Lecture 17. Ohm's law

1. The current in the circuit is the same all the way around

2. The current density $\vec{\mathbf{J}}$ is proportional to the force per unit charge $\vec{\mathbf{f}}$

$$\vec{\mathbf{J}} = \sigma\vec{\mathbf{f}}$$

The proportionality coefficient σ is called **conductivity**

3. Conductivity is a **material** property (i.e. depends solely on the material)

4. For perfect conductors $\sigma = \infty$. For perfect insulators, $\sigma = 0$

5. The reciprocal to σ value is called **resistivity**

$$\rho = \frac{1}{\sigma}$$

6. **Ohm's law** states that the current density $\vec{\mathbf{J}}$ is proportional to the electric field $\vec{\mathbf{E}}$ applied:

$$\vec{\mathbf{J}} = \sigma\vec{\mathbf{E}}$$

7. In many cases, the total current I flowing from one electrode to another, is proportional to the potential difference V between them:

$$I = \frac{V}{R}$$

with the (reciprocal) proportionality coefficient R called the **resistance**

8. The resistance R is the function of **geometry** and **material** of the conductor

9. Ohm's law holds because of frequent collisions of the current carriers (=electrons) with the ions of the lattice. This reduces the **free path** between the collisions to 10's of nm. As a results, the **drift velocity** of electrons (i.e. due to applied electric field) is much lower than their **thermal velocity** (due to thermal motion)

10. In many cases, the resistance R is proportional to temperature T. This proportionality breaks at low temperatures because of decreased thermal vibrations of the lattice ions which leads to a longer free pathway λ. At very low temperatures, the resistance might become zero (**superconductivity**)

Lecture 18. Ohm, emf, Faraday and Lenz

1. Collisions of the electrons with the lattice cause conversion of the work done by the electrical force, into heat in the conductor/resistor (the **Joule heating law**):

$$P = IV = \frac{V^2}{R}$$

2. The **work done per unit charge** to push it through the loop is called the **electromotive force** $\mathcal{E}$ (emf):

$$\mathcal{E} = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{l}} = \oint \frac{\vec{\mathbf{F}}}{q} \cdot d\vec{\mathbf{l}}$$

3. emf is **not** the force!

4. **Internal resistance** is that of the source of emf. Because of the internal resistance, the voltage across the terminals of the source of emf in a circuit is no longer the **open-circuit voltage** (i.e. the emf) but lower

5. **Motional emf** appears when one moves a wire through a magnetic field. This is by far the most common way to produce electricity

6. The **flux rule**: the emf $\mathcal{E}$ generated in the loop, is minus the rate of change of magnetic field flux Φ through the loop

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}}$$

7. **Lentz's rule** accounts for the minus sign in Faraday's law: the induced current flowS in such a direction that the magnetic flux it produces tends to cancel the ***change*** in flux

Lecture 19. Eddy currents, Faraday's law, inductance

1. **Faraday's law in the integral form**: circulation of the electric field along a contour equals to minus the rate of change of the magnetic flux though the surface limited by the contour:

$$\oint \vec{\mathbf{E}} \cdot d\vec{\boldsymbol{l}} = -\frac{d}{dt}\int_S \vec{\mathbf{B}} \cdot d\vec{\mathbf{a}}$$

2. **Faraday's law in the differential form**: Changing in time magnetic field creates electric field, which curl equals to minus time derivate of magnetic field

$$\nabla \times \vec{\mathbf{E}} = -\frac{\partial \vec{\mathbf{B}}}{\partial t}$$

3. Similarities of Faraday's law with Ampère's law allows for efficient solution of problems with build-in symmetry, applying the tricks associated with Ampere's law:

$$-\frac{d\Phi}{dt} \leftrightarrow \mu_0 I_{encl}$$

4. **Eddy currents** are loops of electrical current induced within conductors by a changing magnetic field in the conductor according to Faraday's law of induction

5. Eddy currents have numerous applications in industry, entertainment and household

6. The flux of magnetic field Φ_2 through a loop is proportional to the current I_1 run in another loop. The coefficient of proportionality is called **mutual inductance** M_{21}

$$\Phi_2 = I_1 M_{21}$$

7. Mutual inductance $M_{21}$is a **symmetric** ($M_{21} = M_{12}$ and therefore simply inductance) **and purely geometrical quantity** which depends on sizes, shapes, and relative positions of the loops.

