Calculus 2 (for Physics)
Midterm Exam (with solutions)
Exam Date: December 21, 2022 (18:30 – 20:30)

university of groningen

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1) A particle moves through space. Its position vector at time t (seconds) is

$$\vec{r}(t) = 3\cos(\alpha t^2 + \omega t)\,\hat{\imath} + 3\sin(\alpha t^2 + \omega t)\,\hat{\jmath} + 4(\alpha t^2 + \omega t)\,\hat{k},$$

where α, ω are positive constants.

a) Find the velocity, speed, and acceleration of the particle at time t.

b) Calculate the length of the particle's trajectory for $0 \le t \le 1$.

c) Verify that the particle's motion satisfies the following equations

$$\begin{cases} \vec{r}\,'(t) = (2\alpha t + \omega)\big(\hat{k}\times\vec{r}(t) + 4\hat{k}\big), \\ \vec{r}(0) \;\, = 3\,\hat{\imath}. \end{cases}$$

d) Find the tangential and normal components of acceleration.

<u>Solution.</u> Let $x(t), y(t), z(t)$ stand for the components of $\vec{r}(t)$, that is

$$x(t) = 3\cos(\alpha t^2 + \omega t), \quad y(t) = 3\sin(\alpha t^2 + \omega t), \quad z(t) = 4(\alpha t^2 + \omega t).$$

The components satisfy the equation

$$x^2 + y^2 = 3^2$$

as per the trigonometric identity $\cos^2 p + \sin^2 p = 1$, therefore the particle's trajectory lies on the surface of the cylinder of radius 3 with the z-axis as axis. Furthermore, all three components depend on time via the expression $p(t) = \alpha t^2 + \omega t = \alpha(t + \frac{\omega}{2\alpha})^2 - \frac{\omega^2}{4\alpha}$ which, due to $\alpha > 0$, has a (global) minimum at $t = -\frac{\omega}{2\alpha}$ and assumes the same value at $t = -\frac{\omega}{2\alpha} \pm \delta$, namely $p(-\frac{\omega}{2\alpha} \pm \delta) = \alpha\delta^2 - \frac{\omega^2}{4\alpha}$ for all $\delta \ge 0$.

Therefore we can restrict the parameter t to values $\boxed{t \ge -\frac{\omega}{2\alpha}}$ without loss of generality.

a) The velocity vector $\vec{v}(t)$ of the particle is the time derivative of its position vector $\vec{r}(t)$, which can be found by differentiating each component of $\vec{r}(t)$ with respect to t (cf. Formula 2 in Section 13.2), that is

$$\vec{v}(t) = \vec{r}\,'(t) = x'(t)\,\hat{\imath} + y'(t)\,\hat{\jmath} + z'(t)\,\hat{k}.$$

By using the rules of differentiation we obtain

$$\begin{aligned} x'(t) &= [3\cos(\alpha t^2 + \omega t)]' \\ &\overset{(1)}{=} 3[\cos(\alpha t^2 + \omega t)]' \\ &\overset{(2)}{=} 3[-\sin(\alpha t^2 + \omega t)][\alpha t^2 + \omega t]' \\ &\overset{(3)}{=} 3[-\sin(\alpha t^2 + \omega t)][(\alpha t^2)' + (\omega t)'] \\ &\overset{(1)}{=} 3[-\sin(\alpha t^2 + \omega t)][\alpha (t^2)' + \omega (t)'] \\ &\overset{(4)}{=} 3[-\sin(\alpha t^2 + \omega t)][\alpha(2t) + \omega(1)] \\ &= -3(2\alpha t + \omega)\sin(\alpha t^2 + \omega t), \end{aligned}$$

$$\begin{aligned} y'(t) &= [3\sin(\alpha t^2 + \omega t)]' \\ &\overset{(1)}{=} 3[\sin(\alpha t^2 + \omega t)]' \\ &\overset{(2)}{=} 3[\cos(\alpha t^2 + \omega t)][\alpha t^2 + \omega t]' \\ &\overset{(3)}{=} 3[\cos(\alpha t^2 + \omega t)][(\alpha t^2)' + (\omega t)'] \\ &\overset{(1)}{=} 3[\cos(\alpha t^2 + \omega t)][\alpha (t^2)' + \omega (t)'] \\ &\overset{(4)}{=} 3[\cos(\alpha t^2 + \omega t)][\alpha(2t) + \omega(1)] \\ &= 3(2\alpha t + \omega)\cos(\alpha t^2 + \omega t), \end{aligned}$$

