Midterm Exam Mathematical Physics, Prof. G. Palasantzas

- Date: 23-05-2023
- Total number of points 100
- 10 points bonus for taking the exam
- For all problems, justify your answer

Problem 1 (25 points)

(a: 10 points) Is the series $\sum_{n=1}^{\infty}(n^2)^{(3/n)}$ convergent?

(b: 15 points) Is the series $\sum_{n=1}^{\infty}\frac{\cos^8(13nx)}{3+7^n}$ convergent?

Problem 2 (20 points)

Consider the series $\sum_{n=1}^{\infty} n\,\frac{(x-7)^n}{4^n}$.

(a: 5 points) Find the radius of convergence

(b: 15 points) Find the interval of convergence

Problem 3 (20 points)

If a, b, c are all positive constants and y(x) is a solution of the differential equation

$$ay'' + by' + cy = 0$$

then show that $\mathrm{Iim}_{x\to+\infty} y(x) = 0$

Problem 4 (25 points)

Consider a spring with spring constant k and a mass m attached to it ($k=m\omega^2$). The mass m is assumed to perform motion under the influence of an external force $F(t)=F_1\cos(2\omega t)+F_2\cos(3\omega t)$. The general equation of motion of the mass m reads of the form

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

In the absence of dissipation ($c=0$) find the particular solution of the equation of motion

Problem 1

Problem (1a)

$$a_n = (n^2)^{3/n} = n^{6/n}$$

Take the limit

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} n^{6/n} = \lim_{n\to\infty} e^{\frac{6}{n}\ln(n)} \Rightarrow$$

$$\lim_{n\to\infty} a_n = e^{6\lim_{n\to\infty} \frac{\ln(n)}{n}}$$

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = \underbrace{\lim_{n\to\infty} \frac{1/n}{1}}_{\text{L'hopitals rule}} = 0$$

Thus $\lim_{n\to\infty} a_n = e^0 = 1 \neq 0 \Rightarrow$ the series diverges!

Problem (1b)

$$|a_n| = \left| \frac{\cos^8(13nx)}{3+7^n} \right| \leq \frac{1}{3+7^n} < \frac{1}{7^n}$$

$\sum_{n=1}^{\infty} \frac{1}{7^n}$ is a convergent geometric series $\sum_{n=1}^{\infty} r^{n-1}$ with $r = \frac{1}{7} < 1$ since one can write

$$\sum_{n=1}^{\infty} \frac{1}{7^n} = \frac{1}{7} \sum_{n=1}^{\infty} \left(\frac{1}{7}\right)^{n-1} = \frac{1}{7} \frac{1}{1-\frac{1}{7}}$$

Thus, from the comparison test $\sum |a_n|$ is absolute convergent $\Rightarrow$ $\sum a_n$ is convergent!

Problem 2

a $a_n = \frac{n}{b^n}(x-a)^n$, where $b > 0$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)|x-a|^{n+1}}{b^{n+1}}\cdot\frac{b^n}{n|x-a|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{|x-a|}{b} = \frac{|x-a|}{b}.$$

By the Ratio Test, the series converges when $\frac{|x-a|}{b} < 1 \Leftrightarrow |x-a| < b \Leftrightarrow -b < x-a < b \Leftrightarrow$ $a-b < x < a+b$. When $|x-a| = b$, $\lim_{n\to\infty}|a_n| = \lim_{n\to\infty} n = \infty$, so the series diverges. Thus, $I = (a-b, a+b)$.

b=4, a=7

Problem 3

The auxiliary equation is $ar^2+br+c=0$.

If $b^2-4ac>0$, then any solution is of the form

$y(x)=c_1e^{r_1x}+c_2e^{r_2x}$ where $r_1=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$ and $r_2=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$.

But a , b , and c are all positive so both r_1 and r_2 are negative and $y(x)=0$

If $b^2-4ac=0$, then any solution is of the form

$y(x)=c_1e^{rx}+c_2xe^{rx}$ where $r=-b/(2a)<0$ since a , b are positive. Hence $y(x)=0$.

if $b^2-4ac<0$

then any solution is of the form $y(x)=e^{\alpha x}(c_1\cos\beta x+c_2\sin\beta x)$ where $\alpha=-b/(2a)<0$ since a and b are positive. Thus $y(x)=0$.

Note above: For the case b^2-4ac>0

r1 and r2 are both negative when you have two real roots because:

r2<0 because of the (-) sign for both positive terms

r1<0 because

$\sqrt{b^2-4ac} < b \iff b^2-4ac < b^2$ which is
true because $4ac>0$ so you have
$-b+\sqrt{b^2-4ac} < 0 \Rightarrow r1 < 0$

Problem 4

$$F(t) = \frac{F_1}{2}\sum_{n=\pm 1} e^{i\omega_1 t} + \frac{F_2}{2}\sum_{n=\pm 1} e^{i\omega_2 t}$$

Try a particular solution of the form

$$X_p(t) = \sum_{n=\pm 1} A_n e^{in\omega_1 t} + \sum_{n=\pm 1} B_n e^{in\omega_2 t} \quad (1)$$

$$X_p''(t) = \sum_{n=\pm 1} A_n\left[-(n\omega_1)^2\right] e^{in\omega_1 t} + \sum_{n=\pm 1} B_n\left[-(n\omega_2)^2\right] e^{in\omega_2 t}$$

