MPS.3 Left- and right-normalized MPS

Computation of normalization and matrix elements of local operators is simpler if the MPS is built from tensors with special normalization properties, called 'left-normalized' or 'right-normalized' tensors.

Left-normalization

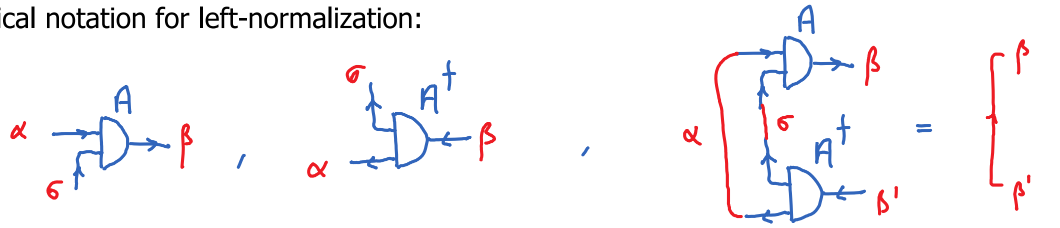
A 3-leg tensor $A^{\alpha\sigma}{}_{\beta}$ is called 'left-normalized' if it is a left isometry, i.e. if it satisfies

$\boxed{A^\dagger A = \mathbb{1}}$. Explicitly: $(A^\dagger A)^{\beta'}{}_{\beta} = A^{\dagger\beta'}{}_{\sigma\alpha}\, A^{\alpha\sigma}{}_{\beta} = \mathbb{1}^{\beta'}{}_{\beta}$ (1)

Such an A defines an 'isometry' from space labeled by its left indices to space labeled by its right indices.

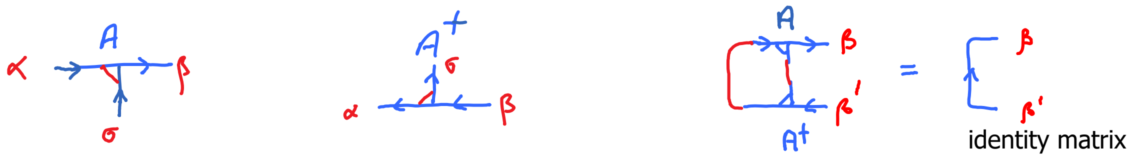
distance-preserving map (in index-free notation: if $y = Ax$, then $y^\dagger y = x^\dagger A^\dagger A x = x^\dagger x$)

Graphical notation for left-normalization:



(2a)

More compact notation: draw 'left-facing diagonals' at vertices



(2b)

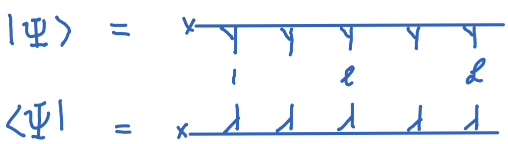
The right-angled triangle contains <u>complete</u> information about all arrows attached to it:

for A , incoming arrows to sharp angles, outgoing arrow from right angle,

for $A^\dagger$, outgoing arrows from sharp angles, incoming to from right angle:

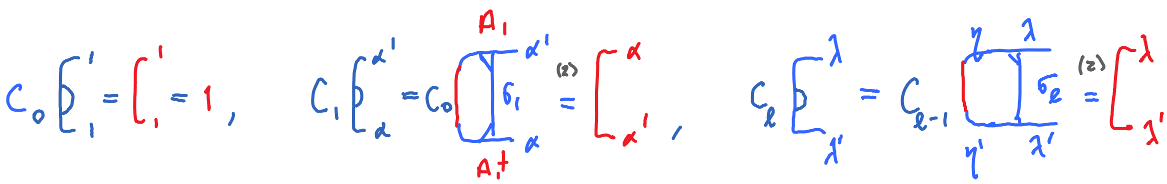
Hence, there is no need to draw arrows explicitly when using , !

Consider a 'left-normalized MPS', i.e. one constructed purely from left isometries:



(3)

Then, closing the zipper left-to-right is easy, since all C_ℓ reduce to identity matrices:



(4a)

We suppress arrows for C, too, since they can be reconstructed from arrows of constitutent As.

(4a)

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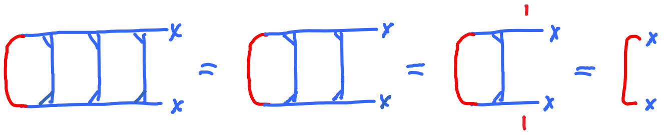
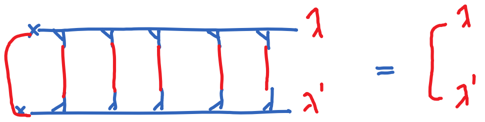
Hence:

$\langle \Psi | \Psi \rangle =$  $= 1$ ☺ (4b)

