Chapter 1 Theory of Interest

1.1 Accumulation Function

- $a(t) = \frac{V(t)}{V(0)}$, where $a(0) = 1$.
- Cash flow diagram,
- y-axis: $a(t)$, x-axis: t.

1.2 Simple and Compound Interest

- Simple Interest:
  $a(t) = 1 + tr\%, \quad \text{for } t \geq 0.$
- Compound Interest:
  $a(t) = (1 + r\%)^t, \quad \text{for } t \geq 0.$
- In QF1100, 'interest' means 'compound interest'.

1.3 Frequency of Compounding

- Nominal interest rate $r\%$ compounded p times annually, interest of each period is $\frac{r\%}{p}$.
- p is frequency of compounding.
- Nominal rate $r = r^{(p)}$, indicating frequency of compounding p.
- Effective annual rate: $r_e\% := \left(1 + \frac{r\%}{p}\right)^p - 1$
- Accumulation function:
  $a(t) = (1 + r_e\%)^t = \left(1 + \frac{r\%}{p}\right)^{pt}$.
- Equivalent if yield same effective interest rate: $\left(1 + \frac{r^{(p)}}{p}\right)^p = \left(1 + \frac{r^{(q)}}{q}\right)^q$.

1.4 Continuous Compounding

- Effective rate satisfies:
  $1 + r_e = \lim_{p \to \infty} \left(1 + \frac{r}{p}\right)^p = e^r$
- Accumulation function:
  $a(t) = (1 + r_e)^t = e^{rt}$.

1.5 Force of Interest

- Definition: $\delta(t) = \frac{a'(t)}{a(t)} = [\ln(a(t))]'$.
- Measures how good the investment is at time t.
- Formula:
  - $a(t) = \exp\left(\int_0^t \delta(u)\, du\right)$,
  - $a(s,t) := \frac{a(t)}{a(s)} = \exp\left(\int_s^t \delta(u)\, du\right)$, if $0 < s < t$
- Principle of Consistency:
  $a(t_0, t_n) = a(t_0, t_1)a(t_1, t_2) \cdots a(t_{n-1}, t_n)$

2.1 Present Value and Time Value

- For a cash flow $\mathbf{C} = \{(c_1, t_1), (c_2, t_2), \ldots, (c_n, t_n)\}$,
- present value satisfies $PV(\mathbf{C}) = \sum_{i=1}^n \frac{c_i}{a(t_i)}$.
- At time t, time value, $TV_t(\mathbf{C}) = PV(\mathbf{C}) \times a(t)$.
- If $a(t) = (1 + r\%)^t$, we have
  $PV(\mathbf{C}) = \sum_{i=1}^n \frac{c_i}{(1+r)^{t_i}}$ and $TV_t(\mathbf{C}) = \sum_{i=1}^n \frac{c_i}{(1+r)^{t_i - t}}$.

2.2 Principle of Equivalence

- Two cash flows are **equivalent** ⇔ They have same present value.

2.3 Internal Rate of Return

- Equation of value: $PV(\mathbf{C}) = \sum_{i=1}^n \frac{c_i}{(1+r)^{t_i}} = 0$.
- Any non-negative solution to the equation is called **yield**, or **internal rate of return (IRR)**.

2 Annuities

- **Annuity**: a series of payments made at regular intervals.
- **Perpetuity**: an annuity with infinite number of payments.

3 Loans

- $L =$ the present value of C, if $\mathbf{C} = \{(c_1, t_1), (c_2, t_2), \ldots, (c_n, t_n)\}$ is series of repayments.

Chapter 2 Bonds

1.1 Basic terminology for bonds

Face Value	The amount based on which the periodic interest payments are computed.	F
Redemption /Maturity Value	The amount to be paid at the end of the loan. Usually same as face value.	R
Coupon Rate	The bond's interest payments, as a percentage to the par value.	$c\%$
Maturity Date	The date on which the loan will be fully repaid.	-
-	The number of coupon payments per year.	m
-	The total number of coupon payments.	n

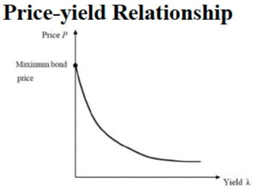
- Cash flow of bond:
  $\left((-F,0), (\frac{c\%F}{m}, \frac{1}{m}), (\frac{c\%F}{m}, \frac{2}{m}), \ldots, (\frac{c\%F}{m} + R, \frac{n}{m})\right)$
- For the rest of this course, unless we explicitly say so, we will always assume that the face value and the redemption value of the bond are the same.

