Name: ______________________ student ID: S

Mechanics Resit Exam

February 16, 2023

18:30-20:30

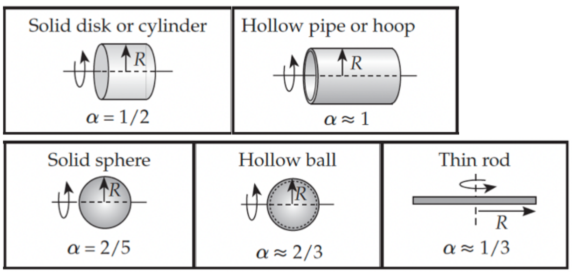
Aletta Jacobs Hall 1 (A1-N3)

About the Exam

1. There are 5 questions on the exam (75 points).
2. Write your answers *clearly* and *legibly*. Use standard notation and units when appropriate.
3. **Show all your work** (e.g., free-body diagrams, intermediate steps used in calculations, presumptions about problem) for full credit.
4. You may bring a *scientific* calculator to use on the exam – one will not be provided for you.
5. If you tear the pages apart, *write your name and student ID at the top of each page*.
6. A scratch sheet of paper is provided at the end of the exam (raise your hand if you need more).

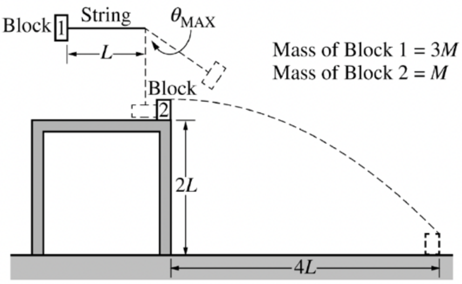
Possibly Useful Information

$\vec{v}(t) \equiv \frac{d\vec{r}}{dt}$	$\vec{r}(t) = \frac{1}{2}\vec{a}t^2 + \vec{v}_0 t + \vec{r}_0$	$\lvert\vec{v}\rvert = \sqrt{v_x^2 + v_y^2 + v_z^2}$	$\vec{r}_{CM} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$	$M\frac{\Delta\vec{r}_{CM}}{\Delta t} = \sum m_i \vec{v}_i$
$\vec{a}(t) \equiv \frac{d\vec{v}}{dt}$	$\vec{v}(t) = \vec{a}t + \vec{v}_0$	$\lvert\vec{a}\rvert = \sqrt{a_x^2 + a_y^2 + a_z^2}$	$\vec{p} = m\vec{v}$	$\vec{F} = \frac{d\vec{p}}{dt} \quad F_x = -\frac{dV}{dx}$
$K = \frac{1}{2}m\lvert\vec{v}\rvert^2$	$dK_{\text{CM}} = \lvert\vec{F}_{\text{ext}}\rvert\lvert d\vec{r}_{\text{CM}}\rvert cos\theta$	$V_g(r) = -\frac{Gm_1m_2}{r}$	$V_g(h) = m\lvert\vec{g}\rvert h$	$W = \int \vec{F}\cdot d\vec{r}$
$\sum \vec{F} = m\vec{a}$	$\lvert\vec{F}_{\text{KF}}\rvert \approx \mu_K\lvert\vec{F}_{\text{N}}\rvert$ $\lvert\vec{F}_{\text{SF}}\rvert \leq \lvert\vec{F}_{\text{SF,max}}\rvert \approx \mu_S\lvert\vec{F}_{\text{N}}\rvert$	$\lvert\vec{F}_{\text{D}}\rvert \approx \frac{1}{2}C\rho A\lvert\vec{v}\rvert^2$	$\vec{F}_{\text{T}}^{\text{A(B)}} = -\vec{F}_{\text{T}}^{\text{B(A)}}$	$m\vec{a}' = -m\vec{A} + \vec{F}_1 + \vec{F}_2 + \cdots$
$\lvert\vec{v}\rvert = \frac{2\pi R}{T}$	$\lvert\vec{a}\rvert = \frac{\lvert\vec{v}\rvert^2}{R}$	$\vec{a} = \frac{d\lvert\vec{v}\rvert}{dt}\hat{v} - \frac{\lvert\vec{v}\rvert^2}{R}\hat{r}$	$\vec{F}_{\text{Sp,x}} = -k_s\vec{x}$	$V_{\text{Sp}}(x) = \frac{1}{2}k_s(x - x_0)^2$
$\frac{d^2x}{dt^2} = -\omega^2 x$	$x(t) = A\cos(\omega t + \theta)$	$\omega \equiv \sqrt{k_s/m}$	$\omega \equiv \sqrt{\lvert\vec{g}\rvert/L}$	$T = \frac{2\pi}{\omega} = \frac{1}{f}$
$\vec{\tau} = \vec{r} \times \vec{F}$	$\theta \equiv \frac{s}{R} \quad \lvert\vec{\omega}\rvert = \left\lvert\frac{d\theta}{dt}\right\rvert = \frac{\lvert\vec{v}\rvert}{R}$	$\vec{L} = I\vec{\omega} = \vec{r} \times \vec{p}$	$\vec{L} = \vec{L}_{CM} + \vec{L}_{rot}$	$I = \sum m_i r_i^2 = \alpha M R^2$
$\vec{\tau} \equiv \frac{[d\vec{L}]}{dt}$	$U^{\text{rot}} = \frac{1}{2}I\lvert\vec{\omega}\rvert^2$	$\lvert\vec{F}_{\text{g}}\rvert = \left\lvert\frac{Gm_1m_2}{r^2}\right\rvert$	$\lvert\vec{v}\rvert = \sqrt{\frac{GM}{R}}$	$r(\theta) = \frac{b}{1 + \varepsilon\cos\theta}$
$T^2 = \frac{4\pi^2}{GM}a^3$	$a = \frac{1}{2}(r_c + r_f) = \frac{b}{1 - \varepsilon^2}$	$\frac{2E}{m} = \lvert\vec{v}\rvert^2 - \frac{2GM}{r}$	$E = \mp\frac{GMm}{2a}$	$\varepsilon^2 = 1 + \frac{1}{(GM)^2}\left(\frac{L}{m}\right)^2\left(\frac{2E}{m}\right)$



