R. Bergmann, M. Ghiradelli, L. Rahm

Submission Deadline:
Tuesday 3. October 2023, 17:00

Project #1

18. September 2023

Notes

There you also finde a `.zip`-file on Blacboard containing `.csv`-files and images, which should be extracted in the same folder as your Jupyter notebook.

This project is an individual project and should be solved that way. You can discuss and work on the problems with other students, but you should write the solutions *in your own words and code.*
Your answer should be contained in a Jupyter notebook that will be uploaded to Inspera.
The project also comes with a `.zip`-file containing a few files, `.csv`-files and images, which should be placed in the same folder as your Jupyter Notebook.
All code —also the tests if necessary— should be in individual cells that can just be run (as soon as the necessary functions are defined). Functions should only be used in cells *after* their definition, such that an evaluation in order of the notebook runs without errors. Try to always explain what a certain code section / cell does *before* the cell, and discuss the results *after* the chunk.

The project is counts 10% on the final grade, is in a group with the second project (20%). and overall you have to pass this part of the exam.

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Introduction

In this project we want to consider numerical methods for signal and image processing that are based on Fourier series. We will first consider the Fourier Transform of *periodic functions* and *periodic signals*. A (complex-valued) function $f\colon \mathbb{R} \to \mathbb{C}$ is periodic with periodic T if

$$f(x) = f(x+T) \qquad \text{holds for all } x \in \mathbb{R}.$$

For such a function we just need to know one *period*, for example $[-\frac{T}{2}, \frac{T}{2})$ or $[0, T)$. Note that these intervals are *half-open*, since e. g. $f(0) = f(T)$ and we only need one of these values to determine f. We will consider periodic functions with a period of $T = 1$. We refer to this period as write $\mathbb{T} \equiv [0, 1)$. We further collect all *square integrable* functions in the set

$$L^2(\mathbb{T}) := \left\{ f\colon \mathbb{T} \to \mathbb{R} \,\middle|\, \int_0^1 |f(x)|^2 \,\mathrm{d}x < \infty \right\}.$$

Based on these functions we can create a vector space consisting of functions. We first define an *inner product* for two functions $f, g \in L^2(\mathbb{T})$ by

$$\langle f\,, g\rangle = \int_0^1 f(x)\overline{g(x)}\,\mathrm{d}x,$$

where $\overline{a}$ denotes the complex conjugate of a complex number $a \in \mathbb{C}$. With a few more details (concerning integration and the Lebesgue measure) we get a norm $\|f\| = \sqrt{\langle f\,, f\rangle} < \infty$ from this inner product as well if we build suitable congruence classes.
Note that in both cases we could —similar to the note above— also integrate from $-\frac{1}{2}$ to $\frac{1}{2}$ instead, which we can also refer to with $\mathbb{T}$ (because the integral is still the same due to periodicity).

Task 1: The (Discrete) Fourier Transform

a) We consider the functions $\mathrm{e}^{2\pi i k x}$, $k \in \mathbb{Z}$, $x \in \mathbb{T}$. Prove that for any $k, h \in \mathbb{Z}$ we have

$$\langle \mathrm{e}^{2\pi \mathrm{i} k \cdot}\,, \mathrm{e}^{2\pi \mathrm{i} h \cdot} \rangle = \begin{cases} 1 & \text{if } k = h \\ 0 & \text{else.} \end{cases}$$

Notation. Since similar to the inner product definition above, we would like to avoid the variable x in the inner product, we write a $\cdot$ instead. This emphasizes that we plug in the complete function (and not the function evaluated at some x). So the best way to "read" $\cdot$ is a variable, that we did not assign a name to, because it is not so important in that place. We do the same with functions when we write just $f = f(\cdot)$ and $g = g(\cdot)$, just that we even leave out the variable completely.

b) We consider the functions of the form $\sqrt{2}\sin(2\pi mx)$, $m = 1, 2, \ldots$, $\cos(2\pi 0x)$ and $\sqrt{2}\cos(2\pi nx)$, $n = 1, 2, \ldots$, $x \in \mathbb{T}$. Prove that for these functions form an orthonormal system, i. e. we have

