QF2104 AY24/25 Sem 1 Final Cheatsheet Prepared by Li Tianze

Chapter 0 Theory of Interest

0.1 Accumulation Function

- $a(t) = \frac{V(t)}{V(0)}$, where $a(0) = 1$.
- Cash flow diagram,
- y-axis: $a(t)$, x-axis: t.

0.2 Simple and Compound Interest

- Simple Interest: $a(t) = 1 + tr\%$, for $t \geq 0$.
- Compound Interest:
  $a(t) = (1 + r\%)^t$, for $t \geq 0$.
- In QF1100, 'interest' means 'compound interest'.

0.3 Frequency of Compounding

- Nominal interest rate $r\%$ compounded p times annually, interest of each period is $\frac{r\%}{p}$.
- p is frequency of compounding.
- Nominal rate $r = r^{(p)}$, indicating frequency of compounding p.
- Effective annual rate: $r_e\% := \left(1 + \frac{r\%}{p}\right)^p - 1$
- Accumulation function:
  $a(t) = (1 + r_e\%)^t = \left(1 + \frac{r\%}{p}\right)^{pt}$
- Equivalent if yield same effective interest rate: $\left(1 + \frac{r^{(p)}}{p}\right)^p = \left(1 + \frac{r^{(q)}}{q}\right)^q$.

0.4 Continuous Compounding

- Effective rate satisfies: $1 + r_e = \lim_{p\to\infty}\left(1 + \frac{r}{p}\right)^p = e^r$
- Accumulation function: $a(t) = (1 + r_e)^t = e^{rt}$.

0.5 Force of Interest

- Definition: $\delta(t) = \frac{a'(t)}{a(t)} = [\ln(a(t))]'$
- Measures how good the investment is at time t.
- Formula:
  $$a(t) = \exp\left(\int_0^t \delta(u)\,du\right),$$
  $$a(s,t) := \frac{a(t)}{a(s)} = \exp\left(\int_s^t \delta(u)\,du\right)$$, if $0 < s < t$

- Principle of Consistency:
  $$a(t_0, t_n) = a(t_0, t_1)a(t_1, t_2)\cdots a(t_{n-1}, t_n)$$

0.6 Present Value and Time Value

- For a cash flow $\mathbf{C} = \{(c_1, t_1), (c_2, t_2), \ldots, (c_n, t_n)\}$,
- present value satisfies $PV(\mathbf{C}) = \sum_{i=1}^{n} \frac{c_i}{a(t_i)}$.
- At time t, time value, $TV_t(\mathbf{C}) = PV(\mathbf{C}) \times a(t)$.
- If $a(t) = (1 + r\%)^t$, we have
  $PV(\mathbf{C}) = \sum_{i=1}^{n} \frac{c_i}{(1+r)^{t_i}}$ and $TV_t(\mathbf{C}) = \sum_{i=1}^{n} \frac{c_i}{(1+r)^{t_i - t}}$.

0.7 Principle of Equivalence

- Two cash flows are **equivalent** ⇔ They have same present value.

0.8 Internal Rate of Return

- Equation of value: $PV(\mathbf{C}) = \sum_{i=1}^{n} \frac{c_i}{(1+r)^{t_i}} = 0$.
- Any non-negative solution to the equation is called **yield**, or **internal rate of return (IRR)**.

0.9 Loans

- $L =$ the present value of C if $\mathbf{C} = \{(c_1, t_1), (c_2, t_2), \ldots, (c_n, t_n)\}$ is series of repayments.

0.10 Newton-Raphson Iterative Method

- $\alpha_{n+1} = \alpha_n - \frac{f(\alpha_n)}{f'(\alpha_n)}$

Chapter 1 Financial Instruments

1.1 Commercial Paper & Treasury Bills

- Nominal yield: $\lambda = \frac{F/P - 1}{M}$, M in years (365 days).
- $P(1 + \lambda M) = R = F$

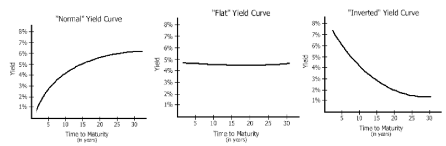
1.2 Basic terminology for bonds

- Cash flow of bond:
  $\left((-F, 0), (\frac{c\%F}{m}, \frac{1}{m}), (\frac{c\%F}{m}, \frac{2}{m}), \ldots, (\frac{c\%F}{m} + R, \frac{n}{m})\right)$

