KVANT KT 1

TAAVI TAMMARU

1.1

Näita, et olekuvektor

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

on normeeritud.

et olekuvektor oleks normeeritud peab kehtima seos

$|\alpha|^2 + |\beta|^2 = 1$ asendame α ja β, näeme et $e^{i\phi}\cdot e^{-i\phi} = 1$

$\cos(\frac{\theta}{2})^2 + \sin(\frac{\theta}{2})^2 = 1$ (teame, et $\cos(\alpha)^2 + \sin(\alpha)^2 = 1$)

seega antud olekuvektor on tõesti normeeritud.

1.2

Olgu $P_\psi = |\psi\rangle\langle\psi|$ projektor eelmises ülesandes antud seisundile $|\psi\rangle$ ja $P_0 = |0\rangle\langle 0|$ projektor kanoonilisele baasivektorile $|0\rangle$. Diraci braket tähistust kasutades, arvuta:

(1) P_ψ^2

(2) $\frac{P_0|\psi\rangle}{\|P_0|\psi\rangle\|}$

kus $|\psi\rangle$ on eelmises ülesandes antud seisund.

1) $P_\psi^2 = |\psi\rangle\langle\psi|\psi\rangle\langle\psi| \Rightarrow \langle\psi|\psi\rangle = 1$

$= |\psi\rangle\langle\psi|$

$= (\cos(\frac{\theta}{2})|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})|1\rangle)\cdot(\cos(\frac{\theta}{2})\langle 0| + e^{-i\phi}\sin(\frac{\theta}{2})\langle 1|)$

$= \cos(\frac{\theta}{2})^2|0\rangle\langle 0| + e^{-i\phi}\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})|0\rangle\langle 1| + e^{i\phi}\sin(\frac{\theta}{2})\cos(\frac{\theta}{2})|1\rangle\langle 0|$
$+ \sin(\frac{\theta}{2})^2|1\rangle\langle 1|$

2)

$P_0|\psi\rangle = |0\rangle\langle 0|\psi\rangle = |0\rangle\langle 0|(\cos(\frac{\theta}{2})|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})|1\rangle)$

$= \cos(\frac{\theta}{2})|0\rangle\langle 0|0\rangle + e^{i\phi}\sin(\frac{\theta}{2})|0\rangle\langle 0|1\rangle)$

$= \cos(\frac{\theta}{2})|0\rangle$

$\|P_0|\psi\rangle\| = \cos(\frac{\theta}{2})$ (pikkus on eelnevalt leitud ühikvektori kordaja) seega $\frac{P_0|\psi\rangle}{\|P_0|\psi\rangle\|} = \frac{\cos(\frac{\theta}{2})|0\rangle}{\cos(\frac{\theta}{2})} = |0\rangle$

1.3 Näita, et valemiga (2.12) antud unitaarne maatriks $R_{\hat{n}}(\theta)$ ei ole hermiitiline.

$$R_{\hat{n}}(\theta) = e^{-i\frac{\theta}{2}\hat{n}\cdot\sigma} = \cos\left(\frac{\theta}{2}\right)\mathbb{I} - i\sin\left(\frac{\theta}{2}\right)\hat{n}\cdot\sigma$$

kui maatriks ei ole hermiitiline siis pärast konjugeerimist ja transponeerimist ei saa me algset maatriksit tagasi.

transponeerime ja konjugeerime antud maatriksit ja teame et nii ühikmaatriks $\mathbb{I}$ kui ka pauli maatriksid σ on hermiitilised ehk need ei muutu, küll aga muutub konjugeerimisel imaginaararvulise kordaja märk.

nii saame:

$$R_{\hat{n}}(\theta)^{\dagger} = e^{-i\frac{\theta}{2}\hat{n}\sigma} = \cos\left(\frac{\theta}{2}\right)\mathbb{I} - i\sin\left(\frac{\theta}{2}\right)\hat{n}\cdot\sigma$$

$$e^{-i\frac{\theta}{2}\hat{n}\sigma} \neq e^{i\frac{\theta}{2}\hat{n}\sigma}$$

$R_{\hat{n}}(\theta)^{\dagger} \neq R_{\hat{n}}(\theta)$ ehk antud maatriks ei ole hermiitiline.

1.4 Kasutades Pauli maatriksite kommutatsiooni eeskirja

$$[\sigma_i, \sigma_j] = 2i \sum_{k=1}^{3} \epsilon_{ijk} \sigma_k$$

arvuta kommutaator σ_3 ja σ_1 vahel.

$$[\sigma_3, \sigma_1] = 2i \sum_{k=1}^{3} \varepsilon_{ijk}\sigma_k$$

$$= 2i(\varepsilon_{311}\sigma_1 + \varepsilon_{312}\sigma_2 + \varepsilon_{313}\sigma_3)$$

$$= 2i(0 + 1 \cdot \sigma_2 + 0)$$

$$= 2i\sigma_2$$

1.5 Kasutades ühikvektori $\hat{n}$ esitust polaarkoordinaatides, kirjuta spinni vaadeldav

$$S_{\hat{n}} = \frac{1}{2}\hbar\hat{n} \cdot \sigma$$

maatrikskujul.