Lecture 20. Transformers, back emf, energy of magnetic field

1. A **transformer** is a device based on Faraday’s induction, to increase ('step up') or decrease ('step down') voltage levels between circuits

2. If a current in a circuit is changed, there is a **back emf**

$$\mathcal{E}_{back} = -L\frac{dI}{dt}$$

induced in the circuit, which direction is such to oppose any change in the current

3. Inductance plays a similar role to mass in mechanical system: **the greater the inductance, harder it is to change the current**

4. Extra work W must be spent to overcome the "**back emf**”. This energy is stored in the coil

$$W = \frac{1}{2}LI^2$$

and is recoverable. The energy does not depend how the current has developed; only its final value matters

5. The regime in which magnetostatic rules can be used to calculate the magnetic field changing in time, is called **a quasistatic approximation**

6. More generally, the **energy stored in a current distribution $\vec{\mathbf{J}}$** is expressed as

$$W = \frac{1}{2}\int_{\mathcal{V}} \left(\vec{\mathbf{A}}\cdot\vec{\mathbf{J}}\right) d\tau$$

7. Another way to express the energy of **magnetic field is via the field $\boldsymbol{B}$ itself**:

$$W = \frac{1}{2\mu_0}\int_{all\ space} B^2 d\tau$$

8. The expressions above for **magnetic field energy** are extraordinarily **similar** to those for **electric field energy**

9. The ”**charging capacitor” paradox**: the magnetic field around a wire depends on the choice of a particular surface in Ampère’s law if the current is not stationary

Lecture 21. Maxwell

1. The ”**charging capacitor” paradox**: the magnetic field around a wire depends on the choice of a particular surface in Ampère’s law if the current is not stationary

2. “**Divergence problem**”: application of the divergence operator to Ampère’s law leads to a contradiction between the before-Maxwell equations for electric and magnetic fields

3. Maxwell suggested that, similarly to Faraday’s law, **changing in time electric field induces the magnetic field**

4. The existence of magnetic monopoles would symmetrized Maxwell equations with respect to electric and magnetic field but none have been found as yet

5. **Displacement current** (density) refers to an additional to the current density term, introduced by Maxwell

$$\vec{\mathbf{J}}_d \equiv \epsilon_0 \frac{\partial \vec{\mathbf{E}}}{\partial t}$$

6. Displacement current is not a real current as there are no charges that move. The name was coined historically; see "polarization current" below

7. Reminder from **Dielectric Materials**:
i). an electric polarization $\vec{\mathbf{P}}$ is the dipole moment per unit of volume
ii). Bound surface charge density $\sigma_b = \vec{\mathbf{P}} \cdot \hat{\boldsymbol{n}}$
iii). Bound charge density $\rho_b = -\boldsymbol{\nabla} \cdot \vec{\mathbf{P}}$
iv). Electric displacement: $\vec{\mathbf{D}} \equiv \epsilon_0 \vec{\mathbf{E}} + \vec{\mathbf{P}}$

8. Reminder from **Magnetic Materials**:
i). Magnetization $\vec{\mathbf{M}}$ is the magnetic dipole moment per unit of volume
ii). Bound surface current $\vec{\mathbf{K}}_b = \vec{\mathbf{M}} \times \hat{\boldsymbol{n}}$
iii). Bound volume current $\vec{\mathbf{J}}_b = \boldsymbol{\nabla} \times \vec{\mathbf{M}}$
iv). The auxiliary (magnetic) field: $\vec{\mathbf{H}} \equiv \frac{1}{\mu_0}\vec{\mathbf{B}} - \vec{\mathbf{M}}$

9. Only one of the equations needs modification for matter: Ampère law $\vec{\boldsymbol{\nabla}} \times \vec{\mathbf{H}} = \vec{\mathbf{J}}_f$

10. **Polarization current** (density) refers to additional "current" due to movement of bound charge:

$$\vec{\mathbf{J}}_p = \frac{\partial \vec{\mathbf{P}}}{\partial t}$$

11. The polarization current has nothing to do with the *bound* current $\vec{\mathbf{J}}_b = \boldsymbol{\nabla} \times \vec{\mathbf{M}}$. The polarization current appears as a reaction of bound charges to applied electric field which changes with time: the bound charges are slightly *displaced* from their previous positions

12. Maxwell's equations in matter **must be supplemented** with the appropriate **constitutive relations**, giving $\vec{\mathbf{D}}$ and $\vec{\mathbf{H}}$ in terms of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$. These should be derived upon considering interactions of the electromagnetic field with a particular material, typically performed at semi-classical or quantum-mechanical level.

13. Maxwell's equations in **linear isotropic media** are similar to those in vacuum with substitution

$$\mu_0 \epsilon_0 \leftrightarrow \mu \epsilon$$

14. B**oundary conditions** take care of the **fields separated by an interface** (which in general might carry charge and current)

15. The boundary conditions are derived as follows

$$\begin{aligned} D_1^{\perp} - D_2^{\perp} &= \sigma_f \qquad & \vec{\mathbf{E}}_1^{\parallel} - \vec{\mathbf{E}}_2^{\parallel} &= \vec{\mathbf{0}} \\ B_1^{\perp} - B_2^{\perp} &= 0 \qquad & \vec{\mathbf{H}}_1^{\parallel} - \vec{\mathbf{H}}_2^{\parallel} &= \vec{\mathbf{K}}_f \times \hat{\mathbf{n}} \end{aligned}$$