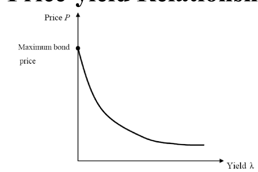
1.2 Bond Yields

- The nominal yield of the bond is the nominal IRR compounded m times per annum of holding the bond from time t to maturity. The price of the bond at time t is
  $$P(t) = \frac{R}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{n-tm}} + \sum_{i=1}^{n-tm} \frac{c\%F/m}{\left(1 + \frac{\lambda(t)\%}{m}\right)^i}.$$

- If $\boldsymbol{R} = \boldsymbol{F}$, then
  $$P(t) = F\left[\frac{1}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{n-tm}} + \frac{c}{\lambda(t)}\left(1 - \frac{1}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{n-tm}}\right)\right]$$
  $$= F + F\left(\frac{c - \lambda(t)}{\lambda(t)}\right)\left[1 - \frac{1}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{n-tm}}\right].$$
- A bond is to be priced at time t,
  1. $P(t) > F$ if and only if $c > \lambda(t)$ at a premium
  2. $P(t) = F$ if and only if $c = \lambda(t)$ at par if $P(t)$
  3. $P(t) < F$ if and only if $c < \lambda(t)$ at a discount i
- Hence,
- Effective yield of the bond satisfies
  $$P(t) = \frac{R}{(1 + \lambda_e(t)\%)^{\frac{n}{m} - t}} + \sum_{i=1}^{n-tm} \frac{c\%F/m}{(1 + \lambda_e(t)\%)^{\frac{i}{m}}},$$
  where $P(\frac{n}{m}) = R$.



1.3 Price-yield Relationship



- Yield = 0 -> max bond price -> cash flow undiscounted -> bond price = Σ(coupon payments) + redemption value

1.4 Spot Rate & Forward Rate

- $(1 + s_k)^k = (1 + s_j)^j (1 + f_{j,k})^{k-j}$
- $(1 + s_n)^n = (1 + s_1)(1 + f_{1,2})(1 + f_{2,3}) \ldots (1 + f_{n-1,n})$
- $P = \frac{F}{(1 + s_n)^n}$

1.5 Common Types of Bonds

- Zero coupon bond: pays no coupon. At any time t, of a N-year zero coupon bond with redemption R satisfies $P(t) = \frac{R}{(1 + \lambda_e(t)\%)^{N-t}}$.
- Perpetual bond/**Consol**: a bond never matures, satisfies $P(t) = \frac{cF}{\lambda(t)}$.

1.6 Pricing a bond

- Always Made Assumptions:
  1. Interest rate constant over the lifetime of the bond.
  2. The yield at any point in time has to equal the interest rates. The price of the bond is equal to its PV. (No significant default or liquidity risks)

1.7 The Macaulay Duration and Average Holding Times

- Macaulay duration is $D = \frac{\sum_{i=1}^{n} t_i \cdot PV(C_i)}{PV(\mathbf{C})} = \sum_{i=1}^{n} w_i t_i$. For a cash flow $\mathbf{C} = \{(c_1, t_1), (c_2, t_2), \dots, (c_n, t_n)\}$.
- Observe that
  1. if $C_i \geq 0$ for all i, then $t_1 \leq D \leq t_n$ (this is the case for any bond)
  2. if $C_i = 0$ for all $i < n$, then $D = t_n$ (this is the case for any zero-coupon bond)
- For infinite cash flow, $D = \frac{\sum_{i=1}^{\infty} t_i \cdot PV(C_i)}{PV(\mathbf{C})}$.
- For a bond redeemable at par and pays total n coupons with m times a year. Suppose bond yield and coupon rate are $\lambda\%$ and $c\%$. The cash flow is $\mathbf{C} = \left(\left(\frac{c\%F}{m}, t_1\right), \dots, \left(\frac{c\%F}{m}, t_{n-1}\right), \left(\frac{c\%F}{m} + F, t_n\right)\right)$, where $t_i = \frac{i}{m}$, so that $D = \frac{1}{P}\left[\sum_{i=1}^{n} \frac{\frac{c\%F}{m}}{(1+\frac{\lambda\%}{m})^i} \frac{i}{m} + \frac{F}{(1+\frac{\lambda\%}{m})^n} \frac{n}{m}\right]$, where $P = \sum_{i=1}^{n} \frac{\frac{c\%F}{m}}{(1+\frac{\lambda\%}{m})^i} + \frac{F}{(1+\frac{\lambda\%}{m})^n}$ **(3.4)**.
- It can be shown that,
  $$D = \frac{1+\frac{\lambda\%}{m}}{\lambda\%} - \frac{1+\frac{\lambda\%}{m} + n\left(\frac{c\%}{m} - \frac{\lambda\%}{m}\right)}{c\%\left[\left(1+\frac{\lambda\%}{m}\right)^n - 1\right] + \lambda\%}.$$
- For perpetual bond, $D = \frac{1+\frac{\lambda\%}{m}}{\lambda\%}$.
- If the bond is priced at par $\lambda = c$,
  $$D = \frac{1+\frac{c\%}{m}}{c\%}\left(1 - \frac{1}{\left(1+\frac{c\%}{m}\right)^n}\right).$$

