NOTES ON

TONAL THEORIES *COMING FROM* REPRESENTATIONS OF ALGEBRAIC GROUPS *OTHER THAN* $\mathbf{GL_1(\mathbb{C})}$

— IN PROGESS —

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ABSTRACT. Lots of the structure of tonal harmony emerges from the representation theory of $\mathrm{GL}_1(\mathbb{C})$. I here begin the project of composing music using tonal structures that emerge from the representation theory of other, higher-dimensional algebraic groups, such as $\mathrm{SL}_2(\mathbb{C})$. This isn't just some theoretical exercise. As these notes should make clear, implementing these tonal structures in code will be very difficult to pull off without the super detailed outline of the general theory that appears below.

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1. NAVIGATING TONALITY THROUGH THE REPRESENTATION THEORY OF $\mathrm{GL}_1(\mathbb{C})$.

1.1. Representation theory of $\mathrm{GL}_1(\mathbb{C})$. [...]

1.2. Tonnetze. [...]

Date: March 29, 2023.

2. Homogeneous polynomials and chords.

2.1. **Decomposing complex numbers for better signal analysis.** Fix an element $z \in \mathbb{C}$. Fixing a positive real *period* $P \in \mathbb{R}_{>0}$ once and for all, we can decompose z into its real and imaginary parts as

$$z \;=\; z(A,\theta) \;=\; \log(A) + i\tfrac{2\pi}{P}\theta, \tag{2.1.0.1}$$

for unique $0 < A < \infty$ and $0 \le \theta < P$. We refer to the positive real number

$$\lambda := \frac{1}{P}$$

as the *frequency* of $z(A,\theta)$. One reason for decomposing z as in Equation (2.1.0.1) is that it gives the exponential of z a form relevant to music. Indeed, from Equation (2.1.0.1) we get

$$e^z \;=\; Ae^{i2\pi\lambda\theta},$$

with real and imaginary parts

$$\mathrm{Re}(e^z) \;=\; A\cos(2\pi\lambda\theta) \qquad \text{and} \qquad \mathrm{Im}(e^z) \;=\; A\sin(2\pi\lambda\theta),$$

respectively. If we fix A and let θ change linearly at the rate of 1 unit per second, then both $\mathrm{Re}(e^z)$ and $\mathrm{Im}(e^z)$ describe a "pure" tone playing with amplitude A at λ Hz. Here A is measured in units of $A_0\, e^{20\text{ dB}}$, where A_0 is some fixed reference amplitude. The tone associated to $\mathrm{Re}(e^z)$ and the tone associated to $\mathrm{Im}(e^z)$ are out of phase by a quarter period $\frac{P}{4}$.

2.2. **Independent complex variables.** Suppose now that we choose two complex numbers $z_1, z_2 \in \mathbb{C}$, independently of one another, with corresponding exponentials

$$e^{z_1} = A_1\, e^{i2\pi\lambda\theta_1} \qquad \text{and} \qquad e^{z_2} = A_2\, e^{i2\pi\lambda\theta_2}. \tag{2.2.0.1}$$

In Equation (2.2.0.1), we assume that we've fixed a single period P, hence a single frequency λ, that z_1 and z_2 share. However, we could just as well choose two different periods, P_1 and P_2 say, and thus two different frequencies $\lambda_1 = \frac{1}{P_1}$ and $\lambda_2 = \frac{1}{P_2}$, to get

$$e^{z_1} = A_1\, e^{i\frac{2\pi}{P_1}\theta_1} = A_1\, e^{i2\pi\lambda_1\theta_1} \qquad \text{and} \qquad e^{z_2} = A_2\, e^{i\frac{2\pi}{P_2}\theta_2} = A_2\, e^{i2\pi\lambda_2\theta_2},$$

where $\lambda_1 = \frac{1}{P_1}$ and $\lambda_2 = \frac{1}{P_2}$.

Given a function $f(x,y)$ of two variables, we can evaluate f at $x = e^{z_1}$ and $y = e^{z_2}$ to obtain the value $f(e^{z_1}, e^{z_2})$. In the special case that $f(x,y)$ is a *Laurent monomial*, i.e., that

$$f(x,y) = x^m y^n,$$

for $m, n \in \mathbb{Z}$, we have

$$f(e^{z_1}, e^{z_2}) \;=\; A_1A_2\, e^{i2\pi(m\lambda_1\theta_1 + n\lambda_2\theta_2)} \;=\; A_1A_2\, e^{i2\pi\left(\frac{m}{P_1}\theta_1 + \frac{n}{P_2}\theta_2\right)}.$$

