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1) Consider the hyperbolic paraboloid H given by the equation

$$x^2 - y^2 - z = 0$$

a) Treating H as a level surface of a function of three variables, find an equation of the tangent plane to H at the point $P(4, 3, 7)$.

b) Use the Implicit Function Theorem to show that near the point P in part a), H can be considered to be the graph of a function f of x and y. Compute the partial derivatives f_x and f_y and show that the tangent plane found in a) coincides with the graph of the linearization $L(x, y)$ of $f(x, y)$ at $(4, 3)$.

c) Use the method of Lagrange multipliers to find the point(s) closest to the origin along the intersection of H with the plane $z = x + y$.

Solution. a) The paraboloid H can be viewed as a level surface for the function $F(x, y, z) = x^2 - y^2 - z$. To obtain a normal vector to the tangent plane we may use the fact that the gradient vector is perpendicular to level surfaces. The gradient vector of $F(x, y, z)$ is

$$\nabla F(x, y, z) = F_x\,\hat{\imath} + F_y\,\hat{\jmath} + F_z\,\hat{k} = 2x\,\hat{\imath} - 2y\,\hat{\jmath} - \hat{k}$$

which at the point $(4, 3, 7)$ becomes

$$\vec{n} = \nabla F(4, 3, 7) = 2(4)\,\hat{\imath} - 2(3)\,\hat{\jmath} - \hat{k} = 8\,\hat{\imath} - 6\,\hat{\jmath} - \hat{k}.$$

For any point $Q(x, y, z)$ in the tangent plane, the vector $\overrightarrow{PQ} = (x-4)\,\hat{\imath} + (y-3)\,\hat{\jmath} + (z-7)\,\hat{k}$ lies in the plane and as such it is perpendicular to $\vec{n}$, i.e. we have

$$\vec{n}\cdot\overrightarrow{PQ} = 0 \quad\Leftrightarrow\quad 8(x-4) - 6(y-3) - (z-7) = 0 \quad\Leftrightarrow\quad 8x - 6y - z = 7.$$

Therefore $8x - 6y - z = 7$ is an equation for the tangent plane to H at $(4, 3, 7)$.

b) Since $F(x, y, z)$ is a polynomial function of x, y, z, its partial derivatives are continuous. Furthermore, we have $F(4, 3, 7) = 0$ and $F_z(4, 3, 7) = -1|_{z=7} = -1 \neq 0$. By the Implicit Function Theorem, there is a neighbourhood of $(4, 3, 7)$ in which a unique function $z = f(x, y)$ exists that satisfies $F(x, y, f(x, y)) = 0$. The partial derivatives of f are found via implicit differentiation

$$f_x = -\frac{F_x}{F_z} = -\frac{2x}{-1} = 2x, \qquad f_y = -\frac{F_y}{F_z} = -\frac{-2y}{-1} = -2y$$

taking the following values at $(4, 3, 7)$:

$$f_x(4, 3) = 2(4) = 8, \qquad f_y(4, 3) = -2(3) = -6.$$

Hence the linearization of $z = f(x, y)$ at $(4, 3)$ is

$$\begin{aligned} L(x, y) &= f_x(4, 3)(x-4) + f_y(4, 3)(y-3) + f(4, 3) \\ &= 8(x-4) - 6(y-3) + 7 \\ &= 8x - 6y - 7. \end{aligned}$$

The graph of the linearization is given by the equation $z = 8x - 6y - 7$ which is equivalent to the equation $8x - 6y - z = 7$ of the tangent plane found in part a).

c) Finding the point(s) on the intersection of the paraboloid $z = x^2 - y^2$ and the plane $z = x + y$ with a minimal distance from the origin is equivalent to minimizing the function $d(x, y) = x^2 + y^2 + (x+y)^2$ (i.e.

