Analysis and Design of Algorithms

Chapter 2: Fundamentals of the Analysis of Algorithm Efficiency

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Fundamentals of the Analysis of Algorithm Efficiency

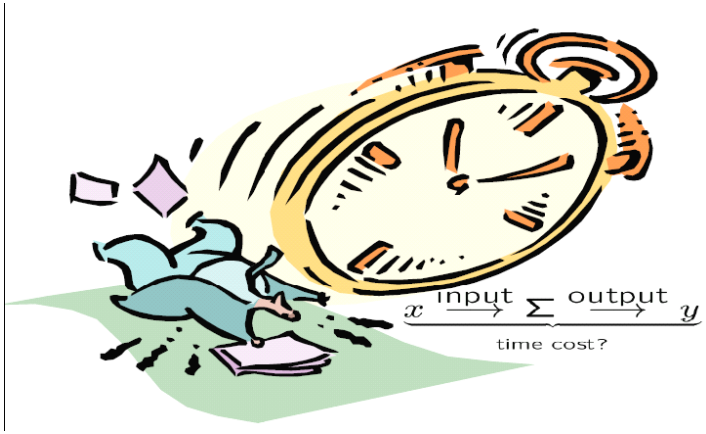
- ***Algorithm analysis framework***
- ***Asymptotic notations***
- ***Analysis of non-recursive algorithms***
- ***Analysis of recursive algorithms***

Analysis of Algorithms

- ***Analysis of algorithms means to investigate an algorithm's efficiency with respect to resources: running time and memory space.***
  - *with respect to input size, input type, and algorithm function*

$$C=F(N, I, A)$$

  - *Time efficiency T(N, I):*
    how fast an algorithm runs.
  - *Space efficiency S(N , I):*
    the space an algorithm requires.



Analysis of Algorithms

Time efficiency T(N, I):

- *time consumed for an algorithm running on an computer*

- *Element operations O_1, O_2, …, O_k,*
  execution time for the element operation t_1, t_2, …, t_k
  number of times element operation O_i is executed: e_i

$$T(N, I) = \sum_{i=1}^{k} t_i e_i(N, I)$$

Analysis of Algorithms

Analysis Framework

- *Measuring running time*
- *Measuring an input's size*
- *Orders of growth (of the algorithm's efficiency function)*
- *Worst-base, best-case and average-case efficiency*

Analysis of Algorithms

Units for Measuring Running Time

- *Should we measure the running time using standard unit of time measurements, such as seconds, minutes?*
  - ➔ Depends on the speed of the computer
  - ➔ Depends on the quality of programing
- *Count the number of times each element operation is executed.*
  - ➔ Difficult and unnecessary

- ***Solution:*** *Count the number of times an algorithm's basic operation is executed.*
  - ***Basic operation****: the operation that contributes the most to the total running time.*
  - *For example, the basic operation is usually the most time-consuming operation in the algorithm's innermost loop.*

Analysis of Algorithms

Measuring Input Size

- *Efficiency is defined as a function of input size*
- *Typically, algorithms run longer as the size of its input increases*
- *Input size depends on the problem.*
  - *Examples*
- *We are interested in how efficiency scales wrt input size*