16. For linear isotropic media, the boundary conditions become

$$\epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \qquad \vec{\mathbf{E}}_1^{\parallel} = \vec{\mathbf{E}}_2^{\parallel}$$
$$B_1^{\perp} = B_2^{\perp} \qquad \frac{1}{\mu_1}\vec{\mathbf{B}}_1^{\parallel} = \frac{1}{\mu_2}\vec{\mathbf{B}}_2^{\parallel}$$

17. The equations for boundary conditions are the basis for the theory of **reflection and refraction** of electromagnetic waves

Lecture 22. Conservation of charge and energy

1. The **continuity equation**

$$\frac{\partial \rho}{\partial t} = -\boldsymbol{\nabla} \cdot \vec{\mathbf{J}}$$

is a mathematical formulation of the local charge conservation law

2. **The conservation of charge** law is already contained in Maxwell's equations so that in this regard it **is not an independent law**

3. **The work dW done by EM forces** to move charges around **in time** dt

$$\frac{dW}{dt} = \int_{\mathcal{V}} \left(\vec{\mathbf{E}} \cdot \vec{\mathbf{J}}\right) d\tau$$

4. To express the work in terms of EM fields only, Maxwell's equations are used to **eliminate $\vec{\mathbf{J}}$ from the expression above**

5. **Poynting's theorem** states that the work done on the charges by the EM force is equal to the decrease in energy remaining in the fields, less the energy that flowed out through the surface

$$\frac{dW}{dt} = -\frac{d}{dt}\left\{\int_{\mathcal{V}} u \, d\tau\right\} - \oint_{\mathcal{S}} \vec{\mathbf{S}} \cdot d\vec{\mathbf{a}}$$

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right); \; \vec{\mathbf{S}} \equiv \frac{1}{\mu_0}\left(\vec{\mathbf{E}} \times \vec{\mathbf{B}}\right)$$

6. The **Poynting vector $\vec{\mathbf{S}}$** is the energy per unit time, per unit area, transported by the fields

7. The Poynting vector $\vec{\mathbf{S}}$ is also the **energy flux density**

8. **Local conservation of EM energy** (no work on charges):

$$\frac{\partial u}{\partial t} = -\boldsymbol{\nabla} \cdot \vec{\mathbf{S}}$$

which is similar to the charge conservation law

Lecture 23. Conservation of momentum

1. EM field itself carries momentum

2. **Force per unit volume** acting on a moving distribution of charges

$$\vec{\mathbf{f}} = \epsilon_0\left[(\nabla\cdot\vec{\mathbf{E}})\vec{\mathbf{E}} + (\vec{\mathbf{E}}\cdot\nabla)\vec{\mathbf{E}} - \frac{1}{2}\nabla(E^2)\right] + \frac{1}{\mu_0}\left[(\nabla\cdot\vec{\mathbf{B}})\vec{\mathbf{B}} + (\vec{\mathbf{B}}\cdot\nabla)\vec{\mathbf{B}} - \frac{1}{2}\nabla(B^2)\right] - \epsilon_0\mu_0\frac{\partial}{\partial t}\left[\frac{1}{\mu_0}(\vec{\mathbf{E}}\times\vec{\mathbf{B}})\right]$$

consists of the contribution from electric field, the contribution from magnetic field and the time-derivative of their cross-product (the Poynting vector)

3. The **Maxwell stress tensor** $\overleftrightarrow{\mathbf{T}}$ allows for more compact expression for force $\vec{\mathbf{F}}$

4. An operation that converts one vector (force) into another (reaction) is called **a tensor**

5. If we know the stress tensor $\overleftrightarrow{\mathbf{T}}$, the force $\vec{\mathbf{F}}$ excerted upon any volume, can be calculated by integrating the stress tensor over the surface which encloses the volume

$$\vec{\mathbf{F}} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}}\cdot d\vec{\mathbf{a}}$$

6. We **define** components of the Maxwell stress tensor $\overleftrightarrow{\mathrm{T}}$ as

$$T_{ij} \equiv \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right)$$

7. With this definition, we can show that

$$\vec{\mathbf{F}} = \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}}\cdot d\vec{\mathbf{a}} - \epsilon_0\mu_0\frac{d}{dt}\int_{\mathcal{V}} \vec{\mathbf{S}}\, d\tau$$

where $\vec{\mathbf{S}}$ is the Poynting vector

8. **Electromagnetic momentum density** is defined as $\vec{\mathbf{g}} = \epsilon_0\mu_0\vec{\mathbf{S}}$

9**. Conservation of momentum law** in electrodynamics: if the mechanical momentum $\vec{\mathbf{p}}_{mech}$ increases, either the field momentum decreases, or else the fields are carrying momentum into the volume through the surface

$$\frac{d\vec{\mathbf{p}}_{mech}}{dt} = -\frac{d}{dt}\int_{\mathcal{V}} \vec{\mathbf{g}}\, d\tau + \oint_{\mathcal{S}} \overleftrightarrow{\mathbf{T}}\cdot d\vec{\mathbf{a}}$$