distance-squared from the origin for points in the plane) subject to the constraint $g(x,y) = x^2 - y^2 - (x+y) = 0$ (i.e. the intersection of the paraboloid and the plane). We use the method of Lagrange multipliers and solve

$$\begin{cases} \nabla d(x,y) = \lambda \nabla g(x,y) \\ \quad g(x,y) = 0 \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} 4x + 2y = \lambda(2x-1) \\ 2x + 4y = -\lambda(2y+1) \\ x^2 - y^2 - (x+y) = 0 \end{cases}$$

for x, y, λ. The third equation can be written as $(x-y-1)(x+y) = 0$ which implies that $y = x-1$ or $y = -x$ (i.e. the intersection consists of two straight lines: $(x, -x, 0)$ and $(x, x-1, 2x-1)$, $x \in \mathbb{R}$). If $y = x-1$, then summing the first two equations results in $12x - 6 = 0$, i.e. $x = 1/2$ and $y = -1/2$. However, no λ would satisfy these equations. If $y = -x$, then the difference of the first two equations is $4x = 0$ whose only solution is $x = 0$ which implies that $y = 0$ and $\lambda = 0$ as well. Therefore there is only one solution, $(x, y, \lambda) = (0, 0, 0)$ which corresponds to the origin so the minimal distance in question is actually 0.

2) Consider the vector field

$$\vec{G}(x,y,z) = (y^2 z + 2xz^2)\,\hat{\imath} + (2xyz)\,\hat{\jmath} + (xy^2 + 2x^2 z)\,\hat{k}$$

a) Show that $\operatorname{curl} \vec{G} = \vec{0}$.

b) Determine a scalar potential for $\vec{G}$.

c) Evaluate the line integral of $\vec{G}$ along the curve of intersection of the sphere $x^2 + y^2 + z^2 = 5$ and the plane $y = 1$.

Solution. a) We have

$$\begin{aligned}
\operatorname{curl} \vec{G} = \nabla \times \vec{G} = &\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (y^2 z + 2xz^2) & (2xyz) & (xy^2 + 2x^2 z) \end{vmatrix} \\
= &\left[\frac{\partial}{\partial y}(xy^2 + 2x^2 z) - \frac{\partial}{\partial z}(2xyz) \right] \hat{\imath} \\
&+ \left[\frac{\partial}{\partial z}(y^2 z + 2xz^2) - \frac{\partial}{\partial x}(xy^2 + 2x^2 z) \right] \hat{\jmath} \\
&+ \left[\frac{\partial}{\partial x}(2xyz) - \frac{\partial}{\partial y}(y^2 z + 2xz^2) \right] \hat{k} \\
= &[2xy - 2xy]\,\hat{\imath} + \left[y^2 + 4xz - y^2 - 4xz\right]\,\hat{\jmath} + [2yz - 2yz]\,\hat{k} \\
= &0\,\hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k} = \vec{0}.
\end{aligned}$$

so in fact $\operatorname{curl} \vec{G} = \vec{0}$ holds everywhere.

b) We need to solve the equation $\nabla g(x,y,z) = \vec{G}(x,y,z)$ for the unknown scalar potential $g(x,y,z)$. Written in component form, the equation reads

$$g_x = y^2 z + 2xz^2, \tag{1}$$

$$g_y = 2xyz, \tag{2}$$

$$g_z = xy^2 + 2x^2 z. \tag{3}$$

Integrating both sides of equation (1) with respect to x, we obtain

$$g(x,y,z) = xy^2z + x^2z^2 + h(y,z), \tag{4}$$

where $h(y,z)$ is a constant of integration depending on y and z (but not x). Differentiating (4) with respect to y, we get

$$g_y = 2xyz + h_y(y,z) \tag{5}$$

Comparing equations (2) and (5) gives

$$h_y(y,z) = 0 \tag{6}$$

which integrated with respect to y yields

$$h(y,z) = k(z). \tag{7}$$

Again, we have constant of integration $k(z)$ that may depend on z (but not y). Plugging this into equation (4) gives us

$$g(x,y,z) = xy^2z + x^2z^2 + k(z). \tag{8}$$

Differentiating (8) with respect to z and comparing the result to (3) yields

$$k'(z) = 0 \quad \Rightarrow \quad k(z) = K \text{ (constant)}. \tag{9}$$

Therefore we find that

$$g(x,y,z) = xy^2z + x^2z^2 + K \tag{10}$$

is a scalar potential of the vector field $\vec{G}$.

c) The intersection of the sphere and the plane is a circle given by the equation $x^2 + z^2 = 4$. Thus it is a circle in the xz-plane with radius 2 and centre at the origin. Let's denote this circle by C.

