Kompleksmuutuja funktsioonid KT   Taavi Tammearu

1

a) kasutame $R(z_0) = \lim\limits_{z \to z_0} (z - z_0) f(z)$

$$\operatorname{Res}\left(\frac{1}{\tan(z)}, z = n\pi\right) = \lim_{z \to n\pi} (z - n\pi) \frac{1}{\tan(z)} \Rightarrow$$

$$\tan(z) \approx z - n\pi$$

$$\lim_{z \to n\pi} (z - n\pi) \frac{1}{\cancel{\tan(z)}\; z - n\pi} = \lim_{z \to n\pi} 1 = 1$$

b) arvutame laurenti rea ja leiame b_1

$\sin(z)$ taylori rida on

$$\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots$$

seega laurenti rida

$$\frac{\sin(z)}{z^2} = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \dots\right) = \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!}$$

ainult üks liige mille ~~kordaja~~ z aste on negatiivne ja selle liikme kordaja on 1

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c) $\mathrm{Res}_{z=\infty} f(z) = -\mathrm{Res}_{z=0}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right)\right)$

$$= \frac{1}{z^2}\,\frac{1}{z^2}\,\frac{1}{\frac{1}{z^2}+1} = \frac{1}{z^2}\cdot\frac{1}{\left(\frac{1}{z^2}+1\right)}$$

$$= \frac{1}{z^2(1+z^2)}$$

laurent'i rea lahendus

$$\frac{1}{z^2(z^2+1)} = \frac{1}{z^2} - 1 + z^2 - z^4$$

siin ei ole $\frac{1}{z}$ liiget seega resiid on 0

d) 3. järku poolus punktis z_0

laurenti reaga

$$\frac{e^{z^2}}{z^3} = \frac{1}{z^3}\sum \left(z^2\right) = \frac{1}{z^3} + \frac{1}{z} + \frac{z}{2} + \dots$$

$b_1 = 1$ ehk $\mathrm{Res}_{z=z_0}(f(z)) = 1$

tuletise meetodiga

~~Res(f(z)~~ $\cancel{\mathrm{Res}(f(z)) = \frac{p(z_0)}{q'(z_0)}}$

$$\mathrm{Res}(f(z)) = \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\left(z^3\cdot\frac{e^{z^2}}{z^3}\right) = 1$$

e) $f(z) = \frac{e^z}{z(z+1)}$

$$Res(f(z)) = \frac{e^z}{2z_0 + 1} = \frac{e^{z_0}}{0+1} = e^{z_0} = e^0 = 1$$

kui $z_0 = -1$

$$Res(f(z)) = \frac{e^{z_0}}{2z_0+1} = \frac{e^{-1}}{-1} = \frac{1}{-e} = -\frac{1}{e}$$

f) $f(z) = \frac{e^z}{z \sin(z)}$

$z_0 = 0$ + lähedal teeme lihtsustuse et

$\sin(z) \approx z$, seega on tegemist 2. järku poolusega

$$Res(f(z)) = 1 \lim_{z \to 0} \frac{d}{dz} \left(z^2 \frac{e^z}{z \cdot \sin(z)} \right)$$

$$= 1$$

2

residuaalide teoreem

$$\int_C f(z)\,dz = 2\pi i \sum \mathrm{Res}(f, z_k)$$

kui $n=-1$, $f(z)=\frac{1}{z}$ esimest järku poolus

$z=0$ juures

$$\mathrm{Res}(z^{-1}, 0) = 1$$

seega

$$\int_C z^{-1}\,dz = 2\pi i \cdot \mathrm{Res}(z^{-1}, 0) = 2\pi i \cdot 1 = 2\pi i$$

kui $n \neq 1$

siis ei ole ühtegi poolust kontuuri C sees, ehk

$$\int_C z^n\,dz = 0$$ Cauchy teoreemi kohaselt

seega

$$\int_C z^n\,dz = \begin{cases} 2\pi i & \text{kui } n=-1 \\ 0 & \text{kui } n \neq -1 \end{cases}$$

3

a) $\int_0^{2\pi} \sin(\theta)^{2n}\, d\theta \qquad \oint_C \left(\frac{z - z^{-1}}{2i}\right)^{2n} \frac{dz}{iz}$

viime selle kujule

$$\frac{1}{i(2i)^{2n}} \oint_C \frac{(z - z^{-1})^{2n}}{z}\, dz$$ rakendame binoomvalemit

$$(z - z^{-1})^{2n} = \sum z^{2n-2k} (-1)^k \binom{2n}{k}$$

$$\frac{1}{i(2i)^{2n}} \oint_C \frac{\sum z^{2n-2k} (-1)^k \binom{2n}{k}}{z}\, dz$$ leiame kuna liige on konstant