Analysis of Algorithms

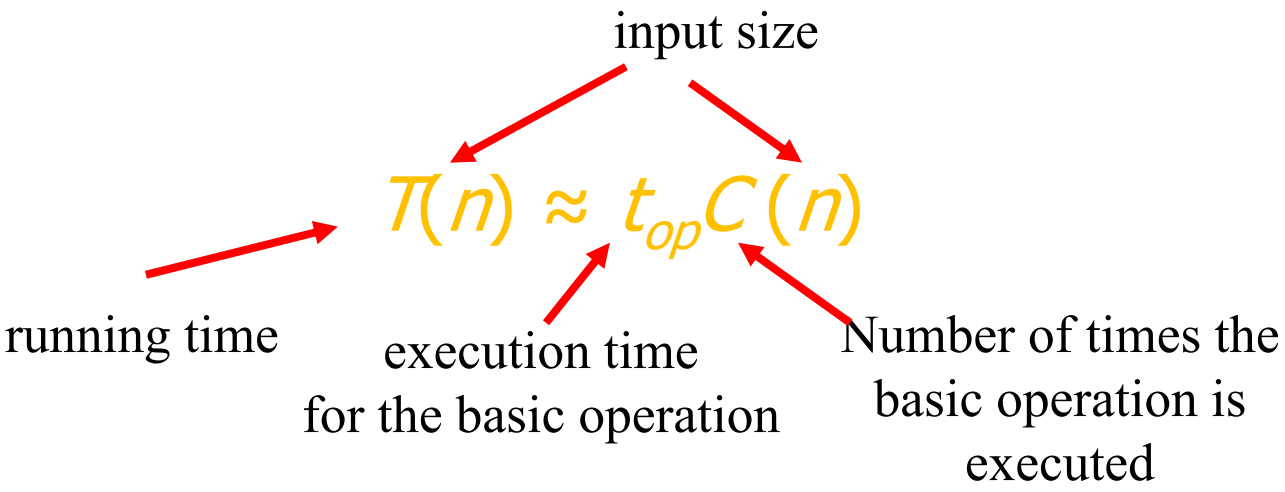
Input size and basic operation examples

Problem	***Input size measure***	***Basic operation***
Searching for key in a list of n items	Number of list's items, i.e. n	Key comparison
Multiplication of two matrices	Matrix dimensions or total number of elements	Multiplication of two numbers
Evaluating $p(x) = a_n x^n + \ldots + a_0$	polynomial's degree *n*	Multiplication, addition
Typical graph problem	#vertices and/or edges	Visiting a vertex or traversing an edge

Analysis of Algorithms

Theoretical Analysis of Time Efficiency

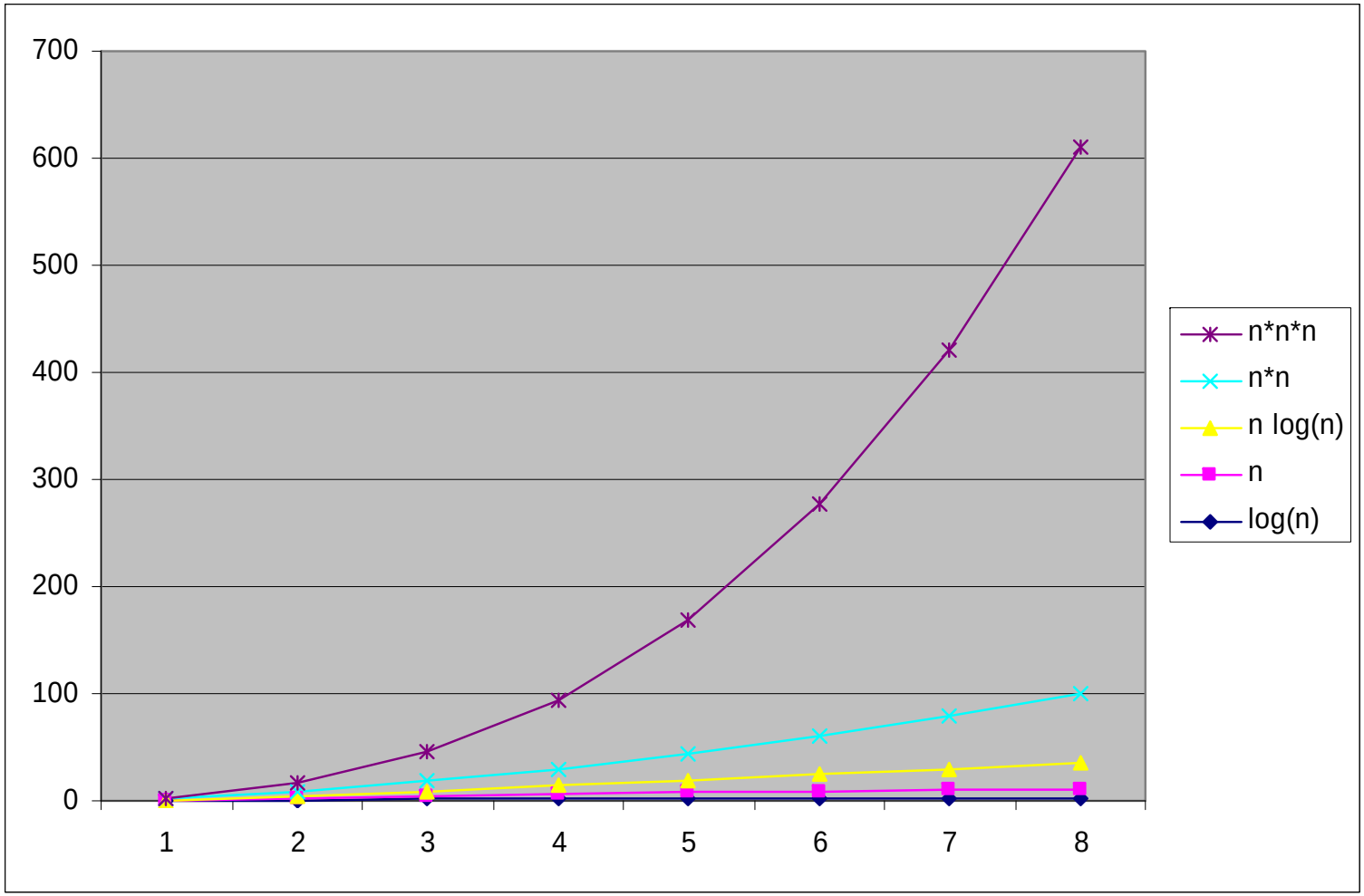
- *Time efficiency is analyzed by determining the number of repetitions of the basic operation as a function of input size.*