10. **The local conservation of field momentum:** the change of electromagnetic momentum density equals divergence of Maxwell's stress tensor

$$\frac{\partial\vec{\mathbf{g}}}{\partial t} = \nabla\cdot\overleftrightarrow{\mathbf{T}}$$

11. The Poynting vector $\vec{\mathbf{S}}$ has appeared in **two quite different roles**:

- $\vec{\mathbf{S}}$ is the energy per unit area, per unit time, transported by the electromagnetic fields (energy flux density)
- $\epsilon_0\mu_0\vec{\mathbf{S}} = \vec{\mathbf{g}}$ is the momentum per unit volume stored in the electromagnetic fields

12. The Maxwell tensor $\overleftrightarrow{\mathbf{T}}$ has appeared in **two quite different roles**:

- $\overleftrightarrow{\mathbf{T}}$ is the electromagnetic stress (force per unit area) acting on a surface of the volume under consideration
- $-\overleftrightarrow{\mathbf{T}}$ describes the flow of momentum (**the momentum current density**) carried by the electromagnetic fields

13. All conservation laws are already contained in Maxwell's equations; we simply made them more explicit – and more convenient for calculations

Lecture 24. Waves

1. A **wave** is a physical phenomenon which amplitude obeys a **wave equation**

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{\mathrm{v}^2}\frac{\partial^2 f}{\partial t^2}$$

where v is the **speed of propagation**

2. Mathematically, the wave equation is a **second-order linear homogeneous differential** equation

3. A **general solution of the wave equation:**

$$f(z,t) = g(z \pm \mathrm{v}t)$$

where only this combination $z \pm \mathrm{v}t$ of the two variables z and t matters

4. The minus sign in the argument $z - \mathrm{v}t$ corresponds to the wave travelling to the $+z$ direction. The plus sign in the argument $z + \mathrm{v}t$ corresponds to the wave travelling to the $-z$ direction

5. **Harmonic (sinusoidal, monochromatic) waves** are describes as

$$f(z,t) = A\,cos[k(z - \mathrm{v}t) + \delta]$$

where k is **the wave number** and δ is **the phase constant** ($0 \le \delta \le 2\pi$).

6. The **wavelength** λ is one period of the wave in the z-direction:

$$\lambda = \frac{2\pi}{k}$$

7. The **period** T is one full cycle in time:

$$T = \frac{2\pi}{k\mathrm{v}}$$

8. The **frequency** ν (aka **linear frequency**) is inversed of the period:

$$\nu = \frac{1}{T} = \frac{\mathrm{v}}{\lambda}$$

9. The **angular frequency** $\omega = 2\pi\nu$ is introduced for not dragging a factor of 2π in calculations

10. **Complex wave function** $\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)}$ is related to a*ctual* wave function as $f(z,t) = Re\left[\tilde{f}(z,t)\right]$

11. Complex notations are introduced for conveniency of the bookkeeping

12. In a **longitudinal wave** the particle displacement is parallel to the direction of wave propagation. In a **transverse wave** the particle displacement is perpendicular to the direction of wave propagation

Lecture 25. EM waves

1. The **wave equation of EM waves**

$$\nabla^2\vec{\mathbf{E}} = \mu_0\epsilon_0\frac{\partial^2\vec{\mathbf{E}}}{\partial t^2};\;\; \nabla^2\vec{\mathbf{B}} = \mu_0\epsilon_0\frac{\partial^2\vec{\mathbf{B}}}{\partial t^2}$$

decouples electric $\vec{\mathbf{E}}$ and magnetic $\vec{\mathbf{B}}$ fields at expense of raising the order from 1st to 2nd

2. Wave equations are derived for **empty space** which implies that no carrier (i.e. the medium in which the wave propagates) is needed

3. All EM waves propagate in free space with identical speed of

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

aka **speed of light**

4. **Hertz's experiments** on propagation of EM waves were the first experimental demonstration of validity of Maxwell's theory

Lecture 26. Plain EM waives. Radiation intensity and radiation pressure.

1. **Plane wave** approximation: the wave travels into the *z*-direction with no *x*- or *y*-dependence

$$\vec{\mathbf{E}}(x, y, z, t) = \vec{\mathbf{E}}(z, t);\ \vec{\mathbf{B}}(x, y, z, t) = \vec{\mathbf{B}}(z, t)$$

2. **Plane monochromatic wave** approximation: both field oscillate as sinusoidal waves

$$\tilde{\vec{\mathbf{E}}}(z, t) = \tilde{\vec{\mathbf{E}}}_0 e^{i(kz-\omega t)};\ \tilde{\vec{\mathbf{B}}}(z, t) = \tilde{\vec{\mathbf{B}}}_0 e^{i(kz-\omega t)}$$

