Lecture 1. Charges

1. **Electric charge** (or simply charge), is a fundamental physical property that causes objects to feel an attractive or repulsive force toward one another

2. Main properties of charges:

• There two types of charges: **positive and negative**

• **Like charges repel** each other and **unlike charges attract** each other

• Charges are **additive:** + plus - = null (no charge or neutral)

• + and - electrical charges are in **perfect balance** so in bulk matter their effect is neutralized

• Charge is **conserved**: the total charge of the universe is fixed (**global conservation** of charge)

• **Local charge** is conserved, too: no charge can appear or disappear without passing along some continuous path (**local conservation** of charge)

• Electric charge is carried by subatomic particles, **electrons (e^-) and protons (p^+)**

• Charge is **quantized**; the basic unit of charge is that of an electron (not that it matters for **classical electricity**)

• Charge is a **relativistic invariant**; it has the same value in all inertial systems

• The unit of charge is SI is called **Coulomb** (C); the charge of electron $e^- = 1.6 \times 10^{-19}$ C

3. **Fundamental problem of electrodynamics**: What force do the source charges exert on the test charge is the positions of the source charges (in space and time) are given (all charges could be in motion).

4. A **test charge** is an idealized model of a charge which is considered to be insufficient to alter the parameters of the rest of the system. The concept of a test charge simplifies problems, and could provide a good approximation of a real system.

5. In **electrostatics**, all the **source charges are stationary** (or slow-moving) but the test charge could be placed in any place

Lecture 2. Coulomb, superposition, electric field and its flux

1. **Coulomb's law** (1875): The electrostatic force $\vec{\mathbf{F}}$ between two **point charges** q and Q is along the straight line joining the two charges; the magnitude of the force is directly proportional to the product of the magnitudes of charges and inversely proportional to the square of the distance between them

$$\vec{\mathbf{F}} = \frac{1}{4\pi\epsilon_0}\frac{qQ}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}$$

where $\vec{\boldsymbol{\mathscr{r}}} = \vec{\mathbf{r}} - \vec{\mathbf{r}}'$ is a **separation vector** (=distance between the charges) and $\hat{\boldsymbol{\mathscr{r}}} = \frac{\vec{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}}$ is **direction of the separation vector**

2. The constant $\epsilon_0 = 8.85 \times 10^{-12}\ \frac{\mathrm{C}^2}{\mathrm{N\cdot m}^2}$ is called **permittivity of free space**

3. If two charges are of the same sign, the force is repulsive $\vec{\mathbf{F}} \propto \hat{\boldsymbol{\mathscr{r}}}$. If two charges are of the opposite sign, the force is repulsive $\vec{\mathbf{F}} \propto -\hat{\boldsymbol{\mathscr{r}}}$

4. **Superposition principle** states that the interaction between any two charges is completely unaffected by the presence of others

5. **Electric field $\vec{\mathbf{E}}$** is a force exerted upon a unit test charge $\vec{\mathbf{F}} = Q\vec{\mathbf{E}}$ if the test charge were at the observation point

6. Electric field is a means how the **charges "talk" to each other**. All we have to know to calculate the force, is the electric field at the point of the charge (**local field**). We don't have to be concerned with the charge configuration (perhaps far far away) which has created the electric field

7. **Units** of electric field are N/C or V(olt)/m

8. Electric field $\vec{\mathbf{E}}(\vec{\mathbf{r}})$ of a **continuous charge distribution**

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_o}\int \frac{\rho(\vec{\mathbf{r}}')}{\mathscr{r}^2}\hat{\boldsymbol{\mathscr{r}}}\, d\tau'$$

where the integration goes over the volume occupied by the charge

9. $\rho(\vec{\mathbf{r}}')$ refers to the **volume charge density** (=charge per a unit volume).

10. Similarly we could introduce **surface charge density** σ (=charge per a unit area) and **linear charge density** λ (=charge per a unit length). Respectively, the integrals above become surface or linear integrals

11. Since there are positive and negative charges, **charge density could be positive or negative**

12. Even though we draw a single integral sign, the volume element $d\tau$ suggests a three-fold integral. Hence, to calculate the electric field $\vec{\mathbf{E}}(\vec{\mathbf{r}})$, we need to take **three three-fold integrals**, one per each component of $\vec{\mathbf{E}}$

13. **Field lines** is a graphical visual aid for **visualizing vector fields**. Field lines consist of imaginary directed lines which are tangent to the field vector at each point along its length. The magnitude of the field is indicated by the **density** of the field lines.

14. The field lines **begin on positive charge, end on negative charge** (or at infinity but NOT terminate in midair). The field lines **can never cross**: at the intersection, the field would have two different directions at once which is not possible

15. **The flux of $\vec{\mathbf{E}}$** through a surface S, $\Phi_E \equiv \int_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}}$, is a measure of the "number of field lines" passing through the surface S.

