track that popularized many of the production techniques that became standard in drill music is Lex Luger and Waka Flocka Flame's *Hard In da Paint*, recorded in 2008 and released in 2010 [LF10]. For an example of how far trap/drill producers have pushed the use of rapid hi-hat patterns, listen to the Lil Durk track *Green Light*, produced by Kid Wond3r and DY Krazy, and released in 2019 [DWK19].

Comparing the hi-hats starting at about the 2 minute mark in *Voice Leading & Beatmatching Study* to the hi-hats in *Hard In da Paint* and in *Green Light* might help to clarify how fundamentally different $\mathbb{S}^1$-tonal and $\mathbb{Z}/\ell\mathbb{Z}$-signals are. In *Green Light*, the rate of the hi-hat hits is changing relative to the real time of the song. The rate doubles, triples, etc., relative to the passage of real time. In *Voice Leading & Beatmatching Study* on the other hand, the rate of the hi-hats is changing in terms of $\mathbb{Z}/36\mathbb{Z}$-modular arithmetic. The real time passage of the increments constituting the $\mathbb{Z}/36\mathbb{Z}$-signals does not change. In *Green Light*, as the rate of the hi-hats changes, their texture changes. In *Voice Leading & Beatmatching Study*, the texture of the hi-hats stays relatively fixed, and it's the their specific feeling of movement that changes as their $\mathbb{Z}/36\mathbb{Z}$-modular rate changes.

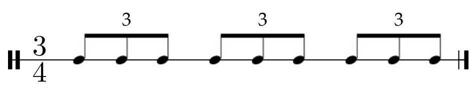
5. BUILDING SECOND SIMPLE MUSICAL IDEA

The present §5 begins with a short development of this second musical proto-idea described in §1.2 , followed by a detailed description of the fully developed, second simple musical idea, and a list of allowed transformations and compositional rules for the idea.

5.1. **Construction for second idea.** Recall from §1.2 that the second proto-idea is to build a simple but compelling $\mathbb{Z}/\ell\mathbb{Z}$-tonal chord for a small ensemble of interfering instruments, and then to use $\mathbb{Z}/\ell\mathbb{Z}$-tonal voice leading to compose for this ensemble.

Because I want to reproduce some of the compelling aspects of *interference_trial_014*, whose core voice leading structure was the tresillo rhythm realized in a measure divided by eighth notes, i.e., realized in $\mathbb{Z}/8\mathbb{Z}$-tonality, it makes sense to try realizing the second simple musical idea in $\mathbb{Z}/\ell\mathbb{Z}$-tonality for relatively small ℓ. In an effort to encourage interaction with the first simple musical idea, provided in §4, we note that $36 = 2^2 \cdot 3^2$, and choose $\ell = 3^2 = 9$ for the second simple musical idea. In other words, the central musical object for the second simple musical idea will be a $\mathbb{Z}/9\mathbb{Z}$-tonal signal.

If the tempo at which we implement it is slow enough, a $\mathbb{Z}/9\mathbb{Z}$-tone is essentially a rhythm. To ensure that it functions well as a rhythm, I construct the $\mathbb{Z}/9\mathbb{Z}$-tone as a rhythm. I'm working in $\mathbb{Z}/9\mathbb{Z}$-tonality at the moment, so I'm trying to construct a musically interesting beat in measure divided into "ninth notes." I imagine the 9 beats in the measure to be collected into 3 sets of consecutive triplets. Said differently, we can notate the measure in $\frac{3}{4}$-time as follows:



(5.1.0.1)

Then the task of coming up with the specific beat amounts, roughly, to choosing which notes in (5.1.0.1) rest and which are accented. I decided to impose one requirement: 2 beats in each triplet must rest, so that only one of the 3 beats in each triplet sounds. Figure 13 depicts the signal I finally settled on. I choose the negative amplitudes in Figure 13 to make the discrete Fourier give better results.

