QF3101 AY24/25 Sem 2 Cheatsheet By Li Tianze

Chapter 0 Day Count Convention

0.3 Day Count Convention

- xxx/yyy where xxx is number of counted days in a month, yyy is number of counted days in a year.
- $actual/actual$ is number of actual days in a month and actual days in a year. This is used in U.S. Treasury notes and bonds.
- $actual/360$, $30/360$ are also popular conventions.
- $actual/actual$ from 1 Jan 22 to 15 March 2022, Day Count is $30 + 28 + 15 = 73$.
- $actual/actual$ from 1 Jan 22 to 15 March 2023, Day Count is $365 + 30 + 28 + 15 = 438$.
- Date 1 is $D_1.M_1.Y_1$, Date 2 is $D_2.M_2.Y_2$. Under 30/360, Daycount is $360(Y_2 - Y_1) + 30\,(M_2 - M_1) + (D_2 - D_1)$.
- $DayCountFactor = \frac{DayCount}{No\ of\ days\ in\ a\ year}$

Chapter 1 Forward, Futures, Hedging

1.1 Forward Formula - Simplest

- At $t = t_0$, current spot price is S_0, the theoretical price is $F_0 = \frac{S_0}{d_{0,T}}$
- Cash-and-carry: Short forward contract, long asset.
- $F_0 > \frac{S_0}{d_{0,T}}$, cash-and-carry arbitrage.
- $F_0 < \frac{S_0}{d_{0,T}}$, reverse cash-and-carry arbitrage

1.2 Forward Price – w. Predictable Income

- C_i' is provided at $t_i \in (0, T]$, we have $S_0 = \sum_1^n C_i' d_{0,t_i} + F_0 d_{0,T}$, rearranging to
- $F_0 = \frac{S_0 - C}{d_{0,T}}$, where $C = \sum_1^n C_i' d_{0,t_i}$

1.3 Forward Price – w. cont. Dividend Yield

- Continuous annualized dividend rate: q.
- $F_0 = S_0 e^{-qT} e^{rT} = S_0 e^{(r-q)T}$

1.4 Forward Price – w. Storage Cost

- Pay storage cost C_j at the beginning of each period j, t_j to t_{j+1}, until K periods. $j \in [0, K-1]$. $t_i = i$th, $i \in [0, K]$.
- $F_0 = \frac{S_0}{d_{0,K}} + \sum_0^{K-1} \frac{C_j}{d_{j,K}} = \frac{S_0 + \sum_0^{K-1} d_{0,j} C_j}{d_{0,K}}$
- $F_0 > \cdots$, short forward contract, borrow C_j at time t_j for $j \in [0, K-1]$.

1.5 Forward Price – w. disc. Dividend Yield

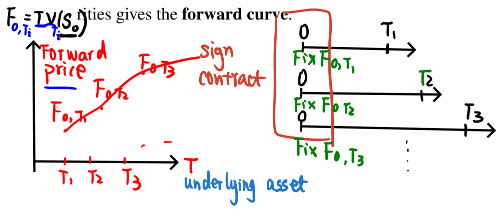
- The yield is negative storage cost.
- $F_0 = \frac{S_0}{d_{0,K}} - \sum_0^{K-1} \frac{C_j}{d_{j,K}} = \frac{S_0 - \sum_0^{K-1} d_{0,j} C_j}{d_{0,K}}$

1.6 Forward Price – w. Convenience d Yield

- The yield is negative storage cost.
- $F_0 = \frac{S_0}{d_{0,K}} + \sum_0^{K-1} \frac{C_j}{d_{j,K}} - \sum_0^{K-1} \frac{y}{d_{j,K}} = \frac{S_0 + \sum_0^{K-1} d_{0,j} C_j - \sum_0^{K-1} d_{0,j} y}{d_{0,K}}$

1.7 Forward Curve

ities gives the **forward curve**.



1.8 Value of Forward

- At time t, when the spot price of forward contract is F_t, the value of long position of is $f_t = (F_t - F_0) d_{t,T}$
- At time T, $f_T = S_T - F_0 = \$saved$
- At time 0, $f_0 = (F_0 - F_0) d_{0,T} = 0$

1.9 Future-Forward Equivalence

- Assuming 0 transaction costs and no margin is required for futures contracts, and supposing the interest rates are known to follow dynamics, then the theoretical futures and forward prices of corresponding contracts are identical.
- $eff.p. = spot\ p. - futures\ profit$ (Long)
- $eff.p. = spot\ p. + futures\ profit$ (Short)

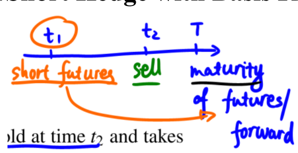
1.10 Hedging by Futures/Forward

- Perfect hedge: Risk **completely eliminated**.
- Short Hedge is taking short pos. of contracts.
- $Net\ CF\ (in) = S_T + (F_0 - S_T) = F_0$
- Long Hedge:
- $Net\ CF\ (out) = S_T - (S_T - F_0) = F_0$

1.11 Basis Risk

- Perfect hedges are often not possible due to:
- (1) Unavailability of contract matching obligation dates.
- (2) Quantity of asset may not be an integer multiple of contract size.
- (3) Lack of liquidity in the futures market.
- (4) Terms of delivery may not completely coincide with obligation.
- (5) Unavailability of contracts with underlying asset exactly match the hedged asset.
- With both prices at a same time, $\boldsymbol{basis} = spot\ price\ of\ asset\ to\ be\ hedged - future\ price\ of\ contract\ used$
- If $\boldsymbol{basis} = 0$ at delivery time, perfect hedge.

