School of Electrical and Information Engineering
University of the Witwatersrand, Johannesburg
ELEN4022A – Fullstack Quantum Computing

Lab 2 Quantum Adders and the Quantum Fourier Transform

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1) Introduction

This will explore the design and implementation of a Quantum Full Adder circuit, as well as the Quantum Fourier Transform (QFT) and the Approximate Quantum Fourier Transform (AQFT). These circuits are building blocks for many quantum algorithms, including the Quantum Fourier Transform Adder (QFTA), which we will also discuss. First, we will present the Quantum Full Adder circuit, which is a key component in many quantum arithmetic circuits. Next, we will introduce the Quantum Fourier Transform (QFT) and the Approximate Quantum Fourier Transform (AQFT). These are important tools for quantum computation, and we will describe their design and implementation. Finally, we will explore the application of the QFT and AQFT in the design of a Quantum Fourier Transform Adder (QFTA). This circuit combines the QFT and Inverse Quantum Fourier Transform (IQFT) to create a fast and efficient quantum arithmetic circuit.

2) Quantum Full Adder

The quantum full adder is a circuit designed to perform the functionality of a classical full adder but acting on qubits. It is made using a combination of the quantum half adder, known as the SUM gate and the quantum carry adder, known as the CARRY gate. The implementations of these gates and their combination into 2-bit and 3-bit adders can be seen in the *ELEN4022_LAB2_2022_Mpano_Iverson.ipynb* file. The adder works as expected, using the example of $11_2 + 11_2$ the 2-bit adder circuit returns 110_2 and using the example of $111_2 + 101_2$ the 3-bit adder returns 1100_2 as desired. These were run through simulation only as the quantum hardware was too busy to return results timeously and reliably.

3) Quantum Fourier Transform

The QFT operates on a quantum register, a collection of qubits representing a quantum state, the QFT maps the amplitudes of the state to the Fourier coefficients of the frequency domain. The Qiskit textbook describes the transformation as going from the computational domain, the 1's and 0's or the "Z basis" to the Fourier basis (the X/Y plane). It converts the amplitudes of the qubits into phase information. For a quantum state $|X\rangle = \sum_{j=0}^{N-1} x_j |j\rangle$ of length n, the QFT maps it to the state $|Y\rangle = \sum_{k=0}^{N-1} x_k |k\rangle$, where $N = 2^n$ with n representing the number of qubits in the register, by the following expression:

$$y_k = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x_j \omega_N^{jk} = \frac{1}{\sqrt{2^n}} \sum_{j=0}^{2^n-1} x_j \omega_{2^n}^{jk}$$

Where $\omega_N^{jk} = e^{2\pi i \frac{jk}{N}} = e^{2\pi i \frac{jk}{2^n}}$. To demonstrate the transform will be computed for a quantum register of 1, 2, and 3-qubits.

The QFT of a single qubit carries the same operation as that of applying the Hadamard (H) gate. Given the state $|X\rangle = x_0|0\rangle + x_1|1\rangle$, and $N = 2$:

$$y_0 = \frac{1}{\sqrt{2}}\left(x_0 e^{2\pi i \frac{0\times 0}{2}} + x_1 e^{2\pi i \frac{1\times 0}{2}}\right) = \frac{1}{\sqrt{2}}(x_0 + x_1)$$

$$y_1 = \frac{1}{\sqrt{2}}\left(x_0 e^{2\pi i \frac{0\times 1}{2}} + x_1 e^{2\pi i \frac{1\times 1}{2}}\right) = \frac{1}{\sqrt{2}}(x_0 - x_1)$$

Thus, the final result $U_{QFT}|X\rangle = |Y\rangle = \frac{1}{\sqrt{2}}(x_0 + x_1)|0\rangle + \frac{1}{\sqrt{2}}(x_0 - x_1)|1\rangle$. This shows the connection between the H gate and the single qubit QFT.

