Calculating Pi: Archimedes' method

Consider the following circle.

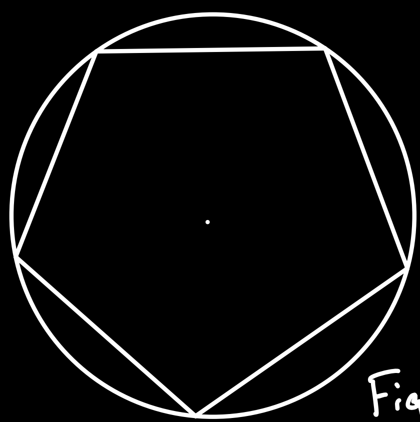


Fig:1

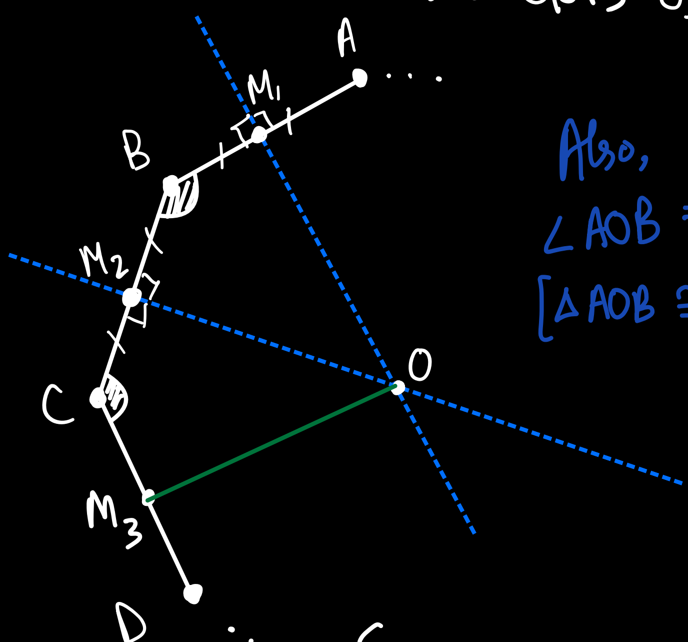
We can try to 'approximate' the circle using a regular pentagon.

→ The 'center' of the pentagon will be the center of the circle.

Defining the 'center' of a regular n-gon:

→ All the vertices of the regular n-gon are equidistant from its 'center'.

Construction: The 'center' of a regular n-gon is the interzection of all the perpendicular bisectors of its sides.



$AB = BC = CD$

$\angle ABC = \angle BCD$

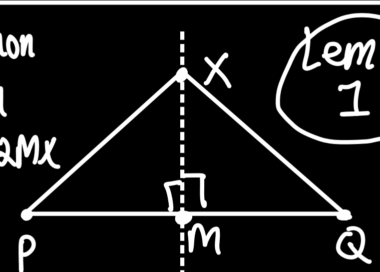
O is the 'center'.