Question	Your Score	Total Points
1		15
2		15
3		15
4		15
5		15
		75

1. Pendulum and Blocks (15 points)



A pendulum (length L) consists of block 1 (mass $3M$) attached to the end of a string; this block is released from rest when the string is horizontal. Block 2 is at the edge of a table (height $2L$) and initially at rest.

Block 1 collides with block 2 (mass M) at the bottom of its swing *without touching the table*; this causes block 2 to launch horizontally from the table and land on the floor at a distance of $4L$ from the base of the table.

After the collision, block 1 continues forward and swings up to its highest point, making an angle of $\theta_{\max}$ with the vertical. Air resistance and friction are negligible. **Express all responses in terms of M, L, and physical constants as appropriate**.

a. (2 points) Determine the speed of block 1 at the bottom of its swing *before* it contacts block 2.

criterion		points
Indicates energy is conserved.	$\sum E_i = \sum E_f \qquad K_i + V_i = K_f + V_f \qquad V_i = K_f$	0.5
Correctly substitutes terms.	$mgh = \frac{1}{2}mv_{1i}^2 \qquad gL = \frac{1}{2}v_{1i}^2$	1.0
Correctly solves for speed.	$v_{1i}^2 = 2gL \qquad v_{1i} = \sqrt{2gL}$	0.5

b. (3 points) Derive an expression for the tension in the string when the string is vertical (before block 1 contacts block 2).

criterion		points
Addresses Newton's second law.	$\sum F = ma \qquad F_T - F_g = ma_c$	1
Correctly substitutes terms and solves for tension.	$F_T = mg + \frac{mv^2}{r} = 3Mg + \frac{(3M)\left(\sqrt{2gL}\right)^2}{L} = 9Mg$	2

Presume the length of the pendulum is $L = 75$ cm.