2.0 Bond Yields

- The nominal yield of the bond is the nominal IRR compounded m times per annum of holding the bond from time t to maturity. The price of the bond at time t is
  $P(t) = \frac{R}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{n-tm}} + \sum_{i=1}^{n-tm} \frac{c\%F/m}{\left(1 + \frac{\lambda(t)\%}{m}\right)^i}$.

- If $R = F$, then
  $$P(t) = F\left[\frac{1}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{n-tm}} + \frac{c}{\lambda(t)}\left(1 - \frac{1}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{n-tm}}\right)\right]$$
  $$= F + F\left(\frac{c - \lambda(t)}{\lambda(t)}\right)\left[1 - \frac{1}{\left(1 + \frac{\lambda(t)\%}{m}\right)^{n-tm}}\right].$$
- A bond is to be priced at time t,
  1. at a *premium* if $P(t) > F$
  2. at *par* if $P(t) = F$
  3. at a *discount* if $P(t) < F$.
     1. $P(t) > F$ if and only if $c > \lambda(t)$
     2. $P(t) = F$ if and only if $c = \lambda(t)$
     3. $P(t) < F$ if and only if $c < \lambda(t)$
- Hence,
- Effective yield of the bond satisfies
  $$P(t) = \frac{R}{(1 + \lambda_e(t)\%)^{\frac{n}{m} - t}} + \sum_{i=1}^{n-tm} \frac{c\%F/m}{(1 + \lambda_e(t)\%)^{\frac{i}{m}}},$$
  where $P(\frac{n}{m}) = R$.

2.1 Price-yield Relationship



- Yield = 0 -> max bond price -> cash flow undiscounted -> bond price = Σ(coupon payments) + redemption value

2.2 Common Types of Bonds

- Zero coupon bond: pays no coupon. At any time t, of a N-year zero coupon bond with redemption R satisfies $P(t) = \frac{R}{(1 + \lambda_e(t)\%)^{N-t}}$.
- Perpetual bond/**Consol**: a bond never matures, satisfies $P(t) = \frac{cF}{\lambda(t)}$.

2.3 Pricing a bond

- <u>Always Made</u> Assumptions:
  1. Interest rate constant over the lifetime of the bond.
  2. The yield at any point in time has to equal the interest rates. The price of the bond is equal to its PV. (No significant default or liquidity risks)