3. **Properties of plain monochromatic EM waves**

- $\vec{\mathbf{E}}$, $\vec{\mathbf{B}}$ and the direction of propagation are mutually perpendicular
- EM waves are transverse
- $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ oscillate in phase
- Amplitudes of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are related as $\mathrm{B}_0 = \mathrm{E}_0/\mathrm{c}$

or in short:

$$\tilde{\vec{\mathbf{E}}}(\mathbf{r}, t) = \tilde{E}_0 e^{i(\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}-\omega t)}\ \hat{\mathbf{n}};\ \ \tilde{\vec{\mathbf{B}}}(\mathbf{r}, t) = \frac{1}{c}\left(\hat{\vec{\mathbf{k}}} \times \tilde{\vec{\mathbf{E}}}\right)$$

4. Polarization of EM waves is the plane of vibrations of $\vec{\mathbf{E}}$. $\hat{\mathbf{n}}$ is called the **polarization vector**

5. The intensity I of polarized light after passing through a polarizer is $I = I_0 \cos^2\theta$, where I_0 is the original intensity and θ is the angle between the direction of polarization and the axis of the filter

6. In the case of *light,* the period so brief (10^{-15} s), that any macroscopic measurement will encompass many cycles of the EM field. That's why we deal with **time-averaged values**

7. The **average energy density** stored in electromagnetic fields

$$\langle u \rangle = \frac{1}{2}\epsilon_0 E_0^2$$

8. The **average power per unit area** transported by an electromagnetic wave is called the **intensity**

$$I \equiv \langle S \rangle = \frac{1}{2} c\epsilon_0 E_0^2$$

9. The **radiation pressure** (average force per unit area) of an EM wave:

$$P = \frac{I}{c}$$

10. Johannes **Kepler** put forward the concept of radiation pressure back in 1619 to explain the observation that a tail of a comet always points away from the Sun. Piotr **Lebedev** was the first to measure the radiation pressure in 1899

11. Radiation pressure from Sun on Earth's surface is extremely weak: 4.5 $\mu N/m^2$

12. Light pressure at the target in the National Ignition Facility (USA) approaches 10^{16} N/m^2 which is 11 decimal orders of magnitude higher than the atmospheric pressure

Lecture 27. Electromagnetic Waves in Materials. Normal incidence at Interfaces

1. Wave equations in linear isotropic matter are identical those in vacuum, with $\mu_0\epsilon_0 \leftrightarrow \mu\epsilon$

2. The propagation speed becomes

$$\mathrm{v} = \frac{c}{n},$$

where

$$n \equiv \sqrt{\frac{\mu\epsilon}{\mu_0\epsilon_0}}$$

is the **index of refraction**

3. General approach to EM waves at an interface:
(i) Write *total* $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ at both sides of the interface
(ii) stitch $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ at the interface via the boundary conditions
(iii) calculate the transmission/reflection coefficients

4. The frequency does not change after the interface; the wavenumber $k = \omega n/c$ does

5. The **reflection coefficient** R is the ratio of the intensity of the reflected wave over the intensity of the incident wave

6. The **transmission coefficient** T is the ratio of the intensity of the transmitted wave over the intensity of the incident wave

7. $R + T = 1$ (conservation of energy)

Lecture 28. Electromagnetic Waves at Interfaces: Oblique Incidence

1. Three fundamental laws of geometrical optics (**plane of incidence, reflection and refraction**) can be derived from Maxwell's equations

2. There are two options for polarization of the EM waves at an interface: **parallel** (*p*) and **orthogonal** (*s*) to plane of incidence

3. For parallel polarization, the transmitted wave is always in phase with the incident one, the reflected wave is either in phase or 180° out of phase

4. For orthogonal polarization, the transmitted wave is always in phase with the incident one, the reflected wave is 180° out of phase.

5. Four **Fresnel's equations** provide connections between incident, reflected and refracted waves at oblique incidence

6. For parallel polarization, the reflected wave totally vanishes at **Brewster's angle** of incidence

$$\theta_B = atan\left(\frac{n_2}{n_1}\right)$$

7. Brewster's angle provides an efficient means of **suppression of the reflected radiation** from e.g. a window or a wet road surface with a polarizer

Lecture 29. Electromagnetic Waves at Interfaces. Potentials and Fields.

1. If $n_1 > n_2$ (e.g., the light travels from water to air), the incident beam might be totally reflected off the surface if the angle of incidence exceeds a critical value

$$\theta_I = \arcsin\left(\frac{n_2}{n_1}\right)$$

This angle is called the **angle of total internal reflection**

2. Brewster's angle provides an efficient means of **suppression of the reflected radiation** from e.g. a window or a wet road surface with a polarizer

3. **Metamaterials** are materials with both $\epsilon_r < 0$ and $\mu_r < 0$ – and therefore $n < 0$

4. **Optical tweezers** are scientific instruments that use a highly focused laser beam to hold and move microscopic objects in a manner similar to tweezers

<u>Potentials and Fields</u>

1. We can use an old definition of the magnetic potential as

$$\vec{\mathbf{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\mathbf{A}}$$

because there are no magnetic charges. However, the scalar potential should be redefined as

$$\vec{\mathbf{E}} = -\vec{\boldsymbol{\nabla}} V - \frac{\partial \vec{\mathbf{A}}}{\partial t}$$

because $\vec{\mathbf{E}}$ is no longer a conservative field

2. Potentials are more useful than the fields when charges and currents move around, and we have to account for the final propagation speed of their disturbances