1.12 Short Hedge with Basis Risk



- old at time t_2 and takes
- t_1: Short Futures, t_2: Close Short pos. & Sell Asset, T: Maturity of Short Futures.
- $eff.price(in) = S_2 + (F_1 - F_2) = F_1 + (S_2 - F_2) = F_1 + b_2$

1.13 Long Hedge with Basis Risk

- t_1: Long Futures, t_2: Close Long pos. & Buy
- $eff.price(out) = S_2 - (F_2 - F_1) = S_2 + (F_1 - F_2) = F_1 + b_2$
- F_1: Entry Future Price, b_2: Cover basis.

1.14 Basis Risk Summary

- $eff.price = F_1 + b_2 = S_1 - (b_1 - b_2)$
- Where $b_1 = S_1 - F_1$, $b_2 = S_2 - F_2$
- b_1: Initial Basis, b_2: Cover Basis.
- Basis Risk increases as $T - t_2$ increases.
- Choose future contract with min. $T - t_2 \geq 0$.

1.15 Cross Hedging

- Use of derivatives on one underlying (proxy) asset to hedge the risk of another asset.
- Proxy asset: S_i^*, F_i^* at time t_i.
- $eff.price = S_2 + (F_1^* - F_2^*) = F_1^* - F_2^* + S_2 = F_1^* + (S_2^* - F_2^*) + (S_2 - S_2^*)$ random
- Two sources of uncertainty in the basis of proxy asset.

1.16 Minimum Variance Hedge

- Hedge ratio is typically equal to 1 when the asset underlying the futures is the same as the asset being hedged.
- With cross hedging, the hedger needs to choose the optimal hedge ratio which minimizes the variance of the cash flow of the hedged position. Having a ratio of 1 in this case does not necessarily lead to the minimum variance. eff price x unit
- At t_1, knowing to purchase Q_a of asset at t_2.
- Must pay $x = Q_a S_2 = Q_a(S_1 + \Delta S)$ at t_2.
- F is futures price of contract used to hedge.
- Q_f is size of long position of F taken.
- Hedge ratio: $h = \frac{Q_f}{Q_a}$
- Neglect interest on margin account by assuming losses/profits settled at t_2.
- At t_2, the CF on hedged position is
- $y = Q_a(S_1 + \Delta S) - Q_f(F_2 - F_1) = Q_a(S_1 + \Delta S - h\Delta F)$, where $\Delta F = F_2 - F_1$
- We want to minimize
- For $f(h) = Var(y)$, take $f'(h) = 0$,
- $h^* = \frac{cov(\Delta S, \Delta F)}{Var(\Delta F)} = \frac{\rho \sigma_{\Delta S} \sigma_{\Delta F}}{\sigma_{\Delta F}^2} = \frac{\rho \sigma_{\Delta S}}{\sigma_{\Delta F}}$
- $Var(y)^* = Q_a^2 \left(Var(\Delta S) - \frac{cov(\Delta S, \Delta F)^2}{Var(\Delta F)} \right)$
- $= Var(x) - \frac{cov(x, \Delta F)^2}{Var(\Delta F)}$
- $Var(x) = Q_a^2 Var(\Delta S)$, since $cov(x, \Delta F)^2 = cov(Q_a(S_1 + \Delta S), \Delta F)^2 = Q_a^2 cov(S_1 + \Delta S, \Delta F) = Q_a^2 cov(\Delta S, \Delta F)$
- $\frac{Var(y)^*}{Var(x)} = 1 - \rho^2$
- When $F_2 = S_2 \to \sigma_{F_2}^2 = \sigma_{S_2}^2 \to \sigma_{F_2 - F_1}^2 = \sigma_{S_2 - S_1}^2 \to \sigma_{\Delta S} = \sigma_{\Delta F}$ and $\rho = 1$, perfect hedge ratio is $h^* = 1$

QF3101 AY24/25 Sem 2 Cheatsheet By Li Tianze

Chapter 2 Forward & Futures in Practice

2.1 Bonds

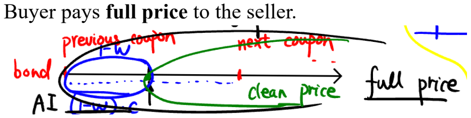
- Cash flow of bond:
  $$\left((-F,0),(\frac{c\%F}{m},\frac{1}{m}),(\frac{c\%F}{m},\frac{2}{m}),\dots,(\frac{c\%F}{m}+R,\frac{n}{m})\right)$$