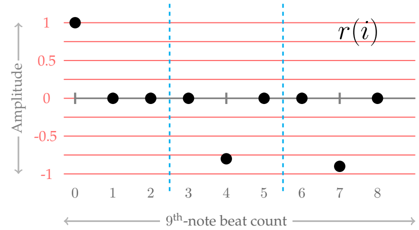


Figure 13. Rhythmic $\mathbb{Z}/9\mathbb{Z}$-tonal signal. The dotted blue lines depict the division of the rhythm into 3 triplets. Two beats in each triplet are silent, and the other is not. The pattern is: 1st beat sounds in 1st triplet, and 2nd beat sounds in the other two triplets.

The 9 characters χ_0, χ_1, through χ_8 of $\mathbb{Z}/9\mathbb{Z}$ constitute the "pure tones" for $\mathbb{Z}/9\mathbb{Z}$-tonality. With respect to these pure tones, the rhythm in Figure 13 is in fact a chord. In mathematical language, letting $r(i)$ denote the signal depicted in Figure 13, we can write $r(i)$ as a sum:

$$r(i) \;=\; \mathrm{Re}\big(c_0\,\chi_0(i)\big) + \mathrm{Re}\big(c_1\,\chi_1(i)\big) + \cdots + \mathrm{Re}\big(c_8\,\chi_8(i)\big). \tag{5.1.0.2}$$

Here, each coefficient c_n, for $0 \le n \le 8$, is a complex number, and $\mathrm{Re}(c_n\,\chi_n(i))$ denotes the real part of the complex-valued function

$$c_n\,\chi_n \,:\, \mathbb{Z}/9\mathbb{Z} \longrightarrow \mathbb{C}.$$

To compute the coefficients c_n that appear in (5.1.0.2), we use the discrete Fourier transform:

$$\begin{aligned} c_n &= \langle r, \overline{\chi_n} \rangle \\ &:= r(0)\cdot\overline{\chi_n(0)} + r(1)\cdot\overline{\chi_n(1)} + \cdots + r(8)\cdot\overline{\chi_n(8)}. \end{aligned} \tag{5.1.0.3}$$

Running the calculation (5.1.0.3) and obtaining the resulting real parts in (5.1.0.2) is a simple application of complex linear algebra. I did the calculation in an Excel sheet.[2] Because some of the characters χ_n come in complex conjugate pairs, whereas real parts are invariant under complex conjugation, the computation results in a 5-term decomposition into a sum of real-valued signals $r_0(i)$ through $r_4(i)$:

$$r(i) = r_0(i) + r_1(i) + r_2(i) + r_3(i) + r_4(i).$$

The signals $r_0(i)$ through $r_4(i)$ in this decomposition appear in visual form in Figure 14 below.

