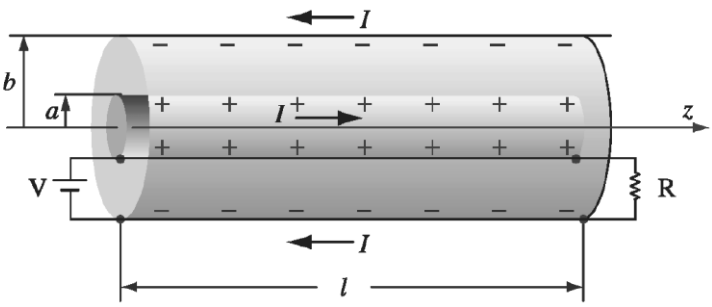
Electricity and Magnetism, Test 5 May 26 2023 18:30-20:30

4 questions, 37 points + 2 bonus points

Write your name and student number on each answer sheet. Use of a calculator is allowed. You may make use of 1 A4 (double sided) with handwritten notes and of the provided formula sheet. The same notation is used as in the book, i.e. a bold-face $\vec{\mathbf{A}}$ is a vector, $\hat{\mathbf{x}}$ is the unit vector in the x-direction. **In your handwritten answers, remember to indicate vectors (unit vectors) with an arrow (hat) above the symbol.**

Question 1 (8 points)

A long coaxial cable, of length l, consists of an inner conductor (radius a) and an outer conductor (radius b). It is connected to a battery at one end and a resistor at the other. The inner conductor carries a uniform charge per unit length λ, and a steady current I to the right; the outer conductor has the opposite charge and current.



A. Calculate the electric field $\vec{\mathbf{E}}$ and show that

$$\vec{\mathbf{E}} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{\mathbf{s}} \text{ between } a \text{ and } b, \text{and 0 outside this region (1 point)}$$

B. Calculate the magnetic field $\vec{\mathbf{B}}$ and show that

$$\vec{\mathbf{B}} = \frac{\mu_0}{2\pi}\frac{I}{s}\hat{\boldsymbol{\phi}} \text{ between } a \text{ and } b, \text{and 0 outside this region (1 point)}$$

C. Using Poynting's vector, find the power (energy per unit time) transported down the cable. Does the direction of Poynting's vector make sense? (4 points)

D. Show that Joule's heat law is satisfied (2 points)

Tip: Express the potential difference via the electric field

Model answers (Example 8.3)

A. From the symmetry, $\vec{\mathbf{E}}$ directed radially from the inner to the outer wire (1/2 point)

Gauss's law with a cylindrical surface

$$E2\pi sl = \frac{Q_{enclosed}}{\epsilon_0} = \frac{1}{\epsilon_0}\lambda l \quad \text{(1/2 point)}$$

$$\vec{\mathbf{E}} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\hat{\mathbf{s}}$$

Outside the cable, the enclosed charge is zero so is the electric field

B. From the symmetry, $\vec{\mathbf{B}}$ directed circumferentially in the clockwise direction if we look from the left end of the wire (1/2 point)

Ampere's law with a circular integration loop:

$B2\pi s = \mu_0 I_{enclosed} = \mu_0 I$ (1/2 point)

$\vec{\mathbf{B}} = \frac{\mu_0}{2\pi}\frac{I}{s}\hat{\boldsymbol{\phi}}$

Outside the cable, the enclosed current is zero so is the magnetic field

C. The Poyning's vector

$$\vec{\mathbf{S}} = \frac{1}{\mu_0}\left(\vec{\mathbf{E}} \times \vec{\mathbf{B}}\right) = \frac{1}{\mu_0}\frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\frac{\mu_0}{2\pi}\frac{I}{s}\hat{\mathbf{s}} \times \hat{\boldsymbol{\phi}} = \frac{\lambda I}{4\pi^2\epsilon_0 s^2}\hat{\mathbf{z}}$$ (1 point)

It makes sense: The energy is flowing down the line, from the battery to resistor (1 point)

The power transported is

$$P = \int \vec{\mathbf{S}} \cdot d\vec{\mathbf{a}} = \frac{\lambda I}{4\pi^2\epsilon_0}\int_a^b \frac{1}{s^2} s ds \int_0^{2\pi} d\phi = \frac{\lambda I}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)$$ (2 points)

D. Potential difference $V = -\int_\infty^a \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{l}} = -\frac{\lambda}{2\pi\epsilon_0}\int_b^a \frac{ds}{s} = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{b}{a}\right)$ (1 point)

$P = VI$ as should be according to Joule's law (1 point)

