CSE417 homework 1

Muzhou Liu

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1. LDF problem 1.3 Prove that the PLA eventually converges to a linear separator for separable data. The following steps will guide you through the proof. Let w^* be an optimal set of weights (one which separates the data). The essential idea in this proof is to show that the PLA weights $w(t)$ get "more aligned" with w^* with every iteration. For simplicity, assume that $w(0) = 0$.

   (a) Let $\rho = \min_{1 \le n \le N} y_n(w^{*T} x_n)$. Show that $\rho > 0$. Because all x(t) are correctly classified by w(t) so:

   - $y(t) = +1 : sign(w^T(t)x(t)) = +1 \Leftrightarrow w^T(t)x(t) > 0$.
     Thus:$y(t)w^T(t)x(t) > 0$.
   - $y(t) = -1 : sign(w^T(t)x(t)) = -1 \Leftrightarrow w^T(t)x(t) > 0$.
     Thus:$y(t)w^T(t)x(t) > 0$. Therefore, as minimum value of $y(t)w^T(t)x(t)$,ρ must larger than 0.

   (b) Show that $w^T(t)w^* \ge w^T(t-1)w^* + \rho$, and conclude that $w^T(t)w^* \ge t\rho$

   Proof. Because we have

   $$w^T(t-1) = (w(t) - y(t)x(t))^T = w^T(t) - y(t)x^T(t)$$

   $\Leftrightarrow w^T(t-1)w^* + \rho = w^T(t)w^* - y(t)x^T(t)w^* + \rho$
   According to the definition of ρ We also have: $\rho \le y(t)x^T(t)w^*$ (as proved in part (a)), which means: $\rho - y(t)x^T(t)w^* \le 0$.
   Then:
   $w^T(t)w^* - y(t)x^T(t)w^* + \rho \le w^T(t)w^*$

   Base case: when $t = 0$: $w^T(0)w^* \ge 0\rho \Leftrightarrow 0 \ge 0$.

   Induction step: Assume $w^T(t)w^* \ge t\rho$ holds $\forall t \ge 0$

   When $t = t+1$: $w^T(t+1)w^* \ge (t+1)\rho$.

   We have $(t+1)\rho = t\rho + \rho \le w^T(t)w^* + \rho \le w^T(t+1)w^*$.
   Thus proofed ◻

(c) Show that $\|w(t)\|^2 \leq \|w(t-1)\|^2 + \|x(t-1)\|^2$.

Proof.

- For y(t - 1) = + 1, we have:

$$\|w(t)\|^2 = \|w(t-1)\|^2 + \|x(t-1)\|^2 - 2\|w(t-1)\|\,\|x(t-1)\|\cos\theta$$

(θ is the angle between $-w(t-1)$ and $x(t-1)$)
Because $y(t-1)w^T(t-1)x(t-1) < 0 \Rightarrow w^T(t-1)x(t-1) < 0$
We have $\rightarrow \cos(180-\theta) < 0 \rightarrow cos(\theta) \geq 0$.

Hence: $-2\|w(t-1)\|\,\|x(t-1)\|\cos\theta \leq 0$.

- For y(t - 1) = - 1, we have:

$$\|w(t)\|^2 = \|w(t-1)\|^2 + \|x(t-1)\|^2 + 2\|w(t-1)\|\,\|x(t-1)\|\cos\theta$$

(θ is the angle between $w(t-1)$ and $x(t-1)$)
Because $y(t-1)w^T(t-1)x(t-1) > 0 \Rightarrow w^T(t-1)x(t-1) > 0$
We have $\rightarrow \cos(\theta) > 0$

Hence: $2\|w(t-1)\|\,\|x(t-1)\|\cos\theta \geq 0$.

□

(d) Show by induction that $\|w(t)\|^2 \leq tR^2$, where $R = \max_{1\leq n\leq N}\|x_n\|$.

