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Incompressible quantum Hall states

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Considering each electron as a superposition of fictitious fractionally charged particles allows, with the help of a natural ansatz, a systematic identification and characterization of the incompressible quantum Hall states at noninteger filling factors. Explicit Laughlin-type wave functions are obtained for all the incompressible states and their quasiparticles. The order of stability of the various states predicted on the basis of physically plausible rules is in agreement with experiments. Although in principle all rational fractions are observable, these rules imply that the even-denominator fractions are in general much less stable than the odd-denominator ones.

Fractionally charged excitations are a truly remarkable outcome of the theories of the fractional quantum Hall effect (FQHE).[1,2] In this paper we assert that a decomposition of the electrons into fictitious fractionally charged particles *at the very outset* straightaway brings out the complete elementary structure of FQHE. This is made possible by a natural ansatz relating incompressibility in the electron system to that in the fictitious particle system. This theory reproduces in a very transparent and unified fashion all the essential characteristics of the FQHE, including identification and the order of stability of various fractions, experimental scarcity of the even-denominator fractions, the quantized Hall resistance, and the charge and statistics of the quasiparticles. It also produces explicit Laughlin-type wave functions for not only all the incompressible quantum Hall states but also their quasiparticles. It emphasizes a fundamental connection between the FQHE and the integer quantum Hall effect (IQHE), and makes it clear that even though repulsive interactions are required for FQHE (gapless excitations would exist in the absence of repulsive interactions), the driving mechanism is, as in IQHE, the Fermi statistics. Unlike previous theories, it predicts a possibility of FQHE at *all* rational filling factors, but at the same time provides reasons for why odd-denominator fractions are experimentally so much more abundant.

We artificially divide each electron into m particles of m distinct species (labeled by λ), solve the problem for these fictitious particles, and in the end enforce the constraint that the coordinates of the m particles belonging to the same electron be equal, i.e., $z_j^{(\lambda)}=z_j$ for all λ and j. We start by writing the following Hamiltonian for the fictitious particles.

$$H(\alpha)=\sum_{\lambda=1}^{m}(H_0^\lambda+\alpha V^\lambda)\,, \tag{1}$$

$$H_0^\lambda=\sum_{j=1}^{N}\frac{1}{2m_\lambda}\left[\mathbf{p}_j^\lambda+\frac{e_\lambda}{c}\mathbf{A}_j^\lambda\right]^2, \tag{2}$$

$$V^\lambda=e_\lambda^2\sum_{j<k}|z_j^{(\lambda)}-z_k^{(\lambda)}|^{-1}\,. \tag{3}$$

There is interaction only between particles of the same species, and the choice $\alpha=e^2/(\sum_\lambda e_\lambda^2)$ produces the physical Coulomb interaction. We will not need the explicit form of interactions in this paper. The masses m_λ are of no relevance.

Solution of the problem amounts to specifying the state of the particles of each individual species. We identify the state obtained after setting $z_j^{(\lambda)}=z_j$ for all λ and j with the electron state. This state is formally an eigenstate of

$$\mathcal{H}(\alpha)\equiv\int\left[\prod_{i,\lambda}dz_i^{(\lambda)}\delta(z_i-z_i^{(\lambda)})\right]H(\alpha)\,. \tag{4}$$

Here we make the following physically plausible ansatz: *An incompressible electron state is obtained if and only if the particles of each individual species are in an incompressible state.* The rest of the paper will examine the consequences of this ansatz and show how it leads to a natural and consistent description of FQHE.