- $\langle \sqrt{2}\sin(2\pi n\cdot)\,, \sqrt{2}\cos(2\pi m\cdot)\rangle = 0$, $n \in \{1, 2, \ldots\}$, $m \in \{0, 1, \ldots\}$
- $\langle \sqrt{2}\sin(2\pi n\cdot)\,, \sqrt{2}\sin(2\pi m\cdot)\rangle = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n, \end{cases} \qquad m, n \in \{1, 2, \ldots\}$
- $\langle \sqrt{2}\cos(2\pi n\cdot)\,, \sqrt{2}\cos(2\pi m\cdot)\rangle = \begin{cases} 0 & \text{if } m \neq n, \\ 1 & \text{if } m = n \neq 0, \\ 2 & \text{if } m = n = 0, \end{cases} \qquad m, n \in \{0, 1, \ldots, \}$

c) We introduce the two spaces. The first one is

$$\mathcal{T}_n := \operatorname{span}(\mathrm{e}^{-2\pi \mathrm{i} n\cdot}, \ldots, \mathrm{e}^{2\pi \mathrm{i} n\cdot}) = \Big\{ f \,\Big|\, f(x) = \sum_{k=-n}^{n} c_k \mathrm{e}^{2\pi \mathrm{i} kx}, \quad \text{where } c_{-n}, c_{-n+1}, \ldots, c_n \in \mathbb{C}, \Big\},$$

where we further restrict the coefficients to $c_k = \overline{c_{-k}}$, $k = 0, \ldots, n$, where $\overline{z}$ denotes the complex conjugate of a complex number $z \in \mathbb{C}$. The second one is

$$\begin{aligned} \mathcal{S}_n &:= \operatorname{span}(\cos(0\cdot), \cos(2\pi\cdot), \ldots, \cos(2\pi n\cdot), \sin(2\pi\cdot), \sin(2\pi 2\cdot), \ldots, \sin(2\pi n\cdot)) \\ &= \Big\{ f \,\Big|\, f(x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos(2\pi kx) + b_k \sin(2\pi kx), \text{ where } a_0, a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R} \Big\} \end{aligned}$$

Use the results from Item a) and Item b) to find orthonormal bases for these spaces. User Euler's identity to argue that both spaces are the same, i.e. $\mathcal{T}_n = \mathcal{S}_n$. What is the dimension of $\mathcal{T}_n$?

d) Use the representation of $\mathcal{S}_n$ from Item c) to prove that the *Fourier coefficients* $a_0, a_1, \ldots, a_n, b_1, \ldots, b_n$ of a function $f \in \mathcal{S}_n$ can be computed as

$$a_k = 2\langle f\,, \cos(2\pi k\cdot)\rangle = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\cos(2\pi kx)\,\mathrm{d}x, \qquad k = 0, 1, \ldots, n$$

and

$$b_k = 2\langle f\,, \sin(2\pi k\cdot)\rangle = 2\int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\sin(2\pi kx)\,\mathrm{d}x, \qquad k = 1,\ldots,n,$$

where we might write $a_k(f)$ and $b_k(f)$ to emphasize that these are the coefficients belonging to f.
Remarks.

- The same way works for the space $\mathcal{T}_n$ where

$$c_k = c_k(f) = \langle f\,, \mathrm{e}^{2\pi\mathrm{i}k\cdot}\rangle = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\mathrm{e}^{-2\pi\mathrm{i}kx}\,\mathrm{d}x, \qquad k = -n,\ldots,n. \tag{1}$$

  which is more often used, since we just have to keep track of one set of coefficients. These are also referred to as *Fourier coefficients.*

- We can actually compute these coefficients for arbitrary $k \in \mathbb{Z}$ and for an arbitrary periodic function f. Computing $c_{-n}(f),\ldots,c_n(f)$ for an arbitrary function, we obtain the *best approximation* $f_n \in \mathcal{T}_n$, i. e. $f_n = \operatorname*{arg\,min}_{g\in\mathcal{T}_n}\lVert f - g\rVert$.