1.8 Modified Duration and Sensitivity

- Modified duration: $D_M := -\frac{\left(\frac{dP}{d\lambda}\right)}{P}$.
- Differentiating **(3.4)**, we have $\frac{dP}{d\lambda} = -\frac{1}{1+\frac{\lambda}{m}} DP$
- The price sensitivity formula: $D_M = \frac{1}{1+\frac{\lambda}{m}} D$, which holds for any cash flow. Or $D_M = D$, if λ is a continuously compounded yield.
- By linear approximation, $P(\lambda + \Delta\lambda) \approx P(\lambda) - (D_M P) \cdot \Delta\lambda$. or $\Delta P \approx -(D_M P) \cdot \Delta\lambda$, if $\Delta\lambda$ denotes a small change in λ.

1.9 Convexity

- Convexity: $C = \frac{1}{\left(1+\frac{\lambda}{m}\right)^2} \times \left(\frac{PV(t^2 C)}{PV(C)} + \frac{1}{m} \times \frac{PV(tC)}{PV(C)}\right)$, or $C = \frac{PV(t^2 C)}{PV(C)}$ if λ is continuously compounded.
- By quadratic approximation, $P(\lambda + \Delta\lambda) = P(\lambda)[1 - D_M \Delta\lambda + \frac{1}{2} C (\Delta\lambda)^2]$.
- Macaulay Convexity: $MacC = \frac{PV(t^2 C)}{PV(C)}$.
- $C(\lambda) = \frac{1}{\left(1+\frac{\lambda}{m}\right)^2} \times \left(MacC(\lambda) + \frac{D(\lambda)}{m}\right)$
- $C(\lambda) = MacC(\lambda)$ if λ is continuously compounded.

1.10 Duration of Bond Portfolio

- Assume all the bonds in the portfolio have constant effective yield λ_e, all of which are equal to current effective interest rate.
- If $w_i := \frac{\alpha_i P_i}{\sum_{i=1}^{n} \alpha_i P_i}$, then $D_\Pi = \sum_{i=1}^{n} w_i D_i$.

Chapter 2 Expected Utility Theory

2.1 Expected Utility

- $EU(X + w_0) = \sum_{i=1}^{n} p_i U(x_i + w_0) = \int_a^b f(x) U(x_i + w_0)$
- where $f: (a, b) \to (0, \infty)$
- **Decision:** Invest if $EU(X + w_0) > U(w_0)$ else indifferent or avoid.

2.2 Risk Attitude & Jensen's Inequality

Risk attitude	Concavity of U	U''	Jensen's inequality
Risk averse	Strictly concave	$U'' < 0$	$E(U(X)) < U(E(X))$
Risk neutral	Linear	$U'' = 0$	$E(U(X)) = U(E(X))$
Risk loving	Strictly convex	$U'' > 0$	$E(U(X)) > U(E(X))$

2.3 Positive Affine Transformation

- $\forall \alpha > 0$, $\alpha U + \beta$ is a positive affine transformation of U, sharing same attitude.

2.4 Certainty Equivalence

- $c = CE(X; U) = U^{-1}(E(U(w_0 + X)))$
- $U(c) = E(U(w_0 + X))$
- $c > w_0$, invest; Else indifferent or avoid.
- $CE(X; \alpha U + \beta) = CE(X; U)$

2.5 Risk Premium

- $r = RP(X; U) = w_0 - CE(X; U)$
- $U(w_0 - r) = EU(w_0 + X)$
- Invest if $r < 0$, else indifferent or avoid.

