Name: Student Number:

Mechanics and Relativity: R1

September 30, 2022, Aletta Jacobshal

Duration: 90 mins

Before you start, read the following:

- There are 3 problems, for a total of 58 points.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate. If needed, draw your spacetime diagrams on the provided paper.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

	Points
Problem 1:	18
Problem 2:	18
Problem 3:	22
Total:	58
GRADE (1 + # Total/(58/9))	

Useful equations:

$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$
$$\Delta t \geq \Delta s \geq \Delta \tau$$

Possibly relevant equations:

$$F = G\frac{Mm}{r^2}; \qquad F = ma; \qquad PV \propto k_b T; \qquad F = \frac{dp}{dt}$$

Possibly relevant numbers:

$$c = 299792458 \text{ m/s} \tag{1}$$

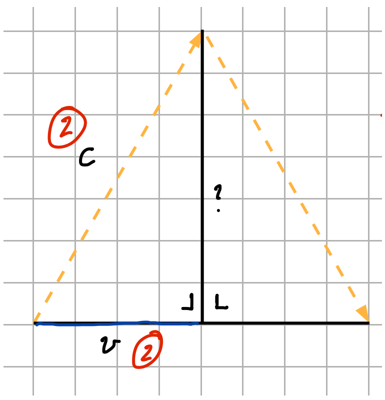
Question 1: Lightclock 2.0 (18 pts)
Consider a lightclock as discussed during the lecture and in Moore. Now instead of deriving the spacetime interval, we aim to measure the time it takes for a ray of light (or a photon) to propagate from the source to the mirror and back.

(a) **(3 pts)** Write down an expression for the time it takes for a ray of light to move from the source to the mirror and back. Assume the vertical length of the clock is given by L. Label this time as t_A.

The total distance is $2L$ **(1 pt)**. Light moves at a speed c **(1 pt)**. The total time is thus $t_A = 2L/c$ **(1 pt)**

(b) **(4 pts)** Next, consider a frame in which the clock is moving to the right with velocity v. Sketch the trajectory of the lightray. Label the vertices of the lightray with the velocities v and c. **Possible Hint: recall that the speed of light is frame independent**

The student should realize that the length of the vertices is simply $\Delta t \times$ velocity. Δt is determined by when the light hits the mirror. We can thus label the horizontal vertice to midpoint as v (the speed of the clock) **(2 pts)**. The other vertice is the light moving to the mirror. Independent of frame, this should still be at the speed of light c **(2 pts)**.



(c) **(4 pts)** Use Pythagoras theorem to obtain the vertical velocity as observed in the frame where the clock is moving.

$A^2 + B^2 = C^2$**(1pt)** $\rightarrow v^2 + B^2 = c^2$**(1pt)** $\rightarrow B^2 = c^2 - v^2$**(1pt)** $\rightarrow B = \sqrt{c^2 - v^2}$**(1pt)**

(d) **(1 pt)** Show that the time it takes for the light ray to move back and forth between the source and the mirror in the moving frame is given by $t_B = 2L/\sqrt{c^2 - v^2}$.

$t_B = 2L/v_{\text{vertical}} = 2L/\sqrt{c^2 - v^2}$**(1pt)**

(e) **(2 pts)** Compute the ratio between time t_B and t_A. Express your answer in terms of the so-called Lorentz factor $\gamma = 1/\sqrt{1 - (v^2/c^2)}$

$\frac{t_B}{t_A} = \frac{2L/\sqrt{c^2-v^2}}{2L/c} = \frac{2L}{2L}\frac{c}{\sqrt{c^2-v^2}} \equiv \gamma$**(2pts)**

(f) **(4 pts)** The relation between t_A and t_B is known as time dilation.We already encountered this during the lecture when we discussed the twin paradox when talking about proper time. In fact this is very similar to the relation we derived for proper time if we assume the velocity is constant (direction of vector is unchanged at all times and is constant), i.e. $\Delta\tau = \sqrt{1 - (|\vec{v}|/c)^2}\Delta t = \gamma^{-1}\Delta t$. For an observer at rest (i.e. measuring t_B), we find $\Delta t = \gamma\Delta\tau$, where the τ is the time measured in the comoving frame (i.e. t_A). Now consider $v = c$. What does this mean for time dilation? Explain your answer using the lightclock.