16. For the field lines that originate on a positive charge must either pass out through the surface or else terminate on a negative charge inside. A charge outside the surface will contribute nothing to the total flux, since its field lines pass in one side and out the other. Hence, **the flux through any closed surface is a measure of the total charge inside (Gauss's law)**

Lecture 3. Gauss, div $\vec{\mathbf{E}}$, $\boldsymbol{\delta}$-function

1. **Gauss law (integral form):** the flux of electric field through any surface enclosing the charge is q/ϵ_o

$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{a}} = \frac{1}{\epsilon_o} Q_{enc}$$

2. Gauss's law contains no information that was not present in **Coulomb's law and the superposition principle** but it is often more convenient for calculations

3. **Gauss law in differential form** states that the divergence of the electric field $\vec{\mathbf{E}}(\vec{\mathbf{r}})$ at any point in space is equal to $1/\epsilon_o$ times the volume charge density $\rho(\vec{\mathbf{r}})$ at that point:

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{E}}(\vec{\mathbf{r}}) = \frac{\rho(\vec{\mathbf{r}})}{\epsilon_o}$$

4. Dirac's **delta-function** is defined as

$$\delta(x) = \begin{cases} 0, & if\ x \neq 0 \\ \infty, & if\ x = 0 \end{cases} \text{ and } \int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

5. Technically, $\delta(x)$ is not a function, since its value is not finite at $x = 0$. It is known as a generalized function, or **distribution**

6. You can imagine δ-function as a limiting case of a Gaussian function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

When $\sigma \to \infty$ (i.e. the gaussian function becomes more and more narrow while its peak value increases), $f(x) \to \infty$ but $\int_{-\infty}^{\infty} \delta(x)\, dx = 1$.

7. It follows from the definition of δ-function that

$$\int_{-\infty}^{\infty} f(x)\delta(x-a)\, dx = f(a)$$

Note that the integration limits should not necessarily run from $-\infty$ to ∞; it is enough if they include the point $x = a$. However, if the point $x = a$ is not included, the integral is zero

8. The **three dimensional $\boldsymbol{\delta}$-function**: $\delta^3(\vec{\mathbf{r}}) = \delta(x)\delta(y)\delta(z)$. Subsequently,

$$\int_{all\ space} f(\vec{\mathbf{r}})\delta^3(\vec{\mathbf{r}} - \vec{\mathbf{a}})\, d\tau = f(\vec{\mathbf{a}})$$

9. δ-function is used to describe the **volume charge density of a point charge** q:

$$\rho(\vec{\mathbf{r}}) = q\delta^3(\vec{\mathbf{r}})$$

10. Divergence of $\hat{\boldsymbol{r}}/r^2$ is not defined at $r = 0$ so

$$\vec{\boldsymbol{\nabla}} \cdot \left(\frac{\hat{\boldsymbol{r}}}{r^2}\right) = 4\pi\delta^3(\vec{\boldsymbol{r}})$$

Lecture 4. Gauss, curl, potential

1. **Symmetry** is vital for applications of Gauss's law to charged objects

2. Curl of electric field is zero: $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = \vec{\mathbf{0}}$

3. **The electric potential** is defined as $V(\vec{\mathbf{r}}) \equiv -\int_{\mathcal{O}}^{\mathbf{r}} \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{l}}$, where $\mathcal{O}$ is a **reference point.** Electric potential is a **scalar quantity**

3a. Later we learn that the electric potential is **potential energy per unit charge**

4. The reference point could be any as soon as we all agree upon it. Quite often, $\mathcal{O} = \infty$

5. **Potential difference** $V(\vec{\mathbf{b}}) - V(\vec{\mathbf{a}}) = -\int_a^b \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{l}}$ is reference-independent

6. How come that a scalar V defines the vector $\vec{\mathbf{E}}$? $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{E}} = \vec{\mathbf{0}}$ imposes a number of conditions on the three components of $\vec{\mathbf{E}}$. In other words, they are not independent

7. Potential obeys the **scalar superposition principle** $V = V_1 + V_2 + V_3 + \cdots$. It's an ordinary (not vector) sum

8. **Units** of potential are **V**[olts]

9. Potential of a localized charge distribution: $V(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_o}\int \frac{\rho(\vec{\mathbf{r}'})}{r} d\tau'$

10. **Poisson's equation**: $\nabla^2 V(\vec{\mathbf{r}}) = -\frac{\rho(\vec{\mathbf{r}})}{\epsilon_o}$ where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the **Laplacian**

11. The **uniqueness theorem** sates that an electric field derived from a potential satisfying Poisson's equation under the boundary conditions (either $V = 0$ or $\nabla V = \mathbf{0}$ on the boundary of the region), is unique

12. In regions where there is no charge $\rho(\vec{\mathbf{r}}) = 0$ and $\nabla^2 V(\vec{\mathbf{r}}) = 0$ (**Laplace's equation**)

Lecture 5. Potential energy, conductors

1. The **work** one must do by moving a test charge *Q* from point *a* to point *b* is

$$W = -Q\int_a^b \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = Q[V(b) - V(a)]$$

where $V(a)$ and $V(b)$ are potentials at points a and b, respectively

2. The work **is path-independent** so $\vec{\mathbf{E}}$ is a **conservative field**

3. **Potential** is potential **energy** (the work it takes to create the system) **per unit charge**

4. The **energy stored** in a charge configuration of n charges q_i is

$$W = \frac{1}{2}\sum_{i=1}^{n} q_i V(\vec{\mathbf{r}}_i)$$

where $V(\vec{\mathbf{r}}_i)$ is the potential at position of q_i due to **all the other charges**

5. The total energy stored in **a charge distribution**

$$W = \frac{1}{2} \int \rho V d\tau$$

6. The total energy stored **in the field**

$$W = \frac{\epsilon_0}{2} \int_{\text{All space}} E^2 \, d\tau$$

7. In electrostatics, we could say that the energy is stored **in the charge**, with **energy density** (energy per unit volume) $u = \rho V/2$. In electrodynamics is general, the energy is located in space, where **the electric field** is, with **energy density** $u = \epsilon_0 E^2/2$

8. In a **conductor** as metals, one or more **electrons** per atom **are free to roam**. A **perfect conductor** would contain an **unlimited supply of free charges**

9. Perfect conductors have **the following properties**:

• Electric field inside conductors is zero: $\vec{\mathbf{E}} = \mathbf{0}$

• The volume charge density is zero *inside* conductors: $\rho = 0$

• Any net charge resides on the surface of a conductor

• A conductor is equipotential, i.e. the potential is constant throughout the conductor

• The electric field just outside a conductor is perpendicular to its surface

• The surface charge is $\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$

10. **Induced charges** are the charges “induced” by an external electric field in a conductor. For instance, the field drives any free positive charges to the right, and negative ones to the left.

