Correspondence between surface integral and electromagnetic phenomenon

🧮 prerequisite for Calc 3

knowledge	use
Vector Field	Describe the electric field/magnetic field
Surface Integral	Describe the magnetic flux through the closed loop
Stokes theorem	Connect line and surfaces integral, supported by core theory
The chapter of divergence and curvature	The direction of the magnetic field

⚡ prerequisite for physics

Knowledge	Use
Faraday's Law of Electromagnetic Induction	Changing electric field → magnetic field → induced magnetic flux
Ampere’s law	Changing magnetic field→ Changing electric field→ induced electric current
Maxwell equations	predict the existence of electromagnetic waves, such as light. And made the expression more valid

Note

Surface Integral

Review:

$$\frac{\partial \vec{r}}{\partial s} = \lim_{\Delta s \to 0} \frac{\vec{r}(s+\Delta s, t) - \vec{r}(s,t)}{\Delta s}$$

correspond to: $\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

$$\boxed{\frac{\partial \vec{r}}{\partial s} = \frac{\partial x}{\partial s}\hat{i} + \frac{\partial y}{\partial s}\hat{j} + \frac{\partial z}{\partial s}\hat{k}}$$

Paramatrization of surface:

$$\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$

Expression of S & $\vec{S}$:

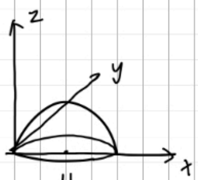
$$\boxed{\begin{aligned} dS &= \|\vec{r}_u \times \vec{r}_v\| \, du \, dv \\ d\vec{S} &= (\vec{r}_u \times \vec{r}_v) \, du \, dv \end{aligned}}$$

scalar field surface integral:

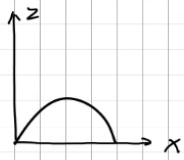
$$\iint_S f(x,y,z) \, dS = \iint_D f(\vec{r}(u,v)) \cdot \|\vec{r}_u \times \vec{r}_v\| \, du \, dv$$

$$= \iint_D f(x,y,z(x,y)) \cdot \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dx \, dy.$$

Eg: $\vec{r}(\phi, \theta) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} R\sin\phi\cos\theta \\ R\sin\phi\sin\theta \\ R\cos\phi \end{bmatrix}$

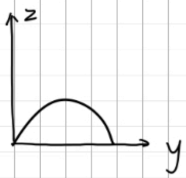


$\Rightarrow$



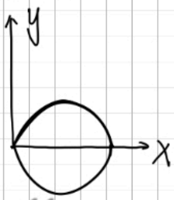
polar region.
angle : ϕ
$\phi \in [0, \frac{\pi}{2}]$

$\Downarrow$



polar region.
angle : ϕ
$\phi \in [0, \frac{\pi}{2}]$

$\Rightarrow$



polar region
angle : θ
$\theta \in [0, 2\pi]$

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$$\iint_S f(x,y,z)\, dS = \iint_D f(x,y,z(x,y))\, dS$$

$$|\vec{r}_x \times \vec{r}_y| = R^2 \sin\phi = |\vec{r}_x|\,|\vec{r}_y| \sin\phi.$$

$$dS = R^2 \sin\phi \, d\phi \, d\theta.$$

Vector field surface integral

$$\iint_S \vec{F} \cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

$$\vec{r}_x \times \vec{r}_y = \left\langle \frac{\partial \vec{r}}{\partial x}, \frac{\partial \vec{r}}{\partial y}, \frac{\partial \vec{r}}{\partial z} \right\rangle.$$

$$\boxed{\vec{r}_x \times \vec{r}_y = \left\langle -\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1 \right\rangle}$$ specific situation