for $v = c$ we have $t_B = \gamma t_A = \frac{1}{0} t_A = \infty$**(2pts)** It simply means that if $v = c$ the lightray is observed to move horizontally forever (the vertical velocity is 0). The lightray will take an infinite time to reach the mirror (and therefore an infinite time to come back to the source).**(2 pts)**

Question 2: Dimensional Analysis and Kepler's Laws (18 pts)
As you will learn in the Mechanics part of the course, Kepler's laws describe the elliptical orbit of a planet around the sun. His third law describes the relation between the period of the orbit Θ and the semi-major axis of the elliptical orbit a, this is (half) the largest distance between two points on the ellipse. You will derive the relation between Θ and a using dimensional analysis. The relation also involves the mass of the sun M and Newton's constant G.

Instruction: Use T, L and M for the *dimensions* of time, length and mass, respectively.

(a) **(5 pts)** Explain why Kepler's third law depends on Newton's constant G and give its dimension.

Kepler's third law describes the dynamics of a planet around the sun due to the effect of gravity. The fundamental constant associated with gravity is Newton's (gravitational) constant G. Its dimensions can be derived from Newton's law of gravitation, given at the front page:

$$G = \frac{Fr^2}{mM} \quad \textbf{(1pts)},$$

since the dimension of force is $[F] = \mathrm{M\,Ł\,T^{-2}}$ **(2 pts)**, we find $[G] = \mathrm{Ł^3T^{-2}M^{-1}}$ **(2 pts)**.

(b) **(10 pts)** Use dimensional analysis to *derive* an expression for Θ in terms of a, G and M.[1]

The dimensions are $[G] = \mathrm{Ł^3T^{-2}M^{-1}}$ **(1/2 pts)**, $[\Theta] = \mathrm{T}$ **(1/2 pts)**, $[M] = \mathrm{M}$ **(1/2 pts)** and $[a] = \mathrm{Ł}$ **(1/2 pts)**. Let us write **(3/2 pts, 1 for equation, 1/2 for prop constant)**:

$$\Theta = kG^\alpha M^\beta a^\gamma,$$

where k is a dimensionless constant of order unity. In terms of the units we find **(2 pts)**:

$$\mathrm{T}^1 = \mathrm{Ł}^{3\alpha}\mathrm{T}^{-2\alpha}\mathrm{M}^{-\alpha}\mathrm{M}^{\beta}\mathrm{Ł}^{\gamma} = \mathrm{Ł}^{3\alpha+\gamma}\mathrm{T}^{-2\alpha}\mathrm{M}^{\beta-\alpha}.$$

Since the left hand side has dimension T, we find **(3 pts. 1 pt for each correct equation)**:

$$1 = -2\alpha, \quad 3\alpha + \gamma = 0, \quad \beta - \alpha = 0.$$

Solving this system of equations gives $\alpha = \beta = -1/2$ **(1/2 pt)** and $\gamma = 3/2$ **(1/2 pt)** so that **(1/2 pts)**:

$$\Theta = k(GM)^{-1/2}a^{3/2}.$$

[1] *Derive* means to propose a power-law relationship between the variables and solve for the coefficients based on the required dimensions.

(c) **(3 pts)** In the relation you derived in above, only the mass of the sun M enters. Technically, this is only correct in case the mass of the planet m is much smaller than the sun's, $m \ll M$. How could the relation be adapted to account for the *additional* mass of the planet?

