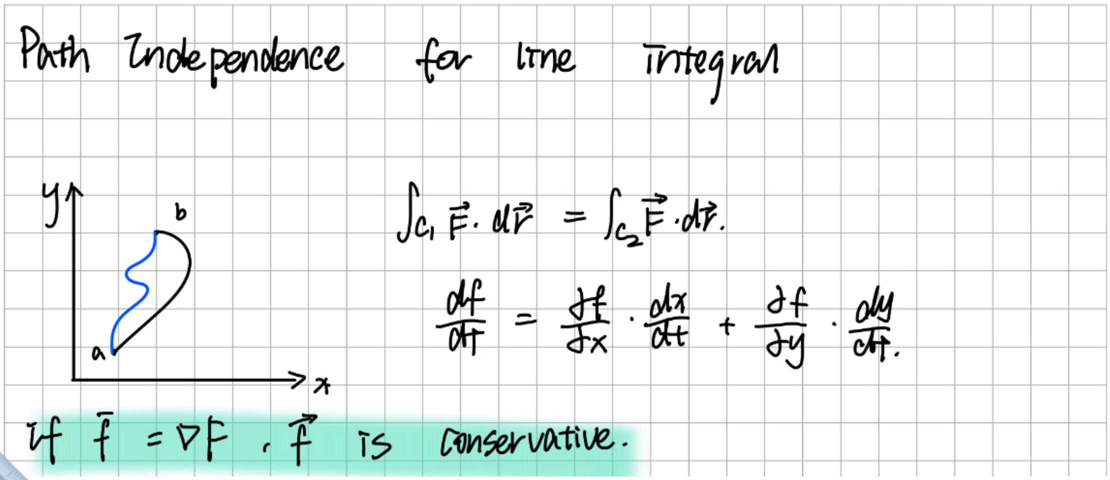


Modeling Work and Electric Fields Using Line Integrals in Vector Fields

🎯 Motivation

At my school, traditional science education was dominated by rote problem-solving and low-efficiency lectures, with virtually no access to research opportunities. Driven by curiosity and a desire to go beyond test prep, I launched a self-guided theoretical research project that connected multivariable calculus with AP Physics C concepts.

My goal was to explore how vector calculus—specifically line integrals—could be used to model work done by electric fields, and what happens when these fields are not static.

📐 Conceptual Framework

This project bridges mathematics and physics:

- **Mathematical Concepts:**
    - Line integrals of vector fields
    - Conservative fields and path independence
    - Parametrized paths and closed-loop integrals
- **Physics Concepts:**

- Work done by electric fields on particles
- Electric potential and energy conservation
- Circulation of electric fields in non-static conditions

Notes:

Parametrization of a reverse path

C:

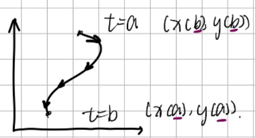
$x = x(t)$

$y = y(t)$.

$a \le t \le b$

$W = \int_C f(x,y)\, ds$

−C:



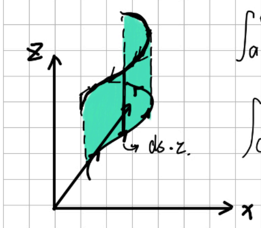
$x = x(a+b-t)$.

$y = y(a+b-t)$

$a \le t \le b$.

$W = \int_{-C} f(x,y)\, ds$.

Scalar field line integral



$$\int_a^b f(x) = -\int_b^a f(x)\, dx.$$

$$\int_C f(x,y)\, ds = \int_a^b f(x(t), y(t)) \sqrt{x'(t)^2 + y'(t)^2}$$

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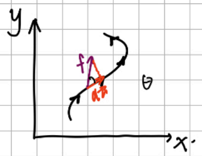
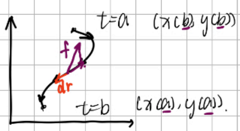
Vector field line Integrals

$\vec{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}$





$\vec{r}(t) = x(a+b-t)\,\hat{i} + y(a+b-t)\,\hat{j}$

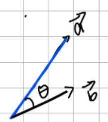
$$\int_C \vec{F}\cdot d\vec{r} = -\int_{-C} \vec{F}\cdot d\vec{r}.$$

Connect with physics

Summary: if $\theta < \frac{\pi}{2}$, object does positive work.

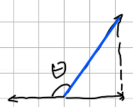
$\boxed{0 \le \theta \le \pi}$ if $\theta > \frac{\pi}{2}$ object does negative work.