Typical mistakes

A. Some students simply stated the result with no derivation. This does not count!

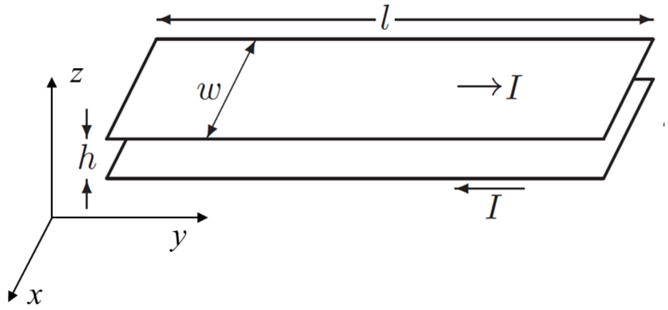
B. Some used Biot-Savart's law to derive the magnetic field, which is very difficult to get right

C. Taking Poynting's vector out of the integral, as if it were independent of the coordinates. Many students explained the direction of Poynting's vector by saying it was the same as the direction of the current. However, the current goes both ways, as it is looped.

D. Using the wrong integration bounds for the calculation of the potential difference.

Question 2 (12 points)

A transmission line is constructed from two thin metal "ribbons", of width w, a very small distance $h \ll w$ apart. The current I travels down one strip and back along the other. In each case, it spreads out uniformly over the surface of the ribbon. Neglect any fringe fields.



A. Calculate the magnetic field $\vec{\mathbf{B}}$ between the plates and show that

$\vec{\mathbf{B}} = -\mu_0 \frac{I}{w}\hat{\mathbf{x}}$ (2 points)

B. Calculate Maxwell's stress tensor (related to the magnetic field) and show that

$\overleftrightarrow{\mathbf{T}} = \frac{\mu_0 I^2}{2w^2}\begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ (3 points)

Tip: the B-components of Maxwell's stress tensor read as

$$T_{ij} \equiv \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2\right)$$

C. Using Maxwell's stress tensor, find the electromagnetic force on the bottom plate (4 points)

Tip: The force as calculated from the Maxwell stress tensor is given by:

$$\vec{\mathbf{F}} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}} \cdot d\vec{\mathbf{a}}$$

D. Now do the same using the Lorentz force law. Are the results identical? Explain why (3 point)

Model answers **(12 points)**

A. Calculate the magnetic field for one plate first (btw, you did this a few times earlier). Draw a rectangular ampèrian loop with the length of l in the xz plane as we did considering the boundary conditions. Use Ampère-Maxwell's law with the time-derivative term nullified (the currents are stationary):

$$\oint_{\mathcal{P}} \vec{\mathbf{B}} \cdot d\vec{\boldsymbol{l}} = \mu_0 I_{fenc} \qquad \text{1/2 point}$$

As the $\vec{\mathbf{B}}$ field is in the $-x$ direction,

$2Bl = \mu_0 K l$, where K is surface current density $K = I/w$

$$\vec{\mathbf{B}} = -\frac{\mu_0 I}{2w}\hat{\mathbf{x}} \qquad \text{1/2 point}$$

In between the plates, the fields add up:

$$\vec{\mathbf{B}} = -\frac{\mu_0 I}{w}\hat{\mathbf{x}} \qquad \text{1 point}$$

Note that by the virtue of the same argument, $\vec{\boldsymbol{B}}$ out of space between the plates is zero. If it helps: the setting begins to resemble a plane capacitor (Problem 8.7 which we considered at the lecture), with the electric field substituted by magnetic field. Therefore, the further solution goes along similar lines – or simply use a brute force approach.

B. The Maxwell stress tensor (no electric field) is given by:

$$T_{ij} \equiv \frac{1}{\mu_0}\left(B_i B_j - \frac{1}{2}\delta_{ij}B^2\right)$$

All off diagonal components of $\overleftrightarrow{\mathbf{T}}$ are zero, because both B_y and B_z are zero:

$T_{ij} = 0$ if $i \neq j$ 1 point

The diagonal components are

$$T_{xx} \equiv \frac{1}{\mu_0}\left(B_x B_x - \frac{1}{2}B_x^2\right) = +\frac{\mu_0 I^2}{2w^2}$$ 1/2 point

$$T_{yy} \equiv \frac{1}{\mu_0}\left(B_y B_y - \frac{1}{2}B_x^2\right) = -\frac{\mu_0 I^2}{2w^2}$$ 1/2 point

$$T_{zz} \equiv \frac{1}{\mu_0}\left(B_z B_z - \frac{1}{2}B_x^2\right) = -\frac{\mu_0 I^2}{2w^2}$$ 1/2 point

$$\overleftrightarrow{\mathbf{T}} = \frac{\mu_0 I^2}{2w^2}\begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$ 1/2 point

only if all previous steps are correct (otherwise it's a copy – paste from the formulation)

