General solution to cubic equations

→ At first, we will see how to solve a 'depressed cubic' (cubic equation without the x^2 term). Then we will introduce a method to convert a general cubic equation into a depressed cubic.

Solving the Depressed Cubic (completing the cube) – Tartaglia's Method:

Suppose we have the following depressed cubic:

$$ax^3 + bx + c = 0$$

$$\Rightarrow x^3 + \frac{b}{a}x = -\frac{c}{a}$$

Think of this in terms of volumes: We have a cube of side length x, and we must add a volume of $\frac{b}{a}x$ to it SUCH THAT the total volume is $-\frac{c}{a}$.

→ Suppose we extend the sides of the cube by length y.

$\therefore$ The new side length, $z = x + y$.

So, the new volume of the cube is:

$$\begin{aligned}
&(x+y)^3 \\
&= (x+y)(x^2 + 2xy + y^2) \\
&= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3 \\
&= x^3 + 3x^2y + 3xy^2 + y^3
\end{aligned}$$

Neglecting the x^3 and y^3 term for the time being:

$$3x^2y + 3xy^2$$
$$= 3xy(x+y)$$
$$= (3yz)x \quad [\because z = x+y]$$

→ Notice, our original depressed cubic was:

$$x^3 + \frac{b}{a}x = -\frac{c}{a}$$

If we let $\boxed{\frac{b}{a} = 3yz \quad \text{—— (i)}}$, our equation becomes:

$$x^3 + (3yz)x = -\frac{c}{a}$$

$$\Rightarrow \underbrace{x^3 + (3yz)x + y^3}_{z^3} = y^3 - \frac{c}{a}$$

$$\therefore \boxed{z^3 = y^3 - \frac{c}{a} \quad \text{—— (ii)}}$$

We can now solve (i) and (ii) simultaneously:

From (i), $z = \dfrac{b}{3ay}$

Substituting this into (ii), $\left(\dfrac{b}{3ay}\right)^3 = y^3 - \dfrac{c}{a}$

$$\Rightarrow \frac{b^3}{27a^3y^3} = y^3 - \frac{c}{a}$$

$$\Rightarrow 27a^3y^3\left(y^3 - \frac{c}{a}\right) = b^3$$

$$\Rightarrow (27a^3)y^6 - (27a^2c)\,y^3 = b^3$$

Now, let $\boxed{p = y^3}$: $(27a^3)p^2 - (27a^2c)p - b^3 = 0$

This is a quadratic equation, which we can solve using the quadratic formula:

$$p = \frac{27a^2c \pm \sqrt{729a^4c^2 - 4\cdot 27a^3 \cdot - b^2}}{2\cdot 27a^3}$$

$$\therefore p = \frac{27a^2c \pm \sqrt{729a^4c^2 + 108a^3b^2}}{54a^3}$$

Now, since $\boxed{p = y^3}$:

$$y = \left(\frac{27a^2c \pm \sqrt{729a^4c^2 + 108a^3b^2}}{54a^3}\right)^{\frac{1}{3}}$$

→ For the next step, you can take any of the two possible values for y. Suppose you take the larger one (+ sign before radical).

We know, $z = \dfrac{b}{3ay}$.

<u>Also:</u> $z = x + y \Rightarrow x = z - y = \dfrac{b}{3ay} - y$

So, we have one possible solution for x:

$$x_0 = \frac{b}{3ay} - y \quad , \text{ where:}$$

$$y = \left(\frac{27a^2c + \sqrt{729a^4c^2 + 108a^3b^2}}{54a^3} \right)^{\frac{1}{3}}$$

→ Now, we use the fact that since this particular x (let's call it x_0) is a root of the function $ax^3 + bx + c$, so $(x - x_0)$ must be a factor of $ax^3 + bx + c$.

$$\therefore \; x - x_0 \mid ax^3 + bx + c$$

$$\therefore \; \frac{ax^3 + bx + c}{x - x_0} \mid ax^3 + bx + c$$

This is ANOTHER factor of $ax^3 + bx + c$

$$\begin{array}{r|l} & ax^2 + ax_0x + (b + ax_0^2) \\ \hline x - x_0 & ax^3 + bx + c \\ & ax^3 - ax_0x^2 \\ \hline & bx + ax_0x^2 \\ & ax_0x^2 - ax_0^2x \\ \hline & bx + ax_0^2x + c \\ & bx + ax_0^2x - bx_0 - ax_0^3 + c \\ \hline & -bx_0 - ax_0^3 + c \end{array}$$

This is the quotient we are looking for

this is actually 0.