Specifically:



$0 < \theta < \frac{\pi}{2}$

$\text{work} = |\vec{a}|\,|\vec{b}|\cos\theta > 0$



$\frac{\pi}{2} < \theta < \pi$

$\text{work} = |\vec{a}|\,|\vec{b}|\cos\theta < 0.$

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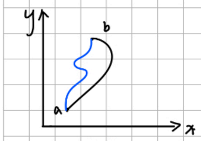
Path Independence for line Integral



$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}.$$

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dt}.$$

if $\vec{f} = \nabla F$, $\vec{f}$ is conservative.

$$\nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}\right)$$

$$\vec{f}(x,y) = (P(x), Q(y))$$

if $\frac{\partial P(x)}{\partial y} = \frac{\partial Q(y)}{\partial y}$, $\vec{f}$ is conservative.

Proof: 梯度 (gradient).

$$\vec{f}(x,y) = \nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (P(x), Q(y))$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

$$\boxed{\frac{P(x)}{\partial y} = \frac{Q(x)}{\partial x}}$$

Connect to physics

Calculate work done by a object. which is followed by a specific trace that can be explained by function.

Scope:

electric field magnetic field

gravitational field.

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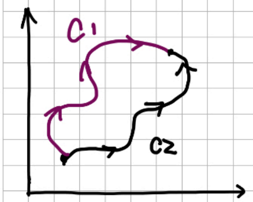
Closed curve line integrals (vector fields)



$$\int_{C_2} \vec{F}\cdot d\vec{r} = -\int_{-C_1} \vec{F}\cdot d\vec{r}$$

$\vec{F}(x,y) = \nabla F = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} \Rightarrow \vec{F}$ is a $\boxed{\text{conservative}}$ field

$$\underbrace{\int_{C_1}\vec{F}\cdot d\vec{r} + \int_{-C_2}\vec{F}\cdot d\vec{r}}_{} = 0$$

$$\boxed{\oint_{C_1+-C_2}\vec{F}\cdot d\vec{r}} = 0. \quad \text{if conservative field.}$$

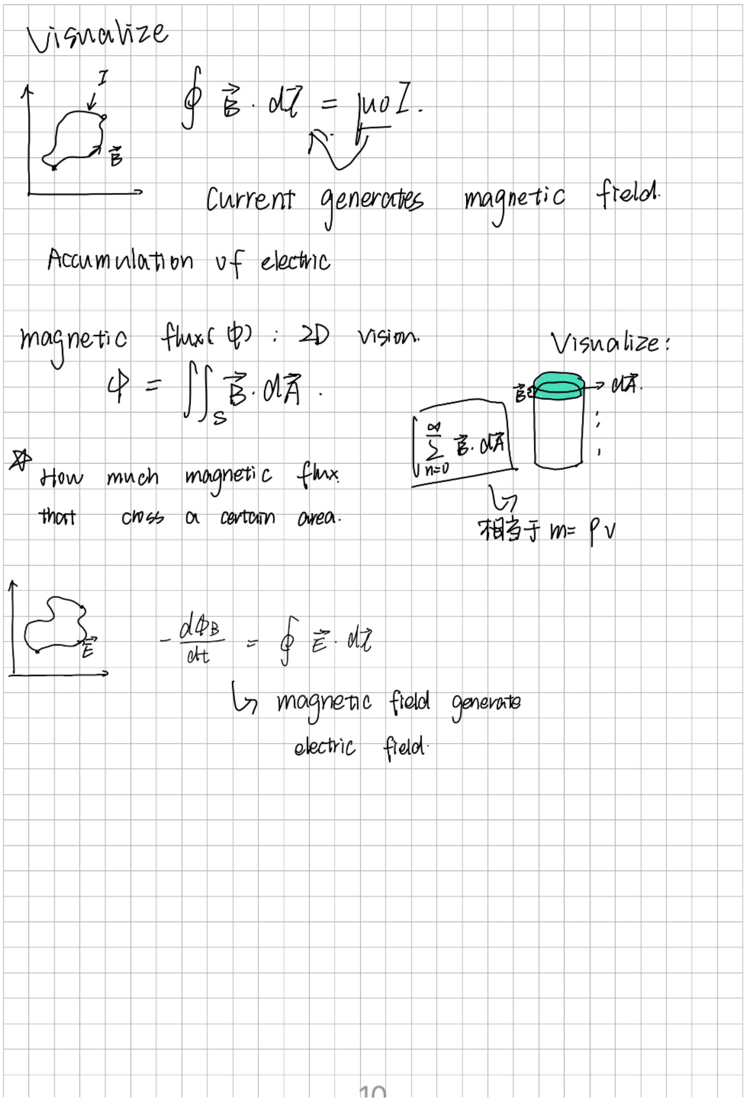
Connect to physics.

Ampère's law:

$$\oint_C \vec{B}\cdot d\vec{l} = \mu_0 I$$

Faraday's law :

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d\phi_B}{dt}.$$



I started with a core question:

"If electric fields are modeled as vector fields, can we use line integrals to determine whether they are truly conservative?"

$$W = \int_C \vec{E} \cdot d\vec{r}$$

Through parametrizing the path and computing the dot product between electric field vectors and displacement vectors, I verified that the total work done is path-independent in conservative fields.