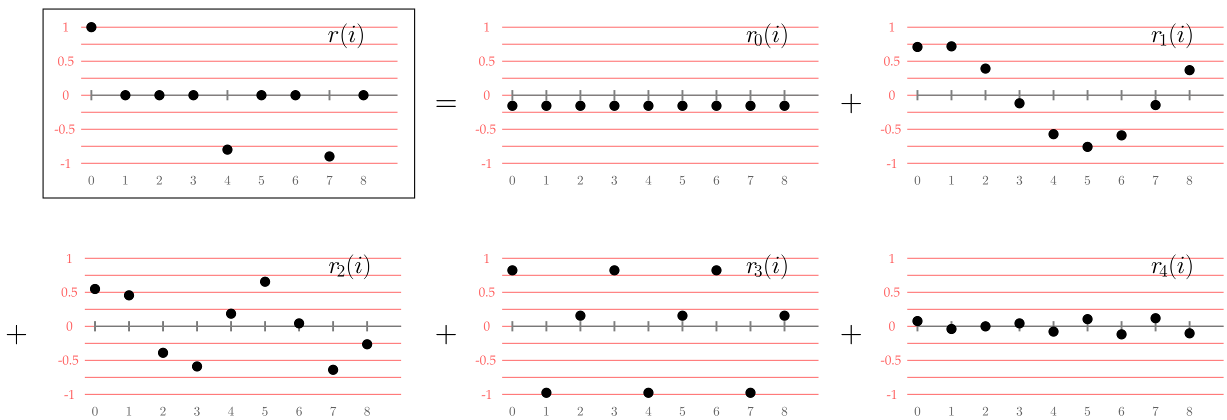


Figure 14. The rhythm from Figure 13, decomposed as an interference pattern / chord. Here $r_0(i) = \mathrm{Re}(c_0\,\chi_0)$, and the other 4 signals $r_1(i)$ through $r_4(i)$ that appear are all of the form $r_n(i) = 2 \cdot \mathrm{Re}(c_n\,\chi_n)$, where all of the coefficients c_0 through c_4 are computed using the discrete Fourier transform (5.1.0.3). For $1 \le n \le 8$, we have $\mathrm{Re}(c_n\,\chi_n) = \mathrm{Re}(c_{9-n}\,\chi_{9-n})$, which is why the decomposition in this figure has only 5 terms, rather than the 9 terms we expect from (5.1.0.2).

5.1.1. Interference at the scale of individual strikes In §3.2.3 and Problem 3.2.4, we explained that the time scale at which standard audio signals interfere is not conducive to a tonal theory of rhythm that makes use of $\mathbb{Z}/\ell\mathbb{Z}$-signals. Conversely, interference patterns produced by a percussion ensemble with interference at the scale of individual notes will behave like chords in tonal harmony.

We can hypothesize that important aspects of rhythmic composition will emerge if we work with percussion ensembles wherein the instruments interfere with one another at the scale of individual strikes. Ensembles with interference at the scale of individual strikes should provide a strong analogues of voice leading, cadences, and modulations of "key," as well as musical possibilities that function like a "sub-tonal timbre."

Figure 15 depicts a way to realize such an ensemble by feeding signals through an interlaced network of side-chained signals coming from "±-pairs" of instruments running through a bank of corresponding "±-pairs" of compressors. This setup is relatively easy to implement in a D.A.W., and is easy to extend to any number of instruments, all dampening one another according to rules given in the caption to Figure 15.

[2] An Excel sheet helps me see more of the details of the calculation, which can be helpful for implementing the results in a D.A.W..

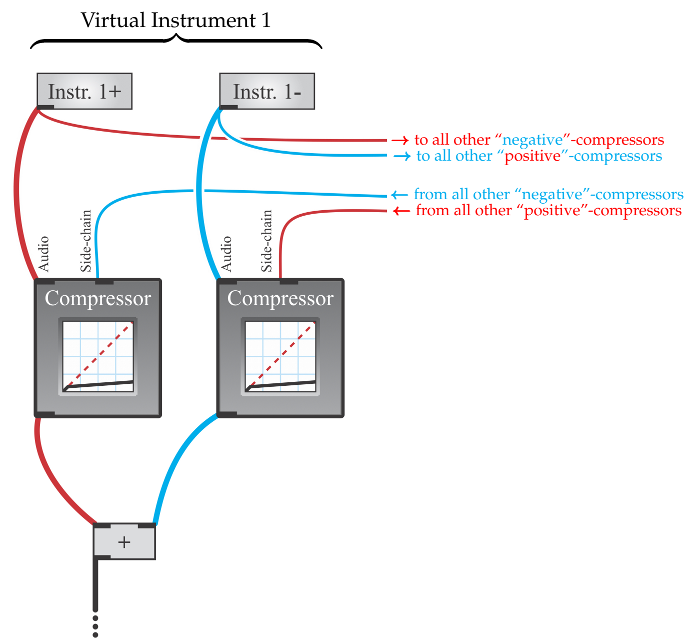


Figure 15. A single ±-instrument pair for interferening at individual strikes. As indicated, this pair plugs into a network of such ±-pairs. Together, the network of ±-pairs forms an ensemble of virtual instruments that interfere at the scale of individual strikes.

5.2. **Simple musical idea 2.** Use the $\mathbb{Z}/9\mathbb{Z}$-tones $r_n(i)$ in the 5-note chord from Figure 14 as a set of basic musical materials. Voice each of the $\mathbb{Z}/9\mathbb{Z}$-tones $r_n(i)$, for $0 \leq n \leq 4$, with a ±-pair of electric pianos with underlying $\mathbb{S}^1$-tonal structure as developed in §6. Run these ±-pairs through the interference compressor system depicted in Figure 15. Use the resulting ensemble of $\mathbb{Z}/9\mathbb{Z}$-tones for *voice leading with untuned percussion*, i.e., voice leading with respect to the $\mathbb{Z}/9\mathbb{Z}$-nature of the signals, rather than the $\mathbb{S}^1$-nature of the signals.