2.6 Arrow-Pratt Absolute Risk Aversion

- $U_{ARA} = -\frac{U''}{U'} = -(\ln(U'))' > 0$
- $U_{ARA} = V_{ARA}$, if they are positive affine transformation.
- Larger ARA → Greater risk aversion
- U is more risk averse, if g is strictly increasing and strictly concave, such that $U(w) = g(V(w))$

2.7 Portfolio Selection

- EU of final wealth:
  $$EU(W) = EU(w_0(1 + \alpha R))$$
  $$= pU(w_0(1 + \alpha a)) + (1 - p)U(w_0(1 - \alpha b)),$$
  If may gain a with probability p, or lose b with probability $1 - p$.
- The individual will choose an α to maximize $EU(W)$.

Chapter 3 Mean Variance Analysis

- Correlation coefficient: $\rho_{ij} = \frac{\sigma_{ij}}{\sigma_i \sigma_j}$
- Covariance: $\sigma_{ij} = cov(r_i, r_j)$
- Portfolio mean: $\mu_p = E(r_p) = \sum_{i=1}^{n} w_i \mu_i = \boldsymbol{w^T \mu}$
- Portfolio variance: $\sigma_p^2 = Var(r_p) = \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$
- And $Var(r_p) = \boldsymbol{w^T C w}$.
- GMVP of two assets: $\alpha = \alpha^* = \frac{\sigma_2(\sigma_2 - \rho_{12}\sigma_1)}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$
  $(\sigma_p^2)^* = \frac{\sigma_1^2 \sigma_2^2 (1 - \rho_{12}^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$ $(\mu_p)^* = \alpha^* \mu_1 + (1 - \alpha^*)\mu_2$

- Portfolio graph: $\sigma_p^2 = A\mu_p^2 + B\mu_p + C$.
- Properties for portfolio with more than 2 assets:

  It can be shown that the feasible set, F of a portfolio with n $(n > 2)$ assets has the following properties:
  i. For any fixed $\mu \in R$, $(\sigma, \mu) \epsilon F$ for some $\sigma > 0$
  ii. For each $(\sigma, \mu) \epsilon F$, $(\sigma', \mu) \epsilon F$ for all $\sigma' > \sigma$
  iii. For each pair of points (σ, μ) and (σ', μ') in the feasible set F and for any $\lambda \epsilon [0,1]$, the point $\lambda(\sigma, \mu) + (1-\lambda)(\sigma', \mu')$ lies in the set F. We say that F is a convex set.
  iv. For any fixed $\mu \in R$, there exists $\sigma^* > 0$ s.t.
     a) We have $(\sigma^*, \mu) \epsilon F$
     b) If $(\sigma, \mu) \epsilon F$, then $\sigma^* < \sigma$
     We call the point (σ^*, μ) the minimum-variance point with mean μ

Chapter 4 Portfolio Theory & CAPM

4.1 Markowitz's Portfolio Theory

- **Problem 1**: Fix μ, we seek a portfolio of minimum σ_p such that $\mu_p = \mu$.
- Constraints: $\sum_{i=1}^{n} w_i = 1$, $\sum_{i=1}^{n} \mu_i w_i = \mu$
- **Problem 2**: We seek a portfolio of min. σ_p. We call this GMV portfolio.
- **Lagrange Method**:

$$L = \frac{1}{2}\sigma_p^2 - \lambda_1\left(\sum_{i=1}^{n} w_i - 1\right) - \lambda_2\left(\sum_{i=1}^{n} w_i\mu_i - \mu\right)$$

$$= \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{n} w_i w_j \sigma_{ij} - \lambda_1\left(\sum_{i=1}^{n} w_i - 1\right) - \lambda_2\left(\sum_{i=1}^{n} w_i\mu_i - \mu\right)$$

$$\sum_{j=1}^{n} \sigma_{ij} w_j - \lambda_1 - \lambda_2\mu_i = 0, i = 1,2,\dots,n \qquad \boldsymbol{Cw} = \lambda_1\boldsymbol{1} + \lambda_2\boldsymbol{\mu}$$