3.1 The Macaulay Duration and Average Holding Times

- Macaulay duration is $D = \frac{\sum_{i=1}^n t_i \cdot PV(C_i)}{PV(\mathbf{C})} = \sum_{i=1}^n w_i t_i$.
  For a cash flow
  $\mathbf{C} = \{(c_1, t_1), (c_2, t_2), \ldots, (c_n, t_n)\}$.
- Observe that
  1. if $C_i \geq 0$ for all i, then $t_1 \leq D \leq t_n$ (this is the case for any bond)
  2. if $C_i = 0$ for all $i < n$, then $D = t_n$ (this is the case for any zero-coupon bor
- For infinite cash flow, $D = \frac{\sum_{i=1}^\infty t_i \cdot PV(C_i)}{PV(\mathbf{C})}$.
- For a bond redeemable at par and pays total n coupons with m times a year. Suppose bond yield and coupon rate are $\lambda\%$ and $c\%$. The cash flow is
  $\mathbf{C} = \left(\left(\frac{c\%F}{m}, t_1\right), \ldots, \left(\frac{c\%F}{m}, t_{n-1}\right), \left(\frac{c\%F}{m} + F, t_n\right)\right)$,
  where $t_i = \frac{i}{m}$, so that
  $$D = \frac{1}{P}\left[\sum_{i=1}^n \frac{\frac{c\%F}{m}}{(1 + \frac{\lambda\%}{m})^i}\frac{i}{m} + \frac{F}{(1 + \frac{\lambda\%}{m})^n}\frac{n}{m}\right],$$ where
  $$P = \sum_{i=1}^n \frac{\frac{c\%F}{m}}{(1 + \frac{\lambda\%}{m})^i} + \frac{F}{(1 + \frac{\lambda\%}{m})^n} \quad \textbf{(3.4)}.$$
- It can be shown that,
  $$D = \frac{1 + \frac{\lambda\%}{m}}{\lambda\%} - \frac{1 + \frac{\lambda\%}{m} + n(\frac{c\%}{m} - \frac{\lambda\%}{m})}{c\%\left[\left(1 + \frac{\lambda\%}{m}\right)^n - 1\right] + \lambda\%}.$$
- For perpetual bond, $D = \frac{1 + \frac{\lambda\%}{m}}{\lambda\%}$.
- If the bond is priced at par $\lambda = c$,
  $$D = \frac{1 + \frac{c\%}{m}}{c\%}\left(1 - \frac{1}{(1 + \frac{c\%}{m})^n}\right).$$

3.2 Modified Duration and Sensitivity

- Modified duration: $D_M := -\frac{\left(\frac{dP}{d\lambda}\right)}{P}$.
- Differentiating **(3.4)**, we have
  $\frac{dP}{d\lambda} = -\frac{1}{1 + \frac{\lambda}{m}} DP$
- The price sensitivity formula: $D_M = \frac{1}{1 + \frac{\lambda}{m}} D$, which holds for any cash flow.
- By linear approximation,
  $P(\lambda + \Delta\lambda) \approx P(\lambda) - (D_M P) \cdot \Delta\lambda$. or
  $\Delta P \approx -(D_M P) \cdot \Delta\lambda$, if $\Delta\lambda$ denotes a small change in λ.

3.3 Duration of Bond Portfolio

- Assume all the bonds in the portfolio have constant effective yield λ_e, all of which are equal to current effective interest rate.
- If $w_i := \frac{\alpha_i P_i}{\sum_{i=1}^n \alpha_i P_i}$, then $D_\Pi = \sum_{i=1}^n w_i D_i$.

Appendix: Series Formula	
$\sum_{i=1}^\infty y^i = \frac{y}{1-y}$	$\sum_{i=1}^\infty i y^i = \frac{y}{(1-y)^2}$
$\sum_{i=1}^n y^i = \frac{1 - y^{n+1}}{1-y}$	$\sum_{i=1}^n i y^i = \frac{y(1-y^n) - ny^{n+1}(1-y)}{(1-y)^2}$
$\sum_{i=0}^n r^i = \frac{1 - r^{n+1}}{1-r}$	$\sum_{i=0}^n i r^{i-1} = \frac{1 - (n+1)r^n + nr^{n+1}}{(1-r)^2}$
$\sum_{i=1}^n c = cn$	$\sum_{i=1}^n i = \frac{n(n+1)}{2}$
$\sum_{i=1}^n i^3 = \frac{n^2(n+1)^2}{4}$	$\sum_{i=1}^n i^2 = \frac{n(n+1)(2n+1)}{6}$

Chapter 3 Mean Variance Analysis

- Finite Probability Space is a pair(omega, P), where omega is a finite, non-empty set, called the sample space. P: omega -> [0, 1] is the probability function.
- Random variable X on omega is a function: X: omega -> R
- **Proposition 1.5.** *Let a, b, c be constants and X, Y be random variables on a finite probability space. Then*

$$\mathbf{E}(c) = c, \qquad \mathbf{E}(cX) = c\mathbf{E}(X),$$
$$\mathbf{E}(X+c) = \mathbf{E}(X)+c, \qquad \mathbf{E}(aX+bY) = a\mathbf{E}(X)+b\mathbf{E}(Y).$$

