Part 2

☆ $\boxed{\text{Ex. 4.}}$ ☆.

According to the formula

① $r_t(k) = \log\{1+R_t(k)\} = \log\{(1+R_t)\cdots(1+R_{t-k+1})\}$
$= r_t + r_{t-1} + \cdots + r_{t-k+1}$

② $r_t = \log(1+R_t) = \log\left(\frac{P_t}{P_{t-1}}\right) = p_t - p_{t-1}$

$r_3(2) = r_3 + r_2$
$= \log\left(\frac{P_3}{P_2}\right) + \log\left(\frac{P_2}{P_1}\right)$
$= (p_3 - p_2) + (p_2 - p_1) = p_3 - p_1$
$= \log\left(\frac{P_3}{P_1}\right) = \boxed{\log\left(\frac{98}{95}\right) = 0.03109059}$

Part 2

Exercise 7

(a) $r_t(4)$ is $\boxed{N(\overset{0.24}{4 \times 0.06}, \overset{1.88}{4 \times 0.47})}$

(b) Using R, $P\{r_1(4) < 2\} = $ pnorm (2, 4×0.06, sd = sqrt(4×0.47))

$= \boxed{0.9004}$

(c) $r_2(1) = r_2 \quad r_2(2) = r_2 + r_1$

Thus, $cov(r_2(1), r_2(2)) = var(r_2) = \boxed{0.47}$

(d) $r_t(3) = r_t + r_{t-1} + r_{t-2}$

Thus, given $r_{t-2} = 0.6$

$r_t(3)$ is $N(0.6 + 2 \times 0.06, 2 \times 0.47)$

$\Rightarrow \boxed{N(0.72, 0.94)}$

Part 2

Exercise 8

(a) $P(X_2 > 1.3X_0) = P(r_1 + r_2 > \log(1.3))$

where $r_1 + r_2$ is $N(2\mu, 2\sigma^2)$

using R:

```
pnorm( log(1.3), mean = 2*mu, sd = sqrt(2)* sigma, lower.tail = FALSE)
```

where mu & sigma is given by the problem.

(b) In (A.4), use $g(x) = \exp(x)$ and $h(y) = \log(y)$

The density is at x

$$\frac{1}{\sqrt{2\pi}\,\sigma x} \exp\left[-\frac{1}{2\sigma^2}\left\{\log(x) - \left(\log(X_0) + \mu\right)\right\}^2\right]$$

(c)

$$\exp\left[\{\log(X_0) + k\mu\} + \sigma\sqrt{k}\,\Phi^{-1}(0.9)\right]$$

where Φ is the standard normal cumulative distribution function

(d) Use the fact that X_k^2 is lognormal$(2k\mu, 4k\sigma^2)$. Then

$$E(X_k^2) = X_0^2 \exp(2k\mu + 2k\sigma^2)$$

(e) Using $Var(X_k) = E(X_k^2) - (E(X_k))^2$ ~~wh~~ Then:

$$X_0^2 \exp(2k\mu + k\sigma^2)\{\exp(k\sigma^2) - 1\}$$

Part 2

Exercise 9

(a) $P(X_3 > 1.2 X_0) = P\left(\log\left(\frac{X_3}{X_0}\right) > \log 1.2\right)$

using R $\underline{=}$ pnorm(log(1.2), mean = 0.1*3, sd = sqrt(3)*0.2, lower.tail = FALSE)

$= \boxed{0.633}$

(b) $Var(Y|X) = E(Y^2|X) - (E(Y|X))^2$

$\frac{X_k}{k} = X_0 \frac{\exp(r_1 + \cdots + r_k)}{k}$

$E\left(\frac{X_k}{k} \mid X_0\right) = E\left(\frac{X_0 \exp(r_1 + \cdots + r_k)}{k} \mid X_0\right)$

~~since X_0 given, fixed~~

$\cancel{= \frac{X_0}{k} E[\exp(r_1 + \cdots + r_k)]}$

(c). $P\left(\frac{X_t}{X_0} \geq 2\right) = P\left(\log\left(\frac{X_t}{X_0}\right) \geq \log(2)\right)$

use R $\underline{=}$ pnorm(log(2), mean = t*0.1, sd = sqrt(t)*0.2, lower.tail = FALSE)

set as $:=$ prob.

```
t = 1:1000
min(which(prob > 0.9))
```
$\Rightarrow$ output: 18

The minimum number is $\boxed{18}$

Part 3.

3. If X is a continuous random variable with a strictly increasing distribution function F, find the distribution of $U = F(X)$

Sol: Let $F_Y(y)$ be the CDF of $Y = F(X)$

Obviously, $y \in [0,1]$ by the nature of CDF

Then, for any $y \in [0,1]$ we have:

$$F_Y(y) = Pr(Y \le y) = Pr(F(X) \le y) = Pr(X \le F^{-1}(y)) = F(F^{-1}(y)) = y$$

$F_Y(y) = y \Rightarrow F_Y$ is a uniform distribution.

