The Ninth Grade Math Competition Class
Divisors
Anthony Wang

1. Find the product of the positive divisors of 2400 that are multiples of 6.

$$2400 = 2^5 \cdot 3^1 \cdot 5^2$$

$\times 6$ $\quad$ $6\times$

$$n = 400 = 2^4 \cdot 3^0 \cdot 5^2$$

15 div.

$$n^{\frac{t(n)}{2}} = 400^{\frac{15}{2}} = 20^{15}$$

$$\boxed{20^{15} \cdot 6^{15}}$$

2. Find the product of the divisors of 3200 that are perfect squares.

16 $\quad 3200 = 2^7 \cdot 5^2$

$\sqrt{x}$ $\quad x^2$ $\quad \sqrt{3200} = 2^{\frac{7}{2}} \cdot 5^1$

$4 \cdot 2 =$ 8 div

4 $\quad 40 = 2^3 \cdot 5^1$

$n^{\frac{t(n)}{2}} = 40^{\frac{8}{2}} = 40^4$

$(40^4)^2 = \boxed{40^8}$

3. A proper divisor of a number is a divisor of the number that is not the number itself. What is the smallest positive integer that is less than the sum of its positive proper divisors?

$8 = 2^3$

$1, 2, 4, \cancel{8}$

$1+2+4 = 7$

$7 =$

$1, \cancel{7}$

1

$4, 6, 8, 9, 10,$

$\boxed{12} = 2^2 \cdot 3^1$

$1, 2, 3, 4, 6, \cancel{12}$

$1+2+3+4+6 = 16$

4. How many positive cubes divide $3! \cdot 5! \cdot 7!$?
(A) 2 (B) 3 (C) 4 (D) 5 (E) 6

$$3! \cdot 5! \cdot 7! = (3 \cdot 2 \cdot 1) \cdot (5 \cdot 4 \cdot 3 \cdot 2 \cdot 1) \cdot (7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$$

$$= 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1$$

2^0 3^0 5^0 7^0

2^3 3^3

2^6

$$3 \cdot 2 \cdot 1 \cdot 1 = \boxed{6}$$

5. How many of positive divisors of 3200 are not multiples of any perfect square greater than 1?

$3200 = 32 \cdot 100$

$2^5 \cdot 10^2$

$2^5 \cdot 2^2 \cdot 5^2$

$2^7 \cdot 5^2$

$2^0 \quad 5^0$

$2^1 \quad 5^1$

$2 \cdot 2 = \boxed{4}$

6. How many positive integers have exactly three proper divisors, each of which is less than 50?

4 div.

1 prime $n = p^3$ $\boxed{2^3 \quad 3^3 \quad 5^3 \quad 7^3}$ $\cancel{11^3}$

1, 7, 49 $\quad$ 1, 11, 121

$n = p^x q^y$

$1, p, q$

2 primes $n = p^1 q^1$

$(1+1)(1+1) = 4$ ✓

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31

37, 41, 43, 47 $\quad$ 15 primes

$\frac{15 \cdot 14}{2} = 105$ $\quad$ $2 \cdot 3$ $\quad$ $3 \cdot 2$

$105 + 4 = 109$

7. Jan is thinking of a positive integer. Her integer has exactly 16 positive divisors, two of which are 12 and 15. What is Jan's number?

$$12 = 2^2 \cdot 3^1$$

$$15 = 3^1 \cdot 5^1$$

$$n = 2^x \cdot 3^y \cdot 5^z$$

$$3\geq \quad 2\geq \quad 2\geq$$

$$(x+1)(y+1)(z+1) = 16$$

$$4 \cdot 2 \cdot 2$$

$$= 2^3 \cdot 3^1 \cdot 5^1 = \boxed{120}$$

8. What is the sum of all positive integers less than 100 that have exactly 12 divisors?

$12 \quad p^{11} \quad 2^{11} = 2048$

$6 \cdot 2 \quad p^5 q^1 \quad 2^5 3^1 = 96$

$3^5 2^1 = 486$

$2^5 5^1 = 160$

$4 \cdot 3 \quad p^3 \cdot q^2 \quad 2^3 \cdot 3^2 = 72$

$3 \cdot 2 \cdot 2 \quad p^2 \cdot q \cdot r \quad 2^2 \cdot 3 \cdot 5 = 60$

$2^2 \cdot 3 \cdot 7 = 84$

$2^2 \cdot 3 \cdot 11 = 132$

$3^2 \cdot 2 \cdot 5 = 90$

402

9. Dentoe p_k be the k^{th} prime number. Show that $p_1 p_2 \cdots p_n + 1$ cannot be the perfect square of an integer.

$2 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots + 1$

$p_1 p_2 \cdots p_n + 1 \neq x^2$

$p_1 p_2 \cdots p_n = x^2 - 1 = (x-1)(x+1)$

$p_1 = 2$ $\quad 2k \quad 2j$

10. Prove that it is impossible for three consecutive squares to sum to another perfect squares.

$$3^2 + 4^2 + 5^2 = 50$$

$$x^2 + (x+1)^2 + (x+2)^2 = x^2 + x^2 + 2x + 1 + x^2 + 4x + 4$$
$$= 3x^2 + 5x + 5$$

$$(x-1)^2 + x^2 + (x+1)^2 = x^2 - \cancel{2x} + 1 + x^2 + x^2 + \cancel{2x} + 1$$
$$= 3x^2 + 2$$

$$a^2 = 3x^2 + 2$$

$$1^2 + 2^2 + 3^2 = 14$$
$$2^2 + 3^2 + 4^2 = 29$$
$$3^2 + 4^2 + 5^2 = 50$$

$$1^2 = 1 = 3 \cdot 0 + 1$$
$$2^2 = 4 = 3 \cdot 1 + 1$$
$$3^2 = 9 = 3 \cdot 3$$
$$4^2 = 16 = 3 \cdot 5 + 1$$
$$5^2 = 25 = 3 \cdot 8 + 1$$
$$6^2 = 36 = 3 \cdot 12$$

$$(3n+1)^2 = 9n^2 + 6n + 1$$
$$3(3n^2 + 2n) + 1$$

$$(3n+2)^2 = 9n^2 + 12n + 4$$
$$3(3n^2 + 4n + 1) + 1$$

11. A positive integer n is nice if there is a positive integer m with exactly four positive divisors (including 1 and m) such that the sum of the four divisors is equal to n. How many numbers in the set $\{2010, 2011, 2012, \cdots, 2019\}$ are nice?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 5 .

$m = p^3 \quad n = 1 + p + p^2 + p^3 = (p+1)(p^2+1)$

$1+2$

$m = pq \quad n = 1 + p + q + pq = (1+p)(1+q)$

$2 \Rightarrow 3(1+q)$

$3, 5, 7$

$2010 = 2 \cdot 3 \cdot 5 \cdot 67 = 3 \cdot (670)$

$3 \cdot (669 + 1)$

$2012 = 2^2 \cdot 503$

$2014 = 2 \cdot 1007$

$2019 = 3 \cdot 673$

$2016 = 2^5 \cdot 3^2 \cdot 7 = (2^5 \cdot 3 \cdot 7) \cdot 3 \qquad 2^2 \cdot (2^3 \cdot 3^2 \cdot 7)$

$672 \cdot 3 \qquad 2^2 \cdot (504)$

$2018 = 2 \cdot 1009 \qquad (671+1) \cdot 3 \qquad (3+1) \cdot (503+1)$