Proof. Base case:
$t = 0$: $\|w(0)\|^2 \leq 0R^2 \rightarrow 0 \leq 0$.
Induction step:
Assume: $\|w(t)\|^2 \leq tR^2$ holds for $t \geq 0$
when $t = t+1$ $\|w(t+1)\|^2 \leq (t+1)R^2$.
According to the conclusion of part c) we have: $\|w(t+1)\|^2 \leq (t+1)R^2 = tR^2 + R^2 \geq \|w(t)\|^2 + R^2 \geq \|w(t)\|^2 + \|x(t-1)\|^2$. So the statement follows. □

(e) Using (b) and (d), show that:

$$\frac{w^T(t)}{\|w(t)\|}w^* \geq \sqrt{t}.\frac{\rho}{R}$$

and hence prove that:

$$t \leq \frac{R^2\|w^*\|^2}{\rho^2}$$

Proof. According to part b) we have

$$\frac{w^T(t)}{\|w(t)\|}w^* \geq \frac{w^T(t-1)w^* + \rho}{\|w(t)\|} \geq \frac{t\rho}{\|w(t)\|}$$

combining with the conclusion of part d):-

$$\frac{t\rho}{\|w(t)\|} \geq \frac{t\rho}{\sqrt{t}R} = \frac{\sqrt{t}\rho}{R}$$

Therefore:

$$\frac{w^T(t)}{\|w(t)\|}w^* \geq \sqrt{t}.\frac{\rho}{R}$$

by extract t to one side we have:

$$t \leq \frac{R^2(w^T(t)w^*)^2}{\rho^2\left\|w(t)\right\|^2} = \frac{R^2}{\rho^2} \cdot \frac{(w^T(t)w^*)^2}{\left\|w(t)\right\|^2\left\|w^*\right\|^2} \cdot \left\|w^*\right\|^2$$

Because:

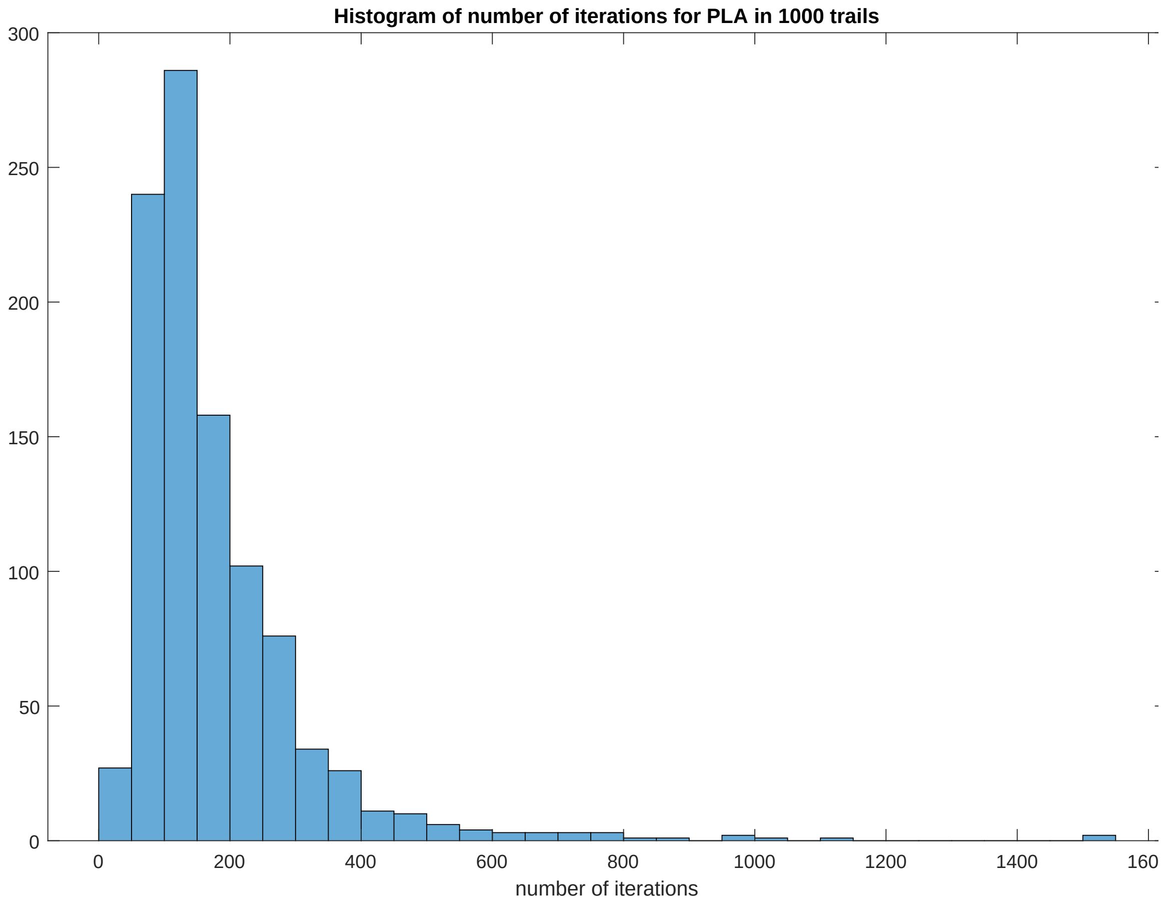
$$\frac{(w^T(t)w^*)^2}{\left\|w(t)\right\|^2\left\|w^*\right\|^2} = cos(\theta)^2 \leq 1$$