$$\begin{aligned} z'(t) &= [4(\alpha t^2 + \omega t)]' \\ &\overset{(1)}{=} 4[\alpha t^2 + \omega t]' \\ &\overset{(3)}{=} 4[(\alpha t^2)' + (\omega t)'] \\ &\overset{(1)}{=} 4[\alpha (t^2)' + \omega (t)'] \\ &\overset{(4)}{=} 4[\alpha(2t) + \omega(1)] \\ &= 4(2\alpha t + \omega). \end{aligned}$$

(1) Constant Multiple Rule $(cf)' = cf'$, (2) Chain Rule $(f \circ g)' = (f' \circ g)g'$, Trigonometric Derivatives $(\cos t)' = -\sin t$, $(\sin t)' = \cos t$, (3) Sum Rule $(f + g)' = f' + g'$, (4) Power Rule $(t^n)' = nt^{n-1}$.

All three components $x'(t), y'(t), z'(t)$ contain $(2\alpha t + \omega)$ as a factor therefore the velocity of the particle at time t can be written as follows

$$\boxed{\vec{v}(t) = (2\alpha t + \omega)(-3\sin(\alpha t^2 + \omega t)\,\hat{\imath} + 3\cos(\alpha t^2 + \omega t)\,\hat{\jmath} + 4\hat{k})}$$

Introducing the shorthand notation

$$f(t) = 5(2\alpha t + \omega), \quad \hat{u}(t) = -\tfrac{3}{5}\sin(\alpha t^2 + \omega t)\,\hat{\imath} + \tfrac{3}{5}\cos(\alpha t^2 + \omega t)\,\hat{\jmath} + \tfrac{4}{5}\hat{k}$$

lets us write

$$\vec{v}(t) = f(t)\hat{u}(t).$$

Note that $\hat{u}(t)$ is a unit vector as we have

$$\begin{aligned}
|\hat{u}(t)| &\overset{(5)}{=} \sqrt{(-\tfrac{3}{5})^2\sin^2(\alpha t^2+\omega t) + (\tfrac{3}{5})^2\cos^2(\alpha t^2+\omega t) + (\tfrac{4}{5})^2} \\
&= \sqrt{(-\tfrac{3}{5})^2\sin^2(\alpha t^2+\omega t) + (\tfrac{3}{5})^2\cos^2(\alpha t^2+\omega t) + (\tfrac{4}{5})^2} \\
&= \sqrt{\tfrac{9}{25}\big[\sin^2(\alpha t^2+\omega t) + \cos^2(\alpha t^2+\omega t)\big] + \tfrac{16}{25}} \\
&\overset{(6)}{=} \sqrt{\tfrac{9}{25}(1) + \tfrac{16}{25}} \\
&= 1
\end{aligned}$$

(5) length formula $|\hat{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$, (6) trigonometric identity $\sin^2 p + \cos^2 p = 1$.
The speed $v(t)$ of the particle is the magnitude of its velocity vector $\vec{v}(t)$, i.e.

$$v(t) = |\vec{v}(t)| = |f(t)\hat{u}(t)| \overset{(7)}{=} |f(t)|\,|\hat{u}(t)| = |f(t)|(1) = 5|2\alpha t + \omega| \overset{(\dagger)}{=} 5(2\alpha t + \omega).$$

(7) length & scalar multiplication $|f\hat{u}| = |f|\,|\hat{u}|$, ($\dagger$) parameter restriction $t \geq -\frac{\omega}{2\alpha}$. Therefore the speed of the particle at time t is

$$\boxed{v(t) = 5(2\alpha t + \omega)} \qquad t \geq -\tfrac{\omega}{2\alpha}$$

The acceleration $\vec{a}(t)$ of the particle is the time derivative of its velocity $\vec{v}(t)$. We may compute this derivative using the Product Rule (cf. Formula 3.3 in Section 13.2), i.e.