Since $\vec{G}$ is defined everywhere and conservative, i.e. we have $\vec{G}(x,y,z) = \nabla g(x,y,z)$ with the potential $g(x,y,z) = xy^2z + x^2z^2 + K$ and the curve along which we integrate $\vec{G}$ is closed we have

$$\oint_C \vec{G} \cdot d\vec{r} = 0$$

by the Fundamental Theorem of Line Integrals.

Alternatively, we could parametrize the curve in question by the vector function

$$\vec{r}(t) = (2\cos t)\,\hat{\imath} + \hat{\jmath} + (2\sin t)\,\hat{k}, \qquad 0 \le t \le 2\pi$$

whose derivative is

$$\vec{r}\,'(t) = (-2\sin t)\hat{\imath} + (2\cos t)\,\hat{k}, \qquad 0 \le t \le 2\pi$$

and along which the vector fields takes the values

$$\vec{G}(\vec{r}(t)) = (2\sin t + 16\cos t\sin^2 t)\,\hat{\imath} + (8\cos t\sin t)\,\hat{\jmath} + (2\cos t + 16\cos^2 t\sin t)\,\hat{k}.$$

Hence the line integral can also be directly evaluated as follows

$$\begin{aligned}
\int_C \vec{G} \cdot d\vec{r} &= \int_0^{2\pi} \vec{G}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^{2\pi} \big(4(\cos^2 t - \sin^2 t) + 32(\cos^3 t\sin t - \sin^3 t\cos t)\big)\,dt \\
&= \int_0^{2\pi} \big(4\cos 2t + 32(\cos^3 t\sin t - \sin^3 t\cos t)\big)\,dt \\
&= \big[2\sin 2t - 8(\cos^4 t + \sin^4 t)\big]_{t=0}^{t=2\pi} = 0 - 0 = 0.
\end{aligned}$$

3) Consider the velocity field given by

$$\vec{V}(x,y,z) = (-2yz)\,\hat{\imath} + (y)\,\hat{\jmath} + (3x)\,\hat{k}$$

and the surface $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 + z = 5,\ 1 \le z \le 4\}$ with upward normal vectors and positively-oriented boundary ∂S.

a) Describe and sketch the surface S and its boundary ∂S.
Draw the orientation of ∂S.

Verify Stokes' Theorem by directly computing

b) the circulation of $\vec{V}$ along ∂S, i.e. $\displaystyle\int_{\partial S} \vec{V} \cdot d\vec{r}$ and

c) the flux of $\operatorname{curl} \vec{V}$ across S, that is $\displaystyle\iint_S \operatorname{curl} \vec{V} \cdot d\vec{S}$.

Solution. a) The surface S is a piece of the paraboloid obtained by taking the parabolic arc $z = 5 - x^2$, $1 \le x \le 2$ in the xz-plane and rotating it about the z-axis. The surface can be seen in Figure 1. The boundary ∂S consists of the two circles (C_1 and C_2):

$$C_1 = \{(x, y, 1) \mid x^2 + y^2 = 4\}, \quad C_2 = \{(x, y, 4) \mid x^2 + y^2 = 1\}.$$

Since the normal vectors point upward (meaning a positive z-component), the boundary becomes positively-oriented if the lower circle C_1 is traversed counter-clockwise and the upper circle C_2 is traversed clockwise when viewed from above.