The real and imaginary parts of this are

$$\mathrm{Re}\, f(e^{z_1}, e^{z_2}) \;=\; A_1A_2\cos\Big(2\pi\big(\tfrac{m}{P_1}\theta_1 + \tfrac{m}{P_2}\theta_2\big)\Big) \tag{2.2.0.2}$$

and

$$\mathrm{Im}\, f(e^{z_1}, e^{z_2}) \;=\; A_1A_2\sin\Big(2\pi\big(\tfrac{m}{P_1}\theta_1 + \tfrac{m}{P_2}\theta_2\big)\Big). \tag{2.2.0.3}$$

This is a situation ripe for techniques from frequency modulation. For instance, if we let t denote our time variable, in units of seconds, and we define

$$\theta_1(t) \;=\; t \qquad \text{and} \qquad \theta_2(t) \;=\; \sin(\omega t) \quad \text{for some } \omega \in \mathbb{R}_{>0},$$

then the formulas in Equations (2.2.0.2) and (2.2.0.3) become instances of *FM synthesis*. We can also see that the formulas in Equations (2.2.0.2) and (2.2.0.3) give us a broad generalization of FM synthesis, in that we can use any pair of real-valued functions

$$\theta_1(t) \qquad \text{and} \qquad \theta_2(t)$$

of t that we like. In this way, a kind of generalized FM synthesis realizes one version of the notion of "pitch movement in 2 dimensions." See Figure 1.

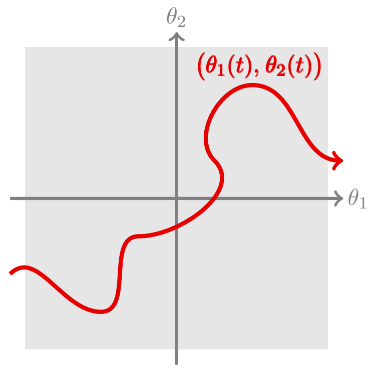

Figure 1. General FM-synthesis from curves in the real plane.

Example 2.2.1. Ring modulation as a special case. Let us briefly remark here that, although it is not usually presented in this way, *ring modulation* in signal processing arises when we evaluate the real and imaginary parts of the monomial xy at $x = e^{z_1}$ and $y = e^{z_2}$. In this way, ring modulation becomes a special case of the above discussion.

Example 2.2.2. Logarithmic representations. Consider the case of a single complex-valued function $z(t) = i2\pi\lambda t$ of the real variable t. How might we transform such a function. One obvious way is to consider all $\mathbb{R}$-*affine transformations* of t, that is, all transformations that can be described by an $\mathbb{R}$-linear polynomial:

$$t \longmapsto a + bt.$$

The group of all such transformations is denoted $\mathrm{Aff}(\mathbb{R})$. It admits a normal decomposition

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathrm{Aff}(\mathbb{R}) \longrightarrow \mathbb{R}^\times \longrightarrow 1$$

that gives this group a semidirect product decomposition

$$\mathrm{Aff}(\mathbb{R}) \;\cong\; \mathbb{R} \rtimes \mathbb{R}^\times.$$

We can effectively "mod out" the *translation* action by the normal subgroup $\mathbb{R} \lhd \mathrm{Aff}(\mathbb{R})$ by focus on the action of the non-normal, multiplicative subgroup $\mathbb{R}^\times \subset \mathrm{Aff}(\mathbb{R})$.

If we work with $e^z = e^{i2\pi\lambda t}$, the translation action $t \mapsto a + t$ by the normal subgroup $\mathbb{R} \rtimes \mathrm{Aff}(\mathbb{R})$ amounts to phase shifting, as it takes

$$e^{i2\pi\lambda t} \longmapsto e^{i(2\pi\lambda t + \phi)}, \quad \text{where } \phi = 2\pi\lambda a.$$

If we take a musical perspective that ignores the effects of phase shifting, then we can focus on the action of the non-normal multiplicative subgroup $\mathbb{R}^\times \subset \mathrm{Aff}(\mathbb{R})$. It is common to use logarithmic coordinates for this group by writing $b = e^s$. We obtain transformations of the form

$$e^{i2\pi\lambda t} \longmapsto e^{i2\pi\lambda e^s t}.$$

Writing $\lambda = e^{s_0}$, this becomes

$$e^{i2\pi e^{s_0}t} \longmapsto e^{i2\pi e^{s_0+s}t}.$$

The charged particle dynamics in *Voice Leader — DISPL.* take place in something like the Lie algebra of the multiplicative group $\mathbb{R}^{\times}$ that acts on the frequency specturm. Ignoring the negative component of this Lie algebra amounts to the assertion that we do not consider backward movement through time in musical contexts.

This leads us to the following questions for the case of 2 independent variables z_1 and z_2.

Question 2.2.3. What is the right "space" to do dynamics in when there are two variables?... (frequency or log-frequency).

There are going to be competing pictures, throughout these notes, for what level "space" should be taken at...

