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Problem 1: Urgent care center

a)

The conditions of a queueing system to be a M/M/1 queue with arrival rate $\lambda > 0$ and expected service time $\frac{1}{\mu} > 0$ are as follows:

1. The interarrival times are iid $Exp(\lambda)$.
2. The service times are distributed $Exp(\mu)$.
3. There is one server. Service times are independent of the arrival process.

The arrival of patients follows a Poisson process with rate $\lambda > 0$. Thus, the interarrival time of patients follows an Exponential distribution with parameter λ and the first condition is satisfied.

The treatment times of patients follow independent Exponential distributions with expected value $\frac{1}{\mu} > 0$. Thus, the parameter of the Exponential distribution is μ and the second condition is satisfied.

The third condition is satisfied directly from the introduction text of this exercise.

The stochastic process $\{X(t) : t \geq 0\}$ can be viewed as a birth-and-death process because the arrival of patients can be seen as births and the departure of patients can be seen as deaths. The state space of the process is discrete and all conditions for a birth-and-death-process are satisfied.

The birth rates of this birth-and-death process, i.e., the arrival of a patient, is independent of the number of patients in the queue and follows a Poisson process with rate $\lambda > 0$. Thus the birth rates are $\lambda_i = \lambda, i = 0, 1, ...$

The death rates of this process, i.e., the treatment time of a patient, again is independent of the number of patients in the queue and has an expected service time $\frac{1}{\mu} > 0$. Thus, the death rates are $\mu_0 = 0, \mu_i = \mu, i = 1, 2, ...$

By Little's Law, the expected time in the UCC is given by $W = \frac{L}{\lambda}$ where L denotes the number of patients in the UCC. The expected number of patients in the UCC is given by $L = \frac{\lambda}{\mu - \lambda}$.

Thus, $W = \frac{1}{\lambda}(\frac{\lambda}{\mu-\lambda}) = \frac{1}{\mu-\lambda}, \lambda < \mu$.

b)

Figure 1 shows one realization of the birth-and-death process $\{X(t) : t \geq 0\}$ for the time 0-12 hours. The steps used to calculate the expected time spent in the process by a patient is as follows: After the simulation, we created the probability mass of the number of patients in the system which serves as an estimate for the limiting distribution

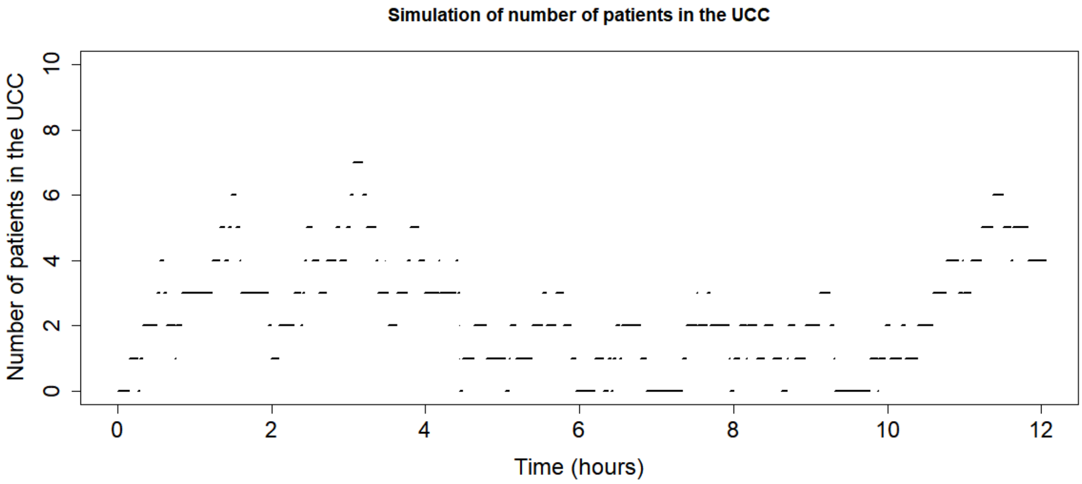


Figure 1: One simulation of the process $\{X(t) : t \geq 0\}$ for the time 0-12 hours.