$$\sum_{i=1}^{n} w_i - 1 = 0 \qquad \boldsymbol{1^T w} = 1$$

- $$\sum_{i=1}^{n} w_i\mu_i - \mu = 0 \qquad \boldsymbol{\mu^T w} = \mu$$
- $$\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \mu \end{pmatrix} \quad \begin{pmatrix} a & b \\ b & c \end{pmatrix} = \begin{pmatrix} \boldsymbol{1^T C^{-1} 1} & \boldsymbol{1^T C^{-1}\mu} \\ \boldsymbol{1^T C^{-1}\mu} & \boldsymbol{\mu^T C^{-1}\mu} \end{pmatrix}$$
- $$\lambda_1 = \frac{c - b\mu}{ac - b^2} \text{ and } \lambda_2 = \frac{a\mu - b}{ac - b^2}$$
- Optimal weights: $\boldsymbol{w}_\mu^* = \frac{c - b\mu}{ac - b^2}\boldsymbol{C^{-1}1} + \frac{a\mu - b}{ac - b^2}\boldsymbol{C^{-1}\mu}$
- Min. var frontier: $\sigma^2 = \frac{a\mu^2 - 2b\mu + c}{ac - b^2}$
- $\sigma^2 = \frac{a(\mu - \frac{b}{a})^2 + (c - \frac{b^2}{a})}{\Delta}$, $\Delta = ac - b^2$.
- GMV portfolio:
- $\sigma_{GMV} = \sqrt{\frac{1}{a}}$, $\mu_{GMV} = \frac{b}{a}$, $\boldsymbol{w_{GMV}} = \frac{\boldsymbol{C^{-1}1}}{\boldsymbol{1^T C^{-1} 1}}$
- Efficient portfolio: $\mu_p > \frac{b}{a}$

- **Two fund Theorem**:

  $w_{Fund1} = \frac{\boldsymbol{C^{-1}1}}{\boldsymbol{1^T C^{-1}1}}$ and $w_{Fund2} = \frac{\boldsymbol{C^{-1}\mu}}{\boldsymbol{1^T C^{-1}\mu}}$
- $\boldsymbol{w}_\mu^* = \alpha\frac{\boldsymbol{C^{-1}1}}{\boldsymbol{1^T C^{-1}1}} + (1-\alpha)\frac{\boldsymbol{C^{-1}\mu}}{\boldsymbol{1^T C^{-1}\mu}}$, where $\alpha = a\frac{c - b\mu}{ac - b^2}$

  Theorem (Two-fund Theorem) : Let $\mathbf{w_1}$ and $\mathbf{w_2}$ be the weight vectors of any two distinct portfolios on the minimum-variance frontier. The minimum variance set of portfolios is the set
- $$\{\alpha\boldsymbol{w_1} + (1-\alpha)\boldsymbol{w_2} : \alpha \epsilon \boldsymbol{R}\}$$

  Corollary A: Let $\boldsymbol{w_1}$ and $\boldsymbol{w_2}$ be two distinct efficient portfolios (i.e. lying on the efficient frontier). Then any two convex combination $\alpha\boldsymbol{w_1} + (1-\alpha)\boldsymbol{w_2}, \alpha \epsilon [0,1]$ of $\boldsymbol{w_1}$ and $\boldsymbol{w_2}$ is an
- efficient portfolio.

  Corollary B: If $b = \boldsymbol{1^T C^{-1}\mu} > 0$, then Fund 2 $(\frac{\boldsymbol{C^{-1}\mu}}{\boldsymbol{1^T C^{-1}\mu}})$ is efficient. In this case, any portfolio of the form $\alpha\boldsymbol{w}_{Fund1} +$
- $(1-\alpha)\boldsymbol{w}_{Fund2}, \alpha < 1$, is an efficient portfolio.

