$$S_p = f(n,p) = 1^p + 2^p + 3^p + \dots + n^p$$

$\boxed{n, p \in \mathbb{N}}$ ↳ what is the explicit formula for this?

<u>For $p=0$:</u>

$$S_0 = 1^0 + 2^0 + 3^0 + \dots + n^0$$
$$= \underbrace{1 + 1 + 1 + \dots + 1}_{n \text{ times}}$$
$$= n$$

$\therefore \boxed{S_0 = n}$

<u>For $p=1$:</u>

$$S_1 = 1 + 2 + 3 + \dots + n$$
$$+ \; S_1 = n + (n-1) + (n-2) + \dots + 1$$
$$2S_1 = \underbrace{(n+1) + (n+1) + (n+1) + \dots + (n+1)}_{n \text{ times}}$$
$$= n(n+1)$$
$$\therefore S_1 = \frac{n(n+1)}{2}$$

<u>For $p=2$:</u> (This will be used to illustrate the approach for a general power, p.)

Consider the difference of two consecutive cubes. → power 3 ↙ 1 more than the power we are currently working with.

$$(m+1)^3 - m^3 = \cancel{m^3} + 3m^2 + 3m + 1^3 - \cancel{m^3}$$
$$= 3m^2 + 3m + 1$$

Now, we construct the following set of equations:

$$
\begin{aligned}
& \cancel{2^3} - 1^3 &&= 3(1)^2 + 3(1) + 1 \\
+\ & \cancel{3^3} - \cancel{2^3} &&= 3(2)^2 + 3(2) + 1 \\
+\ & \cancel{4^3} - \cancel{3^3} &&= 3(3)^2 + 3(3) + 1 \\
& \quad \vdots \\
+\ & \cancel{n^3} - \cancel{(n-1)^3} &&= 3(n-1)^2 + 3(n-1) + 1 \\
+\ & (n+1)^3 - \cancel{n^3} &&= 3(n)^2 + 3(n) + 1 \\
\hline
& (n+1)^3 - 1^3 &&= 3(1^2 + 2^2 + 3^2 + \ldots + n^2) \\
& && \quad + \\
& && \quad 3(1 + 2 + 3 + \ldots + n) \\
& && \quad + \\
& && \quad 1(1^0 + 2^0 + 3^0 + \ldots + n^0) \\
& && = \underbrace{3S_2 + 3S_1 + 1S_0}
\end{aligned}
$$

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 1 & & \\
 & 1 & & 2 & & 1 & \\
1 & & 3 & & 3 & & 1 \\
 & & \uparrow & & \uparrow & & \uparrow \\
 & & 3_{c_1} & & 3_{c_2} & & 3_{c_3}
\end{array}
$$

In general:

$$(n+1)^{p+1} - 1^{p+1} = \binom{p+1}{1} S_p + \binom{p+1}{2} S_{p-1} + \binom{p+1}{3} S_{p-2} + \ldots + \binom{p+1}{r} S_{p-r+1} + \ldots + \binom{p+1}{p+1} S_0$$

choose operator

$$\therefore \boxed{(n+1)^{p+1} - 1 = \sum_{r=1}^{p+1} \binom{p+1}{r} S_{(p+1)-r}}$$

where,

$$\boxed{S_x = 1^x + 2^x + 3^x + \ldots + n^x \quad \text{for } x \in \mathbb{N}}$$

Substituting $p = 2$:

$$(n+1)^3 - 1 = \binom{3}{1} S_2 + \binom{3}{2} S_1 + \binom{3}{3} S_0$$

$$= 3S_2 + 3S_1 + S_0$$

$$= 3S_2 + 3 \cdot \frac{n(n+1)}{2} + n$$

$$= 3S_2 + \frac{3n^2 + 3n}{2} + n$$

$$\Rightarrow n^3 + 3n^2 + 3n \cancel{+1} \cancel{-1} = 3S_2 + \frac{3n^2 + 3n + 2n}{2}$$

$$\Rightarrow n(n^2 + 3n + 3) = 3S_2 + \frac{3n^2 + 5n}{2}$$

$$\Rightarrow 2n(n^2 + 3n + 3) = 6S_2 + 3n^2 + 5n$$

$$\Rightarrow 6S_2 = 2n(n^2 + 3n + 3) - n(3n + 5)$$

$$= n(2n^2+6n+6) - n(3n+5)$$

$$= n(2n^2+6n+6-3n-5)$$

$$= n(2n^2+3n+1)$$

$$2n^2+3n+1 = 2n^2+2n+n+1$$

$$= 2n(n+1)+(n+1)$$

$$= (2n+1)(n+1)$$

$$\therefore 6S_2 = n(2n+1)(n+1)$$

$$\therefore \boxed{S_2 = \frac{n(2n+1)(n+1)}{6}}$$

In general:

$$\boxed{S_p = \frac{1}{p+1} \cdot \left[(n+1)^{p+1} - 1 - \sum_{r=1}^{p} \binom{p+1}{r+1} S_{p-r} \right]}$$

This can be expressed more clearly as:

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Recurrence for the sum of first n natural numbers, (each) raised to the p^{th} power:

$$\boxed{S_p = \frac{1}{p+1} \cdot \left[\left(\sum_{a=1}^{p+1} \binom{p+1}{a} n^a \right) - \left(\sum_{b=1}^{p} \binom{p+1}{b+1} S_{p-b} \right) \right]}$$

where,

$S_x = 1^x + 2^x + 3^x + \ldots + n^x$, for $x \in \mathbb{N}$

and,

$S_0 = n$