Also,

$\angle AOB = \angle BOC$

$[\triangle AOB \cong \triangle BOC]$

- XM common
- $PM = QM$
- $\angle PMX = \angle QMX = 90°$



Lemma 1

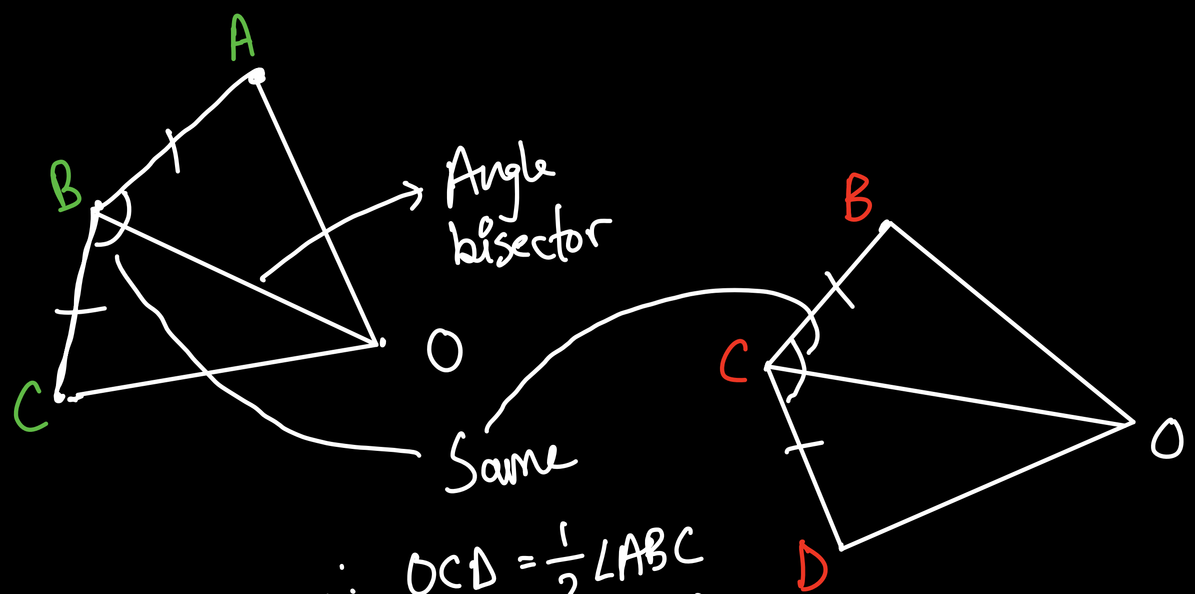
Any point on the perp. bisec. of PQ is equidistant from P & Q.

So,

$\boxed{OA = OB = OC}$

Now, we have to prove that, if we construct OM_3 (M_3 is the midpoint of CD), then $OM_3 \perp CD$.

<u>Consider the quadrilaterals ABCOA and BCDOB:</u>



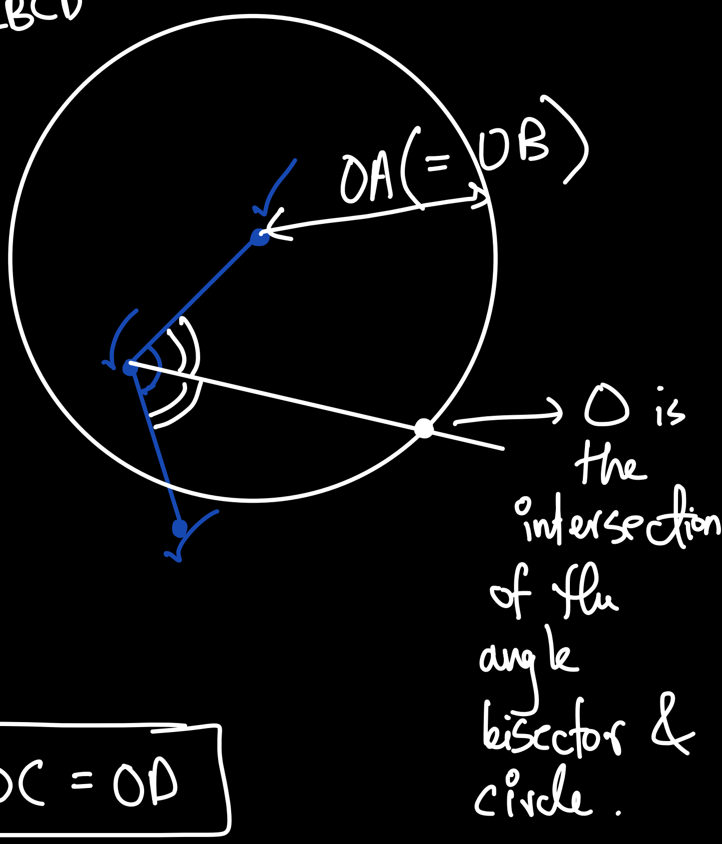
$\therefore OCD = \frac{1}{2}\angle ABC$
$= \frac{1}{2}\angle BCD$