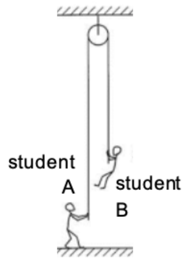
c. (4 points) Calculate the speed of block 1 just after it collides with block 2.

criterion		points
Indicates momentum is conserved.	$\sum p_i = \sum p_f \qquad m_1v_{1i} + m_2v_{2i} = m_1v_{1f} + m_2v_{2f}$	0.5
Correctly substitutes terms.	$(3M)v_{1i} + 0 = (3M)v_{1f} + (M)v_{2f}$	0.5
Calculates speed for block 2 as it leaves table…will need to calculate time for block 2 to leave the table and contact the floor to solve this. May use other kinematic equations.	$\Delta y = \frac{1}{2}at^2 + v_2t \qquad t = \sqrt{\frac{2\Delta y}{a}} = \sqrt{\frac{2(2L)}{g}} = \sqrt{\frac{4(0.75\text{ m})}{\left(9.8\frac{m}{s^2}\right)}} = 0.55\text{ s}$ $v_2 = \frac{\Delta x}{t} = \frac{4L}{t} = \frac{4(0.75\text{ m})}{0.55\text{ s}} = 5.45\frac{\text{m}}{\text{s}}$	2
Correctly solves for v_{1f} by substituting v_{2i} into conservation of momentum equation.	$v_{1f} = \frac{(3M)v_{1i} - (M)v_{2f}}{(3M)} = \frac{3v_{1i} - v_{2f}}{3} = \frac{3\sqrt{2gL} - \frac{4L}{t}}{3}$ $= \frac{3\sqrt{2\left(9.8\frac{m}{s^2}\right)(0.75\text{ m})} - 5.45\frac{\text{m}}{\text{s}}}{3} = 2.02\frac{\text{m}}{\text{s}}$	1

d. (6 points) Calculate the angle $\theta_{\max}$ the string makes with the vertical when block 1 is at its highest point after the collision.

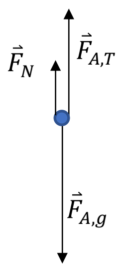
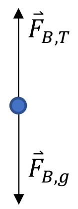
criterion		points
Indicates energy is conserved and identifies relevant non-zero terms.	$\sum E_i = \sum E_f \qquad K_i + V_i = K_f + V_f \qquad K_1 = V_2$	0.5
Correct substitutions of terms into conservation of energy equation.	$\frac{1}{2}mv^2 = mgh \qquad K_1 = \frac{1}{2}mv_{1f}^2 \qquad h = L(1 - \cos\theta)$ $\frac{1}{2}mv_{1f}^2 = mgL(1 - \cos\theta)$	3.5
Correctly solves for angle $\theta_{\max}$.	$1 - \cos\theta = \frac{mv_{1f}^2}{2mgL} = \frac{v_{1f}^2}{2gL} \qquad \theta = \cos^{-1}\left(1 - \frac{v_{1f}^2}{2gL}\right)$ $\theta = \cos^{-1}\left(1 - \frac{\left(2.02\frac{\text{m}}{\text{s}}\right)^2}{2\left(9.8\frac{m}{s^2}\right)(0.75\text{ m})}\right) = 43.5°$	2

2. Pulley (15 points)



Student A ($m = 70$ kg) stands on the ground and holds one end of a rope.
Student B ($m = 60$ kg) is at rest, hanging onto the other end of the rope above the ground.
The rope passes over a pulley that is attached to the ceiling. Presume that the masses of the rope and pulley are negligible.

a. (5 points) Clearly sketch and label the forces acting on student A and student B using the dots below. The lengths of the force vectors should be consistent with their relative magnitudes.