$\pi = (\pi_0, \pi_1, ...)$. From this probability mass function we calculated the average which serves as the estimate of the expected value of the limiting distribution, i.e., the expected value of how many patients are in the UCC. From the expected number of patients in the UCC, we calculated the expected waiting time via Little's Law.

The CI calculated based on 30 simulations of the birth-and-death process is $W \in [0.945, 1.102]$. This is perfectly reasonable since the exact value of the expected time by Little's Law is $W = \frac{1}{\mu - \lambda} = \frac{1}{6-5} = 1$.

c)

The arrival of patients follows a Poisson process with rate $\lambda > 0$. From the arriving patients, only a fraction $0 < p < 1$ is considered urgent and thus the arrival of urgent patients follows a Poisson process with rate $p\lambda > 0$ because of the thinning phenomenon. Thus, the interarrival times of urgent patients is iid $Exp(p\lambda)$. The second and third condition from subtask a) remains unchanged and thus the stochastic process $\{U(t) : t \geq 0\}$ satisfies the conditions of an M/M/1 queue. We acknowledge that this description is not a formal proof but it outlines the structure of the derivation.

Thus, the arrival rate is $p\lambda > 0$.

The long-run mean number of urgent patients in the UCC is given by the expected value of the limiting distribution. This is given by $L = \frac{\lambda}{\mu - \lambda}$ where λ is the arrival rate and $\frac{1}{\mu}$ is the expected service time. Thus, in this case, the long-run mean number of urgent patients in the UCC is given by $L = \frac{p\lambda}{\mu - p\lambda}$.

d)

The stochastic process $\{N(t) : t \geq 0\}$ does not behave like an M/M/1 queue because the second condition of an M/M/1 queue is not satisfied. The service times are NOT distributed $Exp(\mu)$. The service times are distributed $Exp(\mu)$ conditional on $\{U(t)\} = 0$. If $\{U(t)\} > 0$, it is deterministic that $\{N(t)\}$ will not be served.

The long-run mean number of patients overall in the UCC is given by $L_{tot} = \frac{\lambda}{\mu-\lambda}$ and the long-run mean number of urgent patients in th UCC is given by $L_U = \frac{p\lambda}{\mu-p\lambda}$. Since the number of urgent patients and the number of normal patients in the UCC together sum up to the number of patients overall in the UCC, the long-run mean number of normal patients in the UCC is given by $L_N = L_{tot} - L_U$:

$$L_N = \frac{\lambda}{\mu-\lambda} - \frac{p\lambda}{\mu-p\lambda} = \frac{\lambda(\mu-p\lambda)}{(\mu-\lambda)(\mu-p\lambda)} - \frac{p\lambda(\mu-\lambda)}{(\mu-\lambda)(\mu-p\lambda)} = \frac{\mu\lambda(1-p)}{(\mu-\lambda)(\mu-p\lambda)} \tag{1}$$

e)

By Little's Law, $W = \frac{L}{\lambda}$ where W is the expected time spent by a patient in the UCC, λ is the interarrival rate of new patients and L is the average number of patients in the UCC.

The respective values for urgent patients in the UCC is given by $\lambda_U = p\lambda$ and $L_U = \frac{p\lambda}{\mu-p\lambda}$. Thus,

$$W_U = \frac{L_U}{\lambda_U} = \frac{p\lambda}{\mu-p\lambda} * \frac{1}{p\lambda} = \frac{1}{\mu-p\lambda} \tag{2}$$

The respective values for normal patients in the UCC is given by $\lambda_N = (1-p)\lambda$ and $L_N = \frac{\mu\lambda(1-p)}{(\mu-\lambda)(\mu-p\lambda)}$. Thus,

$$W_N = \frac{L_N}{\lambda_N} = \frac{\mu\lambda(1-p)}{(\mu-\lambda)(\mu-p\lambda)} * \frac{1}{(1-p)\lambda} = \frac{\mu}{(\mu-\lambda)(\mu-p\lambda)} \tag{3}$$

f)

With $\lambda = 5$ and $\mu = 6$, expected times in the UCC are $W_U = \frac{1}{6-5p}$ and $W_N = \frac{6}{6-5p}$. Figure 2 shows W_U and W_N as a function of p jointly in one figure. It is worth noticing that although the situation around $p \approx 1$ seems much more severe than the situation at $p \approx 0$, the overall expected time of any patient with unknown type in the UCC does not change and stays at $W = 1$. This is because the distribution of patients with type urgent and with type normal varies with p.