The efficiency analysis framework ignores the multiplicative constants of C(n) and **focuses on the orders of growth of the C(n).**

Analysis of Algorithms

Order of growth



Analysis of Algorithms

- ***Order of growth***

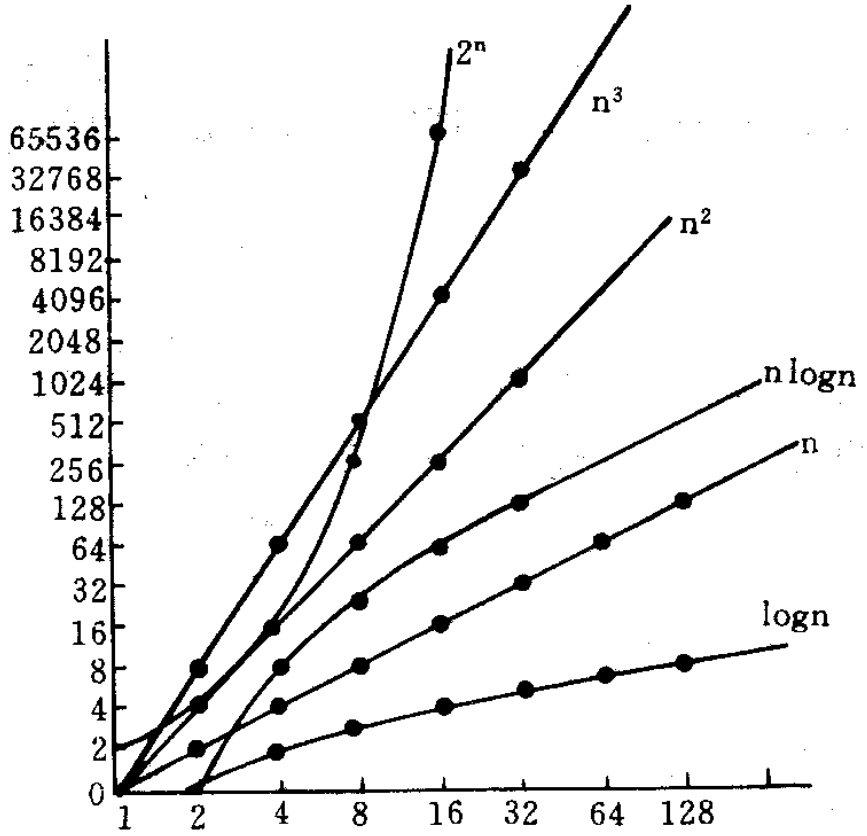


图 1.1　一般计算时间函数的曲线

Analysis of Algorithms

Basic Efficiency classes

The time efficiencies of a large number of algorithms fall into only a few classes.

fast

slow

1	constant
log *n*	logarithmic
n	linear
n log *n*	*n* log *n*
n^2	quadratic
n^3	cubic
2^n	exponential
n!	factorial

High time efficiency

low time efficiency

Analysis of Algorithms

Worst-Case, Best-Case, and Average-Case Efficiency

- *For some algorithms efficiency depends on type of input.*

  - ***Example: Sequential Search***
    - *Problem:* Given a list of *n* elements and a search key *K*, find an element equal to *K*, if any.
    - *Algorithm:* Scan the list and compare its successive elements with *K* until either a matching element is found (*successful search*) or the list is exhausted (*unsuccessful search*)

Given a sequential search problem of an input size of n,
what kind of input would make the running time the longest?
How many key comparisons?

Analysis of Algorithms

- ***Example: Sequential Search***

```
ALGORITHM SequentialSearch(A[0..n-1], K)
    //Searches for a given value in a given array by sequential
    search
    //Input: An array A[0..n-1] and a search key K
    //Output: Returns the index of the first element of A that
    matches K or –1 if there are no matching elements
    i ←0
    while i < n and A[i]≠K do
      i ← i + 1
    if i < n                        //A[i] = K
      return i
    else
      return -1
```

Analysis of Algorithms

- ***Example: Sequential Search***

  - probability for successful search is p ($0 \le p \le 1$);
  - probability for successful search on each position i ($0 \le i < n$) in an array is equal, p/n。

