

## An elementary derivation of Faulhaber's formula:

For a given  $p \in \mathbb{Z}^+$ , we aim to obtain a polynomial of  $n \in \mathbb{N}$ ,  $S_p$ , such that:

$$S_p = 1^p + 2^p + 3^p + \dots + n^p$$

To start, observe that:

$$S_0 = 1^0 + 2^0 + 3^0 + \dots + n^0 = \underbrace{1 + 1 + 1 + \dots + 1}_{n \text{ times}} = n$$

For finding  $S_1$ , we can use the well-known pairing technique due to Gauss:

$$\begin{aligned} S_1 &= 1 + 2 + 3 + \dots + n \\ + S_1 &= n + (n+1) + (n+2) + \dots + 1 \\ \hline 2S_1 &= \underbrace{(n+1) + (n+1) + (n+1) + \dots + (n+1)}_{n \text{ times}} \end{aligned}$$

$$\Rightarrow 2S_1 = n(n+1) = n^2 + n$$

$$\therefore S_1 = \frac{n^2 + n}{2}$$

For finding  $S_2, S_3, \dots$ , this pairing trick is no longer useful, since  $a^k + b^k \neq (a+1)^k + (b-1)^k$  for any  $k > 1$ . However, analysis of the following expression, will prove to be useful shortly:

$$(m+1)^{p+1} - m^{p+1}$$

The binomial formula tells us:

$$(a+b)^n = \binom{n}{0} a^0 b^n + \binom{n}{1} a^1 b^{n-1} + \dots + \binom{n}{n} a^n b^0$$
$$= \sum_{r=0}^n \binom{n}{r} a^r b^{n-r}$$

So,

$$(m+1)^{p+1} - m^{p+1} = \left( \sum_{r=0}^{p+1} \binom{p+1}{r} m^r \right) - m^{p+1}$$
$$= \sum_{r=0}^p \binom{p+1}{r} m^r$$

Now, we construct the following set of equations and add them up:

$$\begin{aligned} 2^{p+1} - 1^{p+1} &= \sum_{r=0}^p \binom{p+1}{r} 1^r \\ + \quad 3^{p+1} - 2^{p+1} &= \sum_{r=0}^p \binom{p+1}{r} 2^r \\ &\vdots \\ + \quad n^{p+1} - (n-1)^{p+1} &= \sum_{r=0}^p \binom{p+1}{r} (n-1)^r \\ + \quad (n+1)^{p+1} - n^{p+1} &= \sum_{r=0}^p \binom{p+1}{r} n^r \end{aligned}$$

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$$(n+1)^{p+1} - 1^{p+1} = \sum_{r=0}^p \binom{p+1}{r} (1^r + 2^r + 3^r + \dots + n^r)$$

Or,

$$(n+1)^{p+1} - 1 = \sum_{r=0}^p \binom{p+1}{r} S_r$$

$$\left| \begin{aligned} \binom{p+1}{p} &= \frac{(p+1)!}{(p!)(p+1-p)!} \\ &= \frac{(p+1)\cancel{(p)} \dots \cancel{(2)}\cancel{(1)}}{\cancel{(p)} \dots \cancel{(2)}\cancel{(1)}} \\ &= p+1 \end{aligned} \right.$$

$$= \binom{p+1}{p} S_p + \sum_{r=0}^{p-1} \binom{p+1}{r} S_r$$

$$= (p+1) S_p + \sum_{r=0}^{p-1} \binom{p+1}{r} S_r$$

$$\therefore S_p = \frac{1}{p+1} \cdot \left[ (n+1)^{p+1} - 1 - \sum_{r=0}^{p-1} \binom{p+1}{r} S_r \right]$$

$$= \frac{1}{p+1} \cdot \left[ \sum_{r=0}^{p+1} \binom{p+1}{r} n^r - 1 - \sum_{r=0}^{p-1} \binom{p+1}{r} S_r \right]$$

$$= \frac{1}{p+1} \cdot \left[ \left( \sum_{r=1}^{p+1} \binom{p+1}{r} n^r \right) \underbrace{+ 1}_{\substack{\uparrow \\ r=0 \text{ gives } 1 \\ \text{as summand}}} - 1 - \left( \sum_{r=1}^{p-1} \binom{p+1}{r} S_r \right) \underbrace{- n}_{\substack{\uparrow \\ r=0 \text{ gives } n \\ \text{as summand} \\ (\text{since } S_0 = n)}} \right]$$

$$= \frac{1}{p+1} \left[ \sum_{r=1}^{p+1} \binom{p+1}{r} n^r - \left( \sum_{r=1}^{p-1} \binom{p+1}{r} S_r \right) - n \right]$$

$$= \frac{1}{p+1} \left[ \left( \sum_{r=1}^{p-1} \binom{p+1}{r} n^r \right) + (p+1)n^p + n^{p+1} - n - \sum_{r=1}^{p-1} \binom{p+1}{r} S_r \right]$$

$$= \frac{1}{p+1} \left[ \sum_{r=1}^{p-1} \binom{p+1}{r} (n^r - S_r) + (p+1)n^p + n^{p+1} - n \right]$$

$$= n^p + \frac{1}{p+1} \left[ \sum_{r=1}^{p-1} \binom{p+1}{r} (n^r - S_r) + n(n^p - 1) \right]$$

$$\therefore S_p = n^p + \frac{1}{p+1} \left[ \left( \sum_{r=1}^{p-1} \binom{p+1}{r} (n^r - S_r) \right) + n(n^p - 1) \right]$$

where,

$$S_0 = n$$

□

Applying the formula:

This formula tells us how to find the formula for  $S_p$  given we know the formula for  $S_0, S_1, \dots, S_p$ .

Let's find out the formula for  $S_2$ :

$$S_2 = n^2 + \frac{1}{3} \left( \binom{3}{1} \left( n - \frac{n^2+n}{2} \right) + n(n^2-1) \right)$$

$$= n^2 + \frac{1}{3} \left( 3 \left( \frac{2n-n^2-n}{2} \right) + n^3 - n \right)$$

$$= n^2 + \frac{1}{3} \left( 3 \left( \frac{n-n^2}{2} \right) + n^3 - n \right)$$

$$= n^2 + \frac{1}{6} (3n - 3n^2 + 2n^3 - 2n)$$

$$= \frac{6n^2}{6} + \frac{2n^3 - 3n^2 + n}{6}$$

$$= \frac{2n^3 + 3n^2 + n}{6}$$

$$= \frac{n(2n^2 + 3n + 1)}{6}$$

$$= \frac{(n)(2n^2 + 2n + n + 1)}{6}$$

$$= \frac{(n)(2n(n+1) + (n+1))}{6}$$

$$\therefore S_2 = \frac{(n)(n+1)(2n+1)}{6}$$



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