The situation at $p \approx 0$ means that quasi every patient arriving at the UCC is of type normal. Then, $W_N \approx 1$ as it is the case without types and $W_U \approx \frac{1}{6}$ because an urgent patient will be treated without delay at quasi all points in time and thus the expected time in the UCC is the expected time of treatment.

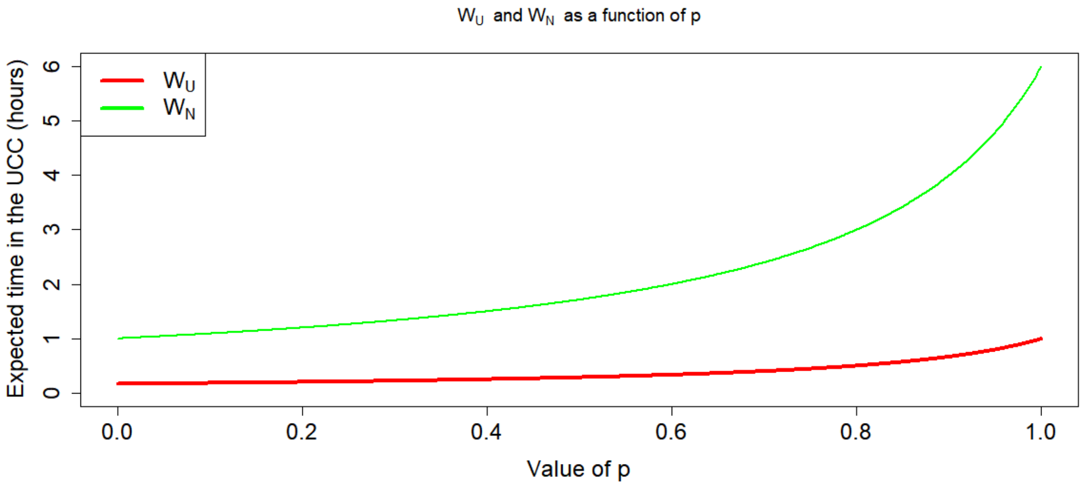


Figure 2: W_U and W_N as a function of p.

The situation at $p \approx 1$ means that quasi every patient arriving at the UCC is of type urgent. Then, $W_U \approx 1$ as it is the case without types and $W_N \approx 6$ because a normal patient will only be treated if the state of the queue is 0 and no urgent patients are left in the UCC which means that in expectation, they will wait very long for treatment.

The expected time spend at the UCC by a normal patient W_n in the extreme cases $p \approx 0$ and $p \approx 1$ is given by:

$$\begin{aligned} W_N(0) &= \frac{6}{6-5p} = \frac{6}{6-0} = 1 \\ W_N(1) &= \frac{6}{6-5p} = \frac{6}{6-5} = 6 \end{aligned} \tag{4}$$

The value of p for which the expected time spent at the UCC for a normal patient is 2 hours is given by:

$$\begin{aligned} W_N(p) &= 2 \\ \frac{6}{6-5p} &= 2 \\ 6 &= 10p \\ p &= \frac{3}{5} \end{aligned} \tag{5}$$

g)

Figure 3 shows one realization of the the processes $\{U(t) : t \geq 0\}$ and $\{N(t) : t \geq 0\}$ for the time 0-12 hours jointly in one figure.