2.2 US Treasury Bills (T-Bills)

- Highly liquid, initially sold at auctions.
- No coupon payments.
- Discount basis, interest subtracted from par value to derive purchase price.
- Dollar price quoted per \$100 of face value.
- Quotations in discount yield/rate, or **Ask**, d.
- No. of days to maturity, t.
- $d = \frac{F - Purchase\ price}{F} \times \frac{360\ days}{t}$
- Coupon Equivalent Yield/Investment rate/**Ask Yield**, i, is measure of the true yield to T-bill holder.
- Number of days in a year, y, 355 or 356.
- **Case 1:** $t \le half\ a\ year$, $i = \frac{F-P}{P} \times \frac{y}{t}$.
- **Case 2:** $t > half\ a\ year$,
- $P\left(1+\frac{1}{2}i\right)\left(1+\frac{t-0.5y}{y}i\right) = F$
- $\left(\frac{t}{2y} - 0.25\right)i^2 + \frac{t}{y}i + \frac{P-F}{P} = 0$
- US Treasury Notes: 1~10 years, Bonds: > 10 years. They pay semi-annual coupons.

2.3 Accrued Interest

- If next coupon is exactly one period from now,
  $$P = \frac{F}{\left(1+\frac{r}{m}\right)^n} + \sum_{i=1}^{n} \frac{C}{\left(1+\frac{r}{m}\right)^i}.$$
- Otherwise, **full price**, **dirty price** or **invoice price** is
  $$P = \frac{F}{\left(1+\frac{r}{m}\right)^{w+n-1}} + \sum_{i=0}^{n-1} \frac{C}{\left(1+\frac{r}{m}\right)^{w+i}}.$$
- Where $w = \frac{\text{no of days between pricing date and the next coupon payment date}}{\text{no of days in the current coupon period}}$
- **Quoted price** for bonds in US does not include accrued interest, which is **clean price** or **flat price**.
- $Dirty\ Price = (1-w) \times c + clean\ price$
- Buyer pays **full price** to the seller.



-

2.4 Interest Rate Future & Forward

- Have the value of underlying asset dependent solely on value of an interest rate.
- Long pos. profits when interest rate falls.

2.5 Repurchase Agreements (Repo)

- At time 0, a trader can finance the purchase of a security worth S_0 from the security market by simultaneously entering into a T-day repo with Mr. Y using the same security just bought as collateral.
- Mr. Y lends $S_0(1-h\%)$ to the trader. Since S_0 amount is paid for the security, the net outflow at time 0 for the trader is $S_0 \times h\%$.
- At time T, the trader pays Mr. Y $S_0(1-h\%)(1+r\times T/360)$ and receives the security.
- Payoff: $S_T - S_0(1-h\%)\left(1 + r \times \frac{T}{360}\right) - TV_T(S_0 \times h\%)$

2.6 Secured Overnight Financing Rate (SOFR)

- Reflect cost of overnight loans in US Treasury repo market.
- Obtained from volume-weighted median of transactional-level data over the course of a business day and is published on FRBNY website on the next business day.
- Day count convention: $actual/360$

2.7 SOFR Averages

- Simple SOFR: $r = \left(\sum_{i=1}^{T}\frac{r_i n_i}{Y}\right)\frac{Y}{d_c} = \frac{1}{d_c}\sum_{i=1}^{T} r_i n_i$
- Compound SOFR: $r = \left[\prod_{i=1}^{T}\left(1+\frac{r_i n_i}{Y}\right)\right]\frac{Y}{d_c}$
- Where n_i for most days is 1, for Fridays is generally 3.
- In **advance** structure: Reference to an avg SOFR observed before interest period begins.
- In **arrears** structure: Reference to an avg SOFR over the interest period. Conventions include:
- Payment delays: Use the current SOFR rate for each day in the interest period, but the payment is made "x" days after the last day of the accrual period.
- Lookbacks: For each day in the interest period, the SOFR rate from "x" business days earlier is used.
- Lockouts: The SOFR rates applied for the last days of the interest period are frozen at the rate observed "x" days before the period ends.

2.8 One-Month SOFR Futures

- Underlying asset: \$ 5,000,000 loan for one-month period.
- Cash settled at expiry, using final settlement price.
- $F_t = 100(1 - simple\ avg\ SOFR\ for\ contract\ mth)$
- r_m simple avg daily SOFRs for the contract delivery month (per annum, actual/360)
- Final settlement price: $100 \times (1 - r_m)$
- "Long" pos gains when expected one-month simple SOFR falls (Future prices rises).
- "Long" 1 contract ↔ Lend \$5mil at rate $(100 - F_0)\%$ for the contract month.
- Changes: Future price decreases 0.01, rise by 1 basis point, margin of long position decreased \$41.67

