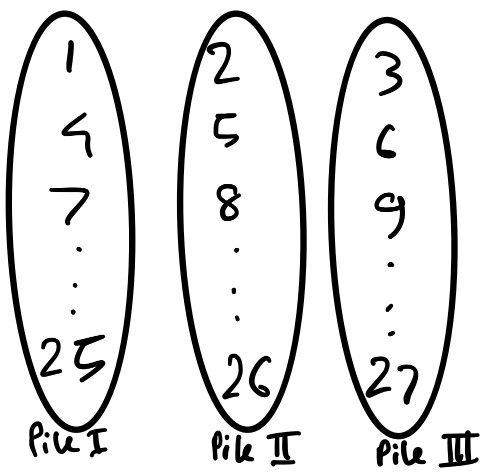
Martin Gardner's 27 card trick:

27 cards are shuffled and arranged in a pile, among which one card is previously selected by an audience. The audience also names his/her favourite number, n, such that $1 \leqslant n \leqslant 27$.

The following two operations are performed three times:

→ The pile of cards (from top down: 1, 2, 3, ⋮, 27) is dealt into three piles of 9 cards as follows:



→ You (the magician) ask the audience to point out the pile in which his/her card belongs to.

Then, you organise the three piles in a particular order to again form one complete pile of 27 cards.

At the end of all three iterations, it is revealed that the audience's selected card is the nth card from the top of the final pile.

How the trick is performed:

→ At first, convert $(n-1)_{10}$ to base 3. Suppose you get something like: $(d_0 d_1 d_2)_3$ [where $0 \leq d_0 \leq 2$, $0 \leq d_1 \leq 2$, $0 \leq d_2 \leq 2$]

→ Next, reverse the base 3 number: $(d_2 d_1 d_0)_3$.

→ Then use the following table:

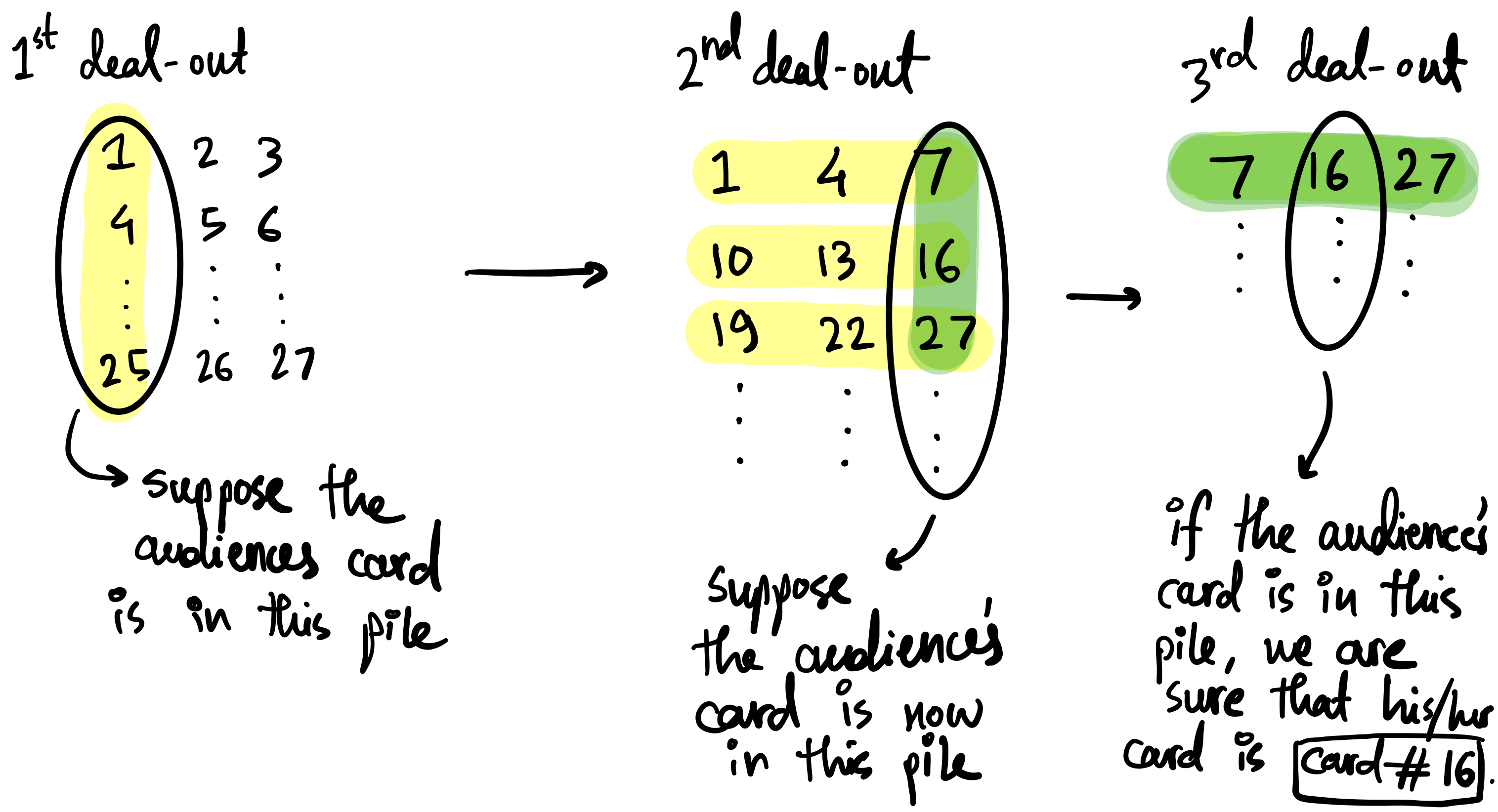
$$\begin{cases} 0: \text{Top} \\ 1: \text{Middle} \\ 2: \text{Bottom} \end{cases}$$