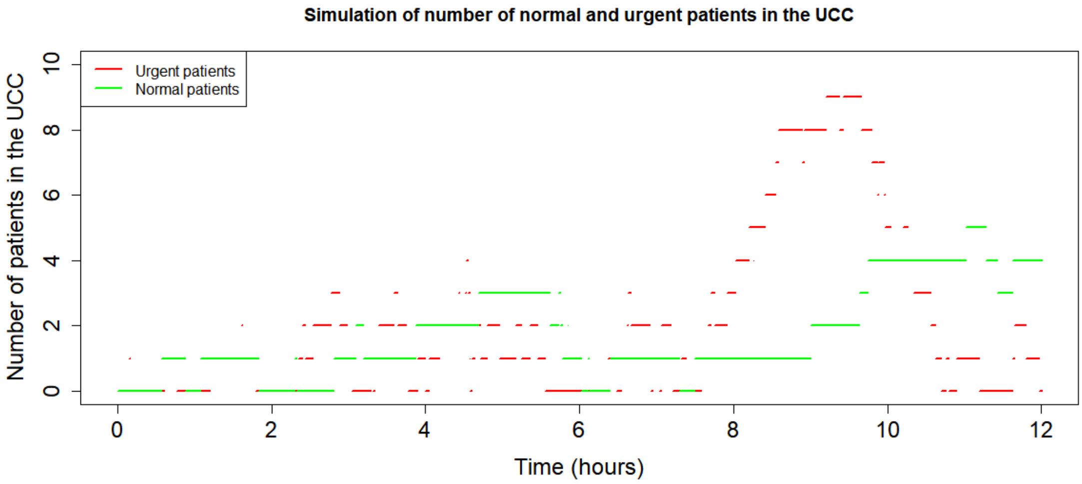


Figure 3: One simulation of the processes $\{U(t) : t \geq 0\}$ and $\{N(t) : t \geq 0\}$ for the time 0-12 hours.

The CIs of the expected time in the UCC calculated based on 30 simulations of the birth-and-death processes is $W_U \in [0.483, 0.508]$ for an urgent patient and $W_N \in [2.863, 3.308]$ for a normal patient. This is perfectly reasonable since the exact value of the expected times by Little's Law is $W_U = 0.5$ and $W_N = 3$, respectively.

Generally, the confidence intervals were computed using the same approach as in subtask b) via the mass probability of the stochastic processes. But, there is one trick to apply to avoid a skewed solution since the two birth-and-death processes $\{U(t) : t \geq 0\}$ and $\{N(t) : t \geq 0\}$ are related to each other, one must not consider each of the stochastic processes in isolation. Concretely, the underlying time index must not be used twice: We therefore assigned each high-level time window of the simulation to either $U(t)$ or $N(t)$ and limited the time basis, increasing the average and the confidence interval.

Problem 2: Calibrating climate models

In this scenario, we are predicting "the albedo of sea ice". To do that we have parameters θ, which represents the amount of sun light reflected by sea ice, and $Y(\theta)$ which is the score of a model generated by θ. To accomplish this, we aim to employ a Gaussian process $\{Y(\theta) : \theta \in [0, 1]\}$, to represent the unknown connection between the parameter value θ and the corresponding score $Y(\theta)$.

a)

First of all, we begin this task by creating a grid with 51 points between the values 0.25 and 0.50. After that, we desire finding the conditional mean and covariance values of the five points we were given.

Then we utilize a Gaussian distribution, T, with two random variables. The variables are T_1, for the values of the grid with 51 points, and T_2 for the values of the five given points. From this we get this distribution for a conditional variable T_con:

$$\overrightarrow{T} = (\overrightarrow{T_1}, \overrightarrow{T_2}) \sim N_{n_1+n_2}\left(\begin{bmatrix} \overrightarrow{\mu_1} \\ \overrightarrow{\mu_2} \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \right) \tag{6}$$

$$T_1 \mid T_2 = T_{\text{con}} \sim N_{n_1}(\overrightarrow{\mu_{\text{con}}}, \Sigma_{\text{con}}) \tag{7}$$

Where:

$$\overrightarrow{\mu_{\text{con}}} = \overrightarrow{\mu_1} + \Sigma_{12}\Sigma_{22}^{-1}(\overrightarrow{T_2} - \overrightarrow{\mu_2}) \tag{8}$$

$$\Sigma_{\text{con}} = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \tag{9}$$

To calculate the covariance matrix we operate with this formula for the covariance of two random variables:

$$Cov(X,Y) = Corr(X,Y)\sqrt{Var(X)Var(Y)} \tag{10}$$

When this is calculated, we can finally find a prediction interval with respect to its probability, which has this appearance:

$$P(l < X < u) = 0.90 \tag{11}$$

$$[\mu_{\text{con}} - z\sigma_{\text{con}},\ \mu_{\text{con}} + z\sigma_{\text{con}}] \tag{12}$$

Where l is the interval's lower bound and u is the upper bound. Altogether we have this distribution and these intervals shown below.