From the previous part, we found $T \propto M^{-1/2}$ **(1 pt)**. To keep the relationship dimensionally consistent, the natural choice is to replace $M \to M + m$ so that we find **(2 pts)**:

$$T = k(G(M+m))^{-1/2} a^{3/2}$$

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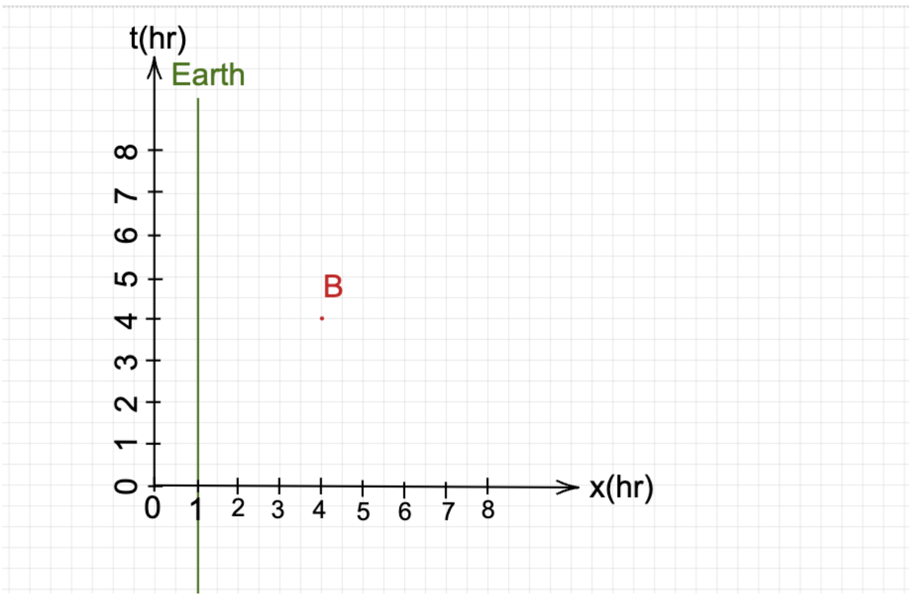
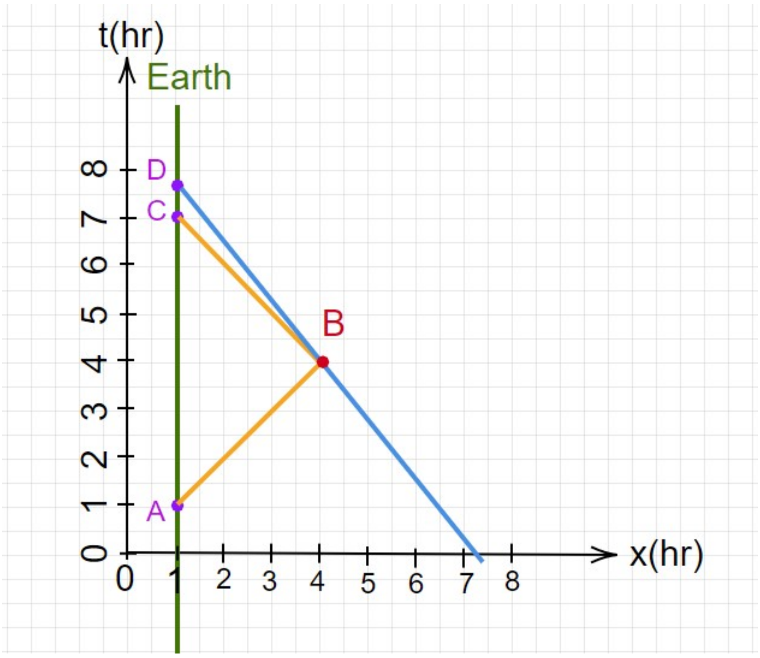
Question 3: An Asteroid on the Way! (22 pts)
Dr. Kate Dibiasky discovers an asteroid at a constant speed of $0.8c$ that is on its way to Earth. Dr. Randall Mindy discovers it will wipe out most of life on Earth unless action is taken. From Earth, a radar signal is sent out to detect it (event A), which reflects from the asteroid (event B), and is detected (=seen) on Earth (event C). The time difference measured by a stationary observer on Earth between events A and C is 6 hours . We want to find out when the asteroid hits Earth (D).