Moreover, the matrices for site 1 to any site $\ell = 1, \dots, \mathcal{L}$ define an <u>orthonormal state space</u>:

$$|\Psi_\lambda\rangle_\ell = |\sigma_1\rangle \otimes |\sigma_2\rangle \otimes \dots \otimes |\sigma_\ell\rangle \left[A_1^{\sigma_1} A_2^{\sigma_2} \dots A_\ell^{\sigma_\ell}\right]^1{}_\lambda \qquad (5)$$



close the zipper

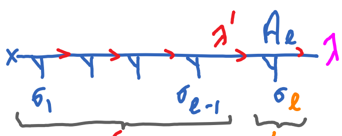
$${}_\ell\langle \Psi^{\lambda'} | \Psi_\lambda \rangle_\ell = \mathbb{1}^{\lambda'}{}_\lambda \quad ☺ \qquad (6)$$

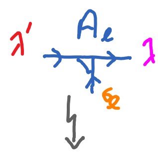
Call this state space $\quad \mathbb{V}_\ell = \text{span}\{|\Psi_\lambda\rangle_\ell\} \subseteq \mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \dots \otimes \mathbb{V}_\ell \qquad (7)$

where $\quad \mathbb{V}_\ell = \text{span}\{|\sigma_\ell\rangle\}$ is local state space of site ℓ

These state spaces are built up iteratively from left to right through left-isometric maps:

Each  defines an isometric map to a new (possibly smaller) basis:





$$A_\ell : \mathbb{V}_{\ell-1} \otimes \mathbb{V}_\ell \to \mathbb{V}_\ell, \qquad \underbrace{|\Psi_{\lambda'}\rangle_{\ell-1} |\sigma_\ell\rangle}_{\text{old basis}} \mapsto \underbrace{|\Psi_\lambda\rangle_\ell}_{\text{new basis}} = |\Psi_{\lambda'}\rangle_{\ell-1} |\sigma_\ell\rangle [A_\ell]^{\lambda'\sigma_\ell}{}_\lambda \qquad (8)$$

If A_ℓ is a unitary, then $\quad \underbrace{\dim(\mathbb{V}_\ell)}_{D_\ell} = \underbrace{\dim(\mathbb{V}_{\ell-1}) \cdot \dim(\mathbb{V}_\ell)}_{D_{\ell-1} \cdot d} \Rightarrow$ no truncation (9)

If A_ℓ is a (non-unitary) isometry, then $D_\ell < D_{\ell-1} \cdot d \Rightarrow$ truncation was involved! (10)

Hence $\quad \mathbb{V}_\ell = \mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \dots \otimes \mathbb{V}_\ell$ only if all A's are not only isometries but unitaries.

$D_\ell = d^\ell$ truncation possible no truncation!

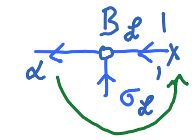
Even if truncation is involved, the resulting MPS are useful, precisely because they are parametrized by a limited number of parameters (namely elements of A tensors). E.g., they can be optimized variationally by minimizing energy $\Rightarrow$DMRG). $D^2 d \cdot \ell$

Right-normalization

So far we have viewed an MPS as being built up from left to right, hence used right-pointing arrows on ket diagram. Sometimes it is useful to build it up from right to left, using left-pointing arrows.

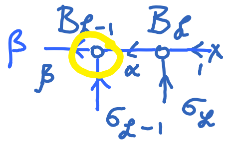
Building blocks:

$$|\alpha\rangle = |\sigma_\ell\rangle [B_\ell]_\alpha{}^{\sigma_\ell 1} \tag{11}$$

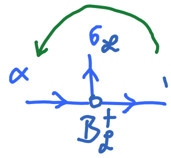


left-to-right index order as in diagram

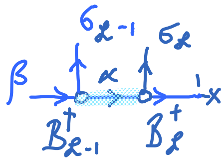
$$|\beta\rangle = |\sigma_{\ell-1}\rangle|\sigma_\ell\rangle [B_{\ell-1}]_\beta{}^{\sigma_{\ell-1}\alpha} [B_\ell]_\alpha{}^{\sigma_\ell 1} \tag{12}$$



$$\langle\alpha| = \underbrace{[B_\ell^\dagger]_{1\sigma_\ell}{}^{\alpha}}_{:= [B_\ell]_\alpha{}^{\sigma_\ell 1}} \langle\sigma_\ell| \tag{13}$$

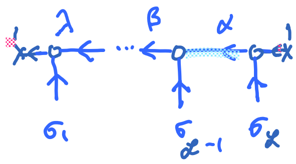


$$\langle\beta| = [B_\ell^\dagger]_{1\sigma_\ell}{}^{\alpha} [B_{\ell-1}^\dagger]_{\alpha\sigma_{\ell-1}}{}^{\beta} \langle\sigma_\ell|\langle\sigma_{\ell-1}| \tag{14}$$