The physics of the problem makes the following demands: (i) The particles of all the species must be fermions, because only then can they use their Fermi statistics to produce incompressible states, in analogy to IQHE. This implies that m must be an odd integer. (ii) Since each fictitious particle sees the physical magnetic field, $\mathbf{A}_j^\lambda$ must produce the physical magnetic field for all λ. (iii) The density of each species must be equal to the density of electrons. As the density is related to the filling factor ν_λ by density $=Be_\lambda\nu_\lambda/hc$, the filling factor of a given species is inversely related to its charge e_λ. The same density of all the fictitious particles requires the product $e_\lambda p_\lambda$ to be the same. (iv) Let us denote the filling factors corresponding to incompressible states of the fictitious λ particles by p_λ, and those corresponding to the incompressible electron states by p. Clearly, p_λ's can assume any integer values, which will be our starting point. As we will see later, integer values of p_λ generate incompressible states at fractional filling factors, and more incompressible states can be obtained by allowing p_λ's to also assume the fractional values thus obtained. In this paper, for simplicity of illustration, we will assume p_λ's to be integers unless mentioned otherwise. A given set of filling factors $p_1,\ldots,p_m$ occurs simultaneously only for the system with $e_\lambda p_\lambda$ $=$const, or, with the condition $\sum_\lambda e_\lambda=e$, when

$$e_\lambda=pep_\lambda^{-1}\,, \tag{5}$$

$$p\equiv\left[\sum_{\lambda=1}^{m}p_\lambda^{-1}\right]^{-1}\,. \tag{6}$$

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The eigenstates of H_0^λ when the fictitious λ particles fill p_λ Landau levels (LL's) are

$$\chi_{p_\lambda} \equiv \Phi_{p_\lambda}[\{z_j^{(\lambda)}\}] \exp\left(-\frac{1}{4}\frac{e_\lambda}{e}\sum_i |z_i^{(\lambda)}|^2\right), \quad (7)$$

where the z's are measured in units of the magnetic length $(\hbar c/eB)^{1/2}$. This yields the electron states

$$[p_1,\ldots,p_m] = \prod_\lambda \chi_{p_\lambda} = \prod_\lambda \Phi_{p_\lambda}[\{z_j\}] \exp\left(-\tfrac{1}{4}\sum_i |z_i|^2\right). \quad (8)$$

These are the incompressible fractional quantum Hall states according to our ansatz. We emphasize that since χ_{p_λ} are also approximate eigenstates of $H_0^\lambda + \alpha V^\lambda$, the states $[p_1,\ldots,p_m]$ are also valid in the presence of interactions [i.e., are approximate eigenstates of the fully interacting Hamiltonian $\mathcal{H}(\alpha)$]. We adopt the convention $p_1 \geq p_2 \geq \cdots \geq p_m$.

These states are the main result of our paper. These can also be regarded as generalizations of the Laughlin states; the Laughlin states are a product of an odd number of incompressible integer quantum Hall state χ_1, whereas the states in Eq. (8) are a product of an odd number of *any* incompressible states. All the results of this paper can be derived by taking these states as the starting point.

The states in Eq. (8) can be easily shown to be translationally and rotationally invariant. The filling factor of a state can be determined by counting the number of occupied single particle states in each LL. Taking a disk-shaped geometry,[2] this amounts to determining the largest power of a coordinate z_j in the product $\prod_\lambda \Phi_{p_\lambda}[\{z_j\}]$. Since the largest power of z_j in each factor $\Phi_{p_\lambda}[\{z_j\}]$ is Np_λ^{-1}, the largest power in the product is $\sum_\lambda Np_\lambda^{-1}$, which yields the filling factor to be p defined in Eq. (6).

In order to determine the Hall resistance, we apply the gauge argument of Laughlin[3] to the fictitious particles. For a Hall voltage of V_H, the Hall current carried by each species is $I_\lambda = cp_\lambda e_\lambda V_H/\phi_\lambda$ where $\phi_\lambda = hc/e_\lambda$. The total current is then $I = \sum_\lambda p_\lambda e_\lambda^2 V_H/h$ which is equal to $pe^2 V_H/h$, because $p_\lambda e_\lambda = pe$ and $\sum_\lambda e_\lambda = e$. This yields the Hall resistance $R_H = h/pe^2$, where p is fractional as defined in Eq. (6). In analogy with IQHE, impurities and inhomogeneities present in the physical sample produce a plateau at this value of the Hall resistance so long as the Fermi level lies in a mobility gap.[3]

The quasiparticles can be obtained trivially in this framework. The lowest-energy quasihole, which is an excitation of the smallest possible charge, is obtained by creating a hole in the state χ_{p_1}. Similarly, the lowest-energy quasielectron is obtained by adding a particle in the (p_1+1)th LL of this state. States containing an arbitrary number of quasiparticles can be constructed analogously. Due to charge conservation the charge of a quasiparticle is simply e_1. As explained later, the quasiparticle gaps will be determined by interactions, and not by the cyclotron energy.