(a) **(4 pts)** Draw a spacetime diagram. Draw and label the worldline of Earth and event B with respect to an observer at rest relative to the Earth. Make sure that the diagram can be read unambiguously by adding the necessary markings to the axes. You can use the graphing paper attached. Use space below for clarification if needed.



(1 pt): drawing a diagram
(1 pt): drawing labels and ticks
(1 pt): worldline earth
(1 pt): drawing event B

(b) **(3 pts)** In the same diagram, draw the worldlines of the asteroid and the radar signal and indicate the events A, B, and D.



(1 pt): drawing worldline to event B
(1 pt): drawing worldline to event C
(1 pt): drawing worldline to event D

(c) **(1 pt)** Using the diagram, find the time interval (Δt) measured by a stationary observer on Earth between event C and the moment the asteroid strikes Earth (event D).

$\Delta t = 0.75\,\text{hr}$ (± 0.1 is acceptible)**(1 pt)**

(d) **(2 pts)** Write down an expression for the coordinates of event B in terms of the coordinates of event A and C.

You can use the radar method: $t_B = \frac{1}{2}(t_A + t_C))$ and $x_E = \frac{1}{2}(t_C - t_A))$ **(2 pt)**. If the diagram is drawn with an offset as the example above, you need to add x_A to the spatial position of event B, i.e. $x_E = \frac{1}{2}(t_C - t_A)) + x_A$.

(e) **(5 pts)** Write down an expression for the coordinates of event D in terms of the coordinates of event A and C.

The asteroid is moving in the direction of earth, i.e. $v = -0.8c = \frac{\Delta x}{\Delta t} \rightarrow \Delta t = -\frac{5}{4}\Delta x$ **(2 pts)**. The coordinates of event D are thus given by: $x_D = x_A = x_C$ and $t_D = \frac{1}{2}(t_A + t_C) + \Delta t = \frac{1}{2}(t_A + t_C) - \frac{5}{4}(\Delta x) = \frac{1}{2}(t_A + t_C) + \frac{5}{4}(\frac{1}{2}(t_C - t_A)) = \frac{9}{8}t_C - \frac{1}{8}t_A$ **(3 pts)**. Note that an offset in the diagram as above is irrelevant since $\Delta x = x_A - x_B$.

(f) **(1 pt)** Using the derived expression, check your answer in question (c).

So we use $\Delta t = t_D - t_C = \frac{1}{8}t_C - \frac{1}{8}t_A = \frac{1}{8}(t_C - t_A)$. The radar signal is seen 6 hrs after it was sent, i.e. $\Delta t = 6/8$hrs $= 0.75$hr **(1 pt)**

(g) **(1pt)** What kind of time does each observer (consider a stationary observer on Earth and an observer on the asteroid) measure between events B and D?

An observer on the asteroid measures the proper time, whereas an observer on Earth measures the coordinate time. **(1 pt)**

Half an hour after the signal is received with respect to a stationary observer on Earth, Kate and Randall leave Earth at a constant speed of $0.6c$ in a spaceship (event E) and they move away from the asteroid (in the opposite direction to the asteroid). Assume that the ship accelerates instantly, so both the ship and a stationary observer on Earth are in an inertial frame.

(h) **(5 pts)** Calculate the time measured between events E and D by the ship's clock explaining the formula you need to use. Hint: consider the discussion around 1(f), but realize who is present at both events.

$\Delta t' = \gamma \Delta t (\mathbf{1\ pt})$

Δt is the time registered for the clock on Earth **(1 pt)**

$\Delta t'$ is the time registered for the clock in the ship

$\gamma = 5/4(\mathbf{1\ pt})$, $\Delta t = 0.25(\mathbf{0.5\ pt})\,\mathrm{hr}$ therefore $\Delta t' = 0.31\mathrm{hr}(\mathbf{0.5\ pt})$

The ship clock runs faster than the Earth clock because the Earth clock reads the proper time between E and D.**(1 pt)**

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