$$\vec{a}(t) = \vec{v}\,'(t) = [f(t)\hat{u}(t)]' = f'(t)\hat{u}(t) + f(t)\hat{u}\,'(t).$$

We have

$$f'(t) = 10\alpha, \quad \hat{u}\,'(t) = -\tfrac{3}{5}(2\alpha t + \omega)(\cos(\alpha t^2 + \omega t)\,\hat{\imath} + \sin(\alpha t^2 + \omega t)\,\hat{\jmath}).$$

Therefore the acceleration of the particle at time t is

$$\boxed{\begin{aligned}
\vec{a}(t) = &-3\big[2\alpha\sin(\alpha t^2+\omega t) + (2\alpha t+\omega)^2\cos(\alpha t^2+\omega t)\big]\,\hat{\imath} \\
&+3\big[2\alpha\cos(\alpha t^2+\omega t) - (2\alpha t+\omega)^2\sin(\alpha t^2+\omega t)\big]\,\hat{\jmath} \\
&+8\alpha\,\hat{k}
\end{aligned}}$$

b) The arc length L of the curve traced out by the position vector $\vec{r}(t)$ for $0 \leq t \leq 1$ is given by the definite integral (cf. Formula 3 in Section 13.3)

$$L = \int_0^1 |\vec{r}\,'(t)|\,dt = \int_0^1 |\vec{v}(t)|\,dt = \int_0^1 v(t)\,dt = \int_0^1 5(2\alpha t + \omega)\,dt = \big[5(\alpha t^2 + \omega t)\big]_{t=0}^{t=1} = 5(\alpha + \omega).$$

Therefore the length of the particle's trajectory is $\boxed{L = 5(\alpha + \omega)}$.

c) By the properties of the vector product (cf. Formula 11 in Section 12.4) we have

$$\begin{aligned}
\hat{k} \times \vec{r}(t) &= 3\cos(\alpha t^2+\omega t)\,(\hat{k}\times\hat{\imath}) + 3\sin(\alpha t^2+\omega t)\,(\hat{k}\times\hat{\jmath}) + 4(\alpha t^2+\omega t)\,(\hat{k}\times\hat{k}) \\
&= 3\cos(\alpha t^2+\omega t)\,\hat{\jmath} + 3\sin(\alpha t^2+\omega t)\,(-\hat{\imath}) + 4(\alpha t^2+\omega t)\,(\vec{0}) \\
&= -3\sin(\alpha t^2+\omega t)\,\hat{\imath} + 3\cos(\alpha t^2+\omega t)\,\hat{\jmath}
\end{aligned}$$

therefore the first equation follows

$$(2\alpha t+\omega)(\hat{k}\times\vec{r}(t)+4\hat{k}) = (2\alpha t+\omega)(-3\sin(\alpha t^2+\omega t)\,\hat{\imath}+3\cos(\alpha t^2+\omega t)\,\hat{\jmath}+4\hat{k}) = \vec{v}(t) = \vec{r}\,'(t).$$

Setting $t=0$ we obtain

$$\vec{r}(0) = 3\cos(0)\,\hat{\imath}+3\sin(0)\,\hat{\jmath}+4(0)\,\hat{k} = 3\,\hat{\imath}$$

as $\cos(0)=1$ and $\sin(0)=0$, thus the second equation is verified.

d) By introducing the notation

$$\hat{n}(t) = \cos(\alpha t^2+\omega t)\,\hat{\imath}+\sin(\alpha t^2+\omega t)\,\hat{\jmath}$$

we can write the acceleration $\vec{a}(t)$ in the following compact form

$$\vec{a}(t) = 10\alpha\,\hat{u}(t) - 3(2\alpha t+\omega)^2\,\hat{n}(t).$$

Note that $\hat{n}(t)$ is a unit vector as

$$|\hat{n}(t)| = \sqrt{\cos^2(\alpha t^2+\omega t)+\sin^2(\alpha t^2+\omega t)} = 1$$

and $\hat{u}(t)$ and $\hat{n}(t)$ are orthogonal as

$$\hat{u}(t)\cdot\hat{n}(t) = -\tfrac{3}{5}\sin(\alpha t^2+\omega t)\cos(\alpha t^2+\omega t)+\tfrac{3}{5}\cos(\alpha t^2+\omega t)\sin(\alpha t^2+\omega t) = 0.$$