- For $f, g \in L_2(\mathbb{T})$ the *Parseval identity* holds, that is

$$\langle f\,, g\rangle = \sum_{k\in\mathbb{Z}} c_k(f)\overline{c_k(g)}. \tag{2}$$

e) We want to use equidistant points $x_0,\ldots,x_{N-1}$, $x_j = \frac{j}{N}$, $j = 0,\ldots,N$, for some $N \in \mathbb{N}$ to approximate the integral required for the Fourier coefficients $c_k(f)$ of a function f from (1). We introduce for ease of notation $f_j = f(x_j)$ and $\boldsymbol{f} = (f_0,\ldots,f_{N-1})$.
Show that using the composite trapezoidal rule, we obtain

$$c_k(f) \approx \hat{f}_k := \frac{1}{N}\sum_{j=0}^{N-1} f_j\mathrm{e}^{-2\pi\mathrm{i}jk/N}.$$

Show further, that the $\hat{f}_k$ are N periodic, that is $\hat{f}_k = \hat{f}_{k+N}$ for all $k \in \mathbb{Z}$.
What does that mean for the approximation in the equation above?

Remark. The coefficients $\hat{\boldsymbol{f}} = (\hat{f}_0,\ldots,\hat{f}_{N-1})$ are called the *Discrete Fourier Transform (DFT)* of f. There exist fast transforms, we will see in the lecture, called the *Fast Fourier Transform (FFT)* to compute these

It is easier for this task to consider the Fourier coefficients on the interval $[0, 1)$ instead of the symmetric interval $[-\frac{1}{2}, \frac{1}{2}]$.

f) Let $N \in \mathbb{N}$ and $k \in \mathbb{Z}$ be given. Prove that

$$\frac{1}{N}\sum_{j=0}^{N-1} \mathrm{e}^{-2\pi \mathrm{i} jk/N} = \begin{cases} 1 & \text{if } k \bmod N \equiv 0,\\ 0 & \text{else.}\end{cases}$$

g) We collect the discrete Fourier transform into a matrix, that is we want to write $\hat{\boldsymbol{f}} = \frac{1}{N}\mathcal{F}_N\boldsymbol{f}$ with $\mathcal{F} \in \mathbb{C}^{N\times N}$ given by

$$\mathcal{F}_N = \left(\mathrm{e}^{-2\pi ikl/N}\right)_{k,l=0}^{N-1}$$

We further introduce for a vector $\boldsymbol{a} = (a_0, ..., a_{N-1})^{\mathrm{T}}$ the *circulant matrix*

$$\operatorname{circ}\boldsymbol{a} = \left(a_{k-l \bmod N}\right)_{k,l=0}^{N-1} = \begin{pmatrix} a_0 & a_{N-1} & \cdots & a_2 & a_1\\ a_1 & a_0 & \cdots & a_3 & a_2\\ \vdots & & \ddots & & \vdots\\ a_{N-1} & a_{N-2} & \cdots & a_1 & a_0\end{pmatrix}$$

Prove that the Fourier matrix $\mathcal{F}_N$ diagonalizes the circulant matrix, i. e. using $\hat{\boldsymbol{a}} = \mathcal{F}_N\boldsymbol{a}$ we get

$$\operatorname{circ}\boldsymbol{a} = \frac{1}{N}\overline{\mathcal{F}_N}\operatorname{diag}(\hat{\boldsymbol{a}})\mathcal{F}_N,$$

where $\operatorname{diag}(\cdot)$ denotes the diagonal matrix.

Hint. Consider a single entry of the result from the matrix product on the right and use the result from Item f)

Derive further a formula for the inverse $\mathcal{F}_N^{-1}$ using the just proven property with the help of 1f).

h) Write a function `transform(f,N,start=0.0)` that takes a function f as its first parameter and the number of samples N as its second and returns the vector $\boldsymbol{f} = (f_0, \ldots, f_{N-1})^{\mathrm{T}}$ of function values. The optional third parameter `start` would by default be zero, and state where to start the sampling. This way the function can perform sampling on both intervals we discussed until now. In this task we want to use this to compute the DFT $\hat{\boldsymbol{f}}$ of $\boldsymbol{f}$, which is available as `scipy.fft.fft`.

