An elementary derivation of Faulhaber's formula:

For a given $p \in \mathbb{Z}^+$, we aim to obtain a polynomial of $n \in \mathbb{N}$, S_p, such that:

$$S_p = 1^p + 2^p + 3^p + \ldots + n^p$$

To start, observe that:

$$S_0 = 1^0 + 2^0 + 3^0 + \ldots + n^0 = \underbrace{1 + 1 + 1 + \ldots + 1}_{n \text{ times}} = n$$

For finding S_1, we can use the well-known pairing technique due to Gauss:

$$S_1 = 1 + 2 + 3 + \ldots + n$$

$$+ \; S_1 = n + (n+1) + (n+2) + \ldots + 1$$

$$2S_1 = \underbrace{(n+1) + (n+1) + (n+1) + \ldots + (n+1)}_{n \text{ times}}$$

$$\Rightarrow 2S_1 = n(n+1) = n^2 + n$$

$$\therefore S_1 = \frac{n^2 + n}{2}$$

For finding $S_2, S_3, \ldots$, this pairing trick is no longer useful, since $a^k + b^k \neq (a+1)^k + (b-1)^k$ for any $k > 1$. However, analysis of the following expression, will prove to be useful shortly:

$$(m+1)^{p+1} - m^{p+1}$$

The binomial formula tells us:

$$(a+b)^n = \binom{n}{0}a^0b^n + \binom{n}{1}a^1b^{n-1} + \ldots + \binom{n}{n}a^nb^0$$

$$= \sum_{r=0}^{n} \binom{n}{r} a^r b^{n-r}$$

So,

$$(m+1)^{p+1} - m^{p+1} = \left(\sum_{r=0}^{p+1} \binom{p+1}{r} m^r \right) - m^{p+1}$$

$$= \sum_{r=0}^{p} \binom{p+1}{r} m^r$$

Now, we construct the following set of equations and add them up:

$$\cancel{2^{p+1}} - 1^{p+1} = \sum_{r=0}^{p} \binom{p+1}{r} 1^r$$

$$+ \quad \cancel{3^{p+1}} - \cancel{2^{p+1}} = \sum_{r=0}^{p} \binom{p+1}{r} 2^r$$

$$\vdots$$

$$+ \quad \cancel{n^{p+1}} - \cancel{(n-1)^{p+1}} = \sum_{r=0}^{p} \binom{p+1}{r} (n-1)^r$$

$$+ \quad (n+1)^{p+1} - \cancel{n^{p+1}} = \sum_{r=0}^{p} \binom{p+1}{r} n^r$$

$$(n+1)^{p+1} - 1^{p+1} = \sum_{r=0}^{p} \binom{p+1}{r} \left(1^r + 2^r + 3^r + \ldots + n^r\right)$$

Or,

$$(n+1)^{p+1} - 1 = \sum_{r=0}^{p} \binom{p+1}{r} S_r$$

$$\binom{p+1}{p} = \frac{(p+1)!}{(p!)(\cancel{p+1}-\cancel{p})!} = \frac{(p+1)\cancel{(p)\cdots(2)(1)}}{\cancel{(p)\cdots(2)(1)}} = p+1$$

$$= \binom{p+1}{p} S_p + \sum_{r=0}^{p-1} \binom{p+1}{r} S_r$$

$$= (p+1) S_p + \sum_{r=0}^{p-1} \binom{p+1}{r} S_r$$

$$\therefore \quad S_p = \frac{1}{p+1} \cdot \left[(n+1)^{p+1} - 1 - \sum_{r=0}^{p-1} \binom{p+1}{r} S_r \right]$$

$$= \frac{1}{p+1} \cdot \left[\sum_{r=0}^{p+1} \binom{p+1}{r} n^r - 1 - \sum_{r=0}^{p-1} \binom{p+1}{r} S_r \right]$$

$$= \frac{1}{p+1} \cdot \left[\left(\sum_{r=1}^{p+1} \binom{p+1}{r} n^r \right) \underbrace{+\,1} - 1 - \left(\sum_{r=1}^{p-1} \binom{p+1}{r} S_r \right) - n \right]$$

$r=0$ gives 1 as summand

$r=0$ gives n as summand (since $S_0 = n$)

$$= \frac{1}{p+1} \left[\sum_{r=1}^{p+1} \binom{p+1}{r} n^r - \left(\sum_{r=1}^{p-1} \binom{p+1}{r} S_r \right) - n \right]$$

$$= \frac{1}{p+1} \left[\left(\sum_{r=1}^{p-1} \binom{p+1}{r} n^r \right) + (p+1) n^p + n^{p+1} - n - \sum_{r=1}^{p-1} \binom{p+1}{r} S_r \right]$$

$$= \frac{1}{p+1}\left[\sum_{r=1}^{p-1} \binom{p+1}{r}\left(n^r - S_r\right) + (p+1)n^p + n^{p+1} - n \right]$$

$$= n^p + \frac{1}{p+1}\left[\sum_{r=1}^{p-1} \binom{p+1}{r}\left(n^r - S_r\right) + n\left(n^p - 1\right)\right]$$

$$\therefore \quad \boxed{S_p = n^p + \frac{1}{p+1}\left[\left(\sum_{r=1}^{p-1}\binom{p+1}{r}\left(n^r - S_r\right)\right) + n\left(n^p - 1\right)\right]}$$

where,

$$S_0 = n$$

$\square$

Applying the formula:

This formula tells us how to find the formula for S_p given we know the formula for $S_0, S_1, \dots, S_{p-1}$.

Let's find out the formula for S_2:

$$S_2 = n^2 + \frac{1}{3}\left(\binom{3}{1}\left(n - \frac{n^2+n}{2}\right) + n\left(n^2 - 1\right)\right)$$

$$= n^2 + \frac{1}{3}\left(3\left(\frac{2n - n^2 - n}{2}\right) + n^3 - n\right)$$

$$= n^2 + \frac{1}{3}\left(3\left(\frac{n - n^2}{2}\right) + n^3 - n\right)$$

$$= n^2 + \frac{1}{6}\left(3n - 3n^2 + 2n^3 - 2n\right)$$

$$= \frac{6n^2}{6} + \frac{2n^3 - 3n^2 + n}{6}$$

$$= \frac{2n^3 + 3n^2 + n}{6}$$

$$= \frac{n(2n^2 + 3n + 1)}{6}$$

$$= \frac{(n)(2n^2 + 2n + n + 1)}{6}$$

$$= \frac{(n)(2n(n+1) + (n+1))}{6}$$

$$\therefore \boxed{S_2 = \frac{(n)(n+1)(2n+1)}{6}}$$

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18/9/2023