Name: Student Number:

Mechanics and Relativity: R3

October 31st, 2022, Aletta Jacobshal
Duration: 120 mins

Before you start, read the following:

- There are 3 problems, for a total of 50 points.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate. Draw your spacetime diagrams on the provided hyperbolic paper.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

	Points
Problem 1:	20
Problem 2:	20
Problem 3:	10
Total:	50
GRADE (1 + # Total/(50/9))	

Useful equations:

$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$
$$\Delta t \geq \Delta s \geq \Delta \tau$$

The Lorentz transformation equations with $\gamma \equiv (1-\beta^2)^{-1/2}$:

$$t' = \gamma(t - \beta\, x)\,,$$
$$x' = \gamma(x - \beta t)\,,$$
$$y' = y\,,$$
$$z' = z\,.$$

The relativistic Doppler shift formula and Esintein velocity transformations:

$$\frac{\lambda_R}{\lambda_E} = \sqrt{\frac{1+v_x}{1-v_x}} \quad v'_x = \frac{v_x - \beta}{1-\beta v_x} \quad v'_{y,z} = \gamma^{-1}\frac{v_{y,z}}{1-\beta v_x}\,.$$

Possibly relevant equations:

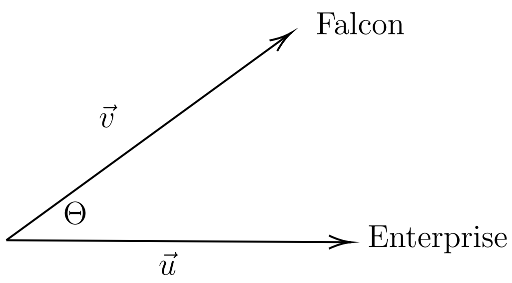
$$F = G\frac{Mm}{r^2}; \qquad F = ma; \qquad PV \propto k_b T; \qquad F = \frac{dp}{dt}$$

Possibly relevant numbers:

$$L_{\text{sun}} = 3.83 \times 10^{24}\ \text{kg m}^2\ \text{s}^{-3} \qquad c = 299792458\ \text{m/s} \qquad 1\ \text{eV} = 1.602176565 \times 10^{-19}\ \text{J}$$

Question 1: The Enterprise and the Falcon (20 pts)

Consider two spaceships, the Enterprise and the Falcon, moving with velocities $\vec{u}$ and $\vec{v}$ as shown in the figure below. The velocity vectors make an angle Θ. The aim is to obtain the speed $|\vec{V}| = V$ of the other spaceship (Falcon) as measured by the Enterprise in terms of Θ, $|\vec{v}| = v$ and $|\vec{u}| = u$ (note that the answer is relative: the speed of the Falcon as observed from the Enterprise is the same as the speed of the Enterprise as observed from the Falcon). Consider a frame where we measure $\vec{u}$ and $\vec{v}$ velocities (we will refer to this as the rest frame). Take the horizontal direction to align with the x-axis and the vertical direction with the y-axis.



(a) **(2 pts)** Write down the velocity components of the two spaceships in the frame at rest.

In this frame the Enterprise (A) $v_x^A = u$ **(1/2 pts)** and $v_y^A = 0$ **(1/2 pts)**. For the Falcon (B) $v_x^B = v\cos\Theta$**(1/2 pts)** and $v_y^B = v\sin\Theta$**(1/2 pts)**.

(b) **(2 pts)** In the rest frame of the Enterprise, what are the velocity components of the Enterprise and the Falcon?

By construction, since the Enterprise is at rest, $v_x^{A\prime} = v_y^{A\prime} = 0$ **(1/2 pts)**. For the Falcon we can apply the velocity transformations, i.e. $v_x^{B\prime} = (v_x^B - \beta)/(1 - \beta v_x^B)$**(1/2 pts)** and $v_y^{B\prime} = v_y^B/(\gamma(1 - \beta v_x^B))$**(1/2 pts)** Here $\beta = v_x^A u$. We thus find **(1/2 pts)**

$$v_x^{B\prime} = \frac{v\cos\Theta - u}{(1 - uv\cos\Theta)}, \tag{1}$$

$$v_y^{B\prime} = \frac{v\sin\Theta(1 - u\cos\Theta)^{1/2}}{(1 - uv\cos\Theta)}. \tag{2}$$