Substitue into the equation or motion and you get

$$\sum_{n=\pm 1}\left[k - m(n\omega_1)^2\right] A_n e^{in\omega_1 t} + \sum_{n=\pm 1}\left[k - m(n\omega_2)^2\right] B_n e^{in\omega_2 t} =$$

$$= \frac{F_1}{2}\sum_{n=\pm 1} e^{in\omega_1 t} + \frac{F_2}{2}\sum_{n=\pm 1} e^{in\omega_2 t} \Rightarrow$$

$$\sum_{n=\pm 1}\left\{\left[k - m(n\omega_1)^2\right]A_n - \frac{F_1}{2}\right\} e^{in\omega_1 t} + \sum_{n=\pm 1}\left\{\left[k - m(n\omega_2)^2\right]B_n - \frac{F_2}{2}\right\} e^{in\omega_2 t} = 0$$

$$\Rightarrow \left[k - m(n\omega_1)^2\right]A_n - \frac{F_1}{2} = 0 \Rightarrow A_n = \frac{F_1}{2}\,\frac{1}{k - m(n\omega_1)^2} \quad (2)$$

$$\left[k - m(n\omega_2)^2\right]B_n - \frac{F_2}{2} = 0 \Rightarrow B_n = \frac{F_2}{2}\,\frac{1}{k - m(n\omega_2)^2} \quad (3)$$

since $\omega_1, \omega_2 \neq \omega \left(= \sqrt{k/m}\right)$ $k - m(n\omega_{1,2})^2 \neq 0$

(1), (2), (3) $\Rightarrow$

$$X_p(t) = \frac{F_1}{k - m\omega_1^2}\,\underbrace{\frac{1}{2}\left[e^{i\omega_1 t} + e^{-i\omega_1 t}\right]}_{\cos(\omega_1 t)} + \frac{F_2}{k - m\omega_2^2}\,\underbrace{\frac{1}{2}\left[e^{i\omega_2 t} + e^{-i\omega_2 t}\right]}_{\cos(\omega_2 t)}$$

$k = m\omega^2$

$$\boxed{\Rightarrow X_p(t) = \frac{F_1}{m(\omega^2 - \omega_1^2)}\cos(\omega_1 t) + \frac{F_2}{m(\omega^2 - \omega_2^2)}\cos(\omega_2 t)}$$

$$\boxed{\omega_1 = 2\omega\,,\quad \omega_2 = 3\omega}$$

You can also solve this problem, if you wish, using Fourier series

Take below as non-zero terms for the force i=2 and i=3

$$m x'' + kx = \sum_{i=1}^{100} F_{0i}\, e^{i w_i t} \quad , w_i \neq w_0 \;\; \forall\, i = 1, \cdots 100$$

$$w = \sqrt{k/m}$$

F_i : constant, $\quad w_i = i\, w_0$

Try solution $\quad x_p = \sum_{i=1}^{100} \tilde{A}_i\, e^{i w_i t}$

$$x_p' = \sum_{i=1}^{100} \tilde{A}_i\, (i w_i)\, e^{i w_i t}$$

$$x_p'' = \sum_{i=1}^{100} \tilde{A}_i\, (-w_i^2)\, e^{i w_i t}$$

substitution into the equation:

$$m \sum_{i=1}^{\infty} \tilde{A}_i\, (-w_i^2)\, e^{i w_i t} + k \sum_{i=1}^{\infty} \tilde{A}_i\, e^{i w_i t} = \sum_{i=1}^{100} F_{0i}\, e^{i w_i t}$$

$$\rightarrow \sum_{i=1}^{\infty} \left[(k - m w_i^2) \tilde{A}_i - F_{0i} \right] e^{i w_i t} = 0 \quad (1)$$

To separate the contributions of each w_i multiply (1) by $e^{-i w_j t}$ and then integrate from 0 to $T_0 = \frac{2\pi}{w_0}$

$$\sum_{i=1}^{100} \left[(k - m w_i^2) \tilde{A}_i - F_{0i} \right] \underbrace{\int_0^{\frac{2\pi}{w_0}} e^{i [w_i - w_j] t}\, dt}_{?} = 0$$

$$\int_0^{\frac{2\pi}{w_0}} e^{i [w_i - w_j] t}\, dt \; \begin{cases} \rightarrow \dfrac{2\pi}{w_0} & i = j \\[2ex] \rightarrow \dfrac{1}{i [w_i - w_j]} \Bigg[\underbrace{e^{i 2\pi \frac{w_i - w_j}{w_0}} - 1}_{e^{i 2\pi (i-j)} - 1 \,\equiv\, 0} \Bigg] & i \neq j \end{cases}$$

$$e^{i\,2\pi\frac{w_i - w_j}{w_0}} - 1 = e^{i\,2\pi(i-j)} - 1 = 0$$

$\Rightarrow$ only the $i=j$ survives $\Rightarrow$

$$(k - m\omega_i^2)\tilde{A}_i - F_{oi} = 0 \Rightarrow$$

$$\tilde{A}_i = \frac{F_{oi}}{k - m w_i^2}$$

$$X_p(t) = \sum_{i=1}^{100} \frac{F_{oi}}{k - m w_i^2} e^{i w_i t}$$

Real part $\Rightarrow X_p(t) = \sum_{i=1}^{100} \frac{F_{oi}}{k - m w_i^2} \cos(w_i t)$

$F_{o2}=F_1$, $F_{o3}=F_2$ $\omega_{2,3}=(2, 3)\omega$ and all other Foi=0 with i#2 and 3