2.9 Three-Month SOFR Futures

- Underlying asset: \$ 1,000,000 loan for three-month period.
- Cash settled at expiry, using final settlement price.
- $F_t = 100(1 - compound\ avg\ SOFR\ for\ contract\ qtr)$
- r_m compound avg daily SOFRs for the contract delivery month (per annum, actual/360)
- Referencing quarter:
  $[3^{rd} Wed\ of\ 3^{rd}\ mth\ prior\ delivery\ mth,$
- $3^{rd}\ Wed\ of\ delivery\ mth)$
- Three-month SOFR futures have delivery dates for the nearest 39 March-quarterly months (Mar, Jun, Sep, Dec)
- Dec 22 three-mth SOFR future:$[14/Dec/22,\ 15/Mar/23)$
- r_m simple avg daily SOFRs for the contract delivery month (per annum, actual/360)
- Final settlement price: $100 \times (1 - r_m)$
- "Long" pos gains when expected three-month compound SOFR falls (Future prices rises).
- "Long" 1 contract ↔ Lend \$1mil at rate $(100 - F_0)\%$ for the contract quarter.
- Changes: Future price decreases 0.01, rise by 1 basis point, margin of long position decreased \$25

2.10 Treasury Bond Futures

- The underlying for T-bond futures is a hypothetical 30-year 6% (semi-annual) coupon bond with par value \$100,000.
- The future price is quoted using 'thirty-second' notation for each \$100 face value. If futures settle at $104\ 15/32$ or $104-15$, the long position can buy this face value \$100 bond at $\$104 + \$\left(\frac{15}{32}\right) = \$104.46875$.
- The bond delivered must have remaining maturity of 15~25 years. The short position can choose any business day in the delivery month to deliver. Cash settlements are allowed.
- The deliverable bonds have various maturities, coupon rates and yields.
- $P_{inv} = N \times F_T \times CF + AI$. P_{inv}: invoice(dirty) price. N: nominal value of the bond in the contract. F_T: future settlement(clean) price. CF: conversion factor of the deliverable bond. AI: the bond's accrued interest to delivery.
- Conversion factor, CF: Clean price of \$1 (face value) coupon bond with a maturity (1^{st} day of delivery month) equal to the deliverable bond if priced to yield 6%, compounded semi-annually.
- Maturities are rounded down to the nearest quarter, as coupon are of 6-mth length, $w = \frac{1}{2}$ or 1.
- $19y\ 2mth \to n = 38, w = 1$; $19y\ 4mth \to n = 39, w = \frac{1}{2}$
- $CF = \frac{1}{1.03^{w+n-1}} + \sum_{i=0}^{n-1}\frac{c}{1.03^{w+i}} - c(1-w)$

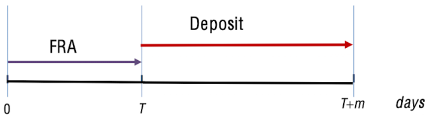
2.11 T-Bond Futures – Cheapest to Deliver (CTD)

- The short pos delivers bond to maximize the profit $\max_i F_T \times CF(i) - CPrice(i)$
- To cost nothing deliver, $GB(i^*) = 0$
- Future price should be $F_T = \frac{CPrice(i^*)}{CF(i^*)} = \min_i \frac{CPrice(i)}{CF(i)}$
- At $t \le T$, **Gross Basis**: $GB(i) = CPrice(i) - F_T \times CF(i)$.
- In fact, the CTD bond is not identified by GB, but by **net basis** or the **implied repo rate**.
- At $t < T$, **Net Basis**:$NB(i) = DPrice(i) \times \left(1 + \frac{rD}{360}\right) -$
  $F_t \times CF(i) - AI$, where $DPrice(i)$: Dirty price of bond i at t per \$100 face value. r: repo rate to fund bond purchase.
- At $t < T$, **Internal Repo Rate** satisfies
- $DPrice(i) \times \left(1 + \frac{IRR \times D}{360}\right) = F_t \times CF(i) + AI$

QF3101 AY24/25 Sem 2 Cheatsheet By Li Tianze

- Thus, $IRR = \frac{(F_t \times CF(i) + AI) - DPrice(i)}{DPrice(i)} \times \frac{360}{D}$, valid only when no interim coupon.

2.12 Forward Rate Agreements (FRAs)



- The underlying is hypothetical deposit, settled in cash.
- T, settlement day. X fixed rate (actual/360). y actual reference rate(actual/360). N principle. m deposit period.
- "Long" position to hedge interest rate rise.
- At $t = T + m$, payoff of "Long" side is
- $\max((y - X) \times N \times \frac{m}{360}, 0)$
- At $t = T$, payoff of "Long" position is
- $\max((y - X) \times N \times \frac{m}{360} \times \frac{1}{1 + \frac{ym}{360}}, 0)$
- FRA covering period starting A months from now and ending B months from now are referred to as $A \times B$ (A-by-B) contract, referencing rate is $(B - A)$-month SOFR.
- At time t, the forward rate for $[T, T + m]$ is
- $f(t, T, T + m) = \left(\frac{d_{t,T}}{d_{t,T+m}} - 1\right) \times \frac{360}{m}$
- At time t, the value for "long" position is
- $v = (f(t, T, T + m) - X) \times N \times \frac{m}{360} \times d_{t,T+m}$
- Which is $v = \left[d_{t,T} - \left(1 + X \times \frac{m}{360}\right) \times d_{t,T+m}\right] \times N$

