Linear Algebra - Final Exam

April 4, 2023 11.45-13.45

Write your name and student number on all pages. Number the pages sequentially.
At the end of the exam, put your papers in the envelope, write your name and student number.
DO NOT SEAL THE ENVELOPE!
You **can** have a "cheat sheet": This is an A4 paper written on **one** side. You are **NOT** allowed to have books, course notes, homework assignments, etc., laptops, e-readers, tablets, telephones, etc., during the exam!.
You can also use a normal calculator (not a programmable/graphic one).
Give a clear explanation of your answer and show the relevant computations.
You will not get points for results without calculations/explanations.

QUESTIONS:

1. $\boxed{2}$ Given the vectors $\mathbf{v} = \begin{bmatrix} \beta \\ 3 \\ 3 \end{bmatrix}$, $\mathbf{a} = \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix} -1 \\ \alpha \\ 2 \end{bmatrix}$, use **only row reduction** to determine for which value(s) of α and β the vector $\mathbf{v}$ is in Span$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$.

2. For a linear transformation $L: \mathbb{R}^3 \to \mathbb{R}^2$ it is given that:
$L \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$, $L \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$ and $L \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

   (a) $\boxed{1}$ Find the matrix A (defined by $L(\mathbf{x}) = A\mathbf{x}$) and show that $\begin{bmatrix} -9 \\ -2 \end{bmatrix}$ is the image of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. **You get no points for this part if you do not explain how you got** A**.**

   (b) $\boxed{1}$ Three 3D points are given by $P(2,1,3)$, $Q_1(1,1,1)$ and $Q_2(1,1,2)$. Determine and describe the images of the segment through P and Q_1 and segment through P and Q_2.

3. Check whether the following are eigenvalues and eigenvectors of a matrix. Justify your answer. If the answer is yes, give the matrix.

   (a) $\boxed{1}$ $\lambda_1 = \frac{7}{2}$, $\lambda_2 = -3$, $\lambda_3 = \frac{1}{8}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 5 \\ -7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

   (b) $\boxed{1}$ $\lambda_1 = -4$, $\lambda_2 = 2$, $\lambda_3 = 2$, $\lambda_4 = 1$, $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ -6 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$.

   You have to explain how you reached the conclusion and show your calculations for that.

4. Determine whether the following sets of vectors are linearly independent:

   (a) $\boxed{0.5}$ $\{1, x^2, x^2 - 2\} \in P_3$

   (b) $\boxed{0.5}$ $\{x + 2, x + 1, x^2 - 1\} \in P_3$

   (c) $\boxed{0.5}$ $\{x + 2, x^2 - 1\} \in P_3$

   (d) $\boxed{0.5}$ $\{\cos \pi x, \sin \pi x\} \in C[0, 1]$.

   You have to explain how you reached the conclusion and show your calculations for that.

Please turn over

5. $\boxed{\mathbf{2}}$ Find the **real** solution of the following initial-value problem:

$$\begin{aligned} x_1' &= \ \ 2x_1 + \ x_2 + \ x_3 \\ x_2' &= -2x_3 \\ x_3' &= \ \ 2x_2 \end{aligned}$$

with $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$.

Hint: Here you have the formulae from the book that give linearly independent **real** *solutions:*
$\mathbf{y}_1 = e^{at}\left[\mathbb{R}\mathrm{e}(\mathbf{x})\cos(bt) - \mathbb{I}\mathrm{m}(\mathbf{x})\sin(bt)\right]$ *and* $\mathbf{y}_2 = e^{at}\left[\mathbb{I}\mathrm{m}(\mathbf{x})\cos(bt) + \mathbb{R}\mathrm{e}(\mathbf{x})\sin(bt)\right]$.
But notice that you still need to explain how you did the rest and why you use these formulae.
You have to compute them yourself, but for you to check, 2 of the 3 e-values and corresponding e-vectors are:
$\lambda_1 = 2$, $\lambda_2 = 2i$, $\mathbf{v}_1 = [\,1 \ \ 0 \ \ 0\,]^T$, $\mathbf{v}_2 = [\,-i \ \ 2i \ \ 2\,]^T$.

