Name: Student Number:

# Test 2: Mechanics and Relativity

October 14, 2021, Aletta Jacobshal

Duration: 90 mins

**Before you start, read the following:**

- There are 4 problems, for a total of 90 points.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate. Draw your spacetime diagrams on the provided hyperbolic paper.
- Write your answers in the boxes provided.
- Use the lined sheets to do your scratch work. Don't hand them in.
- Write in a readable manner, illegible handwriting will not be graded.

| | Points |
|---|---|
| Problem 1: | |
| Problem 2: | |
| Problem 3: | |
| Problem 4: | |
| Total: | |
| GRADE (1 + # Total/10) | |

**Useful equations:**

The Lorentz transformation equations with $\gamma \equiv (1-\beta^2)^{-1/2}$:

$$\begin{aligned} t' &= \gamma(t - \beta\, x)\,, \\ x' &= \gamma(x - \beta t)\,, \\ y' &= y\,, \\ z' &= z\,. \end{aligned}$$

The relativistic Doppler shift formula:

$$\frac{\lambda_R}{\lambda_E} = \sqrt{\frac{1+v_x}{1-v_x}}\,.$$

## Problem 1: Moving Triangle (24 pt)

In the home frame (HF), a triangle moves at speed $\beta$ in the positive $x$ direction. In the other frame (OF), triangle $ABC$ is an isosceles right triangle where line segments $AB$ and $BC$ are congruent and line segments $AN$, $BN$, and $CN$ all have a length of $\ell$.

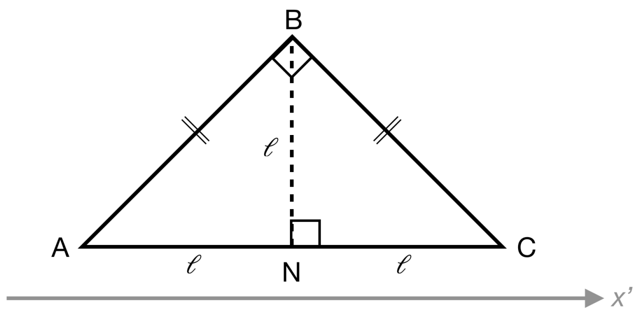



Figure 1: Isosceles right triangle $ABC$ in the other frame.

(a) You are a stationary observer in the HF. Is the length of line segment $BN$ affected due to Lorentz contraction according to you? Briefly explain your answer. (4 pt)

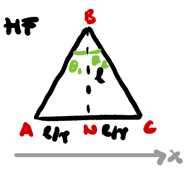
No: Lorentz contraction only occurs in the direction of motion. Segment BN is perpendicular to the direction of motion.
(2pt)
reasoning (2pt)

(b) Presume angle $\angle B$ is $\theta$ in the HF. Prove that

$$\theta = 2\arctan\left(\sqrt{1-\beta^2}\right). \tag{1}$$

(8 pt)
Note: if $y = \tan(x)$, then $x = \arctan(y)$.

In the HF, the length of segments AN and CN will be reduced by $\gamma$: $\ell/\gamma$ (2pt)




$\tan\theta_1 = \tan\theta_2 = \frac{\ell/\gamma}{\ell} = \frac{1}{\gamma}$ (2pt)

Invert to get: $\theta_1 = \theta_2 = \arctan(1/\gamma)$ (1pt)

Use that $\angle B \equiv \theta = \theta_1 + \theta_2$ (1pt)

Finally:
$\angle B \equiv \theta = \theta_1 + \theta_2 = 2\arctan(\frac{1}{\gamma}) = 2\arctan(\sqrt{1-\beta^2}) = \theta(\beta)$ (2pts)

(c) (i) When $\beta \ll 1$, show how the formula from (b) becomes:

$$\theta = \frac{\pi}{2} - \frac{\beta^2}{2} \tag{2}$$

(ii) Determine the result for $\beta = 0$.