4.2 Portfolio with Risk-free Asset

- $(1 - \sum_{i=1}^{n} w_i, w_1, w_2, \dots, w_n)^T = (1 - \boldsymbol{1^T w}, w_1, w_2, \dots, w_n)^T$
- $r_p = (1 - \boldsymbol{1^T w})r_f + \boldsymbol{w^T r}$
- **Problem 3**: Fix μ. We seek a portfolio in the $n+1$ assets (n and the risk-free asset) with min. risk σ_p such that $\mu_p = \mu$.
- We minimize: $\boldsymbol{w^T C w} = \sum_{j=1}^{n}\sum_{i=1}^{n} w_i w_j \sigma_{ij}$
- Constraint: $\mu_p = \left(1 - \sum_{i=1}^{n} w_i\right) r_f + \sum_{i=1}^{n} w_i\mu_i = \mu$
- Optimal weight: $\boldsymbol{w} = \frac{(\mu - r_f)\boldsymbol{C^{-1}(\mu} - r_f\boldsymbol{1})}{(\boldsymbol{\mu} - r_f\boldsymbol{1})^T\boldsymbol{C^{-1}}(\boldsymbol{\mu} - r_f\boldsymbol{1})}$
- Variance: $\sigma^2 = \boldsymbol{w^T C w} = \frac{(\mu - r_f)^2}{(\boldsymbol{\mu} - r_f\boldsymbol{1})^T\boldsymbol{C^{-1}}(\boldsymbol{\mu} - r_f\boldsymbol{1})}$
- **CML**: $\sigma = \frac{|\mu - r_f|}{\sqrt{c - 2br_f + ar_f^2}}$
- **Tangency Portfolio**:
- It lies on the efficient frontier line.
- It lies on the efficient frontier for risky assets.
- It invests only in risky assets, $\boldsymbol{1^T w_{tan}} = 1$.
- Let $\boldsymbol{w_{tan}} = k\boldsymbol{w}$, solve k for $\boldsymbol{w_{tan}} = \frac{\boldsymbol{C^{-1}(\mu} - r_f\boldsymbol{1})}{\boldsymbol{1^T C^{-1}(\mu} - r_f\boldsymbol{1})}$
- Mean: $\mu_{tan} = \frac{\boldsymbol{\mu^T C^{-1}(\mu} - r_f\boldsymbol{1})}{\boldsymbol{1^T C^{-1}(\mu} - r_f\boldsymbol{1})} = \frac{c - r_f b}{b - r_f a}$
- Variance: $\sigma_{tan}^2 = \frac{(\boldsymbol{\mu} - r_f\boldsymbol{1})^T\boldsymbol{C^{-1}}(\boldsymbol{\mu} - r_f\boldsymbol{1})}{(\boldsymbol{1^T C^{-1}(\mu} - r_f\boldsymbol{1}))^2} = \frac{c - 2r_f b + ar_f^2}{(b - r_f a)^2}$

- CML equation: $\mu_p = r_f + \frac{\sigma_p}{\sigma_{tan}}(\mu_{tan} - r_f)$.

4.3 Sharpe Ratio

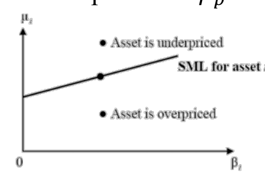
- $SR_p = (\mu_p - r_f)/\sigma_p$

4.4 One Fund Theorem

  Theorem (One-Fund Theorem) In a financial market with risky assets and a risk-free asset, an investor will choose to hold only the risk-free asset and the tangency portfolio. Investors differ only
- in the proportion of total wealth allocated to the tangency portfolio.

4.5 Capital Asset Pricing Model

- Market portfolio: $w_i = \frac{u_i p_i}{\sum_{i=1}^{n} u_i p_i}$, p_i is price.
- CML for tan portfolio: $\mu_p - r_f = \frac{\sigma_p}{\sigma_m}(\mu_m - r_f)$
- **Theorem**:
- $\forall$ portfolio: $\mu_i - r_f = \frac{\sigma_{im}}{\sigma_m^2}(\mu_m - r_f)$
- Beta of asset i: $\beta_i = \frac{\sigma_{im}}{\sigma_m^2}$
- Beta of portfolio: $\beta_p = \boldsymbol{w^T\beta}$



- **SML**: $f(\beta_i) = \mu_i = r_f + (\mu_m - r_f)\beta_i$
- Alpha = Estimated Return – CAPM Return
- $\boldsymbol{\beta} = \frac{1}{\sigma_m^2}\boldsymbol{Cw_n}$, $\boldsymbol{w_m} = \frac{\boldsymbol{C^{-1}\beta}}{\boldsymbol{1^T C^{-1}\beta}}$, $\boldsymbol{1^T C^{-1}\beta} = \boldsymbol{\beta^T C^{-1}\beta}$.

