Math 540: Project 6

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Consider the Dittus-Boelter equation

$$Nu = \theta_1 Re^{\theta_2} Pr^{\theta_3} \tag{1}$$

where Nu, Re and Pr respectively denote the Nusselt, Reynolds and Prandtl numbers. Reported nominal parameter values are

$$\theta^0 = [0.023, 0.8, 0.4] \tag{2}$$

and data is provided in the file `db_data.txt` where $Re =$ `db_data(:,1)`; $Pr =$ `db_data(:,2)`; $Nu =$ `db_data(:,3)`.

Construct the Fisher information matrix and discuss the identifiability of the parameters. Now use DRAM to compute posterior densities for the parameters. Are your pairwise plots consistent with the Fisher information results?

Solution.

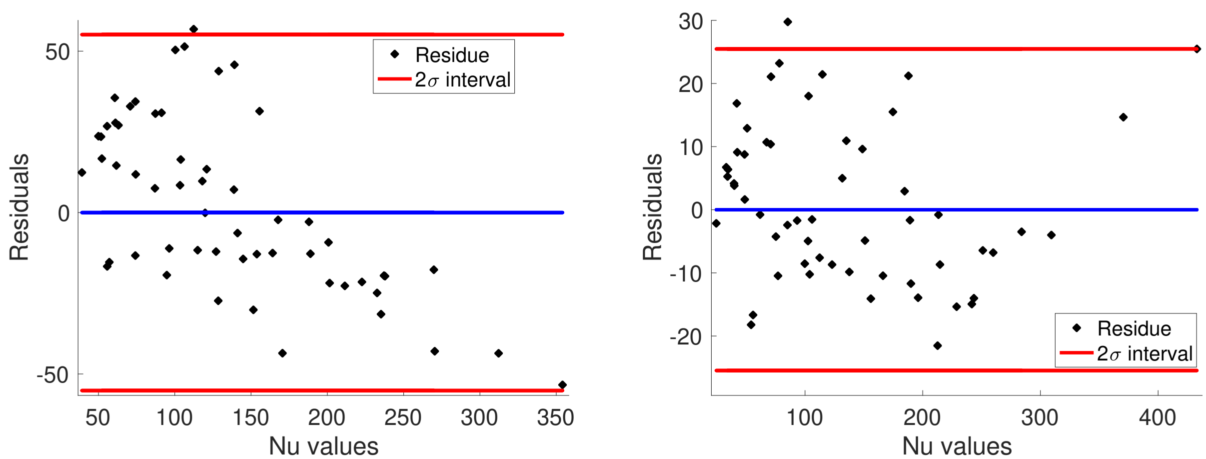
Here we have given data and nominal parameter values. First, we'd like to check relative residuals with given parameter values. Using Figure 1(a) as a reference, we observe that the residuals are as opposed to being distributed in an identical and independent manner. Using MATLAB's **lsqnonlin.m** command, we obtained the optimal parameter values,

$$\theta = [0.004, 0.986, 0.411].$$

Next, using these optimal parameters, we obtained an error variance estimate

$$\sigma^2 = \frac{1}{n-p} \boldsymbol{R}^\top \boldsymbol{R} = 162.1507$$

where $\boldsymbol{R}$ is the vector of the residuals and $n = 56, p = 3$. The residual plot using optimal parameters given in Figure 1(b) is identically and independently distributed.



(a) (b)

Figure 1: (a) Residuals with given nominal parameter values; (b) Residuals with optimal parameter values;

Following that, we obtained the sensitivity matrix,

$$\mathbf{X} = \begin{bmatrix} Re_1^{\theta_2} Pr_1^{\theta_3} & \theta_1 Re_1^{\theta_2} Pr_1^{\theta_3} \ln(Re_1) & \theta_1 Re_1^{\theta_2} Pr_1^{\theta_3} \ln(Pr_1) \\ Re_2^{\theta_2} Pr_2^{\theta_3} & \theta_1 Re_2^{\theta_2} Pr_2^{\theta_3} \ln(Re_2) & \theta_1 Re_2^{\theta_2} Pr_2^{\theta_3} \ln(Pr_2) \\ \vdots & \vdots & \vdots \\ Re_n^{\theta_2} Pr_n^{\theta_3} & \theta_1 Re_n^{\theta_2} Pr_n^{\theta_3} \ln(Re_n) & \theta_1 Re_n^{\theta_2} Pr_n^{\theta_3} \ln(Pr_n) \end{bmatrix}_{n \times p}$$

and Fisher's information matrix

$$\mathcal{F} = \mathbf{X}^\top \mathbf{X} = \begin{bmatrix} 9.3892 \times 10^{10} & 3.5764 \times 10^{9} & 1.3305 \times 10^{9} \\ 3.5764 \times 10^{9} & 1.3700 \times 10^{8} & 4.9713 \times 10^{7} \\ 1.3305 \times 10^{9} & 4.9713 \times 10^{7} & 2.1098 \times 10^{7} \end{bmatrix}.$$

The eigenvalues of the Fisher's matrix are

$$\lambda = \left[2.9325 \times 10^{5}, 2.7239 \times 10^{6}, 9.4047 \times 10^{10}\right]$$

Because all of the eigenvalues are sufficiently large, we infer that all of the parameters are identifiable. The covariance matrix is

$$\mathbf{V} = \sigma^2 (\mathbf{X}^\top \mathbf{X})^{-1} = \begin{bmatrix} 9.0474 \times 10^{-7} & -2.0102 \times 10^{-5} & -9.6884 \times 10^{-6} \\ -2.0102 \times 10^{-5} & 4.5482 \times 10^{-4} & 1.9604 \times 10^{-4} \\ -9.6884 \times 10^{-6} & 1.9604 \times 10^{-4} & 1.5674 \times 10^{-4} \end{bmatrix}.$$