→ $\angle ABC = \angle BCD$

→ $AB = BC$

→ $BC = CD$

→ $OA = OB$



$\therefore ABCOA \cong BCDOB$

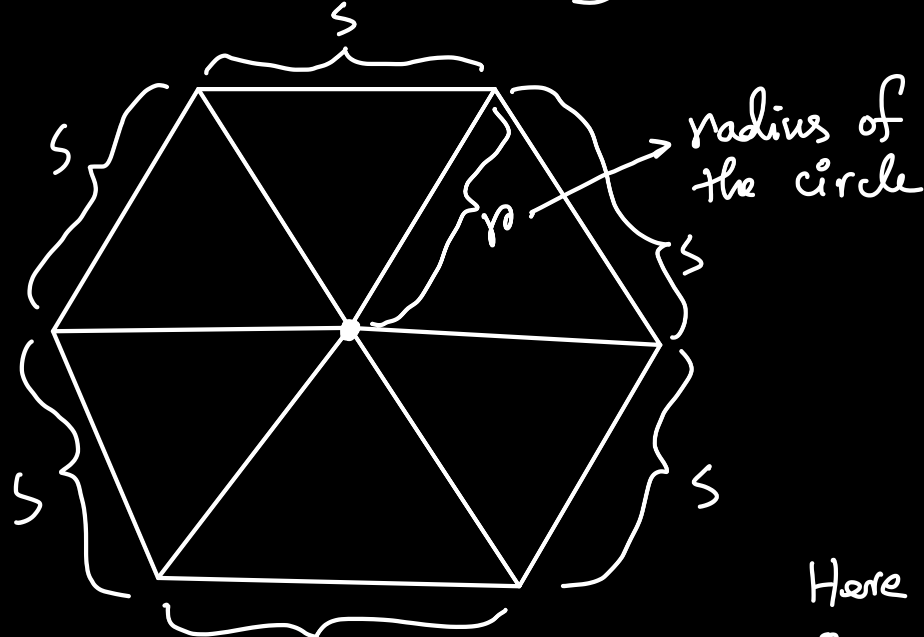
$\therefore OC = OD$
$\quad\;\; \| $
$\quad OA$
$\quad\;\; \|$
$\quad OB$

$\boxed{OA = OB = OC = OD}$

$\therefore$ If $OC = OD$, the converse of Lemma 1 applies, and thus O lies on the $\perp$ bisector of CD.

But, since $CM_3 = DM_3$, OM_3 IS the perpendicular bisector of CD. $\blacksquare$

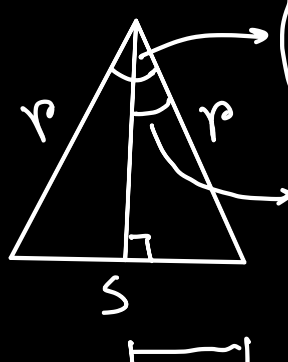
Now that we have proved the existence of the 'center', observe the following:



Here, a regular hexagon is split into 6 congruent isosceles triangle.

In general, a regular n-gon will be split up into n congruent isosceles triangle, the tips of which is the 'center'.

Looking at one of the isosceles triangles:



$\left(\frac{360}{n}\right)^\circ$

$\frac{360^\circ}{n} \times \frac{1}{2} = \frac{180^\circ}{n}$

$\frac{s}{2}$

$$\therefore \sin\left(\frac{180^\circ}{n}\right) = \frac{\left(\frac{s}{2}\right)}{r} = \frac{s}{2r}$$

$$\therefore \boxed{s = 2r \cdot \sin\left(\frac{180^\circ}{n}\right)}$$

So, the perimeter of the regular n-gon is:

$$P = ns = 2r \cdot n \cdot \sin\left(\frac{180^\circ}{n}\right)$$

Now, as in fig: 1, if we keep r fixed and let $n \to \infty$, the regular polygon approaches the circle. As such, the perimeter of the regular polygon BECOMES the circumference of the circle (i.e., $2\pi r$).