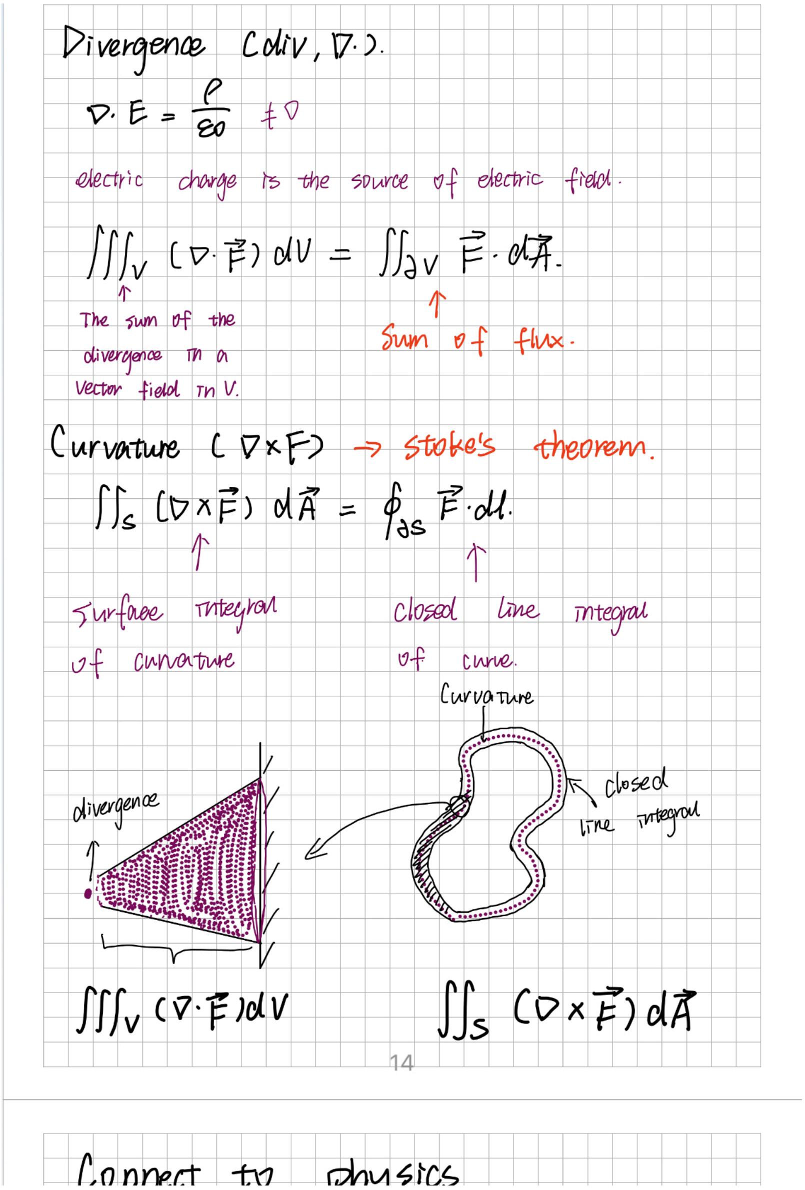
Review:

$\vec{a} = \langle a_1, a_2, a_3 \rangle$ $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$.

$\vec{b} = \langle b_1, b_2, b_3 \rangle$ $b_1 \neq 0$ $b_2 \neq 0$ $b_3 \neq 0$.

$$\vec{a} \times \vec{b} = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

$$= \begin{bmatrix} a_2 & a_3 \\ b_2 & b_3 \end{bmatrix} \hat{i} + \begin{bmatrix} a_3 & a_1 \\ b_3 & b_1 \end{bmatrix} \hat{j} + \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix} \hat{k}.$$



Divergence (div, $\nabla\cdot$).

$\nabla\cdot E = \frac{\rho}{\varepsilon_0} \neq 0$

electric charge is the source of electric field.

$\iiint_V (\nabla\cdot\vec{F})\,dV = \iint_{\partial V} \vec{F}\cdot d\vec{A}$.

The sum of the divergence in a vector field in V.

Sum of flux.

Curvature ($\nabla\times F$) → stoke's theorem.

$\iint_S (\nabla\times\vec{F})\,d\vec{A} = \oint_{\partial S} \vec{F}\cdot dl$.

Surface integral of curvature

closed line integral of curve.

$\iiint_V (\nabla\cdot\vec{F})\,dV$

$\iint_S (\nabla\times\vec{F})\,d\vec{A}$

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Connect to physics

🧠 Inspiration & Conceptual Reflection

When I first learned the formula for surface integrals, especially the general form:

$$\Phi = \iint_S \vec{F}\cdot d\vec{A}$$

realized this was exactly how electric flux and magnetic flux are calculated in electromagnetism. The vector field F⃗ becomes the electric field E⃗ or magnetic field B⃗, and dA⃗ describes the orientation of the surface. It immediately connected physics and multivariable calculus in a way that felt elegant and powerful.

Later, when I studied **divergence** and **curl**, I found even deeper connections:

- **Divergence** tells us how much a field “spreads out” from a point — this matches the idea of sources or sinks in electric fields (like charges).
- **Curl** shows the local “circulation” of a field — just like how a changing electric field creates a circulating magnetic field in Maxwell's equations.