Following that, we'd like to analyse the same model via the DRAM algorithm. We utilized optimal parameter values as initial values and the covariance matrix derived by the Frequentist technique as the initial covariance matrix for this approach. Figure 2 shows the chain plots for each parameter as well as the paired samples plots. We can see from these plots that the MCMC method has converged. Furthermore, the paired sample plots demonstrate the correlation between parameters, and all parameters can be identified. As a consequence, they coincide with the results produced using the Frequentist technique. The marginal distributions of each parameter are presented in Figure 3. Using DRAM we obtained the Bayesian parameter estimations,

$$\theta = [0.004, 0.982, 0.409],$$

error of variance estimate,

$$\sigma^2 = 168.5046.$$

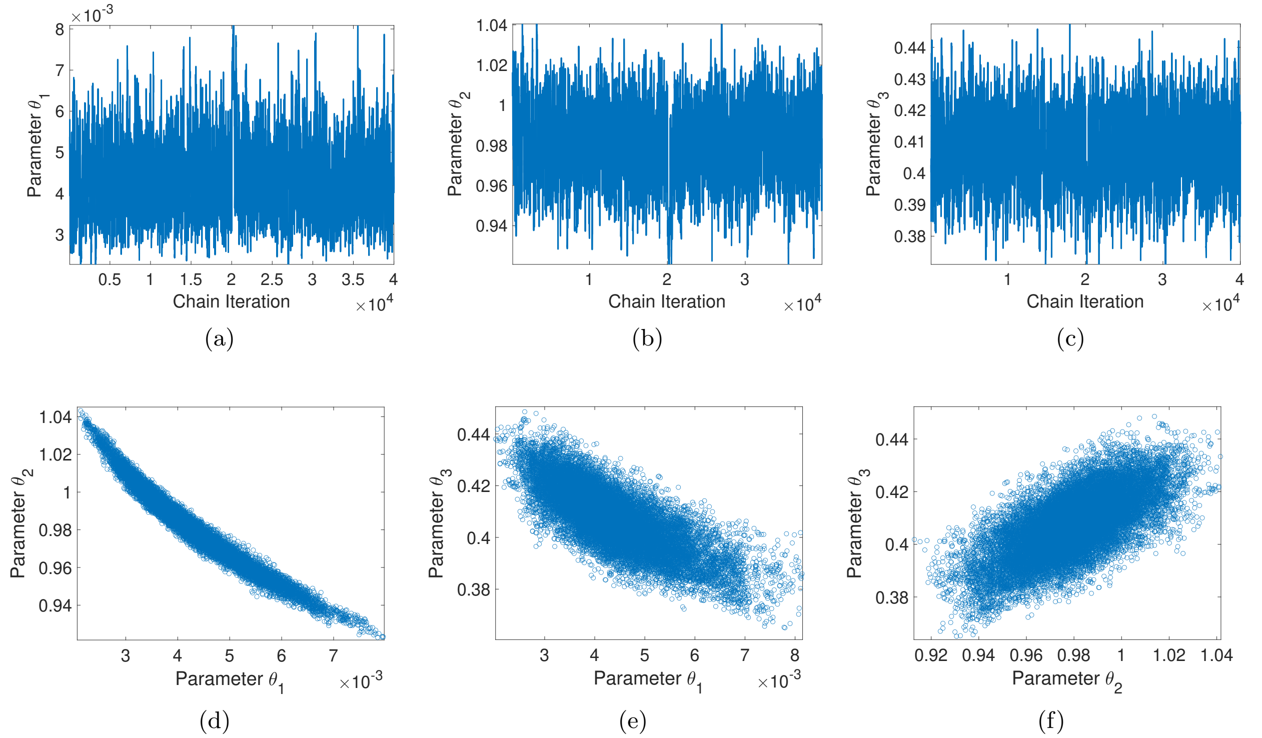


Figure 2: Chain plot of (a) θ_1; (b) θ_2; (c) θ_3; The pairwise joints samples (d) θ_1 & θ_2; (e) θ_1 & θ_3; (f) θ_2 & θ_3

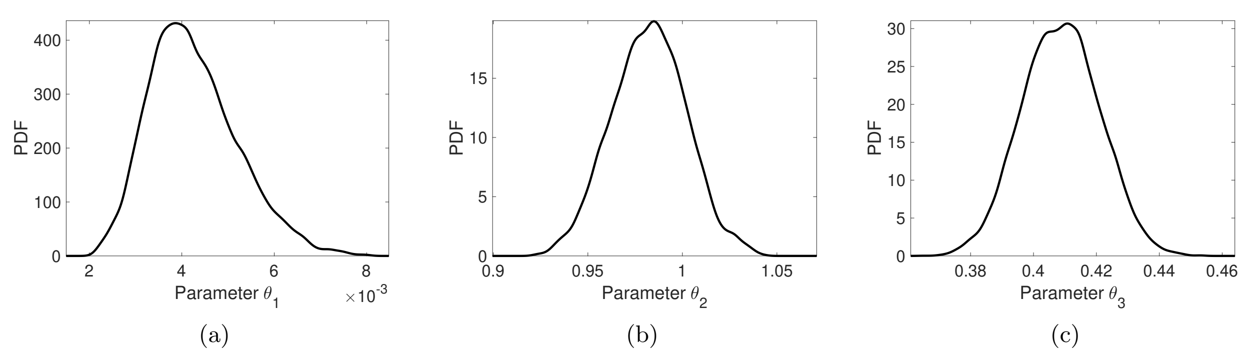


Figure 3: Marginal distribution of (a) θ_1; (b) θ_2; (c) θ_3;