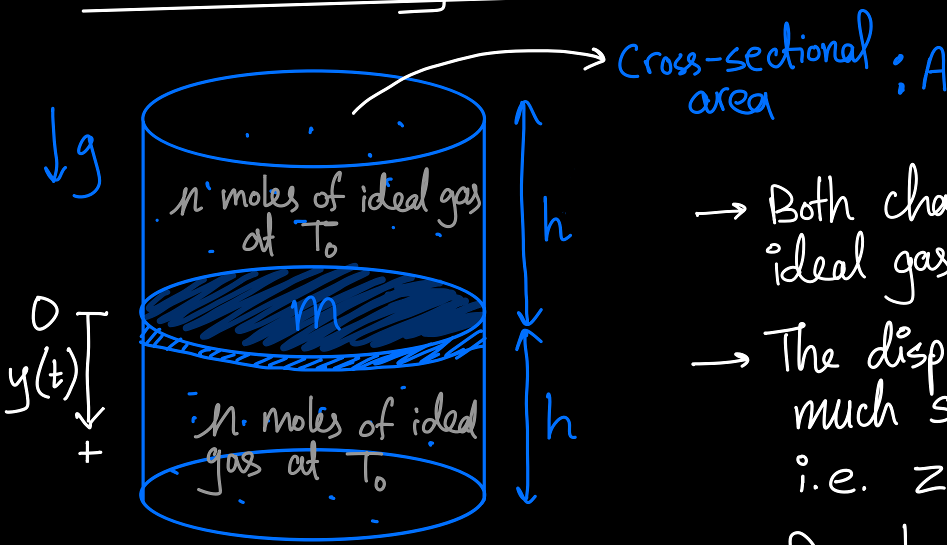
An interesting SHM !



→ Both chambers contain n moles of ideal gas at T_0 temperature.

→ The displacements of mass m are much smaller compared to ℓ, i.e. $z(t) \ll \ell$.

→ Due to the previous fact along with the fact that there is no friction between the cylinder walls & m, we can consider the expansion and contractions of the gas chambers to be isothermal.

Question:

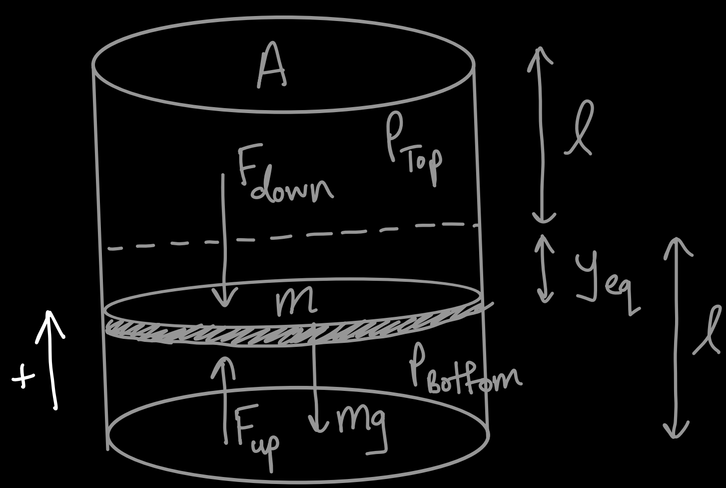
Find an expression for $y(t)$.

Solution:

As we can all guess, m will move in some kind of oscillatory motion. Initially, force due to pressures from both top and bottom chambers will be equal, as a result, m will accelerate downwards due to mg. However, this compresses the bottom chamber and increases the upwards force on m. Hence, after reaching the equilibrium position, m "overshoots" due to inertia and compresses the bottom chamber even more.

After reaching a turning point, it accelerates upwards towards the equilibrium position, only to "overshoot" upwards yet again.

It is always helpful to consider displacements of m with respect to the EQUILIBRIUM POSITION (acceleration $= 0$) in the cases of oscillatory motion.



The state equation of an ideal gas:

$$PV = nRT \Rightarrow P = \frac{nRT}{V}$$

In our case, all expansions and contractions are isothermal and thus temp. of both chambers are T_0.

$$F_{down} = P_{Top} \cdot A = \frac{nRT_0}{A(h+y_{eq})} \cdot A$$

$$\therefore \boxed{F_{down} = \frac{nRT_0}{h+y_{eq}}}$$

$$F_{up} = P_{Bottom} \cdot A = \frac{nRT_0}{A(h-y_{eq})} \cdot A$$

$$\therefore \boxed{F_{up} = \frac{nRT_0}{h-y_{eq}}}$$

$$\therefore \sum F_y = F_{up} - F_{down} - mg \qquad \text{[Upwards is positive]}$$

$$= \frac{nRT_0}{h-y_{eq}} - \frac{nRT_0}{h+y_{eq}} - mg$$

$$= nRT_0\left(\frac{1}{h-y_{eq}} - \frac{1}{h+y_{eq}}\right) - mg$$

$$= nRT_0 \left(\frac{h + y_{eq} - (h - y_{eq})}{h^2 - y_{eq}^2} \right) - mg$$

$$= nRT_0 \left(\frac{2y_{eq}}{h^2\left(1 - \frac{y_{eq}^2}{h^2}\right)} \right) - mg$$

Now, notice that since $y(t) \ll l$, the ratio $\frac{y_{eq}}{h} \approx 0$.
As such, the ratio $\frac{y_{eq}^2}{h^2} = \left(\frac{y_{eq}}{h}\right)^2$ is even smaller.