- $\sigma_X^2 = \mathbf{Var}(X) = \mathbf{E}[(X-\mu_x)^2] = \mathbf{E}(X^2) - 2\mathbf{E}(X)\mu_X + \mu_X^2 = \mathbf{E}(X^2) - [\mathbf{E}(X)]^2$

$\sigma_{X,Y} = \mathbf{Cov}(X,Y) := \mathbf{E}[(X-\mathbf{E}(X))(Y-\mathbf{E}(Y))] = \mathbf{E}(XY) - \mathbf{E}(X)\mathbf{E}(Y).$

- *Correlation Coefficient*: $\rho_{X,Y} = \dfrac{\mathbf{Cov}(X,Y)}{\sqrt{\mathbf{Var}(X)\,\mathbf{Var}(Y)}}.$
- Let X, Y be random variables on a finite probability space.
  - If $P(X=x, Y=y) = P(X=x)P(Y=y)$ for all $(x,y) \in \Omega$, then we say that X and Y are *independent*.
  - If $\mathbf{Cov}(X,Y) = 0$, we say that X and Y are *uncorrelated*.
- **Proposition 1.9.** *Let a, b be constants and X, Y be random variables on a finite probability space. Then*

$$\mathbf{Cov}(X,Y) = \mathbf{Cov}(Y,X),$$
$$\mathbf{Cov}(X,X) = \mathbf{Var}(X),$$
$$\mathbf{Var}(aX+bY) = a^2\mathbf{Var}(X) + b^2\mathbf{Var}(Y) + 2ab\,\mathbf{Cov}(X,Y),$$
$$-1 \le \rho(X,Y) \le 1,$$
$$\mathbf{Cov}(aX+bY,Z) = a\,\mathbf{Cov}(X,Z) + b\,\mathbf{Cov}(Y,Z).$$

Furthermore,

1. *if X and Y are independent, then X and Y are uncorrelated.*
2. *X and Y are uncorrelated if and only if $\mathbf{Var}(X+Y) = \mathbf{Var}(X) + \mathbf{Var}(Y)$.*

-*Rate of return*: $r = \dfrac{W_1 - W_0}{W_0} = \dfrac{W_1}{W_0} - 1 = \dfrac{P_1 - P_0}{P_0} = \dfrac{P_1}{P_0} - 1.$

-*Portfolio weight vector*: $w = (w_1, w_2, \dots, w_n)^T$

-*Portfolio mean*: $\mu_p = \mathbf{E}(r_p) = \sum_{i=1}^n w_i\mu_i$

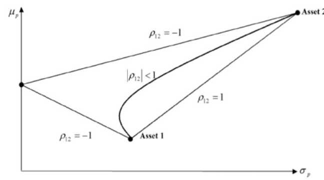
-*Portfolio variance*: $= \sum_{i=1}^n\sum_{j=1}^n w_iw_j\sigma_{i,j} = \sum_{i=1}^n w_i^2\sigma_i^2 + 2\sum_{i=1}^n\sum_{j<i} w_iw_j\sigma_{i,j}$

$= \sum_{i=1}^n w_i^2\sigma_i^2 + \sum_{i=1}^n\sum_{j\ne i} w_iw_j\sigma_{i,j}$

- Portfolio average variance(Unsystematic risk): $\bar{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n \sigma_i^2$
- Portfolio average covariance(Systematic Risk): $\bar{\phi} = \frac{1}{n(n-1)}\sum_{\substack{i,j=1\\ i\ne j}}^n \sigma_{i,j}$

*** Portfolio of two assets**

- Portfolio mean: $\mu_p = \alpha\mu_1 + (1-\alpha)\mu_2$
- Portfolio variance: $\sigma_p^2 = \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\alpha(1-\alpha)\sigma_{1,2} = \alpha^2\sigma_1^2 + (1-\alpha)^2\sigma_2^2 + 2\alpha(1-\alpha)\rho_{1,2}\sigma_1\sigma_2.$
- GMVP weight: $\alpha = \alpha^* = \dfrac{\sigma_2(\sigma_2 - \rho_{1,2}\sigma_1)}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2}.$
- GMVP variance: $(\sigma_p^2)^* = \dfrac{\sigma_1^2\sigma_2^2(1-\rho_{1,2}^2)}{\sigma_1^2 + \sigma_2^2 - 2\rho_{1,2}\sigma_1\sigma_2}$
- The graph of *mean* against *std* for all allowable weights, *w*, is the feasible set for two given assets