Hint: You can use the binomial expansion inside the argument of the arctan first - before applying $\arctan(1 + x) = \pi/4 + x/2$ for $|x| \ll 1$. (8 pt)

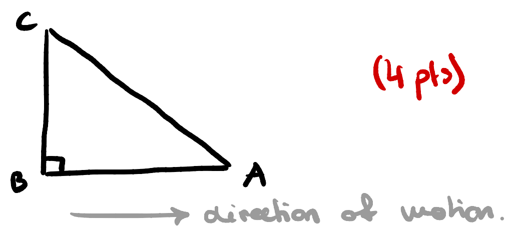
Binomial expansion: $\sqrt{1-\beta^2} \approx 1 - \frac{1}{2}\beta^2$ (3pts)

$\theta(\beta) = 2\arctan(\sqrt{1-\beta^2}) \approx 2\arctan(1 \underbrace{- \tfrac{1}{2}\beta^2}_{\equiv x}) \approx 2\left(\frac{\pi}{4} + \frac{x}{2}\right) = \frac{\pi}{2} - \frac{\beta^2}{2}$ (3pts)

In the limit $\beta = 0$, we get $\theta = \frac{\pi}{2} = 90°$ since then HF = OF. (2pts)

(d) Find an orientation for triangle $ABC$ in the OF such that it will remain ~~as~~ an ~~isosceles~~ right triangle in the HF. Sketch the orientation of this triangle and make sure to indicate its direction of motion. (4 pt)

For example:




(4 pts)

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## Problem 2: Space Oddity (24 pt)

A space probe is launched from Mission Control, which is on planet Athshe. After a brief period of acceleration which you can neglect, it moves at a constant speed of 0.7c with respect to Athshe. The batteries keeping the data transmitter on the probe active have a lifetime of 15.0 yr, as measured in a rest frame.

(a) How long do the batteries on the space probe last as measured by Mission Control on Athshe? (6 pt)

$\Delta t = \gamma \Delta t_p$ (1) where $\Delta t$ is measured by Mission Control, $\Delta t_p$ is measured in a rest frame and $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$. (3pt)

plugging $\gamma$ and $\Delta t_p$ which is 15 ly into (1);

$$\Delta t = \frac{10}{\sqrt{51}} \cdot 15\,\text{yr}$$ (3pt)

$$= 21.0\,\text{yr}$$

(b) Determine the distance of the probe from Athshe when its batteries fail, as measured by Mission Control. (6 pt)

$d$ is the distance of the probe from Athshe when its batteries fail, $v$ is the speed.

$d = v\Delta t$ (3pt)

$= (0.7c)(21\,\text{ly}) = 14\,\text{ly}.$ (3pt)

(c) Determine the distance of the the probe from Athshe when its batteries fail in the probe frame. (6 pt)

$d_p$ is the distance of the probe from Athshe when its batteries fail in the probe frame.

$d_p = v\,\Delta t_p$ (3pt)

$= (0.7c)(15.0\,\text{ly}) = 10.5\,\text{ly}$ (3pt)

(d) The probe sends a light ray of wavelength 650 nm to Mission Control as long as the data transmitter is active. Calculate the wavelength of the light received by Mission Control. Is the light received by Mission Control in the visible spectrum according to the electromagnetic spectrum below? If not, in which segment of the spectrum is the light received by Mission Control? (6 pt)

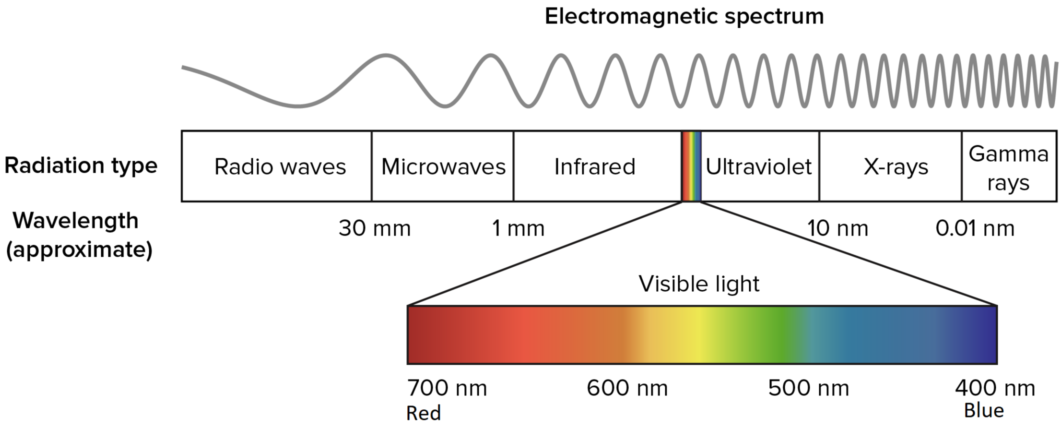
The relativistic Doppler shift formula is given as