2.13 Currency Forwards & Futures

- Prepaid currency forward price: $F_0^P = x_0 d_{0,T}(r_u)$
- Where x_0 is current exchange rate, r_f is foreign interest rate.
- Currency forward price: $F_0 = \frac{F_0^P}{d_{0,T}(r_f)} = x_0 \frac{d_{0,T}(r_u)}{d_{0,T}(r_f)}$
- Currency pair convention: base-currency/terms-currency.
- Terms-currency is also the quote-currency. RMB/USD=0.14

2.14 Stock Index Futures

- Tracks value of hypothetical portfolio of stocks in a market.
- All positions are settled in cash.
- S&P500: Tick size 0.05, value \$250 $\times$ $index$ (from 1 Nov 97, previous \$500 $\times$ $index$), closed based on 'open' on the 3^{rd} Friday of the expiry month.
- Dollar returns of hedged portfolio:
- $D = r_p P_0 + nm(F_0 - F_T) = r_p P_0 - nmI_0(r_M - r_f)$
- Min. variance hedge: $n^* = \frac{P_0}{mI_0} \frac{cov(r_p, r_M)}{\sigma_M^2} = \frac{P_0}{mI_0} \beta_p$
- Variance of optimally hedged portfolio: $\sigma_D^2 = \sigma_p^2 P_0^2 (1 - \rho^2)$
- Dollar return: $D^* = P_0\left(r_p - \beta_p\left(r_M - r_f\right)\right) = P_0 r_D^*$
- By CAPM, $r_p = \alpha + r_f + \beta_p(r_M - r_f) + \epsilon$
- where estimation error $\alpha = r_p - \mu_p$, ϵ is random variable.
- **Portfolio Immunization**: no systematic risk as $r_D^* = \alpha + r_f + \epsilon$ has zero beta.

Chapter 3 Swaps

- Life span of swap: **Swap term** or **Swap tenor**

3.1 Commodity Swaps

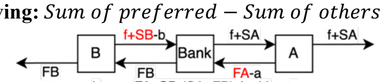
- Exchanged cash flows determined by price of commodity.
- A receives cash flow based on spot price and pays fixed CF.
- Net CF: $N \times [S_1 - X, S_2 - X, \dots, S_K - X]$, S_i is price at t_i.
- At time t_0, $PV(CF) = \sum_{i=1}^{K} d_{0,i} N_i (F_{0,i} - X)$.
- As initial swap value is zero, Swap Price
- $X = \sum_{i=1}^{K} \left(\frac{d_{0,i} N_i}{\sum_{j=1}^{K} d_{0,j} N_j}\right) F_{0,i}$.

3.2 Interest Rate Swaps

- Pay CF determined by fixed interest rate (fixed leg), receive CF determined by floating interest rate (floating leg).
- **Swap rate**: the rate causes the initial value of swap to zero.
- Swap rate usually in 30/360.
- **"30 over"** for 5-year swap: r=5-year Treasury yield+30 bp.
- Floating rate is typically a short-term rate from 3-month LIBOR (obsolete) or 3-month term SOFR.
- A receives CF on floating leg and receives on fixed leg.
- Net CF: $N \times [c_0 - r, c_1 - r, \dots, c_{K-1} - r]$.
- Assume c_i quoted in advance in this Chapter, so are pmts.
- **Reset**: At start of each coupon period, the coupon rate is set to the floating rate for that period, except for first period.
- **Initial Floating-leg-value:** $B_{fl} = N(1 - d_{0,K})$, **at par**.
- **Initial Fixed-leg-value:** $B_{fix} = Nr \sum_{i=1}^{K} d_{0,i}$.
- **Value of swap (A)**: $B_{fl} - B_{fix}$.
- **Swap rate**: $r = \frac{1 - d_{0,K}}{\sum_{i=1}^{K} d_{0,i}}$, when $B_{fl} = B_{fix}$.
- **Past Leg Values**: To determine swap value initiated $(coupon\ period - f_1)$ ago:
- $B_{fl} = (N + f_1)e^{-r_1 t_1}$, $B_{fix} = Ne^{-r_n t_n} + \sum_{i=1}^{n} ke^{-r_i t_i}$.
- Example:

	Fixed Rate	Floating Rate
A	F_A	$f + S_A$
B	F_B	$f + S_B$

- **Total Saving:** $Sum\ of\ preferred - Sum\ of\ others > 0$



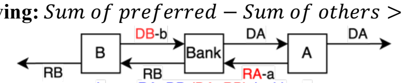
- Savings: b FA+SB-(SA+FB)-(a+b) a

3.3 Currency Swaps

- Example:

	USD	RMB
A	D_A	R_A
B	D_B	R_B

- **Total Saving:** $Sum\ of\ preferred - Sum\ of\ others > 0$



- Savings: b RA+DB-(DA+RB)-(a+b) a
- **Values for A (USD):** $PV(CF_{USD}) - X \times PV(CF_{RMB})$
- **Values for B (USD):** $X \times PV(CF_{RMB}) - PV(CF_{USD})$