✓ dividing all instances of size n into several classes so that for each instance of the class the number of times the basic operation is executed is the same;
✓ a probability distribution of inputs is obtained or assumed

$$T_{avg}(n) = \sum_{size(I)=n} p(I)T(I)$$

$$= \left(1 \cdot \frac{p}{n} + 2 \cdot \frac{p}{n} + 3 \cdot \frac{p}{n} + \cdots + n \cdot \frac{p}{n}\right) + n \cdot (1-p)$$

$$= \frac{p}{n}\sum_{i=1}^{n} i + n(1-p) = \frac{p(n+1)}{2} + n(1-p)$$

  - T worst $(n)=n$
  - T bset $(n)=1$

Analysis of Algorithms

- ***Problem:*** *impossible to calculate e_i for every legal input I*
- ***Solution:*** *to calculate e_i for some representative input*

- *Worst case Efficiency*
  - ***Efficiency (# of times the basic operation will be executed) for the worst case input of size n***
  - *The algorithm runs the* ***longest*** *among all possible inputs of size n*
  - *To see what kind of inputs yield the* ***largest*** *value of the basic operation's count C(n) among all possible inputs of size n*
  - ***Bounding an algorithm's efficiency from above***

Analysis of Algorithms

- *Best case*
  - ***Efficiency (# of times the basic operation will be executed) for the best case input of size n.***
  - *The algorithm runs the **fastest** among all possible inputs of size n.*
  - *To see what kind of inputs yield the **smallest** value of the basic operation's count C(n) among all possible inputs of size n*
  - ***Bounding an algorithm's efficiency from above***
  - *If the best-case efficiency of an algorithm is unsatisfactory, we can immediately discard it.*

Analysis of Algorithms

- *Average case:*
  - ***Efficiency (#of times the basic operation will be executed) for a typical/random input of size n.***
  - ***NOT the average of worst and best case***
  - ***How to find the average case efficiency?***

$$T_{avg}(N) = \sum_{I \in D_N} P(I)T(N,I) = \sum_{I \in D_N} P(I)\sum_{i=1}^{k} t_i e_i(N,I)$$

Tavg cannot be obtained by taking the average of *Tworst* and *Tbest*

Analysis of Algorithms

Summary of the Analysis Framework

- *Time efficiency is measured by counting the number of basic operations executed in the algorithm. The space efficiency is measured by the number of extra memory units consumed.*
- *Both time and space efficiencies are measured as functions of input size.*
- *The framework's primary interest lies in the order of growth of the algorithm's running time (space) as its input size goes infinity.*
- *The efficiencies of some algorithms may differ significantly for inputs of the same size. For these algorithms, we need to distinguish between the worst-case, best-case and average case efficiencies.*

Asymptotic notations

Asymptotic complexity

- *If*

  $T(n) \to \infty$, as $n \to \infty$;

  $(T(n) - t(n))/ T(n) \to 0$, as $n \to \infty$;

  Then, $t(n)$ is called asymptotic state of $T(n)$, $n \to \infty$

  $t(n)$ is called asymptotic complexity of algorithm A, $n \to \infty$

  - ***Example:***

    for $T(n)=3n^2+4n\log n+7$, $t(n)=3n^2$

- *$t(n)$ consider only the leading term of $T(n)$*
- *ignore the constant coefficient*
- *only consider the rank of t(n)*

Asymptotic notations

- ***Three notations used to compare orders of growth of an algorithm's basic operation count***

  - *O(g(n)): class of functions t(n) that grow no faster than g(n)*

    Upper Bound

  - *Ω(g(n)): class of functions t(n) that grow at least as fast as g(n)*

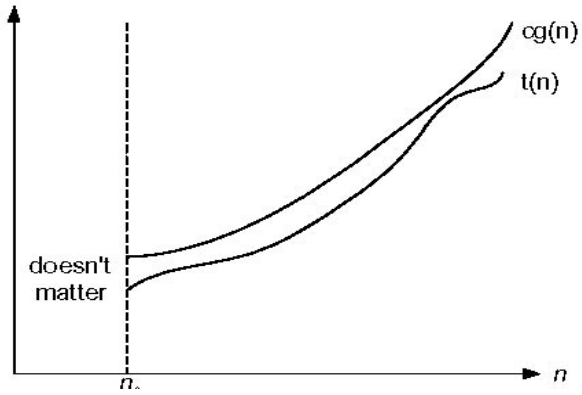
  - *Θ (g(n)): class of functions t(n) that grow at same rate as g(n)*

Asymptotic notations

O-notation

- *Formal definition:*
  - ***A function t(n) is said to be in O(g(n)), denoted t(n) ∈ O(g(n)), if t(n) is bounded above by some constant multiple of g(n) for all large n, i.e., if there exist some positive constant c and some nonnegative integer n_0 such that***