Look at the highlighed part on the previous page. If $(d_2 d_1 d_0)_3$ is (for example) 102, then, in the first iteration, you put the pile containing the audience's card in the middle for the recompiling. In the second iteration, you put the pile containing the audience's card on top; and in the third iteration, you put it at the bottom.

→ The chosen card, after the final recompiling, will be the **n**th card from the top.

Why the aforementioned method works:

It is fairly obvious why we can confidently identify the chosen card after the three iterations. It is a matter of simple process of elimination. Suppose at each iteration, we place the pile containing the audience's card on top:



$\boxed{\text{9 possibilities} \longrightarrow \text{3 possibilities} \longrightarrow \text{1 possibility}}$

Now, in this demonstration, the pile (containing the chosen card) was always placed on top in all three iterations. As a result, the selected card appeared in position 1 of the final recompiled pile.

→ By varying whether we place the pile containing the chosen card in top, middle, or bottom in iteration 1, 2, and 3, we can make the chosen card go to ANY position we like in the final pile.

We will construct a table for our convenience. Suppose we want the chosen card to be the n^{th} card of the final pile. That means there needs to be $\boxed{n-1}$ cards on top of it in the final pile.

Crux idea: Our ordering (top, middle, bottom placement) of the pile containing the chosen card in EACH iteration constitutes to a certain number of cards ending on TOP of the chosen card in the final pile.

So, our table might look something like:

(next page)

First iteration	Top :	
	middle :	
	Bottom :	
Second iteration	Top :	
	middle :	
	Bottom :	
Third iteration	Top :	
	middle :	
	Bottom :	

We need to insert values in the highlighted parts of the table that says how many cards that certain placement (done in a certain iteration) adds on top of the chosen card in the final pile.