$$\frac{\lambda_R}{\lambda_E} = \sqrt{\frac{1+v}{1-v}}$$

, the emitter (in our case, the space probe) moves away from the receiver (Mission Control). So, $v$ is positive. (2pt)

$$\frac{\lambda_R}{\lambda_E} = 650\,\mathrm{nm}\sqrt{\frac{1.7}{0.3}} = 650\,\mathrm{nm}\sqrt{\frac{17}{3}} = 1547\,\mathrm{nm} \quad (3\,\mathrm{pt})$$

$\Rightarrow$ not visible, it is in the IR sector. (1 pt)



## Problem 3: Discovery of the Spacion (26 pt)

Consider the following scenario:

- Charlie is creating an unstable subatomic particle (*Spacion*) on his spaceship.
- Once the *Spacion* particle is produced (event P), it moves in a circular orbit at a constant *speed*.
- The *Spacion* particle travels one complete revolution before it starts to decay (event D).
- Charlie measures the time between events P and D to be 500 ns.
- A second spaceship passes near Charlie's spaceship; in this second spaceship, Alice is sitting in the front and Bob is sitting in the back.
- Alice passes Charlie's spaceship just as the *Spacion* particle is produced while Bob passes Charlie's spaceship when the particle starts to decay.

(a) Assume the *Spacion* particle has an internal clock that can measure time. For questions i-iii, choose from the following options: S=*Spacion* particle, C=Charlie, AB=Alice+Bob. More than one option may be possible.

(i) Which option(s) measure a proper time between events P and D? (2 pt)

S, C ( C measures $\Delta s$, which is a form of $\Delta\tau$).

(ii) Which option(s) measure a spacetime interval between these events? (2 pt)

C

(iii) Which option(s) measure the shortest time interval between these events? (2 pt)

$\Delta t \geq \Delta s \geq \Delta\tau$ : S measures shortest time interval.

(b) The time interval between events P and D is measured by the *Spacion* particle to be 400 ns. Calculate the radius of the *Spacion* particle's circular orbit in Charlie's frame. (Hint: First calculate the speed of the *Spacion* particle in Charlie's frame.) Show all your work. (6 pt)

Relation $\Delta\tau$ and $\Delta t$:

$$\Delta\tau = \int_{t_1}^{t_2} dt \sqrt{1-v^2(t)}$$ circular orbit $v \neq v(t)$ (1 pt)

Constant speed: $\Delta\tau = \Delta t\sqrt{1-v^2}$ (1 pt)

$$\left.\begin{array}{l}\Delta\tau = 400 \text{ ns}\\ \Delta t = 500 \text{ ns}\end{array}\right\} 1-v^2 = \frac{\Delta\tau^2}{\Delta t^2} = \frac{16}{25} \rightarrow v^2 = \frac{9}{25} \rightarrow v = \frac{3}{5}$$ (2 pt)

We must have: $v\Delta t = 2\pi R$ (1 pt) $\rightarrow R = \frac{v \cdot \Delta t}{2\pi} = \frac{1}{2\pi}\frac{3}{5}\cdot 500$ ns

$R = \frac{300}{2\pi}$ ns $= 47.7$ ns } (1 pt)

(c) Sketch a two-observer diagram for this scenario on the provided hyperbolic paper: Charlie (located at $x = 0$ ns) observes event P at $t = 0$ ns in the home frame (HF). In the other frame (OF), the second spaceship (with Alice and Bob) moves at $\beta = 2/5$ in the positive $x$ direction relative to the HF. Presume the origins of the HF and OF coincide. In the diagram, make sure to: (8 pt) → see graph paper.

