Quest to the Cubics!

Exploring depressed cubics and Tartaglia's Method

Md Faiyaz Siddiquee

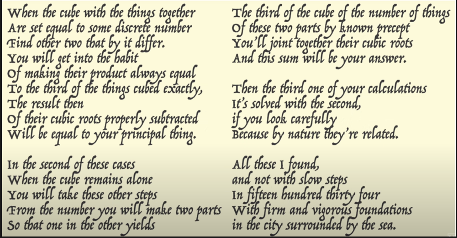
September 2023

History and background

Scipione Del Ferro: Mathematics Professor at University of Bologna *(Solves the Depressed Cubic in 1510, which he kept secret until 1526. On his deathbed in 1526, he reveals it to his student Antonio Fior)*.

Antonio Fior: challenges **Nicollo Tartaglia** to a math duel in 1535, where Tartaglia wins.

Later, *Tartaglia* becomes the second person in history to solve the Depressed Cubic.



When the cube with the things together
Are set equal to some discrete number
Find other two that by it differ.
You will get into the habit
Of making their product always equal
To the third of the things cubed exactly,
The result then
Of their cubic roots properly subtracted
Will be equal to your principal thing.

In the second of these cases
When the cube remains alone
You will take these other steps
From the number you will make two parts
So that one in the other yields

The third of the cube of the number of things
Of these two parts by known precept
You'll joint together their cubic roots
And this sum will be your answer.

Then the third one of your calculations
It's solved with the second,
if you look carefully
Because by nature they're related.

All these I found,
and not with slow steps
In fifteen hundred thirty four
With firm and vigorous foundations
in the city surrounded by the sea.

Figure: Tartaglia's solution encoded in a poem

Introduction

Standard high school "Completing the Square":

$$ax^2 + bx + c = 0$$
$$\implies x^2 + \frac{b}{a}x = -\frac{c}{a}$$
$$\implies x^2 + 2 * \frac{b}{2a} * x + (\frac{b}{2a})^2 = -\frac{c}{a} + (\frac{b}{2a})^2$$
$$\implies (x + \frac{b}{2a})^2 = \frac{-4ac}{(2a)^2} + \frac{b^2}{(2a)^2}$$
$$\implies x + \frac{b}{2a} = \frac{\pm\sqrt{b^2-4ac}}{2a}$$
$$\boxed{\implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

This motivates us to "Complete the Cube", which we will explore in the next slide when dealing with Depressed Cubics.

The Depressed Cubic: Part 1

- A Depressed Cubic is a cubic equation without the x^2 term.

To begin, consider the following Depressed Cubic:
$2x^3 - 6x - 3 = 0 \implies x^3 - 3x = \frac{3}{2}$

Now, think of this in terms of volumes: We have a *cube* of side length x, and we must add a volume of $-3x$ to get a total volume of $\frac{3}{2}$.

- Suppose that we *extend* the sides of the cube by length y. The new size length is now $z = x + y$

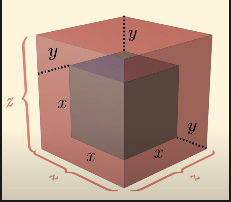


Figure: Extending a cube

The Depressed Cubic: Part 2

The new volume of the cube then becomes:
$z^3 = (x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = x^3 + 3xy(x+y) + y^3 = x^3 + (3yz)x + y^3$.

Notice that our original depressed cubic was: $x^3 - 3x = \frac{3}{2}$. Thus, equating the coefficients of x, we get: $-3 = 3yz$. Now, our depressed cubic becomes:

$$x^3 + (3yz)x = \frac{3}{2}$$
$$\implies x^3 + (3yz)x + y^3 = \frac{3}{2} + y^3$$
$$\implies z^3 = \frac{3}{2} + y^3$$

We have completed the cube!

The Depressed Cubic: The Final Showdown

So far we have the following important results:

$-3 = 3yz$, and
$z^3 = \frac{3}{2} + y^3$

We will solve these simultaneously!
$-3 = 3yz \implies yz = -1 \implies z = -\frac{1}{y}$

Substituting:
$(-\frac{1}{y})^3 = \frac{3}{2} + y^3$
$\implies -\frac{1}{y^3} = \mathrm{y}^3 + \frac{3}{2}$
$\implies -1 = y^6 + \frac{3}{2}y^3$ here, letting $p = y^3$, we finally have:
$\boxed{\mathrm{p}^2 + \frac{3}{2}p + 1 = 0}$
(This can now be solved using the usual quadratic formula!)

Putting things together and encountering the imaginary

One of the solutions of the quadratic $p^2 + \frac{3}{2}p + 1 = 0$ is $p = \frac{-3+\sqrt{7}i}{4}$. We know, $p = y^3 \implies y = p^{\frac{1}{3}}$. So, one of the possible values of y is:

$y = (\frac{-3+\sqrt{7}i}{4})^{\frac{1}{3}}$
We know that $z = -\frac{1}{y}$ and $z = x + y$. So,

$$\boxed{x = -\frac{1}{y} - y, \quad \text{where } y = (\frac{-3+\sqrt{7}i}{4})^{\frac{1}{3}}}.$$

Manipulating this using complex analysis (namely, using Euler's Identity $r * e^{i\theta} = r * (cos(\theta) + i * sin(\theta))$, we obtain the precise (and quite long!) answer. Notably, the imaginary part cancels out!

$x \approx -1.38467$
After this, the rest of the roots can be obtained by algebraic long division.

Thank You!

Any questions will be appreciated.