KT3 Taavi Tammaru

- muutujate eraldamise meetod

$$\beta \frac{\partial u}{\partial t} + qu = g x^2 D \frac{\partial^2 u}{\partial x^2} + g x D \frac{\partial u}{\partial x}$$

teeme arenduse $u(x,t) = X(x)T(t)$

$$\beta X(x)\, T'(t) + q X(x) T(t) = g x^2 D X''(x) T(t) + g x D X'(x) T(t)$$

$$X(x)\left(\beta T'(t) + q T(t)\right) = T(t)\left(g x^2 D X''(x) + g x D X'(x)\right)$$

$$\frac{\beta T'(t) + q T(t)}{T(t)} = \frac{g x^2 D X''(x) + g x D X'(x)}{X(x)}$$

$$\begin{cases} \dfrac{\beta T'(t) + q T(t)}{g T(t) D} = -\lambda \\ \dfrac{x^2 X''(x) + x X'(x)}{X(x)} = -\lambda \end{cases}$$

ja nüüd saame neid lahendada sama moodi nagu kahte tavalist ODE'd!

$$\begin{cases} \beta T'(t) + q T(t) + \lambda T(t) g D = 0 \\ x^2 X''(x) + x X'(x) + \lambda X(x) = 0 \end{cases}$$

- Formuleerige Strum-Liouville ülesanne rajatingimuste arvesse võtmise.

$X(1) = 0 \qquad t \geq 0$

$$\frac{\partial X(x)}{\partial x}\Big|_{x=e^{3L}} = 0 \qquad t \geq 0$$

seega need võrrandid ja

$$x^2 X''(x) + x X'(x) + \lambda X(x) = 0$$

defineerivad meie Strum-Liouvilli rajaülesande.

- Nüüd leiame λ_n ja $X_n(x)$

nagu soovitatud teeme muutujavahetuse

$$\eta = \ln(\sqrt[3]{x})$$

$$\frac{\partial X}{\partial x} = \frac{\partial \eta}{\partial x}\frac{\partial x}{\partial \eta} = \frac{1}{3x}\frac{\partial x}{\partial \eta}$$

$$X'' = \frac{\partial}{\partial x}\left(\frac{1}{3x}\frac{\partial x}{\partial \eta}\right) = -\frac{1}{3x^2}\frac{\partial x}{\partial \eta} + \frac{1}{9x^2}\frac{\partial^2 X}{\partial \eta^2}$$

asendame saadud tulemused tagasi diferentsiaalvõrrandisse

$$x^2\left(-\frac{1}{3x^2}\frac{\partial x}{\partial \eta}+\frac{1}{9x^2}\frac{\partial^2 X}{\partial \eta^2}\right)+x\left(\frac{1}{3x}\frac{\partial x}{\partial \eta}\right)=-\lambda X$$

erinevate tuletistega liikmed taanduvad välja ja saame.

$$\frac{1}{9}\frac{\partial^2 X}{\partial \eta^2}=-\lambda X$$

lahend on $X(x)= C_1 e^{-\ln(\sqrt[3]{x})\sqrt{-9\lambda}} + C_2 e^{-\ln(\sqrt[3]{x})\sqrt{-9\lambda}}$

$$X(x)= C_1 e^{-3\ln(\sqrt[3]{x})\sqrt{-\lambda}} + C_2 e^{3\ln(\sqrt[3]{x})\sqrt{-\lambda}}$$

$X(1)=0$

seega

$$C_1 e^{-3\ln(\sqrt[3]{1})\sqrt{-\lambda}} + C_2 e^{3\ln(\sqrt[3]{1})\sqrt{-\lambda}} = 0$$

e^x ei saa võrduda nulliga

nüüd vaatame teist rajatingimust

$$\frac{1}{e^{3L}}\sqrt{-\lambda}\,C_1\,e^{\sqrt{-\lambda}\,3L}-\frac{1}{e^{3L}}\sqrt{-\lambda}\,C_2\,e^{-\sqrt{-\lambda}\,3L}=0$$

$$\sqrt{-\lambda}\,C_1\,e^{\sqrt{-\lambda}\,3L}-\sqrt{-\lambda}\,C_2\,e^{-\sqrt{-\lambda}\,3L}=0$$

Kasutame determinandi reeglit

$$\begin{vmatrix} \varepsilon_{11\lambda} & \varepsilon_{12\lambda} \\ \varepsilon_{21\lambda} & \varepsilon_{22\lambda} \end{vmatrix} = 0 \qquad \varepsilon_{11\lambda}\varepsilon_{22\lambda} - \varepsilon_{12\lambda}\varepsilon_{21\lambda} = 0$$

$$\begin{vmatrix} 1 & 1 \\ \sqrt{-\lambda}\, e^{\sqrt{-\lambda}\,3L} & -\sqrt{-\lambda}\, e^{-\sqrt{-\lambda}\,3L} \end{vmatrix} = e^{\sqrt{-\lambda}\,3L} + e^{-\sqrt{-\lambda}\,3L} = 0$$

kui $\lambda < 0$ siis lahendeid ei ole, sest
sest mõlemad liikmed on positiivsed

kui $\lambda = 0$ $\quad e^{3L} + e^{-3L} = 0$ ei ole ka lahendeid

kui $\lambda > 0$

$$e^{i\sqrt{\lambda}\,3L} + e^{-i\sqrt{\lambda}\,3L} = 0$$

$$e^{ix} + e^{-ix} = 2\cos(x)$$

$$2\cos(3\sqrt{\lambda}L) = 0 \implies 3L\sqrt{\lambda} = (2n+1)\frac{\pi}{2}$$

$$\lambda_n = \left(\frac{(2n+1)\pi}{6L}\right)^2$$ $n \in \mathbb{Z}$ need on omaväärtused

$$X_n(x) = C_{1n} X_{1n}(x) + C_{2n} X_{2n}(x)$$

Nüüd saame, üldkujulise lahendi

i tuleb arvesse sest $\sqrt{-\lambda}$ $\lambda > 0$, kasutades euleri valemit saame.