- Logarithmic return: $R = \ln\left(\frac{W_1}{W_0}\right)$
- *r* is the rate of return, *R* is the log return, $\exp(R) = 1 + r$, $R = \ln(1+r)$
- Mean log return: $\sum p_i \ln\frac{W_i}{W}$, variance: $\sum p_i \ln^2\frac{W_i}{W} - \left(\sum p_i \ln\frac{W_i}{W}\right)^2$

Chapter 4 Forwards and Futures

***** Forward contract *****

- Long forward contract: $S(T) - F(0)$
- Short forward contract: $F(0) - S(T)$
- Arbitrage opportunity arises when:
  1. $V(0) \le 0$,
  2. $\mathbb{P}(V(t) < 0) = 0$ for some $t > 0$
  3. If $V(0) = 0$, then $\mathbb{P}(V(t) > 0) < 1$

*** No carrying cost**

- When $F_0 > S(0)e^{r\%T}$, Long underlying, Short forward, Borrow $S(0)$
- When $F_0 < S(0)e^{r\%T}$, Short underlying, Long forward, Deposit $S(0)$
- By no-arbitrage principle, $F_0 = S(0)e^{r\%T}$.

*** With carrying cost**

- By no-arbitrage principle, $F_0 = S(0)e^{r\%T} + \sum_{i=1}^n c_i e^{r\%(T-t_i)}$

***** Future contract *****

- **Daily Cash Flows of a Futures Contract**

Date	Futures Price	Spot Price	Cash Flow from Futures
0	$F(0)$	$S(0)$	-
1	$F(1)$	$S(1)$	$F(1) - F(0)$
2	$F(2)$	$S(2)$	$F(2) - F(1)$
.	.	.	.
.	.	.	.
.	.	.	.
$T-1$	$F(T-1)$	$S(T-1)$	$F(T-1) - F(T-2)$
T	$F(T)$	$S(T)$	$F(T) - F(T-1)$

Long position

- At maturity date, T, $S(T) = F(T)$, otherwise arbitrage exists.
- Sum of entries of profit for long position are $S(T) - F(0)$.
- Let $R = exp(\frac{r}{365})$, the accrued total profit for long position is $\sum_{i=1}^T [F(i) - F(i-1)]R^{T-i}$.
- Margin account, typically accumulate at risk free interest rate.
- Initial margin, such as, 50 % of total contract value.
- Maintenance margin level, such as 25% of total contract value.
- If value of margin account drops below the maintenance margin, a margin call will be issued. If additional margin is not provided, the futures position will be closed out.

***** Hedging *****

- Perfect hedge: Take an equal and opposite position in the futures market.
- Perfect hedge is rare and only possible when: (i) Trader has to match the delivery date of asset and delivery date of the futures. (ii) The commodity to be hedged must exactly match the commodity underlying the futures contract. (iii) The amount of the asset obligated must be an integral multiple of the contract size.
- Suppose selling k of Asset A, hedge by shorting k' futures of Asset B.
- Suppose buying k of Asset A, hedge by longing k' futures of Asset B.
- Units of asset A, *k*. Units of Asset B, *k'*.
- $S_A(T)$: spot price of A at time T, $F_B(T)$: future price of B at time T.
- Hedge ratio, $h = \frac{k'}{k}$
- Basis of portfolio: $X_T := S_A(T) - F_B(T)$
- Cash flow at time *T*,

$$Y_T := kS_A(T) - k'(F_B(T) - F_B(0))$$
$$= k(S_A(T) - hF_B(T) - k'F_B(0))$$
$$= KX_T - k'F_B(0)$$