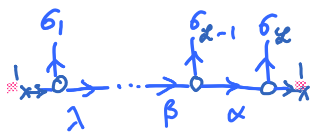


Iterating this, we obtain kets and bras of the form

$$|\psi\rangle = |\sigma_1\rangle|\sigma_2\rangle\dots|\sigma_\ell\rangle [B_1]_1{}^{\sigma_1\lambda} \dots [B_{\ell-1}]_\beta{}^{\sigma_{\ell-1}\alpha} [B_\ell]_\alpha{}^{\sigma_\ell 1}$$
$$= |\vec{\sigma}\rangle [B_1^{\sigma_1}\dots B_{\ell-1}^{\sigma_{\ell-1}} B_\ell^{\sigma_\ell}]_1{}^1 \tag{15}$$



$$\langle\psi| = [B_1^\dagger]_{1\sigma_\ell}{}^{\alpha} [B_{\ell-1}^\dagger]_{\alpha\sigma_{\ell-1}}{}^{\beta} \dots [B_1]_{\lambda\sigma_1}{}^1 \langle\sigma_\ell|\dots\langle\sigma_2|\langle\sigma_1|$$
$$= [B_\ell^{\dagger\sigma_\ell} B_{\ell-1}^{\dagger\sigma_{\ell-1}}\dots B_1^{\dagger\sigma_1}]_1{}^1 \langle\vec{\sigma}| \tag{16}$$

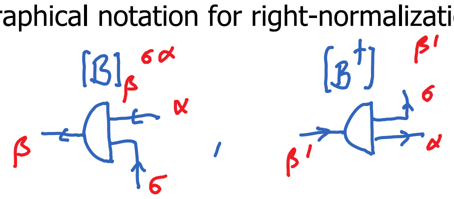


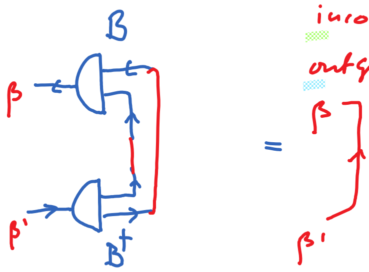
A three-leg tensor $B_\beta{}^{\sigma\alpha}$ is called right-normalized if it is a right isometry, i.e. if it satisfies

$$BB^\dagger = \mathbb{1}. \qquad \text{Explicitly:} \quad (BB^\dagger)_\beta{}^{\beta'} = B_\beta{}^{\sigma\alpha} B^\dagger{}_{\alpha\sigma}{}^{\beta'} = \mathbb{1}_\beta{}^{\beta'} \tag{17}$$

Such a $B^\dagger$ defines an 'isometry' from space labeled by its right indices to space labeled by its left indices.

Graphical notation for right-normalization:





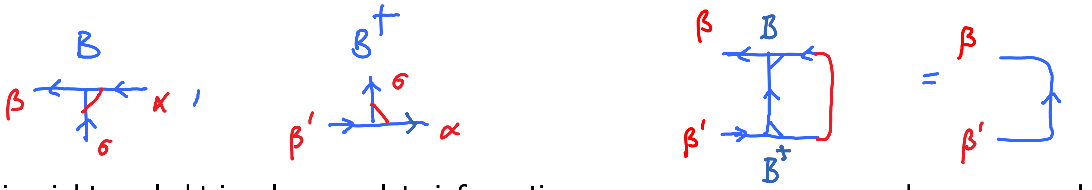
incoming = high

outgoing = low

(18a)

More compact notation: draw 'right-facing diagonals' at vertices

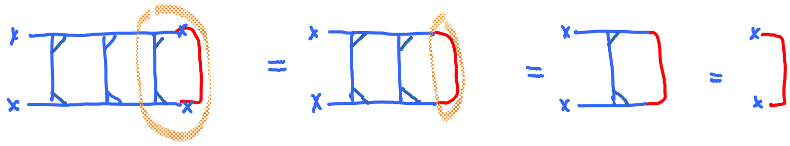
(18b)



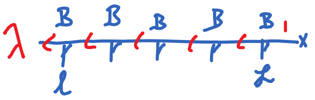
(18b)

Again, right-angled triangles complete information on arrows, so arrows can be suppressed.

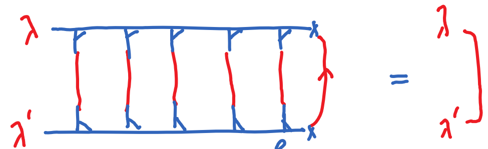
For 'right-normalized MPS', constructed purely from right isometries, closing zipper right-to-left is easy:

$\langle\psi|\psi\rangle =$  $= 1$ (19)

Moreover, the matrices for site $\mathcal{L}$ to any site $\ell = 1, \dots, \mathcal{L}$ define an orthonormal state space:



$$|\Phi_\lambda\rangle_\ell = |\sigma_\ell\rangle\otimes\dots\otimes|\sigma_{\mathcal{L}}\rangle \left[B_\ell^{\sigma_\ell} B_{\ell+1}^{\sigma_{\ell+1}}\cdots B_{\mathcal{L}}^{\sigma_{\mathcal{L}}}\right]_\lambda , \qquad (20)$$



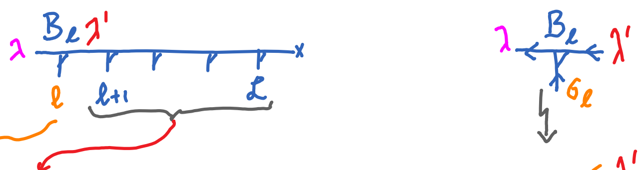
close the zipper

$${}_\ell\langle\Phi^{\lambda'}|\Phi_\lambda\rangle_\ell = \mathbb{1}^{\lambda'}{}_{\lambda} \qquad (21)$$