C. The force as calculated from the Maxwell stress tensor is given by:

$$\vec{\mathbf{F}} = \oint_S \overleftrightarrow{\mathbf{T}} \cdot d\vec{\mathbf{a}}$$

The area element is $d\vec{\mathbf{a}} = dx\, dy\, \hat{\mathbf{z}}$ because the force is on the bottom side of the top plate (outside the field is zero). 1 point

$$\vec{\mathbf{F}} = \frac{\mu_0 I^2}{2w^2}\int_S \begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ dx\, dy \end{pmatrix}$$ 1 point

$$= -\frac{\mu_0 I^2}{2w^2}\int_S dx\, dy\, \hat{\mathbf{z}}$$ 1 point

$$= -\frac{\mu_0 I^2}{2w^2} lw\, \hat{\mathbf{z}} = -\frac{\mu_0 I^2 l}{2w}\, \hat{\mathbf{z}}$$ 1 point

Note that conductors carrying anti-parallel currents repel each other so that the force on top plate should be in -z direction as we derived.

D. The force (per unit area) due to the field of the lower plate is given by:

$$\vec{\mathbf{f}} = \vec{\mathbf{K}}_{Lower} \times \vec{\mathbf{B}}_{Upper}$$

$$= \frac{I}{w}(-\hat{\mathbf{y}}) \times \left(-\frac{\mu_0 I}{2w}\right)\hat{\mathbf{x}}$$ 1 point (0.5 pts if B is for both plates)

$$= -\frac{\mu_0 I^2}{2w^2}\hat{\mathbf{z}}$$ 1 point (0.5 pts if direction is wrong)

Multiplying by the area lw, we arrive at the result of (C)

The result must be the same because the Maxwell stress tensor formalism is simply a more convenient way of expression the Lorentz force, exactly what we did at the lectures. (1 point)

Typical mistakes

A. Setting up the Amperian loop in the region between the plates, or around both plates, instead of having it centered on one of them, resulting in non zero current enclosed.
B. Not justifying why off diagonals are 0.

Simply stating the final tensor without explanation of minus signs on T_yy and T_zz, and no calculations shown.

C. Choosing a wrong direction for the surface element da_z, giving an attractive force between the plates.

Integrating over the volume between the plates, instead of over the area of the bottom plate.

D. Using the magnetic field of both plates (in the region between them) in the Lorentz force formulation

Neglecting the “sheet current” K by substituting it with I

Question 3 (10 points)

Consider two waves which electric fields are expressed as

$\vec{\mathbf{E}}_1 = E_0 \cos(kz - \omega t)\,\hat{\mathbf{x}}$ and $\vec{\mathbf{E}}_2 = E_0 \cos(kz + \omega t)\,\hat{\mathbf{x}}$.

A. Show that the electric field of the resulting wave is

$\vec{\mathbf{E}} = 2E_0 \cos(kz)\cos(\omega t)\,\hat{\mathbf{x}}$

What type of wave is it? (1 point)

Tip: you might find it useful: $\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$

B. Calculate the magnetic field associated with this electric field by finding the magnetic fields associated with $\vec{\mathbf{E}}_1$ and $\vec{\mathbf{E}}_2$, and then adding them, to show that

$\vec{\mathbf{B}} = \frac{2E_0}{c}\sin(kz)\sin(\omega t)\,\hat{\mathbf{y}}$ (3 points)

Tip: you might find it useful: $\sin x \sin y = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$

C. What is the phase difference between $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$? (1 point)

D. Find energy density of this wave. Draw plots of the energy density as a function of kz at the following times:

$$\omega t = 0, \pi/4, \pi/2$$

Don't forget to mark axis labels and indicate which ωt is shown (5 points)

Tip 1: be careful as you need to use a general expression for energy density via electric and magnetic fields

Tip 2: check if the result is consistent with (C)

Model answers

A. $\vec{\mathbf{E}} = E_0[\cos(kz - \omega t) + \cos(kz + \omega t)]\hat{\mathbf{x}} = 2E_0 \cos(kz)\cos(\omega t)\,\hat{\mathbf{x}}$ (1/2 point)

It's a *standing wave* (1/2 point)