- Variance of X_T: $\sigma_{X_T}^2 = \sigma_{S_A(T)}^2 + h^2\sigma_{F_B(T)}^2 - 2h\sigma_{S_A(T)}\sigma_{F_B(T)}\rho,$
- Variance of Y_T: $\sigma_{Y_T}^2 = k^2\sigma_{X_T}^2$
- To reduce variance of Y_T: $\frac{d\sigma_{X_T}^2}{dh} = 2h\sigma_{F_B(T)}^2 - 2\sigma_{S_A(T)}\sigma_{F_B(T)}\rho = 0.$
- Optimal hedge ratio: $h = \frac{\sigma_{S_A(T)}}{\sigma_{F_B(T)}}\rho.$
- Minimum variance hedge: $\sigma_{Y_T} = k\sigma_{X_T} = k\sigma_{S_A(T)}\sqrt{1-\rho^2}$
- The stronger the correlation between $S_A(T)$ and $F_B(T)$, the lower the risk.
- In perfect hedge, this risk is driven to 0.

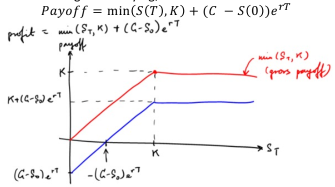
Chapter 5 Options

- Option price == Option premium
- Option seller/writer == Option shorter
- Long Call profit: $\max(0, -K + S(T)) - TV(C_K)$
- Short Call profit: $\min(0, -S(T) + K) + TV(C_K)$
- Long Put profit: $\max(0, -S(T) + K) - TV(P_K)$
- Short Put profit: $\min(0, -K + S(T)) + TV(P_K)$

***** Option Trading Strategy *****

- **Covered call**: Long 1 underlying, Short 1 call.

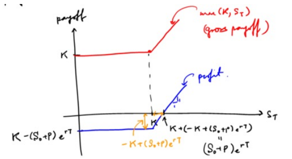
$Payoff = \min(S(T), K) + (C - S(0))e^{rT}$



The spread gives, $C \le S(0)$

- **Protective puts**: Long 1 underlying. Long 1 put.

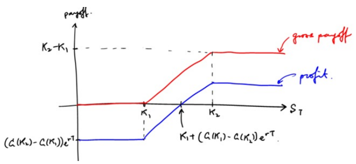
$Payoff = \max(K, S(T)) - (S(0) + P)e^{rT}$



The spread gives, $P \ge Ke^{-rT} - S(0)$

- **Bull spread using calls**: Long 1 K_1-call. Short 1 K_2 -call. K_1 is typically chosen to be close to the current value of the underlying.

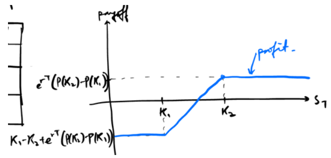
$pf = \max(0, -K_1 + S(T)) + \min(0, K_2 - S(T)) + (C_{K_1} - C_{K_2})\,e^{rT}$



The spread gives, $C_{K_2} - C_{K_1} \le 0$, $C_{K_1} - C_{K_2} \le e^{-rT}(K_2 - K_1)$

- **Bull spread using puts**: Long 1 K_1-put. Short 1 K_2 -put.

$pf = \max(0, K_1 - S(T)) + \min(0, -K_2 + S(T)) + (P_{K_2} - P_{K_1})\,e^{rT}$

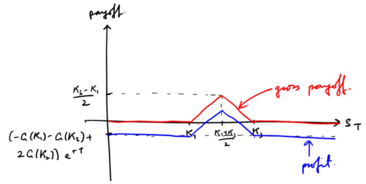


The spread gives, $P_{K_2} - P_{K_1} > 0$

- **Symmetric butterfly spread using calls**: Long 1 K_1-call. Short 2 K_2 -calls. Long 1 K_3-call.