Now we consider some specific cases. We define m_0 to be the number of p_λ's different from 1.

(i) $m=1$: This corresponds to IQHE. The quasiparticles have charge e.

(ii) $m_0=0$: In this case the filling factor is $1/m$ and the state $[1,1,\ldots]$ is identical to the Laughlin state[2] because $\Phi_1 = \prod_{j<k}(z_j - z_k)$. The quasiparticles have charge e/m in agreement with Laughlin's theory. The quasihole is obtained by creating a hole in χ_1. The wave function for the state χ_1 with a hole at the origin is $(\prod_j z_j)\chi_1$ so that the quasihole state is

$$\left(\prod_j z_j\right)\Phi_1^m \exp\left(-\tfrac{1}{4}\sum_i |z_i|^2\right). \quad (9)$$

Encouragingly, this is precisely Laughlin's trial wave function for a quasihole at the origin.[2] The quasielectron at the origin is similarly given by

$$\exp\left(-\tfrac{1}{4}\sum_i |z_i|^2\right)\begin{vmatrix} z_1^* & z_2^* & z_3^* & \cdots \\ 1 & 1 & 1 & \cdots \\ z_1 & z_2 & z_3 & \cdots \\ z_1^2 & z_2^2 & z_3^2 & \cdots \\ \vdots & \vdots & \vdots & \end{vmatrix}\Phi_1^{m-1}. \quad (10)$$

This is different from the trial states considered in the past.[1,2] We note here that this is the first approach[4] that allows an exact construction of the quasielectron state, which illustrates its power. We now ask how many electrons there are in the higher LL in this state. First expand the polynomial multiplying the exponential in Eq. (10). Each term contains one z^*, and the terms with the largest coefficients are such that almost all z_j's appear with a large power of order $\frac{1}{2}(m-1)N$. Now express the wave function in terms of single-particle eigenstates

$$\zeta_{0,s}(z) = (2\pi 2^s s!)^{-1/2} z^s \exp(-\tfrac{1}{4}|z|^2), \quad (11)$$

$$\zeta_{1,s}(z) = (2\pi 2^{s+1} s!)^{-1/2} z^{s-1}[2s - |z|^2]\exp(-\tfrac{1}{4}|z|^2), \quad (12)$$

where $\zeta_{l,s}$ corresponds to the lth LL with s being the angular momentum index. If each term contained exactly one $\zeta_{1,s}$ (none can have more than one), there would be exactly one electron in the higher LL. However this is clearly not the case, since the coordinate z_j^* of the jth particle appears in the combination $z_j^* z_j^s$, which expands as

$$z_j^* z_j^s \exp(-\tfrac{1}{4}|z_j|^2) = (2\pi 2^{s+1} s! s)^{1/2}[\zeta_{0,s-1}(z_j) - s^{-1/2}\zeta_{1,s}(z_j)], \quad (13)$$

and, in general, has finite amplitude in the lowest LL. In fact, when s is a big number, which is typically the case, $z_j^* z_j^s \exp(-|z_j|^2/4)$ lies almost entirely in the lowest LL. This strongly suggests that the number of electrons in the quasielectron state of Eq. (10) in the higher LL vanishes in the limit $(m-1)N \to \infty$. Explicit analytic calculations for few-particle systems indeed show a rapid decrease in the number of electrons (<1) in the $l=1$ LL as either m or N is increased,[5] thus lending support to this assertion. This would imply that the quasielectron energy is not the cyclotron energy of the fictitious particles, but is instead determined by interactions, and vanishes in the absence of interactions. With the help of the explicit wave functions, computation of the quasiparticle energies is in principle straightforward.