11. Induced charges are responsible for **mutual attraction of a charge and uncharged metal shell**

12. The electric field of an uncharged spherical conductor with a carved cavity with a charge q inside the cavity, is the same as of a sphere with the charge **regardless to the shape of the cavity and the position of the charge inside**.

Lecture 6. Capacitance. Laplace equation, uniqueness theorems, method of images. Dipoles

1. As both electric field and potential are proportional the charge, it can be factorized out. The proportionality coefficient is called **capacitance**:

$$C \equiv \frac{Q}{V}$$

2. Capacitance is a **purely geometrical quantity**, determined by the sizes, shapes, and separation of the two conductors. Capacitance is intrinsically **positive**

3. Capacitance is measured in **Farads** (F); a farad is a coulomb-per-volt (C/V). More practical units are the microfarad ($1\ \mu\text{F} = 10^{-6}\ \text{F}$), nanofarad ($1\ n\text{F} = 10^{-9}\ \text{F}$) and the picofarad ($1\ \text{pF} = 10^{-12}\ \text{F}$).

4. A **capacitor** is an electronic component consistent of a pair of conductors separated by a dielectric.

5. **Solution of a 1D Laplace equation** is a liner function

$$V(x) = mx + b$$

1D Laplace's equation tolerates **no local maxima or minima**; extreme values of V must occur at the end points

6. The **method of images** relies on finding an equipotential surface of **some** (not given!) **combination of “image” (imaginary) charges** which is the same as the metal surface in the problem in hand

7. The uniqueness of the solution found in the method of images, is warranted by the uniqueness theorem: if you find **any solution** with the given boundary conditions, **this solution is unique**

8. For instance, a configuration of a charge in a distance from an infinitely large conducting surface is substituted by this charge and another charge (imaginary charge), which is the mirror image of the real charge in respect to the surface but has the opposite sign

9. There are **two rules of the method of images**: (i) you must never put an image charge in the region where you are computing the potential and (ii) the image charges must add up to the correct total in each region

10. A (physical) **electric dipole** consists of two equal and opposite charges $\pm q$ separated by a distance d

11. The **potential of a dipole** is

$$V(\vec{\mathbf{r}}) = \frac{1}{4\pi\epsilon_0}\frac{qdcos\theta}{r^2}$$

12. The potential of a dipole **decreases as $\mathbf{1/r^2}$** at large r; it falls off more rapidly than the potential of a point charge

13. The potential of a dipole **depends on the polar angle $\boldsymbol{\theta}$**; $V = 0$ at $\theta = \pi/2$ (the two charges perfectly compensate each other).

13. Similarly to a dipole, we could construct **a quadrupole** (two dipoles with the opposite orientation of charges); the quadrupole potential decreases as $1/r^3$ at large r. An **octapole** combines two quadrupoles with opposite orientations; the octapole potential decreases as $1/r^4$

Lecture 7. Multipole expansion, monopoles and dipoles, electric field of a dipole

1. **Multipole expansion** uses binominal expansion of a potential of any charge configuration $\rho\left(\overrightarrow{\mathbf{r}'}\right)$ using r'/r as a “smallness” parameter: all terms that contain the same power of r'/r, are collected together

2. Multipole expansion is **exact**; however, oftentimes only few first terms are used. These are: monopoles, dipoles and quadrupoles

3. The **dipole moment $\vec{\mathbf{p}}$** of a charge configuration $\rho\left(\overrightarrow{\mathbf{r}'}\right)$ is defined as

$$\vec{\mathbf{p}} = \int \vec{\mathbf{r'}} cos\alpha \, \rho(\vec{\mathbf{r'}}) \, d\tau'$$

where α is the angle between the source vector $\vec{\mathbf{r'}}$ and the position vector $\vec{\mathbf{r}}$ of the field point

4. For a **physical dipole** (equal and opposite charges $\pm q$ at distance d) the dipole moment is*:*

$$\vec{\mathbf{p}} = q\vec{\mathbf{d}}$$

where $\vec{\mathbf{d}}$ is the vector from the **negative** charge to the **positive** one

5. Dipole moments are **vectors** so they add accordingly

6. **Moving the origin** (or the charge distribution) can radically alter the multipole expansion so you need to be careful with it.

7. One **notable exception from #6** is that if the total charge is zero, the dipole moment is independent of the choice of coordinates

8. **Electric field of a dipole $\vec{\mathbf{p}}$** oriented along z (in spherical system of coordinates):

$$\vec{\mathbf{E}}_{dip}(r,\theta) \; = \frac{p}{4\pi\epsilon_o r^3}\left(2 \, cos\theta \, \hat{\mathbf{r}} + sin\theta \, \hat{\boldsymbol{\theta}}\right)$$

9. The electric field of a dipole **decays as $\boldsymbol{r^{-3}}$** with the distance off the dipole

10. There is no $\hat{\mathbf{r}}$-component of $\vec{\mathbf{E}}_{dip}$ at $\theta = \pi/2$ (i.e. at the direction orthogonal to $\vec{\mathbf{p}}$) while the $\hat{\boldsymbol{\theta}}$-component is maximized. Similarily, there is no $\hat{\boldsymbol{\theta}}$-component of $\vec{\mathbf{E}}_{dip}$ at $\theta = 0$ while the $\hat{\mathbf{r}}$-component is maximized.