(c) **(5 pts)** Show that the speed of Falcon as observed in the frame where the Enterprise is at rest is given by

$$V = \sqrt{1 - \frac{(1-u^2)(1-v^2)}{(1-uv\cos\Theta)^2}}. \tag{3}$$

We calculate the speed between the spaceships via **(1 pt)**

$$|V| = \sqrt{(v_x^{B\prime})^2 + (v_y^{B\prime})^2}. \tag{4}$$

Inserting the found solution for the components above we obtain **(3/2 pts)**

$$V = \sqrt{\frac{v^2\cos^2\Theta + u^2 - 2uv\cos\Theta + v^2\sin^2\Theta - u^2v^2\sin^2\Theta}{(1-uv\cos\Theta)^2}} \tag{5}$$

$$= \sqrt{\frac{v^2 + u^2 - 2uv\cos\Theta - u^2v^2\sin^2\Theta}{(1-uv\cos\Theta)^2}}. \tag{6}$$

Here we used $\cos^2\Theta + \sin^2\Theta = 1$. We can simplify this further as **(5/2 pts)**

$$V = \sqrt{\frac{v^2 + u^2 - 2uv\cos\Theta + u^2v^2(\cos^2\Theta - 1)}{(1-uv\cos\Theta)^2}} \tag{7}$$

$$= \sqrt{\frac{v^2 + u^2 - 1 - u^2v^2 - 2uv\cos\Theta + u^2v^2\cos^2\Theta + 1}{(1-uv\cos\Theta)^2}} \tag{8}$$

$$= \sqrt{\frac{(1-uv\cos\Theta)^2 - (1-u^2)(1-v^2)}{(1-uv\cos\Theta)^2}} \tag{9}$$

$$= \sqrt{1 - \frac{(1-u^2)(1-v^2)}{(1-uv\cos\Theta)^2}}. \tag{10}$$

Note that in this question and the questions below students might not always show the same steps or even skip a few steps. As long as the answer is correct and it is evident that the student understood, give full points.

(d) **(2 pts)** Show what happens to Eq. (3) when $\Theta = 0$? Explain if this answer makes sense.

When $\Theta = \pi \to \cos\Theta = 1$. We thus find that **(1 pt)**

$$V_{\Theta=0} = \sqrt{1 - \frac{(1-u^2)(1-v^2)}{(1-uv)^2}} \tag{11}$$
$$= \sqrt{\frac{u^2+v^2-2u^2v^2}{(1-uv)^2}} \tag{12}$$
$$= \frac{v-u}{1-uv}. \tag{13}$$

This makes sense since the spaceships are now moving in the same directions and their resulting net speed should be minimized (and identical to the usual transformation) **(1 pt)**.

(e) **(3 pts)** In the lecture we have defined a four-momentum $\mathbf{p} = (E, \vec{p}) = (\gamma m, \gamma m\vec{v})$. Similarly (as explained in the lecture) we can define four-velocity $\mathbf{v} = (\gamma, \gamma\vec{v})$. In addition, in the lecture clips and the lecture, we defined the inner product of two four-vectors which is similar to the usual Euclidean inner product, except that its signature is now Minkowski:

$$\mathbf{A}\cdot\mathbf{B} \equiv A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3, \tag{14}$$

for two four-vectors $\mathbf{A} = (A_0, A_1, A_2, A_3)$ and $\mathbf{B} = (B_0, B_1, B_2, B_3)$. Show that the inner product is invariant under Lorentz transformations, i.e. $\mathbf{A}\cdot\mathbf{B} = \mathbf{A}'\cdot\mathbf{B}'$. Assume the other frame is moving in the $+x$ directions. HINT: time (space) components of the four-vector transform as t (x) under Lorentz transformations.

Applying the Lorentz tranformations we obtain

$$\mathbf{A}'\cdot\mathbf{B}' = \gamma(A_0 - \beta A_1)\gamma(B_0 - \beta B_1) - \gamma(A_1 - \beta A_0)\gamma(B_1 - \beta B_0) - A_2B_2 - A_3B_3 \ (\mathbf{1/2pts}) \tag{15}$$
$$= \gamma^2(A_0B_0 - \beta A_1B_0 - \beta B_1A_0 - \beta^2 A_1B_1 - A_1B_1 + \beta A_0B_1 + \beta B_0A_1 - \beta^2 B_0A_0) - A_2B_2 - A_3B_3 \ (\mathbf{1/2pt}) \tag{16}$$
$$= \gamma^2((A_0B_)(1-\beta^2) - A_1B_1(1-\beta^2)) - A_2B_2 - A_3B_3 \ (\mathbf{1/2pts}) \tag{17}$$
$$= A_0B_0 - A_1B_1 - A_2B_2 - A_3B_3 = \mathbf{A}\cdot\mathbf{B} \ (\mathbf{1pt}). \tag{18}$$

Here we used that $\gamma^2 = (1-\beta^2)^{-1}$ **(1/2 pts)**.

