The Ninth Grade Math Competition Class
Quadratic Formula and Polynomial
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1. Find the value of x if x is positive and $x - 1$ is the reciprocal of $x + \frac{1}{2}$.

$$x-1 = \frac{1}{x+\frac{1}{2}}$$

$$(x-1)\left(x+\frac{1}{2}\right) = 1$$

$$x^2 - x + \frac{1}{2}x - \frac{1}{2} = 1$$

$$x^2 - \frac{1}{2}x - \frac{3}{2} = 0$$

$$2x^2 - x - 3 = 0$$

$$(2x-3)(x+1) = 0$$

$$\boxed{x = \frac{3}{2}} \qquad \cancel{x = -1}$$

2. It is given that one root of $2x^2 + rx + s = 0$, with r and s real numbers, is $3 + 2i$. Find s.

$3+2i$

$3-2i$

$$(3+2i)(3-2i) = \frac{s}{2}$$

$$13 = \frac{s}{2} \Rightarrow \boxed{s = 26}$$

3. Find all values of k such that $x^2 + kx + 27 = 0$ has two distinct real solutions for x.

$$k^2 - 4 \cdot 27 \cdot 1 > 0$$

$$k^2 > 108$$

$$\boxed{k > \sqrt{108} = 6\sqrt{3}}$$

$$\boxed{k < -\sqrt{108} = -6\sqrt{3}}$$

4. Find all real solutions to $(x^2 - 5x + 5)^{x^2-9x+20} = 1$.

$1^a = 1$

$a^0 = 1$

even
$-1^a = 1$

$x^2 - 5x + 5 = 1$

$x^2 - 5x + 4 = 0$

$(x-4)(x-1) \Rightarrow \boxed{x = 1, 4}$

$x^2 - 9x + 20 = 0$

$(x-4)(x-5) \Rightarrow \boxed{x = 4, 5}$

$x^2 - 5x + 5 = -1$

$x^2 - 5x + 6 = 0$

$(x-3)(x-2) \Rightarrow \boxed{x = 2, 3}$

5. Find all real solutions (x, y) of the system $x^2 + y = 12 = y^2 + x$.

$$x^2+y=12$$
$$y^2+x=12$$
$$x^2+y=y^2+x$$
$$x^2-y^2=x-y$$
$$(x-y)(x+y)=x-y$$

$\Rightarrow x=y$

$$x^2+x=12$$
$$x^2+x-12=0$$
$$\frac{-1\pm\sqrt{1+48}}{2}$$
$$\frac{-1\pm 7}{2}$$
$$\boxed{x=3,-4}$$

or $x+y=1$

$$y=1-x$$
$$x^2+1-x=12$$
$$x^2-x-11=0$$
$$\frac{1\pm\sqrt{1+44}}{2}=\frac{1}{2}\pm\frac{\sqrt{45}}{2}=\frac{1}{2}\pm\frac{3\sqrt{5}}{2}$$

$$\boxed{\begin{array}{l}\left(\frac{1}{2}+\frac{3\sqrt{5}}{2}, \frac{1}{2}-\frac{3\sqrt{5}}{2}\right)\\ \left(\frac{1}{2}-\frac{3\sqrt{5}}{2}, \frac{1}{2}+\frac{3\sqrt{5}}{2}\right)\end{array}}$$

6. Find all values of m for which the zeros of $2x^2 - mx - 8$ differ by $m - 1$.

r, s

$$s - r = m - 1$$

$$x = \frac{m \pm \sqrt{m^2 - 4 \cdot (-8) \cdot 2}}{4}$$

$$r = \frac{m - \sqrt{m^2 + 64}}{4} \qquad s = \frac{m + \sqrt{m^2 + 64}}{4}$$

$$\frac{m + \sqrt{m^2+64}}{4} - \frac{m - \sqrt{m^2+64}}{4} = m - 1$$

$$\frac{\cancel{2}\sqrt{m^2+64}}{\cancel{4}\,2} = m - 1$$

$$\sqrt{m^2+64} = 2(m-1)$$

$$m^2 + 64 = 4(m^2 - 2m + 1)$$

$$m^2 + 64 = 4m^2 - 8m + 4$$

$$0 = 3m^2 - 8m - 60$$

$$0 = (3m + 10)(m - 6)$$

$$\Rightarrow m = 6, -\frac{10}{3}$$

$P(x)$

7. A polynomial of degree four with leading coefficient 1 and integer coefficients has two zeros, both of which are integers. Which of the following can also be a zero of the polynomial?