11. There is no $\hat{\boldsymbol{\varphi}}$-component of $\vec{\mathbf{E}}_{dip}$

12. Coulomb meter $(\mathrm{C} \cdot \mathrm{m})$ is used to denote the dipole moment.

13. Dipoles become crucial in describing electric field in matter

Lecture 8. Dielectrics, induced dipole moments, polarizability

1. In dielectrics **all charges are attached** to atoms or molecules. All they can do is move a bit *within* the atom or molecule

2. In the presence of an external field $\vec{\mathbf{E}}$, **the nucleus and the electron cloud are displaced** with respect to each other. This displacement causes the **induced dipole moment**

3. In a simple model, the induced dipole moment $\vec{\mathbf{p}}$ per atom (molecule) is proportional to the electric field applied $\vec{\mathbf{E}}$ and the volume of the atom/molecule of a size of a

$$\vec{\mathbf{p}} = 4\pi\epsilon_o a^3 \, \vec{\mathbf{E}}$$

4. The constant of proportionality α between the induced dipole moment and the electric filed is called **atomic (electronic) polarizability** (=ability to get polarized)

$$\vec{\mathbf{p}} = \alpha\vec{\mathbf{E}}$$

5. The polarizability is a **property of the material.** Its value depends on the detailed structure of the atom/molecule in question, but the general rule is: **larger the size, stronger the polarizability**

6. For molecules, the polarizability could be different in different directions so that the polarizability is a **tensor**. In simple words, it is a 3x3 matrix that projects (=rotates and scales) the magnitude and direction of the electric field onto the polarization

7. The **units of the polarizability** are $\mathrm{C \cdot m^2 \cdot V^{-1}}$

8. Another mechanism of polarizability is characteristic for the molecules that have built-in permanent dipole moments, the so-called **polar molecules**

9. A polar molecule in the uniform electric field experiences a **torque**:

$$\vec{\mathbf{N}} = \vec{\mathbf{p}} \times \vec{\mathbf{E}}$$

10. The molecules rotate until their dipole moments point in the direction of the applied field so that **all polar molecules are aligned** (as opposed to no-field situation when thermal motion scrambles molecular orientations thereby making the net dipole moment zero)

11. In the nonuniform electric field, the molecules also experience **a net force**

$$\vec{\mathbf{F}} = (\vec{\mathbf{p}} \cdot \boldsymbol{\nabla})\vec{\mathbf{E}}$$

assuming that the dipole is a point dipole

12. More generally, polarizability is a property of matter **to acquire an electric dipole moment in proportion to the applied field** – the material becomes **polarized**

13. The polarized material is characterized by its **polarization $\vec{\mathbf{P}}$ – dipole moment per unit volume**

14. The unit of polarization is given as coulomb per square meter ($\mathrm{C \cdot m^{-2}}$)

Lecture 9. Bound charges, the electric displacement, Gauss's law in matter

0. Now we will study the field that a **polarized material itself produces** (regardless to the cause of polarization), and then we will put together the **original field $\vec{\mathbf{E}}$**, which was responsible for $\vec{\mathbf{P}}$, **plus the new field**, which is due to $\vec{\mathbf{P}}$

1. The potential of a polarized object is the same as that produced by a **volume charge density** $\rho_b \equiv -\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{P}}$ plus a **surface charge density** $\sigma_b \equiv \vec{\mathbf{P}} \cdot \hat{\mathbf{n}}$, where $\vec{\mathbf{P}}$ is polarization of the material

2. The **bound surface charge** is the charge piled up at the surface of the dielectric, given by the dipole moment perpendicular to the surface

3. A **volume charge density** originates from accumulation of bound charge within the material if the polarization in not uniform (i.e. $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{P}} \neq 0$)

4. To calculate the potential produced by a polarized object, we **find the bound charges**, and then **calculate the potential *they* produce**, in the same way we calculate the potential of any other volume and surface charges

5. Electric field inside a **uniformly polarized sphere** is uniform and directed opposite to the polarization. Outside the sphere the potential is identical to that of a perfect dipole at the origin, whose dipole moment is equal to the total dipole moment of the sphere

6. Now we could divide **all charges** into the **bound charges** ρ_b (over which we have no control) and **free charges** ρ_f (which we could control – for instance, we can change its amount): $\rho = \rho_b + \rho_f$

7. **Electric displacement** is defined as $\vec{\mathbf{D}} \equiv \epsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}}$

8. **Gauss's law** in matter is formulated as $\nabla \cdot \vec{\mathbf{D}} = \rho_f$, or $\oint \vec{\mathbf{D}} \cdot d\vec{\mathbf{a}} = Q_{f_{encl}}$ where the **free charge only** enters.

