Prime Factorization

Primes: numbers with no divisors except 1 and itself

7   2, 3, 4, 5, 6

$\sqrt{7}$

$7 = ab$

$8 = ab$ $25 = a \cdot b$

$1 \cdot 8,\ 2 \cdot 4,\ 4 \cdot 2,\ 8 \cdot 1$ $1 \cdot 25,\ 5 \cdot 5,\ 25 \cdot 1$

You only need to check numbers up to $\sqrt{n}$

Q: Is 1 prime?

No, 1 is not prime or composite

Ex: Is 209 prime?

$209 = 11 \cdot 19$

$\sqrt{209} \approx 14$

2, 3, 5, 7, 11, 13

Ex: What's the smallest composite # that has no prime factors < 10?

$121 = 11 \cdot 11$

Prime factorization: $72 = 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 = 2^3 \cdot 3^2$

Def: LCM - Least common multiple

$\mathrm{lcm}(8, 12) = 24$

$2^3 \cdot 3^0 \quad 2^2 \cdot 3^1 \quad 2^3 \cdot 3^1$

$\mathrm{lcm}(12, 30) = 60$

$2^2 \cdot 3^1 \cdot 5^0 \quad 2^1 \cdot 3^1 \cdot 5^1 \quad 2^2 \cdot 3^1 \cdot 5^1$

Def: GCD - Greatest Common Divisor

$\gcd(12, 45) = 3$

$2^2 \cdot 3^1 \cdot 5^0 \quad 2^0 \cdot 3^2 \cdot 5^1 \quad 2^0 \cdot 3^1 \cdot 5^0$

$\gcd(\underbrace{4}_{2^2 \cdot 3^0}, \underbrace{6}_{2^1 \cdot 3^1})$, $\gcd(4,6) = 2^1 \cdot 3^0$, $\mathrm{lcm}(4,6) = 2^2 \cdot 3^1$

$\gcd(4, 6) \cdot \mathrm{lcm}(4,6) = 24 = 4 \cdot 6$

$\gcd(8, 12) \cdot \mathrm{lcm}(8,12) = 96$

$\gcd(6, 8) \cdot \mathrm{lcm}(6,8) = 48$

$\overbrace{\gcd(4,6)}^{2^1 \cdot 3^0} \cdot \overbrace{\mathrm{lcm}(4,6)}^{2^2 \cdot 3^1} = \overbrace{4}^{2^2 \cdot 3^0} \cdot \overbrace{6}^{2^1 \cdot 3^1}$

$5^1 \qquad 5^1$

$$\boxed{\gcd(a,b) \cdot \mathrm{lcm}(a,b) = ab}$$

Ex: $\gcd(70, n) = 10 \qquad \mathrm{lcm}(70, n) = 210$

$\gcd(70, n) \cdot \mathrm{lcm}(70, n) = 10 \cdot 210 = 70n$

$\boxed{\Rightarrow n = 30}$

$\gcd(ac, bc) = c \cdot \gcd(a,b)$
$\operatorname{lcm}(ac, bc) = c \cdot \operatorname{lcm}(a,b)$

$$\gcd(\underbrace{800}_{100\cdot 8}, \underbrace{1800}_{100\cdot 18}) = 100 \cdot \underbrace{\gcd(8,18)}_{2} = \boxed{200}$$

Q: How many divisors does a certain # have?

12 $1, 2, 3, 4, 6, 12$

$2^2 \cdot 3^1$ $2^0 3^0$, $2^1 3^0$, $2^0 3^1$, $2^2 3^0$, $2^1 3^1$, $2^2 3^1$

$d = 2^a 3^b$ 2^a $0 \le a \le 2$ $a = 0, 1, 2$ 3

divisor of 12 3^b $0 \le b \le 1$ $b = 0, 1$ 2

$2^a 3^b$ 6

3

	2^0	2^1	2^2
3^0	1	2	4
3^1	3	6	12

2

$$n = \underbrace{p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}}_{\text{prime factorization}}$$

$p_1^0 \, p_1^1 \, p_1^2 \dots p_1^{e_1}$

d $p_1^{a_1}$ $0 \le a_1 \le e_1$ $0, 1, 2 \dots e_1$ $e_1 + 1$

d is div. of n $p_2^{a_2}$ $0 \le a_2 \le e_2$ $e_2 + 1$

$\vdots$ $\vdots$

$p_k^{a_k}$ $e_k + 1$

Total # of div. $t(n) = (e_1+1)(e_2+1)\cdots(e_k+1)$

Ex: # of div. of 5400

$$5400 = 2^3 \cdot 3^3 \cdot 5^2$$

$$(3+1)(3+1)(2+1) = \boxed{48}$$

$$\boxed{\begin{array}{l} \text{If } \gcd(m,n) = 1 \\ t(mn) = t(m)\,t(n) \end{array}}$$

$m = 16 = 2^4$ $\quad t(m) = 5$

$n = 21 = 3^1 \cdot 7^1$ $\quad t(n) = 4$

$mn = 2^4 \cdot 3^1 \cdot 7^1$ $\quad t(mn) = 20$

Q: which #'s have an odd of divisors?

$$n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$$

$$t(n) = \underbrace{(\overset{\text{even}}{e_1}+1)}_{\text{odd}}\underbrace{(\overset{\text{even}}{e_2}+1)}_{\text{odd}} \cdots \underbrace{(\overset{\text{even}}{e_k}+1)}_{\text{odd}} \qquad \text{odd}$$

$$n = p_1^{2f_1} p_2^{2f_2} \cdots p_k^{2f_k}$$

$$n = (p_1^{f_1} p_2^{f_2} \cdots p_k^{f_k})^2$$