(So, to find out the total number of cards on top of the chosen card in the final pile, we need to ADD three entries (one from each iteration))

Since $\underbrace{\text{Top}}_{1^{st}\text{ iter.}}$, $\underbrace{\text{Top}}_{2^{nd}\text{ iter.}}$, $\underbrace{\text{Top}}_{3^{rd}\text{ iter.}}$ obviously puts a total of zero cards on top of the chosen card in the final pile, Top = 0 for all three iterations:

First iteration	Top :	0
	middle :	
	Bottom :	
Second iteration	Top :	0
	middle :	
	Bottom :	
Third iteration	Top :	0
	middle :	9
	Bottom :	18

Also, in the Third iteration, placing the pile (contaning chosen card) in the middle means placing one extra pile (9 cards) on top of our pile of interest.

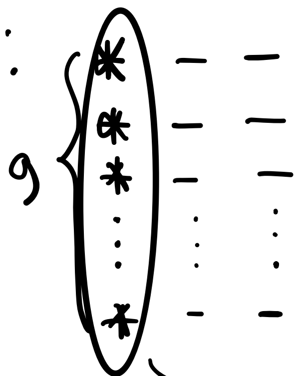
→ Similarly, placing it at the bottom means adding 9+9 = 18 extra cards on top of our pile of interest.

Now, to find out the rest of the values for First iteration, suppose we place the pile with chosen card on TOP for both the 2nd & 3rd iteration. This allows us to investigate the impact of the ordering of the First iteration alone:

Key:
* : Cards which might be the chosen card
— : cards which are surely not the chosen card

First iteration → Middle:

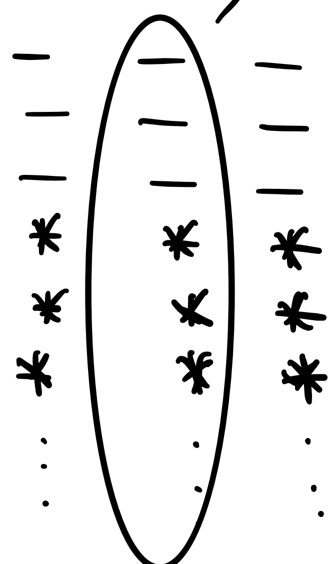
1st deal-out:



this pile is placed in the middle (sandwitched between the other two piles)

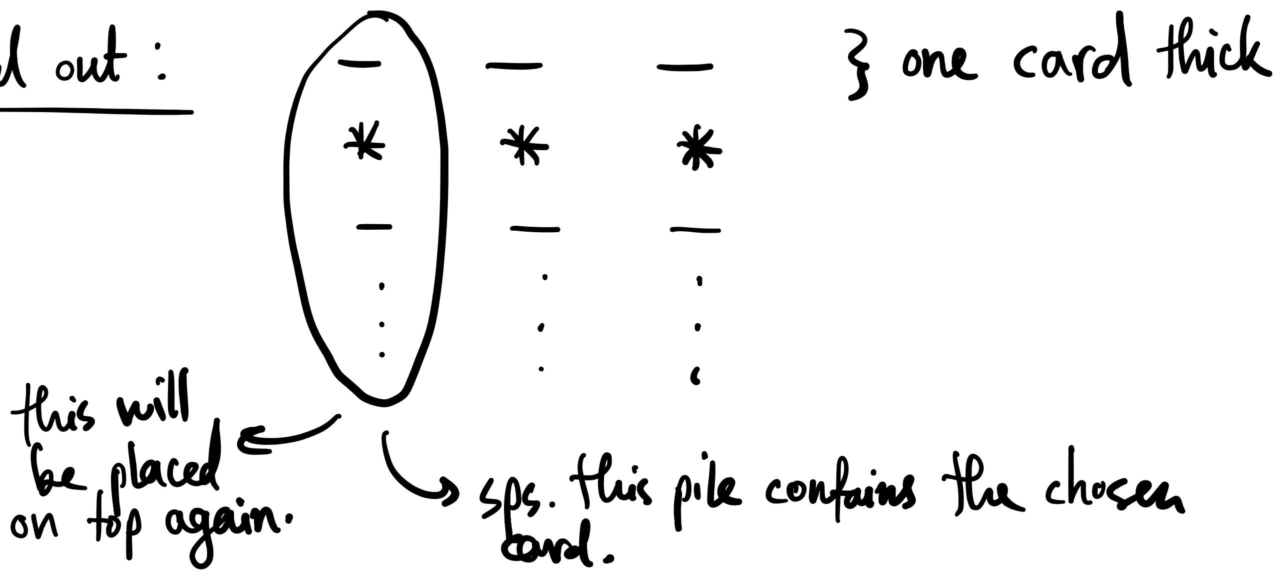
↳ sps. the audience says that this pile contains the chosen card.

