An interesting puzzle: A gentle introduction to problem solving using induction

Faiyaz Siddiquee

September 5, 2023

1 Problem Statement

You are given all $\mathbb{N}$ from 1 to n arranged in a row in a random order. An 'operation' on the row is defined as:

Let the first number of the row be k. We select the first k numbers of the row and reverse their order (first element becomes k*th* element, second element becomes $k-1$*th* element, and so on).

Prove that after a finite number of successive application of the 'operation' on our given row, we will obtain 1 as the first number.

2 Solution

Out of all the possible approaches, possibly the most simple and easy-to-find approach will be outlined in this section.

As with any mathematical problem, we should start with experimenting and getting a feel for the problem. Consider the following:

$$\begin{gathered}(5,3,4,1,6,2)\\ \downarrow\\ (6,1,4,3,5,2)\\ \downarrow\\ (2,5,3,4,1,6)\\ \downarrow\\ (5,2,3,4,1,6)\\ \downarrow\\ (\mathbf{1},4,3,2,5,6)\end{gathered}$$

And hence we get 1 as the first number of the row.

One little thing to notice in the last step is that if we were to apply the 'operation' to the last configuration of the row, we would get back the same thing (since the leading 1 is reversed back to the same place).

From the example shown, we can quite confidently guess that each of the different configurations of the row after successive application of the 'operation' will be different from the rest of the configurations. In other words, any two configurations of the row obtained in the process of successively applying the 'operation' will be different from each other.

If we can prove this conjecture, our problem will essentially be solved, since the set $1, 2, 3, \cdots, n$ has $n!$ (a finite number) permutations. So, we must get 1 as the leading number at some point, and, after that, the configuration of the row does not change anymore as we keep on applying the 'operation'.

Before moving on to induction, let's concretely write down a slightly rephrased version of our claim:

Claim 1: Given any configuration of the row, C_0 , all the configurations that follow from the successive application of the 'operation' will be different from C_0.

It is left as an exercise to the reader to find out why this statement and the previously described conjecture are equivalent.

Now, we can move on to induction:

Base cases:

For $n = 1$, there is nothing to show. For $n = 2$, if the configuration is $(1, 2)$, we are done. Otherwise,
$(2, 1) \to (1, 2)$, and we are done.

Inductive hypothesis:

Suppose that *Claim 1* is true for $n = k$. We are required to show that *Claim 1* is also true for $n = k + 1$.

Inductive step:

We are given a row of all $\mathbb{N}$ from 1 to $n + 1$ arranged in a random order. We now consider three different possibilities:

Possibility 1: $n + 1$ is the first of the row. Then:

$$
\begin{gathered}
(n+1, a_1, a_2, \cdots, a_n) \\
\downarrow \\
(a_1, a_2, \cdots, a_n, n+1) \\
\downarrow \\
\vdots
\end{gathered}
$$

(all configurations contain $n+1$ as the last number, since $(a_1, a_2, \cdots, a_n)$ are all less than $n+1$ and hence cannot 'reach out' to the last number in the reversing process)

Now, comparing the initial configuration with the configurations that follow, we can see that all the following configurations have $n+1$ as the last element, whereas the initial configuration does not. Hence, we have completed our inductive step for *Possibility 1*.

Possibility 2: $n+1$ is the last of the row. Then:

$$
\begin{gathered}
(a_1, a_2, \cdots, a_n, n+1) \\
\downarrow \\
(\underline{a_{i_1}, a_{i_2}, \cdots, a_{i_n}}, n+1) \\
\downarrow \\
\vdots
\end{gathered}
$$

(all configurations contain $n+1$ as the last number)

Now, since there are only n numbers in the underlined part, by the inductive hypothesis, the configuration of the underlined part will be different for each of the following configurations compared to the initial configuration. Hence, we have completed our inductive step for *Possibility 2*.

Possibility 3: $n+1$ is between the first and last element of the row. Then:

$$
\begin{gathered}
(a_1, a_2, \cdots, n+1, \cdots, a_{n-1}, a_n) \\
\downarrow \\
(\underline{a_{i_1}, a_{i_2}, \cdots, n+1, \cdots, a_{i_{n-1}}}, a_n) \\
\downarrow \\
\vdots
\end{gathered}
$$

(a_n remains in the last place until $n+1$ becomes the first element)

$$
\begin{gathered}
\vdots \\
\downarrow \\
(\underline{n+1, a_{j_1}, a_{j_2}, \cdots, a_{j_{n-1}}}, a_n) \\
\downarrow \\
(a_n, a_{j_{n-1}}, \cdots, a_{j_1}, \underline{n+1}) \\
\downarrow \\
\vdots
\end{gathered}
$$

(all of the following configurations will have $n+1$ in the last place)

Using arguments similar to what were used in *Possibility 2* (using the inductive hypothesis for n numbers) and *Possibility 1* ($n+1$ remains in the last place as we proceed further from a certain point), we can say that the underlined parts in the configurations that follow will be different compared to the initial configuration.

So, we have proven *Claim 1*, and hence the required result follows.

$\square$

3 Source

This problem appeared as Problem 6 in the Camp Test of Jamilur Reza Choudhury High Performance Camp 2023 (phase one), which was arranged by Bangladesh Mathematical Olympiad Committee. I happened to be one of those types of students to whom the solution of an approachable problem only became clear after the conclusion of the camp test.