3. Potentials are defined on basis of **noname and Faraday** equations so these two are automatically taken care of. The other two Maxwell's equations (**Gauss' and Ampère-Maxwell's** laws) form the basis for inhomogeneous wave equations

4. The **potentials are not define uniquely**; there are an infinite number of ways to define the potentials for a certain field

5. **Gauge transformations** impose extra conditions upon the potentials

6. The **Coulomb gauge $\vec{\boldsymbol{\nabla}} \cdot \vec{\mathbf{A}} = 0$** is widely used in magnetostatics as the problem of finding $\vec{\mathbf{E}}$ reduces to solving Poisson's equation

$$\nabla^2 V = -\frac{1}{\epsilon_0}\rho \text{ with a known solution } V(\vec{\boldsymbol{r}}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\boldsymbol{r}}', t)}{\mathscr{r}} d\tau' \;\; (V(\infty) = 0)$$

7. In electrodynamics, **the Lorenz gauge** is more convenient

$$\vec{\nabla}\cdot\vec{\mathbf{A}} = -\mu_0\epsilon_0\frac{\partial V}{\partial t}$$

8. The problem of finding $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ is now reduced to solving two **inhomogeneous wave equations** with the source term on the right:

$$\Box^2 V = -\frac{1}{\epsilon_0}\rho; \quad \Box^2\vec{\mathbf{A}} = -\mu_0\vec{\mathbf{J}}$$

where $\Box^2 \equiv \nabla^2 - \mu_0\epsilon_0\frac{\partial^2}{\partial t^2}$ is **d'Alembertian operator**

9. The generalized solutions are called **retarded potentials**:

$$V(\vec{\mathbf{r}},t) = \frac{1}{4\pi\epsilon_0}\int\frac{\rho(\vec{\mathbf{r}}',t_r)}{\mathscr{r}}d\tau', \qquad \vec{\mathbf{A}}(\vec{\mathbf{r}},t) = \frac{\mu_0}{4\pi}\int\frac{\vec{\mathbf{J}}(\vec{\mathbf{r}}',t_r)}{\mathscr{r}}d\tau'$$

as they relate potentials at time t to the configuration of charges and currents at the **retarded** (=preceding) **time** $t_r \equiv t - \frac{\mathscr{r}}{c}$.

10. If you apply the logic of retarded time to the fields, you'll get the entirely wrong answer!

11. The expressions for $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$, besides retarded times, contain **time-derivatives** of ρ and $\vec{\mathbf{J}}$ (Jefimenko's equations):

$$\vec{\mathbf{E}}(\vec{\mathbf{r}},t) = \frac{1}{4\pi\epsilon_0}\int\left[\frac{\rho(\vec{\mathbf{r}}',t_r)}{\mathscr{r}^2}\hat{\mathscr{r}} + \frac{1}{c\mathscr{r}}\frac{\partial\rho(\vec{\mathbf{r}}',t_r)}{\partial t_r}\hat{\mathscr{r}} - \frac{1}{c^2\mathscr{r}}\frac{\partial\vec{\mathbf{J}}(\vec{\mathbf{r}}',t_r)}{\partial t_r}\right]d\tau'$$

$$\vec{\mathbf{B}}(\vec{\mathbf{r}},t) = \frac{\mu_0}{4\pi}\int\left[\frac{\vec{\mathbf{J}}(\vec{\mathbf{r}}',t_r)}{\mathscr{r}^2} + \frac{1}{c\mathscr{r}}\frac{\partial\vec{\mathbf{J}}(\vec{\mathbf{r}}',t_r)}{\partial t_r}\right]\times\hat{\mathscr{r}}\,d\tau'$$

12. The (quasi)static approximation we used earlier, works much better than we had any right to expect because the two errors involved – neglecting time retardation and the time-dependent terms – **cancel each other** to the first approximation.

Lecture 30. Radiation

1. When charges accelerate, their fields can transport energy irreversibly out to infinity – a process we call **radiation**

2. Static sources do not radiate but the sources which involve time derivatives $\dot{\rho}$ and $\dot{\vec{\mathbf{J}}}$, do

3. While considering the radiation we should pick up the parts of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ that decrease like r^{-1} at large distances from the source, to obtain the r^{-2} dependence in $\vec{\mathbf{S}}$

4. Properties of the dipole radiation:
- The dipole oscillating at frequency ω, radiates a monochromatic wave at frequency ω
- Waves travel in the radial direction at the speed of light
- $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are in phase
- $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ are mutually perpendicular and transverse
- The ratio of the amplitudes is $E_0/B_0 = c$
- The wave amplitude decreases like $1/r$ – this is called a **spherical wave**

5. The (time-averaged) **intensity** emitted by the oscillation dipole

$$\langle \vec{\mathbf{S}} \rangle = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\left(\frac{sin^2\theta}{r^2}\right)\hat{\mathbf{r}}$$

