The Ninth Grade Math Competition Class
Complex Numbers
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1. Suppose $(-3+8i)(-3+Ai)$ is a real number, find the value of A where A is real.

$9 + (-24)i - 3Ai + 8Ai^2$

$(9-8A) + \underbrace{(-24-3A)i}_{=0}$

$$\frac{1}{-3+8i} \frac{(-3-8i)}{(-3-8i)}$$

$-24 - 3A = 0$

$-3A = 24$

$A = -8$

2. Find all complex numbers whose squares equal $7 - 24i$.

$z^2 = 7 - 24i$

$\boxed{\begin{array}{l} a = -4,\ b = 3 \\ a = 4,\ b = -3 \end{array}}$

$z = a + bi$

$(a+bi)^2 = 7 - 24i$

$\underline{a^2} + \underline{2abi} - \underline{b^2} = \underline{7} - \underline{24i}$

$a^2 - b^2 = 7$

$2ab = -24$

3. Let $a = \frac{(2+i)^2}{3+i}$, find $1 + \frac{1}{a}$.

$$1 + \frac{1}{\frac{(2+i)^2}{3+i}} = 1 + \frac{3+i}{(2+i)^2} = 1 + \frac{3+i}{4+4i-1} = 1 + \frac{3+i}{3+4i}$$

$$= \frac{3+4i}{3+4i} + \frac{3+i}{3+4i} = \frac{6+5i}{3+4i} \frac{(3-4i)}{(3-4i)} = \frac{18+15i-24i+20}{9+16}$$

$$= \frac{38-9i}{25} = \frac{38}{25} - \frac{9}{25} i$$

4. Find all x such that $x^5 = x^3$ (What if $x^5 = x^{-3}$).

$x^5 = x^3$

$x^5 - x^3 = 0$

$x^3(x^2-1) = 0$

Case 1: $x^3 = 0$, $\Rightarrow \boxed{x = 0}$

2: $x^2 - 1 = 0$, $\Rightarrow \boxed{x = \pm 1}$

$x^5 = \frac{1}{x^3}$

$x^8 = 1$

$1, -1, i, -i, \pm\sqrt{i}, \pm\sqrt{-i}$

$(\sqrt{i})^8 = ((\sqrt{i})^2)^4 = i^4 = 1$

$\sqrt{i} = a + bi$

5. If $x = \frac{1-\sqrt{3}i}{2}$, what is $\frac{1}{x^2-x}$. $\frac{1}{-1} = -1$

$= \frac{1}{2} - \frac{\sqrt{3}}{2}i$

$$x^2 - x = \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^2 - \left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$$

$$x^2 - x = \frac{1}{4} - \cancel{\frac{\sqrt{3}}{2}i} - \frac{3}{4} - \frac{1}{2} + \cancel{\frac{\sqrt{3}}{2}i} = -1$$