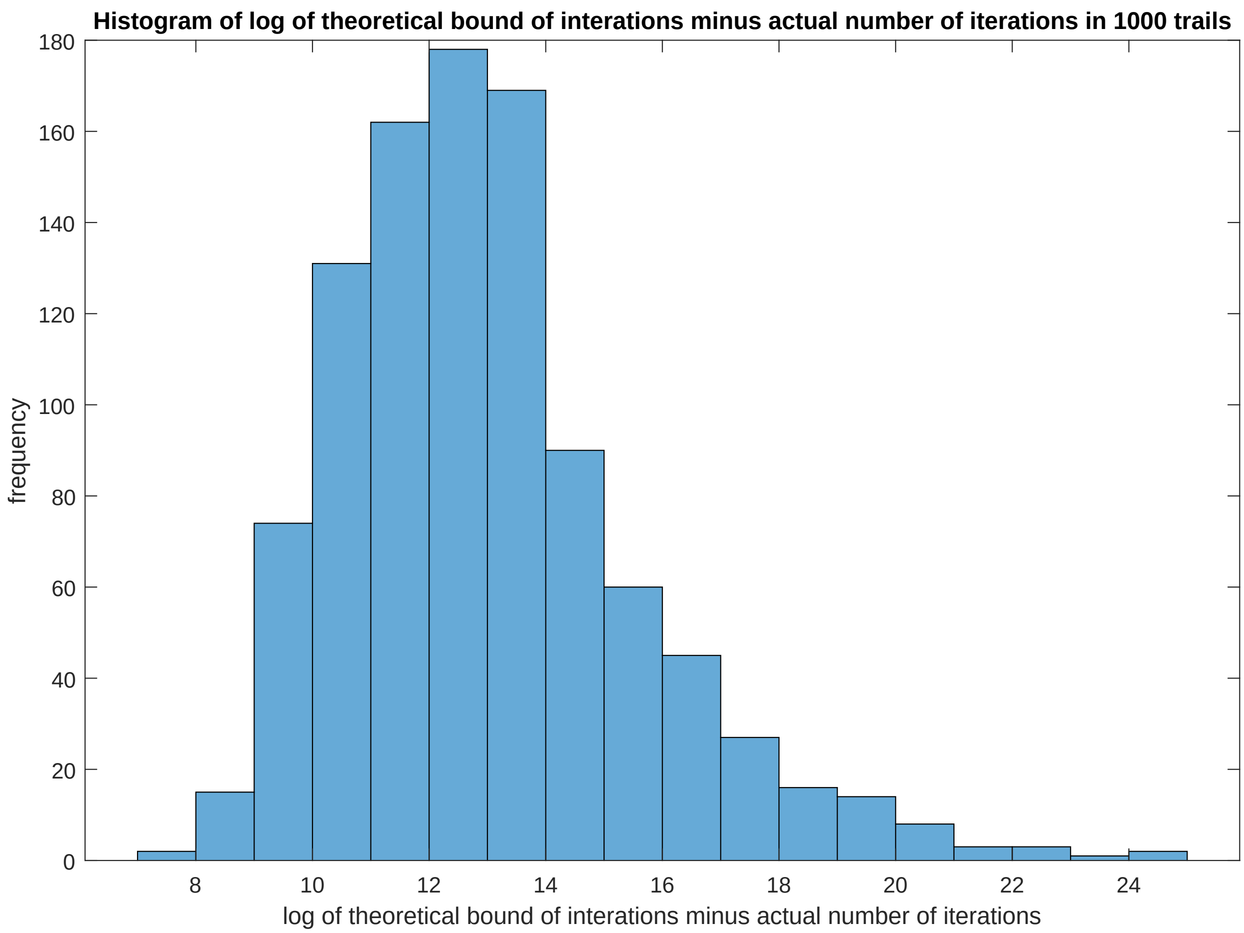
We get that:

$$t \leq \frac{R^2\left\|w^*\right\|^2}{\rho^2}$$

□

2. Problem 2


Histogram of number of iterations for PLA in 1000 trails
300
250
200
150
100
50
0
0
200
400
600
800
1000
1200
1400
160
number of iterations


Histogram of log of theoretical bound of interations minus actual number of iterations in 1000 trails
frequency
0
20
40
60
80
100
120
140
160
180
8
10
12
14
16
18
20
22
24
log of theoretical bound of interations minus actual number of iterations

The iterations of the PLA in 1000 trails looked approximately normal but with some right skew. The theoretical upper bound are mostly way larger than the steps it actually takes.

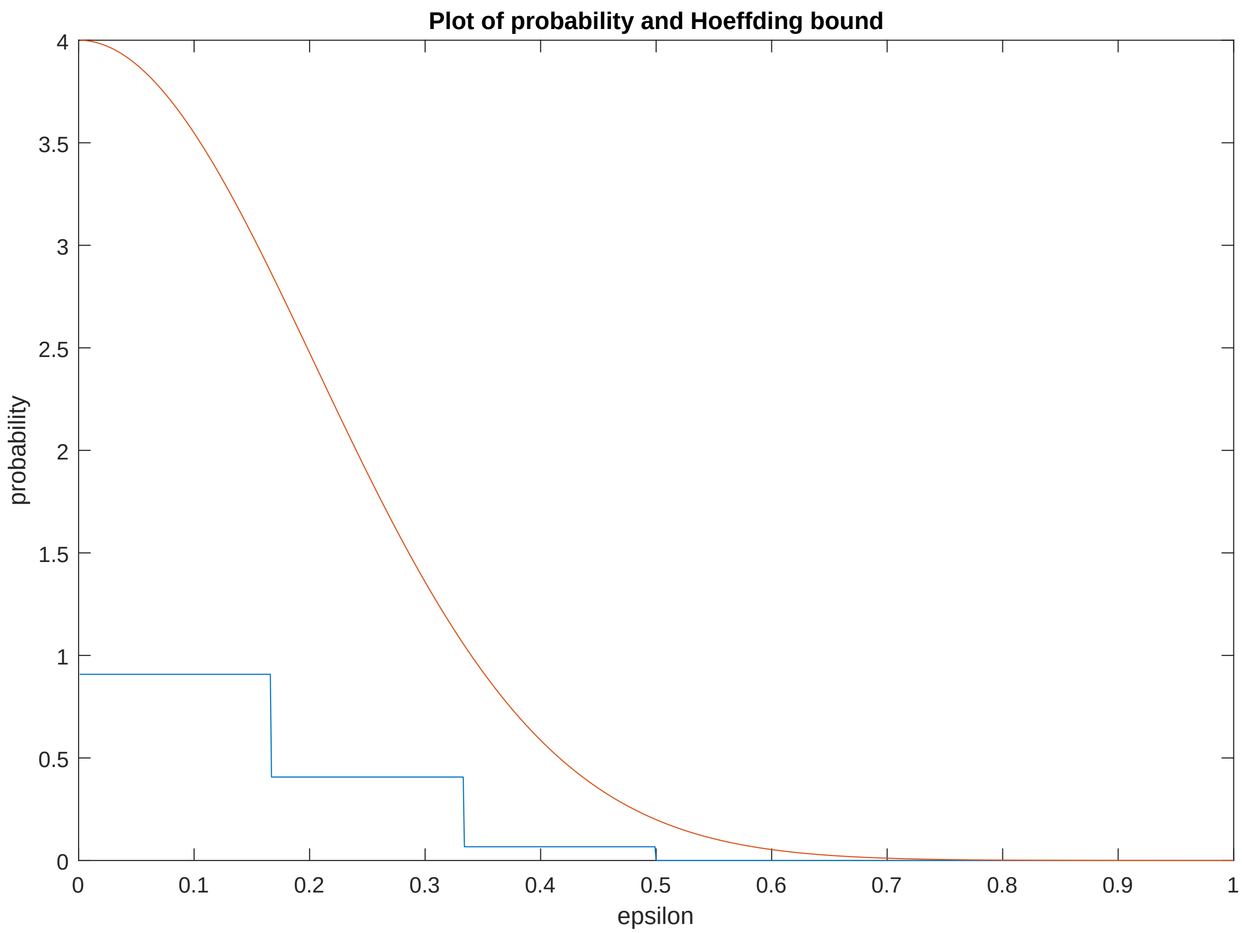
3. LFD problem 1.7

   (a) Set event E as at least one coin will have $\nu = 0$, event F as no coin will have $\nu = 0$ and c denote number of coins

      - when $\mu = 0.05$:
        $\mathbb{P}(k = 0, N = 10, \mu = 0.05) = \binom{10}{0} \cdot 0.05^0 \cdot (1 - 0.05)^{10-0} = 0.5987$
        $\mathbb{P}(E|c = n) = 1 - \mathbb{P}(F|c = n)$
        Because every coin follows a binomial distribution with p =0.5987 to get $\nu = 0$
        When n=1: $\mathbb{P}(E|c = n) = 1 - (1 - p)^n = 0.5987$
        Similarly, when n=1,000: $\mathbb{P}(E|c = n) = 1 - (1 - p)^n = 1$
        When n=1,000,000: $\mathbb{P}(E|c = n) = 1 - (1 - p)^n = 1$

      - $\mu = 0.8$:
        $\mathbb{P}(k = 0, N = 10, \mu = 0.8) = \binom{10}{0} \cdot 0.8^0 \cdot (1 - 0.8)^{10-0} = 1.024 \cdot 10^{-7}$
        $\mathbb{P}(E|c = n) = 1 - \mathbb{P}(F|c = n)$
        Because every coin follows a bernoulli distribution with $p = 1.024 \cdot 10^{-7}$ to get $\nu = 0$
