華中科技大學
HUAZHONG UNIVERSITY OF SCIENCE AND TECHNOLOGY

随机过程

Stochastic Process

§ 4.4 C-K方程的证明

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C-K方程

定理4.3.4

设$\{X_n, n \in \mathbb{N}_0\}$为一Markov链，则

$$p_{ij}^{(n)} = \sum_{k \in I} p_{ik}^{(l)} p_{kj}^{(n-l)}, \qquad l = 0,1,\cdots,n.$$

C-K方程的证明

证明 对任意$l = 1,2,\cdots,n-1$,

$$
\begin{aligned}
p_{ij}^{(n)} &= P(X_{m+n} = j|X_m = i) \\
&= P\left(X_{m+n} = j, \sum_{k\in I}\{X_{m+l} = k\} \mid X_m = i\right) \\
&= \sum_{k\in I} P(X_{m+n} = j, X_{m+l} = k \mid X_m = i) \\
&= \sum_{k\in I} P(X_{m+l} = k \mid X_m = i) P(X_{m+n} = j \mid X_m = i, X_{m+l} = k) \\
&= \sum_{k\in I} P(X_{m+l} = k \mid X_m = i) P(X_{m+n} = j|X_{m+l} = k)\ .
\end{aligned}
$$

两状态Markov链

例4.4.1

设Markov链$\{X_n, n \in \mathbb{N}_0\}$的状态空间$I = \{0,1\}$, 转移概率矩阵$\mathbb{P} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$.

$\mathbb{P}$的两个特征值为$\lambda_1 = 1, \lambda_2 = 1-p-q$, 将$\mathbb{P}$对角化，得到

$$\mathbb{P}^{(n)} = \begin{pmatrix} \dfrac{q}{p+q} + \dfrac{p}{p+q}(1-p-q)^n & \dfrac{p}{p+q} - \dfrac{p}{p+q}(1-p-q)^n \\ \dfrac{q}{p+q} - \dfrac{p}{p+q}(1-p-q)^n & \dfrac{p}{p+q} + \dfrac{q}{p+q}(1-p-q)^n \end{pmatrix}.$$

当$0 < p+q < 2$时，有

$$\lim_{n\to+\infty} \mathbb{P}^{(n)} = \begin{pmatrix} \dfrac{q}{p+q} & \dfrac{p}{p+q} \\ \dfrac{q}{p+q} & \dfrac{p}{p+q} \end{pmatrix}.$$

作业

构造10个状态的随机矩阵，用Matlab求其幂，看看有什么极限性质.

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谢 谢 ！