Thus our simple model has reproduced the Laughlin states, and its essential characteristics, including its insensitivity to interactions and its fractionally charged quasiparticles. We believe that our derivation of the Laughlin states gives further insight into the origin of their incompressibility.

(iii) $m_0=1$: In this case the states $[p_1,1,1,\ldots]$ correspond to filling factors

$$p=\frac{p_1}{(m-1)p_1+1}\,. \tag{14}$$

These have odd denominators. Interestingly, these states are precisely the trial states proposed in my composite fermion approach for the FQHE.[5] In fact, the present work has resulted from a desire to gain a better understanding of these trial states. In analogy with the quasielectron state in Eq. (10), these states are also expected to lie largely in the lowest LL.[6] The charge of the quasiparticles is $e/[(m-1)p_1+1]$, which is in agreement with the charge of the quasiparticles of the hierarchical states[7–9] for these filling factors. The quasiparticle states can be written exactly as was done for the Laughlin states.

(iv) $m_0>1$: Even denominator fractions now appear. It is worth emphasizing that FQHE at even-denominator fractions is possible in our theory even for spinless electrons, in contrast to the earlier studies.[10–12] For example, the state

$$[2,2,1]=\Phi_1\Phi_2^2\exp\left(-\tfrac{1}{4}\sum_i |z_i|^2\right), \tag{15}$$

corresponds to filling factor $\nu=\frac{1}{2}$. Notice that when $m_0>1$, the charge of the quasiparticles of the P/Q state is not e/Q. For example, the charge of the quasiparticles of the state [2,2,1] at $\nu=\frac{1}{2}$ is $e/4$.

from e_lambda p_p_lambda = ep
with p = 1/2, p_lambda = 2

Incompressible states exist for all rational fractions. We prove this by explicit construction. A state at P/Q, where Q is odd, can be constructed by choosing in Eq. (6) $m=Q$ and $p_\lambda=P$. Similarly, a state at P/Q, where P is odd, Q is even, and $P<rQ$, where r is an even number, can be obtained by choosing $m=rQ+r-P$, r of the p_λ's equal to r^2, and the rest equal to rP. These states are obviously not unique. Given an incompressible state at a fraction p, one can construct states at $1-p$ (Ref. 13) and at $n+p$ (Refs. 4 and 11) where n is an integer. As indicated earlier, more states can be generated by choosing χ_{p_λ} in Eq. (7) to be one of the states obtained above corresponding to a noninteger filling factor. In general, there will be several incompressible candidates for a given rational fraction, but usually the most stable one can be determined uniquely with the help of the rules given below. It must be emphasized that only rational filling factors are generated in this theory; incompressibility is not possible at nonrational filling factors.

A state with a larger quasiparticle gap is more stable than a state with a smaller quasiparticle gap. Since one important parameter in the calculation of the quasiparticle gap is the charge of the quasiparticles,[14] we assume that in the case of two sufficiently similar states the state with the smaller quasiparticle charge has smaller gap and is therefore less stable. This implies that the state $[p_1,\ldots p_m]$ is more stable than $[p_{1+1},\ldots p_m]$ and the state $[\{p_\lambda\}]$ is more stable than $[\{p_\lambda\},1,1]$. The strongest correlations are expected to be due to the binding of electrons and the zeros of the wave function,[2,12] indicating that the change in stability is most pronounced when the two states differ in their m_0. Thus the most stable states are the ones with small m and m_0. Analogous rules hold[5] for noninteger values of p_λ. In general, if $\chi_{p_\lambda'}$ is less stable than χ_{p_λ}, the state $[p_1,\ldots p_\lambda',\ldots p_m]$ is less stable than $[p_1,\ldots p_\lambda,\ldots p_m]$.