For a 2-qubit QFT, the input state $|X\rangle = |x_1 x_2\rangle$:

First the H gate is applied on x_1: $H_1|X\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\frac{2\pi i}{2}x_1}|1\rangle\right] \otimes |x_2\rangle$

Followed by the controlled rotation gate on x_1 controlled by x_2:

$$CROT_2|X\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\left(\frac{2\pi i}{2^2}x_2 + \frac{2\pi i}{2}x_1\right)}|1\rangle\right] \otimes |x_2\rangle$$

Last is the H gate applied to x_2: $U_{QFT}|X\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\left(\frac{2\pi i}{2^2}x_2 + \frac{2\pi i}{2}x_1\right)}|1\rangle\right] \otimes \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\frac{2\pi i}{2}x_2}|1\rangle\right]$

For a 3-qubit QFT with the input state $|X\rangle = |x_1 x_2 x_3\rangle$:

First the H gate is applied on x_1: $H_1|X\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\frac{2\pi i}{2}x_1}|1\rangle\right] \otimes |x_2 x_3\rangle$

Followed by the controlled rotation (CROT) gate on x_1 controlled by x_2:

$$CROT_2|X\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\left(\frac{2\pi i}{2^2}x_2 + \frac{2\pi i}{2}x_1\right)}|1\rangle\right] \otimes |x_2 x_3\rangle$$

Then another CROT gate on x_1 controlled by x_3:

$$CROT_3|X\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\left(\frac{2\pi i}{2^3}x_3 + \frac{2\pi i}{2^2}x_2 + \frac{2\pi i}{2}x_1\right)}|1\rangle\right] \otimes |x_2 x_3\rangle$$

H gate applied to x_2: $H_2|X\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\left(\frac{2\pi i}{2^3}x_3 + \frac{2\pi i}{2^2}x_2 + \frac{2\pi i}{2}x_1\right)}|1\rangle\right] \otimes \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\frac{2\pi i}{2}x_2}|1\rangle\right] \otimes |x_3\rangle$

Followed by the controlled rotation (CROT) gate on x_2 controlled by x_3:

$$CROT_2|X\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\left(\frac{2\pi i}{2^3}x_3 + \frac{2\pi i}{2^2}x_2 + \frac{2\pi i}{2}x_1\right)}|1\rangle\right] \otimes \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\left(\frac{2\pi i}{2^2}x_3 + \frac{2\pi i}{2}x_2\right)}|1\rangle\right] \otimes |x_3\rangle$$

Last is the H gate applied to x_3:

$$U_{QFT}|X\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\left(\frac{2\pi i}{2^3}x_3 + \frac{2\pi i}{2^2}x_2 + \frac{2\pi i}{2}x_1\right)}|1\rangle\right] \otimes \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\left(\frac{2\pi i}{2^2}x_3 + \frac{2\pi i}{2}x_2\right)}|1\rangle\right] \otimes \frac{1}{\sqrt{2}}\left[|0\rangle + e^{\frac{2\pi i}{2}x_3}|1\rangle\right]$$

4) Approx. Quantum Fourier Transform

When applying the QFT to registers of larger numbers of qubits, the rotations begin diminishing in size to the point where they are indistinguishable from noise or interreference. Not only that but the complexity of the circuit at those sizes means that the finer rotations bring diminishing returns in terms of usefulness vs efficiency. To aid with this the approximate QFT was designed where unlike in the standard QFT, not every CROT gate is applied recursively. Instead, the AQFT takes in an additional argument, the degree. The degree determines the maximum number of rotations to be applied (Prokopenya, 2015). This can be seen by the functions *aqft* and *iqft* in the attatched code files. This is useful not only for simulation purposes but for building these circuits they are less complex and the degree of accuracy is configurable allowing for applications to suit their own tolerances while saving on complexity and time.

5) Quantum Fourier Adders

The quantum Fourier adders were built and examples for 2-bit and 3-bit systems were shown and validated. The AQFT adders were not built as with the length of these bits it would not be useful or demonstrative of the AQFT since it's applications are mainly useful for larger numbers of bits (Prokopenya, 2015).

6) Conclusion

The aim of this lab was to build familiarity with Quantum Full Adders, the QFT, the AQFT and their respective adders. This has been achieved and demonstrated through this report and within the code files. The QFT is a powerful tool that has shown it can perform well in application where it is used well like with the QFTA.

References

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