Calculus 2

Final Exam

April 11, 2022 (16:00 – 18:00)

university of groningen

Please turn over and read the instructions!

1) Consider the hyperboloid of one sheet H given by the equation

$$x^2 + \frac{y^2}{9} - \frac{z^2}{4} = 2$$

$\boxed{8}$ a) Treating H as a level surface of a function of three variables, find an equation of the tangent plane to H at the point $P(3, 9, 8)$.

$\boxed{8}$ b) Use the Implicit Function Theorem to show that near the point P in part a), H can be considered to be the graph of a function f of x and z. Compute the partial derivatives f_x and f_z and show that the tangent plane found in a) coincides with the graph of the linearization $L(x, z)$ of $f(x, z)$ at $(3, 8)$.

$\boxed{8}$ c) Use the method of Lagrange multipliers to find the point $Q(x_*, y_*, z_*)$ on the tangent plane in part a) that is closest to the origin. Determine the distance of between the tangent plane and the origin.

2) Consider the vector field

$$\vec{G}(x, y, z) = \frac{Ax}{x^2 + y^2 + 1}\,\vec{\imath} + \left(\frac{2y}{x^2 + y^2 + 1} + Bze^y\right)\vec{\jmath} + e^y\,\vec{k}$$

with parameters $A, B \in \mathbb{R}$.

$\boxed{9}$ a) Determine the values of A and B for which $\vec{G}$ is conservative.

$\boxed{8}$ b) For A and B found in part a), determine a scalar potential for $\vec{G}$.

$\boxed{4}$ c) For A and B found in part a), compute the line integral of $\vec{G}$ along the curve of intersection of the paraboloid $z = x^2 + y^2$ and the plane $y = 1$ from the point $P_0(0, 1, 1)$ to the point $P_1(1, 1, 2)$.

3) Consider a fluid with the velocity field

$$\vec{V}(x, y, z) = \frac{-y}{\sqrt{x^2 + y^2}}\,\vec{\imath} + \frac{x}{\sqrt{x^2 + y^2}}\,\vec{\jmath} + (x^2 + y^2)\,z\,\vec{k}$$

and the surface $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1,\ -y - 3 \le z \le y + 3\}$ with outward normal vectors and positively-oriented boundary ∂S.

$\boxed{3}$ a) Describe and sketch the surface S and its boundary ∂S (draw orientation).

Verify Stokes' Theorem by

$\boxed{8}$ b) calculating the circulation of $\vec{V}$ along ∂S, i.e. $\int_{\partial S} \vec{V} \cdot d\vec{r}$ and

$\boxed{12}$ c) computing the flux of $\operatorname{curl} \vec{V}$ across S, that is $\iint_S \operatorname{curl} \vec{V} \cdot d\vec{S}$.

4) Consider the vector field

$$\vec{F}(x, y, z) = xz\,\vec{\imath} + yz\,\vec{\jmath} + z^2\,\vec{k}$$

over the solid region $E = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le 4,\ z \ge \sqrt{x^2 + y^2}\}$ and its outward-oriented boundary surface ∂E.

$\boxed{2}$ a) Describe and sketch the region E and the surface ∂E (draw orientation).

Verify the Divergence Theorem by

$\boxed{12}$ b) computing the flux of $\vec{F}$ across ∂E, that is $\iint_{\partial E} \vec{F} \cdot d\vec{S}$ and

$\boxed{8}$ c) evaluating the triple integral of $\operatorname{div} \vec{F}$ over E, i.e. $\iiint_E \operatorname{div} \vec{F}\, dV$.

Instructions

- **write your name and student number on the envelope and on the top of each sheet!**
- use the writing and scratch paper provided, raise your hand if you need more paper
- start each question on a new page
- use a pen with black or blue ink
- do not use any kind of correcting fluid or tape
- any rough work should be crossed through neatly so it can be seen
- this is a closed-book exam, it is not allowed to use the textbook or the lecture notes
- you are allowed to use the formula sheet provided or a simple pocket calculator
- programmable calculators are not allowed, nor the use of electronic devices (tablet, laptop, mobile phone, etc.) to solve the exercises
- your work should be clearly and logically structured
- **explain your reasoning using words**
- show all your calculations, an answer without any computation will not be rewarded
- you can achieve 100 points (including the 10 bonus points)
- upon completion[1] place your worksheets in the envelope and submit them at the front desk

[1] At the end of the exam or after you finished whichever is sooner.

Formula sheet

Gradient of $g(x,y,z)$: $\operatorname{grad} g(x,y,z) = \nabla g(x,y,z) = \langle g_x, g_y, g_z \rangle$

Linearization of $f(x,z)$ at (a,c): $L(x,z) = f_x(a,c)(x-a) + f_z(a,c)(z-c) + f(a,c)$

Implicit Differentiation: Suppose an equation of the form $F(x,y,z)=0$ is used to define y implicitly as a function $f(x,z)$ in a neighbourhood of the point $P(a,b,c)$ such that $F(a,b,c)=0$ and $F_y(a,b,c) \neq 0$. Then

$$f_x = -\frac{F_x}{F_y}, \qquad f_z = -\frac{F_z}{F_y}$$

Method of Lagrange Multipliers: To find the max./min. of f along the level surface $g(x,y,z)=k$ solve $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ for (x,y,z,λ) while making sure that $g(x,y,z)=k$ is satisfied.

Cylindrical Coordinates (r,θ,z): $x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z.$

Spherical Coordinates (ρ,ϕ,θ): $x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi.$

Change of Variables in a Triple Integral (T: $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$):

$$\iiint\limits_{T(S)} f(x,y,z)\,dV = \iiint\limits_{S} f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

where $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$ is the Jacobian of T.

Jacobian of the Cylindrical-Cartesian transformation: $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$

Jacobian of the Spherical-Cartesian transformation: $\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2\sin\phi$

Line Integral of a Vector Field: $\displaystyle\int\limits_C \vec{F}(x,y,z)\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$

Surface Integral of a Scalar Function: $\displaystyle\iint\limits_S f(x,y,z)\,dS = \iint\limits_D f(\vec{r}(u,v))|\vec{r}_u \times \vec{r}_v|\,du\,dv$

Surface Integral of a Vector Field: $\displaystyle\iint\limits_S \vec{F}(x,y,z)\cdot d\vec{S} = \iint\limits_S \vec{F}\cdot\vec{n}\,dS = \iint\limits_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u \times \vec{r}_v)\,du\,dv$

Curl of a Vector Field $\vec{F}(x,y,z) = P(x,y,z)\,\vec{\imath} + Q(x,y,z)\,\vec{\jmath} + R(x,y,z)\,\vec{k}$:

$$\operatorname{curl}\vec{F} = \nabla\times\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

Divergence of a Vector Field $\vec{F}(x,y,z) = P(x,y,z)\,\vec{\imath} + Q(x,y,z)\,\vec{\jmath} + R(x,y,z)\,\vec{k}$:

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Definition: The vector field $\vec{G}$ is *conservative*, if there exists a function g such that $\vec{G} = \operatorname{grad} g$.

Theorem: If $\vec{G}$ is conservative, then $\operatorname{curl}\vec{G} = \vec{0}$.

Theorem: If $\vec{G}$ is defined everywhere on $\mathbb{R}^3$, has continuously differentiable components and $\operatorname{curl}\vec{G} = \vec{0}$, then $\vec{G}$ is conservative.