The order of stability predicted by these rules can be shown to be in excellent agreement with experiments.[5] The experimental scarcity of the even-denominator states can be understood because they are quite rare in the favorable parameter range. For example, in the parameter space $m_0\le 2$, $m\le 5$, and $p_\lambda<10$, there are only four even-denominator states ([2,2,1] at $\nu=\frac{1}{2}$; [2,2,1,1,1] at $\nu=\frac{1}{4}$; [6,6,1] at $\nu=\frac{3}{4}$; and [6,6,1,1,1] at $\nu=\frac{3}{10}$) compared to 95 odd-denominator states. The most stable even-denominator state is predicted to be $\nu=\frac{1}{2}$. The states [6,6,1] and [6,6,1,1,1] are extremely unlikely to be observed, because the much stronger state [6,1,1] at $\nu=\frac{6}{13}$ is barely observable in the best available samples of the day.[15] Thus there is little likelihood of the observation of any even-denominator fractions other than $\frac{1}{2}$, $\frac{1}{4}$, and its hole analog state $\frac{3}{4}$.

Electron spin is included in the formalism by assuming that when an electron is divided into m particles, one of them, say the $\lambda=1$ particle, carries the spin with it, and the rest of the particles are spinless. Let $q=q_\uparrow+q_\downarrow$ be the filling factor of the $\lambda=1$ particles such that $q_\uparrow$ ($q_\downarrow$) spin-up (spin-down) Landau bands are occupied. The filling factor of the resulting state, denoted by $[q_\uparrow,q_\downarrow;p_2,\ldots]$, is given by Eq. (6) with $p_1=q$. When $q_\uparrow=q_\downarrow$ ($q_\uparrow\neq q_\downarrow$) this state is spin-unpolarized (partially spin polarized). The states $[1,1;1,\ldots]$ are precisely the spin-unpolarized states at $2/(2m-1)$ considered by Halperin[10] and Haldane.[4] The state [1,1;2,1] is a spin-unpolarized state at $\nu=\frac{1}{2}$ and is expected to be more favorable than the spin-polarized state [2,2,1] for sufficiently small Zeeman energy because of its smaller m_0, and by analogy with the numerical work that shows that the spin-unpolarized state $[1,1;1,\ldots]$ has lower energy than the corresponding spin-polarized state for small Zeeman energy.[16] This is in agreement with experiments.[15,17] Note that the states obtained here satisfy the Fock criterion.[4,5,11]

Finally we calculate the adiabatic statistics of the quasiparticles. Even though the fictitious particles are fermionic, the quasiparticles are obtained *after* the identification $z_j^{(\lambda)}=z_j$ and hence their statistics must be calculated from first principles. Following Arovas, Schrieffer, and Wilczek,[18] the excitation described by the wave function $\prod_i(z_i-z_0)[\{p_\lambda\}]$ has charge $-ep$ and obeys p statistics (i.e., interchange of two of these excitations produces a phase πp). It can be shown that $\prod_i(z_i-z_0)\chi_{p_1}$ consists of p_1 holes at z_0, one in each of the p_1 LL's, so that the above excitation has p_1 quasiholes at z_0, a special case of which was encountered in Eq. (9). Thus from simple counting arguments we expect a single quasihole to have charge $-ep/p_1=-e_1$ and obey (p/p_1^2) statistics. A similar analysis for quasielectrons is difficult because of their more complicated wave functions, but since a quasielectron-quasihole pair is a neutral boson, the quasielectron must have charge e_1 and obey $(-p/p_1^2)$ statistics. We would like to point out that wave functions for the many electron state containing an arbitrary number of quasiparticles can be written without regard to their statistics.

It is greatly satisfying that our simple approach is successful in explaining the phenomenology of QHE not only at the odd-denominator rational filling factors but also at even-denominator fractions. This success combined with the fact that we recover the Laughlin states, as well as the spin-unpolarized states of Halperin and Haldane, lends necessary credibility to our approach and strongly suggests that the states obtained in this work represent the physics of the FQHE. However, it indeed remains to be shown that these states are in fact legitimate representations of the *true* ground states. Further work in this direction is in progress.

In summary, incompressibility in a partially filled LL obtains when parts of electrons individually occupy integer quantum Hall (or, in general any incompressible) states.

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