General solution to cubic equations

→ At first, we will see how to solve a 'depressed cubic' (cubic equation without the x^2 term). Then we will introduce a method to convert a general cubic equation into a depressed cubic.

Solving the Depressed Cubic (completing the cube) – Tartaglia's Method:

Suppose we have the following depressed cubic:

$$ax^3 + bx + c = 0$$

$$\Rightarrow x^3 + \frac{b}{a}x = -\frac{c}{a}$$

Think of this in terms of volumes: We have a <u>cube</u> of side length x, and we must add a volume of $\frac{b}{a}x$ to it SUCH THAT the total volume is $-\frac{c}{a}$.

→ Suppose we extend the sides of the cube by length y.

$\therefore$ The new side length, $z = x + y$.

So, the new volume of the cube is:

$$
\begin{aligned}
&(x+y)^3 \\
&= (x+y)(x^2 + 2xy + y^2) \\
&= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3 \\
&= x^3 + 3x^2y + 3xy^2 + y^3
\end{aligned}
$$

Neglecting the x^3 and y^3 term for the time being:

$$3x^2y + 3xy^2$$
$$= 3xy(x+y)$$
$$= (3yz)x \quad [\because z = x+y]$$

→ Notice, our original depressed cubic was:

$$x^3 + \frac{b}{a}x = -\frac{c}{a}$$

If we let $\boxed{\frac{b}{a} = 3yz \quad \text{— (i)}}$, our equation becomes:

$$x^3 + (3yz)x = -\frac{c}{a}$$
$$\Rightarrow \underbrace{x^3 + (3yz)x + y^3}_{z^3} = y^3 - \frac{c}{a}$$

$$\therefore \boxed{z^3 = y^3 - \frac{c}{a} \quad \text{— (ii)}}$$

We can now solve (i) and (ii) simultaneously:

From (i), $z = \dfrac{b}{3ay}$

Substituting this into (ii), $\left(\dfrac{b}{3ay}\right)^3 = y^3 - \dfrac{c}{a}$

$$\Rightarrow \frac{b^3}{27a^3y^3} = y^3 - \frac{c}{a}$$

$$\Rightarrow 27a^3y^3\left(y^3 - \frac{c}{a}\right) = b^3$$

$$\Rightarrow (27a^3)y^6 - (27a^2c)y^3 = b^3$$

Now, let $\boxed{p = y^3}$: $(27a^3)p^2 - (27a^2c)p - b^3 = 0$

This is a quadratic equation, which we can solve using the quadratic formula:

$$p = \frac{27a^2c \pm \sqrt{729a^4c^2 - 4 \cdot 27a^3 \cdot -b^2}}{2 \cdot 27a^3}$$

$$\therefore p = \frac{27a^2c \pm \sqrt{729a^4c^2 + 108a^3b^2}}{54a^3}$$

Now, since $\boxed{p = y^3}$:

$$y = \left(\frac{27a^2c \pm \sqrt{729a^4c^2 + 108a^3b^2}}{54a^3}\right)^{\frac{1}{3}}$$

$\rightarrow$ For the next step, you can take any of the two possible values for y. Suppose you take the larger one (+ sign before radical).

We know, $z = \frac{b}{3ay}$.

<u>Also:</u> $z = x + y \Rightarrow x = z - y = \frac{b}{3ay} - y$

So, we have one possible solution for x:

$$x_0 = \frac{b}{3ay} - y \quad , \text{ where :}$$

$$y = \left(\frac{27a^2c + \sqrt{729a^4c^2 + 108a^3b^2}}{54a^3} \right)^{\frac{1}{3}}$$

→ Now, we use the fact that since this particular x (let's call it x_0) is a root of the function $ax^3 + bx + c$, so $(x - x_0)$ must be a factor of $ax^3 + bx + c$.

$$\therefore \; x - x_0 \mid ax^3 + bx + c$$

$$\therefore \; \frac{ax^3 + bx + c}{x - x_0} \mid ax^3 + bx + c$$

→ This is ANOTHER factor of $ax^3 + bx + c$

$$\begin{array}{r|l}
 & ax^2 + ax_0x + (b + ax_0^2) \\ \hline
x - x_0 & ax^3 + bx + c \\
 & ax^3 - ax_0x^2 \\ \hline
 & bx + ax_0x^2 \\
 & ax_0x^2 - ax_0^2x \\ \hline
 & bx + ax_0^2x + c \\
 & bx + ax_0^2x - bx_0 - ax_0^3 + c \\ \hline
 & -bx_0 - ax_0^3 + c
\end{array}$$

→ This is the quotient we are looking for

→ this is actually 0.

