

$$S_p = f(n, p) = 1^p + 2^p + 3^p + \dots + n^p$$

$$\boxed{n, p \in \mathbb{N}}$$

↳ what is the explicit formula for this?

For  $p=0$ :

$$S_0 = 1^0 + 2^0 + 3^0 + \dots + n^0$$

$$= \underbrace{1 + 1 + 1 + \dots + 1}_{n \text{ times}}$$

$$= n$$

$$\therefore \boxed{S_0 = n}$$

For  $p=1$ :

$$S_1 = 1 + 2 + 3 + \dots + n$$

$$+ S_1 = n + (n-1) + (n-2) + \dots + 1$$


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$$2S_1 = \underbrace{(n+1) + (n+1) + (n+1) + \dots + (n+1)}_{n \text{ times}}$$

$$= n(n+1)$$

$$\therefore S_1 = \frac{n(n+1)}{2}$$

For  $p=2$ : (This will be used to illustrate the approach for a general power,  $p$ .)

Consider the difference of two consecutive cubes. power 3

$$(m+1)^3 - m^3 = \cancel{m^3} + 3m^2 + 3m + 1^3 - \cancel{m^3}$$

$$= 3m^2 + 3m + 1$$

1 more than the power we are currently working with.

Now, we construct the following set of equations:

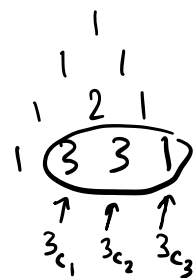
$$\begin{aligned}
 2^3 - 1^3 &= 3(1)^2 + 3(1) + 1 \\
 + \quad 3^3 - 2^3 &= 3(2)^2 + 3(2) + 1 \\
 + \quad 4^3 - 3^3 &= 3(3)^2 + 3(3) + 1 \\
 &\vdots \\
 + \quad n^3 - (n-1)^3 &= 3(n-1)^2 + 3(n-1) + 1 \\
 + \quad (n+1)^3 - n^3 &= 3(n)^2 + 3(n) + 1
 \end{aligned}$$

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$$\begin{aligned}
 (n+1)^3 - 1^3 &= 3(1^2 + 2^2 + 3^2 + \dots + n^2) \\
 &\quad + \\
 &\quad 3(1 + 2 + 3 + \dots + n) \\
 &\quad + \\
 &\quad 1(1^0 + 2^0 + 3^0 + \dots + n^0)
 \end{aligned}$$

$$= \underbrace{(3)S_2 + (3)S_1 + (1)S_0}_{\text{Pascal's Triangle Diagram}}$$

In general:



$$\begin{aligned}
 (n+1)^{p+1} - 1^{p+1} &= \binom{p+1}{1} S_p + \binom{p+1}{2} S_{p-1} + \binom{p+1}{3} S_{p-2} \\
 &\quad + \dots + \binom{p+1}{r} S_{p-r+1} + \dots + \binom{p+1}{p+1} S_0
 \end{aligned}$$

choose operator

$$\therefore (n+1)^{p+1} - 1 = \sum_{r=1}^{p+1} \binom{p+1}{r} S_{(p+1)-r}$$

where,

$$S_x = 1^x + 2^x + 3^x + \dots + n^x \quad \text{for } x \in \mathbb{N}$$

Substituting  $p=2$ :

$$(n+1)^3 - 1 = \binom{3}{1} S_2 + \binom{3}{2} S_1 + \binom{3}{3} S_0$$

$$= 3S_2 + 3S_1 + S_0$$

$$= 3S_2 + 3 \cdot \frac{n(n+1)}{2} + n$$

$$= 3S_2 + \frac{3n^2 + 3n}{2} + n$$

$$\Rightarrow n^3 + 3n^2 + 3n + 1 - 1 = 3S_2 + \frac{3n^2 + 3n + 2n}{2}$$

$$\Rightarrow n(n^2 + 3n + 3) = 3S_2 + \frac{3n^2 + 5n}{2}$$

$$\Rightarrow 2n(n^2 + 3n + 3) = 6S_2 + 3n^2 + 5n$$

$$\Rightarrow 6S_2 = 2n(n^2 + 3n + 3) - n(3n + 5)$$

$$\begin{aligned}
 &= n(2n^2 + 6n + 6) - n(3n + 5) \\
 &= n(2n^2 + 6n + 6 - 3n - 5) \\
 &= n(2n^2 + 3n + 1)
 \end{aligned}$$

$$\begin{aligned}
 2n^2 + 3n + 1 &= 2n^2 + 2n + n + 1 \\
 &= 2n(n+1) + (n+1) \\
 &= (2n+1)(n+1)
 \end{aligned}$$

$$\therefore 6S_2 = n(2n+1)(n+1)$$

$$\therefore S_2 = \frac{n(2n+1)(n+1)}{6}$$

In general :

$$S_p = \frac{1}{p+1} \cdot \left[ (n+1)^{p+1} - 1 - \sum_{r=1}^p \binom{p+1}{r+1} S_{p-r} \right]$$

This can be expressed more clearly as :

Recurrence for the sum of first  $n$  natural numbers, (each) raised to the  $p^{\text{th}}$  power:

$$S_p = \frac{1}{p+1} \cdot \left[ \left( \sum_{a=1}^{p+1} \binom{p+1}{a} n^a \right) - \left( \sum_{b=1}^p \binom{p+1}{b+1} S_{p-b} \right) \right]$$

where,

$$S_x = 1^x + 2^x + 3^x + \dots + n^x \quad , \text{ for } x \in \mathbb{N}$$

and,

$$S_0 = n$$