FBD for Student A	**FBD for Student B**	**criterion**	**points**
		Correctly sketches and labels tension, normal, and gravitational forces for student A.	1
		Sum of magnitudes for tension and normal forces for student A is equal to magnitude of gravitational force for student A.	2
		Correctly sketches and labels tension and gravitational forces for student B.	1
		Magnitudes of tension and gravitational forces for student B are equal and opposite to each other.	1

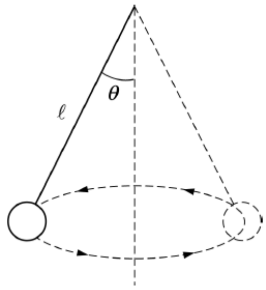
b. (5 points) Calculate the magnitude of the force exerted **by the floor on Student A.**

criterion		**points**
Equates rope tension to weight of student B.	$F_{B,T} = m_B g$	1
Writes correct expression for sum of forces on student A.	$\sum F_A = F_{A,T} + F_{A,N} - F_{A,g} = F_{A,T} + F_{A,N} - m_A g = 0$	1
Indicates tension in rope for student B is equal to tension in rope for student A.	$F_{B,T} = F_{A,T}$	1
Correctly solve for normal force.	$F_{A,N} = m_A g - F_{A,T} = m_A g - m_B g$	1
Correct calculation of normal force for student A.	$F_{A,N} = g(m_A - m_B) = \left(9.8\,\frac{\text{m}}{\text{s}^2}\right)(70\text{ kg} - 60\text{ kg}) = 98\text{ N}$	1

c. (5 points) Calculate the **minimum acceleration** student B must climb up the rope to lift student A up off the floor.

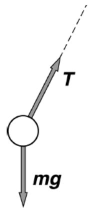
criterion		**points**
Writes Newton's second law for student B.	$\sum F_B = m_B a$	1
Writes correct expression for the sum of the forces on student B.	$\sum F_B = F_{B,T} - F_{B,g} = F_{B,T} - m_B g$	1
Correctly solves for acceleration.	$m_B a = F_{B,T} - m_B g \qquad a = \frac{F_{B,T}}{m_B} - \frac{m_B g}{m_B} = \frac{F_{B,T}}{m_B} - g$	1.5
Indicates minimum tension needed to lift student A is equal to the weight of student A.	$F_{A,T} = F_{B,T} = (70\text{ kg})\left(9.8\,\frac{\text{m}}{\text{s}^2}\right) = 686\text{ N}$	0.5
Correctly substitutes rope tension to solve for acceleration.	$a = \frac{F_{B,T}}{m_B} - g = \frac{686\text{ N}}{60\text{ kg}} - 9.8\,\frac{\text{m}}{\text{s}^2} = 1.63\,\frac{\text{m}}{\text{s}^2}$	1

3. Ball on a String (15 points)



A ball (mass m), attached to the end of a string (length ℓ), swings in a horizontal circle with a constant speed. The string makes an angle θ with the vertical. **Express responses in terms of the m, ℓ, forces, and physical constants as appropriate**.

a. (5 points) On the left ball in the diagram (solid lines), **sketch and label the forces** acting on it. Make sure the lengths of the force vectors are consistent with the relative magnitude of the forces.

criterion		points
	Tension and gravitational forces are appropriately labeled.	1
	Tension force sketched in direction that is upward and to the right.	2
	Gravitational force sketched in downward direction.	1
	Does not sketch any other force.	1

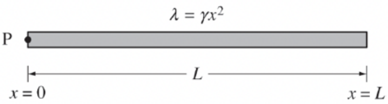
b. (3 points) Derive an expression for the mass of the ball.

criterion		points
Indicates that the sum of the vertical forces is zero.	$\sum F_y = 0$	1
Writes correct expression for the sum of the vertical forces.	$T\cos\theta - mg = 0$	1
Correctly determines expression for the mass of the ball.	$mg = T\cos\theta \qquad m = \dfrac{T\cos\theta}{g}$	1

c. (7 points) Derive an expression for the speed of the ball.