$\therefore$ As $n \to \infty$:

$$2\pi r = 2r \cdot n \cdot \sin\left(\frac{180^\circ}{n}\right)$$

$$\therefore \pi = n \cdot \sin\left(\frac{180^\circ}{n}\right)$$

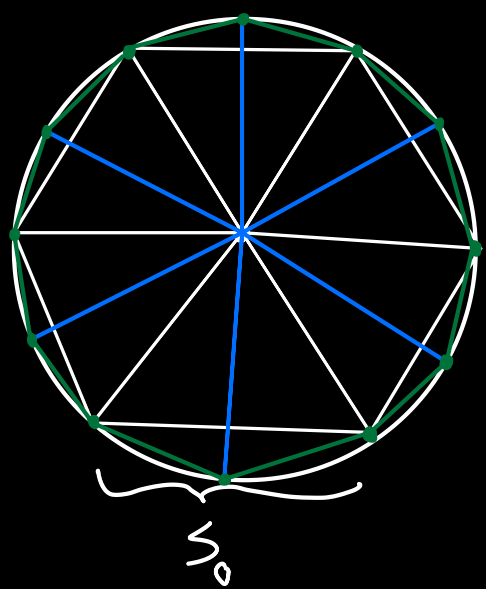
Hence, we have Archimedes' Pi Formula:

$$\boxed{\pi = \lim_{n\to\infty} n \cdot \sin\left(\frac{180^\circ}{n}\right)}$$

Now, while doing calculations by hand, computations involving sine might be difficult. In fact, the way Archimedes' did the calculations was by disecting the inscribed regular polygon.

→ Suppose we have a circle with radius $\frac{1}{2}$.

$$C = 2\pi r = 2\pi \cdot \frac{1}{2} = \pi$$



We start with a hexagon.

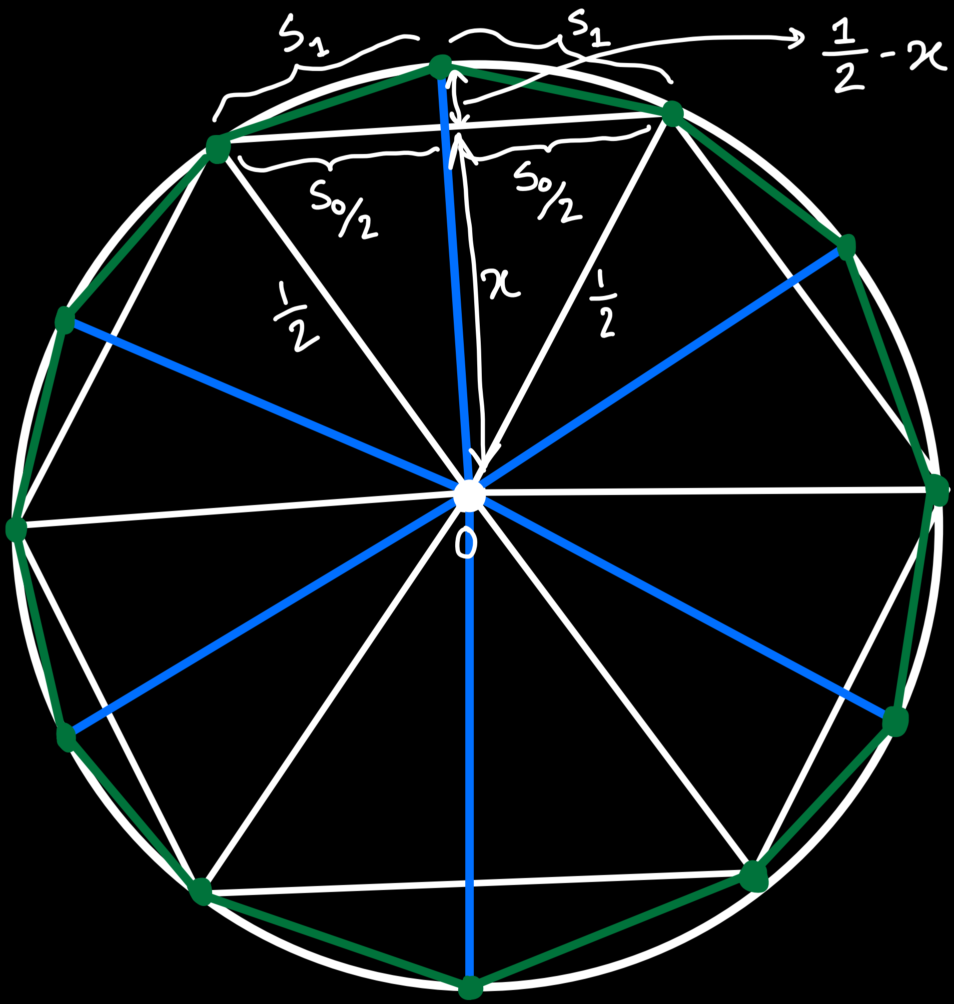
$$s_0 = 2r \cdot \sin\left(\frac{180^\circ}{n}\right)$$
$$= 2 \cdot \frac{1}{2} \cdot \sin\left(\frac{180^\circ}{6}\right)$$
$$= \sin(30^\circ)$$

