The Ninth Grade Math Competition Class
Decimals
Anthony Wang

1. Convert repeating decimal $0.\overline{3123}$ to fraction.

$x = 0.312331233123\ldots$

$1000x = 312.3312331 2 3$

$10000x = 3123.3123312 3 3123$

$9999x = 3123$

$$\boxed{x = \frac{3123}{9999} = \frac{347}{1111}}$$

2. Compute $\frac{4!+3!}{3!+2!}$. Express your answer as a decimal to the nearest hundredth.

$$\frac{\overset{24}{4!}+3!}{\underset{6}{3!}+\underset{2}{2!}} = \frac{30}{8} = \frac{15}{4} = 3\frac{3}{4} = 3.75$$

3. What is the 4037^{th} digit following the decimal point in the expansion of $\frac{1}{111}$?

$$\frac{1}{111} = \frac{9}{999} = 0.\overline{009}\ldots$$

$$120\,120$$

$$0.009009$$

$$111\overline{)1.000}$$
$$-999$$
$$1000$$
$$999$$

$$1345 \; 1222$$

$$3\overline{)4037}$$
$$-3$$
$$10$$
$$13$$
$$17$$

4. Evaluate the infinite geometric series

0.01 0.0007 0.000049

$$\frac{7^0}{100} + \frac{7^1}{100^2} + \frac{7^2}{100^3} + \cdots \quad \frac{7^3}{100^4}$$

as a fraction and find the first 6 digits in its decimal expansion.

0.01
0.0007
0.000049
0.00000343
0.010749
0.010752

$$x = \frac{7^0}{100} + \frac{7^1}{100^2} + \frac{7^2}{100^3} + \cdots$$

$$\frac{7}{100}x = \frac{7^1}{100^2} + \frac{7^2}{100^3} + \frac{7^3}{100^4} + \cdots$$

$$\frac{93}{100}x = \frac{7^0}{100}$$

$$x = \boxed{\frac{1}{93}}$$

5. Let S be the set of real numbers that can be represented as repeating decimals of the form $0.\overline{abc}$, where a, b, c are distinct digits. Find the sum of the elements of S.

$0.\overline{999}$

$0.\overline{012}$, $0.\overline{013}$, $0.\overline{014}$ . . . . .

. . . $0.\overline{986}$, $0.\overline{987}$

$0.\overline{999}$

$10 \cdot 9 \cdot 8 = 720$

$= 360 \cdot 0.\overline{999}$

$\boxed{=360}$

$x = 0.9999999\ldots$

$10x = 9.99999\ldots$

$9x = 9$

$x = 1$

6. The rational number r is the largest number less than 1 whose base-7 expansion consists of two distinct digits, i.e., $r = 0.\overline{AB}$. Written as a reduced fraction, $r = \frac{p}{q}$, find $p + q$.

BASE 10: $0.\overline{AB}$

$$0.\overline{98} = \frac{98}{99}$$

BASE 7: $0.\overline{AB}_7$

$$x = 0.\overline{65}_7$$

$$7^2 x = 65.\overline{65}_7 \ldots$$

$$66_7 x = 65_7$$

$$x = \frac{65_7}{66_7} = \frac{47}{48} \Rightarrow p + q = \boxed{95}$$

$\sim p$

$\sim q$

7. Express $0.72\overline{45}$ as a common fraction.

$$x = 0.72\overline{45} \quad (1)$$

$$100x = 72.45\overline{45} \quad (2)$$

$$99x = 71.73 \quad (2)-(1)$$

$$x = \frac{71.73}{99} = \frac{7173}{9900} = \frac{797}{1100}$$

8. Let p be a prime number other than 2 or 5. What is the maximum possible number of digits in the repeating block of digits in $\frac{1}{p}$?

$\frac{1}{3} = 0.3333\ldots$ Block size: 1

$\frac{1}{7} = 0.\overline{142857}\ldots$: 6

$\frac{1}{11} = 0.0909\ldots$: 2

$\frac{1}{13} = 0.\ldots$: 6

$\frac{1}{p} = 0.\overline{a_1 a_2 a_3 a_4 \ldots a_k}$: k

$$\frac{1}{p} = \frac{\overbrace{a_1 a_2 a_3 \ldots a_k}^{a}}{\underbrace{99999\ldots9}_{10^k - 1}}$$

$$\frac{10^k - 1}{a} = p$$

8. Let p be a prime number other than 2 or 5. What is the maximum possible number of digits in the repeating block of digits in $\frac{1}{p}$?

$\frac{1}{p}$

$0.a_1a_2 \cdots a_k\, a_1$

$p \overline{)1.0}$

0.142857

$7 \overline{)1.0}$

$$\begin{array}{r} 1.0 \\ -7 \\ \hline 30 \\ 28 \\ \hline 20 \\ -14 \\ \hline 60 \\ -56 \\ \hline 40 \\ 35 \\ \hline 50 \\ 49 \\ \hline 1 \end{array}$$

possible rem: $0, 1, 2, 3, \ldots, p-1$

not a rem.

$p-1$