ühikvektori esitus polaarkordinaatides

$$\hat{n} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$$

aga see ei sobi kuna siin on ainult kaks elementi,

järeldame siin et võeldakse Blochi ühikvektorit

$$\hat{n} = \begin{pmatrix} \cos\phi\sin\theta \\ \sin\phi\sin\theta \\ \cos\theta \end{pmatrix}$$

$$\hat{n} \cdot \sigma = \cos\phi\sin\theta\,\sigma_x + \sin\phi\sin\theta\,\sigma_y + \cos\theta\,\sigma_z$$

$$= \begin{bmatrix} 0 & \cos\phi\sin\theta \\ \cos\phi\sin\theta & 0 \end{bmatrix} + \begin{bmatrix} 0 & -\sin\phi\sin\theta i \\ \sin\phi\sin\theta i & 0 \end{bmatrix} + \begin{bmatrix} \cos\theta & 0 \\ 0 & -\cos(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\theta) & \cos\varphi \sin\theta - i \sin \sin\theta \\ \cos(\varphi)\sin\theta + i\sin\varphi & -\cos(\theta) \end{bmatrix}$$

Spinni vaadeldav maatrikskujul

$$S_{\hat{n}} = \frac{1}{2}\hbar \begin{bmatrix} \cos\theta & \cos\varphi\sin\theta - i\sin\varphi\sin\theta \\ \cos\varphi\sin\theta + i\sin\varphi & -\cos\theta \end{bmatrix}$$

1.6

Näidata, et ülesandes 1.1 antud kvantbiti suvalise seisundi $|\psi\rangle$ jaoks avalduvad Pauli σ operaatorite keskväärtused järgmiselt:

$$\langle\psi|\sigma_1|\psi\rangle = \sin\theta\cos\phi$$
$$\langle\psi|\sigma_2|\psi\rangle = \sin\theta\sin\phi$$
$$\langle\psi|\sigma_3|\psi\rangle = \cos\theta$$

ning nendest kolmest keskväärtusest moodustatud vektor on normeeritud.

$$\langle\psi|\sigma_1|\psi\rangle = \langle\psi|\left(\cos\frac{\theta}{2}\sigma_1|0\rangle + e^{i\varphi}\sin\frac{\theta}{2}\sigma_1|1\rangle\right)\rangle$$

Siin toimub kvantbiti vahetus.

$$\langle\psi|\left(\cos\frac{\theta}{2}\sigma_1|1\rangle + e^{i\varphi}\sin\frac{\theta}{2}\sigma_1|0\rangle\right)\rangle =$$

$$= \left(\cos\left(\frac{\theta}{2}\right)\langle 0| + e^{-i\varphi}\sin\left(\frac{\theta}{2}\right)\langle 1|\right)\left(\left(\cos\frac{\theta}{2}\sigma_1|1\rangle + e^{i\varphi}\sin\frac{\theta}{2}\sigma_1|0\rangle\right)\rangle\right)$$

$$= \cos\left(\frac{\theta}{2}\right)^2\langle 0|1\rangle + e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\langle 0|0\rangle + e^{-i\varphi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\langle 1|1\rangle$$
$$+ \sin\left(\frac{\theta}{2}\right)^2\langle 0|1\rangle$$

$$= e^{i\varphi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\langle 0|0\rangle + e^{-i\varphi}\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\langle 1|1\rangle$$

$$= \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)\left(e^{i\varphi} + e^{-i\varphi}\right)$$

$$= \sin(\theta)\cos(\varphi)$$

2)

$$\langle\psi|\sigma_2|\psi\rangle = \langle\psi|(\cos(\tfrac{\theta}{2})\sigma_2|0\rangle + e^{i\varphi}\sin(\tfrac{\theta}{2})\sigma_2|1\rangle)\rangle$$

$$= \langle\psi|(i\cos(\tfrac{\theta}{2})|1\rangle - ie^{i\varphi}\sin(\tfrac{\theta}{2})|0\rangle) =$$

$$= -ie^{i\varphi}\sin(\tfrac{\theta}{2})\cos(\tfrac{\theta}{2}) + ie^{-i\varphi}\sin(\tfrac{\theta}{2})\cos(\tfrac{\theta}{2})$$

$$= \sin(\tfrac{\theta}{2})\cos(\tfrac{\theta}{2})(\sin\varphi - i\cos\varphi + \sin\varphi + i\cos\varphi)$$

$$= 2\sin(\tfrac{\theta}{2})\cos(\tfrac{\theta}{2})\sin\varphi = \sin\theta\sin\varphi$$

3) $$\langle\psi|\sigma_3|\psi\rangle = \langle\psi|\cos(\tfrac{\theta}{2})\sigma_3|0\rangle + e^{i\varphi}\sin(\tfrac{\theta}{2})\sigma_3|1\rangle)\rangle$$

$$= \langle\psi|\cos(\tfrac{\theta}{2})|0\rangle - e^{i\varphi}\sin(\tfrac{\theta}{2})|1\rangle)\rangle = \cos(\tfrac{\theta}{2})^2 - \sin(\tfrac{\theta}{2})^2$$

$$= \cos(\tfrac{\theta}{2})^2 - \sin(\tfrac{\theta}{2})^2 = \cos(\theta)$$

kolme keskväärtuse vektori norm

$$\| |a\rangle \| = \sqrt{\langle a|a\rangle}$$

$$\sqrt{\sin(\theta)^2\cos(\varphi)^2 + \sin(\theta)^2\sin(\varphi)^2 + \cos(\theta)^2}$$

$$\sqrt{\sin(\theta)^2(\cos(\varphi)^2 + \sin(\varphi)^2) + \cos(\theta)^2} = 1$$

vektor on normeeritud!