NOTES ON

TONAL THEORIES *COMING FROM* REPRESENTATIONS OF ALGEBRAIC GROUPS *OTHER THAN* $\mathbf{GL_1(\mathbb{C})}$

— IN PROGESS —

TYLER FOSTER

ABSTRACT. Lots of the structure of tonal harmony emerges from the representation theory of $\mathrm{GL}_1(\mathbb{C})$. I here begin the project of composing music using tonal structures that emerge from the representation theory of other, higher-dimensional algebraic groups, such as $\mathrm{SL}_2(\mathbb{C})$. This isn't just some theoretical exercise. As these notes should make clear, implementing these tonal structures in code will be very difficult to pull off without the super detailed outline of the general theory that appears below.

CONTENTS

1. HOMOGENEOUS POLYNOMIALS AND CHORDS.

1.1. **Decomposing complex numbers for better signal analysis.** Fix an element $z \in \mathbb{C}$. Fixing a positive real *period* $P \in \mathbb{R}_{>0}$ once and for all, we can decompose z into its real and imaginary parts as

$$z \;=\; z(A,\theta) \;=\; \log(A) + i\tfrac{2\pi}{P}\theta, \tag{1.1.0.1}$$

for unique $0 < A < \infty$ and $0 \leq \theta < P$. We refer to the positive real number

$$\lambda := \frac{1}{P}$$

as the *frequency* of $z(A,\theta)$. One reason for decomposing z as in Equation (1.1.0.1) is that it gives the exponential of z a form relevant to music. Indeed, from Equation (1.1.0.1) we get

$$e^z \;=\; Ae^{i2\pi\lambda\theta},$$

with real and imaginary parts

$$\mathrm{Re}(e^z) \;=\; A\cos(2\pi\lambda\theta) \qquad \text{and} \qquad \mathrm{Im}(e^z) \;=\; A\sin(2\pi\lambda\theta),$$

respectively. If we fix A and let θ change linearly at the rate of 1 unit per second, then both $\mathrm{Re}(e^z)$ and $\mathrm{Im}(e^z)$ describe a "pure" tone playing with amplitude A at λ Hz,[1] such that the

Date: March 27, 2023.

[1] Here A is measured in units of $A_0\, e^{20\mathrm{dB}}$, where A_0 is some fixed reference amplitude.

tone associated to $\mathrm{Re}(e^z)$ and the tone associated to $\mathrm{Im}(e^z)$ are out of phase by a quarter period $\frac{P}{4}$.

1.2. **Independent complex variables.** [...]

$$e^{z_1} = A_1\, e^{i2\pi\lambda\theta_1} \qquad \text{and} \qquad e^{z_2} = A_2\, e^{i2\pi\lambda\theta_2}.$$

Note the possibility here of choosing two different values for our period, P_1 and P_2 say, to get

$$e^{z_1} = A_1\, e^{i\frac{2\pi}{P_1}\theta_1} = A_1\, e^{i2\pi\lambda_1\theta_1} \qquad \text{and} \qquad e^{z_2} = A_2\, e^{i\frac{2\pi}{P_2}\theta_2} = A_2\, e^{i2\pi\lambda_2\theta_2},$$

where $\lambda_1 = \frac{1}{P_1}$ and $\lambda_2 = \frac{1}{P_2}$.

Given a function $f(x,y)$ of two variables, we can evaluate f at $x = e^{z_1}$ and $y = e^{z_2}$ to obtain the value $f(e^{z_1}, e^{z_2})$. In the special case that $f(x,y)$ is a *Laurent monomial*, i.e., that

$$f(x,y) = x^m y^n,$$

for $m, n \in \mathbb{Z}$, we have

$$f(e^{z_1}, e^{z_2}) \;=\; A_1 A_2\, e^{i2\pi(m\lambda_1\theta_1 + n\lambda_2\theta_2)} \;=\; A_1 A_2\, e^{i2\pi\left(\frac{m}{P_1}\theta_1 + \frac{n}{P_2}\theta_2\right)}.$$

The real and imaginary parts of this are

$$\mathrm{Re}\, f(e^{z_1}, e^{z_2}) \;=\; A_1 A_2 \cos\Big(2\pi\big(\tfrac{m}{P_1}\theta_1 + \tfrac{m}{P_2}\theta_2\big)\Big) \tag{1.2.0.1}$$

and

$$\mathrm{Im}\, f(e^{z_1}, e^{z_2}) \;=\; A_1 A_2 \sin\Big(2\pi\big(\tfrac{m}{P_1}\theta_1 + \tfrac{m}{P_2}\theta_2\big)\Big). \tag{1.2.0.2}$$

This is a situation ripe for techniques from frequency modulation. For instance, if we let t denote our time variable, in units of seconds, and we define

$$\theta_1(t) \;=\; t \qquad \text{and} \qquad \theta_2(t) \;=\; \sin(\omega\, t) \quad \text{for some } \omega \in \mathbb{R}_{>0},$$

then the formulas in Equations (1.2.0.1) and (1.2.0.2) become instances of *FM synthesis*. We can also see that the formulas in Equations (1.2.0.1) and (1.2.0.2) give us a broad generalization of FM synthesis, in that we can use any pair of real-valued functions

$$\theta_1(t) \qquad \text{and} \qquad \theta_2(t)$$

of t that we like. In this way, a kind of generalized FM synthesis realizes one version of the notion of "pitch movement in 2 dimensions." It is important to keep in mind here that the linear independence between the 2 dimensions here is happening in the logarithm here. In other words, the linear independence is in the variable z, not in the value e^z.

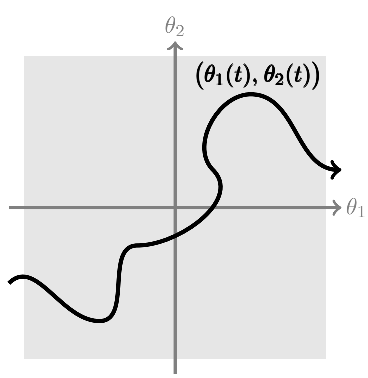

Figure 1. General FM-synthesis from curves in the real plane.

This begins to move into the realm of harmony. We pause our development of harmony here, and pick it back up in §[...]

2. Representations of $\mathrm{SL}_2(\mathbb{C})$.

We let $\mathrm{SL}_2(\mathbb{C})$ act on $\mathbb{C}[x, y]$ through its inverse action on the argument (x, y) of each function $f(x, y) \in \mathbb{C}[x, y]$. Thus the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$ acts on each $f(x, y) \in \mathbb{C}[x, y]$ via the action of its inverse $\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$