Call this state space $W_\ell = \mathrm{span}\{|\Phi_\lambda\rangle_\ell\} \subseteq V_\ell \otimes V_{\ell+1} \otimes \dots \otimes V_{\mathcal{L}}$ (22)

These state spaces are built up iteratively from right to left through right-isometric maps:

Each B_ℓ defines an isometric map to a new (possibly smaller) basis:

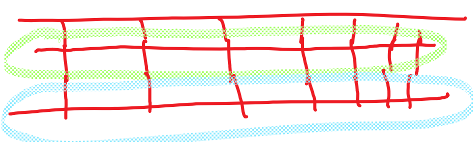


$$B_\ell : V_\ell \otimes W_{\ell+1} \to W_\ell , \qquad |\sigma_\ell\rangle|\Phi_{\lambda'}\rangle_{\ell+1} \mapsto |\Phi_\lambda\rangle_\ell = |\sigma_\ell\rangle|\Phi_{\lambda'}\rangle_{\ell+1}[B_\ell]_\lambda{}^{\sigma_\ell\lambda'} \qquad (23)$$

old basis new basis

$W_\ell = V_\ell \otimes V_{\ell+1} \otimes \dots \otimes V_{\mathcal{L}}$ only if all B's are not only isometries but unitaries. (24)

Summary: MPS built purely from left-normalized A's or purely from right-normalized B's are automatically normalized to 1. Shorter MPSs built on subchains automatically define orthonormal state spaces.

$U(t)\,U(t)\,|\psi\rangle =$ 

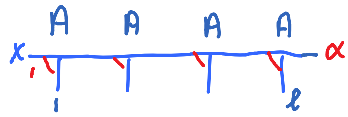
MPS.4 Various canonical MPS forms

Any matrix product can be expressed in infinitely many different ways without changing the product:

$$M\,M' = (M\,\cancel{U}\,\cancel{U^{-1}}\,M') = \tilde{M}\,\tilde{M}' \qquad \text{'gauge freedom'} \qquad (1)$$

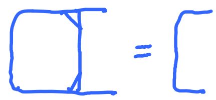
Gauge freedom can be exploited to 'reshape' MPSs into particularly convenient, 'canonical' forms:

(i) Left-canonical (lc-) MPS:
[all tensors are left-normalized, denoted A]



(2)

$$|\Psi_\alpha\rangle_\ell = |\sigma_1\rangle \dots |\sigma_\ell\rangle \left[A_1^{\sigma_1} \dots A_\ell^{\sigma_\ell}\right]^1{}_\alpha \qquad A^\dagger A = \mathbb{1}$$

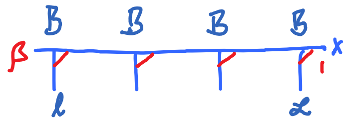

(3)

These states form an orthonormal set: $\quad {}_\ell\langle\Psi^{\alpha'}|\Psi_\alpha\rangle_\ell \overset{\text{(MPS-I.2.6)}}{=} \mathbb{1}^{\alpha'}{}_\alpha \qquad (4)$

In general, $\mathbb{V}_\ell \subset \mathcal{H}^\ell$.

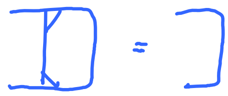
true subset

(ii) Right-canonical (rc-) MPS:
[all tensors are right-normalized, denoted B]



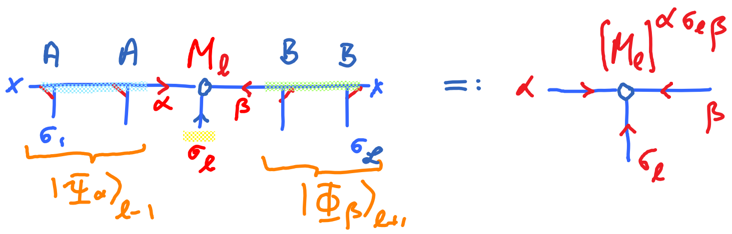
(5)

$$|\Phi_\beta\rangle_\ell = |\sigma_\ell\rangle \dots |\sigma_{\mathcal{L}}\rangle \left[B_\ell^{\sigma_\ell} \dots B_{\mathcal{L}}^{\sigma_{\mathcal{L}}}\right]_\beta{}^1 \qquad B B^\dagger = \mathbb{1}$$


(6)

These states form an orthonormal set: $\quad {}_\ell\langle\Phi^{\beta'}|\Phi_\beta\rangle_\ell \overset{\text{(MPS-I.2.18)}}{=} \mathbb{1}^{\beta'}{}_\beta \qquad (7)$

(iii) Site-canonical (sc-) MPS:
[left-normalized to left of site ℓ ,
right-normalized to right of site ℓ]



(8)