For finding other roots of our original equation, it is just sufficient to find roots of the NEW factor:

$$ax^2 + ax_0x + (b + ax_0^2) = 0$$

$$\therefore\ x = \frac{-ax_0 \pm \sqrt{a^2x_0^2 - 4a(b + ax_0^2)}}{2a}$$

Hence, ALL the solutions to the depressed cubic $ax^3 + bx + c = 0$ are:

$$\boxed{x_0 = \frac{b}{3ay} - y \ , \text{ where: } \quad y = \left(\frac{27a^2c + \sqrt{729a^4c^2 + 108a^3b^2}}{54a^3} \right)^{\frac{1}{3}}}$$

$$\boxed{x_1 = \frac{-ax_0 \pm \sqrt{a^2x_0^2 - 4a(b + ax_0^2)}}{2a}}$$

$$\boxed{x_2 = \frac{-ax_0 - \sqrt{a^2x_0^2 - 4a(b + ax_0^2)}}{2a}}$$

Now, we will explore the method to convert a general cubic equation into a depressed cubic.

Cardano's method of converting general cubic equation to depressed cubic:

We have: $ax^3 + bx^2 + cx + d = 0$

$\rightarrow$ Subsitute $\boxed{x = z - \frac{b}{3a}}$ — Think about the possible motivation behind the substitution.

So, our equation becomes:

$$a\left(z - \frac{b}{3a}\right)^3 + b\left(z - \frac{b}{3a}\right)^2 + c\left(z - \frac{b}{3a}\right) + d = 0$$

$$\Rightarrow a\left(z^3 - 3z^2\frac{b}{3a} + 3z\frac{b^2}{9a^2} - \frac{b^3}{27a^3}\right) + b\left(z^2 - 2z\frac{b}{3a} + \frac{b^2}{9a^2}\right) + c\left(z - \frac{b}{3a}\right) + d = 0$$

$$\Rightarrow az^3 - \cancel{z^2 b} + \frac{\cancel{3}zb^2}{\cancel{9a}\,3a} - \frac{b^3}{27a^2} + \cancel{bz^2} - \frac{2zb^2}{3a} + \frac{b^3}{9a^2} + cz - \frac{bc}{3a} + d = 0$$

$$\Rightarrow \boxed{az^3 + \left(c - \frac{b^2}{3a}\right)z + \left(d + \frac{2b^3}{27a^2} - \frac{bc}{3a}\right) = 0}$$

This can now be solved by Tartaglia's Method.

→ Then, for each solution for z: $\boxed{x = z - \frac{b}{3a}}$

Hence, if we are given a cubic equation: $ax^3 + bx^2 + cx + d = 0$,

→ At first, make the following considerations:

$$\boxed{\begin{aligned} a' &= a \\ b' &= c - \frac{b^2}{3a} \\ c' &= d + \frac{2b^3}{27a^2} - \frac{bc}{3a} \end{aligned}}$$

Then, we have the following solutions:

$$x_0 = \frac{b'}{3a'y} - y - \frac{b}{3a}, \quad \text{where:}$$

$$y = \left(\frac{27a'^2c' + \sqrt{729a'^4c'^2 + 108a'^3b'^2}}{54a'^3} \right)^{\frac{1}{3}}$$

$$\boxed{x_1 = \frac{-a'x_0 \pm \sqrt{a'^2x_0'^2 - 4a'(b' + a'x_0^2)}}{2a'}}$$

$$\boxed{x_2 = \frac{-a'x_0 - \sqrt{a'^2x_0^2 - 4a'(b' + a'x_0^2)}}{2a'}}$$

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