- The intensity profile takes the form of a donut
- The maximum is in the equatorial plane (where $\sin\theta = 1$)
- No radiation along the *axis* of the dipole (where $\sin\theta = 0$)
- The intensity has strong frequency dependence: ω^4

Lecture 31. Radiation

1. The **power** radiated by a dipole

$$\langle P \rangle = \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$

2. The ω^4 dependence results in more intense radiation in the blue than in the red (**blue sky**) and stronger scattering of the blue colors (**red sunset**)

3. Because dipoles do not radiate along the oscillation axis, light received by the observer is polarized perpendicular to the sun's rays (**polarized radiation from the blue sky**)

4. The same phenomenon is responsible for **Brewster's angle**: the dipoles behind the interface do not re-emit light at the direction perpendicular to the refracted wave (for *p*-polarized light)

5. Accelerated a (non-relativistic) point charge q radiates power as an EM wave according to the **Larmor formula:**

$$P = \frac{\mu_0}{6\pi c}\ddot{p}^2 = \frac{\mu_0 q^2 a^2}{6\pi c}$$

where $\ddot{p}$ is the second time-derivative of the dipole moment and a is the acceleration

6. Radiation emitted by an accelerated (relativistic) charge is called the **synchrotron radiation**.

7. Synchrotron radiation is used to produce radiation of wavelengths from 100 µm to 0.1 nm in linear accelerators

8. Synchrotron radiation is believed to be generated by **astronomical objects**, typically when relativistic electrons spiral through magnetic fields

Lecture 32. Electrodynamics and relativity

1. First Einstein's postulate (aka the **principle of relativity**): The laws of physics apply in all inertial reference systems. This elevates Galileo's observation about classical mechanics to the status of a general law, applying to *all* of physics.

2. Second Einstein's postulate (aka the **universal speed of light):** The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.

3. Unlike Newtonian mechanics, classical electrodynamics is *already* consistent with special relativity – in fact, it led Einstein to his formulation of the special relativity theory.

4. **Charge is invariant**. The charge of a particle is a fixed number, independent of how fast it happens to be moving

5. Considering a number of simple situations allows us to learn how the fields are transformed

6. The orthogonal components (i.e. *perpendicular* to the direction of motion of $\mathcal{S}$) of the electric field are scaled with a γ_0 factor

$$\vec{\mathbf{E}}^{\perp} = \gamma_0 \vec{\mathbf{E}}_0^{\perp}$$

where $\gamma_0 = 1/\sqrt{1 - v_0^2/c^2}$

7. Parallel component of the electric field is invariant

$$E^{\parallel} = E_0^{\parallel}$$

8. The $\vec{\mathbf{E}}$ field of a charge points away from its *instantaneous* (as opposed to the retarded) position because the γ_0-factor from Lorentz transformations of the *x*-coordinate balances the γ_0-factor from field transformation of $\vec{\mathbf{E}}^{\perp}$.

9. To derive the general rule of $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$ transformations, we must start out in a system where there are both of them. For this purpose, the system $\mathcal{S}$ (which moves to the right with speed v_0) itself will serve nicely.

10. But now we need the third system $\bar{\mathcal{S}}$ traveling to the right with speed v relative to $\mathcal{S}$, where we need to find the transformed fields

Lecture 33. Electrodynamics and relativity

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2. But now we need the third system $\bar{\mathcal{S}}$ traveling to the right with speed v relative to $\mathcal{S}$, where we need to find the transformed fields

3. **Transformation of the EM fields** are described by the following relations:

$$\bar{E}_x = E_x; \bar{E}_y = \gamma\left(E_y - \mathrm{v}B_z\right); \quad \bar{E}_z = \gamma\left(E_z + \mathrm{v}B_y\right)$$

$$\bar{B}_x = B_x;\ \bar{B}_y = \gamma\left(B_y + \frac{\mathrm{v}}{c^2}E_z\right);\ \bar{B}_z = \gamma\left(B_z - \frac{\mathrm{v}}{c^2}E_y\right)$$

with $\gamma = \dfrac{1}{\sqrt{1 - \mathrm{v}^2/c^2}}$

4. The components of the fields parallel to motion, stay unchanged while the orthogonal components are interwound.

5. If in any system $\vec{\mathbf{B}} = 0$, then

$$\vec{\bar{\mathbf{B}}} = -\frac{1}{c^2}\left(\vec{\mathbf{v}} \times \vec{\bar{\mathbf{E}}}\right)$$

6. If in any system $\vec{\mathbf{E}} = 0$, then

$$\vec{\bar{\mathbf{E}}} = \vec{\mathbf{v}} \times \vec{\bar{\mathbf{B}}}$$

7. EM field of a point **charge in uniform motion**:

$$\vec{\mathbf{E}}(\vec{\mathbf{r}}, t) = \frac{q}{4\pi\epsilon_0}\frac{1 - \mathrm{v}^2/c^2}{(1 - (\mathrm{v}^2/c^2)\ sin^2\theta)^{3/2}}\frac{\hat{\mathbf{R}}}{R^2}$$

$$\vec{\mathbf{B}} = \frac{1}{c^2}\left(\vec{\mathbf{v}} \times \vec{\mathbf{E}}\right)$$