2nd deal-out:
(remember that the chosen pile was placed in the middle in the prev. step)



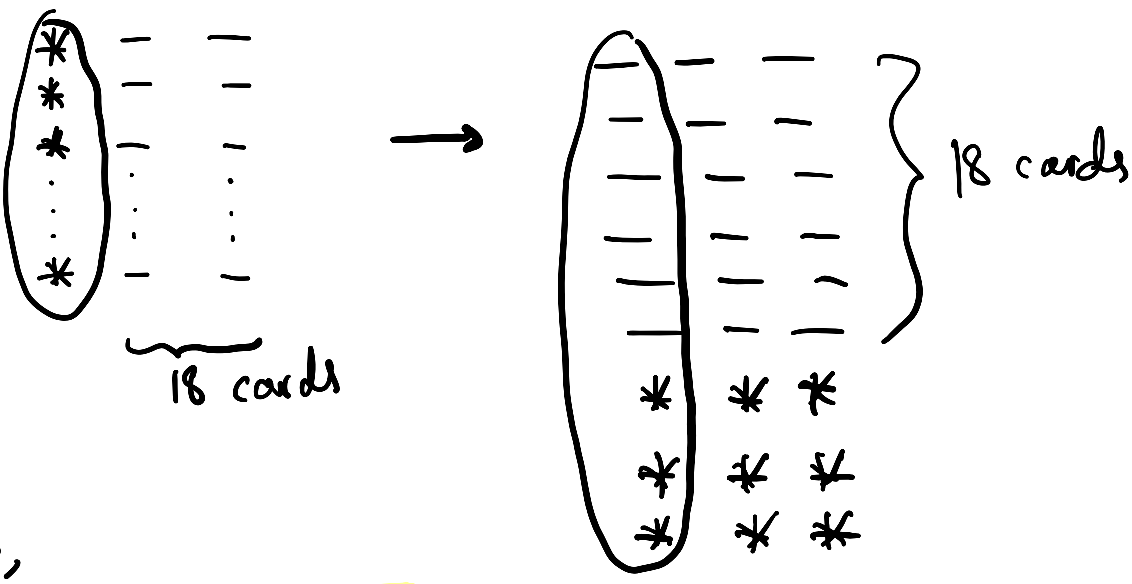
Now this pile is placed on top.

3rd deal out :

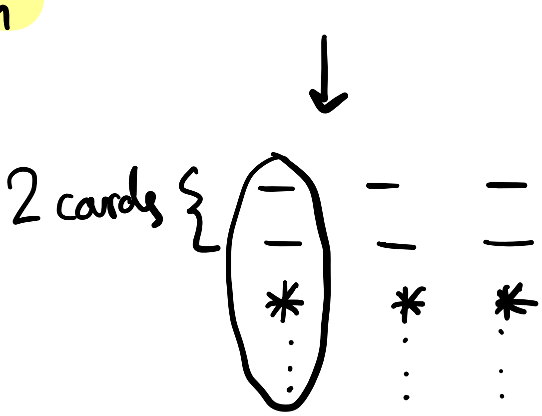


So, we have seen that $\boxed{\text{First iteration} \rightarrow \text{Middle}}$ adds 1 card on top of the chosen card in the final pile.