Therefore $\hat{u}(t)$ and $\hat{n}(t)$ are, up to sign, the unit tangent and principal normal vectors. Indeed, we have

$$\vec{T}(t) = \tfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \tfrac{\vec{v}(t)}{v(t)} = \tfrac{5(2\alpha t+\omega)}{5(2\alpha t+\omega)}\hat{u}(t) = \hat{u}(t), \qquad t\geq -\tfrac{\omega}{2\alpha}$$

and since

$$\vec{T}\,'(t) = \hat{u}\,'(t) = -\tfrac{3}{5}(2\alpha t+\omega)\hat{n}(t)$$

meaning that $|\vec{T}\,'(t)| = \frac{3}{5}(2\alpha t+\omega)$ and

$$\vec{N}(t) = \tfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = \frac{-\frac{3}{5}(2\alpha t+\omega)}{\frac{3}{5}(2\alpha t+\omega)}\hat{n}(t) = -\hat{n}(t)$$

Thus the acceleration reads

$$\vec{a}(t) = 10\alpha\,\vec{T}(t)+3(2\alpha t+\omega)^2\,\vec{N}(t)$$

and the tangential and normal components of acceleration are

$$\boxed{a_T = 10\alpha, \quad a_N = 3(2\alpha t+\omega)^2}.$$

2) Consider a thin metal disk corresponding to $x^2+y^2\leq 8$ in the xy-plane. The temperature (measured in degrees Celsius) at point (x,y) is

$$T(x,y) = x^3+3x^2-3(x+1)y^2.$$

a) Compute the rate of change of temperature at $(1,-1)$ in the direction of $\vec{v}=-3\hat{\imath}+4\hat{\jmath}$. What is the maximum rate of change at $(1,-1)$?

b) Find and classify (min/max/saddle) the stationary points of $T(x,y)$.

c) Locate the points of the highest and lowest temperatures on the disk.
[Hint: These points are on the edge/circumference of the disk.]

Solution. a) The rate of change of temperature in question is the directional derivative $D_{\hat{u}}T(1,-1)$, where $\hat{u}$ is the unit vector parallel to $\vec{v}$. Since the magnitude of $\vec{v}$ is $|\vec{v}| = \sqrt{(-3)^2+4^2} = \sqrt{25} = 5$, we have $\hat{u} = \frac{\vec{v}}{|\vec{v}|} = -\frac{3}{5}\,\hat{\imath} + \frac{4}{5}\,\hat{\jmath}$. The directional derivative at an arbitrary point (x,y) can be computed as the dot product

$$D_{\hat{u}}T(x,y) = \nabla T(x,y)\cdot\hat{u}$$

(cf. Formula 9 Section 14.6) where $\nabla T(x,y)$ is the gradient vector of T at (x,y), that is

$$\nabla T(x,y) = T_x\,\hat{\imath} + T_y\,\hat{\jmath} = (3x^2+6x-3y^2)\,\hat{\imath} - 6(x+1)y\,\hat{\jmath}$$

(cf. Definition 8 Section 14.6). At $(1,-1)$ the gradient vector becomes

$$\nabla T(1,-1) = (3(1)^2+6(1)-3(-1)^2)\,\hat{\imath} - 6(1+1)(-1)\,\hat{\jmath} = 6\,\hat{\imath} + 12\,\hat{\jmath}.$$

Therefore the rate of change of temperature at $(1,-1)$ is

$$D_{\hat{u}}T(1,-1) = \nabla T(1,-1)\cdot\hat{u} = \langle 6,12\rangle\cdot\langle -\tfrac{3}{5},\tfrac{4}{5}\rangle = 6(-\tfrac{3}{5}) + 12(\tfrac{4}{5}) = \tfrac{-18+48}{5} = \tfrac{30}{5} = 6.$$

The maximum rate of change is the magnitude of the gradient vector (Theorem 15 Section 14.6), i.e.