NOTE: Maximum points possible, $p = 10$. Grade, $g = 0.9p + 1$

Solutions

General consideration: In each problem the students should get some points if they explain correctly what they have to do (the method) to solve the problem even if either the final result is incorrect because they make algebraic mistakes or they did not manage to solve the problem.
I therefore give separately the points for the method as "X points for the method" and the points for the result as "Y points for the result".
If a student does not state explicitly the method, but they use it in a way that is clear what they do, you should add the X points of the method at the end.
They do not get points for a solution if they use a different method than the one asked for, even if the result is correct. I may not write this in all cases, but keep this in mind.

1. I need to see whether I can find numbers x_1, x_2, x_3 to write $\mathbf{v} = x_1\mathbf{a} + x_2\mathbf{b} + x_3\mathbf{c}$:

   0.6 points for this or similar statement; give the points if they do not make the statement explicitly but use it to solve this problem

   Because the question states explicitly that they have to use row reduction, they get no points for this exercise if they use any other method

   $x_1 \begin{bmatrix} -3 \\ -2 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ \alpha \\ 2 \end{bmatrix} = \begin{bmatrix} \beta \\ 3 \\ 3 \end{bmatrix}$, or equivalently $\begin{bmatrix} -3 & 1 & -1 \\ -2 & 4 & \alpha \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \beta \\ 3 \\ 3 \end{bmatrix}$.

   Then write the augmented matrix and operate on the rows:

   $$\begin{bmatrix} -3 & 1 & -1 & | & \beta \\ -2 & 4 & \alpha & | & 3 \\ 2 & 1 & 2 & | & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} -3 & 1 & -1 & | & b \\ 0 & 10/3 & (3a+2)/3 & | & (9-2b)/3 \\ 0 & 0 & (-a+2)/2 & | & (2b+3)/2 \end{bmatrix},$$

   Give 0.8 points for the row reduction

   If $\alpha \neq 2$ there is 1 solution. So yes, in this case $\mathbf{v} \in \text{Span}\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$

   Give 0.2 points for the result

   If $\alpha = 2$ **and** $\beta = -3/2$ the last row is all 0's, there is a free variable and so there are infinite solutions. So yes, in this case $\mathbf{v} \in \text{Span}\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$

   Give 0.2 points for the result

   If $\alpha = 2$ **and** $\beta \neq -3/2$ the last row is inconsistent, so no solutions. So no, in this case $\mathbf{v} \notin \text{Span}\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$

   Give 0.2 points for the result

   Here it is important that they say (somehow) that both conditions (hence the "and") must hold. They get 0 points if they do not say that either explicitly or implicitly (in the latter case it must be clear that they do so), or if they say "or".

2. (a) I need to find the effect of the linear transformation of the unit vectors.

   They get 0.2 points for making this or similar statement, even if later they make algebraic mistakes. They get the points also if they do not say this explicitly, but use it to get A

   $\mathbf{v}_1 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} = -2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = -2\mathbf{e}_2$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix} = -\mathbf{e}_1 + 2\mathbf{e}_2$. $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3 \Rightarrow$

   $L(-2\mathbf{e}_2) = -2L(\mathbf{e}_2) = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

   $L(-\mathbf{e}_1 + 2\mathbf{e}_2) = -L(\mathbf{e}_1) + 2L(\mathbf{e}_2) = \begin{bmatrix} 6 \\ 0 \end{bmatrix}$

   $L(\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3) = L(\mathbf{e}_1) + 2L(\mathbf{e}_2) + 3L(\mathbf{e}_3) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

   From the first one $L(\mathbf{e}_2) = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$

   From the second one $L(\mathbf{e}_1) = -\begin{bmatrix} 6 \\ 0 \end{bmatrix} + 2L(\mathbf{e}_2) = -\begin{bmatrix} 6 \\ 0 \end{bmatrix} + 2\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} -12 \\ -2 \end{bmatrix}$

From the third one $3L(\mathbf{e}_3) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} - L(\mathbf{e}_1) - 2L(\mathbf{e}_2) = \begin{bmatrix} 0 \\ -1 \end{bmatrix} - \begin{bmatrix} -12 \\ -2 \end{bmatrix} - 2\begin{bmatrix} -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 18 \\ 3 \end{bmatrix} \Rightarrow L(\mathbf{e}_3) = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$.

$A = [L(\mathbf{e}_1)\ L(\mathbf{e}_2)\ L(\mathbf{e}_3)] = \begin{bmatrix} -12 & -3 & 6 \\ -2 & -1 & 1 \end{bmatrix}$.