        When n=1: $\mathbb{P}(E|c = n) = 1 - (1 - p)^n = 1.024 \cdot 10^{-7}$
        Similarly, when n=1,000: $\mathbb{P}(E|c = n) = 1 - (1 - p)^n = 0.0001$
        When n=1,000,000: $\mathbb{P}(E|c = n) = 1 - (1 - p)^n = 0.097$

   (b) Denote A as the event that $|\nu_1 - \mu|_1 > \epsilon$ and B as the event that $|\nu_2 - \mu_2| > \epsilon$
      Then:
      $\mathbb{P}(Max|\nu_i - \mu_i| > \epsilon) \leq \mathbb{P}(AorB) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(A \textit{ and } B)$
      Because $\mathbb{P}(A \textit{ and } B)$ always greater or equal to 0, we have:
      $\mathbb{P}(Max|\nu_i - \mu_i| > \epsilon) \leq \mathbb{P}(A) + \mathbb{P}(B)$
      Thus:
      $\mathbb{P}(Max|\nu_i - \mu_i| > \epsilon) \leq 2 \cdot 2e^{-2N\epsilon^2} = 4e^{-12\epsilon^2}$


Plot of probability and Hoeffding bound
4
3.5
3
2.5
2
1.5
1
0.5
0
probability
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
epsilon

4. LFD problem 1.8

(a) *Proof.* For a non-negative random variable $\alpha > 0$ We can have a non-negative variable t that satisifys

$$\alpha \mathbb{I}_{t \geq \alpha} \leq t$$

As when $t \geq \alpha$, $\alpha \cdot 1 \leq t$; when $t < \alpha$, $\alpha \cdot 0 \leq t$
Taking expectation on both sides we have:

$$\begin{aligned} \mathbb{E}(\alpha \mathbb{I}_{t \geq \alpha}) &\leq \mathbb{E}(t) \\ \Leftrightarrow \alpha \cdot 1 \cdot \mathbb{P}(t \geq \alpha) + \alpha \cdot 0 \cdot \mathbb{P}(t < \alpha) = \alpha \mathbb{P}(t \geq \alpha) &\leq \mathbb{E}(t) \\ \Leftrightarrow \mathbb{P}(t \geq \alpha) &\leq \frac{\mathbb{E}(t)}{\alpha} \end{aligned}$$