$$|\nabla T(1,-1)| = \sqrt{6^2+12^2} = \sqrt{180} = \sqrt{36\cdot 5} = 6\sqrt{5}.$$

b) To find the stationary points of $T(x,y)$ we need to solve the system of equations

$$\begin{cases} T_x(x,y) = 3x^2+6x-3y^2 = 0 \\ T_y(x,y) = -6(x+1)y = 0 \end{cases}$$

The second equation implies that $x=-1$ or $y=0$. With $x=-1$ the first equation turns into $y^2=-1$. This has no real number solutions, therefore $x=-1$ does not yield a stationary point. With $y=0$ the first equation reads $3x^2+6x=0$. This has two solutions $x=0$ and $x=-2$. Thus $T(x,y)$ has two stationary points $(0,0)$ and $(-2,0)$. (Both of these points are on the disk $x^2+y^2\le 8$.) To classify these points we perform the Second Derivative Test (Formula 3 Section 14.7). We computed the second partial derivatives of T:

$$T_{xx} = (T_x)_x = 6x+6 = 6(x+1), \quad T_{xy} = (T_x)_y = -6y,$$
$$T_{yx} = (T_y)_x = -6y, \quad T_{yy} = (T_y)_y = -6(x+1).$$

Therefore we have

$$D(x,y) = T_{xx}T_{yy} - [T_{xy}]^2 = -36[(x+1)^2+y^2].$$

At the stationary points we have

$$D(0,0) = D(-2,0) = -36 < 0$$

hence $T(x,y)$ has saddle points at both by the second derivative test.

c) The extreme values of temperature on the disk $x^2+y^2\le 8$ are determined as follows:

1. find the values at stationary points inside the region, i.e. for (x,y) satisfying $x^2+y^2<8$
2. find the extrema on the boundary of the region, i.e. for (x,y) satisfying $x^2+y^2=8$.
3. select the largest and smallest values found in steps 1&2.

(cf. Formula 9 Section 14.7)

In part b), we saw that the function $T(x,y)$ has two stationary points $(0,0)$, $(-2,0)$. These points are inside the region. However, we also showed that these are saddle points so T has no extreme values there. Therefore the points of maximum and minimum temperature values must be on the boundary of the disk. (This agrees with the hint given.)

Let us continue with step 2 and locate the extrema along the boundary of the disk. We use the method of Lagrange multipliers. The boundary curve of the disk is the circle $x^2 + y^2 = 8$, hence we set $g(x, y) = x^2 + y^2 - 8$ (so $g_x = 2x$, $g_y = 2y$) and consider the following system of equations

$$\begin{cases} \nabla T(x, y) = \lambda \nabla g(x, y) \\ g(x, y) = 0 \end{cases}$$

which in the current setting reads

$$\begin{cases} \mathrm{I}: & 3x^2 + 6x - 3y^2 = \lambda(2x) \\ \mathrm{II}: & -6(x+1)y = \lambda(2y) \\ \mathrm{III}: & x^2 + y^2 - 8 = 0 \end{cases}$$

We may use the constraint, i.e. equation III to express y in terms of x. We get

$$y = \pm\sqrt{8 - x^2}.$$

Plugging this into equation I results in the following

$$3x^2 + 6x - 24 + 3x^2 = 2\lambda x.$$

Finally, equation II implies that $-6(x + 1) = 2\lambda$ or $y = 0$.

- If $-6(x + 1) = 2\lambda$, then the previous equation turns into

  $$3x^2 + 6x - 24 + 3x^2 = -6(x + 1)x,$$

  or equivalently

  $$x^2 + x - 2 = 0.$$

  The quadratic formula yields two real solutions

  $$x = \frac{-1 \pm \sqrt{1^2 - 4(-2)}}{2} = \frac{-1 \pm \sqrt{9}}{2} = \frac{-1 \pm 3}{2},$$

  i.e. $x = 1$, $x = -2$. Recall that $y = \pm\sqrt{8 - x^2}$, resulting in four solutions for (x, y):

  $$(1, \sqrt{7}), \quad (1, -\sqrt{7}), \quad (-2, 2), \quad (-2, -2).$$

- If $y = 0$, then equation III yields $x = \pm\sqrt{8 - y^2} = \pm\sqrt{8} = \pm 2\sqrt{2}$. These give us two more solutions for (x, y). Namely, we have

  $$(2\sqrt{2}, 0), \quad (-2\sqrt{2}, 0).$$

We have exhausted all possibilities and have found six points on the boundary circle where $T(x, y)$ has local maxima/minima:

$$(1, \sqrt{7}), \quad (1, -\sqrt{7}), \quad (-2, 2), \quad (-2, -2), \quad (2\sqrt{2}, 0), \quad (-2\sqrt{2}, 0).$$