4.6 Systematic Risk and Non-systematic Risk

- $r_p = r_f + \beta_p(r_m - r_f) + \epsilon_p$
- Where $E(\epsilon_p) = 0$, $Cov(\epsilon_p, r_m) = 0$
- $Var(r_p) = \beta_p^2\sigma_m^2 + Var(\epsilon_p)$.
- Systematic risk cannot be diversified since all assets has non-zero beta.
- Idiosyncratic/specific/non-systematic risk can be reduced through diversification.
- A portfolio lies on CML (efficient) has $\sigma_p^2 = \beta_p^2\sigma_m^2$. We conclude that efficient portfolios contain only systematic risk.

Chapter 5 Options

5.1 Notations

T: maturity date of option

τ : exercise date of options

➢ for American options, $0 \le \tau \le T$; for European options, $\tau = T$

K: strike price

S_t: price of underlying asset at time t

C: current call option (either European or American) price

P: current put option (either European or American) price

C_A: current American call option price

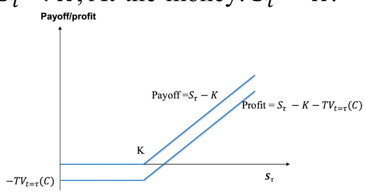
C_E: current European call option price

P_A: current American put option price

P_E: current European put option price

- American options can be exercised any time before the expiration date.
- European options can ONLY be exercised on the expiration date.

5.2 Call Options

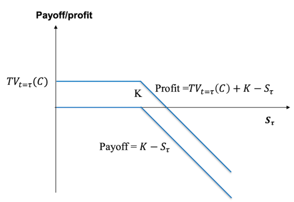
- In-the-money: $S_\tau > K$; Out-of-the-money: $S_\tau < K$; At-the-money: $S_\tau = K$.
- 

- Payoff of **long** call:

$$\max\{S_\tau - K, 0\} = \begin{cases} 0 & if\ S_\tau < K \\ S_\tau - K & if\ S_\tau \ge K \end{cases}$$

- Profit of **long** call:

$$\max\{S_\tau - K, 0\} - TV_{t=\tau}(C)$$

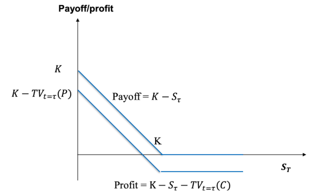
- 

- Payoff of **short** call:

$$-\max\{S_\tau - K, 0\} = \min\{K - S_\tau, 0\} = \begin{cases} 0 & if\ S_\tau < K \\ K - S_T & if\ S_\tau \ge K \end{cases}$$

- Profit of **short** call: $TV_{t=\tau}(C) - \max\{S_\tau - K, 0\}$

5.3 Put Options

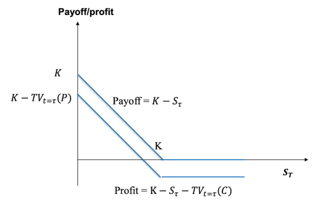
- In-the-money: $S_\tau < K$; Out-of-the-money: $S_\tau > K$; At-the-money: $S_\tau = K$.
- 

- Payoff of **long** put:

$$\max\{K - S_\tau, 0\} = \begin{cases} 0 & if\ S_\tau > K \\ K - S_\tau & if\ S_\tau \le K \end{cases}$$

- Profit of **long** put:

$$\max\{K - S_\tau, 0\} - TV_{t=\tau}(C)$$

- 

- Payoff of **short** put:

$$-\max\{K - S_\tau, 0\} = \min\{S_\tau - K, 0\} = \begin{cases} 0 & if S_\tau > K \\ S_\tau - K & if S_\tau \le K \end{cases}$$

- Profit of **short** put: $TV_{t=\tau}(P) - \max\{K - S_\tau, 0\}$

5.4 Arbitrage Opportunity

- Exists if conditions hold:
  i. $V(0) \le 0$
  ii. $V(\tau) \ge 0$
  iii. it is not the case $V(0) = V(\tau) = 0$

5.5 Bounds on Options

- $C_E \le C_A$, $P_E \le P_A$
- $\max\{S_0 - Ke^{-rT}, 0\} \le C_E \le C_A \le S_0$
- $\max\{0,\ Ke^{-rT} - S_0\} \le P_E \le P_A \le K$
- If $K_2 > K_1$, then $C(K_2) \le C(K_1)$.
- If $K_2 > K_1$, then $P(K_2) \ge P(K_1)$.

5.6 Put-Call Parity

- $C_E + Ke^{-rT} = P_E + S_0$
- $S_0 - K \le C_A - P_A \le S_0 - Ke^{-rT}$