□

(b) *Proof.* According to part a) we easily get that $((u - \mu)^2 \geq 0)$:

$$\mathbb{P}((u - \mu)^2 \geq \alpha) \leq \frac{\mathbb{E}((u - \mu)^2)}{\alpha}$$

Because u has a mean μ and variance σ^2; according to the definition of variance, we have:

$$\mathbb{P}((u - \mu)^2 \geq \alpha) \leq \frac{\sigma^2}{\alpha}$$

□

(c) *Proof.* Because $u = \frac{1}{N} \sum_{n=1}^{N} \mu$, According to the property of expected value and variance we have:

$$\mathbb{E}(u) = \frac{1}{N} \sum_{n=1}^{N} \mu = \mu$$

$$Var(u) = \frac{1}{N} \sum_{n=1}^{N} \sigma^2 = \frac{\sigma^2}{N^2}$$

Referring the conclusion of part b) we easily have:

$$\mathbb{P}((u - \mu)^2 \geq \alpha) \leq \frac{\sigma^2}{N\alpha}$$

□

1 Appendix

```
function [ w iterations ] = perceptron_learn( data_in )

[nrow, ncol] = size(data_in);
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```
true_label = data_in(1:nrow,ncol);
true_label = true_label';
w = zeros(1,ncol-1);
iterations = 0 ;
while isequal( sign(w*data_in(1:nrow, 1:ncol-1)'), true_label )==0
    index = find(sign(w*data_in(1:nrow, 1:ncol-1)')-true_label ~= 0);
    w = w+ data_in(index(1),ncol)*data_in(index(1),1:ncol-1);
    iterations = iterations+1;
end


end

function [ num_iters bound_minus_ni] =
  perceptron_experiment ( N, d, num_samples )
num_iters = zeros(1,num_samples);
bounds = zeros(1,num_samples);
for i= 1:num_samples
    rng(i);

    w_star = rand(1,d+1);
    w_star(1)= 0;
    x = 2* rand(d+1,N)-1 ;
    x(1,1:N)= ones(1,N);
    correct_tag = sign(w_star*x);
    data_in = [x',correct_tag']; % nrow = N, ncol=d+2
    f =@precptron_learn;
    [w,iter] = perceptron_learn(data_in);
     num_iters(i)= iter;

    rho=min(correct_tag.*(w_star*x));           %from problem 1.3
    R2=max(sum(x.*x));
    W2=sum(w_star.*w_star);
    bounds(i)=R2*W2/rho^2;
    bound_minus_ni = bounds - num_iters

end

  [c,d]= perceptron_experiment(100,10,1000)
    histogram(c);
    xlabel('number of iterations');
    ylabel('frequency');
    title('Histogram of number of iterations for PLA in 1000 trails')
    histogram(log(d));
    xlabel('log of theoretical bound of interations minus actual number of
```

```
    iterations')
    ylabel('frequency')
     title('Histogram of log of theoretical bound of interations minus actual
     number of iterations in 1000 trails')

  function[proba probb] = max_abs(N,mu)

proba = zeros(1,1000);
probb = zeros(1,1000);
record = zeros(1,1000);
nnnn = 0;
for j = 1:1000
    epsilon = j/1000;

    for i = 1:1000
        rng(i);
        c1 = randi([0,1],1,N);
        nu1 = mean(c1);
        rng(i+1000);
        c2 = randi([0,1],1,N);
        nu2= mean(c2);
        m = max(abs(nu1-mu),abs(nu2-mu));
        record(i) = m > epsilon ;

    end

    proba(j) = mean(record);
    probb(j) = 4 * exp(-12 * epsilon^2);
    nnnn = nnnn+1
end

end


 [aa,bb] = max_abs(6,0.5)
 epsilon = (1:1000)/1000;
 plot(epsilon, aa, epsilon,bb)
 xlabel('epsilon')
 ylabel('probability')
 title('Plot of probability and Hoeffding bound')
```