And particle moves in a path(function r(x)), we first should parametrize function r(x) and write in the form of :

$$\vec{r} = x(t)\,\hat{i} + y(t)\,\hat{j}, \quad \frac{d\vec{r}}{dt} = \frac{dx}{dt}\,\hat{i} + \frac{dy}{dt}\,\hat{j}$$

For example,

A grain of charge q=2μC along the path C from the point A(0,0),

B(1,1), the path is a straight line. There is an electric field in space:

$$\vec{E}(x,y) = 2x\hat{i} + 3y^2\hat{j}$$

Calculate the charge in the path in the process of movement on C, the electric field does to it.

Solution:

$$W = q\int_C \vec{E}\cdot d\vec{r}$$

1. Path parameterization:

The path is a straight section. A(0,0)→B(1,1), can be expressed as:

$$\vec{r} = t\,\hat{i} + t\,\hat{j}$$

$$d\vec{r} = (\hat{i} + \hat{j})dt$$

2. **Substitute into the electric field expression along the path:**

There is on the path x=t. y=t, so:

$$\vec{E}(t) = 2t\hat{i} + 3t^2\hat{j}$$

3. Calculate the point product:

$$E(t)\cdot dr = (2t)(1) + (3t^2)(1) = 2t + 3t^2$$

4. Integral calculation:

$$W = q\int(2t + 3t^2)dt = 2q = 4\mu J$$

2. Electric Circulation and Closed Paths

When the motion happens along a **closed loop**, such as in a conducting wire loop, we write:

$$\oint_C \vec{E} \cdot d\vec{r}$$

This models the **electric circulation** (the total work done by the electric field around a closed path). In Faraday's Law, this is related to **changing magnetic flux**.

For example,

C is a unit circle (radius 1), circled around the origin in the counterclockwise direction. Calculate the induced electromotive potential along the path

The known distribution of the induced electric field is:

$$\vec{E}(x, y) = k(-y\hat{i} + x\hat{j})$$

Path parameterization:

The unit circle, the polar coordinates are parameterized as:

$$\vec{r}(t) = cost\hat{i} + sint\hat{j}$$

$$d\vec{r} = (-sint\hat{i} + cost\hat{j})dt$$

Substitute electric field expression:

$$\vec{E}(\vec{r}(t)) = k(-sint\hat{i} + costj)$$

Calculate for product:

$$E(t) \cdot d\vec{r}(t) = [k(-sint)i + kcostj] \cdot [-sinti + cost\hat{j}] = k$$

Calculate integral:

$$\oint_C \vec{E} \cdot d\vec{r} = 2\pi k$$

3. Linking Line Integral and Surface Integral (Faraday's Law)

Faraday's Law says that a **changing magnetic flux** induces an **electric field** around a loop. In math:

$$\oint_C \vec{E} \cdot d\vec{r} = -\frac{d\Phi_B}{dt}$$

This equation links:

- **Left**: a line integral (circulation of electric field around a loop)
- **Right**: the **rate of change** of magnetic flux (a surface integral)

4. **Negative gradient(Lenz's law):**

$$\mathcal{E} = -\frac{d\Phi_B}{dt}$$

Why Lenz's law is considered as negative gradient?

> The induced electric potential is equal to the negative value of the rate of change of magnetic flux over time, reflecting the physical idea of "obstructing the original change"

Visualization Analogy：

- **Line integral** = Accumulation **along a curve** → like walking along a path and summing small steps of force.
- The **negative sign** in Faraday's Law reflects **Lenz's Law**: the induced electric field **opposes** the change in magnetic flux(negative gradient)

Where I got stuck / What I want to improve

I didn't rely on any outside coursework or videos—this was entirely self-driven. But I encountered real difficulties:

- I couldn't initially understand why $\partial \vec{E}/\partial t$ appeared in Maxwell's equations. Why not total derivative?
- I wasn't clear on the meaning of **displacement current**, or how it fits into Ampère's law.
- I didn't know how to visualize vector fields beyond simple 2D paths—no Desmos or Python tools used.

But these challenges only deepened my curiosity. I recorded all questions and connected them to topics like vector curl and divergence, which I plan to explore after completing Calc 3.

💡Next Steps

- Develop a visualization system using Python/Matplotlib for time-varying field paths.
- Expand into Maxwell's equations and attempt my own derivations for displacement current.

🌟Takeaway

This project wasn't about solving a textbook problem—it was about designing my own way to *ask* one. It's one of the first times I saw math and physics converge into a single, elegant system—built not with labs or scripts, but through questioning, modeling, and pure logic.