The temperature function $T(x, y) = x^3 + 3x^2 - 3(x + 1)y^2$ at these points evaluates to

$$T(1, \sqrt{7}) = T(1, -\sqrt{7}) = -38, \quad T(-2, 2) = T(-2, -2) = 16$$

$$T(2\sqrt{2}, 0) = 24 + 16\sqrt{2} \approx 46.6, \quad T(-2\sqrt{2}, 0) = 24 - 16\sqrt{2} \approx 1.4.$$

In summary, the highest temperature (approximately $46.6°\mathrm{C}$) is assumed at the point $(2\sqrt{2}, 0)$, whereas the lowest temperature ($-38°\mathrm{C}$) is assumed at the points $(1, \pm\sqrt{7})$.

3) Find the moments of inertia I_x, I_y, I_0 for the homogeneous lamina with density $\rho(x,y) = 1$ $(\mathrm{kg/m^2})$ occupying the region that is above the x-axis but under the parabola $y = 1 - x^2$ in the xy-plane.

Solution. Let D denote the region in question. It can be viewed as type I or type II. To describe D as a type I region we need the x-intercepts (i.e. zeros) of the parabola $y = 1 - x^2$. These are $x = -1$ and $x = 1$. For every $x \in [-1, 1]$ we have $0 \le 1 - x^2$, therefore a point $(x, y) \in D$ with a fixed x-coordinate has a y-coordinate between 0 and $1 - x^2$. Thus the region can be written as follows

$$D = \{(x,y) \mid -1 \le x \le 1,\ 0 \le y \le 1 - x^2\}.$$

To describe D as a type II region we need the y-intercept (value at $x = 0$) of the parabola $y = 1 - x^2$. This is $y = 1$. For every $y \in [0, 1]$ a point $(x, y) \in D$ has an x-coordinate between $-\sqrt{1-y}$ and $\sqrt{1-y}$ (these are the two branches of the parabola $y = 1 - x^2$ with $x < 0$ and $x > 0$). Thus the region can also be written as follows

$$D = \{(x,y) \mid 0 \le y \le 1,\ -\sqrt{1-y} \le x \le \sqrt{1-y}\}.$$

Therefore the double integral of a continuous function $f(x, y)$ on D can be written in the following ways

$$\iint_D f(x,y)\, dA = \int_{-1}^{1} \int_{0}^{1-x^2} f(x,y)\, dy\, dx = \int_{0}^{1} \int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y)\, dx\, dy.$$

Looking at the limits of integration we see that the type I integral (i.e. first integrating with respect to y, then with respect to x) seems simpler. The moment of inertia about the x-axis is

$$\begin{aligned} I_x &= \iint_D y^2 \rho(x,y)\, dA = \int_{-1}^{1} \int_{0}^{1-x^2} y^2(1)\, dy\, dx = \int_{-1}^{1} \left[\tfrac{1}{3} y^3\right]_{y=0}^{y=1-x^2} dx = \int_{-1}^{1} \tfrac{1}{3}(1-x^2)^3\, dx \\ &= \int_{-1}^{1} \tfrac{1}{3}(1 - 3x^2 + 3x^4 - x^6)\, dx = \tfrac{1}{3}\left[x - x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7\right]_{x=-1}^{x=1} = \tfrac{2}{3}\left(1 - 1 + \tfrac{3}{5} - \tfrac{1}{7}\right) \\ &= \tfrac{2}{3}\left(\tfrac{16}{35}\right) = \tfrac{32}{105}. \end{aligned}$$