They get 0.6 points for the three transforms of the unit vectors (0.2 points for each), or for the final matrix

Then $L\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -12 & -3 & 6 \\ -2 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ -2 \end{bmatrix}$.

0.2 points for this one

They get no points for part (a) if they do not explain how they got A, with some calculation

(b) P: $L\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} -9 \\ -2 \end{bmatrix}$, Q1: $L\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -9 \\ -2 \end{bmatrix}$, Q2: $L\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$.

Both P and Q1 are mapped onto $\begin{bmatrix} -9 \\ -2 \end{bmatrix}$; the segment from P to Q1 in 3D is projected onto the same point in 2D.

They get 0.4 points for this result

The segment from P to Q2 in 3D is projected onto the segment from $(-9, -2)$ and $(-3, -1)$ in 2D.

They get 0.6 points for this result

3. (a) In this case it is easy to see that the vectors are not linearly independent because, if I call $P = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$, $\det P = 0$.

0.2 points for noticing that if $\det P = 0$ is enough to show no independence

0.4 points for calculating the determinant. They get the full point also if they use a different, valid, method to prove that the vectors are not independent

Then P^{-1} does not exist, and I won't be able to calculate $A = PDP^{-1}$. So the answer is no, these are not the eigenvalues and eigenvectors of a matrix.

0.4 points for the answer

(b) In this case the vectors are linearly independent. To see this I call $P = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]$, which is a block diagonal matrix. It is easy to calculate the determinant of the two (2×2) blocks and multiply them by each other to get that $\det P = \det \begin{bmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 \\ -6 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \end{bmatrix} = 6$.

0.2 points for the determinant

So the answer is yes, these are the eigenvalues and eigenvectors of a matrix.

0.2 points for noticing that if $\det P \neq 0$ is enough to show independence

To find this matrix, A, I need to invert P, so I can write $A = PDP^{-1}$ where D is a diagonal matrix with the eigenvalues in the diagonal. To invert P I take advantage from the fact that P is block diagonal. The blocks, and corresponding inverses are:

$P_{11} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}$, $P_{22} = \begin{bmatrix} -6 & 0 \\ 5 & 1 \end{bmatrix}$, $P_{11}^{-1} = \begin{bmatrix} -1/6 & 0 \\ 5/6 & 1 \end{bmatrix}$, and $P_{22}^{-1} = \begin{bmatrix} 1 & -1 \\ 1 & -2 \end{bmatrix}$, and hence:

$$P^{-1} = \begin{bmatrix} 0 & 0 & -1/6 & 0 \\ 0 & 0 & 5/6 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix}.$$

$\boxed{\text{0.3 points for the inverse of } P}$

Then:

$$A = PDP^{-1} = \begin{bmatrix} 0 & 0 & 2 & -1 \\ 0 & 0 & 1 & -1 \\ -6 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1/6 & 0 \\ 0 & 0 & 5/6 & 1 \\ 1 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 5 & 2 \end{bmatrix}$$

$\boxed{\text{0.3 points for } A}$

4. To do this I need to calculate the Wronskian to see whether the vectors are linearly independent. If the Wronskian is $\neq 0$ I know the vectors are linearly independent. If the Wronskian is $= 0$ I need to do more to find out.

In all cases give 0.1 points for this or similar statement; give the points if they do not make the statement explicitly but use it to solve this problem. In all cases it is also okay and give full points if they skip the Wronskian part and go directly to test whether, for given values of the functions, they can build a system of linear homogeneous equations that has only the trivial solution. Make sure that you do not give more than the maximum points, 0.5, per item. I do not give notes for each item since they are all similar, and this grading notes should suffice.

Subtract 0.1 points if they find that the Wronskian is 0 and stop there, since that is not enough to assess linear independence (see for instance example 8 in the book).

Give no points further if they do not show how they calculated the Wronskian, or if they say correctly whether the vectors are linearly independent, but they do not show the steps.

(a) $W(1, x^2, x^2 - 2) = 0$. I need to do more to find out. Write $c_1 + c_2 x^2 + c_3(x^2 - 2) = 0$. Take:

$$\begin{aligned} x = 0 &: c_1 \phantom{{}+2c_2} - 2c_3 = 0 \\ x = \sqrt{2} &: c_1 + 2c_2 \phantom{{}- c_3} = 0 \\ x = 1 &: c_1 + c_2 - c_3 = 0 \end{aligned}$$

The augmented matrix is: $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 1 & -1 & 0 \end{bmatrix}$ and after row operations I get $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

So there is a free variable, hence the vectors are not linearly independent.

