<u>Newton and Gravity: A Historical Note on the Mathematics behind it</u>

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In nature, we often seen the so-called *Inverse Square Law* pop up everywhere. For example, in isotropic radiation from stars (symmetrical in all directions), we have the relation:

$$F_{received} = \frac{E}{4\pi d^2}$$

(Where $F_{received}$ is the Energy flux (Energy per unit area) received at a distance d from a source radiating energy of amount E at a given time). **These types of equations are categorised as inverse square laws because a quantity decreases by a factor of the square of the distance.**

In 1666, Sir Issac Newton came up with his Universal Law of Gravitation. Apart from quantitative significance, his law reshaped the view of 17th century physics on gravity and introduced the idea that gravity is universal; the force that makes an apple fall to the ground is the same as the force that makes Moon go around the Earth!

The Universal Law of Gravitation states that any two masses in the universe, M and m separated by a distance of d exerts equal attractive forces on each other of magnitude F, where:

$$F = \frac{GMm}{d^2}$$

Here, G is known as the Universal Gravitational constant (which is about $6.67 * 10^{-11}$ in SI units).

Again, we can see that the gravitational attraction force follows the inverse square law.

Using Newton's Second Law, $F_{net} = ma$, we can find the acceleration of mass m when it is at a distance of d from M:

$$a = \frac{GM}{d^2} \qquad \text{- - - (i)}$$

Legends mention the story of Newton beginning to contemplate the clockwork of gravity after 'being occasioned by the fall of an apple'.

Newton's genius lies in the fact that he was the first to realise that the force which makes an apple fall onto Earth's surface and make the Moon go around are both the same; the Earth's gravitational attraction! The only difference is that the Moon has a velocity tangential to its orbit, so it doesn't crash onto the Earth's surface.

Given the information that the Moon orbits the Earth in an almost circular path with an angular speed of 360°/28 days, and that the distance to the moon is about 60 times the radius of the Earth (determined from Lunar Eclipses), we can perform a reasonably accurate calculation to verify Newton's Universal Law of Gravitation.

From our one dimensional motion equations given by Galileo, we can find the distance an object (or an apple!) falls in 1 second close to the Earth's surface (when released from rest) by the equation:

$s = \frac{1}{2}gt^2$ (here, since t = 1 sec, the t^2 term becomes 1).

Using g = 9.8 m/s^2:

$$s_{1sec} = 4.9 \text{ meters}$$

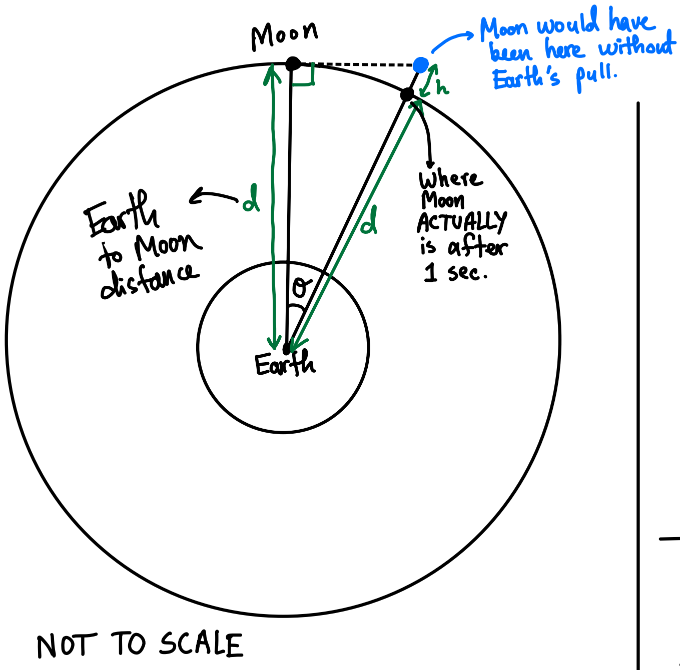
Now, since the Moon is 60 times as far away as an apple close to the Earth's surface, its acceleration due to Earth's gravitational pull is 60^2 = 3600 times weaker [by equation (i)].

So in 1 second, the Moon falls by:

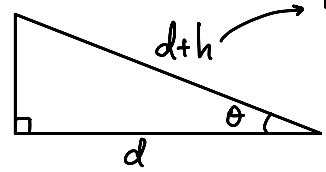
$$s_{1sec} = 4.9/3600 = 1.3 * 10^{-3} \text{ meters}$$

The calculation that Newton performed to validate this result is purely geometrical and illustrated in the next page:

The Geometrical Argument



We get the right angled triangle:



Using the definition of the trigonometric function Cosine:

$$\cos(\theta) = \frac{d}{d+h}$$

$$\Rightarrow d\cdot\cos(\theta) + h\cdot\cos(\theta) = d$$

$$\Rightarrow h\cdot\cos(\theta) = d - d\cdot\cos(\theta)$$

$$\therefore h = \frac{d\left(1-\cos(\theta)\right)}{\cos(\theta)} \quad \text{—— (i)}$$

Now,

In 28 days $\xrightarrow{\text{the moon revolves}}$ 360°

$\therefore$ " 1 day $\longrightarrow \frac{360^\circ}{28}$

$\therefore$ " $\underbrace{86400 \text{ sec}}_{1 \text{ day}} \longrightarrow \frac{360^\circ}{28}$

$\therefore$ " 1 sec $\longrightarrow \frac{360^\circ}{28 \times 86400}$

$= (1.49 \times 10^{-4})^\circ$

So, we have:

$\theta = 1.49 \times 10^{-4}$

We know:

$d = 3.84 \times 10^{8}$ m

$\therefore$ From (i):

$h = 1.3 \times 10^{-3}$ m

(Which is as predicted by Newton!)