B. $\vec{\mathbf{E}}_1 = E_0 \cos(kz - \omega t)\,\hat{\mathbf{x}};\ \vec{\mathbf{B}}_1 = \frac{E_0}{c}\cos(kz - \omega t)\,\hat{\mathbf{y}}$ (1 point)

$\vec{\mathbf{E}}_2 = E_0 \cos(kz + \omega t)\,\hat{\mathbf{x}};\ \vec{\mathbf{B}}_2 = -\frac{E_0}{c}\cos(kz + \omega t)\,\hat{\mathbf{y}}$ (1 point)

$\vec{\mathbf{B}} = \frac{E_0}{c}\cos(kz - \omega t)\,\hat{\mathbf{y}} - \frac{E_0}{c}\cos(kz + \omega t)\,\hat{\mathbf{y}} = \frac{2E_0}{c}\sin(kz)\sin(\omega t)\,\hat{\mathbf{y}}$ (1 point)

C. $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are shifted by $\pi/2$ in phase (sine vs cosine) (1 point)

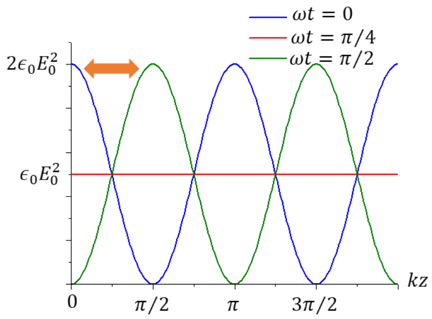
D. $u = \frac{\epsilon_0}{2}E^2 + \frac{\mu_0}{2}B^2 = 2\epsilon_0 E_0^2(\cos^2(kz)\cos^2(\omega t) + \sin^2(kz)\sin^2(\omega t))$ (2 points)

$\omega t = 0$: $u = 2\epsilon_0 E_0^2 \cos^2(kz)$ (1/2 point plus 1/2 point for the graph)

$\omega t = \pi/4$: $u = \epsilon_0 E_0^2(\cos^2(kz) + \sin^2(kz)) = \epsilon_0 E_0^2$ (1/2 point plus 1/2 point for the graph)

$\omega t = \pi/2$: $u = 2\epsilon_0 E_0^2 \sin^2(kz)$ (1/2 point plus 1/2 point for the graph)

Note: if the graphs are OK but no expressions are given, 1 point is awarded for each correct graph



This graph shows that the energy sloshes back and forth between the regions where kz is closed to even multiples of $\pi/2$ and regions where it's close to odd multiples of $\pi/2$ (orange arrow in the figure above). However, there's no energy flux to any direction at the longer scale

To expose this better, you might calculate Poynting's vector, too.

Typical mistakes

A. Many students gave an answer different from "standing wave" even despite the fact that the standing wave was discussed at lectures and tutorials. Some answers were correct but beside the point (e.g. "transverse", "polarized"), some were wrong ("plane monochromatic wave")

C. Many students say "0 / no phase difference" even though the calculated a cos for the electric field and sine for the magnetic field. Apparently, their justification was based on properties of plane monochromatic waves.

D. Some students do not remember that $\mu_0\epsilon_0 = c^2$ meaning that their result could have been further simplified. In some cases, this led to a wrong graph.

Another mistake: graphs look more like abs(sin(x)) instead of sin^2(x).

Few students computed irrelevant entities (e.g. the Poynting vector, the Maxwell stress tensor)

Question 4 (7 points + 2 bonus points)

You are an astronomer studying the origin of the solar system, and you are evaluating a hypothesis that sufficiently small particles were blown out of the solar system by the force of sunlight. To see how small such particles must be, compare the force of sunlight with the force of solar gravity. Assume spherical, fully reflecting particles with density $\rho = 2\ \mathrm{g/cm^3}$. The total power emitted by the Sun to all directions, is approximately $P = 4 \cdot 10^{26}$ W, mass of the Sun is $M = 2 \cdot 10^{30}$ kg, and the gravitational constant is $G = 6.7 \cdot 10^{-11}\ \mathrm{m^3\ kg^{-1} s^{-2}}$

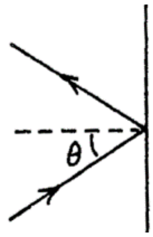
A. Does the distance from the Sun matter? Explain why (2 points)

B. Solve the particle radius at which the two forces (gravitational and electromagnetic) are equal. Consider the spherical particle as a flat disk of radius $r/\sqrt{2}$, oriented normally to the sun rays. (5 points)

C. (**2 points bonus**) Justify the tip for (B), i.e. calculate electromagnetic pressure exerted on a perfectly reflecting sphere, and compare it with the EM pressure on a perfectly reflecting disk.