Chapter 4 Options

4.1 European Vanilla Options

	Payoff at T (φ)	Profit at T
Call	$\max(0, S_T - K)$	$\max(0, S_T - K) - c/d_{0,T}$
Put	$\max(0, K - S_T)$	$\max(0, K - S_T) - p/d_{0,T}$

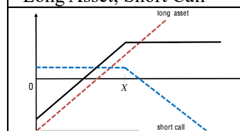
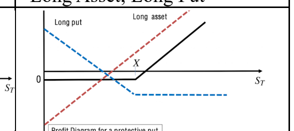
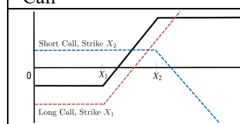
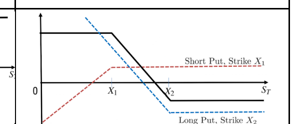
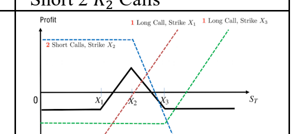
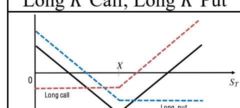
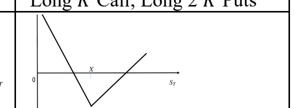
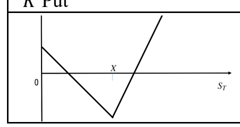
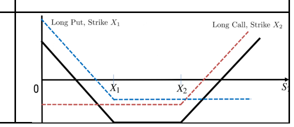
- **Moneyness**:

In-the-money	Out-of-the-money	At-the-money
$\varphi > 0$	$\varphi < 0$	$\varphi = 0$

- **Intrinsic Value:** φ if option exercised immediately.

4.2 Trading Strategies

- Options and Underliers; Spreads; Combinations.
- Given $K_i < K_{i+1}$:

Covered Call	**Protective Put**
Long Asset; Short Call	Long Asset; Long Put
	
Bull Spreads	**Bear Spreads**
Long K_1 Call; Short K_2 Call	Short K_1 Put; Long K_2 Put
	
Box Spreads	**Butterfly Spreads**
(Bull + Bear) Spreads	Long K_1, K_3 Calls; Short 2 K_2 Calls
Flat line: $Profit = K_2 - K_1$	
Straddles	**Strips**
Long K Call; Long K Put	Long K Call; Long 2 K Puts
	
Straps	**Strangles**
Long 2 KCalls; Long K Put	Long K_1Put; Long K_2 Call
	

- **Top Vertical = −Strangles**

QF3101 AY24/25 Sem 2 Cheatsheet By Li Tianze

4.3 Fixed Income and Interest Rate Options

- Rate ↑, fixed income ↓, bearish for u. asset, $\varphi(p) > 0$.
- Rate ↑, rate-based income ↑, bullish for u. asset, $\varphi(c) > 0$.
- **Options on Treasury Bills:**
- Underlying: T-bill with maturity of m days when option expires. The strike "discount rate", $x\%$, is with the corresponding dollar price of
- Strike Price: $K = X = \left(1 - x\% \times \frac{m}{360}\right) \times F$
- European Call Payoff: $\varphi = \max(P_T - X, 0)$, where P_T is the date-T dollar price of T-bill with face value F, maturing at date $T + m$. Payment made at time $T < T + m$.
- **Options on Interest Rates:**
- At maturity of rate, $\boldsymbol{t = k + m}$,
- Payoff of caplet: $c_k = \max(R_k - R_K) \times \frac{m}{360} \times N$
- Payoff of floorlet: $p_k = \max(R_K - R_k) \times \frac{m}{360} \times N$
- Where N is the nominal principle, R_k is the m-day interest rate (actual/360), R_K is the cap/floor rate (actual/360).
- At maturity of option, $\boldsymbol{t = k}$,
- Value of the option: $c_k/(1 + \frac{rm}{360})$ or $p_k/(1 + \frac{rm}{360})$.
- Payoff is given at maturity of rate, $t = k + m$.
- **Interest Rate Collar**: Long 1 Caplet; Short 1 Floorlet.
- Let $\alpha = 1 + R_K(m/360)$,
- A caplet is equivalent to α puts on m-day zero coupon bond, strike price $\frac{1}{\alpha}$.
- A floorlet is equivalent to α calls on m-day zero coupon bond, strike price $\frac{1}{\alpha}$.

Chapter 5 Value at Risk

- **Def:** The maximum loss that will not be exceeded with a given probability (confidence level) during risk horizon.
- Confidence level: c; Significance level: $1 - c$; Loss: L; Smallest absolute loss: VaR; Risk horizon: h (time interval).
- We have $P(L > VaR) \leq 1 - c$.

