NOTES ON

GALOIS ACTIONS ON ALGEBRAICALLY TEMPERED SCALES

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CONTENTS

1. MATHEMATICAL SETTING

1.1. **Tonic frequency as a vector.** At the outset, we take a *frequency* to be some kind of unit that we can rescale, as in the operation

$$\lambda\,\mathrm{Hz} \quad\longmapsto\quad 2\lambda\,\mathrm{Hz}, \tag{1.1.0.1}$$

and that we can add, as in the operation

$$\lambda\,\mathrm{Hz},\quad \mu\,\mathrm{Hz} \quad\longmapsto\quad \lambda+\mu\,\mathrm{Hz}. \tag{1.1.0.2}$$

The ability to rescale corresponds to the ability to change pitch, to create musical movement along intervals. The ability to add pitches together is perhaps a bit suspect from a musical perspective, but it does correspond to nonlinear distortion phenomenon that occur in music, called *Tartini tones*.[1]

The operations (1.1.0.1) and (1.1.0.2) make frequencies look like elements in vector spaces. We will treat them as such.

Fix a frequency

$$\lambda\,\mathrm{Hz} \tag{1.1.0.3}$$

once and for all.

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[1] "In a power chord, the ratio between the frequencies of the root and fifth are very close to the just interval 3:2. When played through distortion, the intermodulation leads to the production of partials closely related in frequency to the harmonics of the original two notes, producing a more coherent sound. The intermodulation makes the spectrum of the sound expand in both directions, and with enough distortion, a new fundamental frequency component appears an octave lower than the root note of the chord played without distortion, giving a richer, more bassy and more subjectively 'powerful' sound than the undistorted signal. Even when played without distortion, the simple ratios between the harmonics in the notes of a power chord can give a stark and powerful sound, owing to the resultant tone (combination tone) effect." — Wikipedia, *Power chord*

1.2. **Just intoned scales as $\mathbb{Q}$-vector spaces.** This frequency (1.1.0.3) will serve as a *generator* for various scales that we will construct. Initially, we want to be somewhat agnostic about what it means to "generate a scale from a frequency λ Hz," as we will end up giving several incompatible interpretations of what it means to "generate a scale."

Example 1.2.1. Just intoned scale with tonic $\boldsymbol{\lambda}$ Hz. If we take λ Hz to be our tonic, then the commonly encountered *just intervals over this tonic* occur at the frequencies

$$2\lambda \text{ Hz}, \quad \tfrac{3}{2}\lambda \text{ Hz}, \quad \tfrac{4}{3}\lambda \text{ Hz}, \quad \tfrac{5}{4}\lambda \text{ Hz}, \quad \tfrac{6}{5}\lambda \text{ Hz}, \quad \text{and} \quad \tfrac{9}{8}\lambda \text{ Hz}.$$

These are all instances of multiples $\frac{m}{n}\lambda$ Hz by rational numbers $\frac{m}{n} \in \mathbb{Q}$ with relatively small *naive multiplicative height*. The set of all $\mathbb{Q}$-multiples of the frequency λ Hz is the $\mathbb{Q}$-vector space

$$V \;=\; \mathbb{Q}\lambda \text{ Hz}. \tag{1.2.1.1}$$

If we think of "a just-intoned scale with tonic λ Hz" as any collection of rational multiples of λ Hz, then the vector space (1.2.1.1) is the smallest just-intoned scale containing λ Hz and containing all just intervals over all of its pitches. We call (1.2.1.1) the *complete just scale generated by the pitch λ Hz.*

1.3. **Alternate perspective: Frequencies are vectors in a Lie algebra.** The real reason we use frequencies is

$$f(t) \;=\; Ae^{i\,2\pi\,\lambda\,t}$$

[...]

$$Ae^{i\,2\pi\,\lambda\,t} \;=\; e^{\log A + i\,2\pi\,\lambda\,t}$$

[...]

$$\begin{aligned} u &:= \log A \\ v &:= 2\pi\lambda t \end{aligned}$$

[...]

$$e^{u+iv} = e^{z}.$$

[...]

$$e^{(-)} : \mathbb{C} \twoheadrightarrow \mathbb{C}^{\times}$$

[...]

$$\exp : \mathfrak{g} \longrightarrow G$$

[...]