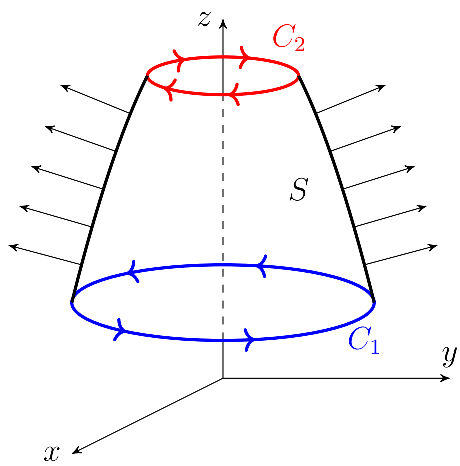

Figure 1: The surface S and its boundary $\partial S = C_1 \cup C_2$ (with positive orientation).

b) Based on part a), the following vector functions can be used to represent the (oriented) boundary $\partial S = C_1 \cup C_2$:

$$\begin{aligned} C_1: &\quad \vec{r}_1(t) = (2\cos t)\,\hat{\imath} + (2\sin t)\,\hat{\jmath} + \hat{k}, \quad 0 \le t \le 2\pi, \\ C_2: &\quad \vec{r}_2(t) = \cos t\,\hat{\imath} - \sin t\,\hat{\jmath} + 4\hat{k}, \quad 0 \le t \le 2\pi, \end{aligned}$$

The circulation of $\vec{V}$ along the boundary $\partial S = C_1 \cup C_2$ is the sum of the line integrals of $\vec{V}$ along each of the curves:

$$\int_{\partial S} \vec{V} \cdot d\vec{r} = \int_{C_1} \vec{V} \cdot d\vec{r} + \int_{C_2} \vec{V} \cdot d\vec{r}.$$

To compute these line integrals, we need the tangent vectors as well as the values $\vec{V}$ takes along the curves. Along C_1, we obtain

$$\vec{r}_1{}'(t) = (-2\sin t)\,\hat{\imath} + (2\cos t)\,\hat{\jmath}$$

and

$$\vec{V}(\vec{r}_1(t)) = (-4\sin t)\,\hat{\imath} + (\sin t)\,\hat{\jmath} + (6\cos t)\,\hat{k}$$

hence the line integral is

$$\int_{C_1} \vec{V} \cdot d\vec{r} = \int_0^{2\pi} \vec{V}(\vec{r}_1(t)) \cdot \vec{r}_1{}'(t)\, dt = \int_0^{2\pi} (8\sin^2 t + 2\sin t\cos t)\, dt$$

$$= \int_0^{2\pi} \big(4(1-\cos 2t) + \sin 2t\big)\, dt = [4t - 2\sin 2t - \tfrac{1}{2}\cos 2t]_{t=0}^{t=2\pi} = 8\pi.$$

Along C_2 we get

$$\vec{r}_2{}'(t) = -\sin t\,\hat{\imath} - \cos t\,\hat{\jmath}$$

and

$$\vec{V}(\vec{r}_2(t)) = (8\sin t)\,\hat{\imath} + (-\sin t)\,\hat{\jmath} + (3\cos t)\,\hat{k}.$$

hence the line integral equals

$$\int_{C_2} \vec{V} \cdot d\vec{r} = \int_0^{2\pi} \vec{V}(\vec{r}_2(t)) \cdot \vec{r}_2{}'(t)\, dt = \int_0^{2\pi} (-8\sin^2 t - \sin t\cos t)\, dt$$

$$= \int_0^{2\pi} \big(-4(1-\cos 2t) - \tfrac{1}{2}\sin 2t\big)\, dt = [-4t + 2\sin 2t + \tfrac{1}{4}\cos 2t]_{t=0}^{t=2\pi} = -8\pi.$$

Thus we see that

$$\int_{\partial S} \vec{V} \cdot d\vec{r} = 8\pi - 8\pi = 0,$$

that is the circulation of $\vec{V}$ along the boundary of S equals 0.