My advice: try to solve this **only if you have done all other questions**. The bonus points will be awarded only if the solutions are fully correct.

Tip 2: If you decide to accept the challenge, first consider a pressure by an EM wave which is incident to a perfectly reflecting surface at some angle θ, into the direction orthogonal to the surface. With this knowledge in hand, move to the sphere and perform integration over the surface.



Model answers

A. Gravity scales as R^{-2} so does the Sun intensity. Hence, they cancel (2 points)

B. $F_{light} = PA = 2\frac{I}{c}\frac{\pi r^2}{2} = \frac{P}{c4\pi R^2}\pi r^2$ (1 point)

A factor of 2 in front is because of full reflection (1 point)

$$F_{grav} = G\frac{Mm}{R^2} = G\frac{M}{R^2}\rho\frac{4}{3}\pi r^3 \quad \text{(1 point)}$$

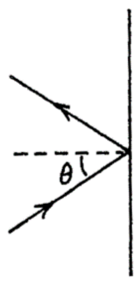
$$\frac{P}{c4R^2}r^2 = G\frac{M}{R^2}\rho\frac{4}{3}\pi r^3$$

$$r = \frac{3P}{16\pi cGM\rho} \quad \text{(1 point)}$$

$$= \frac{3 \cdot 4 \cdot 10^{26}}{16\pi \cdot 3 \cdot 10^8 \cdot 6.7 \cdot 10^{-11} \cdot 2 \cdot 10^{30} \cdot 2 \cdot 10^3} = \frac{10^{-4}}{16\pi \cdot 6.7} = 0.3\ \mu\mathrm{m} \quad \text{(1 point)}$$

so particles smaller than this size can be blown away from the Sun by radiation pressure

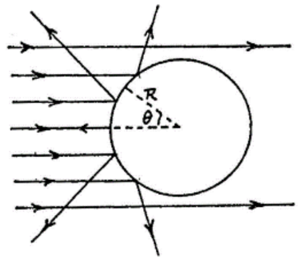
C. To justify the tip for (B), we need to calculate electromagnetic pressure exerted on a perfectly reflecting sphere.



We begin as following. When the wave is incident obliquely on a target surface, at angle θ from the normal, then a unit area of the surface intercepts an area $\cos\theta$ of the wave front. The component of the resulting force normal to the surface brings in a second factor of $\cos\theta$. And the force is doubled by the reaction from the reflected radiation. Thus, the pressure normal to the perfectly reflecting surface is

$$p_n(\theta) = 2\cos^2\theta\ p = \frac{2I}{c}\cos^2\theta \quad \text{(1 point)}$$

When the plane wave is incident on a reflecting sphere, the radiation pressure on an area element $da = 2\pi r\sin\theta \cdot r d\theta = 2\pi r^2 \sin\theta\, d\theta$ produces the pressure above in the inward-radial direction. From symmetry, the only contribution to the net force on the sphere is the component of this radial force in the direction of travel of the incident wave, which brings in yet another factor of $\cos\theta$. Thus



$$F = \frac{2I}{c}2\pi r^2\int_0^{\pi/2}\cos^3\theta\sin\theta\, d\theta = \frac{4I\pi r^2}{c}\int_0^1 x^3 dx$$
$$= \frac{I}{c}\pi r^2 \quad \text{(1 point)}$$

i.e. the same as the solution for a perfectly reflecting disk of radius $r/\sqrt{2}$

Let's evaluate this result. The reflecting sphere reflects the incoming radiation in the "backward" direction for polar angles less than $\pi/4$, but in the "forward" direction for angles between $\pi/4$ and $\pi/2$. The net momentum of the reflected radiation turns out to be zero. Because of this fact, the solution for perfectly absorbing particle is exactly the same but the adequacy of the model becomes more questionable (do you understand why?)

The bonus points are awarded only if the solutions are fully correct.

https://cohengroup.ccmr.cornell.edu/courses/phys3327/HW6/sol6.pdf

https://www.youtube.com/watch?v=ANJMdoFXM4U

Typical mistakes

A. Most students realized that either one between Intensity and Gravitational force are inversely proportional to the square of r but didn't combine them to cancel them out in the final formula

B. Confusing the two R's (r of the perfect disk and distance from the Sun) as well as the two P's (Power and Pressure).

The area of the shell (4 pi R^2) was in many cases wrong or substituted by the volume.

Three formulae for the area of the circle, the area of the spherical shell and the volume of the sphere were confused

C. Plugging in cos theta rather than cos^2 theta.