Consider the following functions defined on $\mathbb{T} = [-\frac{1}{2}, \frac{1}{2})$ as periodic functions (by periodic continuation),

- $f_1(x) = \sin(8\pi x), \qquad x \in \mathbb{T},$
- $f_2(x) = \sin(32\pi x) + \cos(128\pi x), \qquad x \in \mathbb{T},$
- $f_3(x) = x, \qquad x \in \mathbb{T},$
- $f_4(x) = 1 - |x|, \quad x \in \mathbb{T},$

with $N = 5, 17, 257$ and plot $\boldsymbol{f}$ and $\hat{\boldsymbol{f}}$ for these functions side by side. Take into account that these signals might be complex.

Note that the $\hat{f}_k$ are complex values, so you have to plot the real and imaginary parts in the same plot in different colors.

For which cases does $\boldsymbol{f}$ approximate f well?

i) Take f_2 from Item h) and plot its discrete Fourier coefficients $\hat{\boldsymbol{f}}$ after applying `fftshift`. Plot the result again for $N = 17, 65, 257$.

- State $a_k(f_2)$ and $b_k(f_2)$ without solving an integral.
- Compute $c_k(f_2)$ (again without computing the integral)
- What does the `fftshift`-function do?
- Using Euler's identity – Can you use the coefficients to "remove" the second summand of f_2 by just modifying $\hat{\boldsymbol{f}} \in \mathbb{C}^{257}$
- Can you do the same for the case $N = 17$, i. e. $\hat{\boldsymbol{f}} \in \mathbb{C}^{17}$? Try to find a reason for your answer.

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NTNU

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Task 2: Signal Processing

In this task we want to consider signal processing tasks. These appear for example when processing audio signals. We write signals as vectors, i.e. $\boldsymbol{a} = (a_0, \ldots, a_{N-1}) \in \mathbb{R}^N$ and would like to work with the, in frequency domain $\hat{\boldsymbol{a}} = \frac{1}{N}\mathcal{F}_N\boldsymbol{a}$.

We can also generalize convolution to these signals. Since we consider them as samples from a periodic function in this project, first remember that for two functions $f, g \in L^2(\mathbb{T})$ the (periodic) convolution reads

$$(f * g)(x) = \int_{\mathbb{T}} f(y)g(x-y)\,\mathrm{d}x = \int_0^1 f(y)g(x-y)\,\mathrm{d}y$$

We can do the same for our signal $\boldsymbol{a}$, but we then have to take into account the (assumed) periodicity of the signal. namely for the function $g(x-y)$ any value outside of $[0,1]$ can just be found my considering periodicity. For $\boldsymbol{a}$ we have to do this explicitly: Let $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^N$. Then the *cyclic convolution* of $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined entry wise for all $j = 0, \ldots, N-1$ by

$$(\boldsymbol{a} * \boldsymbol{b})_j = \sum_{k=0}^{N-1} a_k b_{j-k \bmod N},$$

where $m \bmod N$ is the nonnegative residue modulo N, i.e. the unique number $n \in \{0, \ldots, N-1\}$ such that there exists a multiple $p \in \mathbb{Z}$ with $m = n + pN$ – or in other words that N is a divisor of $m - n$.