c) Let us first compute the curl of $\vec{V}$:

$$\operatorname{curl}\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (-2yz) & (y) & (3x) \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(3x) - \frac{\partial}{\partial z}(y)\right]\,\hat{\imath} + \left[\frac{\partial}{\partial z}(-2yz) - \frac{\partial}{\partial x}(3x)\right]\,\hat{\jmath} + \left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(-2yz)\right]\,\hat{k}$$

$$= [(0) - (0)]\,\hat{\imath} + [(-2y) - (3)]\,\hat{\jmath} + [(0) - (-2z)]\,\hat{k}$$

$$= -\,(2y+3)\,\hat{\jmath} + (2z)\,\hat{k}.$$

The surface S is a piece of the graph of the function $f(x,y) = 5 - x^2 - y^2$, one can find the normal vector using Cartesian coordinates as follows

$$\vec{n} = (1, 0, f_x) \times (0, 1, f_y) = (-f_x, -f_y, 1) = (2x, 2y, 1).$$

Therefore we have

$$\operatorname{curl}\vec{V}(x, y, f(x,y)) \cdot \vec{n} = -6y - 4y^2 + 2(5 - x^2 - y^2) = 10 - 6y - 2x^2 - 6y^2.$$

Thus the flux of $\operatorname{curl}\vec{V}$ equals

$$\iint_S \operatorname{curl}\vec{V} \cdot d\vec{S} = \iint_S \operatorname{curl}\vec{V} \cdot \vec{n}\, dS = \iint_D (10 - 6y - 2x^2 - 6y^2)\, dx\, dy,$$

where D is the projection of S to the xy-plane, i.e. the annulus $1 \leq x^2 + y^2 \leq 4$. Computing this double integral will be more convenient if we switch to polar coordinates:

$$D = \{(x, y) \mid 1 \leq x^2 + y^2 \leq 4\} = \{(r\cos\theta, r\sin\theta) \mid 1 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}.$$

The integral then reads

$$\begin{aligned}
\iint_S \operatorname{curl}\vec{V} \cdot d\vec{S} &= \iint_D (10 - 6y - 2x^2 - 6y^2)\, dx\, dy \\
&= \int_1^2 \int_0^{2\pi} (10 - 6r\sin\theta - 2r^2\cos^2\theta - 6r^2\sin^2\theta) r\, d\theta\, dr \\
&= \int_1^2 \int_0^{2\pi} \big(10 - 6r\sin\theta - r^2(1 + \cos 2\theta) - 3r^2(1 - \cos 2\theta)\big) r\, d\theta\, dr \\
&= \int_1^2 \left[10\theta + 6r\cos\theta - r^2(\theta + \tfrac{1}{2}\sin 2\theta) - 3r^2(\theta - \tfrac{1}{2}\sin 2\theta)\right]_{\theta=0}^{\theta=2\pi} r\, dr \\
&= \int_1^2 \big(10(2\pi) - r^2(2\pi) - 3r^2(2\pi)\big) r\, dr \\
&= 2\pi \int_1^2 \big(10r - 4r^3\big)\, dr = 2\pi\big[5r^2 - r^4\big]_{r=1}^{r=2} = 2\pi[5(2^2) - 2^4 - 5(1^2) + 1^4] = 0.
\end{aligned}$$

Thus we see that the flux of $\operatorname{curl}\vec{V}$ across S is 0.

4) Consider the force field given by

$$\vec{F}(x, y, z) = (2x)\,\hat{\imath} + (2y)\,\hat{\jmath} + (z^2)\,\hat{k}$$

over the solid region $E = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq 1,\ x \geq 0,\ y \geq 0\}$ and its outward-oriented boundary surface ∂E.

a) Describe and sketch the solid region E and the boundary surface ∂E. Draw the orientation of ∂E.

Verify the Divergence Theorem by directly calculating

b) the flux of $\vec{F}$ across ∂E, that is $\oiint_{\partial E} \vec{F} \cdot d\vec{S}$ and

c) the triple integral of $\operatorname{div}\vec{F}$ over E, i.e. $\iiint_E \operatorname{div}\vec{F}\, dV$.