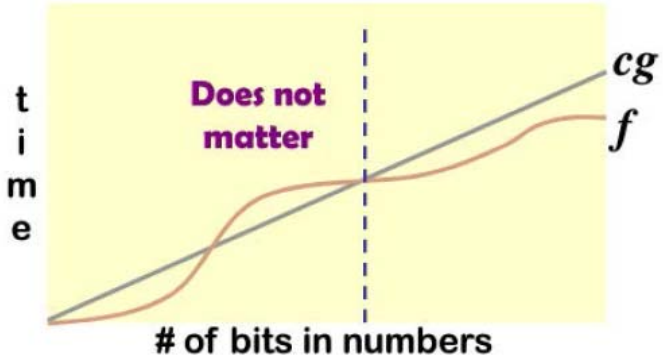
    $t(n) \leq cg(n) \quad \text{for all} \quad n \geq n_0$



- ***E.g.:***

$100n + 5 <= 100n + n = 101n <= 101n^2$

- ***Example:***



$n^2 \in O(n^2)$

$n^2 + 2n \in O(n^2)$

$n + 5 \in O(n^2)$

$20 \in O(n)$

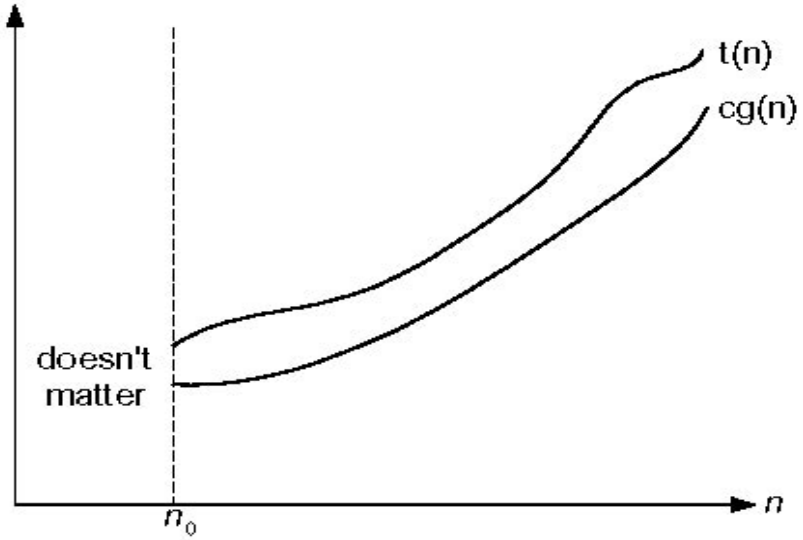
Asymptotic notations

Ω-notation

Formal definition:

- ***A function t(n) is said to be in Ω(g(n)), denoted t(n) ∈ Ω(g(n)), if t(n) is bounded below by some constant multiple of g(n) for all large n, i.e., if there exist some positive constant c and some nonnegative integer n_0 such that***

$$t(n) \geqslant cg(n) \quad \text{for all } n \geqslant n_0$$



- ***Example:***
  - $10n^2 \in \Omega(n^2)$
  - $10n^2 + 2n \in \Omega(n^2)$
  - $10n^3 \in \Omega(n^2)$

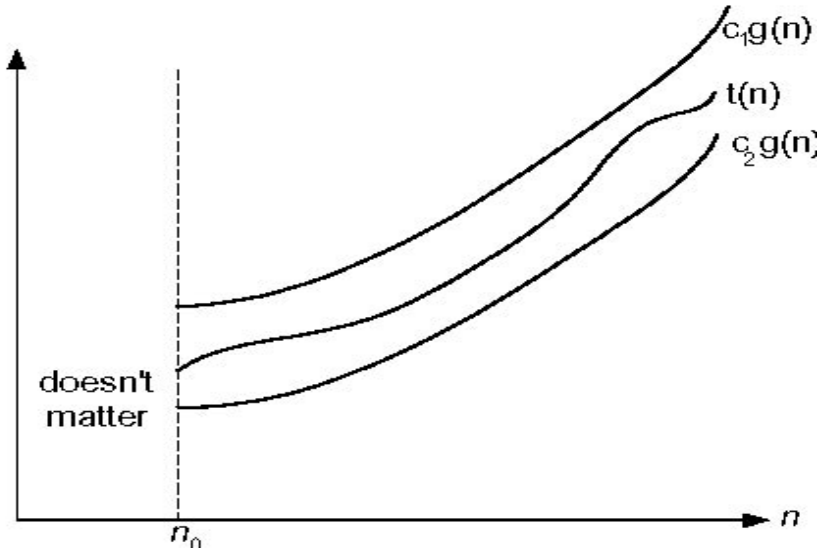
Asymptotic notations

Θ-notation

Formal definition:

- ***A function t(n) is said to be in Θ(g(n)), denoted t(n) ∈ Θ(g(n)), if t(n) is bounded both above and below by some positive constant multiples of g(n) for all large n, i.e., if there exist some positive constant c_1 and c_2 and some nonnegative integer n_0 such that***

$$c_2 g(n) \leq t(n) \leq c_1 g(n) \quad \text{for all } n \geq n_0$$



- ***Example:***
  - $10n^2 \in \Theta(n^2)$
  - $an^2 + bn + c \in \Theta(n^2)$ with a>0
  - $(1/2)n(n-1) \in \Theta(n^2)$
  - $n^2 + \log n \in \Theta(n^2)$

Asymptotic notations

Other notations

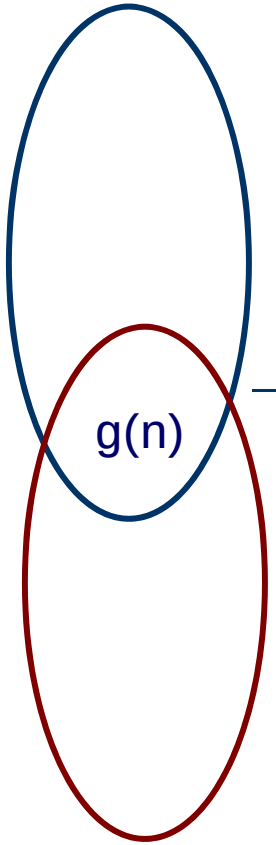
- *o(g(n)):*
  - ***A function t(n) is said to be in o(g(n)), denoted t(n) ∈ o(g(n)), if t(n) is bounded above by some positive constant multiples of g(n) for all large n, i.e., if there exist some positive constant c and some nonnegative integer n0 such that***

    $t(n) < cg(n) \quad \text{for all} \quad n \geqslant n_0$

- *ω(g(n)):*
  - ***A function t(n) is said to be in ω (g(n)), denoted t(n) ∈ ω(g(n)), if t(n) is bounded below by some positive constant multiples of g(n) for all large n, i.e., if there exist some positive constant c and some nonnegative integer n0 such that***

    $t(n) > cg(n) \quad \text{for all} \quad n \geqslant n_0$

Asymptotic notations



>= higher or same order of growth

$\Omega(g(n))$, functions that grow at least as fast as g(n)

=

$\Theta(g(n))$, functions that grow at the same rate as g(n)

<= smaller or same order of growth

$O(g(n))$, functions that grow no faster than g(n)

Asymptotic notations

Some Properties of Asymptotic Order of Growth

- $f(n) \in O(f(n))$ 反身性
- $f(n) \in O(g(n)), g(n) \in O(h(n)) \Rightarrow f(n) \in O(h(n))$；传递性

- $f(n) \in O(g(n)) \Leftrightarrow g(n) \in \Omega(f(n))$ 互对称性

  $f(n) \in \Theta(g(n)) \Leftrightarrow g(n) \in \Theta(f(n))$ 对称性

- $O(g_1(n))+O(g_2(n)) = O(g_1(n)+g_2(n))$； 数学计算
- $O(g_1(n))*O(g_2(n)) = O(g_1(n)*g_2(n))$；
- $O(cf(n)) = O(f(n))$；
- $g(n)= O(f(n)) \Rightarrow O(f(n))+O(g(n)) = O(f(n))$

The analogous assertions are true for the Ω-notation and Θ-notation.

Asymptotic notations

Some Properties of Asymptotic Order of Growth

- *If $t_1(n) \in O(g_1(n))$ and $t_2(n) \in O(g_2(n))$, then*

  $t_1(n) + t_2(n) \in O(\max\{g_1(n), g_2(n)\})$

- *Implication:*

  for an algorithm that comprises two consecutively executed parts,

  The algorithm's overall efficiency will be determined by the part with a larger order of growth.

  - ***Example:***

    - $5n^2 + 3n\log n \in O(n^2)$
    - *check whether an array has identical elements:*

      first, sort the array by some sorting alg.,

      ——*no more than* $(1/2)n(n-1)$ comparisons

      then, scan the sorted array to check its consecutive elements for equality

      ——*no more than* $(n-1)$ comparisons

Asymptotic notations

规则$O(g_1(n)) + O(g_2(n)) = O(\max\{g_1(n), g_2(n)\})$的证明：

- $t_1(n) \in O(g_1(n))$， there exist some positive constant c_1 and nonnegative integer n_1 , such that, for all $n \geq n_1$， $t_1(n) \leq c_1 g_1(n)$.
- $t_2(n) \in O(g_2(n))$， there exist some positive constant c_2 and nonnegative integer n_2 , such that, for all $n \geq n_2$， $t_2(n) \leq c_2 g_2(n)$.
- denote $c_3 = \max\{c_1, c_2\}$， $n_3 = \max\{n_1, n_2\}$， $h(n) = \max\{g_1(n), g_2(n)\}$ 。
- for all $n \geq n_3$， we have
- $t_1(n) + t_2(n) \leq c_1 g_1(n) + c_2 g_2(n)$

  $\leq c_3 g_1(n) + c_3 g_2(n) = c_3(g_1(n) + g_2(n))$

  $\leq c_3 2 \max\{g_1(n), g_2(n)\}$

  $= 2c_3 h(n) = O(\max\{g_1(n)\}, g_2(n)\})$.