$(A)\frac{1+i\sqrt{11}}{2}$ $(B)\frac{1+i}{2}$ $(C)\frac{1}{2}+i$ $(D)1+\frac{i}{2}$ $(E)\frac{1+i\sqrt{13}}{2}$

(B) $r = \frac{1}{2} + \frac{i}{2}$

$s = \frac{1}{2} - \frac{i}{2}$

r, s, t, u

$P(x) = (x-r)(x-s)(x-t)(x-u)$

$P(x) = ((x-\frac{1}{2}) - \frac{i}{2})((x-\frac{1}{2}) + \frac{i}{2})(x-t)(x-u)$

$(x-\frac{1}{2})^2 + \frac{1}{4}$

$x^2 - x + \frac{1}{4} + \frac{1}{4}$

$(x^2 - x + \frac{1}{2})(x-t)(x-u)$

(A) $r = \frac{1}{2} + \frac{\sqrt{11}}{2} i$

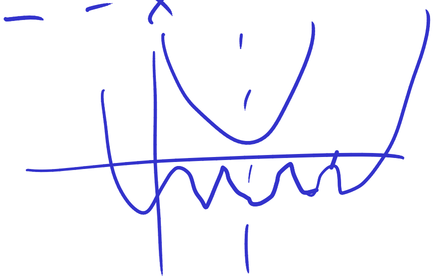
$s = \frac{1}{2} - \frac{\sqrt{11}}{2} i$

$P(x) = ((x-\frac{1}{2}) - \frac{\sqrt{11}}{2} i)((x-\frac{1}{2}) + \frac{\sqrt{11}}{2} i)(x-t)(x-u)$

$= ((x-\frac{1}{2})^2 + \frac{11}{4})(x-t)(x-u)$

$(x^2 - x + \underbrace{\frac{1}{4} + \frac{11}{4}}_{=3})(x-t)(x-u)$

$(-x)^{2001} = -x^{2001}$



8. Find the sum of all the roots of the equation $x^{2001} + (\frac{1}{2} - x)^{2001} = 0$.

a is root

$a^{2001} + (\frac{1}{2} - a)^{2001} = 0$

$x = \frac{1}{2} - a \Rightarrow (\frac{1}{2} - a)^{2001} + a^{2001} = 0$

$\frac{1}{2} - a$ is a root

deg 2000 $\Rightarrow$ 2000 roots $\Rightarrow$ 1000 pairs

Each pair sums to $a + \frac{1}{2} - a = \frac{1}{2}$

$\frac{1}{2} \cdot 1000 = \boxed{500}$

9. Three of the roots of $f(x) = x^4 + 0x^3 + ax^2 + bx + c = 0$ are $-2, -3, 5$. Find the value of $a + b + c$.

$$-\frac{0}{1} = -2 - 3 + 5 + r$$

$$0 = 0 + r \Rightarrow r = 0$$

$$f(x) = (x-0)(x-(-2))(x-(-3))(x-5)$$

$$= x(x+2)(x+3)(x-5)$$

$$f(1) = 1^4 + a \cdot 1^2 + b \cdot 1 + c = 1 + a + b + c$$

$$f(1) = 1 \cdot (3) \cdot (4) \cdot (-4) = -48$$

$$-48 = 1 + a + b + c$$

$$\Rightarrow a + b + c = \boxed{-49}$$

10. One root of the quadratic $x^2 + bx + c = 0$ is $1 - 3i$. If b and c are real numbers, then what are b and c?

$1-3i$

$1+3i$

$1-3i+1+3i = \frac{-b}{1} \Rightarrow 2 = -b$

$\boxed{b=2}$

$(1-3i)(1+3i) = \frac{c}{1}$

$1-3i+3i+9 = \frac{c}{1} \Rightarrow \boxed{c=10}$

$a+b+c = \frac{-3}{1} = -3$

$f(x) = (x-a)(x-b)(x-c)$ $\qquad$ $g(x)$

11. Suppose the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b and c, and the roots of $x^3 + rx^2 + sx + t = 0$ are $a+b$, $b+c$, and $c+a$, find the value of t.