- **Main premise.** A scales is a vector space V that comes with natural "exponentiation" map $\exp : V \longrightarrow \mathbb{C}^{\times}$ to

2. The Galois group of the 12-ET scale

[...]

$$\mathbb{Q} \hookrightarrow \mathbb{Q}(i)$$

[...]

$$\mathbb{Q}(i) \xhookrightarrow{2:1} \mathbb{Q}\big(i, \sqrt{2}\big) \xhookrightarrow{2:1} \mathbb{Q}\big(i, \sqrt[4]{2}\big)$$

[...]

$$\mathbb{Q} \xhookrightarrow{2:1} \mathbb{Q}(\zeta_3) \xhookrightarrow{3:1} \mathbb{Q}\big(\zeta_3, \sqrt[3]{2}\big)$$

[...]

$$e^{\zeta_3} \;=\; e^{-1/2}\, e^{i\sqrt{3}/2}$$

3. THE GALOIS GROUP OF THE 5-ET SCALE

[...]

$$\mathbb{Q} \xhookrightarrow{4:1} \mathbb{Q}(\zeta_5) \xhookrightarrow{5:1} \mathbb{Q}(\zeta_5, \sqrt[5]{2})$$

[...]

$$\zeta_5 \;=\; \tfrac{-1+\sqrt{5}}{4} \;+\; i\,\tfrac{\sqrt{10+2\sqrt{5}}}{4}$$

[...]

$$\zeta_5 \;\longmapsto\; \zeta_5^2$$

[...]

$$\begin{array}{ccccc}
1 & \zeta_5 & \zeta_5^2 & \zeta_5^3 & \zeta_5^4 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \zeta_5^2 & \zeta_5^4 & \zeta_5 & \zeta_5^3 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \zeta_5^4 & \zeta_5^3 & \zeta_5^2 & \zeta_5 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
1 & \zeta_5^3 & \zeta_5 & \zeta_5^4 & \zeta_5^2 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathbf{1} & \boldsymbol{\zeta_5} & \boldsymbol{\zeta_5^2} & \boldsymbol{\zeta_5^3} & \boldsymbol{\zeta_5^4}
\end{array}$$

[...]

$$A$$

4. IMPLEMENTATION PROPOSAL 1: GALOIS-ORBITS OF COMPLEX EMBEDDINGS

Here we provide examples of how one might implement the above ideas.

4.1. **Absolute amplitude- and time-scales.** Given a audio signal $f(t)$, we can rescale it amplitude via

$$f(t) \;\longmapsto\; A \cdot f(t)$$

for any $A \in \mathbb{R}_{\geq 0}$. We can also rescale its rate via

$$f(t) \;\longmapsto\; f(\rho \cdot t)$$

[...]

$$e^{\log A + u} e^{i2\pi v \rho t}$$

5. IMPLEMENTATION PROPOSAL 2: LIFTS OF TATE'S ADDITIVE ζ-FUNCTION

5.1. **$\mathfrak{p}$-Adic fractional parts in number fields.** Before we described $\mathfrak{p}$-adic fractional parts of elements in number fields, it 's worthwhile to spend some time reviewing the theory for elements of $\mathbb{Q}$.

5.1.1. Fractional parts in $\mathbb{Q}$. Fix a nonzero element $a \in \mathbb{Q}$. Put this element in reduced form, so that

$$a \;=\; \pm\frac{M}{N} \quad \text{for elements} \quad M, N \in \mathbb{Z}_{>0}.$$