Composing with these signals is subject to the transformation rules that appear in §5.2.1, and subject to the further composition rules that appear 4.2.2.

5.2.1. **Allowed transformations & compositional moves for simple idea 2.** Items 5.2.1.i through 5.2.1.iv constitute a complete, self-imposed list of allowed transformations and compositional moves for writing the piece.

5.2.1.i. Changing the real time duration of any $\mathbb{Z}/9\mathbb{Z}$-tone by cutting or concatenating by fundamental increments. This amounts to the basic cut-and-paste operations for building longer and shorter signal durations. We provide two simple examples immediately below.

Repeated application of this transformation can be used to create the effect of any discrete time shift $r_n(i) \mapsto r_n(i+j)$, for $j \in \mathbb{Z}/9\mathbb{Z}$.

5.2.1.ii. Argument "rescaling" $r_n(i) \mapsto r_n(mi)$, for $m \in \mathbb{Z}/9\mathbb{Z}$. This is the $\mathbb{Z}/36\mathbb{Z}$ analogue of moving up or down tonal intervals. The cadential diagram, or "tonnetz" for $\mathbb{Z}/9\mathbb{Z}$-tonality appears in Figure 16. Use this to guide choices of multiplying factor m.

5.2.1.iii. Putting down $\mathbb{Z}/9\mathbb{Z}$-tones anywhere on the real musical timeline, along with any continuous time shift in the starting moment of any $\mathbb{Z}/9\mathbb{Z}$-tone.

5.2.1.iv. Changes in the $\mathbb{S}^1$-tone underlying one of the signals $r_n(i)$.

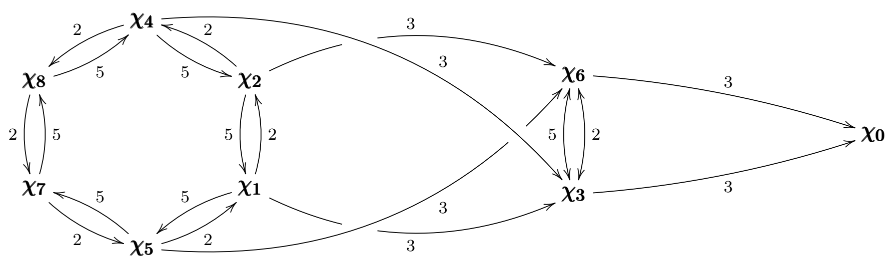


Figure 16. Cadential movement in order-9 tonality. Compare to Figures 7, 8, and 11. While composing, I use this diagram as a tonal guide for how transformations $r_n(i) \mapsto r_n(mi)$ might interact in $\mathbb{Z}/9\mathbb{Z}$-melodies and -chords.

5.2.2. Further composition rules for simple idea 2. The following additional rules guide use of the first simple musical idea as compositional material. They are intended, in part, to restrain and manage the potential explosion in complexity that can result from combinations of the transformations and moves that appear in §4.2.1.

5.2.2.i. Do not include any more variation than seems absolutely necessary!
5.2.2.ii. Some transformation $r_1(m_1 i) + r_2(m_2 i) + r_3(m_3 i) + r_4(m_4 i)$, for $m_1, m_2, m_3, m_4 \in \mathbb{Z}/9\mathbb{Z}$, must occur in each of the 8 larger "cells" in the piece. See §6.

6. Building the $\mathbb{S}^1$-tonal structure of the piece.

I use a slow sequence of $\mathbb{S}^1$-tonal chords, i.e., chords in standard tonal harmony, to underly the development of the second simple musical idea over the course of the piece. This sequence of chords appears in a sketch that I never did anything with. I don't think I've shared it. The sketch has a spectral music kind of flavor, playing with voice leading that runs up into the overtone series. I employ the sequence rather differently for the present piece, as explained below.

