The Ninth Grade Math Competition Class
Factorials
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$8^n = (2^3)^n = 2^{3n}$

1. Find the largest integer value of n for which 8^n evenly divides $100!$.

$10!$

$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ $\quad 8^2$

$\left\lfloor \frac{10}{8} \right\rfloor \Rightarrow \left\lfloor \frac{1}{8} \right\rfloor$

97

$\left\lfloor \frac{100}{2} \right\rfloor \Rightarrow \left\lfloor \frac{50}{2} \right\rfloor \Rightarrow \left\lfloor \frac{25}{2} \right\rfloor \Rightarrow \left\lfloor \frac{12}{2} \right\rfloor \Rightarrow \left\lfloor \frac{6}{2} \right\rfloor \Rightarrow \left\lfloor \frac{3}{2} \right\rfloor \Rightarrow \left\lfloor \frac{1}{2} \right\rfloor$

$\left\lfloor \frac{97}{3} \right\rfloor = 32$

$2^{97} \mid 100!$

$2^{3 \cdot \frac{97}{3}} \mid 100!$

2. Find the prime factorization of $10!$.

$$10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

$$2 \cdot 5 \quad 3^2 \cdot 2^3 \cdot 7 \cdot 2 \cdot 3 \cdot 5 \cdot 2^2 \cdot 3 \cdot 2$$

$$\boxed{2^8 \cdot 3^4 \cdot 5^2 \cdot 7}$$

3. What is the product of the positive divisors of 7!.

$7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$= 7 \cdot (2 \cdot 3) \cdot 5 \cdot (2^2) \cdot 3 \cdot 2$

$= 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1$

$5 \cdot 3 \cdot 2 \cdot 2 = 60$

$$\boxed{\text{prod. of div. of } n = n^{\frac{t(n)}{2}}}$$

$(7!)^{\frac{60}{2}} = \boxed{7!^{30}}$

4. How many positive cubes divide $3!5!7!$.

$$3!\,5!\,7! = (3\cdot 2\cdot 1)\cdot(5\cdot 4\cdot 3\cdot 2\cdot 1)\cdot(7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)$$

$$= 2^8\cdot 3^4\cdot 5^2\cdot 7$$

$2^0 \quad 3^0 \quad 5^0 \quad 7^0$

$2^3 \quad 3^3$

2^6

$$3\cdot 2 = \boxed{6}$$

5. For how many positive integers n less than or equal to 24 is $n!$ evenly divible by $\overbrace{1+2+\cdots+n}^{(n+1)\frac{n}{2}}$?

$1 + 2 + 3 + 4 + \cdots (n-3) + (n-2) + (n-1) + n$

$n+1$ $n+1$ $n+1$ $n+1$

divides

$$\frac{n(n+1)}{2} \Big| \ n! = n \cdot (n-1)!$$

$$\Rightarrow \quad n+1 \mid 2 \cdot (n-1)!$$

$n=4$ $\quad 5 \mid 2 \cdot 3!$ $\quad \times$

Primes: 2, 3, 5, 7, 11, 13, 17, 19, 23

1, 2, 4, 6, ... 22

$n=1$ $\quad 2 \mid 2 \cdot 0! = 2$

$$24 - 9 + 1 = \boxed{16}$$

6. In how many zeros does the decimal expansion of $100^{100} - 100!$ end?

$$
\begin{array}{r}
10000000 \\
-\ 123000 \\
\hline
9877000
\end{array}
$$

$$\left\lfloor \frac{100}{5} \right\rfloor = \left\lfloor \frac{20}{5} \right\rfloor = \left\lfloor \frac{4}{5} \right\rfloor = 0$$

$$20 + 4 = \boxed{24}$$

7. Let P be the product of the first 100 positive odd integers. Find the largest integer k such that P is divisible by 3^k.

$$P = 1 \cdot 3 \cdot 5 \cdot 7 \cdots 199$$

$$2 \cdot 4 \cdot 6 \cdot 8 \cdots \; P = 1 \cdot 3 \cdot 5 \cdot 7 \cdots 199 \cdot 200$$

$$2 \cdot 4 \cdot 6 \cdot 8 \cdots 200 = 2^{100} \cdot 100!$$

$$2^{100} \cdot 100! \, P = 200!$$

$$P = \frac{200!}{2^{100} \cdot 100!}$$

$$\left\lfloor \frac{200}{3} \right\rfloor = \left\lfloor \frac{66}{3} \right\rfloor = \left\lfloor \frac{22}{3} \right\rfloor = \left\lfloor \frac{7}{3} \right\rfloor = \left\lfloor \frac{2}{3} \right\rfloor \quad 97$$

$$\left\lfloor \frac{100}{3} \right\rfloor = \left\lfloor \frac{33}{3} \right\rfloor = \left\lfloor \frac{11}{3} \right\rfloor = \left\lfloor \frac{3}{3} \right\rfloor = \left\lfloor \frac{1}{3} \right\rfloor \quad 48$$

$$97 - 48 = \boxed{49}$$