$\boxed{\text{0.5 points}}$

I used row reduction, but they can use any valid method to solve the system of equations to find the c_i.

(b) $W(x + 2, x + 1, x^2 - 1) = 2 \neq 0$, so yes.

$\boxed{\text{0.5 points}}$

(c) $W(x + 2, x^2 - 1) = x^2 + 4x + 1 \neq 0$ for infinite values of x, so yes.

$\boxed{\text{0.5 points}}$

(d) $W(\cos \pi x, \sin \pi x) = \pi \neq 0$, so yes.

$\boxed{\text{0.5 points}}$

5. The system can be written in matrix form as $\mathbf{x}' = A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{bmatrix}$ and $\mathbf{x}_0 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$.

I first find the evals/evects of A (I do not show the steps, but the students have to).

The evals are $\lambda_1 = 2$, $\lambda_2 = 2i$, $\lambda_3 = -2i$, and the corresponding events are: $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -i \\ 2i \\ 2 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} i \\ -2i \\ 2 \end{bmatrix}$.

0.9 for the 3 evals and 3 events; 0.1 points for each eval and 0.2 points for each evect

It is okay if they calculate only $\mathbf{v}_1$ and $\mathbf{v}_2$, and say that $\mathbf{v}_3$ must be the complex conjugate of $\mathbf{v}_2$

They get no points for the above if they use the eigenvalues/eigenvectors given in the Hint; they get no points for the third eigenvalue/eigenvector (the one not given in the hint) either if they simply use the fact that the third ones should be the complex conjugates of the second ones, unless they calculate it themselves

At this point they may go as I do below, or they can decide to change variables to decouple the equations as I explained in the class (see my notes). Both ways are fine. if they choose to decouple the equations, at the end they **must** transform back to the original variables

I write $\lambda_2 = a + ib$, where $a = 0$ and $b = 2$.
The general **real** solution will then be:

They are allowed to take the formula directly from the Hint in the question.

$$\mathbf{x} = c_1 \mathbf{v}_1 e^{\lambda_1 t} + c_2 \left[\mathbb{R}\mathrm{e}(\mathbf{v}_2)\cos bt - \mathbb{I}\mathrm{m}(\mathbf{v}_2)\sin bt\right] e^{at} + c_3 \left[\mathbb{R}\mathrm{e}(\mathbf{v}_2)\sin bt + \mathbb{I}\mathrm{m}(\mathbf{v}_2)\cos bt\right] e^{at}.$$

It is also possible to write the general solution $\mathbf{x} = \sum_{i=1}^{3} c_i \mathbf{v}_i e^{\lambda_i t}$, where λ_i and $\mathbf{v}_i$ are possibly complex, and keep only the real part of the solution. In all cases the results should be the same, but some of the intermediate steps may differ from what I have below.

$$\mathbf{x} = c_1 \begin{bmatrix}1\\0\\0\end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix}0\\0\\2\end{bmatrix} \cos 2t - \begin{bmatrix}-1\\2\\0\end{bmatrix} \sin 2t \right) + c_3 \left(\begin{bmatrix}0\\0\\2\end{bmatrix} \sin 2t + \begin{bmatrix}-1\\2\\0\end{bmatrix} \cos 2t \right) = \begin{bmatrix} c_1 e^{2t} + c_2 \sin 2t - c_3 \cos 2t \\ -2c_2 \sin 2t + 2c_3 \cos 2t \\ 2c_2 \cos 2t + 2c_3 \sin 2t \end{bmatrix}.$$

0.6 points for the general solution.

Applying the initial condition:

$$\begin{bmatrix} c_1 - c_3 \\ 2c_3 \\ 2c_2 \end{bmatrix} = \begin{bmatrix}2\\2\\2\end{bmatrix}.$$

I can solve this to find $c_1 = 3$, $c_2 = 1$, $c_3 = 1$.

0.3 points for the c_i, 0.1 points for each.

The solution of the initial-value problem is then:

$$\mathbf{x}(t) = \begin{bmatrix} 3e^{2t} + \sin 2t - \cos 2t \\ -2\sin 2t + 2\cos 2t \\ 2\cos 2t + 2\sin 2t \end{bmatrix}.$$

0.2 points for the solution of the initial-value problem.