The moment of inertia about the y-axis is

$$\begin{aligned} I_y &= \iint_D x^2 \rho(x,y)\, dA = \int_{-1}^{1} x^2 \int_{0}^{1-x^2} (1)\, dy\, dx = \int_{-1}^{1} x\, [y]_{y=0}^{y=1-x^2} dx = \int_{-1}^{1} x^2(1-x^2)\, dx \\ &= \int_{-1}^{1} (x^2 - x^4)\, dx = \left[\tfrac{1}{3}x^3 - \tfrac{1}{5}x^5\right]_{x=-1}^{x=1} = 2\left(\tfrac{1}{3} - \tfrac{1}{5}\right) = 2\left(\tfrac{2}{15}\right) = \tfrac{4}{15}. \end{aligned}$$

The moment of inertia about the origin

$$I_0 = \iint_D (x^2 + y^2)\rho(x,y)\, dA = I_x + I_y = \tfrac{32}{105} + \tfrac{4}{15} = \tfrac{32+28}{105} = \tfrac{60}{105} = \tfrac{4}{7}.$$

4) Use triple integrals to locate the center of mass $(\overline{x}, \overline{y}, \overline{z})$ of the uniform i.e. constant density $\rho(x, y, z) = k$ $(\mathrm{kg/m^3})$ solid cone

$$E = \{(x,y,z) \mid 0 \le z \le 1 - \sqrt{x^2 + y^2}\}.$$

[Hint: Work in cylindrical coordinates.]

Solution. The cone E can be described using cylindrical coordinates (r, θ, z) as follows. Using $x = r\cos\theta$ and $y = r\sin\theta$ the defining inequalities $0 \le z \le 1 - \sqrt{x^2 + y^2}$ for E read

$$0 \le z \le 1 - \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = 1 - \sqrt{r^2(\cos^2\theta + \sin^2\theta)} = 1 - \sqrt{r^2} = 1 - r,$$

i.e.

$$0 \leq z \leq 1 - r.$$

These also restrict r, since the inequalities only make sense if the lower bound (0) is less than or equal to the upper bound ($1 - r$). Thus we need $0 \leq 1 - r$, i.e. $r \leq 1$. There is no restriction on the polar angle θ. Therefore

$$E = \{(r, \theta, z) \mid 0 \leq \theta < 2\pi,\ 0 \leq r \leq 1,\ 0 \leq z \leq 1 - r\}$$

and the triple integral of a continuous function $f(x, y, z)$ over E in cylindrical coordinates reads

$$\iiint_E f(x, y, z)\, dV = \int_0^{2\pi} \int_0^1 \int_0^{1-r} f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta.$$

The center of mass $(\overline{x}, \overline{y}, \overline{z})$ of the cone E is given by

$$\overline{x} = \frac{M_{yz}}{m}, \quad \overline{y} = \frac{M_{xz}}{m}, \quad \overline{z} = \frac{M_{xy}}{m},$$

where m is the mass of the cone, that is

$$m = \iiint_E \rho(x, y, z)\, dV$$

and M_{yz}, M_{xz}, M_{xy} are the moments of E about the three coordinate planes, i.e.

$$M_{yz} = \iiint_E x\rho(x, y, z)\, dV, \quad M_{xz} = \iiint_E y\rho(x, y, z)\, dV, \quad M_{xy} = \iiint_E z\rho(x, y, z)\, dV.$$

Since the density is constant the mass of the cone is its volume time that constant:

$$\begin{aligned}
m &= \iiint_E \rho(x, y, z)\, dV = \iiint_E k\, dV = k \iiint_E 1\, dV = k \int_0^{2\pi} \int_0^1 \int_0^{1-r} r\, dz\, dr\, d\theta \\
&= k \int_0^{2\pi} d\theta \int_0^1 r \int_0^{1-r} dz\, dr = k\, [\theta]_{\theta=0}^{\theta=2\pi} \int_0^1 r[z]_{z=0}^{z=1-r}\, dr = k(2\pi) \int_0^1 r(1 - r)\, dr \\
&= k(2\pi) \int_0^1 (r - r^2)\, dr = k(2\pi) \left[\tfrac{1}{2}r^2 - \tfrac{1}{3}r^3\right]_{r=0}^{r=1} = k(2\pi)\left(\tfrac{1}{2} - \tfrac{1}{3}\right) \\
&= k\pi\left(1 - \tfrac{2}{3}\right) = \tfrac{1}{3}k\pi.
\end{aligned}$$