9. You need to be careful as $\vec{\mathbf{D}}$ is **not** "like $\vec{\mathbf{E}}$". In general,

$$\vec{\mathbf{D}}(\vec{\mathbf{r}}) \neq \frac{1}{4\pi}\int \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\rho_f(\vec{\mathbf{r}}')\,d\tau' \text{ and } \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{D}} \neq 0$$

10. The unit of electric displacement is Coulomb per square meter ($\mathrm{C} \cdot \mathrm{m}^{-2}$)

11. But wait… why **did we left out the surface bound charge** σ_b in deriving $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{D}} = \rho_f$? In fact, we did implicitly by including then into ρ_b. If we take a more realistic model of a **gradual change** of $\vec{\mathbf{P}}$ at the interface there **will be no surface bound charge**, and Gauss's law can be safely applied *everywhere.*

12. The integral form of Gauss's law is free from this "defect"

Lecture 10. Linear dielectrics

1. In many instances, the induced polarization $\vec{\mathbf{P}}$ is proportional to the electric field $\vec{\mathbf{E}}$:

$$\vec{\mathbf{P}} = \epsilon_0 \chi_e \vec{\mathbf{E}}$$

The coefficient of proportionality χ_e is called the **electric susceptibility**. It is a **dimensionless** quantity (the dimensions are taken care of by ϵ_0)

2. The materials that obey $\vec{\mathbf{P}} = \epsilon_0 \chi_e \vec{\mathbf{E}}$ are called **linear dielectrics**. They are also **isotropic** (i.e. the dielectric properties are identical in all direction) as χ_e is a **scalar** but not a tensor

3. More generally, $\vec{\mathbf{P}} = \epsilon_0 \chi_e \vec{\mathbf{E}}$ can be regarded as the **first (non-zero) term** in the Taylor expansion of $\vec{\mathbf{P}}(\vec{\mathbf{E}})$, i.e. when the electric field is sufficiently weak. If not, higher-order terms become increasingly important. The field of science called **nonlinear optics** deals with such phenomena.

4. In linear media,

$$\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}}$$

where the proportionality coefficient $\epsilon = \epsilon_0(1 + \chi_e)$ is called the **permittivity** of the material.

5. The units of the permittivity are $\mathrm{C}^2 \cdot \mathrm{N}^{-1} \cdot \mathrm{m}^{-2}$ or $\mathrm{F} \cdot \mathrm{m}^{-1}$

6. The relation $\vec{\mathbf{D}} = \epsilon \vec{\mathbf{E}}$ is called the **constitutive relation**

7. The **relative permittivity**, or **dielectric constant**

$$\epsilon_r \equiv \frac{\epsilon}{\epsilon_0} = 1 + \chi_e$$

indicates the factor over which how the permittivity of the material exceeds ϵ_0. The dielectric constant is a dimensionless quantity.

8. If the space entirely filled with a homogeneous linear dielectric, the **field everywhere is simply reduced by** ϵ_r

9. The electric field inside a dielectric is reduced because the polarization of the medium partially "shields" the charge by surrounding it with bound charge of the opposite sign

10. The **energy** stored of a dielectric-filled system

$$W = \frac{\epsilon_0}{2}\int \vec{\mathbf{D}}\cdot\vec{\mathbf{E}}\,d\tau$$

It accounts for both **electric field energy** and **the energy stored in the dielectric** (by e.g. inducing the dipole moments or rotating the already existing dipoles)

Lecture 11. Magnetic fields, Lorentz law, currents

1. A moving charge generates **a magnetic field $\vec{\mathbf{B}}$** (together of course with the electric field)

2. The **Lorentz force law** postulates that

$$\vec{\mathbf{F}} = q\big[\vec{\mathbf{E}} + \big(\vec{\mathbf{v}}\times\vec{\mathbf{B}}\big)\big]$$

where $\vec{\mathbf{v}}$ is the velocity of the charge q

3. The Lorentz force law is a **fundamental axiom** whose justification is found in experiments

4. The **cyclotron frequency**

$$\omega = \frac{QB}{m}$$

is the angular frequency with which a charge Q revolves in a uniform magnetic field B

5. **Magnetic forces do not work** but redirect forces. Sometimes it is a subtle matter to find out who does the work

6. The **current** in a wire is the charge per unit time passing a given point:

$$I = \frac{dQ}{dt}$$

7. Negative charges moving e.g. to the left count the same as positive ones to the right

8. Typically, it is the negatively charged electrons that do the moving in the **direction *opposite* to the electric current**

9. Current is **measured** in $\mathrm{C/s}$, or amperes (A)

10. In fact, **current is a vector**:

$$\vec{\mathbf{I}} = \lambda\vec{\mathbf{v}}$$

where λ is a line charge travelling at speed $\vec{\mathbf{v}}$ along the wire. For wires, it doesn't matter as the direction is determined by the shape of the wire but it does for surface and volume currents

11. The magnetic force on a segment of current-carrying wire is

$$\vec{\mathbf{F}}_{mag} = \int I\big(d\vec{\boldsymbol{l}}\times\vec{\mathbf{B}}\big)$$

12. When charge flows over a *surface,* we describe it by the **surface current density** *(= current per unit width*)

$$\vec{\mathbf{K}} \equiv \frac{d\vec{\mathbf{I}}}{dl_{\perp}}$$

13. When the flow of charge is distributed throughout a 3D region, we describe it by the **volume current density** (= *current per unit area*)

$$\vec{\mathbf{J}} \equiv \frac{d\vec{\mathbf{I}}}{da_{\perp}}$$

14. Local charge conservation is expressed as **continuity equation**

$$\vec{\nabla} \cdot \vec{\mathbf{J}} = -\frac{\partial \rho}{\partial t}$$

15. **Magnetostatics** is the theory of steady currents which produce steady magnetic fields

16. Magnetostatics is based on the **Lorentz force la**w (force in a given magnetic field), **Biot-Savart law** (magnetic field produced by a steady current) and the **superposition principle**

Lecture 12. Magnetic fields, Lorentz law, currents

1. A moving charge generates **a magnetic field $\vec{\mathbf{B}}$** (together of course with the electric field)

2. The **Lorentz force law** postulates that

$$\vec{\mathbf{F}} = q\left[\vec{\mathbf{E}} + \left(\vec{\mathbf{v}} \times \vec{\mathbf{B}}\right)\right]$$

where $\vec{\mathbf{v}}$ is the velocity of the charge q

3. The Lorentz force law is a **fundamental axiom** whose justification is found in experiments

4. The **cyclotron frequency**

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is the angular frequency with which a charge Q revolves in a uniform magnetic field B

5. **Magnetic forces do not work** but redirect forces. Sometimes it is a subtle matter to find out who does the work

6. The **current** in a wire is the charge per unit time passing a given point:

$$I = \frac{dQ}{dt}$$

7. Negative charges moving e.g. to the left count the same as positive ones to the right

8. Typically, it is the negatively charged electrons that do the moving in the **direction *opposite* to the electric current**. This also causes a minus sign for the charge in the Lorentz force. Therefore, for the sake of simplicity, we consider the current to be caused by positively charged carriers.