5.1 Non-parametric VaR

- Portfolio value: V; Rate of return: $R \sim N(\mu, \sigma^2)$.
- Portfolio value at risk horizon: $V = V_0(1 + R)$.
- Portfolio value for particular rate, R^*: $V^* = V_0(1 + R^*)$.
- Relative VaR: $VaR_r = E(V) - V^* = -V_0(R^* - \mu)$, $\mu = \bar{R}$.
- Absolute VaR: $VaR_a = V_0 - V^* = -V_0 R^*$.
- Short horizon, $\mu \approx 0$, $VaR_a \approx VaR_r$.
- $P\&L = V - V_0 = \Delta V = V_0 R$.
- Risk factor: R; Dollar exposure to risk factor: V_0.

5.2 Parametric VaR

- Portfolio value: V; Rate of return: $R \sim N(\mu, \sigma)$.
- $R \sim N(\mu, \sigma^2), V = V_0 + V_0 R \Rightarrow V \sim N(V_0 + V_0\mu, V_0^2\sigma^2)$
- $\Rightarrow 1 - c = P(V \leq V^*) = P(R < R^*) = P\left(\frac{R-\mu}{\sigma} \leq \frac{R^*-\mu}{\sigma}\right)$
- $= P(Z \leq -\alpha) = \Phi(-\alpha)$, where $\alpha = -\frac{R^*-\mu}{\sigma}$.
- Giving $R^* = \mu - \alpha\sigma$.
- Non-annualized: $VaR_r = \alpha\sigma V_0$; $VaR_a = (\alpha\sigma - \mu)V_0$.
- Annualized: $VaR_r = \alpha\sigma\sqrt{\Delta t}V_0$; $VaR_a = (\alpha\sigma\sqrt{\Delta t} - \mu\Delta t)V_0$.
- Steps: $1 - c \Rightarrow \alpha = \Phi^{-1}(1 - c) \Longrightarrow R^* = \mu - \alpha\sigma \Longrightarrow VaR$.

5.3 Valuation Approaches

- **Full Eval:** P&L: $\Delta V = V(S_h) - V(S_0)$.
- **Local Eval:** P&L: $\Delta V \approx \frac{\partial V}{\partial S}|_0 \Delta S = (\Delta_0)\Delta S = (\Delta_0 S)\frac{\Delta S}{S}$.
- Dollar exposure: $\Delta_0 S$; Risk factor is rate of return in S.
- $VaR = \alpha\sigma|\Delta_0|S_0 = \alpha\sigma_{portfolio}$ (If volatility of portfolio already captured risk factor and dollar exposure).
- **Second order approx**. (**Delta-gamma method**):
- $\Delta V \approx \frac{\partial V}{\partial S}|_0 \Delta S + \frac{\partial^2 V}{\partial S^2}|_0 \Delta S^2 = (\Delta_0)\Delta S + \frac{1}{2}\Gamma_0(\Delta S)^2$,
- $VaR = \alpha\sigma|\Delta_0|S_0 - \frac{1}{2}\Gamma_0(\alpha\sigma S_0)^2$.

5.4 Delta-Normal (Variance-Covariance) Method

- For N risk factors and their delta exposures:

$$P\&L = \Delta V \approx \underbrace{\frac{\partial V}{\partial S_1}\Big|_0}_{\Delta} \cdot \Delta S_1 + \underbrace{\frac{\partial V}{\partial S_2}\Big|_0}_{\Delta} \cdot \Delta S_2 + \cdots + \frac{\partial V}{\partial S_N}\Big|_0 \Delta S_N$$

$$V_t = \overbrace{x_{1,t} + x_{2,t} + \cdots + x_{N,t}}^{\text{portfolio } \Delta_p}$$

$$= \underbrace{\left(\frac{\partial V}{\partial S_1}\Big|_0\right) \cdot S_1}_{\text{dollar exposure}} \cdot \underbrace{\frac{\Delta S_1}{S_1}}_{r_1} + \underbrace{\left(\frac{\partial V}{\partial S_2}\Big|_0\right) \cdot S_2}_{\text{dollar exposure}} \cdot \underbrace{\frac{\Delta S_2}{S_2}}_{r_2} + \cdots + \underbrace{\frac{\partial V}{\partial S_N}\Big| \cdot S_N}_{\text{dollar exposure}} \underbrace{\frac{\Delta S_N}{S_N}}_{r_N}$$

- $x_{i,t}$ as the exposures aggregated across all assets for risk factor i, measured in currency units at time t.
- Weight: $w_{i,t} = \frac{x_{i,t}}{V_t}$.
- $E(r_{p,t+h}) = \sum_{i=1}^N w_{i,t} E(r_{i,t+h}) = \mathbf{w}_t^T \mathbf{r}_{t+h}$,
- $\sigma_{p,t+h}^2 = \sum_{i=1}^N \sum_{j=1}^N w_{i,t} w_{j,t} \sigma_{ij} = \mathbf{w}_t^T \mathbf{C}_{t+h} \mathbf{w}_t$,
- Exposure to risk factor i: $x_{i,t} = w_{i,t} V_t$,
- Exposure vector: $\mathbf{x}_t = (x_{1,t} \quad \cdots \quad x_{N,t})^T = \mathbf{w}_t V_t$.
- $VaR_p = \alpha\sigma_p V_t = \alpha\sqrt{\sigma_{\Delta V}^2} = \alpha\sqrt{\mathbf{x}_t^T \mathbf{C}_{t+h} \mathbf{x}_t}$.
- $Undiversified\ VaR = \sum Individual\ VaR$,
- $Diversified\ VaR = VaR_p$.