(f) **(1 pt)** Write down the four-velocity $\mathbf{u}$ and $\mathbf{v}$ of the two spaceships in the rest frame (as defined above 1 (a)). Use notation where $\gamma_u = (1-u^2)^{-1/2}$ and similarly for v.

The Enterprise is only moving in the x direction, so we can write $\mathbf{u} = (\gamma_u, \gamma_u u, 0, 0)$ **(1/2 pts)**. Here $\gamma_u = (1-u^2)^{-1/2}$. The Falcon is both moving in the x and y directions, $\mathbf{v} = (\gamma_v, \gamma_v v\cos\Theta, \gamma_v v\sin\Theta, 0)$ **(1/2 pts)**.

(g) **(2 pts)** Let us now consider another frame. In this frame, the Falcon is moving in the x' direction and the Enterprise is at rest. We can thus write $\mathbf{u}' = (1, 0)$ and $\mathbf{v}' = (\gamma_{v'}, \gamma_{v'}v')$. We have dropped the y' and z' components for brevity. In this other frame, the Falcon is moving with speed v'. Using the invariance of the inner product to show that

$$\gamma_{v'} = \gamma_u\gamma_v(1 - uv\cos\Theta). \tag{19}$$

Using the invariance of the inner product between the two velocity four-vectors

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} = \mathbf{u}'\cdot\mathbf{v}' &\rightarrow \text{LHS} : (\gamma_u, \gamma_u u, 0, 0)\cdot(\gamma_v, \gamma_v v\cos\Theta, \gamma_v v\sin\Theta, 0) = \gamma_u\gamma_v - \gamma_u\gamma_v uv\cos\Theta\ \mathbf{(1pt)}\\ &\rightarrow \text{RHS} : (\gamma_{v'}, \gamma_{v'}v')\cdot(1, 0) = \gamma_{v'}\ \mathbf{(1pt)}\end{aligned} \tag{20}$$

So we get that $\gamma_{v'} = \gamma_u\gamma_v(1 - uv\cos\Theta)$.

(h) **(3 pts)** Find a solution for v' and show that it is identical to the solution for V in Eq. (3).

Using the definition of the Lorentz factor we have

$$(1 - v'^2)^{-1/2} = \gamma_u\gamma_v(1 - uv\cos\Theta)\ \mathbf{(1/2pts)} \tag{21}$$

Let us square the right hand side and invert:

$$(1 - v'^2) = (\gamma_u\gamma_v)^{-2}(1 - uv\cos\Theta)^{-2}\ \mathbf{(1pt)} \tag{22}$$

or

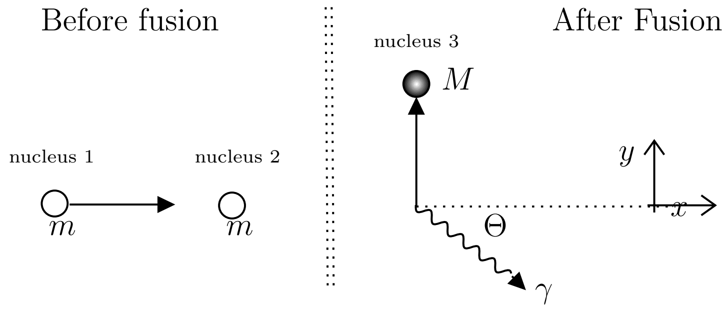
$$v' = \sqrt{1 - (\gamma_u\gamma_v)^{-2}(1 - uv\cos\Theta)^{-2}}\ \mathbf{(1/2pts)}. \tag{23}$$

Writing out the Lorentz factors we obtain

$$v' = \sqrt{1 - \frac{(1-u^2)(1-v^2)}{(1-uv\cos\Theta)^2}}\ \mathbf{(1pt)}, \tag{24}$$

which is indeed the same as we found before.