If $\boldsymbol{a}$ is our input signal and we use $\boldsymbol{b}$ to modify $\boldsymbol{a}$ using convolution, $\boldsymbol{b}$ is usually called a *filter* and the convolution is then called to “filter a signal”

a) What happens with $\boldsymbol{c} = \boldsymbol{a} * \boldsymbol{b}$ if we use a shifted version $\boldsymbol{b}' = (b_{N-1}, b_0, \ldots, b_{N-2})$ instead, i.e. $\boldsymbol{c}' = \boldsymbol{a} * \boldsymbol{b}'$?

b) Prove that for $f, g \in L_1(\mathbb{T})$ (i.e. you can use Fubini) we have that for any $k \in \mathbb{Z}$ that

$$c_k(f * g) = c_k(f)c_k(g)$$

and similarly for signals, we have that (again using from Task 1, Item g), i. e. $\hat{\boldsymbol{a}} = \mathcal{F}_N\boldsymbol{a}$)

$$(\boldsymbol{a} * \boldsymbol{b})\hat{} = \hat{\boldsymbol{a}} \circ \hat{\boldsymbol{b}},$$

where $\circ$ denotes the element wise multiplication, i. e. for two signals $\boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^N$ we have $(\boldsymbol{c} \circ \boldsymbol{d})_j = c_j d_j$, $j = 0, \ldots, N-1$.

Or summarized: *Convolution in time turns into multiplication in frequency domain.*

How does this help/simplify to compute $(\operatorname{circ} \boldsymbol{a})(\operatorname{circ} \boldsymbol{b})$?

c) We want to use sample values of the Dirichlet kernel

$$D_n(x) = 1 + 2\sum_{k=1}^{n} \cos(2\pi k x), n \in \mathbb{N}.$$

Compute the Fourier coefficients $c_k(D_n)$ and state why, when we want to use the equidistant samples $d_j := D_n(\frac{j}{N})$, $j = 0, \ldots, N-1$ as a filter, it is better/easier to define it directly in the discrete Fourier Domain using $\hat{\boldsymbol{d}}$.

d) As a first example – consider $f_2(x) = \sin(32\pi x) + \cos(128\pi x)$ as in Task 1, h), sample this function with $N = 512$ samples. Now we can use the results from before to convolve this samples signal with the Dirichlet kernel D_n, $n = 48$, by sampling this one as well. Note that instead of sampling in time, you can also directly work in frequencies. What happens to the signal after convolution?

Hint. The (fast) Fourier transform of course always works on complex numbers. Here it is enough, to consider the real-valued part of the signal after convolution.

e) You are given the signal from `project1-1e-data.csv`.
We want to convolve this signal with two kernels: The Dirichlet kernel (to be precise: its sampled / discretized variant) D_{92} and the kernel (signal to convolve with) $\boldsymbol{h} = \left(-1,\ 2,\ -1, 0, \ldots, 0\right)^{\mathrm{T}} \in \mathbb{R}^N$, where N is the length of the signal from the file.

What do the results of both filter show. How can you use these to analyse the signal? Check this for each interval.

f) How could we get a filter that does "the opposite of Dirichlet"?

g) If we know a certain set of frequencies, say for the (not yet sampled function) $c_k(f)$ $40 \leq |k| \leq 64$ of interest. How can we "design" a filter to only have these left in our signal?

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Task 3: Image Processing

We now extend these ideas from signals to images. For functions this means, that we consider $f\colon \mathbb{T}^2 \to \mathbb{C}$, i.e. functions $f(\boldsymbol{x})$ that are T-periodic, $T = 1$, in both their arguments. For the sampling points we consider $x_{j_1,j_2} = \bigl(\frac{j_1}{N_1}, \frac{j_2}{N_2}\bigr)$, $j_i = 0,\ldots,N_i-1$, $i = 1,2$. We obtain a sampled *image* $F = \bigl(f(x_{j_1,j_2})\bigr)_{j_1,j_2=0}^{N_1-1,N_2-1} \in \mathbb{R}^{N_1\times N_2}$. Each entry is a *pixel* and if all values are between —say— 0 (black) and 1 (white), we obtain a gray scale image.