6.1. **A 27-note scale & highly symmetric 12-note sub-scale thereof.** As explained in §2.2, the main overtones driving standard tonal harmony, i.e., driving $\mathbb{S}^1$-tonal harmony, are the 2^{nd}-overtone (exactly one octave), the 3^{rd}-overtone (approximately a perfect fifth plus one octave), and the 5^{th}-overtone (approximately a major third plus two octaves). In terms of a continuous periodic audio signal $f(t)$, moving up to the 3^{rd}-overtone of $f(t)$ means carrying out the transformation $f(t) \longmapsto f(3t)$. The signal $f(3t)$ will be approximately one octave plus a perfect fifth above the signal $f(t)$. Moving up to the 5^{th}-overtone of $f(t)$ means carrying out the transformation $f(t) \longmapsto f(5t)$. Movement up to an overtone is also called movement up to a *harmonic*. Moving in the respective opposite directions, i.e., moving to *subharmonics* means carrying out the transformations $f(t) \longmapsto f\left(\frac{1}{3}t\right)$ or $f(t) \longmapsto f\left(\frac{1}{5}t\right)$.

We can produce a short list of all possible tonal movement possible if we're allowed to make the following tonal moves: **(a)** movement up to the 2^{nd}-harmonic or down to the 2^{nd} subharmonic, **(b)** movement up to the 3rd-harmonic or down to the 3^{rd} subharmonic, and **(c)** movement up to the 5^{th}-harmonic or down to the 5^{th} subharmonic. Packaged somewhat differently, this is the collection of all moves $f(t) \longmapsto f(2^a \cdot 3^b \cdot 5^c \cdot t)$, for $a, b, c \in \{-1, 0, 1\}$. There are $3^3 = 27$ possible transformations here. Thinking of the transformation types as "multiplication by $2^{\pm 1}$," "multiplication by $3^{\pm 1}$," and "multiplication by $5^{\pm 1}$," as three independent dimensions, we can decorate a cube with the 27 transformations. Figure 17 depicts this decorated cube. We can think of the 27 transformations decorating this cube as notes in a 27-note scale.

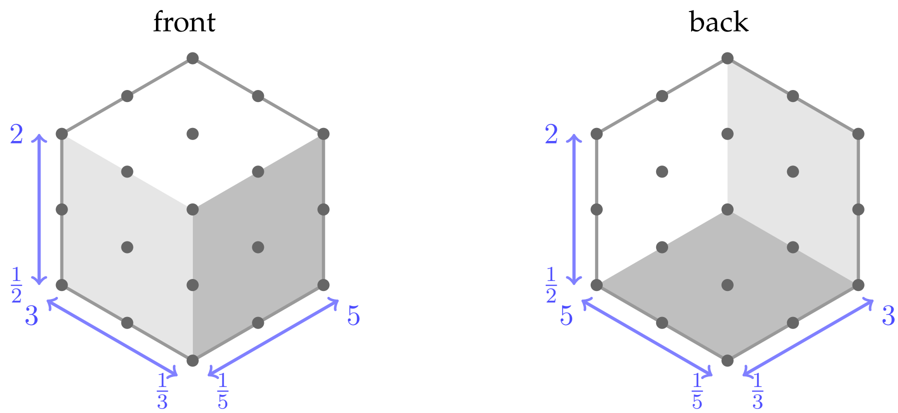
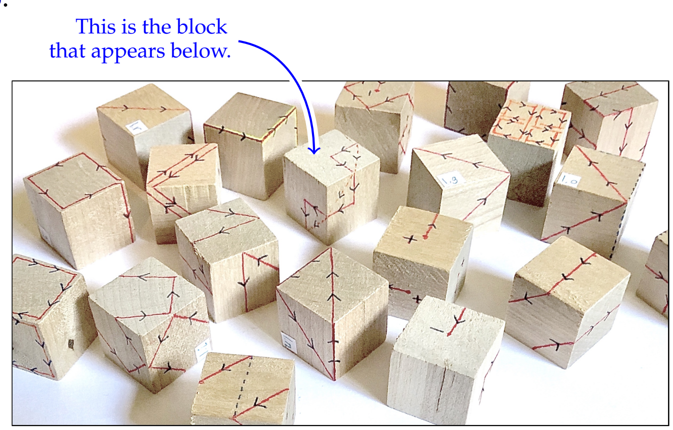


Figure 17. 27-note scale decorating a cube.