Question 2: Nuclear Fusion (20 pts)
Nuclear fusion is the process in which two nuclei are merged together to produce a third nucleus + energy (photon γ, not to be confused with the Lorentz factor). The drawing below suggests such a fusion process. One nucleus of mass m, moving at some speed v to the right, hits a stationary nucleus of the same mass m. The fusion produces one nucleus of mass M and a photon with some energy E_γ. While the nucleus is moving up (along the y axis), the photon moves down with some undetermined angle Θ with respect to the horizontal axis (x axis).



(a) **(2 pts)** Write down the four-momenta of each object in the diagram. Only use the relativistic energies and momenta to describe the total four momenta of the three particles and the photon. You can ignore/drop the z direction.

We have have the incoming four-momenta, $\mathbf{p_1} = (E_1, p_1, 0)$ **(1/2 pt)** and $\mathbf{p_2} = (E_2, p_2, 0)$ **(1/2 pt)**. The outgoing momenta are given by $\mathbf{p_3} = (E_3, 0, p_3)$ **(1/2 pt)** and $\mathbf{p}_\gamma = (E_\gamma, E_\gamma \cos\Theta, -E_\gamma \sin\Theta)$ **(1/2 pt)**. Here we uses that for the photon $E = |\vec{p}|$ and we have projected the components of onto the x and y axis.

(b) **(4 pts)** By equating the four-momenta show that the energy before the fusion is related to the angle Θ, the (3) momentum of the incoming nucleus 1, p_1, and the mass, M, of the new nucleus 3 as

$$E_1 + E_2 = \sqrt{p_1^2(1+\tan^2\Theta)} + \sqrt{p_1^2\tan^2\Theta + M^2}, \tag{25}$$

where E_1 and E_2 are the energies of the two nuclei before the fusion.

Using conservation of momenta $\mathbf{p_1} + \mathbf{p_2} = \mathbf{p_3} + \mathbf{p}_\gamma$ **(1/2 pt)** we derive:

$$p_1 = E_1 \cos\Theta\ (\mathbf{1/2pt}) \tag{26}$$
$$0 = p_3 - E_1 \sin\Theta\ (\mathbf{1/2pt}). \tag{27}$$

Or, $p_3 = p_1 \tan\Theta = -E_1 \sin\Theta$ **(1/2 pt)**. We also know that for the outgoing nuclei we have $E_3^2 = |\vec{p_3}|^2 + M^2 = p_1^2\tan^2\Theta + M^2$ **(1/2 pt)**, while for the massless photon we have $E_\gamma^2 = p_1^2 + p_1^2\tan^2\Theta$ **(1/2 pt)**. Conservation of four-momentum implies $E_1 + E_2 = E_3 + E_\gamma$ **(1/2 pt)**, i.e.

$$E_1 + E_2 = \sqrt{p_1^2(1+\tan^2\Theta)} + \sqrt{p_1^2\tan^2\Theta + M^2}\ (\mathbf{1/2pt}). \tag{28}$$

(c) **(5 pts)** Writing $E_1 + E_2 = E$ and realizing that the first term on the RHS is the energy of the photon E_γ, show that

$$E_\gamma = \frac{E^2 + p_1^2 - M^2}{2E}. \tag{29}$$

Starting with **(1/2 pt)**

$$E = E_\gamma + \sqrt{p_1^2\tan^2\Theta + M^2}. \tag{30}$$

we can put the first terms on the RHS to the LHS and square:

$$(E - E_\gamma)^2 = p_1^2\tan^2\Theta + M^2\ (\mathbf{1/2pt}) \tag{31}$$
$$E^2 - 2EE_\gamma + E_\gamma^2 = p_1^2(1+\tan^2\Theta) - p_1^2 + M^2 = E_\gamma^2 - p_1^2 + M^2\ (\mathbf{2pts}). \tag{32}$$

Here we added and substracted p_1^2 on the RHS and realized that this is again identical to E_γ^2. We can then solve for E_γ as E_γ^2 cancels on the RHS and LHS. We then find

$$E_\gamma = \frac{E^2 + p_1^2 - M^2}{2E}\ (\mathbf{1pt}), \tag{33}$$

which is identical to the solution above.

