The Ninth Grade Math Competition Class
Quadratic Formula and Polynomial
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1. Find the value of x if x is positive and $x - 1$ is the reciprocal of $x + \frac{1}{2}$.

$$x-1 = \frac{1}{x+\frac{1}{2}}$$

$$(x-1)\left(x+\frac{1}{2}\right) = 1$$

$$x^2 - x + \frac{1}{2}x - \frac{1}{2} = 1$$

$$x^2 - \frac{1}{2}x - \frac{3}{2} = 0$$

$$2x^2 - x - 3 = 0$$

$$(2x-3)(x+1) = 0$$

$$x = \frac{3}{2} \qquad \cancel{x=-1}$$

2. It is given that one root of $2x^2 + rx + s = 0$, with r and s real numbers, is $3 + 2i$. Find s.

$3+2i$

$3-2i$

$$(3+2i)(3-2i) = \frac{s}{2}$$

$$13 = \frac{s}{2} \Rightarrow \boxed{s = 26}$$

3. Find all values of k such that $x^2 + kx + 27 = 0$ has two distinct real solutions for x.

$$k^2 - 4 \cdot 27 \cdot 1 > 0$$

$$k^2 > 108$$

$$\boxed{k > \sqrt{108} = 6\sqrt{3}}$$

$$\boxed{k < -\sqrt{108} = -6\sqrt{3}}$$

4. Find all real solutions to $(x^2 - 5x + 5)^{x^2-9x+20} = 1$.

$1^a = 1$

$a^0 = 1$

even

$-1^a = 1$

$x^2 - 5x + 5 = 1$

$x^2 - 5x + 4 = 0$

$(x-4)(x-1) \Rightarrow \boxed{x = 1, 4}$

$x^2 - 9x + 20 = 0$

$(x-4)(x-5) \Rightarrow \boxed{x = 4, 5}$

$x^2 - 5x + 5 = -1$

$x^2 - 5x + 6 = 0$

$(x-3)(x-2) \Rightarrow \boxed{x = 2, 3}$

5. Find all real solutions (x, y) of the system $x^2 + y = 12 = y^2 + x$.

$x^2 + y = 12$

$y^2 + x = 12$

$x^2 + y = y^2 + x$

$x^2 - y^2 = x - y$

$(x-y)(x+y) = x - y$

$\Rightarrow x = y$

$x^2 + x = 12$

$x^2 + x - 12 = 0$

$\frac{-1 \pm \sqrt{1+48}}{2}$

$\frac{-1 \pm 7}{2}$

$\boxed{x = 3, -4}$

or $x + y = 1$

$y = 1 - x$

$x^2 + 1 - x = 12$

$x^2 - x - 11 = 0$

$\frac{1 \pm \sqrt{1+44}}{2} = \frac{1}{2} \pm \frac{\sqrt{45}}{2} = \frac{1}{2} \pm \frac{3\sqrt{5}}{2}$

$$\boxed{\begin{array}{l} \left(\frac{1}{2} + \frac{3\sqrt{5}}{2}, \frac{1}{2} - \frac{3\sqrt{5}}{2}\right) \\ \left(\frac{1}{2} - \frac{3\sqrt{5}}{2}, \frac{1}{2} + \frac{3\sqrt{5}}{2}\right) \end{array}}$$

6. Find all values of m for which the zeros of $2x^2 - mx - 8$ differ by $m - 1$.

r, s

$$s - r = m - 1$$

$$x = \frac{m \pm \sqrt{m^2 - 4\cdot(-8)\cdot 2}}{4}$$

$$r = \frac{m - \sqrt{m^2+64}}{4} \qquad s = \frac{m + \sqrt{m^2+64}}{4}$$

$$\frac{m + \sqrt{m^2+64}}{4} - \frac{m - \sqrt{m^2+64}}{4} = m - 1$$

$$\frac{\cancel{2}\sqrt{m^2+64}}{\cancel{4}\ 2} = m - 1$$

$$\sqrt{m^2+64} = 2(m-1)$$

$$m^2 + 64 = 4(m^2 - 2m + 1)$$

$$m^2 + 64 = 4m^2 - 8m + 4$$

$$0 = 3m^2 - 8m - 60$$

$$0 = (3m+10)(m-6)$$

$$\Rightarrow m = 6, -\frac{10}{3}$$

$P(x)$

7. A polynomial of degree four with leading coefficient 1 and integer coefficients has two zeros, both of which are integers. Which of the following can also be a zero of the polynomial?