criterion		points
Indicates net force is horizontal component of tension and equates it to Newton's second law.	$\sum F = T\sin\theta = ma$	1
Indicates net acceleration is centripetal and substitutes it into Newton's second law.	$a = \dfrac{v^2}{r} \qquad T\sin\theta = m\dfrac{v^2}{r}$	1
Eliminate mass or tension from equation using previously derived relationships.	$T\sin\theta = \left(\dfrac{T\cos\theta}{g}\right)\dfrac{v^2}{r}$ $m\dfrac{v^2}{r} = mg\dfrac{\sin\theta}{\cos\theta}$	2
Indicates relationship between string length and radius.	$r = \ell\sin\theta$	1
Substitutes radius and correctly determines expression for speed.	$T\sin\theta = \left(\dfrac{T\cos\theta}{g}\right)\dfrac{v^2}{(\ell\sin\theta)}$ $m\dfrac{v^2}{(\ell\sin\theta)} = mg\dfrac{\sin\theta}{\cos\theta}$ $v = \sqrt{g\ell\sin\theta\tan\theta}$	2

4. Pivoting Rod (15 points)



A rod (mass M, length L) has a *non-uniform mass density* (λ) given by the $\lambda = \gamma x^2$, where $\gamma = 3M/L^3$ and x is the distance from pivot point P at the left end of the rod.

a. (3 points) Use integral calculus to show that the rotational inertia I of the rod about an axis perpendicular to the page and through pivot point P is $\frac{3}{5}ML^2$.

criterion		points
Uses integral expression for moment of inertia.	$I = \int r^2\, dm$	0.5
Uses correct limits of integration and substitutions into integral expression.	$I = \int x^2\,(\gamma x^2 dx) = \int_0^L \gamma x^4 dx$	1
Sets up integral correctly and calculates correctly.	$I = \gamma \left[\frac{x^5}{5}\right]_{x=0}^{x=L} = \left(\frac{3M}{L^3}\right)\left(\frac{L^5}{5} - 0\right) = \frac{3}{5}ML^2$	1.5

b. (3 points) Determine the horizontal location of the center of mass of the rod relative to pivot point P. Express your answer **only in terms of L**.

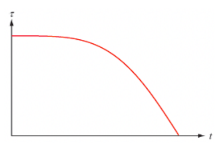
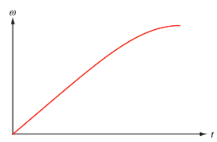
criterion		points
Identifies center-of-mass equation and correctly substitutes integrals.	$\vec{r}_{CM} = \frac{\sum m_i \vec{x}_i}{\sum m_i} = \frac{\int x\, dm}{\int dm}$	1
Correctly substitutes γx^2 into numerator and M into the denominator.	$\vec{x}_{CM} = \frac{\int x\, \lambda\, dx}{M} = \frac{\int x\,(\gamma x^2)\, dx}{M}$	1
Calculates correctly.	$\vec{x}_{CM} = \frac{\int_0^L \gamma x^3 dx}{M} = \frac{\gamma \left[\frac{x^4}{4}\right]_0^L}{M} = \frac{\left(\frac{3M}{L^3}\right)\left(\frac{L^4}{4}\right)}{M} = \frac{3}{4}L$	1

c. (2 points) **Circle one**: For an axis perpendicular to the page, the value of the rod's rotational inertia around pivot point P is (greater than / equal to / less than) the value of the rod's rotational inertia around its center of mass. Justify your choice.

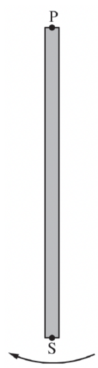
criterion	points
Selects "greater than."	0.5
Sample response could be an explanation that because more of the rod's mass is at the end opposite to point P, then more of the mass is concentrated away from the axis of rotation – which would mean that the rod's rotational inertia would be greater around point P than its center of mass.	1.5

The rod is released from rest in the position shown and starts to rotate about a horizontal axis perpendicular to the page and through pivot point P.

d. (4 points) Use the axes below to sketch graphs representing the magnitudes of the rod's net torque τ and the angular speed ω of the rod as functions of time t – **from the time the rod is released until the time its center of mass reaches its lowest point**.

criterion (2 points for each graph)			
	***τ* vs. *t***: concave downward curve with decreasing slope that approaches zero		***ω* vs. *t***: concave downward curve with increasing slope that approaches a horizontal (positive) value

e. (3 points) Presume the rod has a mass of 3.0 kg and length of 1.0 m. Calculate the linear speed v of point S as the rod swings through the vertical position shown in the diagram to the right.