What amazed me is that these abstract vector operations could **explain the directionality and behavior of physical fields** — they are not just mathematical tools, but physical truths. This realization inspired me to look at each of Maxwell’s equations not only as physical laws, but as a consistent mathematical system describing how information flows in space.

Let’s get into the most essential part:

In a conservative vector field (like a magnetic field), the net flux through any closed surface is zero.

In multi-calc’s point of view, if:

$$\nabla \cdot B = 0$$

Therefore, the flux through any closed curved surface is zero.

Based on this model, we can apply it into magnetic field and electric field to determine whether their flux is zero or not.

Intuitively we can consider the flux in all magnetic fields are zero since there are no magnetic monopoles in nature, the magnetic field is passive, so the magnetic flux through any closed surface is zero.

$$\oint_{\partial V} \vec{B} \cdot d\vec{A} = 0$$

For example: The magnetic flux passes through the cylinder(calculation)

A magnetic field🧲

$$\vec{B}(x, y, z) = \langle y, x, z \rangle$$

Given a **cylindrical surface(closed surface integral)**, the radius is 2, height is from z=0 to z=3. The side formed by the rotation of the z-axis. Find the magnetic flux through the cylinder (side only)

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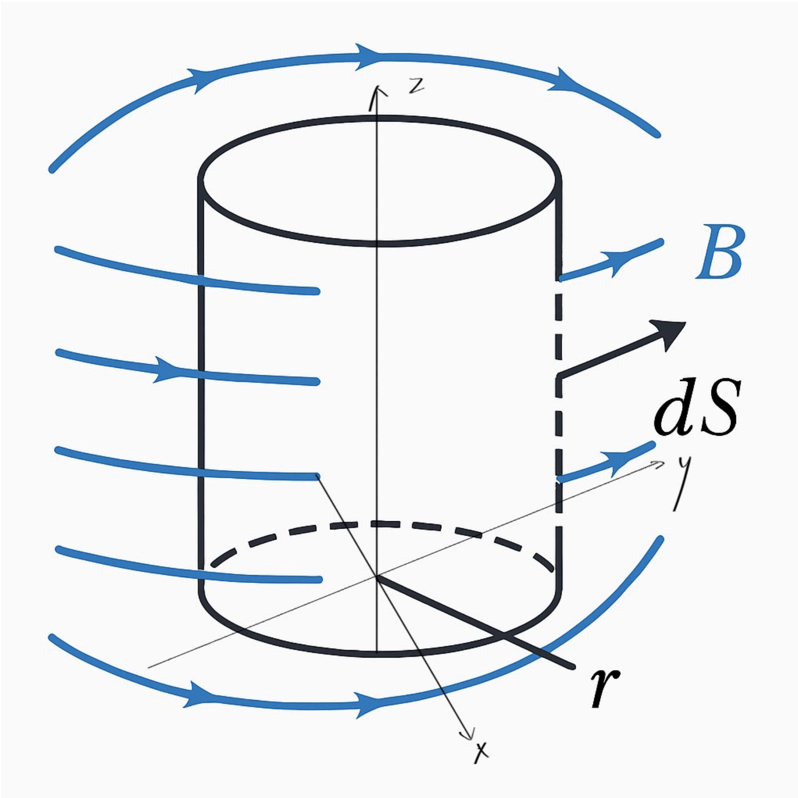


Step 1:Parametrization

Among them, the direction is taken as the outward-facing normal vector (that is, pointing along the outside of the cylinder)

$$r(\theta, z) = \langle 2cos\theta, 2sin\theta, z\rangle(\theta \in [0, 2\pi],\ z \in [0, 3])$$

Step 2: Find the partial derivative and cross product, and get the outward surface direction vector.

$$\frac{\partial r}{\partial \theta} = \langle -2sin\theta, 2cos\theta, 0\rangle, \frac{\partial r}{\partial z} = \langle 0, 0, 1\rangle$$

$$d\vec{S} = \frac{\partial r}{\partial \theta} \times \frac{\partial r}{\partial z} = \langle 2cos\theta, 2sin\theta, 0\rangle dzd\theta$$

Step 3: Plug in vector r function

$$\vec{B}(\vec{r}(\theta,z)) = \langle 2sin\theta, 2cos\theta, z\rangle$$

Step 4: dot product:

$$\vec{B}\cdot d\vec{S} = 8sin\theta cos\ \theta$$

$$\Phi_B = \int_0^3 \int_0^{2\pi} 8sin\theta cos\theta = 0$$

$$Therefore, \varepsilon = \frac{d\Phi_B}{dt} = 0$$

Therefore, we can also conclude that if there is no electric potential induction in the coil(closed surface integral), it means that the magnetic flux does not change over time.