Similarly, the moments are

$$\begin{aligned}
M_{yz} &= \iiint_E x\rho(x, y, z)\, dV = \iiint_E xk\, dV = k \iiint_E x\, dV = k \int_0^{2\pi} \int_0^1 \int_0^{1-r} (r\cos\theta)\, r\, dz\, dr\, d\theta \\
&= k \int_0^{2\pi} \cos\theta\, d\theta \int_0^1 r^2 \int_0^{1-r} 1\, dz\, dr = k[\sin\theta]_{\theta=0}^{\theta=2\pi} \int_0^1 r^2 \int_0^{1-r} 1\, dz\, dr = k(\sin 2\pi - \sin 0) \int_0^1 r^2 \int_0^{1-r} 1\, dz\, dr \\
&= k(0 - 0) \int_0^1 r^2 \int_0^{1-r} 1\, dz\, dr = k(0) \int_0^1 r^2 \int_0^{1-r} 1\, dz\, dr = 0,
\end{aligned}$$

$$\begin{aligned}
M_{xz} &= \iiint_E y\rho(x, y, z)\, dV = \iiint_E yk\, dV = k \iiint_E y\, dV = k \int_0^{2\pi} \int_0^1 \int_0^{1-r} (r\sin\theta)\, r\, dz\, dr\, d\theta \\
&= k \int_0^{2\pi} \sin\theta\, d\theta \int_0^1 r^2 \int_0^{1-r} 1\, dz\, dr = k[-\cos\theta]_{\theta=0}^{\theta=2\pi} \int_0^1 r^2 \int_0^{1-r} 1\, dz\, dr = k(\cos 0 - \cos 2\pi) \int_0^1 r^2 \int_0^{1-r} 1\, dz\, dr \\
&= k(1 - 1) \int_0^1 r^2 \int_0^{1-r} 1\, dz\, dr = k(0) \int_0^1 r^2 \int_0^{1-r} 1\, dz\, dr = 0,
\end{aligned}$$

and

$$\begin{aligned}
M_{xy} &= \iiint\limits_E z\rho(x,y,z)\,dV = \iiint\limits_E zk\,dV = k\iiint\limits_E z\,dV = k\int\limits_0^{2\pi}\int\limits_0^1\int\limits_0^{1-r} z\,r\,dz\,dr\,d\theta \\
&= k\int\limits_0^{2\pi} 1\,d\theta \int\limits_0^1 r \int\limits_0^{1-r} z\,dz\,dr = k[\theta]_{\theta=0}^{\theta=2\pi}\int\limits_0^1 r\left[\tfrac{1}{2}z^2\right]_{z=0}^{z=1-r}\,dr = k(2\pi)\int\limits_0^1 r\tfrac{1}{2}(1-r)^2\,dr \\
&= k\pi\int\limits_0^1 r(1-2r+r^2)\,dr = k\pi\int\limits_0^1 (r-2r^2+r^3)\,dr = k\pi\left[\tfrac{1}{2}r^2-\tfrac{2}{3}r^3+\tfrac{1}{4}r^4\right]_{r=0}^{r=1} \\
&= k\pi\left(\tfrac{1}{2}-\tfrac{2}{3}+\tfrac{1}{4}\right) = k\pi\left(\tfrac{6-8+3}{12}\right) = \tfrac{1}{12}k\pi.
\end{aligned}$$

Therefore the coordinates of the center of mass are

$$\overline{x} = \frac{M_{yz}}{m} = \frac{0}{\frac{1}{3}k\pi} = 0, \quad \overline{y} = \frac{M_{xz}}{m} = \frac{0}{\frac{1}{3}k\pi}m = 0, \quad \overline{z} = \frac{M_{xy}}{m} = \frac{\frac{1}{12}k\pi}{\frac{1}{3}k\pi} = \tfrac{3}{12} = \tfrac{1}{4},$$

i.e. $(\overline{x},\overline{y},\overline{z}) = \left(0,0,\tfrac{1}{4}\right)$. The center of mass of the solid cone E lies on the z-axis a fourth of the way to the apex above the xy-plane.