criterion		points
Indicates energy is conserved.	$\sum E_i = \sum E_f \qquad K_i + V_i = K_f + V_f \qquad V_i = K_f$	1
Correctly substitutes terms.	$mgh_i = \frac{1}{2}I\omega_f^2 \qquad Mg\left(\frac{3}{4}L\right) = \frac{1}{2}\left(\frac{3}{5}ML^2\right)\left(\frac{v}{L}\right)^2$	1.5
Correctly solves for speed.	$v = \sqrt{\frac{5}{2}gL} = \sqrt{\frac{5}{2}\left(9.8\frac{m}{s^2}\right)(1\text{ m})} = 4.9\frac{\text{m}}{\text{s}}$	0.5

5. Binary System (15 points)

A planet orbits around a much heavier star with velocity v_1; there is a distance of r_1 between their centers of mass.

a. (5 points) Derive the velocity of this planet if it was at a distance of r_2 from the star (with respect to r_1). Presume the planetary orbit is circular for both instances.

criteria		points
Identifies relevant forces.	$\vec{F}_g = \frac{GMm}{r^2}$ $\quad \vec{F}_c = \frac{mv_c^2}{r}$	2
Equates forces to each other, recognizing both act towards the centers, and solves for v_1.	$\frac{GMm}{r_1^2} = \frac{mv_1^2}{r_1}$ $\quad v_1 = \sqrt{\frac{GM}{r_1}}$	2
Sets ratio between velocities to solve for v_2.	$v_2 = \sqrt{\frac{GM}{r_1}}$ $\quad \frac{v_2}{v_1} = \sqrt{\frac{r_1}{r_2}}$ $\quad v_2 = v_1\sqrt{\frac{r_1}{r_2}}$	1

b. (6 points) Presume the new orbit is elliptical, where the distance of closest approach is $2r_1$ and the distance of farthest approach is $4r_1$. Derive an expression for the relationship between their velocities **at both positions.**

criteria		points
Indicates relationship between closest and farthest approaches.	$\frac{2E}{m} = \lvert\vec{v}_c\rvert^2 - \frac{2GM}{r} = \lvert\vec{v}_f\rvert^2 - \frac{2GM}{r}$	2
Substitutes correct values for closest and farthest distances.	$\lvert\vec{v}_c\rvert^2 - \frac{2GM}{2r_1} = \lvert\vec{v}_f\rvert^2 - \frac{2GM}{4r_1}$ $\lvert\vec{v}_c\rvert^2 - \frac{GM}{r_1} = \lvert\vec{v}_f\rvert^2 - \frac{GM}{2r_1}$	2
Correctly solves for $\lvert\vec{v}_f\rvert$ or $\lvert\vec{v}_c\rvert$.	$\lvert\vec{v}_f\rvert = \sqrt{\lvert\vec{v}_c\rvert^2 - \frac{GM}{r_1} + \frac{GM}{2r_1}} = \sqrt{\lvert\vec{v}_c\rvert^2 - \frac{GM}{2r_1}}$ $\lvert\vec{v}_c\rvert = \sqrt{\lvert\vec{v}_f\rvert^2 - \frac{GM}{2r_1} + \frac{GM}{r_1}} = \sqrt{\lvert\vec{v}_f\rvert^2 + \frac{GM}{2r_1}}$	2

Note: alternate solutions possible

c. (4 points) Determine the period of the elliptical orbit from part b. Explain whether the period of the elliptical orbit is greater than, equal to, or less than the period of the circular orbit from part a**.**

criteria		points
Addresses relationship between period and radius – exact or proportional.	$T^2 = \frac{4\pi^2}{GM}R^3$ $\quad T^2 \propto R^3$	1
Uses major axis of elliptical orbit to determine radius of new circular orbit.	$2r_1 + 4r_1 = 6r_1$ ("diameter") $\quad r_{circle} = 3r_1$	2
Uses the new circle radius to determine that the period of its orbit would be ***larger***.	$T = \sqrt{R^3} = \sqrt{27} \approx 5.2$	1