So come back to the introduction of ∇·B=0, how does the magnetic flux change when ∇·B≠ 0

If ∇·B≠ 0, according to Gauses' divergence theorem

$$\Phi_B = \iiint(\nabla\cdot B)dV = \oiint B\cdot dA \neq 0$$

There is a magnetic flux that pierces a closed surface, which means that in the volumeV There is a source inside, that is, magnetic monopoles.

But in all experiments, including high-precision detection experiments, so far, the existence of magnetic monopoles has not been found.

Therefore, assuming ∇·B≠0 is wrong, which leads to the contradiction between the conclusion and the experimental results. And thus infer that:

$$\nabla\cdot B = 0$$

It shows that the magnetic field line has no starting point and end point, and can only be a closed loop. However, it failed to explain how the magnetic field is excited or rotates around the current.

So lets analyze the rotation (∇×B) of magnetic flux by curvature in multivariable calculus!

According to the Ampere's law:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

According to the Stokes's theorem:

$$\iint (\nabla \times \vec{B}) d\vec{A} = \oint \vec{B} \cdot d\vec{l} = \mu_0 I$$

We all know:

$$I = \int J \cdot d\vec{A}$$

Plug current into Stokes's theorem:

$$\iint (\nabla \times \vec{B}) d\vec{A} = \mu_0 \int J \cdot d\vec{A}$$

Therefore, we conjure up the expression of ∇×B!

$$\nabla \times \vec{B} = \mu_0 J$$

However, this caused a **problem** when applied to a charging capacitor. Inside the capacitor, **there is no conduction current** (no actual charges moving), but the magnetic field still changes — how is that possible?

Maxwell realized that **a changing electric field** can also create a magnetic field.

And then he conjured up an expression called:

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$

But this expression sounds too complicated to understand so far, we don't know how this expression changed.

However, we can try to connect to electric field because we saw the "E" in this new expression.

$$\oint_{\partial V} \vec{E} \cdot d\vec{A} = \frac{q_n}{\varepsilon_0}$$

We all know:

$$q_n \neq 0$$

Therefore:

$$\oint_{\partial V} \vec{E} \cdot d\vec{A} \neq 0$$

However, we can use divergence to answer this question and then explain that electric fields have charged particles, which will generate electric current in physics language.

First we should find the expression of ∇·E.

∇·E is the "divergence intensity" of a certain point in the vector field.

According to the function

$$\iiint (\nabla \cdot E) dV = \oint E \cdot dA = \frac{q_n}{\varepsilon_0}$$

As we all know

$$\int \rho dV = q$$

$$\frac{1}{\varepsilon_0} \int \rho dV = \int_V (\nabla \cdot E) dV$$

$$\nabla \cdot E = \frac{\rho}{\varepsilon_0}$$

Therefore, the divergence of the electric field at a point is proportional to the local charge density.

If it caused the change of electric flux, would it affect the magnetic flux?

According to Gauses’s law

$$I = \frac{q}{t}$$

$$\Phi_E = \int \vec{E} d\vec{A} = \frac{q}{\varepsilon_0}$$

$$\frac{d\Phi_E}{dt} = \frac{\int \vec{E} d\vec{A}}{dt} = \int \vec{E} dl = \frac{I}{\varepsilon_0}$$

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt}$$

Plug it into formula of the Ampere's law(without electric current), and then we will get:

$$I_{enc} = 0$$

$$\oint B \cdot dl = \mu_0(I_d + 0)$$

$$\oint \vec{B} d\vec{l} = \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

In addition

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_B}{dt}$$

When the electric flux changes with time, it creates a magnetic field — this is described by the Ampère-Maxwell Law.

Similarly, when the magnetic flux changes, it induces an electric field — this is described by Faraday's Law.

So, both types of flux are dynamic and interconnected: one changing flux can cause the other field to appear.