6. Show that $\overline{w+z} = \bar{w} + \bar{z}$, and $\overline{wz} = \bar{w} \cdot \bar{z}$.

$w = a + bi$

$z = c + di$

$\overline{w+z} = \bar{w} + \bar{z}$

$\overline{a+bi + c+di} = \overline{a+bi} + \overline{c+di}$

$\overline{(a+c) + (b+d)i} = a - bi + c - di$

$(a+c) - (b+d)i = (a+c) - (b+d)i$

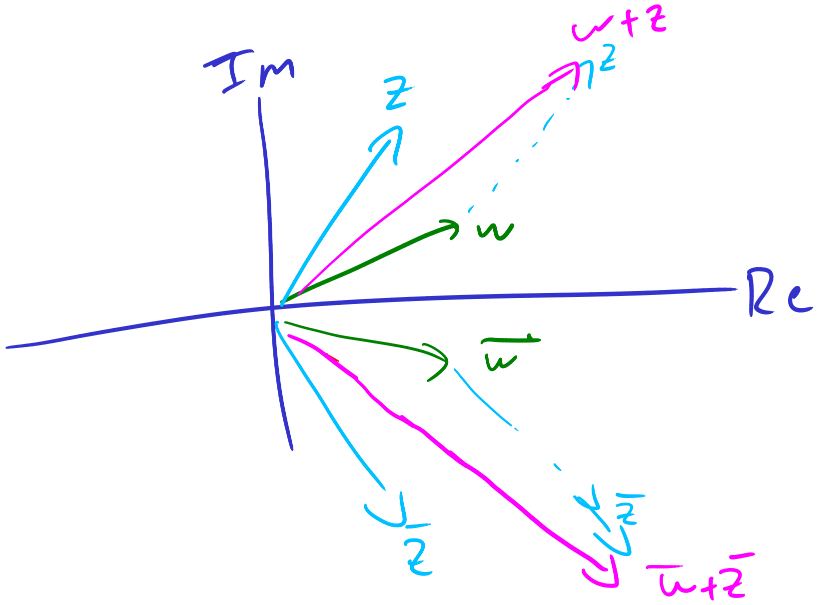
$\overline{wz} = \bar{w} \cdot \bar{z}$

$\overline{(a+bi)(c+di)} = \overline{a+bi}\ \overline{c+di}$

$\overline{ac+bci+adi-bd} = (a-bi)(c-di)$

$(ac-bd) - (bc-ad)i$

$= (ac-bd) - (bc+ad)i$



7. Write $\sqrt{-16 + 30i}$ as a complex number.

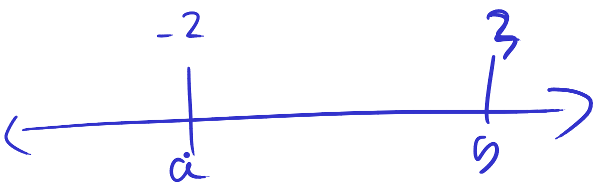
$z = a + bi$

$z = 3 + 5i$

$-z = -3 - 5i$

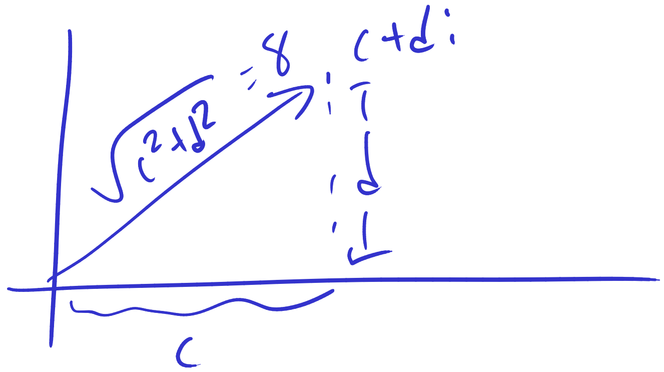
$\sqrt{-16 + 30i} = a + bi$

8. A function f is defined on the complex numbers by $f(z) = (a + bi)z$, where a and b are positive numbers. This function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a + bi| = 8$ and that $b^2 = \frac{m}{n}$, where m and n are positive integers, Find $m + n$.



$|-2-3| = |-5| = 5$

$|a-b|$



$|f(z) - z| = |f(z) - 0|$

$|a+bi-1| = |a+bi| = 8$

$(a-1)^2 + b^2 = 64$

$a^2 + b^2 = 64$

$a^2 - (a-1)^2 = 0$

$a^2 - a^2 + 2a - 1 = 0$

$2a = 1 \Rightarrow a = \frac{1}{2}$

$\frac{1}{4} + b^2 = 64$

$b^2 = 64 - \frac{1}{4} = \frac{255}{4}$ $m = 255$, $n = 4$

$m + n = 255 + 4 = 259$

9. There is a complex number z with imaginary part 164 and a positive integer n such that $\frac{z}{z+n} = 4i$, find n.

$z = a + 164i$

$$\frac{a + 164i}{a + n + 164i} = 4i$$

$a + 164i = 4(a+n)i - 656$

$a = -656$

$164 = 4a + 4n$

$41 = a + n$

$n = 41 - a = 41 - (-656) = 697$

10. Find c if a, b, and c are positive integers which satisfy $c = (a + bi)^3 - 107i$.

$c = a^3 + 3a^2bi - 3ab^2 - b^3i - 107i$

$c = a^3 - 3ab^2$

$0 = 3a^2b - b^3 - 107$

$c = a(a^2 - 3b^2)$

$6 \quad 6^2 \quad -3 \cdot 1^2$

$107 = b(3a^2 - b^2)$

$1 \quad 107$

$107 \quad 1$

$b = 1$

$3a^2 - b^2 = 107$

$3a^2 = 108 \quad a^2 = 36 \quad a = 6$

$c = 6(36 - 3) = 198$

$\boxed{a = 6,\ b = 1,\ c = 198}$