Solution. a) The inequality $x^2 + y^2 + z^2 \leq 1$ yields the closed solid ball of radius 1 centred at the origin. Thus E is a spherical wedge. (see Figure 2). Accordingly, the boundary of E is the union of a spherical lune and two half-discs. More precisely, let S_1 denote of the portion of the sphere of radius 1 centred at the origin for which we have the azimuthal angle $0 \leq \theta \leq \pi/2$. Let S_2 denote the half-disc of unit radius centred at the origin that lies in the xz-plane and contain the positive x-axis. On S_2, we have $\theta = 0$. And let S_3 denote the half-disc of unit radius centred at the origin that lies in the yz-plane and contain the positive y-axis. On S_3, we have $\theta = \pi/2$. In summary, we have $\partial E = S_1 \cup S_2 \cup S_3$.

b) From part a), we deduce that the spherical lune S_1 is given by the vector function

$$S_1: \qquad \vec{r}_1(\phi, \theta) = (\sin\phi\cos\theta)\,\hat{\imath} + (\sin\phi\sin\theta)\,\hat{\jmath} + (\cos\phi)\,\hat{k}, \quad 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq \pi/2.$$

the half-disc S_2 $(\theta = 0)$ is given by the vector function

$$S_2: \qquad \vec{r}_2(\rho, \phi) = (\rho\sin\phi)\,\hat{\imath} + (\rho\cos\phi)\,\hat{k}, \quad 0 \leq \rho \leq 1,\ 0 \leq \phi \leq \pi.$$

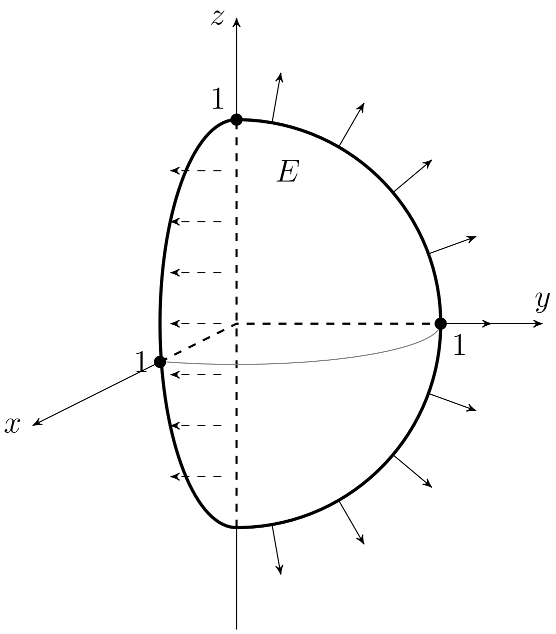

Figure 2: The solid region E (with outward-pointing normal vectors).

and the half-disc S_3 ($\theta = \pi/2$) is given by the vector function

$$S_3: \qquad \vec{r}_3(\rho,\phi) = (\rho\sin\phi)\,\hat{\jmath} + (\rho\cos\phi)\,\hat{k}, \quad 0 \le \rho \le 1,\ 0 \le \phi \le \pi.$$

The flux of $\vec{F}$ across $\partial E = S_1 \cup S_2 \cup S_3$ is the sum of the surface integrals of $\vec{F}$ across S_1, S_2, and S_3:

$$\oiint_{\partial E} \vec{F}\cdot d\vec{S} = \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S} + \iint_{S_3} \vec{F}\cdot d\vec{S}.$$

To evaluate these surface integrals, we need to find (outward) normal vectors to the surfaces as well as the values of $\vec{F}$ on the surfaces. On S_1, we have

$$(\vec{r}_1)_\phi = (\cos\phi\cos\theta)\,\hat{\imath} + (\cos\phi\sin\theta)\,\hat{\jmath} - (\sin\phi)\,\hat{k}$$

and

$$(\vec{r}_1)_\theta = (-\sin\phi\sin\theta)\,\hat{\imath} + (\sin\phi\cos\theta)\,\hat{\jmath}.$$