We know that:

$$I_d = \varepsilon_0 \frac{d\Phi_E}{dt}$$

So plug this in Ampere's law(has electric current):

$$\oint \vec{B} d\vec{l} = \mu_0 I_{enc} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

And then according to Stokes's theorem:

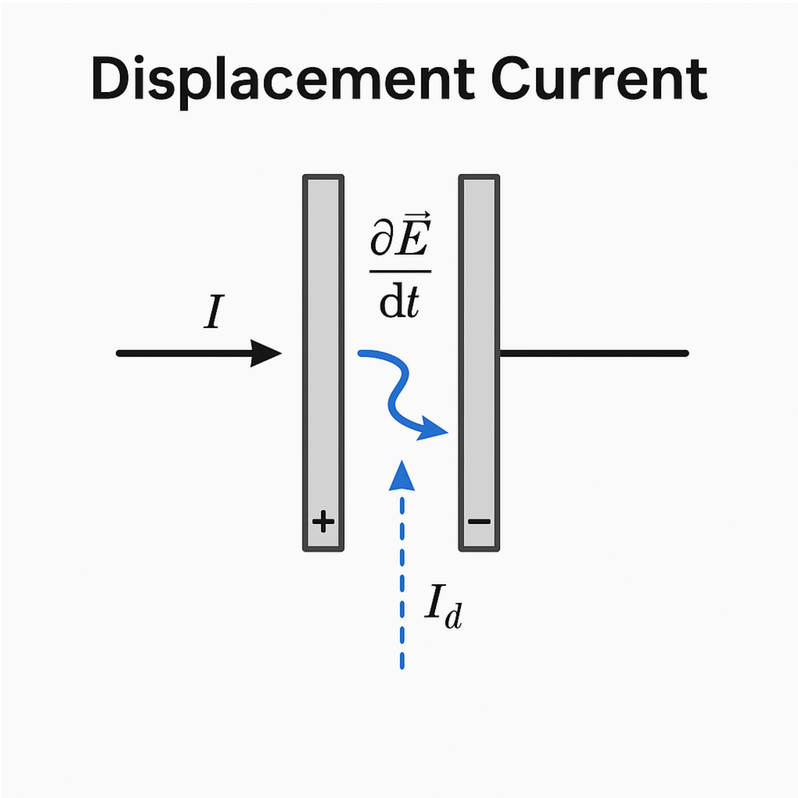
$$\iint_s (\nabla \times B) dl = \oint \vec{B} \cdot dl$$

We will get:

$$\iint_S (\nabla \times B) dl = \mu_0 \iint_S J \cdot dS + \mu_0 \varepsilon_0 \frac{d}{dt} \iint_S E \cdot dS$$

Because this is for any curved surface small sizeS is established, and we can remove the point integral

$$\nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{\partial E}{\partial t}$$



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Here is a question: Aren't capacitors non-conductive? Why can there be "electric current"?

You're right. The capacitor is not conductive inside, so it doesn't conduct current.

Although the "displacement current" is not a substantial electron flow, it is in Maxwell's equation.

It is completely equivalent to the effect of true current on the magnetic field, So it "looks" like a "fictional but real and effective current"，The direction is decided by the partial derivative of E with respect to time (t).

But another question is that why it contains partial derivative rather than derivative?

Because we are concerned about the change of a fixed point in space over time, not a change felt by a particle or observer along the trajectory (that's what requires derivative (d))

Therefore, according to the reasoning above, it not only shows that electric current and electric field can both generate magnetic fields And both "electric"works independently. And Maxwell's theorem explains that Electric field will also be generated without current I and also fixed the failure of the "ampere's law" in the previous electromagnetism near the capacitor, and also enables electromagnetic waves to be established in vacuum!

In addition, this correction made the equation valid **everywhere**, including inside a capacitor, and allowed the four equations to predict the existence of **electromagnetic waves**, such as light.

Next, we will come to apply the Faraday's theorem into Maxwell equations for the perspective of multivariable calculus.

According to Stokes's theorem:

$$\iint_S (\nabla \times E) dS = \oint \vec{E} \cdot dl = -\frac{d\Phi_B}{dt}$$

Because:

$$\Phi_B = \iint \vec{B} \cdot dA$$

$$\iint_S (\nabla \times E) dS = -\frac{d(\iint \vec{B} \cdot dA)}{dt}$$

Therefore

$$\nabla \times E = -\frac{\partial B}{\partial t}$$

As long as the magnetic field changes with time, even if there is no electric current, it will also cause a "rotating" electric field (non-conservative field) in space.