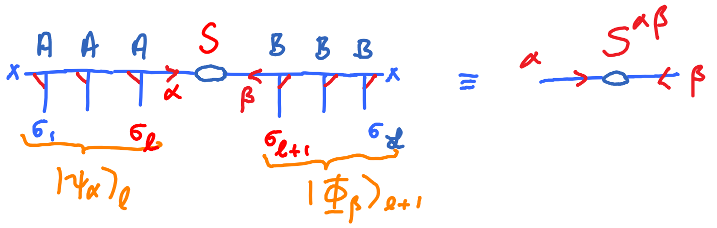
$$|\Psi\rangle = |\vec{\sigma}\rangle_{\mathcal{L}} \left[A_1^{\sigma_1} \dots A_{\ell-1}^{\sigma_{\ell-1}}\right]^1{}_\alpha \left[M_\ell\right]^{\alpha\sigma_\ell\beta} \left[B_{\ell+1}^{\sigma_{\ell+1}} \dots B_{\mathcal{L}}^{\sigma_{\mathcal{L}}}\right]_\beta{}^1 = |\Psi_\alpha\rangle_{\ell-1} |\sigma_\ell\rangle |\Phi_\beta\rangle_{\ell+1} \left[M_\ell\right]^{\alpha\sigma_\ell\beta} \qquad (9)$$

The states $|\alpha,\sigma_\ell,\beta\rangle := |\Psi_\alpha\rangle_{\ell-1} |\sigma_\ell\rangle |\Phi_\beta\rangle_{\ell+1}$ form an orthonormal set: $\langle\alpha',\sigma',\beta'|\alpha,\sigma,\beta\rangle = \mathbb{1}^{\alpha'}{}_\alpha \mathbb{1}^{\sigma'}{}_\sigma \mathbb{1}^{\beta'}{}_\beta$

(10)

$M_\ell^{\alpha\sigma_\ell\beta}$ can be viewed as the wavefunction of $|\Psi\rangle$ in this basis.

(iv) Bond-canonical (bc-) (or mixed) MPS:
[left-normalized from sites 1 to ℓ ,
right-normalized from sites $\ell+1$ to N]



(11)

$$|\psi\rangle = |\vec{\sigma}\rangle_{\ell} \, [A_1 \ldots A_\ell]^{1\sigma_1 \cdots \sigma_\ell}{}_{\alpha} \, S^{\alpha\beta} \, [B_{\ell+1}^{\sigma_{\ell+1}} \ldots B_{\mathscr{L}}^{\sigma_{\mathscr{L}}}]_{\beta}{}^{1} = \underbrace{|\Psi_\alpha\rangle_{\ell}}_{|\Psi_\alpha\rangle_\ell} \underbrace{|\Phi_\beta\rangle_{\ell+1}}_{|\Phi_\beta\rangle_{\ell+1}} S^{\alpha\beta} \qquad (12)$$

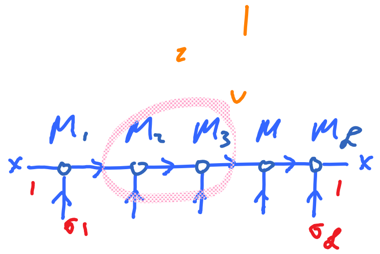
$S^{\alpha\beta}$ can be chosen diagonal

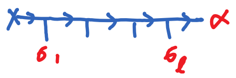
The states $|\alpha,\beta\rangle := |\Psi_\alpha\rangle_\ell \, |\Phi_\beta\rangle_{\ell+1}$ form an orthonormal set: $\langle\alpha',\beta'|\alpha,\beta\rangle = \mathbb{1}^{\alpha'}{}_{\alpha} \, \mathbb{1}^{\beta'}{}_{\beta}$ (13)

How can we bring an arbitrary MPS into one of these forms?

Transforming to site-normalized form

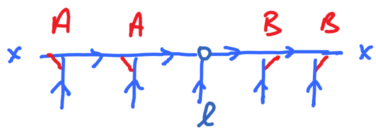
Given: $|\psi\rangle = |\vec{\sigma}\rangle_{\mathscr{L}} \, [M_1 \ldots M_{\mathscr{L}}]^{1\vec{\sigma}}{}_{1}$



[or with index: $|\Psi_\alpha\rangle =$ ]

Goal (i): left-normalize M_1 to $M_{\ell-1}$

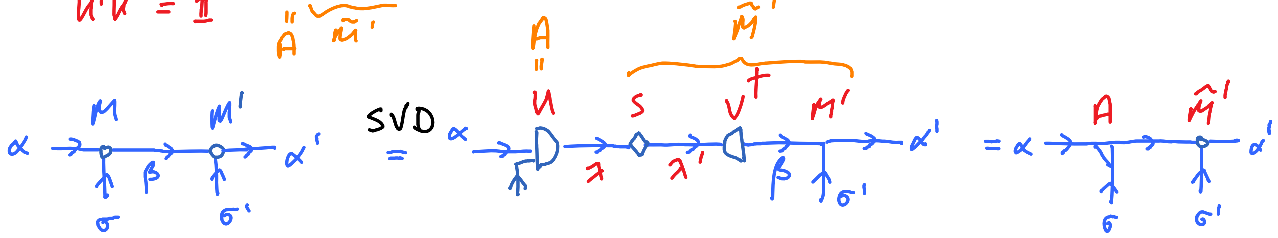
Goal (ii): right-normalize $M_{\mathscr{L}}$ to $M_{\ell+1}$



(i) take a pair of adjacent tensors, $M M'$, and use SVD to yield left isometry on the left:

$$M M' = U(S V^\dagger M') =: \quad , \qquad \text{with} \quad A := U, \quad \tilde{M}' := S V^\dagger M' \qquad (7)$$