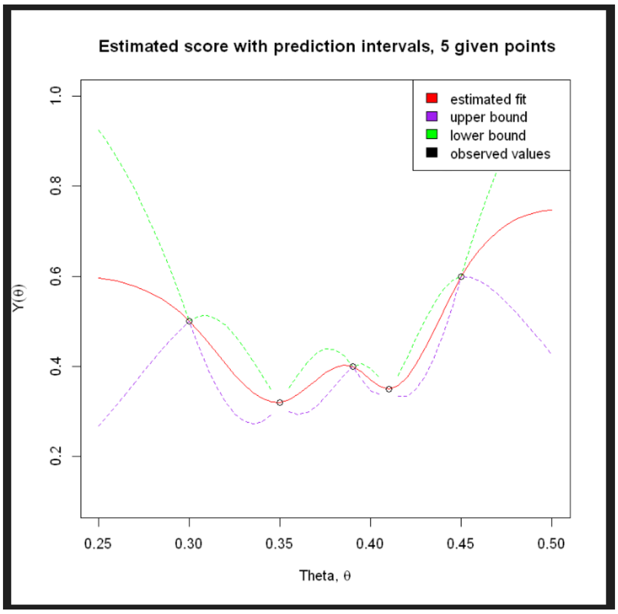


Figure 4: Predicted $Y(\theta)$ and a conditional prediction interval from the given points.

b)

The goal for this task is to score below 0.30, so we use the distribution from the last exercise to find the probability for this to happen. From the figure below, we can clearly see that the probability for any of the given points is equal to zero, as all of these are above $\theta = 0.30$. In addition, we notice that $\theta = 0.34$ gives us the highest probability for the score to be below 0.30.

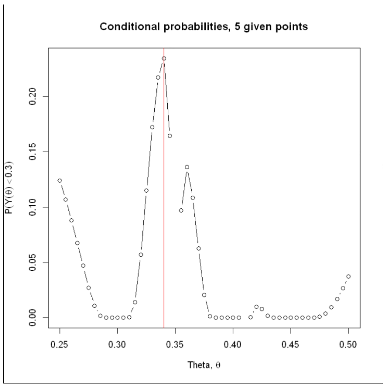
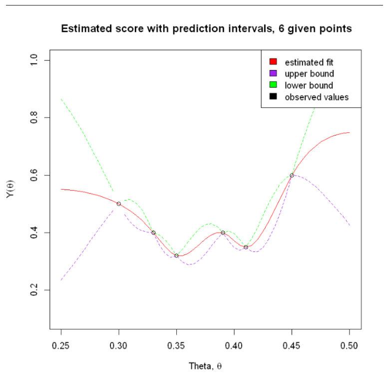


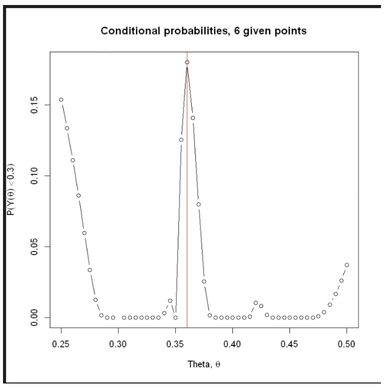
Figure 5: Probability for the score $y(\theta)$ to be smaller than 0.30, for the given points.

c)

Now we have one added point, which means 6 given points. From that, we can find a new conditional distribution, prediction intervals and probability for $y(\theta) < 0.30$ and create two new figures. In the first figure, it is clear that the prediction interval is constrained to the six given points, which aligns perfectly with expectations. In the second figure, we observe that the one added point changes the value that gives the highest probability to be below 0.30, which is $\theta = 0.36$ in this case, so this is the value we suggest the scientists using.



(a) Predicted $Y(\theta)$ and a conditional prediction interval, now with one add given point.



(b) Probability for the score $y(\theta)$ to be smaller than 0.30, with one added point.