The cube provides a convenient way to conceptualize symmetries that occur when we work with three independent parameters of tonal movement. During the fall of 2020, I bought a bunch of little blocks and used them to come up with as many compelling, symmetric closed paths, i.e., *cycles*, on the cube as I could. One of the more compelling cycles I cam up with appears in Figure 18.



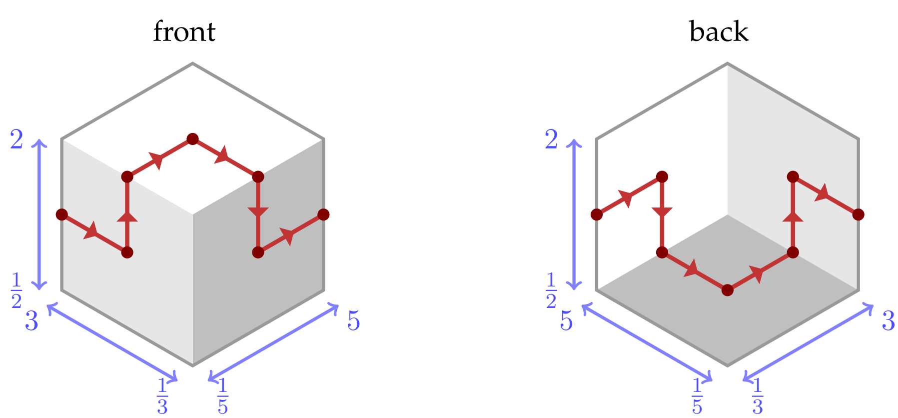


Figure 18. Highly symmetric, 12-note cycle on the cube from Figure 17.

Interpreted as a cycle of pitch classes, the 12-note cycle at bottom in Figure 18 determines a sequence of Major 6, Major 7, and Major triad add minot third chords, via the short connected paths in Figure 2. This sequence of chords appears in Figure 19 below.

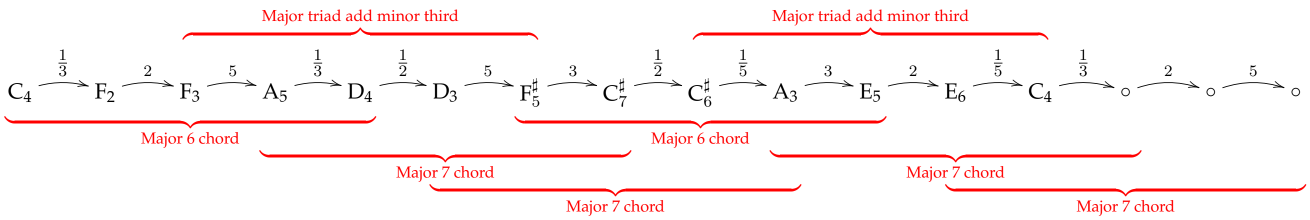


Figure 19. Cycle of Major 6, Major 7, and minor-add-m3 chords from 27-note scale. This is the cycle that appears at bottom in Figure 18, after assigning A_4 as the "tonic" at the center of the cube. Compare to Figure 2. Because the tonic of this 27-note scale is the point at the center of the cube, the 12-note cycle does not contain the tonic of the 27-note scale. This contributes to a sense that the cycle is wandering in kind of a rootless way.

I think I it is relatively easy to hear how I'm using this cycle over the course of the piece.

7. Poetics

By *poetics*, I mean the manner in which the different components of a musical composition interact to effect the listener. [...]

7.1. **Repetition.**

7.1.1. Ostinato Maybe the most obvious form of musical repetition is ostinato, including its extreme contemporary form: digital/electronic looping.

The ideas in §3.1 that suggest that the boundary between tones and loops/vamps is not as distinct as standard musical language might indicate. This suggests an interesting potential for $\mathbb{Z}/\ell\mathbb{Z}$-tonal musical constructions to play with this ambiguity. [...]