$U^\dagger U = \mathbb{1}$ $\qquad$ $\underset{A}{\overset{\shortparallel}{Q}} \, \underbrace{R M'}_{\tilde{M}'}$



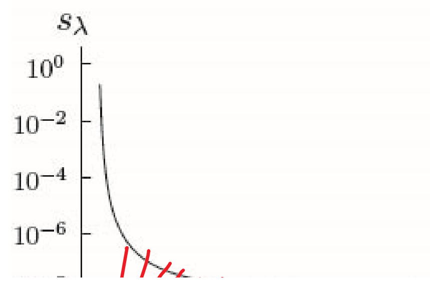
$$M^{\alpha\sigma}{}_{\beta} \, M'^{\beta\sigma'}{}_{\alpha'} = \left(U^{\alpha\sigma}{}_{\lambda}\right)\left(S^{\lambda}{}_{\lambda'} \, V^{\dagger\lambda'}{}_{\beta} \, M'^{\beta\sigma'}{}_{\alpha'}\right) = A^{\alpha\sigma}{}_{\lambda} \, \tilde{M}'^{\lambda\sigma'}{}_{\alpha'} \qquad (9)$$

The property $U^\dagger U = \mathbb{1}$ ensures left-normalization: $A^\dagger A = \mathbb{1}$ (10)

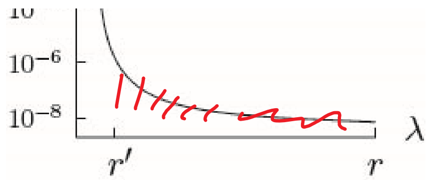
Truncation, if desired, can be performed by discarding some of the smallest singular values, using $U S V^\dagger \approx u s v^\dagger$

$$\sum_{\lambda=1}^{r} \longrightarrow \sum_{\lambda=1}^{r'}$$

(but (10) remains valid!)



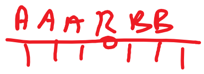
$\sum_{\lambda=1} \longrightarrow \sum_{\lambda=1}$ (but (10) remains valid!)



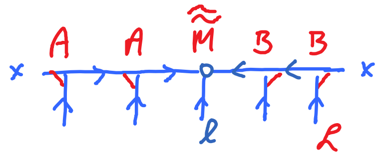
Note: instead of SVD, we could also me QR (cheaper!)

Starting from $M_1 M_2 = A_1 \tilde{M}_2$, move rightward up to $\tilde{M}_{\ell-1} M_\ell = A_{\ell-1} \tilde{M}_\ell$

$A_\ell = QR$

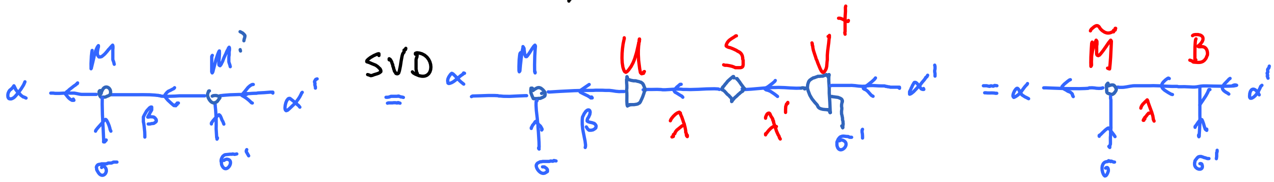


Goal (ii): now right-normalize $M_\mathcal{L}$ to $M_{\ell+1}$



Take a pair of adjacent tensors, $M M'$, and use SVD to yield right isometry on the right:

$$MM' = (\overset{\hat{M}}{M U S}) \overset{B}{V^\dagger} = \tilde{M} B, \quad \text{with} \quad \tilde{M} = M'US, \quad B = V^\dagger. \qquad (13)$$



(14)

$$M_\alpha{}^{\sigma\beta} M'_\beta{}^{\sigma'\alpha'} = \left(M_\alpha{}^{\sigma\beta} U_\beta{}^{\lambda} S_\lambda{}^{\lambda'}\right)\left(V^\dagger{}_{\lambda'}{}^{\sigma'\alpha'}\right) = \tilde{M}_\alpha{}^{\sigma\lambda'} B_{\lambda'}{}^{\sigma'\alpha'} \qquad (15)$$

Here, $V^\dagger V = \mathbb{1}$ ensures right-normalization: $BB^\dagger = \mathbb{1}$. (16)

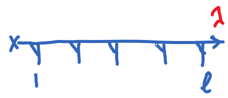
Starting form $M_{\mathcal{L}-1} M_\mathcal{L} = \tilde{M}_{\mathcal{L}-1} B_\mathcal{L}$, move leftward up to $\tilde{M}_\ell \tilde{M}_{\ell+1} = \tilde{\tilde{M}}_\ell B_{\ell+1}$

Summary: using SVD, products of two matrices can be converted into forms containing a left isometry on the left or right isometry on the right:

$$MM' = A\tilde{M}' = \tilde{M}B \qquad (18)$$

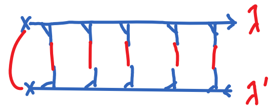
This can be used iteratively to convert any of the four canonical forms into any other one.