$n = k$ puhul on konstant

$$\cancel{\frac{1}{i(2i)^{2n}} \frac{(-1)^n (2n)!}{(n!)^2} = \frac{1}{i\, 4^n (-1)^n} \frac{(-1)^n (2n)!}{n!^2}} =$$

kasutame resiidi teoreemi ja saame

$$\frac{1}{\cancel{i}\, 4^n \cancel{(-1)^n}}\, 2\pi \cancel{i} \frac{\cancel{(-1)^n} (2n)!}{(n!)^2} = 2\pi \frac{(2n)!}{2^{2n} (n!)^2}$$

$$\int_0^{2\pi} \sin(\theta)^{2n}\, d\theta = 2\pi \frac{(2n)!}{2^{2n} (n!)^2}$$

b) $\displaystyle\int_0^{2\pi} \cos(\theta)^{2n} d\theta$

ilmselgelt on koosinuse paarisarvulise astme integraal sama, mis siinuse puhul vahemikus 0 kuni 2π

Seega on vastus sama mis eelmises alapunktis

$$\int_0^{2\pi} \cos(\theta)^{2n} d\theta = \frac{2\pi}{4^n}\binom{2n}{n} = 2\pi \frac{(2n)!}{2^{2n}(n!)^2}$$

c) $\displaystyle\int_0^{2\pi} \sin(\theta)^{2n+1} d\theta$ arendame θ z-iga

$$\oint_C \left(\frac{z - z^{-1}}{2i}\right)^{2n+1} \frac{dz}{iz} = \frac{1}{i(2i)^{2n+1}} \oint_C \frac{(z - z^{-1})^{2n+1}}{z} dz$$

kasutame binoomvalemit

$$= \frac{1}{i(2i)^{2n+1}} \oint_C \frac{\sum z^{2n+1-2k}(-1)^k \binom{2n+1}{k}}{z} dz$$

otsime olukorda, kus integraalis oleks liige mille lugeja oleks konstantne, selleks peab z astme võrduma nulliga

$2n+1-2k \neq 0$

mis on võimatu, Seega $\displaystyle\int_0^{2\pi} \sin(\theta)^{2n+1} d\theta = 0$

terved punktid

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d) $\int_0^{2\pi} \cos(\theta)^{2n+1}\, d\theta$ teeme samamoodi

$$\oint_C \left(\frac{z+z^{-1}}{2}\right)^{2n+1} \frac{dz}{iz} = \frac{1}{2^{2n+1} i} \oint_C \frac{(z+z^{-1})^{2n+1}}{z}\, dz$$

kasutame binoomvalemit

$$\frac{1}{2^{2n+1} i} \oint_C \frac{\sum_{k=0}^{2n+1} z^{(2n+1)-2k} \binom{2n+1}{k}}{z}\, dz$$ samal põhjusel kui siin

et residi oleks olemas peab $(2n+1)-2k=0$

aga see ei ole võimalik, ehk pole ühtegi singulaarsuspunkti

Seega Residi teoreemi kohaselt $\oint_C f(z)\, dz = 2\pi i \sum_{j=0}^{n} b_{-j}$

on integraal võrdne nulliga kuna ühtegi residi pole

$$\int_0^{2\pi} \cos(\theta)^{2n+1}\, d\theta = 0$$

e) $\int_0^{\pi} (\sin(\theta) \cdot \cos(\theta))^{2n}\, d\theta$

$$= \frac{1}{2} \int_0^{2\pi} (\sin(\theta) \cdot \cos(\theta))^{2n}\, d\theta = \frac{1}{2} \oint_C \left(\frac{z+z^{-1}}{2} \cdot \frac{z-z^{-1}}{2i}\right)^{2n} \cancel{d\theta} \frac{dz}{zi}$$

$$= \frac{1}{(2i)^2 \cdot 2} \oint \frac{((z+z^{-1}) \cdot (z-z^{-1}))^{2n}}{z}\, dz = \frac{1}{-8} \oint \frac{(z^2-z^{-2})^{2n}}{z}\, dz$$

$$\frac{1}{-8} \oint \frac{\sum z^{2(2n-k)} (z^{-2})^k \binom{2n}{k}}{z}\, dz = -\frac{1}{8} \oint \frac{\sum z^{(4n-4k)} \binom{2n}{k}}{z}\, dz$$

$4n-4k=0$ ~~et~~ ehk residi leidumiseks

peab $n=k$ ja saame, residi teoreemi kasutades

$$\int_0^{\pi} (\sin(\theta)\cos(\theta))^{2n}\, d\theta = -\frac{4i}{\pi k} \frac{(2n)!}{(n!)^2} (-1)^n$$