Similarly : First iteration → Bottom :

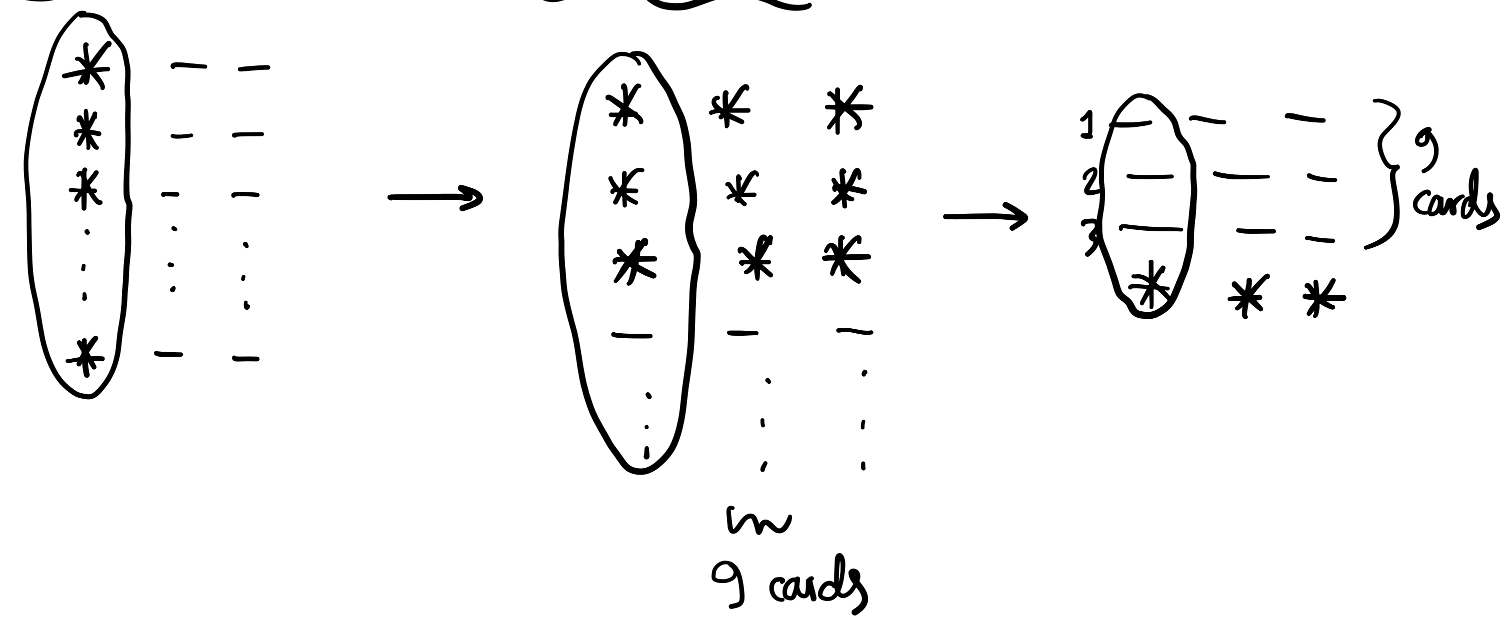


So,
First iteration → Bottom adds 2 cards on top of the chosen card in the final pile.