The cross product of these derivatives yields normal vectors to S_1:

$$\begin{aligned}
(\vec{r}_1)_\phi \times (\vec{r}_1)_\theta =&(\cos\phi\cos\theta\,\hat{\imath} + \cos\phi\sin\theta\,\hat{\jmath} - \sin\phi\,\hat{k}) \times (-\sin\phi\sin\theta\,\hat{\imath} + \sin\phi\cos\theta\,\hat{\jmath})\\
=&(\cos\phi\cos\theta)(\sin\phi\cos\theta)(\hat{\imath}\times\hat{\jmath}) - (\cos\phi\sin\theta)(\sin\phi\sin\theta)(\hat{\jmath}\times\hat{\imath})\\
&+ (\sin\phi)(\sin\phi\sin\theta)(\hat{k}\times\hat{\imath}) - (\sin\phi)(\sin\phi\cos\theta)(\hat{k}\times\hat{\jmath})\\
=&(\sin\phi\cos\phi\cos^2\theta)(\hat{k}) - (\sin\phi\cos\phi\sin^2\theta)(-\hat{k})\\
&+ (\sin^2\phi\sin\theta)(\hat{\jmath}) - (\sin^2\phi\cos\theta)(-\hat{\imath})\\
=&(\sin^2\phi\cos\theta)\,\hat{\imath} + (\sin^2\phi\sin\theta)\,\hat{\jmath} + [\sin\phi\cos\phi(\cos^2\theta + \sin^2\theta)]\,\hat{k}\\
=&(\sin^2\phi\cos\theta)\,\hat{\imath} + (\sin^2\phi\sin\theta)\,\hat{\jmath} + (\sin\phi\cos\phi)\,\hat{k}.
\end{aligned}$$

As for the values of $\vec{F}$ on S_1, we get

$$\vec{F}(\vec{r}_1(\phi,\theta)) = (2\sin\phi\cos\theta)\,\hat{\imath} + (2\sin\phi\sin\theta)\,\hat{\jmath} + (\cos^2\phi)\,\hat{k}.$$

Hence the normal component of $\vec{F}$ is

$$\begin{aligned}
&\vec{F}(\vec{r}_1(\phi,\theta)) \cdot \big((\vec{r}_1)_\phi \times (\vec{r}_1)_\theta\big)\\
&= (2\sin\phi\cos\theta)(\sin^2\phi\cos\theta) + (2\sin\phi\sin\theta)(\sin^2\phi\sin\theta) + (\cos^2\phi)(\sin\phi\cos\phi)\\
&= 2\sin^3\phi(\cos^2\theta + \sin^2\theta) + \cos^3\phi\sin\phi\\
&= 2\sin^3\phi + \cos^3\phi\sin\phi.
\end{aligned}$$

The flux of $\vec{F}$ across S_1 is

$$\iint\limits_{S_1} \vec{F}\cdot d\vec{S} = \int\limits_0^{\pi}\int\limits_0^{\pi/2} \vec{F}(\vec{r}_1(\phi,\theta))\cdot((\vec{r}_1)_\phi \times (\vec{r}_1)_\theta)\,d\theta\,d\phi$$
$$= \int\limits_0^{\pi}\int\limits_0^{\pi/2} (2\sin^3\phi + \cos^3\phi\sin\phi)\,d\theta\,d\phi = \tfrac{\pi}{2}\int\limits_0^{\pi}(2\sin^3\phi+\cos^3\phi\sin\phi)\,d\phi$$
$$= \tfrac{\pi}{2}\int\limits_0^{\pi}\big(2\sin\phi(1-\cos^2\phi)+\cos^3\phi\sin\phi\big)\,d\phi = \tfrac{\pi}{2}[-2\cos\phi+\tfrac{2}{3}\cos^3\phi-\tfrac{1}{4}\cos^4\phi]_{\phi=0}^{\phi=\pi}$$
$$= \tfrac{\pi}{2}\big(-2\cos\pi+\tfrac{2}{3}\cos^3\pi-\tfrac{1}{4}\cos^4\pi\big)-\tfrac{\pi}{2}\big(-2\cos 0+\tfrac{2}{3}\cos^3 0-\tfrac{1}{4}\cos^4 0\big)$$
$$= \tfrac{\pi}{2}\big(-2(-1)+\tfrac{2}{3}(-1)^3-\tfrac{1}{4}(-1)^4\big)-\tfrac{\pi}{2}\big(-2(1)+\tfrac{2}{3}(1^3)-\tfrac{1}{4}(1^4)\big)$$
$$= \tfrac{\pi}{2}\big(2-\tfrac{2}{3}-\tfrac{1}{4}+2-\tfrac{2}{3}+\tfrac{1}{4}\big) = \tfrac{\pi}{2}\big(4-\tfrac{4}{3}\big) = \tfrac{\pi}{2}\tfrac{8}{3} = \tfrac{4\pi}{3}.$$