Asymptotic notations

Using Limits for Comparing Orders of Growth

$\lim_{n\to\infty} T(n)/g(n) =$

- 0 — order of growth of $T(n)$ $<$ order of growth of $g(n)$
- $c>0$ — order of growth of $T(n)$ $=$ order of growth of $g(n)$
- ∞ — order of growth of $T(n)$ $>$ order of growth of $g(n)$

case 1 & 2， $T(n) \in O(g(n))$; *case 3 & 2，* $T(n) \in \Omega(g(n))$;
case 2， $T(n) \in \Theta(g(n))$;

- ***Example:***
  - $5n^2 + 3n\log n \in O(n^2)$
  - $10n$ *vs.* $2n^2$
  - $n(n+1)/2$ *vs.* n^2
  - $\log_b n$ *vs.* $\log_c n$

Asymptotic notations

L'Hôpital's rule

- *If* $\lim_{n\to\infty} f(n) = \lim_{n\to\infty} g(n) = \infty$ and the derivatives f', g' exist, Then

$$\lim_{n\to\infty}\frac{f(n)}{g(n)} = \lim_{n\to\infty}\frac{f'(n)}{g'(n)}$$

$$\lim_{n\to\infty}\frac{\log_2 n}{\sqrt{n}} = \lim\frac{(\log_2 n)'}{(n^{\frac{1}{2}})'} = \lim\frac{\frac{1}{n\ln 2}}{\frac{1}{2}n^{-\frac{1}{2}}} = \frac{2}{\ln 2}\lim\frac{n^{\frac{1}{2}}}{n} = 0$$

$$\Rightarrow \log_2 n \in O(n^{\frac{1}{2}})$$

  - ***Example:***
    - $(1/2)n(n\text{-}1) \in \Theta(n^2)$
    - $\log_2 n \in O(n^{1/2})$
    - $\log_2 n$ *vs.* n
    - 2^n *vs.* $n!$

Asymptotic notations

Orders of growth by some important functions

- *All logarithmic functions $\log_a n$ belong to the same class $\Theta(\log n)$ no matter what the logarithm's base $a > 1$ is.*
- *All polynomials of the same degree k belong to the same class: $a_k n^k + a_{k-1} n^{k-1} + \ldots + a_0 \in \Theta(n^k)$.*
- *Exponential functions a^n have different orders of growth for different a's.*
- *order $\log n$ < order n^α (α>0) < order a^n < order $n!$ < order n^n*

Asymptotic notations

some useful functions

取整函数

- $x-1 < \lfloor x \rfloor \le x \le \lceil x \rceil < x+1$;
- $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$;
- 对于$n \ge 0$，$a,b>0$，有：
- $\lceil \lceil n/a \rceil /b \rceil = \lceil n/ab \rceil$;
- $\lfloor \lfloor n/a \rfloor /b \rfloor = \lfloor n/ab \rfloor$;
- $\lceil a/b \rceil \le (a+(b-1))/b$;
- $\lfloor a/b \rfloor \ge (a-(b-1))/b$;
- $f(x)=\lfloor x \rfloor$, $g(x)=\lceil x \rceil$为单调递增函数。

Asymptotic notations

some useful functions

- 多项式函数
  - $p(n)= a_0+a_1n+a_2n^2+\ldots+a_dn^d$； $a_d>0$;
  - $p(n) = \Theta(n^d)$;
  - $f(n) = O(n^k) \Leftrightarrow f(n)$多项式有界；
  - $f(n) = O(1) \Leftrightarrow f(n) \le c$;
  - $k \ge d \Rightarrow p(n) = O(n^k)$;
  - $k \le d \Rightarrow p(n) = \Omega(n^k)$;
  - $k > d \Rightarrow p(n) = o(n^k)$;
  - $k < d \Rightarrow p(n) = \omega(n^k)$.