Question 2.2.4. What is the Lie bracket on the tangent space $T_{(\mathbf{q},\mathbf{p})}M$ of phase space M at a state $(\mathbf{q},\mathbf{p})$?

Answer. [Poisson algebra structure on the space of functions. The Lie algebra structure on the tangent space itself is known as the symplectic Lie algebra or the Lie algebra of Hamiltonian vector fields...]

Relevance of the question to representations of algebraic groups. **[...]**

2.3. **Homogenous linear combinations of independent complex variables.** If x and y are independent complex variables, then a $\mathbb{C}$-linear combination of monomials in x and y, say

$$a_1x^{m_1}y^{n_1} + a_2x^{m_2}y^{n_2} + \cdots + a_\ell x^{m_\ell}y^{n_\ell}, \quad a_1, a_2, \ldots, a_\ell \in \mathbb{C} \tag{2.3.0.1}$$

is *homogeneous of degree* d if $m_i + n_i = d$ for all $1 \le i \le \ell$, in other words, if all monomials $x^m y^n$ in the linear combination have the same *total degree* $m+n$, equal to d. When the linear combination in Equation (2.3.0.1) is homogeneous of degree d, we refer to it as a *homogeneous polynomial of degree* d in the variables x and y over $\mathbb{C}$.

Notice that, from a musical perspective, a homogeneous linear combination in Equation (2.3.0.1) combines two distinct ideas in a way that isn't possible with a single variable. It packages multiple instances of ring modulation into a single chord-like structure. Indeed,

The general homogeneous polynomial of degree d in x and y can be written

$$f(x,y) \;=\; a_0x^d + a_1x^{d-1}y + a_2x^{d-2}y^2 + \cdots + a_{d-1}xy^{d-1} + a_dy^d,$$

with coefficients $a_0, a_1, \ldots, a_d \in \mathbb{C}$. We can write this more succinctly as

$$f(x,y) \;=\; \sum_{n=0}^{d} a_n x^{d-n}y^n.$$

Evaluating this polynomial at $x = e^{z_1}$ and $y = e^{z_2}$, we obtain

$$f(e^{z_1}, e^{z_2}) \;=\; \sum_{n=0}^{d} a_n A_1^{d-n}A_2^n e^{i2\pi\left(\frac{d-n}{P_1}\theta_1 + \frac{n}{P_2}\theta_2\right)}. \tag{2.3.0.2}$$

To get a slightly clearer picture of this, let us assume that $a_i = 1$ for all $0 \le i \le d$ [change i to different letter...] and that $A_1 = A_2 = 1$. Then the right-hand side of Equation (2.3.0.2) becomes

$$\sum_{n=0}^{d} e^{i2\pi\left(\frac{d-n}{P_1}\theta_1 + \frac{n}{P_2}\theta_2\right)}. \tag{2.3.0.3}$$

We play with this expression a bit more in Example 2.3.1 below.

Example 2.3.1. [...]

Special case: $\theta_1 = \theta_2$ and $P_1 = P_2$. [...] Equation (2.3.0.3) becomes

$$(d+1)e^{i2\pi\frac{d}{P}\theta} \;=\; (d+1)e^{i2\pi d\lambda\theta}$$

Special case: $\theta_1 = 0$. [...] Equation (2.3.0.3) becomes

$$\sum_{n=0}^{d} e^{i2\pi\frac{n}{P}\theta} \;=\; 1 + e^{i2\pi\frac{1}{P}\theta} + e^{i2\pi\frac{2}{P}\theta} + \cdots + e^{i2\pi\frac{d}{P}\theta},$$

or in terms of frequency,

$$f(1, e^{i2\pi\lambda\theta}) \;=\; 1 + e^{i2\pi\lambda\theta} + e^{i2\pi 2\lambda\theta} + \cdots + e^{i2\pi d\lambda\theta}.$$

Ignoring the constant term "1," this is just an equi-voiced[1] overtone chord with root at λ Hz, and including overtones 1 through d.

If we let $A_1 = 1$ and $A_2 = \varepsilon$, where $0 < \varepsilon < 1$, then this becomes

$$f(1, \varepsilon e^{i2\pi\lambda\theta}) \;=\; 1 + \varepsilon e^{i2\pi\lambda\theta} + \varepsilon^2 e^{i2\pi 2\lambda\theta} + \cdots + \varepsilon^d e^{i2\pi d\lambda\theta}.$$

This is the same overtone chord, but with voicing that falls of like a geometric series as we move up the overtone scale.

Question 2.3.2. [Voicing and orchestration questions...]

Remark 2.3.3. [Model movement of several (θ_1, θ_2)-pairs on particle dynamics in 2-dimensional space. This provides an FM version of the particle dynamics experiment from *Voice Leader — DISPL*....]