5.5 Risk Mapping

- Map a large number of assets to a benchmark essential asset.

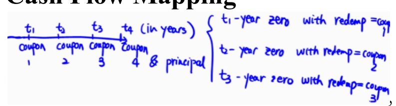
Actual Assets	Mapped Assets
Foreign Exchange (FX)	a number of "core" currencies
FX Forward	fixed income positions in respective currencies
Equity	equity indices in each of the "core" currencies
Fixed Income	combinations of cash flows in a given currency of a limited number of maturities

- **Spot FX Positions**:
- Assume $X \sim N(\mu_X, \sigma_X^2)$ is domestic per foreign exchange rate
- $VaR = \alpha\sigma_X V_0 = \alpha\sigma_X XF$.
- **Equity Positions**:
- Amount x_1 invested in equity of firm 1, $\sigma_1^2 = \beta_1^2\sigma_M^2 + \sigma_e^2$.
- $VaR_1 = \alpha x_1 \sigma_1 = \alpha x_1 \sqrt{\beta_1^2\sigma_M^2 + \sigma_e^2}$.
- σ_e^2 is negligible if portfolio is well-diversified.
- Leading to $VaR_1 = \alpha\sigma_M\beta_1 x_1$.
- Risk factor: μ_M; Dollar exposure: $\beta_1 x_1$.
- $VaR_p = \alpha\sigma_M\beta_p V_0 = \alpha\sigma_M \sum_{k=1}^N \beta_k x_k = \alpha\sigma_M X \frac{\sum_{k=1}^N \beta_k x_k}{X}$
- $= \alpha\sigma_M X \sum_{k=1}^N \beta_k w_k = \alpha\sigma_M \beta_p X = \alpha\sigma_M \beta_p \mathbf{1}^T\mathbf{x}$.
- **Zero Coupon Bonds**:
- Standard Maturity: $\frac{1}{12}, \frac{3}{12}, \frac{6}{12}, 1, 2, 3, 4, 5, 7, 9, 10, 15, 20, 30$.
- Mapped zeros are weighted s.t. MV is preserved, Market Risk is preserved, Sign is preserved $\Leftrightarrow \beta \in [0,1]$.
- Let Z_i denotes PV of i-year zero, where β to be determined.
- $Z_6 = \beta Z_6 + (1 - \beta)Z_6 = Z_5 + Z_7$.
- **To find $\boldsymbol{\beta}$ and $\boldsymbol{VaR}$ of mapped bonds**.
- **Step 1:** Find $\gamma = \frac{n_2 - n}{n_2 - n_1}$
- **Step 2:** Find
- $y_n = \gamma y_{n_1} + (1 - \gamma)y_{n_2}$; $\sigma_n = \gamma\sigma_{n_1} + (1 - \gamma)\sigma_{n_2}$.
- **Step 3:** Find $Z_n = PV(n - year\ zero)$, using y_n.
- **Step 4:** Solve $\beta \in [0,1]$,
- $\sigma_n^2 = \beta^2\sigma_{n_1}^2 + (1 - \beta)^2\sigma_{n_2}^2 + 2\beta(1 - \beta)\rho_{n_1,n_2}\sigma_{n_1}\sigma_{n_2}$
- $\Rightarrow a\beta^2 + b\beta + c = 0$, where

a	b	c
$\sigma_{n_1}^2 + \sigma_{n_2}^2 - 2\rho_{n_1,n_2}\sigma_{n_1}\sigma_{n_2}$	$2\rho_{n_1,n_2}\sigma_{n_1}\sigma_{n_2} - 2\sigma_{n_2}^2$	$\sigma_{n_2}^2 - \sigma_n^2$

- **Step 5:**
- $\mathbf{C} = \boldsymbol{\sigma\rho\sigma}$, where $\boldsymbol{\sigma}$ is $i \times i$ diagonal matrix of individual σ_i.
- $VaR = \alpha\sqrt{\mathbf{x}^T\mathbf{C}\mathbf{x}}$, $\mathbf{x} = [Z_{n_1} \quad Z_{n_2}]^T = [\beta Z_n \quad (1 - \beta)Z_n]^T$.
- $VaR = \alpha\sigma_X V_0$ also good for zero coupon bond.

5.6 Cash Flow Mapping

- 

,
- and t_4-year zero with redemption = coupon4 + principle.
- The position of each zero is then **mapped** into an equivalent position in the **adjacent** standard maturity zeros.
- For each zero-coupon bond i, find β_i, $\mathbf{x}_i$.
- Find $\mathbf{x} = \sum_i \mathbf{x}_i$ and $\mathbf{C} = \boldsymbol{\sigma\rho\sigma}$, thus $VaR = \alpha\sqrt{\mathbf{x}^T\mathbf{C}\mathbf{x}}$.