Asymptotic notations

some useful functions

指数函数

- 对于正整数m,n和实数$a>0$:
- $a^0=1$;
- $a^1=a$;
- $a^{-1}=1/a$;
- $(a^m)^n = a^{mn}$;
- $(a^m)^n = (a^n)^m$;
- $a^m a^n = a^{m+n}$;
- $a>1 \Rightarrow a^n$为单调递增函数;
- $a>1 \Rightarrow \lim_{n\to\infty} \frac{n^b}{a^n} = 0 \Rightarrow n^b = o(a^n)$

Asymptotic notations

some useful functions

指数函数(2)

- $e^x \geq 1+x$;
- $|x| \leq 1 \Rightarrow 1+x \leq e^x \leq 1+x+x^2$;
- $e^x = 1+x+\Theta(x^2)$, as $x \to 0$;

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{i=0}^{\infty} \frac{x^i}{i!}$$

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

Asymptotic notations

some useful functions

对数函数

- $\log n = \log_2 n$; $\lg n = \log_{10} n$; $\ln n = \log_e n$;
- $\log^k n = (\log n)^k$;
- $\log \log n = \log(\log n)$;
- for a>0,b>0,c>0

$$a = b^{\log_b a} \qquad \log_b(1/a) = -\log_b a \qquad \log_b a = \frac{1}{\log_a b}$$

$$\log_b a^n = n \log_b a \qquad \log_b a = \frac{\log_c a}{\log_c b}$$

$$\log_c(ab) = \log_c a + \log_c b \qquad a^{\log_b c} = c^{\log_b a}$$

Asymptotic notations

some useful functions

对数函数(2)

- $|x| \le 1 \Rightarrow \quad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots.$

- for $x > -1$, $\quad \frac{x}{1+x} \le \ln(1+x) \le x$

- for any $a > 0$, $\quad \lim_{n\to\infty} \frac{\log^b n}{(2^a)^{\log n}} = \lim_{n\to\infty} \frac{\log^b n}{n^a} = 0$, $\Rightarrow \log^b n = o(n^a)$

Asymptotic notations

some useful functions

- 阶乘函数

$$n! = \begin{cases} 1 & n = 0 \\ n(n-1)! & n > 0 \end{cases}$$

$$n! = 1 \cdot 2 \cdot 3 \cdots n$$

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1+\Theta\left(\frac{1}{n}\right)\right)$$

Stirling's approximation

Asymptotic notations

some useful functions

- 阶乘函数（2）

$$n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{\alpha_n} \qquad \frac{1}{12n+1} < \alpha_n < \frac{1}{12n}$$

$$n! = o(n^n)$$

$$n! = \omega(2^n)$$

$$\log(n!) = \Theta(n \log n)$$

Asymptotic notations

some useful functions

常用的整数求和公式

算法分析中，在统计语句的频率时，求和公式的一般形式为：

$$\sum_{g(n) \le i \le h(n)} f(i)$$

如：

$$\sum_{1 \le i \le n} i^k = \Theta(n^{k+1})$$

Asymptotic notations

- ***Summary of How to Establish Orders of Growth of an Algorithm's Basic Operation Count***
  - *Method 1: Using limits*
    - ***L'Hôpital's rule***
  - *Method 2: Using the properties*
  - *Method 3: Using the definitions of O-, Ω-, and Θ-notation.*

the time efficiencies of a large number of algorithms fall into a few classes, as see in the list.

Time Efficiency of Non-recursive Algorithms

- ***Steps in mathematical analysis of nonrecursive algorithms:***
  - *Decide on parameter n indicating input size*
  - *Identify algorithm's basic operation*
  - *Check whether the number of times the basic operation is executed depends only on the input size n. If it also depends on the type of input, investigate worst, average, and best case efficiency separately.*
  - *Set up **summation** for C(n) reflecting the number of times the algorithm's basic operation is executed.*
  - *Simplify summation to find a closed-form formula or, at the very least, find its order of growth, using standard formulas (see Appendix A)*

Time Efficiency of Non-recursive Algorithms

- *useful basic rules and standard formulas for sum manipulation*

$$\sum_{i=l}^{u}(a^i \pm b^i) = \sum_{i=l}^{u} a^i \pm \sum_{i=l}^{u} b^i$$

$$\sum_{i=l}^{u} 1 = u - l + 1$$

$$\sum_{i=0}^{n} i = \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \approx \frac{1}{2}n^2 \in \Theta(n^2)$$

Time Efficiency of Non-recursive Algorithms

Example 1: Maximum element

ALGORITHM *MaxElement*$(A[0..n-1])$

```
//Determines the value of the largest element in a given array
//Input: An array A[0..n − 1] of real numbers
//Output: The value of the largest element in A
maxval ← A[0]
for i ← 1 to n − 1 do
    if A[i] > maxval
        maxval ← A[i]
return maxval
```

- ***Basic operation: comparison*** (in the for loop, and executed on each repetition)
- ***Input size: array length n***
- ***time efficiency :***

$$C(n) = \sum_{i=1}^{n-1} 1 = n - 1 = \Theta(n)$$