[...]

3. Representations of $\mathrm{SL}_2(\mathbb{C})$.

3.1. **The Cartan-Weyl basis.** The 3-dimensional Lie algebra

$$\mathfrak{sl}_2(\mathbb{C}) \;=\; \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \right\}$$

admits a standard basis, sometimes called the *Cartan-Weyl basis*, given by

$$E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \text{and} \qquad F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The 1-parameter subgroups gotten by exponentiation of these basis vectors have general element

$$e^{zE} \;=\; \mathrm{I} + zE + \frac{z^2}{2}\cancelto{0}{E^2} + \frac{z^3}{3!}\cancelto{0}{E^3} + \cdots \;=\; \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix},$$

$$e^{zF} \;=\; \mathrm{I} + zF + \frac{z^2}{2}\cancelto{0}{F^2} + \frac{z^3}{3!}\cancelto{0}{F^3} + \cdots \;=\; \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix},$$

[1] We say that a chord is *equi-voiced* if the notes of a chord are equally loud. I don't know if this is standard terminology.

and

$$\begin{aligned} e^{zH} &= I + zH + \tfrac{z^2}{2}H^2 + \tfrac{z^3}{3!}H^3 + \cdots \\ &= \begin{pmatrix} 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots & 0 \\ 0 & 1 - z + \frac{z^2}{2} - \frac{z^3}{3!} + \cdots \end{pmatrix} = \begin{pmatrix} e^z & 0 \\ 0 & e^{-z} \end{pmatrix}, \end{aligned}$$

respectively. One recurring theme in these notes is the question "should we work in the space where our variable is z, or in the space where are variable is $q = e^z$. Note, though, that this distinction gets confused in $\mathrm{SL}_2(\mathbb{C})$, since the complex parameter z acts by multiplication directly in the 1-parameter subgroups

$$\{e^{zE} : z \in \mathbb{C}\} \text{ and } \{e^{zF} : z \in \mathbb{C}\} \subset \mathrm{SL}_2(\mathbb{C}),$$

but acts through multiplication by in the 1-parameter subgroup

$$\{e^{zH} : z \in \mathbb{C}\} \subset \mathrm{SL}_2(\mathbb{C}).$$

Every element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$ admits a factorization

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = e^{uE}e^{vH}e^{wF} = \begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^v & 0 \\ 0 & e^{-v} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}$$

for some $u, v, w \in \mathbb{C}$. To compute u, v, w, observe that the matrix product on the right is equal to

$$\begin{pmatrix} 1 & w \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^v & 0 \\ ue^{-v} & e^{-v} \end{pmatrix} = \begin{pmatrix} e^v + uwe^{-v} & we^{-v} \\ ue^{-v} & e^{-v} \end{pmatrix},$$

hence

$$d = e^{-v}, \qquad w = b/d, \qquad u = c/d$$

[...confusing myself... need to come back to this...]

[Factorizations coming from other permutations of these 3 factors...]

[...]

3.2. **Action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}[x, y]$.** We let $\mathrm{SL}_2(\mathbb{C})$ act on $\mathbb{C}[x, y]$ through its inverse action on the argument (x, y) of each function $f(x, y) \in \mathbb{C}[x, y]$. Thus the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$ acts on each $f(x, y) \in \mathbb{C}[x, y]$ via the action of its inverse $\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$

3.3. **Irreducible representations of $\mathrm{SL}_2(\mathbb{C})$ in $\mathbb{C}[x, y]$.** [...]

3.4. **The irreducible 2-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$.** [...]

3.5. **The irreducible 3-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$.** [...]

3.6. **"Harmonic movement" for $\mathrm{SL}_2(\mathbb{C})$.**

Question 3.6.1. What sort of movement through the category $\mathbf{Rep}(\mathrm{SL}_2(\mathbb{C}))$ is the "correct" generalization of the movement through $\mathbf{Rep}(\mathbb{S}^1)$ that corresponds to $\mathbb{Z}$-multiplicative movement through the terms of a Fourier series?

Temporary answers. There are many reasonable candidates:

Answer 3.6.1.1. Inductive/restrictive functorial movement along homomorphisms $G \longrightarrow \mathrm{SL}_2(\mathbb{C})$ and/or $\mathrm{SL}_2(\mathbb{C}) \longrightarrow G$;
Answer 3.6.1.2. Movement along functors $\mathrm{Hom}(V_n, -)$;
Answer 3.6.1.3. Movement along functors $T^n = (-)^{\otimes n}$;
Answer 3.6.1.4. Movement along functors Sym^n;
Answer 3.6.1.5. Movement along functors Λ^n.

[Explain issue of plethysm/fusion rules...]

[...]

3.7. **Plethysm.** [...]