8. The **Lorentz transformation matrix**:

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\beta \equiv \mathrm{v}/c$ and $\gamma \equiv (1-\beta^2)^{1/2}$

9. **Einstein summation convention:** summation is implied whenever a Greek index is repeated in a product

$$\bar{x}^\mu = \Lambda^\mu_\nu x^\nu \text{ stands for } \bar{x}^\mu = \sum_{\nu=0}^{3} \left(\Lambda^\mu_\nu\right) x^\nu \text{ with } \mu = 0,1,2,3$$

10. A **4-vector** a^ν is any set of *four* components that transform in the same manner as the **coordinates** under Lorentz transformations:

$$\bar{a}^\mu = \Lambda^\mu_\nu a^\nu$$

11. **Proper velocity**

$$\eta^\mu \equiv \frac{dx^\mu}{d\tau}$$

and **relativistic momentum** $\vec{\mathbf{p}} \equiv m\vec{\boldsymbol{\eta}}$ are the 4-vectors

Lecture 34. Electrodynamics and relativity

1. **The field tensor** $F^{\mu\nu}$ and t**he dual field tensor** $G^{\mu\nu}$are antisymmetric second-rank tensors that obey Lorentz's transformations, and represent electromagnetic field in relativistic notations:

$$F^{\mu\nu} = \begin{Bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & B_z & -B_y \\ -E_y/c & -B_z & 0 & B_x \\ -E_z/c & B_y & -B_x & 0 \end{Bmatrix} \qquad G^{\mu\nu} = \begin{Bmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z/c & E_y/c \\ -B_y & E_z/c & 0 & -E_x/c \\ -B_z & -E_y/c & E_x/c & 0 \end{Bmatrix}$$

2. Charge density and current density are combined in **a current density four-vector**

$$J^\mu = \rho_0 \eta^\mu = \left(c\rho,\ J_x,\ J_y,\ J_z\right)$$

3. The current density four-vector is **divergence-less**

$$\frac{\partial J^\mu}{\partial x^\mu} = 0$$

4. The laws of electrodynamics (Maxwell's equations) in tensor notations:

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu} = \mu_0 J^\mu;\ \frac{\partial G^{\mu\nu}}{\partial x^\nu} = 0$$

5. *V* and $\vec{\mathbf{A}}$ together constitute a **four-vector potential**:

$$A^\mu = \left(V/c\ ,\ A_x, A_y, A_z\right)$$

6. The fact that A^μ is a four-vector, makes it more convenient than the fields or field tensors.

7. The **Lorenz gauge in relativistic notations becomes**

$$\frac{\partial A^\nu}{\partial x^\nu} = 0$$

8. The potential definition automatically takes care of the homogeneous Maxwell equation (based on the dual-filed tensor $G^{\mu\nu}$)

9. *Upper* indices designate ***contravariant*** **vectors** as e.g. $x^\mu = (x^0, x^1, x^2, x^3) = (ct,\ x, y, z)$

10. *Lower* indices are for ***covariant* vectors** $x_\mu = (x_0, x_1, x_2, x_3) \equiv (-x^0, x^1, x^2, x^3) = (-ct,\ x, y, z)$

11. Raising or lowering the temporal index costs a minus sign in $x_0 = -x^0$. Raising or lowering a spatial index changes nothing in $x_1 = x^1, x_2 = x^2, x_3 = x^3$

12. The field tensor $F^{\mu\nu}$ can be written via the four-vector potential A^μ as

$$F^{\mu\nu} = \frac{\partial A^\nu}{\partial x_\mu} - \frac{\partial A^\mu}{\partial x_\nu}$$

Note that the differentiation is with respect to the *covariant* vectors ∂x_μ and ∂x_ν

13. The **d'Alembertian in relativistic notations** becomes

$$\Box^2 \equiv \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \frac{\partial}{\partial x_\nu}\frac{\partial}{\partial x^\nu}$$

Note that one differentiation is with respect to the *covariant* vector ∂x_μ and another and with respect to the *contravariant* vector ∂x^ν.

14. The final **formulation of Maxwell's equations** becomes

$$\Box^2 A^\mu = -\mu_0 J^\mu$$

This is generalization of our previous result

$$\Box^2 V = -\frac{1}{\epsilon_0}\rho; \quad \Box^2 \vec{\mathbf{A}} = -\mu_0 \vec{\mathbf{J}}$$

in a single 4-vector equation

15. Classical electrodynamics represents a *classical* field theory approximation of the fundamental theory of **quantum electrodynamics.** Many quantum features, such as photon–photon scattering, quantum optics, quantum entanglement, quantum computing and quantum cryptography, are completely missed