$gross\ payoff$
$= \max(0, S(T) - K_1) + \max(0, S(T) - K_3) + 2\min(0, K_2 - S(T))$

$Payoff = gross\ payoff + (-C_{K_1} - C_{K_3} + 2C_{K_2})\,e^{rT}$



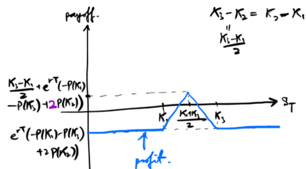
The spread gives,

$-C_{K_1} - C_{K_3} + 2C_{K_2} < 0$, and $C_{K_2} > \frac{K_1 - K_3}{4e^{rT}} + \frac{C_{K_1} + C_{K_3}}{2}$

- **Symmetric butterfly spread using puts**: Long 1 K_1-put. Short 2 K_2 -puts. Long 1 K_3-put.

$gross\ payoff$
$= \max(0, K_1 - S(T)) + \max(0, K_3 - S(T)) + 2\min(0, S(T) - K_2)$

$Payoff = gross\ payoff + (-P_{K_1} - P_{K_3} + 2P_{K_2})\,e^{rT}$

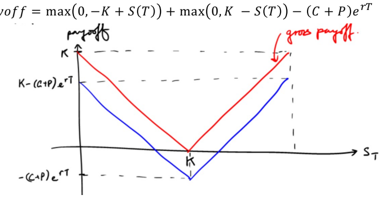


The spread gives,

$-P_{K_1} - P_{K_3} + 2P_{K_2} < 0$, and $P_{K_2} > \frac{K_1 - K_3}{4e^{rT}} + \frac{P_{K_1} + P_{K_3}}{2}$

- **Long straddles**: Long 1 call. Long 1 put.

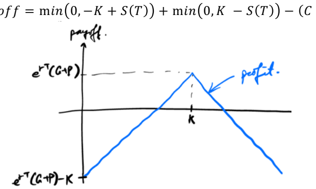
$Payoff = \max(0, -K + S(T)) + \max(0, K - S(T)) - (C + P)e^{rT}$



The spread gives, $C + P > 0$

- **Short straddles**: Short 1 call. Short 1 put.

$Payoff = min(0, -K + S(T)) + min(0, K - S(T)) - (C + P)e^{rT}$



The spread gives, $C + P > 0$

***** Bounds on call option prices *****

- By covered call, $C_A \le S_0$, $C_E \ge \max(0, S_0 - Ke^{-rT})$, hence

$$\max(0, S_0 - Ke^{-rT}) \le C_E \le C_A \le S_0$$

***** Bounds on put option prices *****

- By protective put, $P_A \le K$, $P_E \le Ke^{-rT}$, $P_E \ge \max(Ke^{-rT} - S_0, 0)$. Hence,

$$\max(0, Ke^{-rT} - S_0) \le P_E \le Ke^{-rT} \quad \text{and} \quad \max(0, Ke^{-rT} - S_0) \le P_A \le K.$$

- By bull spread using calls, $C_{K_1} \le C_{K_2}$, $C_{K_1} - C_{K_2} \le e^{-rT}(K_2 - K_1)$. Hence,

$$0 \le C_{K_1} - C_{K_2} \le e^{-rT}(K_2 - K_1).$$

- By bull spread using puts, $P_{K_2} \le P_{K_1}$, $P_{K_2} - P_{K_1} \le e^{-rT}(K_2 - K_1)$. Hence,

$$0 \le P_{K_2} - P_{K_1} \le e^{-rT}(K_2 - K_1)$$

***** Put call parity *****

- Put call parity equation: $C + Ke^{-rT} = P + S_0$
- 凑

***** One-step binomial model*****

$$F_1^u = \begin{cases}\max(uS_0 - K, 0) & \text{if option is a call}\\ \max(K - uS_0, 0) & \text{if option is a put}\end{cases} \quad F_1^d = \begin{cases}\max(dS_0 - K, 0) & \text{if option is a call:}\\ \max(K - dS_0, 0) & \text{if option is a put.}\end{cases}$$