For finding other roots of our original equation, it is just sufficient to find roots of the NEW factor:

$$ax^2 + ax_0x + (b + ax_0^2) = 0$$

$$\therefore \quad x = \frac{-ax_0 \pm \sqrt{a^2x_0^2 - 4a(b + ax_0^2)}}{2a}$$

Hence, ALL the solutions to the depressed cubic $ax^3 + bx + c = 0$ are:

$$\boxed{x_0 = \frac{b}{3ay} - y, \text{ where: } y = \left(\frac{27a^2c + \sqrt{729a^4c^2 + 108a^3b^2}}{54a^3}\right)^{\frac{1}{3}}}$$

$$\boxed{x_1 = \frac{-ax_0 \pm \sqrt{a^2x_0^2 - 4a(b + ax_0^2)}}{2a}}$$

$$\boxed{x_2 = \frac{-ax_0 - \sqrt{a^2x_0^2 - 4a(b + ax_0^2)}}{2a}}$$

Now, we will explore the method to convert a general cubic equation into a depressed cubic.

Cardano's method of converting general cubic equation to depressed cubic:

We have: $ax^3 + bx^2 + cx + d = 0$

$\rightarrow$ Subsitute $\boxed{x = z - \frac{b}{3a}}$ — Think about the possible motivation behind the substitution.

So, our equation becomes:

$$a\left(z - \frac{b}{3a}\right)^3 + b\left(z - \frac{b}{3a}\right)^2 + c\left(z - \frac{b}{3a}\right) + d = 0$$

$$\Rightarrow a\left(z^3 - 3z^2\frac{b}{3a} + 3z\frac{b^2}{9a^2} - \frac{b^3}{27a^3}\right) + b\left(z^2 - 2z\frac{b}{3a} + \frac{b^2}{9a^2}\right) + c\left(z - \frac{b}{3a}\right) + d = 0$$

$$\Rightarrow az^3 \cancel{- z^2 b} + \frac{\cancel{3}zb^2}{\cancel{9a}\,3a} - \frac{b^3}{27a^2} \cancel{+ bz^2} - \frac{2zb^2}{3a} + \frac{b^3}{9a^2} + cz - \frac{bc}{3a} + d = 0$$

$$\Rightarrow \boxed{az^3 + \left(c - \frac{b^2}{3a}\right)z + \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0}$$

This can now be solved by Tartaglia's Method.

→ Then, for each solution for z: $\boxed{x = z - \frac{b}{3a}}$

Hence, if we are given a cubic equation: $ax^3 + bx^2 + cx + d = 0$,

→ At first, make the following considerations:

$$\boxed{\begin{aligned} a' &= a \\ b' &= c - \frac{b^2}{3a} \\ c' &= d + \frac{2b^3}{27a^2} - \frac{bc}{3a} \end{aligned}}$$

Then, we have the following solutions:

$$x_0 = \frac{b'}{3a'y} - y - \frac{b}{3a}, \text{ where:}$$

$$y = \left(\frac{27a'^2c' + \sqrt{729a'^4c'^2 + 108\,a'^3b'^2}}{54a'^3} \right)^{\frac{1}{3}}$$

$$x_1 = \frac{-a'x_0 \pm \sqrt{a'^2x_0'^2 - 4a'(b' + a'x_0^2)}}{2a'} - \frac{b}{3a}$$

$$x_2 = \frac{-a'x_0 - \sqrt{a'^2x_0^2 - 4a'(b' + a'x_0^2)}}{2a'} - \frac{b}{3a}$$

13/6/2023