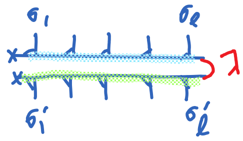
MPS.5 Basis change, projectors

Recall: a set of MPS $|\Psi_\lambda\rangle_\ell = |\sigma_1\rangle\otimes\dots\otimes|\sigma_\ell\rangle\,[A_1^{\sigma_1}\dots A_\ell^{\sigma_\ell}]^1{}_\lambda$ =  (1)

specified by given left-normalized tensors

defines an orthonormal basis for a state space $\mathbb{V}_\ell = \mathrm{span}\{|\Psi_\lambda\rangle_\ell\} \subseteq \mathbb{V}_1\otimes\mathbb{V}_2\otimes\dots\otimes\mathbb{V}_\ell =: \mathbb{V}^{\otimes\ell}$ (2)

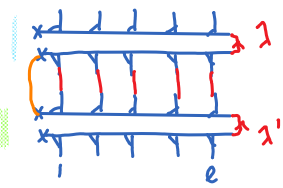
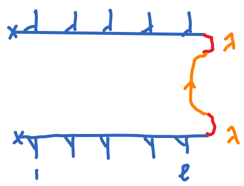
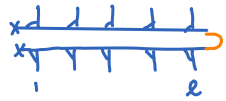
since ${}_\ell\langle\Psi^{\lambda'}|\Psi_\lambda\rangle_\ell$ =  $\overset{\text{close zipper}}{=}$  = $\mathbb{1}^{\lambda'}{}_\lambda$ (3)

Projector onto $\mathbb{V}_\ell$: $\hat P_\ell = |\Psi_\lambda\rangle_\ell\,{}_\ell\langle\Psi^\lambda|$ =  (4)

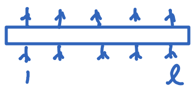
(sum over λ implied)

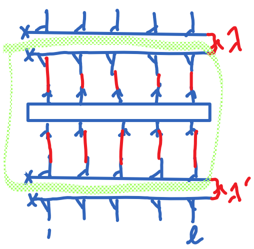
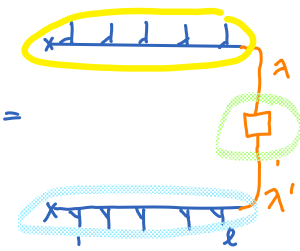
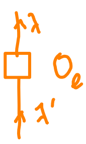
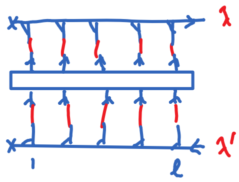
Indeed:

$$\hat P_\ell\,\hat P_\ell = |\Psi_{\lambda'}\rangle_\ell\,\underbrace{{}_\ell\langle\Psi^{\lambda'}|\cdot|\Psi_\lambda\rangle_\ell}_{\mathbb{1}^{\lambda'}{}_\lambda}\,{}_\ell\langle\Psi^\lambda| = \hat P_\ell \quad ✓ \qquad (5)$$

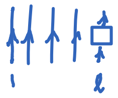
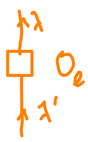
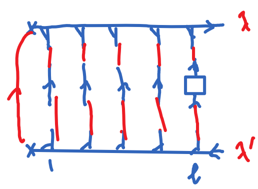
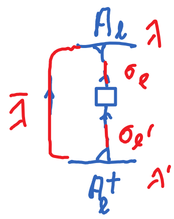
=  =  =  (6)

Operators defined on $\mathbb{V}_\ell^{\otimes\ell}$ can be mapped to $\mathbb{V}_\ell$ using these projectors:

$\hat O$ =  $\xmapsto{\hat P_\ell}$ $\hat O_\ell =: \hat P_\ell\,\hat O\,\hat P_\ell =: |\Psi_{\lambda'}\rangle_\ell\,[O_\ell]^{\lambda'}{}_\lambda\,{}_\ell\langle\Psi^\lambda|$ (7)

$\hat O_\ell$ =  =  , $[O_\ell]^{\lambda'}{}_\lambda :=$  :=  (8)

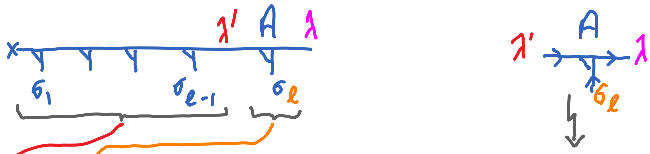
Simplest case: 1-site operator acting only on site ℓ :

$\hat O_\ell$ =  , $[O_\ell]^{\lambda'}{}_\lambda$ =  =  $\overset{\text{close zipper}}{=}$  (9)

During iterative diagonalization, the space $\mathbb{V}_\ell$ is constructed through a sequence of isometric maps:
(possibly involving truncation)

During iterative diagonalization, the space $\mathbb{V}_\ell$ is constructed through a sequence of isometric maps:
(possibly involving truncation)

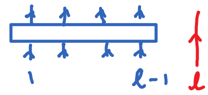
Each A defines an isometric map
to a new (possibly smaller) basis:

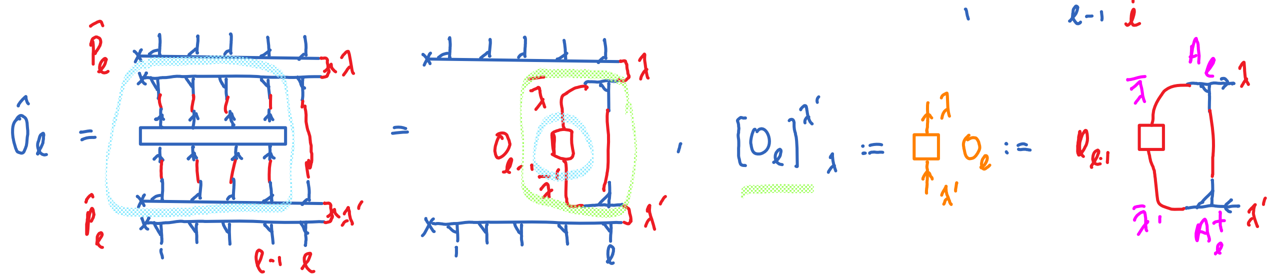


$$A_\ell : \mathbb{V}_{\ell-1} \otimes V_\ell \to \mathbb{V}_\ell , \qquad \underbrace{|\Psi_{\lambda'}\rangle_{\ell-1} |\sigma_\ell\rangle}_{\text{old basis}} \mapsto \underbrace{|\Psi_\lambda\rangle_\ell}_{\text{new basis}} = |\Psi_{\lambda'}\rangle_{\ell-1} |\sigma_\ell\rangle \, A^{\lambda'\sigma_\ell}{}_\lambda \qquad (10)$$

Each such map also induces a transformation of operators defined on its domain of definition.
It is useful to have a graphical depiction for how operators transform under such maps.

Consider an operator defined on $V^{\otimes(\ell-1)}$, represented on $\mathbb{V}_{\ell-1}$ by $\hat O_{\ell-1} = \hat P_{\ell-1} \hat O \hat P_{\ell-1}$ (11)

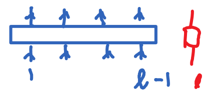
What is its representation on $\mathbb{V}_\ell$? $\hat O_\ell = \hat O_{\ell-1} \otimes \mathbb{1}_\ell =$  (12)



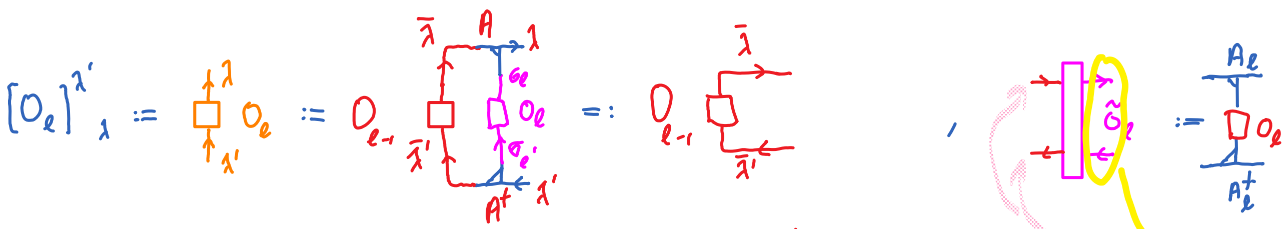
$\hat O_\ell = \ldots = \ldots , \quad [O_\ell]^{\lambda'}{}_\lambda := \ldots := \ldots$ (13)

sum on λ, λ' implied

Explicitly: $\hat O_\ell = |\Psi_{\lambda'}\rangle_\ell \, [O_\ell]^{\lambda'}{}_\lambda \, {}_\ell\langle \Psi^\lambda | , \qquad [O_\ell]^{\lambda'}{}_\lambda = A^{\dagger\lambda'}{}_{\sigma_\ell\bar\lambda'} \, [O_{\ell-1}]^{\bar\lambda'}{}_{\bar\lambda} \, A^{\bar\lambda\sigma_\ell}{}_\lambda$ (14)

Similarly, for operator with non-trivial action also on site : $\hat O_\ell = \hat O_{\ell-1} \otimes \hat o_\ell =$  (15)

Just replace  by in (9):



$[O_\ell]^{\lambda'}{}_\lambda := \ldots := \ldots =: \ldots , \quad \ldots := \ldots$ (16)

$$= A^{\dagger\lambda'}{}_{\sigma'_\ell\bar\lambda'} \, [O_{\ell-1}] \; {}_\lambda \, [o_\ell]^{\sigma'_\ell}{}_{\sigma_\ell} \, A^{\bar\lambda\sigma_\ell}{}_\lambda = [O_{\ell-1}]^{\bar\lambda'}{}_{\bar\lambda} \, [\tilde o_\ell]_{\bar\lambda'}{}^{\bar\lambda}{}_{\lambda} \qquad (17)$$

Thus, the isometry A_ℓ maps the local operator into an effective basis associated with $\mathbb{V}_{\ell-1}$, and $\mathbb{V}_\ell$