9. Current is **measured** in $\mathrm{C/s}$, or amperes (A)

10. In fact, **current is a vector**:

$$\vec{\mathbf{I}} = \lambda \vec{\mathbf{v}}$$

where λ is a line charge travelling at speed $\vec{\mathbf{v}}$ along the wire. For wires, it doesn't matter as the direction is determined by the shape of the wire but it does for surface and volume currents

11. The magnetic force on a segment of current-carrying wire is

$$\vec{\mathbf{F}}_{mag} = \int I(d\vec{\boldsymbol{l}} \times \vec{\mathbf{B}})$$

12. Units of the magnetic field are **Teslas** (T): $1\ \mathrm{T} =\ 1\ \mathrm{N}/(\mathrm{A} \cdot \mathrm{m})$. The cgs unit is the Gauss: 1 T = 10^4G

13. Earth's magnetic field has a strength of ~0.00005 T, or 0.5 G. Most MRI scanners used in hospitals and medical research clinics are 1.5 or 3 T. The strongest magnetic field of 45 T was reached in 2022 in the Steady High Magnetic Field Facility in Hefei, China

Lecture 13. Biot-Savart and Ampère's laws

1. When charge flows over a *surface,* we describe it by the **surface current density** *(= current per unit width*)

$$\vec{\mathbf{K}} \equiv \frac{d\vec{\mathbf{I}}}{dl_{\perp}}$$

2. When the flow of charge is distributed throughout a 3D region, we describe it by the **volume current density** (= *current per unit area*)

$$\vec{\mathbf{J}} \equiv \frac{d\vec{\mathbf{I}}}{da_{\perp}}$$

3. Local charge conservation is expressed as **continuity equation**

$$\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{J}} = -\frac{\partial \rho}{\partial t}$$

4. **Magnetostatics** is the theory of steady currents which produce steady magnetic fields so that $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{J}} = 0$

5. Magnetostatics roots in the **Lorentz force la**w (force in a given magnetic field), the **Biot-Savart law** (magnetic field produced by a steady current) and the **superposition principle**

6. The **Biot-Savart law** states that

$$\vec{\mathbf{B}} = \frac{\mu_0}{4\pi} \int \frac{\vec{\mathbf{I}} \times \vec{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} dl' = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\boldsymbol{l}'} \times \vec{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

where the integration is along the current path, in the direction of the flow; $d\vec{\boldsymbol{l}'}$ is an element of length along the wire, and $\vec{\boldsymbol{\mathscr{r}}}$ is the vector from the source to the point $\vec{\boldsymbol{r}}$

7. μ_0 is the **permeability of free space** (until 2019, $\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$); $\mu_0\ =\ 2 \cdot \alpha \cdot h \cdot e^{-2} \cdot c^{-1}$ since 2019 (where α the fine-structure constant, h the Planck constant, e the elementary charge, c the speed of light)

8. The Biot-Savart law **and** the Lorentz force law in magnetostatics play a similar role as Coulomb's law in electrostatic

8. You cannot recast the Biot-Savart law for a **moving point charge** because a point charge does not constitute a steady current

9. The magnetic field of in infinitely long wire carrying current I

$$\vec{\mathbf{B}} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}$$

10. Ampère's law in integral form states that the line integral of the magnetic field over a closed loop is equal μ_0 times all currents which pierce the loop:

$$\oint \vec{\mathbf{B}} \cdot d\vec{\mathbf{l}} = \mu_0 I_{enc}$$

11. For currents with symmetry, Ampère's law in integral form offers an efficient way of calculating the magnetic field, similarly to Gauss's law in electrostatics

12. Differential form of Ampère's law $\boldsymbol{\nabla} \times \vec{\mathbf{B}} = \mu_0 \vec{\mathbf{J}}$

Lecture 14. Applications of Ampère's law, divergence of $\vec{\mathbf{B}}$, vector potential

1. Magnetic field of an infinitely large surface carrying a surface current density $\vec{\mathbf{K}} = K\,\hat{\mathbf{x}}$:

$$\vec{\mathbf{B}} = \begin{cases} +(\mu_0 K/2)\,\hat{\mathbf{y}} \text{ for } z < 0 \\ -(\mu_0 K/2)\,\hat{\mathbf{y}} \text{ for } z > 0 \end{cases}$$

2. The magnetic field of a long solenoid is homogeneous inside $\vec{\mathbf{B}} = \mu_0 InL\,\hat{\mathbf{z}}$ and zero outside

3. **The magnetic field is divergenceless $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} = 0$** (**noname law**). Such fields are called **solenoidal vector fields**

4. The **integral form of the noname law**

$$\oint_{\mathcal{S}} \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}} = 0$$

where the integral is a surface integral over the closed surface $\mathcal{S}$

5. It states that for each volume element in space, there are exactly the same number of "magnetic field lines" entering and exiting the volume so that the **flux of magnetic field is equal to zero**