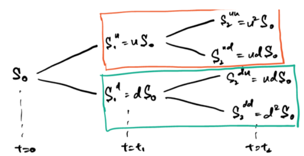
- Replicating portfolio: A portfolio whose end-of-period value is the same as that of the option, consisting of,
- Δ units of underlying asset, and
- an amount B invested in the risk-free asset.
- where Δ and B satisfy, $\begin{cases}\Delta uS_0 + Be^{rt_1} = F_1^u\\ \Delta dS_0 + Be^{rt_1} = F_1^d.\end{cases}$
- Therefore, $\Delta = \dfrac{F_1^u - F_1^d}{S_0(u-d)}$ and $B = \dfrac{uF_1^d - dF_1^u}{e^{rt_1}(u-d)}$
- By no-arbitrage principle, the initial value must be equal to that of the option. Hence
- Initial value: $F_0 = \Delta S_0 + B$

$$= \frac{F_1^u - F_1^d}{u-d} + \frac{uF_1^d - dF_1^u}{e^{rt_1}(u-d)}$$
$$= e^{-rt_1}\left[\frac{e^{rt_1} - d}{u-d}F_1^u + \frac{u - e^{rt_1}}{u-d}F_1^d\right].$$

- Risk-neutral probability: $q := \frac{e^{rt_1} - d}{u-d}$
- Initial value: $F_0 = e^{-rt_1}\left[qF_1^u + (1-q)F_1^d\right]$
- Attention: p, S_0 are not included in the option pricing formula.
- Since $qu + (1-q)d = e^{rt_1}$, we have,

$$S_0 = e^{-rt_1}(qu + (1-q)d)S_0 = e^{-rt_1}(qS_1^u + (1-q)S_1^d).$$

***** Two-step binomial model *****



$$F_2^{uu} := \begin{cases}\max(u^2S_0 - K, 0) & \text{if option is a call}\\ \max(K - u^2S_0, 0) & \text{if option is a put}\end{cases} \quad F_2^{dd} := \begin{cases}\max(d^2S_0 - K, 0) & \text{if option is a call;}\\ \max(K - d^2S_0, 0) & \text{if option is a put.}\end{cases}$$

$$F_2^{ud} = F_2^{du} := \begin{cases}\max(udS_0 - K, 0) & \text{if option is a call;}\\ \max(K - udS_0, 0) & \text{if option is a put,}\end{cases}$$

$$\begin{cases}\Delta^u uS_1^u + B^u e^{r(t_2 - t_1)} = F_2^{uu}\\ \Delta^u dS_1^u + B^u e^{r(t_2 - t_1)} = F_2^{ud}.\end{cases} \quad \begin{cases}\Delta^d uS_1^d + B^d e^{r(t_2 - t_1)} = F_2^{du}\\ \Delta^d dS_1^d + B^d e^{r(t_2 - t_1)} = F_2^{dd}.\end{cases}$$

$$\Delta^u = \frac{F_2^{uu} - F_2^{ud}}{S_1^u(u-d)}, \quad B^u = \frac{u\cdot F_2^{ud} - d\cdot F_2^{uu}}{e^{r(t_2 - t_1)}(u-d)}$$

$$\Delta^d = \frac{F_2^{du} - F_2^{dd}}{S_1^d(u-d)}, \quad B^d = \frac{u\cdot F_2^{dd} - d\cdot F_2^{du}}{e^{r(t_2 - t_1)}(u-d)}.$$

These in turn give

$$F_1^u = e^{-r(t_2 - t_1)}\left[q_2F_2^{uu} + (1-q_2)F_2^{ud}\right] \quad \text{and} \quad F_1^d = e^{-r(t_2 - t_1)}\left[q_2F_2^{du} + (1-q_2)F_2^{dd}\right]$$

where $q_2 := \frac{e^{r(t_2 - t_1)} - d}{u-d}$.

Applying the one-step binomial model once more to period 1, we have

$$F_0 = e^{-rt_1}\left[q_1F_1^u + (1-q_1)F_1^d\right],$$

where $q_1 := \frac{e^{rt_1} - d}{u-d}$. Hence,

$$F_0 = e^{-rt_2}\left[q_1q_2F_2^{uu} + q_1(1-q_2)F_2^{ud} + q_2(1-q_1)F_2^{du} + (1-q_1)(1-q_2)F_2^{dd}\right].$$