(i) Draw and label the axes corresponding to the HF and the OF. (2 pt)

(ii) Calibrate both axes (show scales). (2 pt)

(iii) Draw and label the worldlines for each spaceship (*Charlie* or *Alice and Bob*). (2 pt)

(iv) Locate and label events P and D. (2 pt)

(d) Determine the time difference between P and D as measured by Alice and Bob through two methods - by using the graph and algebraically (through the Lorentz transformations). Do you get the same results? Briefly explain why or why not. (6 pt)

Graphically: $\Delta t'_{PD} = 550$ ns. * (2 pts)

Lorentz transformations: $\Delta t'_{PD} = \gamma\,\Delta t_{PD} - \gamma\beta\,\Delta x_{PD}$ (1 pt) $\beta = 2/5$, $\Delta t_{PD} = 500$ ns, $\Delta x_{PD} = 0$. (1 pt)

$$= \frac{1}{\sqrt{1-4/25}} \cdot 500 \text{ ns}$$
$$= \frac{5}{\sqrt{21}} \cdot 500 \text{ ns} = 545 \text{ ns} \quad **$$
} (1 pt)

* and ** agree sufficiently well! Should be: two means of finding the same thing. (1 pt)

## Problem 4: Dimensional Analysis and Blackbody Radiation (16 pt)

In this dimensional analysis problem, use L as the dimension of length, M as the dimension of mass, T as the dimension of time, and Θ as the dimension of temperature (to avoid confusion with T as the dimension of time).

(a) Using the relations $E = h\nu$ and $PV \propto k_\mathrm{B}T$, where $E$ is energy, $\nu$ is frequency, $P$ is pressure, $V$ is volume and $T$ is temperature, show that $[k_B] = \mathrm{M}\cdot\mathrm{L}^2\cdot\mathrm{T}^{-2}\cdot\Theta^{-1}$ and $[h] = \mathrm{M}\cdot\mathrm{L}^2\cdot\mathrm{T}^{-1}$. (6 pt)

$E = h\nu \rightarrow [h] = \frac{[E]}{[\nu]} = \frac{M\cdot L^2 T^{-2}}{T^{-1}} = M\cdot L^2 T^{-1}$ (3pts)

$k_B T \propto PV \rightarrow [k_B] = \frac{[P][V]}{[T]} = \frac{[F][V]}{[A][T]} = \frac{(M\cdot L T^{-2})(L^3)}{(L^2)\Theta} = M\cdot L^2\cdot T^{-2}\cdot\Theta^{-1}$ (3pts)

Blackbodies are theoretical bodies that absorb and emit radiation over a range of wavelengths. The wavelength at which the intensity of a blackbody is largest ($\lambda_\mathrm{peak}$) depends on the temperature of the blackbody ($T$), speed of light ($c$), Boltzmann's constant ($k_B$), and Planck's constant ($h$).

(b) Derive $\lambda_\mathrm{peak}$ as a function of $h$, $c$, $k_B$, and $T$. (10 pt)

Take: $\lambda_{peak} = C c^\alpha h^\beta T^\gamma k_B^\delta$

1 pt: correct power-law form
1 pt: dimensionless constant C

$[\lambda_{peak}] = L = (L\cdot T^{-1})^\alpha (M\cdot L^2\cdot T^{-1})^\beta \Theta^\gamma (M\cdot L^2 T^{-2}\cdot\Theta^{-1})^\delta$
$= L^{\alpha+2\beta+2\delta}\, T^{-\alpha-\beta-2\delta}\, M^{\beta+\delta}\, \Theta^{\gamma-\delta}$

1 pt: fill in dimensions of quantities
1 pt: group dimensions

Equating powers:
$\alpha + 2\beta + 2\delta = 1 \rightarrow \alpha = 1$ since $\beta = -\delta$ (3)
$-\alpha - \beta - 2\delta = 0 \rightarrow -1 + \beta = 0 \rightarrow \beta = 1$ (4) $\rightarrow \delta = -\beta = -1$ (5) $\rightarrow \gamma = \delta = -1$ (6)
$\beta + \delta = 0 \rightarrow \beta = -\delta$ (2)
$\gamma - \delta = 0 \rightarrow \gamma = \delta$ step (1)

2 pt: 0.5 for each correct equation: 4 x 0.5 pt = 2 pts

3 pts: 0.5 pt for each correct step: 6 x 0.5 pts = 3 pts

Hence we have $\alpha = \beta = 1$ and $\gamma = \delta = -1$: $\lambda_{peak} = C\frac{ch}{k_B T}$ QED. 1 pt for correct functional form.