We can define the multivariate Fourier transform then as

$$\hat{F}_{k_1,k_2} := \sum_{j_1=0}^{N_1-1}\sum_{j_2=0}^{N_2-1} F_{j_1,j_2}\mathrm{e}^{-2\pi\mathrm{i}\left(j_1k_1/N_1 + j_2k_2/N_2\right)} \qquad k_1 = 0,\ldots,N_1-1, k_2 = 0,\ldots,N_2-1, \tag{3}$$

Sometimes it is a bit nicer to write using vectors $\boldsymbol{k} = (k_1,k_2)^{\mathrm{T}}$, $\boldsymbol{j} = (j_1,j_2)^{\mathrm{T}}$, $\boldsymbol{N} = (N_1,N_2)^{\mathrm{T}}$ as

$$\hat{F}_{\boldsymbol{k}} = \sum_{\boldsymbol{j}=(0,0)}^{(N_1,N_2)} F_{\boldsymbol{j}}\mathrm{e}^{-2\pi\mathrm{i}\boldsymbol{j}^{\mathrm{T}}((\operatorname{diag}(\boldsymbol{N}))^{-1}\boldsymbol{k})}.$$

We furthermore know from the 1D case that the Fourier transform (in its fast form) $\boldsymbol{f} = \mathcal{F}_N\boldsymbol{f} \in \mathbb{C}^N$ can be computed in $N\log N$ (see e.g. the lecture or Chapter 5 in Plonka et al. 2018). For this task you may assume that this property is given.

a) Using 1D Fourier transforms, prove that Equation (3) can be computed in the same fast way, namely in $\mathcal{O}(N_1N_2\log(N_1N_2))$

b) Familiarize yourself a bit with the multivariate case by considering the function

$$f(\boldsymbol{x}) = 1 + \frac{1}{2}\sin(2\pi\boldsymbol{x}^{\mathrm{T}}\boldsymbol{k}), \qquad \boldsymbol{x}\in[0,1]^2$$

for the three cases $\boldsymbol{k} \in \{(5,0)^{\mathrm{T}}, (0,10)^{\mathrm{T}}, (8,8)^{\mathrm{T}}\}$.
The values are chosen such that $f(\boldsymbol{x}) \in [0,1]$.
Plot these (as images) F for $N_1 = N_2 = 64$ and their amplitude $|\hat{F}|$, i. e. the absolute value of the discrete Fourier transform. For the spectrum, mention the range of value but for plotting map that into $[0,1]$ to get a gray-scale image.

c) Load the images `barbara.gif` or `klaus.gif`.
We can adapt the filters from Item e) to the 2D case, by defining for example filters like

$$D_{\boldsymbol{N}}(\boldsymbol{x}) = D_{N_1}(x_1)D_{N_2}(x_2)$$

for the 2D Dirichlet kernel and the three variants

$$\begin{pmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} -1 & 2 & -1 \\ -1 & 2 & -1 \\ -1 & 2 & -1 \end{pmatrix}, \qquad \begin{pmatrix} -1 & -1 & -1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{pmatrix}$$

of course extended to the same size of the image with zeros, to make the convolution easier. Theoretically advanced ones are

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ -1 & -2 & -1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & -1 \\ 2 & 0 & -2 \\ 1 & 0 & -1 \end{pmatrix}.$$

Use these and the 2D Fourier transform to do a 2D variant of the convolution here as well. What do the (real parts of) the convolutions look like? Interpret the results for (at least the first three of) the 3×3 filters above and the Dirichlet Kernel $D_{\boldsymbol{N}}$ with $\boldsymbol{N} = (64, 64)^{\mathrm{T}}$.

d) Consider the image `Yarimton.png`. It is a usual image from a newspaper that is printed with so-called “half-toning”. Compute and plot its amplitude (or better the logarithm of the amplitude). Find a way to remove the dot artefacts using the Fourier transform. Do the same with the lighthouse and the Munkholmen image.

Remark. The inverse – that is to generate good half-toning (or dithering-based) images is also not-trivial, see for example `DitherPunk.jl` at `https://github.com/JuliaImages/DitherPunk.jl`.

References

Plonka, Gerlind, Daniel Potts, Gabriele Steidl, and Manfred Tasche (2018). *Numerical Fourier Analysis*. Springer International Publishing. DOI: `10.1007/978-3-030-04306-3`.