$$\frac{(-1)^3 t}{1} = (a+b)(b+c)(c+a)$$

$$-t = (-3-c)(-3-a)(-3-b)$$

$$-t = f(-3)$$

$$-t = -27 + 27 - 12 - 11$$

$$t = 23$$

$f(x)$

12. Let a, b, and c be the roots of $x^3 - 3x^2 + 1$. $= (x-a)(x-b)(x-c)$

$g(x)$

- Find a polynomial whose roots are $a+3$, $b+3$ and $c+3$. $(x-a-3)(x-b-3)(x-c-3)$
- Find a polynomial whose roots are $\frac{1}{a+3}$, $\frac{1}{b+3}$, and $\frac{1}{c+3}$. $f(x-3) = ((x-3)-a)((x-3)-b)((x-3)-c)$
- Compute $\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3}$.
- Find a polynomial whose roots are a^2, b^2 and c^2.

$$f(x-3) = (x-3)^3 - 3(x-3)^2 + 1 = x^3 - 12x^2 + 45x - 53$$

$$x \to \frac{1}{x}$$

$$g\left(\tfrac{1}{x}\right) = \frac{1}{x^3} - \frac{12}{x^2} + \frac{45}{x} - 53$$

$$= 1 - 12x + 45x^2 - 53x^3$$

$$-\frac{45}{-53} = \frac{45}{53}$$

$$(x-a^2)(x-b^2)(x-c^2)$$

$$f(x) = (x-a)(x-b)(x-c)$$

$$f(-x) = (-x-a)(-x-b)(-x-c) = -(x+a)(x+b)(x+c)$$

$$-f(x)f(-x) = (x^2-a^2)(x^2-b^2)(x^2-c^2)$$

$$y = x^2$$

$$-(x^3-3x^2+1)(-x^3-3x^2+1) =$$

$$x^6-9x^4+6x^2-1 = (x^2-a^2)(x^2-b^2)(x^2-c^2)$$

$$y^3-9y^2+6y-1 = (y-a^2)(y-b^2)(y-c^2)$$

13. ~~The equation $2^{333x-2}+2^{111x+2}=2^{222x+1}+1$ has three real roots. Find their sum.~~

$$f(\sqrt{x}) = \underbrace{x^{\frac{3}{2}}}_{b} \underbrace{-3x+1}_{a}$$

$$f(-\sqrt{x}) = -x^{\frac{3}{2}}-3x+1$$

$$f(\sqrt{x})f(-\sqrt{x}) = (a+b)(a-b)$$

$$= a^2-b^2$$

$$= (-3x+1)^2 - x^3$$

$$= -x^3+9x^2-6x+1$$

13. The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Find their sum.

$x = a, b, c$

$a+b+c$

$2^{333x} \cdot 2^{-2} + 2^{111x} \cdot 2^{2} = 2^{222x} \cdot 2 + 1$

$\underbrace{2^{333x}}_{y^3} + \underbrace{2^{111x}}_{y} \cdot 16 = \underbrace{2^{222x}}_{y^2} \cdot 8 + 4$

$y = 2^{111x}$

$\log_2 y = 111x$

$\frac{1}{111} \log_2 y = x$

$y = r, s, t$

$y^3 - 8y^2 + 16y - 4 = 0$

$rst = \frac{-(-4)}{1}$

$a+b+c = \frac{1}{111}\log_2 r + \frac{1}{111}\log_2 s + \frac{1}{111}\log_2 t$

$= \frac{1}{111}\log_2 rst$

$= \frac{1}{111}\log_2 4 = \boxed{\frac{2}{111}}$

14. If $P(x)$ is a polynomial in x such that for all x, $x^{23} + 23x^{17} - 18x^{16} - 24x^{15} + 108x^{14} = (x^4 - 3x^2 - 2x + 9).P(x)$, compute the sum of coefficients of $P(x)$.

15. The real number x satisfies the equation $x + \frac{1}{x} = \sqrt{5}$. What is the value of $x^{11} - 7x^7 + x^3$?

16. All the roots of the polynomial $x^6 - 10z^5 + Az^4 + Bz^3 + cZ^2 + Dz + 16$ are positive integers, possibly repeated. What is the value of B?