(d) **(5 pts)** Show that this implies

$$M < \sqrt{2\gamma_1(1+\gamma_1)}m \tag{34}$$

for this to result in the production of a photon. Here γ_1 is the Lorentz factor associated with the incoming nucleus. Explain that this implies a photon is produced whenever $M < 2m$.

For the incoming nucleus we have $E_1^2 = p_1^2 + m^2 = \gamma_1^2 m^2$ **(1/2 pt)**and for the nucleus at rest, we have $E_2^2 = m^2$ **(1/2 pt)**. For the numerator, we thus find:

$$\begin{aligned}
E^2 + p_1^2 - M^2 &= E_1^2 + 2E_1E_2 + E_2^2 + p_1^2 - M^2 && (35)\\
&= E_1^2 + 2E_1E_2 + m^2 + p_1^2 - M^2 \ (\mathbf{1/2pt})\ (\mathbf{1/2pt}) && (36)\\
&= 2E_1^2 + 2E_1E_2 - M^2 \ (\mathbf{1/2pt}) && (37)\\
&= 2E_1(E_1 + E_2) - M^2 \ (\mathbf{1/2pt}) && (38)\\
&= 2\gamma_1 m(\gamma_1 m + m) - M^2 \ (\mathbf{1/2pt}) && (39)\\
&= 2m^2\gamma_1(1+\gamma_1) - M^2 \ (\mathbf{1/2pt}) && (40)
\end{aligned}$$

In order to have the energy of the photon $E_\gamma > 0$ we need $M < \sqrt{2\gamma_1(1+\gamma_1)}m$ **(1/2 pt)**. Because $\gamma_1 \geq 1$, some energy is produced at all times as long as $M < 2m$ **(1/2 pt)**. This in some sense is classical mass conservation.

(e) **(4 pts)** Finally, for $\gamma_1 = 2$ and $m = 1$ MeV and $M = 2m$ guesstimate how many fusions we need per second to fuel a light bulb. And how many fusions (per second) are needed to power energy release of the sun?

First, let us write down the energy in terms of γ_1 **(1/2 pt)**:

$$E_\gamma = \frac{2m^2\gamma_1(1+\gamma_1) - 4m^2}{2m(1+\gamma_1)} = \frac{m\gamma_1(1+\gamma_1) - 2m}{(1+\gamma_1)} \tag{41}$$

When we insert $\gamma_1 = 2$ and find $E_\gamma = \frac{4}{3}m = \frac{4}{3}$ MeV **(1/2 pt)**. From the front page we have that eV $= \sim 2 \times 10^{-19}$ Joules **(1/2 pt)**. Therefore, at every fusion we generate $\sim 10^{-13}$ Joules/fusion **(1/2 pt)**. 1 Watt = Joule/second **(1/2 pt)**. A typical light bulb has ~ 40 Watts (old school, if led ~ 10 Watts) **(1/2 pt)**. We thus need to produce [Joules/second] times [fusions/joules] $40 \times 10^{13} = 4 \times 10^{14}$ [fusions/second] **(1/2 pt)**. From the cover page we find that the luminosity of the sun is 4×10^{26} Watts, so we need about 4×10^{39} [fusions/second] **(1/2 pt)**. **When not consistently using units subtract 1/2 every time, with max 1 pt after repeated offense.**

Question 3: Wien's Displacement Law, Doppler Shift and Guesstimation (10 pts)
To very good approximation, the emission spectrum of a star resembles that of a blackbody. The intensity of a blackbody varies with the wavelength of the emitted photons. The wavelength at which the most photons are emitted is called peak wavelength λ_{peak}. This peak wavelength is inversely related to the surface temperature of the star T via Wien's displacement law:

$$\lambda_{\text{peak}} = C\frac{hc}{k_{\text{B}}T} \tag{42}$$

Here, h is Planck's constant, c the speed of light and k_B Boltzmann's constant. C is some dimensionless constant of $\mathcal{O}(1)$ (order 1). Since the majority of photons emitted by the star have wavelength λ_{peak}, this wavelength determines the gross color of the star.

(a) **(1 pt)** The star EROS NX11 moves away from earth with a velocity $\beta \ll 1$. Its spectrum is measured on earth. Show that the received peak wavelength $\lambda_{\text{peak,R}}$ is related to the emitted peak wavelength $\lambda_{\text{peak,E}}$ as:

$$\lambda_{\text{peak,R}} = (1+\beta)\lambda_{\text{peak,E}} \tag{43}$$

Hint: Use the Binomial approximation $(1+x)^a = 1 + ax$ to rewrite the fraction $1/(1-\beta)$.