For the half-disc S_2, the outward-pointing normal vector is $-\hat{\jmath}$. and the vector field $\vec{F}$ takes on the following values

$$\vec{F}(\vec{r}_2(\rho,\phi)) = (2\rho\sin\phi)\,\hat{\imath} + (\rho^2\cos^2\phi)\,\hat{k}.$$

Its normal component is zero

$$\vec{F}(\vec{r}_2(\rho,\phi))\cdot(-\hat{\jmath}) = 0$$

and hence the flux of $\vec{F}$ across S_2 is zero

$$\iint\limits_{S_2}\vec{F}\cdot d\vec{S} = \int\limits_0^1\int\limits_0^{\pi}\vec{F}(\vec{r}_2(\rho,\phi))\cdot(-\hat{\jmath})\,d\phi\,d\rho = \int\limits_0^1\int\limits_0^{\pi} 0\,d\phi\,d\rho = 0.$$

For the half-disc S_3, the outward-pointing normal vector is $-\hat{\imath}$. and the vector field $\vec{F}$ takes on the following values

$$\vec{F}(\vec{r}_3(\rho,\phi)) = (2\rho\sin\phi)\,\hat{\jmath} + (\rho^2\cos^2\phi)\,\hat{k}.$$

Its normal component is zero

$$\vec{F}(\vec{r}_3(\rho,\phi))\cdot(-\hat{\imath}) = 0$$

and hence the flux of $\vec{F}$ across S_3 is zero

$$\iint\limits_{S_3}\vec{F}\cdot d\vec{S} = \int\limits_0^1\int\limits_0^{\pi}\vec{F}(\vec{r}_3(\rho,\phi))\cdot(-\hat{\jmath})\,d\phi\,d\rho = \int\limits_0^1\int\limits_0^{\pi} 0\,d\phi\,d\rho = 0.$$

Therefore the total flux of $\vec{F}$ across ∂E is $\frac{4\pi}{3}+0+0=\frac{4\pi}{3}$.

c) The divergence of $\vec{F}$ is

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(2x)+\frac{\partial}{\partial y}(2y)+\frac{\partial}{\partial z}(z^2) = 2+2+2z = 2z+4.$$

In terms of spherical coordinates (ρ,ϕ,θ) the solid E can be expressed as follows

$$E: \qquad 0\le\rho\le 1, \quad 0\le\phi\le\pi, \quad 0\le\theta\le\pi/2.$$

Therefore the triple integral of $\operatorname{div}\vec{F}$ over E is

$$\begin{aligned}
\iiint\limits_E \operatorname{div}\vec{F}\,dV = \iiint\limits_E (2z+4)\,dV &= \int_0^1\int_0^\pi\int_0^{\pi/2} (2\rho\cos\phi+4)(\rho^2\sin\phi)\,d\theta\,d\phi\,d\rho \\
&= \tfrac{\pi}{2}\int_0^1\int_0^\pi (2\rho^3\sin\phi\cos\phi + 4\rho^2\sin\phi)\,d\phi\,d\rho \\
&= \tfrac{\pi}{2}\int_0^1 [\rho^3\sin^2\phi - 4\rho^2\cos\phi]_{\phi=0}^{\phi=\pi}\,d\rho \\
&= \tfrac{\pi}{2}\int_0^1 8\rho^2\,d\rho = 4\pi\int_0^1 \rho^2\,d\rho = 4\pi[\tfrac{1}{3}\rho^3]_{\rho=0}^{\rho=1} = \tfrac{4\pi}{3}.
\end{aligned}$$