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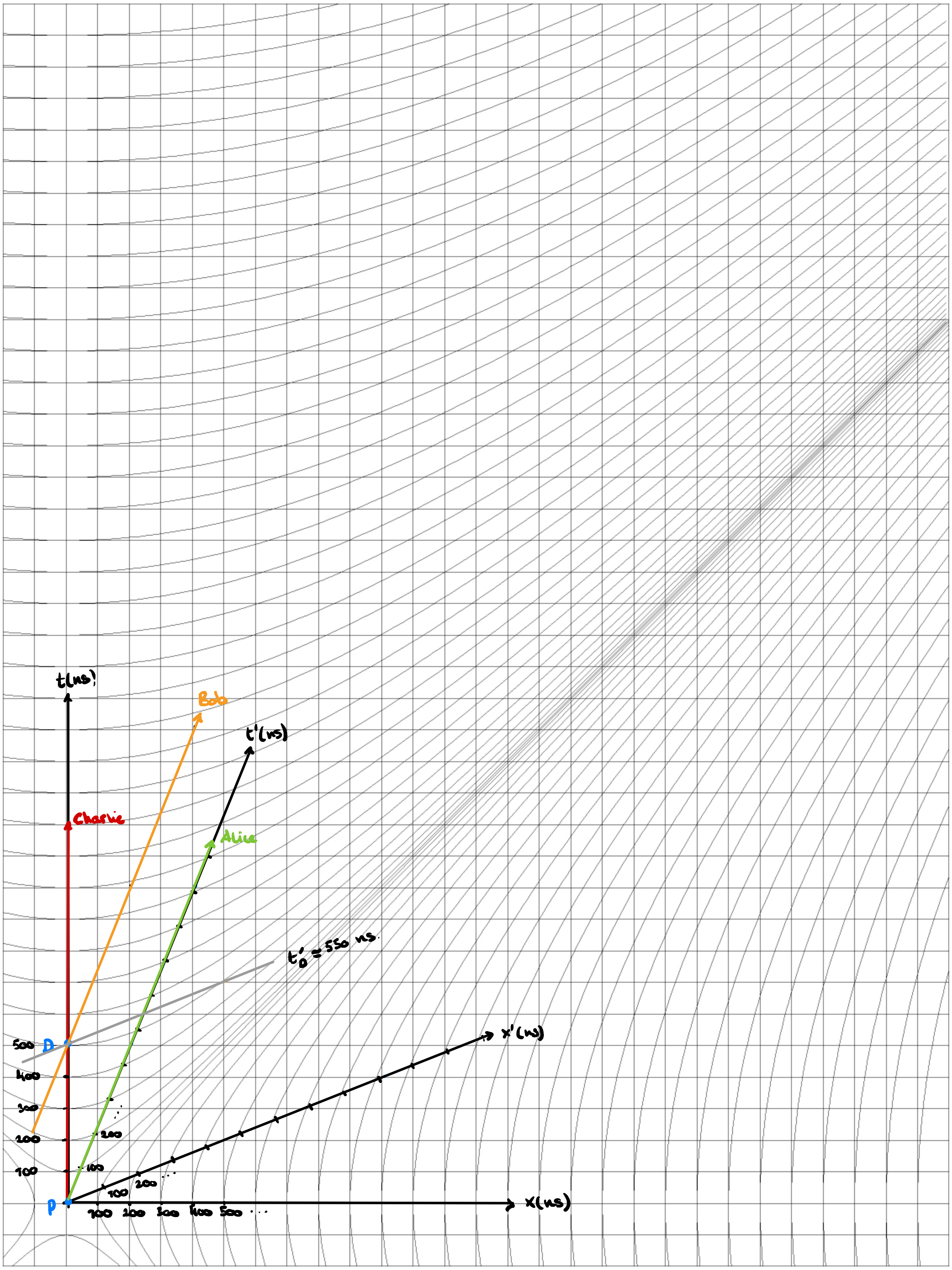

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