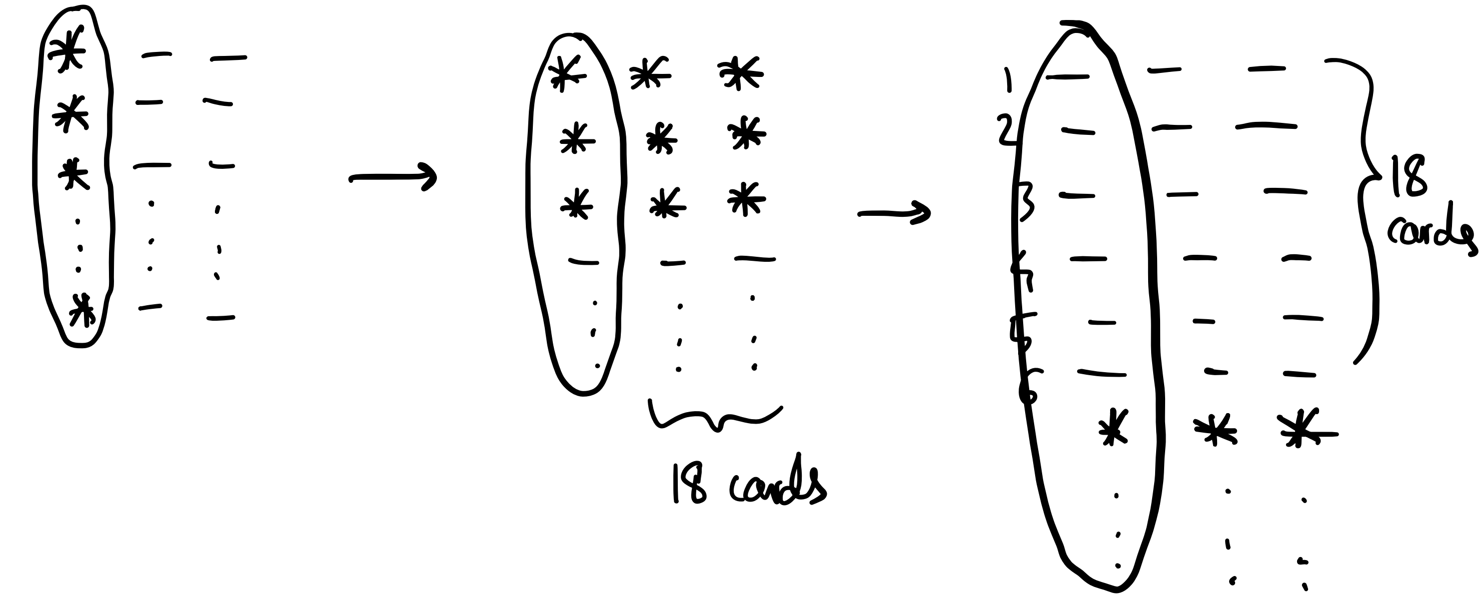
By a similar approach we can work out the entries for the Second Iteration (again, we will pick Top, or 0, for 1^{st} & 3^{rd} iteration):

Second iteration → Middle:



So, Second iteration → Middle adds 3 cards on top.

Second iteration → Bottom:



So, Second iteration $\longrightarrow$ Bottom adds 6 cards on top.

Hence, we get our final table:

First iteration	Top:	0	$= 0 \times 3^0$	Least significant figure (base 3)
	Middle:	1	$= 1 \times 3^0$	
	Bottom:	2	$= 2 \times 3^0$	
Second iteration	Top:	0	$= 0 \times 3^1$	
	Middle:	3	$= 1 \times 3^1$	
	Bottom:	6	$= 2 \times 3^1$	
Third iteration	Top:	0	$= 0 \times 3^2$	Most significant figure (base 3)
	Middle:	9	$= 1 \times 3^2$	
	Bottom:	18	$= 2 \times 3^2$	

Instead of explaining explicitly, the final bit of connection-making is left as an excercise to the reader.

Hint: $(a_n a_{n-1} \cdots a_2 a_1 a_0)_r = (a_n \cdot r^n + a_{n-1} \cdot r^{n-1} + \ldots + a_2 \cdot r^2 + a_1 \cdot r^1 + a_0 \cdot r^0)_{10}$

∎

Fayaz

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