7.1.2. Imitation Imitation [...]

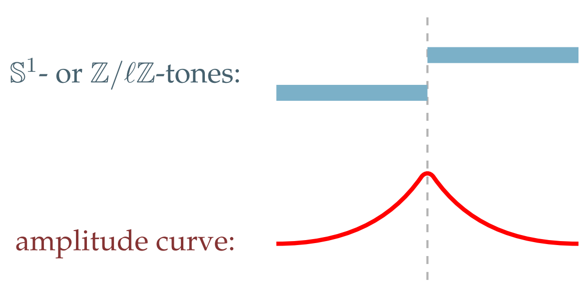


Figure 20. Musical structure imitated throughout the piece. I'm inclined to call this structure "Doppler-like." Instances of it already appear in Figure 10 and Figure [...]. Throughout the piece, loose imitations of this structure are combined with the hand-offs described in §7.2.2.

Another example occurs in the $\mathbb{Z}/9\mathbb{Z}$-voice leading. If we interpret the 4 individual $\mathbb{Z}/9\mathbb{Z}$-tones in this voice leading as individual rhythms played on a single pitch-wavering electric piano note, then permuting the 4 $\mathbb{S}^1$-tones underlying 4 $\mathbb{Z}/9\mathbb{Z}$-tones in a $\mathbb{Z}/9\mathbb{Z}$-chord is equivalent to trading which rhythm a given electric piano note plays. In other words, permuting underlying $\mathbb{S}^1$-tones in a $\mathbb{Z}/9\mathbb{Z}$-chord has the auditory effect of moving an attacco through a fixed set of electric piano "voices."

Imitations with increases or decreases of complexity. [...]

Remark 7.1.3. The virtues & dangers of imitating for the purpose of accenting. [...]

7.2. **Juxtaposition.** [...]

7.2.1. Synchronization & de-sychronization of musical events. [...]

7.2.2. Segue. Hand-off without pronounced imitation. "continue (the next section) without a pause"

Juxtapositions in a small collection of segue types almost functions as one motif throughout the piece.

(i). Elastic collision as a segue. This sort of hand-off already makes a "physical" appearance as transfer-of-momentum in the elastic collisions that occur in the movement triad depicted in Figure 9. It appears in "musical" form in the $\mathbb{Z}/36\mathbb{Z}$-tonal triad in Figure 10.

(ii). $\mathbb{S}^1$-key modulation as segue. A

(iii). Texture segue in a single voice-group. A A

$\mathbb{Z}/\ell\mathbb{Z}$-timbral beat matching = beat matching. A A

[...]

Remark 7.2.3. The technique in point [...]

7.3. **$\mathbb{Z}/\ell\mathbb{Z}$-tonally induced diminution.**

Given a $\mathbb{Z}/\ell\mathbb{Z}$-tonal signal $s(i)$ and an integer $2 \leq m \leq \ell$, the first $\lceil \ell/m \rceil$ increments of the signal $s(mi)$ provide a weak reproduction of the original signal $s(i)$. The extent to which this is perceptible depends on the m-relative detail of $s(i)$. [...]

7.4. **Gradually expanding $\mathbb{Z}/\ell\mathbb{Z}$-tonal movement.** [...]

7.5. **Gradual shifting by $\mathbb{Z}/9\mathbb{Z}$- & $\mathbb{Z}/36\mathbb{Z}$ parallel consecutive 5's.** [...]

7.6. **Phase-shifting slice of a $\mathbb{Z}/\ell\mathbb{Z}$-tone.** [...]

7.7. **Variation.** [...]

7.7.1. Variation of individual strike shape for $\mathbb{Z}/\ell\mathbb{Z}$-signals. [...]

7.8. **Just drums.** [...]

7.9. **Gentleness, Gentility, Harshness, Brutality.** [...]

7.10. **Purposefully making choices that read as slightly "weak" or "poor".** [...]

7.10.1. Choosing voice amplitudes to make one voice struggle against another. [...]

7.10.2. Tempo rubato. [...]

7.11. **Intermittent critical distance.** [...] Writing this text is part of my attempt to force myself to keep stronger critical distance while compositing the piece.
[Par17]

7.11.1. Ignoring immediate / short-term auditory memory. [...]

Remark 7.11.2. After the preparation of the materials above, I constructed the piece in D.A.W. very directly and pretty intuitively. One result of this process is that I can hear myself getting more and more control of the materials as the song progresses. It becomes one developmental dimension of the piece.

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