NOTES ON

GALOIS ACTIONS ON ALGEBRAICALLY TEMPERED SCALES

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Contents

1. Mathematical setting

1.1. **Tonic frequency as a vector.** At the outset, we take a *frequency* to be some kind of unit that we can rescale, as in the operation

$$\lambda\,\mathrm{Hz} \quad\longmapsto\quad 2\lambda\,\mathrm{Hz}, \tag{1.1.0.1}$$

and that we can add, as in the operation

$$\lambda\,\mathrm{Hz},\quad \mu\,\mathrm{Hz} \quad\longmapsto\quad \lambda+\mu\,\mathrm{Hz}. \tag{1.1.0.2}$$

The ability to rescale corresponds to the ability to change pitch, to create musical movement along intervals. The ability to add pitches together is perhaps a bit suspect from a musical perspective, but it does correspond to nonlinear distortion phenomenon that occur in music, called *Tartini tones*.[1]

The operations (**??**) and (**??**) make frequencies look like elements in vector spaces. We will treat them as such.

Fix a frequency

$$\lambda\,\mathrm{Hz} \tag{1.1.0.3}$$

once and for all.

1.2. **Just intoned scales as $\mathbb{Q}$-vector spaces.** This frequency (**??**) will serve as a *generator* for various scales that we will construct. Initially, we want to be somewhat agnostic about what it means to "generate a scale from a frequency λ Hz," as we will end up giving several incompatible interpretations of what it means to "generate a scale."

Example 1.2.1. Just intoned scale with tonic $\boldsymbol{\lambda}$ Hz. If we take λ Hz to be our tonic, then the commonly encountered *just intervals over this tonic* occur at the frequencies

$$2\lambda\,\mathrm{Hz},\qquad \tfrac{3}{2}\lambda\,\mathrm{Hz},\qquad \tfrac{4}{3}\lambda\,\mathrm{Hz},\qquad \tfrac{5}{4}\lambda\,\mathrm{Hz},\qquad \tfrac{6}{5}\lambda\,\mathrm{Hz},\qquad \text{and}\qquad \tfrac{9}{8}\lambda\,\mathrm{Hz}.$$

These are all instances of multiples $\frac{m}{n}\lambda$ Hz by rational numbers $\frac{m}{n}\in\mathbb{Q}$ with relatively small *naive multiplicative height*. The set of all $\mathbb{Q}$-multiples of the frequency λ Hz is the $\mathbb{Q}$-vector space

$$V \;=\; \mathbb{Q}\lambda\,\mathrm{Hz}. \tag{1.2.1.1}$$

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[1] "In a power chord, the ratio between the frequencies of the root and fifth are very close to the just interval 3:2. When played through distortion, the intermodulation leads to the production of partials closely related in frequency to the harmonics of the original two notes, producing a more coherent sound. The intermodulation makes the spectrum of the sound expand in both directions, and with enough distortion, a new fundamental frequency component appears an octave lower than the root note of the chord played without distortion, giving a richer, more bassy and more subjectively 'powerful' sound than the undistorted signal. Even when played without distortion, the simple ratios between the harmonics in the notes of a power chord can give a stark and powerful sound, owing to the resultant tone (combination tone) effect." — Wikipedia, *Power chord*

If we think of "a just-intoned scale with tonic λ Hz" as any collection of rational multiples of λ Hz, then the vector space (**??**) is the smallest just-intoned scale containing λ Hz and containing all just intervals over all of its pitches. We call (**??**) the *complete just scale generated by the pitch λ Hz.*

1.3. **Alternate perspective: Frequencies are vectors in a Lie algebra.** The real reason we use frequencies is

$$f(t) \;=\; Ae^{i\,2\pi\,\lambda\,t}$$

[...]

$$Ae^{i\,2\pi\,\lambda\,t} \;=\; e^{\log A + i\,2\pi\,\lambda\,t}$$

[...]

$$\begin{aligned} u &:= \log A \\ v &:= 2\pi\lambda t \end{aligned}$$

[...]

$$e^{u+iv} = e^{z}.$$

[...]

$$e^{(-)} : \mathbb{C} \twoheadrightarrow \mathbb{C}^{\times}$$

[...]

$$\exp : \mathfrak{g} \longrightarrow G$$

[...]

- **Main premise.** A scales is a vector space V that comes with natural "exponentiation" map $\exp : V \longrightarrow \mathbb{C}^{\times}$ to

2. The Galois group of the 12-ET scale

[...]

$$\mathbb{Q} \hookrightarrow \mathbb{Q}(i)$$

[...]

$$\mathbb{Q}(i) \xhookrightarrow{2:1} \mathbb{Q}\big(i, \sqrt{2}\big) \xhookrightarrow{2:1} \mathbb{Q}\big(i, \sqrt[4]{2}\big)$$

[...]

$$\mathbb{Q} \xhookrightarrow{2:1} \mathbb{Q}(\zeta_3) \xhookrightarrow{3:1} \mathbb{Q}\big(\zeta_3, \sqrt[3]{2}\big)$$

[...]

$$e^{\zeta_3} \;=\; e^{-1/2}\, e^{i\sqrt{3}/2}$$

3. The Galois group of the 5-ET scale

[...]

$$\mathbb{Q} \xhookrightarrow{4:1} \mathbb{Q}(\zeta_5) \xhookrightarrow{5:1} \mathbb{Q}(\zeta_5, \sqrt[5]{2})$$

[...]

$$\zeta_5 \;=\; \tfrac{-1+\sqrt{5}}{4} \;+\; i\,\tfrac{\sqrt{10+2\sqrt{5}}}{4}$$

[...]

$$\zeta_5 \;\longmapsto\; \zeta_5^2$$

[...]

$$\begin{array}{ccccc}
1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^3 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^2 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathbf{1} & \boldsymbol{\zeta_5} & \boldsymbol{\zeta_5^2} & \boldsymbol{\zeta_5^3} & \boldsymbol{\zeta_5^4}
\end{array}$$

[...]

$$A$$

4. Implementation proposals

Here we provide examples of how one might implement the above ideas.

4.1. **Absolute amplitude- and time-scales.** Given a audio signal $f(t)$, we can rescale it amplitude via

$$f(t) \quad \longmapsto \quad A \cdot f(t)$$

for any $A \in \mathbb{R}_{\geq 0}$. We can also rescale its rate via

$$f(t) \quad \longmapsto \quad f(\rho \cdot t)$$

[...]

$$e^{\log A + u} e^{i 2\pi v \rho t}$$