6. **Magnetic field lines do not begin or end anywhere**

7. For the same reason, **there are no magnetic charges** (aka **magnetic monopoles**). The basic entity for magnetism is the **magnetic dipole**

8. We could readily observe magnetic phenomena because it is the *current* that matters: smallish velocity of moving charges is compensated by pouring huge amounts of charge down the *neutral* wire

9. **A vector magnetic potential $\vec{\mathbf{A}}$** is defined as $\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}$. This is possible because $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{B}} = \mathbf{0}$

10. **$\vec{\mathbf{A}}$ is not uniquely defined**: you can add the gradient of any scalar λ with no change in $\vec{\mathbf{B}}$: $\vec{\mathbf{A}}' = \vec{\mathbf{A}} + \vec{\boldsymbol{\nabla}}\lambda$ because $\vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{\nabla}}\lambda = 0$

11. We exploit this freedom by taking $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} = 0$. The definition $\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}$ specifies the *curl* of $\vec{\mathbf{A}}$ but it doesn't say anything about the *divergence* so we are at liberty to pick that as we see fit

12. Ampère's law becomes $\nabla^2 \vec{\mathbf{A}} = -\mu_0 \vec{\mathbf{J}}$ **with solution**

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}) = \frac{\mu_0}{4\pi}\int \frac{\vec{\mathbf{J}}(\vec{\mathbf{r}}')}{\mathcal{r}} d\tau'$$

13. **Remarks** on the magnetic vector potential $\vec{\mathbf{A}}$:
- It's still a vector, i.e. we trade one vector ($\vec{\mathbf{B}}$) for another – still 3 components (3 integrals for $\vec{\mathbf{A}}$)
- While you could find $\vec{\mathbf{A}}$ easier than $\vec{\mathbf{B}}$ (integral of $\mathcal{r}^{-1}$ vs $\mathcal{r}^{-2}$), you still have to find $\vec{\nabla} \times \vec{\mathbf{A}}$
- The true significance of $\vec{\mathbf{A}}$ will become apparent later, when it forms a 4-vector with the scalar potential

Lecture 15. Magnetic dipole moment

1. Multipole expansion of the vector potential give the **magnetic monopole term be always zero**

2. The dominant term in the expansion is **the dipole**:

$$\vec{\mathbf{A}}_{dip}(\vec{\mathbf{r}}) = \frac{\mu_0}{4\pi}\frac{\vec{\mathbf{m}} \times \hat{\mathbf{r}}}{r^2}$$

where $\vec{\mathbf{m}} \equiv I\vec{\mathbf{a}}$ is called **the magnetic dipole moment** ($\vec{\mathbf{a}}$ is "the vector area")

3. The magnetic dipole moment is **independent of the choice of origin** because the monopole term is zero

4. The **"pure" magnetic dipole** should consist of an infinitely small current loop $a \to 0$ which should be compensated by an infinitely large current $I \to \infty$ and **is therefore an approximation**

4. The **magnetic field** of the magnetic dipole

$$\vec{\mathbf{B}}_{dip}(r,\theta) = \frac{\mu_0 m}{4\pi r^3}\left(2\, cos\theta\, \hat{\mathbf{r}} + sin\theta\, \hat{\boldsymbol{\theta}}\right)$$

5. The magnetic field of a magnetic dipole is identical in structure to the electric field of electric dipole; however, the fields of physical dipoles are quite different

6. Magnetic dipoles are instrumental for **describing magnetic phenomena** in matter

Magnetization

1. On an atomic scale, there are **tiny currents in magnetic materials**: electrons orbiting the nuclei and electrons spinning about their axis. This current loops are so small that they could be considered as **magnetic dipoles**

2. When a magnetic field $\vec{\mathbf{B}}$ is applied, a net alignment of the magnetic dipoles occurs so that the medium becomes **magnetized**

3. Some materials acquire a magnetization **parallel to $\vec{\mathbf{B}}$** (**paramagnets**), some **opposite to $\vec{\mathbf{B}}$** (**diamagnets**), and some **retain their polarization** even after the magnetic field is removed (**ferromagnets**)

4. A magnetic dipole experiences **a torque** in a magnetic field $\vec{\mathbf{N}} = \vec{\mathbf{m}} \times \vec{\mathbf{B}}$. The torque is in such a direction as to **line the dipole up parallel to the field**

5. Every electron constitutes a **magnetic dipole due to its spin**. When a magnetic field is applied, the dipoles **tend to align with the applied field**, resulting in a net magnetic moment in the direction of the applied field. It is **the torque on this dipole that accounts for paramagnetism**

6. Quantum mechanics (the Pauli exclusion principle) tends to **lock the electrons** within a given atom **in pairs** which **effectively neutralize the torque** on the combination. As a result, paramagnetism most often occurs in atoms or molecules **with an odd number of electrons** (e.g., aluminum, oxygen, titanium, iron oxide)

7. If there is sufficient **energy exchange between neighboring dipoles**, they may spontaneously **align to form magnetic domains**, resulting in **ferromagnetism**

8. **Random thermal collisions tend to destroy** any order, be it paramagnetic, diamagnetic or ferromagnetic

9. In a uniform magnetic field, the net force on the current loop is zero. In a **nonuniform magnetic field**, the force is $\vec{\mathbf{F}} = \vec{\mathbf{\nabla}}(\vec{\mathbf{m}} \cdot \vec{\mathbf{B}})$

10. The effect of magnetic field on atomic orbits is **two-fold**: (i) **the torq**ue (paramagnetism) and (ii) **speeding up or slowing down of electrons** which leads to the change of the dipole moment:

$$\Delta\vec{\mathbf{m}} = -\frac{e^2R^2}{4m_e}\vec{\mathbf{B}}$$

The former effect is weaker than the latter because **it is hard to tilt the entire orbit**

11. The change of $\vec{\mathbf{m}}$ is opposite to the direction of $\vec{\mathbf{B}}$ so **each atom picks up an extra dipole moment** while all these extra moments are **antiparallel to the field** (diamagnetism).