$\boxed{\text{It's uniform. } U(0,1)}$

Part 4

(a) Find the density $f_Y(y)$

Sol: Use the hint: compute the cdf of Y:
The range of Y is $(0, +\infty)$
(since $Y = e^X$ and $X \in (-\infty, +\infty)$) And for y in this range:

$$F_Y(y) = P(Y \le y) = P(e^X \le y) = P(X \le \ln y) = \Phi\left(\frac{\ln y - 1}{2}\right)$$

Differentiating, we get the density:

$$f_Y(y) = \frac{d}{dy}\Phi\left(\frac{\ln y - 1}{2}\right) = \frac{1}{2y}\Phi'\left(\frac{\ln y - 1}{2}\right)$$

$$= \frac{1}{2y\sqrt{2\pi}} \exp\left(-\left[\frac{1}{2\cdot 2^2}(\ln y - 1)^2\right]\right)$$

$$\boxed{= \frac{1}{2y\sqrt{2\pi}} \exp\left\{-\frac{1}{8}(\ln y - 1)^2\right\} \qquad y \in (0, +\infty)}$$

(b). Find the mean and variance of Y.

(i) mean $= E(Y) = E(e^X)$
use the hint, let $t = 1$ we have: $\boxed{\text{mean of } Y = e^{\mu + \frac{1}{2}\sigma^2}}$

(ii) Variance $= E(Y^2) - (E(Y))^2$

$$= E(e^{2X}) - (E(e^X))^2$$

use hint: let $t = 2$ let $t = 1$

$$= e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}$$

$$\boxed{\text{variance of } Y = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}}$$

Part 5

(a) Show that $E(F_n(x)) = F(x)$

Pf: $P(X_i \le x) = F(x)$ $\quad E(X_i) = \mu \quad Var(X_i) = \sigma^2 \quad \forall i = 1, \dots n)$.

Let $Z_i = I(X_i \le x)$ then $Z_i = \begin{cases} 1 & \text{if } X_i \le x \\ 0 & \text{if } X_i > x \end{cases}$

Then $E(Z_i) = 1 \cdot P(Z_i = 1) + 0 \cdot P(Z_i = 0)$

$= P(Z_i = 1) = P(X_i \le x) = F(x) \quad (*)$

Thus. $E(F_n(x)) = E\left[\frac{1}{n}\sum_{i=1}^{n} I(X_i \le x)\right]$

$= E\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right) \quad \left(\text{since } Z_i = I(X_i \le x)\right)$

$= \frac{1}{n}\sum_{i=1}^{n} E(Z_i) \quad (\because \text{linear property of expectation})$

$= \frac{1}{n}\sum_{i=1}^{n} F(x) \quad (\text{use } (*))$

$= F(x)$

Thus. $E(F_n(x)) = F(x)$ $\square$

(b) Show that $VAR(F_n(x)) = F(x)(1-F(x))/n$

Sol: $Var(F_n(x)) = Var\left[\frac{1}{n}\sum_{i=1}^{n} I(X_i \le x)\right] = Var\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right)$

$= \frac{1}{n^2}\left[\sum_{i=1}^{n} Var(Z_i)\right] \quad (\because Z_1, Z_2 \dots Z_n \text{ are independent})$

$\Rightarrow$. $Var(Z_i) = E(Z_i^2) - (E(Z_i))^2$

$E(Z_i^2) = 1^2 \cdot P(Z_i = 1) + 0^2 \cdot P(Z_i = 0) = P(Z_i = 1) = F(x)$

plug in

$\Rightarrow Var(Z_i) = E(Z_i^2) - (E(Z_i))^2 = F(x) - F^2(x) = F(x)(1 - F(x))$

$Var(F_n(x)) = \frac{1}{n^2}\left(\sum_{i=1}^{n} Var(Z_i)\right) = \frac{1}{n^2}\sum_{i=1}^{n} F(x)(1-F(x)) = \frac{1}{n^{\cancel{2}}} \cdot \cancel{n}\, F(x)(1-F(x))$

$= \boxed{\frac{F(x)(1-F(x))}{n}}$ $\square$

Part 5

(1). Since $z_1, z_2 \cdots z_n$ are iid Bernouli random variable.
where $p = P(z_i = 1) = F(x)$

$\Rightarrow z_1 \cdots z_n \overset{iid}{\sim} Ber(p) \quad p = F(x)$.

By nature of Bernouli:
$\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i \Rightarrow E(\bar{z}) = p \quad var(\bar{z}) = \frac{p(1-p)}{n}$

By $\boxed{\text{Central Limit Theorem}}$:

$$\frac{\bar{z} - p}{\sqrt{\frac{p(1-p)}{n}}} = \frac{\sqrt{n}(\bar{z} - p)}{\sqrt{p(1-p)}} \xrightarrow{L} N(0,1) \text{ as } n \to +\infty$$

Here. $\bar{z} = F_n(x) \quad p = F(x)$

So:
$$\frac{\sqrt{n}(F_n(x) - F(x))}{\sqrt{F(x)(1-F(x))}} \xrightarrow{L} N(0,1) \text{ as } n \to +\infty$$

The asymptotic distribution of $\frac{\sqrt{n}(F_n(x) - F(x))}{\sqrt{F(x)(1-F(x))}}$ is

$\boxed{\text{standard normal distribution.}}$ $N(0,1)$.