$(A)\frac{1+i\sqrt{11}}{2}$ $(B)\frac{1+i}{2}$ $(C)\frac{1}{2}+i$ $(D)1+\frac{i}{2}$ $(E)\frac{1+i\sqrt{13}}{2}$

(B) $r = \frac{1}{2} + \frac{i}{2}$

$s = \frac{1}{2} - \frac{i}{2}$

r, s, t, u

$P(x) = (x-r)(x-s)(x-t)(x-u)$

$P(x) = \left(\left(x-\frac{1}{2}\right)-\frac{i}{2}\right)\left(\left(x-\frac{1}{2}\right)+\frac{i}{2}\right)(x-t)(x-u)$

$\left(x-\frac{1}{2}\right)^2 + \frac{1}{4}$

$x^2 - x + \frac{1}{4} + \frac{1}{4}$

$\left(x^2 - x + \frac{1}{2}\right)(x-t)(x-u)$

(A) $r = \frac{1}{2} + \frac{\sqrt{11}}{2}i$

$s = \frac{1}{2} - \frac{\sqrt{11}}{2}i$

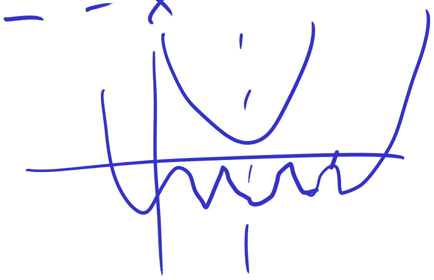
$P(x) = \left(\left(x-\frac{1}{2}\right)-\frac{\sqrt{11}}{2}i\right)\left(\left(x-\frac{1}{2}\right)+\frac{\sqrt{11}}{2}i\right)(x-t)(x-u)$

$= \left(\left(x-\frac{1}{2}\right)^2 + \frac{11}{4}\right)(x-t)(x-u)$

$\left(x^2 - x + \underbrace{\frac{1}{4}+\frac{11}{4}}_{=3}\right)(x-t)(x-u)$

$(-x)^{2001} = -x^{2001}$

8. Find the sum of all the roots of the equation $x^{2001} + (\frac{1}{2} - x)^{2001} = 0$.



a is root

$a^{2001} + (\frac{1}{2} - a)^{2001} = 0$

$x = \frac{1}{2} - a \Rightarrow (\frac{1}{2} - a)^{2001} + a^{2001} = 0$

$\frac{1}{2} - a$ is a root

deg 2000 $\Rightarrow$ 2000 roots $\Rightarrow$ 1000 pairs

Each pair sums to $a + \frac{1}{2} - a = \frac{1}{2}$

$\frac{1}{2} \cdot 1000 = \boxed{500}$

9. Three of the roots of $x^4 + ax^2 + bx + c = 0$ are $-2,\ -3,\ 5$. Find the value of $a + b + c$.

10. One root of the quadratic $x^2 + bx + c = 0$ is $1 - 3i$. If b and c are real numbers, then what are b and c?

$1 - 3i$

$1 + 3i$

$1 - 3i + 1 + 3i = \frac{-b}{1} \Rightarrow 2 = -b$

$(1-3i)(1+3i) = \frac{c}{1}$

$1 - 3i + 3i + 9 = \frac{c}{1} \Rightarrow$

$$\boxed{\begin{array}{l} b = 2 \\ c = 10 \end{array}}$$

11. Suppose the roots of $x^3 + 3x^2 + 4x - 11 = 0$ are a, b and c, and the roots of $x^3 + rx^2 + sx + t = 0$ are $a + b$, $b + c$, and $c + a$, find the value of t.

12. Let a, b, and c be the roots of $x^3 - 3x^2 + 1$.

- Find a polynomial whose roots are $a + 3$, $b + 3$ and $c + 3$.
- Find a polynomial whose roots are $\frac{1}{a+3}$, $\frac{1}{b+3}$, and $\frac{1}{c+3}$.
- Compute $\frac{1}{a+3} + \frac{1}{b+3} + \frac{1}{c+3}$.
- Find a polynomial whose roots are a^2, b^2 and c^2.

13. The equation $2^{333x-2} + 2^{111x+2} = 2^{222x+1} + 1$ has three real roots. Find their sum.

14. If $P(x)$ is a polynomial in x such that for all x, $x^{23} + 23x^{17} - 18x^{16} - 24x^{15} + 108x^{14} = (x^4 - 3x^2 - 2x + 9).P(x)$, compute the sum of coefficients of $P(x)$.

15. The real number x satisfies the equation $x + \frac{1}{x} = \sqrt{5}$. What is the value of $x^{11} - 7x^7 + x^3$?

16. All the roots of the polynomial $x^6 - 10z^5 + Az^4 + Bz^3 + cZ^2 + Dz + 16$ are positive integers, possibly repeated. What is the value of B?