$$X_n(x) = 2 C_n i \sin\left(\left(\frac{(2n+1)\pi}{6L}\right) 3\ln\left(\sqrt[3]{x}\right)\right)$$

Kasutame normeerimistingimust, et leida C_n väärtus.

$$\int_{x_1}^{x_2} \rho(x) X_n^2(x)\, dx = 1$$

$$\int_1^{e^{3L}} 2^2 C_n^2 i^2 \sin^2\left(\left(\frac{(2n+1)\pi}{6L}\right) 3\ln\left(\sqrt[3]{x}\right)\right) \frac{1}{x} dx = 1$$

$$-4 C_n^2 \int_1^{e^{3L}} \sin^2\left(\left(\frac{(2n+1)\pi}{6L}\right) 3\ln\left(\sqrt[3]{x}\right)\right) \frac{1}{x} dx = 1$$

$$-4 C_n^2 \int_1^{e^{3L}} \sin^2\left(\left(\frac{(2n+1)\pi}{6L}\right) 3\ln\left(\sqrt[3]{x}\right)\right) \frac{1}{x}\, dx = 1$$

Kasutades integraali arvutamise tarkvara.

$$-4c_n^2\left(\ln^3(3L)\sin^2\left(\frac{\pi(2n+1)}{6L}\right)\cdot\frac{1}{3} = 1\right.$$

Siinus on alati täisarv π kordne ehk võrdne nulliga.

$$-\frac{4}{3}c_n^2\ln^3(3L) = 1$$

$$c_n = \pm\sqrt{-\frac{3}{4\ln^3(3L)}}$$

$$= \pm i\sqrt{\frac{3}{4\ln^3(3L)}} = i\sqrt{\frac{3}{4\ln^3(3L)}}$$

Seega lahend on

$$X_n(x) = 2C_n i\sin\left(\left(\frac{(2n+1)\pi}{6L}\right)3\ln(\sqrt[3]{x})\right)$$

$$X_n(x) = 2i\sqrt{\frac{3}{4\ln^3(3L)}}\, i\sin\left(\left(\frac{(2n+1)\pi}{6L}\right)3\ln(\sqrt[3]{x})\right)$$

$$X_n(x) = -2\sqrt{\frac{3}{4\ln^3(3L)}}\sin\left(\left(\frac{(2n+1)\pi}{6L}\right)3\ln(\sqrt[3]{x})\right)$$

Ortonormeerimise tingimus

$$\int_{x_1}^{x_2} p(x)\, X_n(x)\, X_{n'}(x)\, dx = \delta_{nn'}$$

$$\int_1^{e^{3L}} \frac{1}{x} \cdot -2\sqrt{\frac{3}{4\ln^3(3L)}} \sin\left(\left(\frac{(2n+1)\pi}{6L}\right) 3\ln(\sqrt[3]{x})\right) \cdot -2\sqrt{\frac{3}{4\ln^3(3L)}} \sin\left(\left(\frac{(2n'+1)\pi}{6L}\right) 3\ln(\sqrt[3]{x})\right) dx = \delta_{nn'}$$

$$\int_1^{e^{3L}} \frac{1}{x} \frac{3}{\ln^3(3L)} \sin\left(\left(\frac{(2n+1)\pi}{6L}\right) 3\ln(\sqrt[3]{x})\right) \sin\left(\left(\frac{(2n'+1)\pi}{6L}\right) 3\ln(\sqrt[3]{x})\right) dx = \delta_{nn'}$$

3.2

$$\int_1^{e^{3L}} \frac{1}{x} \frac{3}{\ln^3(3L)} \sin\left(\left(\frac{(2n+1)\pi}{6L}\right) 3\ln(\sqrt[3]{x})\right) \sin\left(\left(\frac{(2n'+1)\pi}{6L}\right) 3\ln(\sqrt[3]{x})\right) dx = \delta_{nn'}$$

See taandub kujule

$$\int_1^{e^{3L}} \frac{3\sin^2\left(\frac{\pi(2n+1)\ln(x)}{6L}\right)}{\ln^3(3L)\, x} dx = \frac{3}{\ln^3(3L)} \int \frac{\sin^2\left(\frac{\pi(2n+1)\ln(x)}{6L}\right)}{x} dx$$

u-vahetus $u = \frac{\pi(2n+1)\ln(x)}{6L}$

nüüd lahendame

$$\int \sin^2(u)\,du = \frac{u}{2} - \frac{\cos(u)\sin(u)}{2}$$

arendame tagasi sisse ja saame

$$\left. \frac{9L \sin\left(\frac{2\pi n+\pi}{3L}\right) + (-6\pi n - 3\pi)\ln(x)}{(4\pi n + 2\pi)\ln^3(3L)} \right|_1^{e^{3L}}$$

$$= -\frac{3\left(3L\sin\left(\frac{\pi\ln(3L)(2n+1)}{3L}\right) - 2\pi\ln(3L)n - \pi\ln(3L)\right)}{2\pi\ln^3(3L)(2n+1)}$$

$\neq \delta_{nn'}$

mis näitab, et ortonormeerimise tingimus ei kehti.