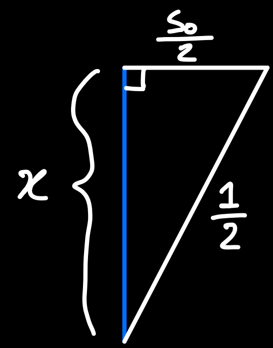
$$\therefore \boxed{s_0 = \frac{1}{2}}$$

Now, for each of the $\frac{180^\circ}{6} = 30^\circ$ angles, we draw the angle bisector and extend it to the circumference of the circle. (In turn, we actually bisect the arc AND the side s, since each of triangles are isosceles).

→ Then, we join all the adjacent points on the circumference of the circle.

See next page for continuation



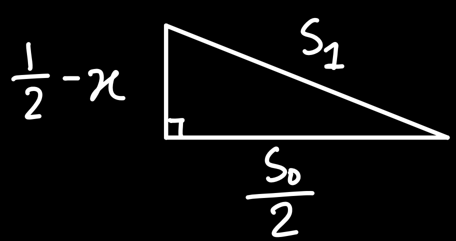


$$x^2 + \left(\frac{S_0}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

$$\Rightarrow x = \sqrt{\frac{1}{4} - \frac{S_0^2}{4}}$$

$$x = \sqrt{\frac{1 - S_0^2}{4}}$$

<u>We have:</u>



$$S_1^2 = \left(\frac{1}{2} - x\right)^2 + \left(\frac{S_0}{2}\right)^2$$

$$\Rightarrow S_1 = \sqrt{\left(\frac{1}{2} - \sqrt{\frac{1 - S_0^2}{4}}\right)^2 + \left(\frac{S_0}{2}\right)^2}$$

$\therefore$ $\boxed{S_0 = \frac{1}{2}}$

$$\boxed{S_{n+1} = \sqrt{\left(\frac{1}{2} - \sqrt{\frac{1 - S_n^2}{4}}\right)^2 + \frac{S_n^2}{4}}}$$

At 0^{th} iteration, $n = 6$ → no. of sides of the inscribed regular polygon.

$\times 2$

At 1^{st} iteration, $n = 12$

$\therefore$ At n^{th} iteration, no. of sides $= (6 \times 2^n)$

$\therefore \boxed{\pi \approx (6 \times 2^n) \cdot S_n}$ (At each iteration)

Circumference

<u>Running our iteration for 100 times gives:</u>

$$\pi \approx \underbrace{3.141592653589793}_{\text{Correct till 15 decimal places}}6\ldots$$

(Not March 14th, unfortunately)