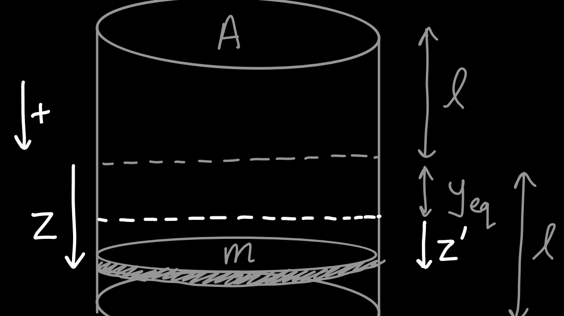
$$\therefore \Sigma F_y = nRT_0 \left(\frac{2y_{eq}}{h^2} \right) - mg \rightarrow \text{(i)}$$ [Note that this is true for all displacement $y(t)$]

For y_{eq}, $\Sigma F_y = 0$.

$$\therefore nRT_0 \left(\frac{2y_{eq}}{h^2} \right) = mg$$

$$\Rightarrow \boxed{y_{eq} = \frac{mgh^2}{2nRT_0}}$$

Now that we have obtained an expression for y_{eq}, we will consider all displacements of m relative to this.



At a displacement of z' from equilibrium, m is at a displacement of:

$$z = y_{eq} + z'$$

from the original (middle) position.

Hence, by (i):

For z and z': Downwards is +

$$\underbrace{\sum F_y}_{\text{upwards is +}} = nRT_0\left(\frac{2(y_{eq}+z')}{h^2}\right) - mg$$

$$= \frac{2nRT_0}{h^2}\left(y_{eq}+z'\right) - mg$$

$$= \frac{2nRT_0}{h^2}\left(\frac{mgh^2}{2nRT_0}+z'\right) - mg$$

$$= \cancel{mg} + \frac{2nRT_0 z'}{h^2} \cancel{-mg}$$

$$\therefore F_{net_{up}} = \left(\frac{2nRT_0}{h^2}\right) z'_{down}$$

Assuming downwards as ultimate + (as in question):

$$F_{net} = -\left(\frac{2nRT_0}{h^2}\right) z'$$

$$\therefore ma = -\left(\frac{2nRT_0}{h^2}\right) z'$$

$$\therefore a = -\left(\frac{2nRT_0}{mh^2}\right) z'$$

This is clearly reminiscent of the equation of SHM!

SHM: $a = -\omega^2 x$ → displacement from equilibrium

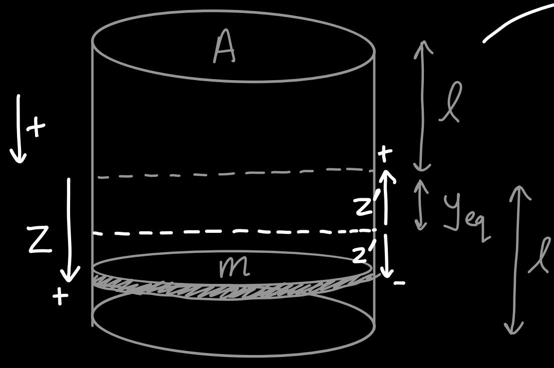
$$\therefore \omega^2 = \frac{2nRT_0}{mh^2}$$

$$\therefore \boxed{\omega = \frac{1}{h}\sqrt{\frac{2nRT_0}{m}}}$$

Now, observe that since this motion has been proven to be simple harmonic, the amplitude of the SHM must be y_{eq}.

<u>Hence</u>: $$z'(t) = A \cdot \cos(\omega t)$$
$$= \frac{mgh^2}{2nRT_0} \cdot \cos\left(\frac{t}{h}\sqrt{\frac{2nRT_0}{m}}\right)$$

Now, before we arrive at an expression for $z(t)$, observe that at $z'(t)$ is positive $\left(= +\frac{mgh}{2nRT_0} = +A\right)$ at $t = 0$. However, at $t = 0$, $z(t) = 0$. As time passes, $z'(t)$ decreases to 0 and then becomes negative below y_{eq}.



Hence, we should negate $z'(t)$ before adding it to y_{eq} to get our required value of $z(t)$.

$$\therefore z(t) = y_{eq} - z'(t)$$
$$= y_{eq} - y_{eq} \cdot \cos(\omega t)$$
$$= y_{eq}(1 - \cos(\omega t))$$

Hence, we have:

$$\boxed{\therefore z(t) = \frac{mgh^2}{2nRT_0} \cdot \left(1 - \cos\left(\frac{t}{h}\sqrt{\frac{2nRT_0}{m}}\right)\right)}$$

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