For each prime ideal $\mathfrak{p} = (p)$ in $\mathbb{Z}$, there exist units $u_{M,p}$ and $u_{N,p}$ in $\mathbb{Z}_p$ so that

$$M \;=\; p^m \cdot u_{M,p} \qquad \text{and} \qquad N \;=\; p^n \cdot u_{N,p}.$$

f Defining $u_p := u_{M,p}/u_{M,p}$ and we have

$$a \;=\; p^{\mathrm{ord}_p(a)} \cdot u_p.$$

Here u_p is a unit in $\mathbb{Z}_p$, so we can write it as

$$u_p \;=\; a_0 + a_1 p + a_2 p^2 + a_3 p^3 + \cdots .$$

Thus

$$a \;=\; a_0\, p^{\mathrm{ord}_p(a)} + a_1\, p^{\mathrm{ord}_p(a)+1} + \cdots + a_{-1-\mathrm{ord}_p(a)}\, p^{-1} + a_{-\mathrm{ord}_p(a)} + a_{1-\mathrm{ord}_p(a)}\, p + a_{2-\mathrm{ord}_p(a)}\, p^2 + \cdots$$

The terms with p-adic absolute value ≤ 1, i.e., the terms with nonnegative power of p in this decomposition, form the *p-adic intergral part of a*, denoted $\lfloor a \rfloor_p$. The terms with p-adic absolute value ≤ 1, i.e., the terms with nonnegative power of p in this decomposition, form the *p-adic fractional part of a*:

$$a \;=\; \underbrace{a_0\, p^{\mathrm{ord}_p(a)} + a_1\, p^{\mathrm{ord}_p(a)+1} + \cdots + a_{-1-\mathrm{ord}_p(a)}\, p^{-1}}_{p\text{-adic fractional part } a - \lfloor a \rfloor_p} + \underbrace{a_{-\mathrm{ord}_p(a)} + a_{1-\mathrm{ord}_p(a)}\, p + a_{2-\mathrm{ord}_p(a)}\, p^2 + \cdots}_{p\text{-adic integral part } \lfloor a \rfloor_p}$$

Example 5.1.2. p-Adic fractional parts of the element $a = \frac{1}{6} \in \mathbb{Q}$. Le

(1) **Case: $p = 2$.** We have $6 = 1{\cdot}2 + 1{\cdot}2^2 = 2 \cdot (1 + 1 \cdot 2)$. Here

$$\begin{array}{r|l}
 & 1 \;+\; 1{\cdot}2 \;+\; 1{\cdot}2^2 \;+\; 1{\cdot}2^3 \;+\; 1{\cdot}2^4 \;+\; 1{\cdot}2^5 \;+\; \cdots \\
\hline
1 \;+\; 1{\cdot}2 & 1 \;+\; 0{\cdot}2 \;+\; 0{\cdot}2^2 \;+\; 0{\cdot}2^3 \;+\; 0{\cdot}2^4 \;+\; 0{\cdot}2^5 \;+\; \cdots \\
 & 1 \;+\; 1{\cdot}2 \\
 & \qquad 1{\cdot}2 \;+\; 1{\cdot}2^2 \\
 & \qquad\qquad\quad 1{\cdot}2^2 \;+\; 1{\cdot}2^3 \\
 & \qquad\qquad\qquad\qquad\quad \ddots
\end{array}$$

In other words, we have

$$\frac{1}{1 + 1{\cdot}2} \;=\; 1 + 2 + 2^2 + 2^3 + \cdots .$$

We can then use this to write $\frac{1}{6}$ in a form that makes the 2-adic integral and fractional parts apparent:

$$\frac{1}{6} \;=\; \underbrace{2^{-1}}_{\substack{\text{2-adic} \\ \text{fractional} \\ \text{part } \frac{1}{6} - \lfloor \frac{1}{6} \rfloor}} + \underbrace{1 + 2 + 2^2 + \cdots}_{\text{2-adic integral part } \lfloor \frac{1}{6} \rfloor} .$$

Thus $\mathrm{frac}_2\left(\frac{1}{6}\right) = \frac{1}{2}$.

(2) **Case: $p = 3$.** We have $6 = 2{\cdot}3$

(3) **Case: $p = 5$.** We have $6 = 1 + 1{\cdot}5$

(4) **Case: $p \geq 7$.** We have $6 = 6{\cdot}p^0$ A