12. Diamagnetism **is universal and affects all atoms**. However, it is much weaker than paramagnetism so that in atoms with odd numbers of electrons, paramagnetism takes over.

13. When a sample is placed in nonuniform magnetic field, the **paramagnet is attracted into the field** while the **diamagnet is repelled away**

14. The state of magnetic polarization is described by the vector quantity called **magnetization $\vec{\mathbf{M}}$**:

$$\vec{\mathbf{M}} = magnetic\ dipole\ moment\ per\ unit\ volume$$

15. The vector potential of a magnetized object is

$$\vec{\mathbf{A}}(\vec{\mathbf{r}}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\vec{\mathbf{J}}_b(\vec{\mathbf{r}}')}{\mathscr{r}}d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\vec{\mathbf{K}}_b(\vec{\mathbf{r}}')}{\mathscr{r}}da'$$

where $\vec{\mathbf{J}}_b = \vec{\mathbf{\nabla}} \times \vec{\mathbf{M}}$ is a **volume (bound) current** and $\vec{\mathbf{K}}_b = \vec{\mathbf{M}} \times \hat{\mathbf{n}}$ is a **surface (bound) current**

16. Instead of integration the contributions of all the dipoles, we first determine **bound currents**, and then calculate the field they produce

Lecture 16. Bound currents, magnetic field $\vec{\mathbf{H}}$, magnetic susceptibility, ferromagnetics

1. **Physical interpretation of the bound surface current** is the following: all the internal currents cancel while the edge there is no adjacent current to do the cancelling

2. If the magnetization is not uniform, **the internal currents no longer cancel** which results in the volume bound current

3. It is a matter of convenience to **separate the total current into the free and bound currents**

$$\vec{\mathbf{J}} = \vec{\mathbf{J}}_f + \vec{\mathbf{J}}_b$$

The **free current can be controlled** by e.g. hooking up a wire to battery; it also involves actual transport of charges. The **bound current is due to magnetization**; it results from a net action of many aligned atomic dipoles

4. **Ampère's law reads** as

$$\vec{\mathbf{\nabla}} \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_f \text{ (differentioal form) or } \quad \oint \vec{\mathbf{H}} \cdot d\vec{\boldsymbol{l}} = I_{f_{enc}} \text{ (integral form)}$$

where $\vec{\mathbf{H}} \equiv \frac{1}{\mu_0}\vec{\mathbf{B}} - \vec{\mathbf{M}}$ is called the **axillary field** or simply the "**$\vec{\mathbf{H}}$-field**"

5. $\vec{\mathbf{H}}$ allows us to express Ampère's law in terms of **something we can control** (free current); magnetizations comes along but we cannot turn it on and off independently

6. $\vec{\mathbf{H}}$ in practical terms (e.g. in engineering) **is more useful then $\vec{\mathbf{B}}$** as it directly originates from the currents we put into the loop while $\vec{\mathbf{B}}$ depends on specific materials and even on the history. For theory, however, $\vec{\mathbf{H}}$ and $\vec{\mathbf{B}}$ are on an equal footing

7. H **is measured in units of** $\mathrm{A/m}$. B is measured in T.

8. $\mu_0\vec{\mathbf{H}}$ is **not** like $\vec{\mathbf{B}}$ when the source is $\vec{\mathbf{J}}_f$ instead of $\vec{\mathbf{J}}$, because $\vec{\mathbf{\nabla}} \cdot \vec{\mathbf{B}} = 0$ whereas $\vec{\mathbf{\nabla}} \cdot \vec{\mathbf{H}} \neq 0$ in general.

9. For many materials, the **magnetization is proportional to the field applied** (provided that the field is not too strong): $\vec{\mathbf{M}} = \chi_m\vec{\mathbf{H}}$. These materials are called **linear media**

10. The proportionality coefficient χ_m is called the **magnetic susceptibility**

11. For **diamagnetics**, the magnetic **susceptibility is negative** while for **paramagnetics it is positive**

12. $\vec{\mathbf{B}}$ is also proportional to $\vec{\mathbf{H}}$ (the **constitutive relation**)

$$\vec{\mathbf{B}} = \mu\vec{\mathbf{H}}$$

13. $\mu \equiv \mu_0(1 + \chi_m)$ is called the **permeability** of the material

14. The **relative permeability** is $\mu_r \equiv 1 + \chi_m = \mu/\mu_0$

15. The volume bound current density in a homogeneous linear material is proportional to the free current density $\vec{\mathbf{J}}_b = \chi_m\vec{\mathbf{J}}_f$

16. In a **ferromagnet**, each dipole "likes" to **point into the same direction** as its neighbors due to mutual interactions of the dipoles. The alignment occurs in relatively small patches, called **domains**.

17. If the field is strong enough, one domain takes over entirely, and the material is said to **be saturated**. The dipoles become aligned even when the external magnetic field is removed. This causes the **permanent magnetization** of the magnets.

18. The magnetization of ferromagnetics depends on pre-history of how exactly the magnetization has been induced (a **hysteresis loop**)

19. Ferromagnetism disappears of the temperature is higher than the so-called **Curie point** above which the material becomes paramagnetic. This occurs abruptly which signifies a **phase transition**.