The relativistic Doppler formula gives **(1/2 pt)**:

$$\lambda_{\text{peak,R}} = \sqrt{\frac{1+\beta}{1-\beta}}\lambda_{\text{peak,E}}, \tag{44}$$

where the Binomial expansion gives $1/(1-\beta) = 1+\beta$ so that the equation simplifies to:

$$\lambda_{\text{peak,R}} = (1+\beta)\lambda_{\text{peak,E}} \ \textbf{(1/2pt)} \tag{45}$$

More precisely you could write

$$\sqrt{\frac{1+\beta}{1-\beta}} = \sqrt{(1+\beta)(1+\beta+\beta^2+\mathcal{O}(\beta^3))} \simeq \sqrt{(1+\beta)^2} = (1+\beta) \tag{46}$$

(b) **(1 pt)** EROS NX11 has a surface temperature of 9000 K and its peak wavelength is 322 nm. It moves away from earth at 10% of the speed of light. What is the peak wavelength as measured on earth?

In that case $\beta = 0.1$ **(1/2 pt)** so that we get $\lambda_{\text{peak,R}} = 1.1 \times 322$ nm $= 354$ nm. **(1/2 pt)**

(c) **(1 pt)** The star ENRA NX31 moves towards the earth at half speed EROS moves at. Its surface temperature is three times as large as that of EROS (with a surface temperature of 9000 K). What is the peak wavelength of ENRA as received on earth?

Since peak wavelength and surface temperature are inversely related, its peak wavelength will be three times as small, so that $\lambda_{\mathrm{peak,E}} = 1/3 \times 322$ nm $= 107$ nm **(1/2 pt)**. In this case $\beta = -0.05$ since it moves towards earth at 5% of the speed of light and $1 + \beta = 0.95$ so that the received peak wavelength is 102 nm **(1/2 pt)**.

(d) **(7 pts)** The number of stars in the observable universe is estimated to be of the order 10^{24}. Use guesstimation to find out if there are more stars in the universe than there are grains of sand on all the world's beaches. Some ingredients to help you along the way:

- Guesstimate the total area of land on earth
- Assume land can be divided into similar size continents
- Guesstimate the length of shoreline
- Guesstimate the size of a grain of sand

The earth has a radius of 5000 km **(1/2 pt)**, so its area is about **(1/2 pt)**:

$$A_{earth} = 4\pi R_{earth}^2 = 2.5 \times 10^8 \text{ km}^2.$$

About 30% of this area is land, so that

$$A_{land} = 7 \times 10^7 \text{ km}^2.$$

(1/2 pt). Assume that all this land is divided into 7 equal sized circles **(1/2 pt)** (1: North America/Canada, 2: South America, 3: Europe, 4: Africa, 5: Russia+Middle East, 6: Central Asia, 7: Oceania). The area of these 'continents' is then $A_{cont} = 10^7$ km^2 **(1/2 pt)**. The radius of these continents is then approximately

$$R_{cont} = \sqrt{A_{cont}/\pi} = 2000 \text{ km}.$$

(1/2 pt) The circumference of these continents is then

$$C_{cont} = 2\pi R_{cont} = 12000 \text{ km}.$$

(1/2 pt) Multiplying by the number of continents gives about 85000 km of shoreline **(1/2 pt)**. However, islands increase the amount of shoreline, so to improve our estimate we can increase the above number by a factor of 4 **(1/2 pt)** or so, yielding a total shoreline $L_{beach} = 360000$ km. Assuming the average beach has a width of 30 m and a depth of 10 m **(1/2 pt)**, the total volume of beach is

$$V_{beach} = 30 \times 10 \times L_{beach} = 10^{11} \text{ m}^3.$$

(1/2 pt) Assume that grain of sand is cube with side length of 0.1 mm, so that its volume $V_{grain} = 10^{-3}$ mm^3 **(1/2 pt)**. We then find

$$N_{grains} = V_{beach}/V_{grain} = 10^{11}/10^{-12} = 10^{23}$$

grains of sand **(1/2 pt)**. This is approximately a factor 10 smaller than the number of stars in the universe **(1/2 pt)**. So there are slightly more stars in the observable universe than there are grains of sand on Earth's beaches.