Re-Exam Mathematical Physics, Prof. G. Palasantzas

- Date: 13-07-2023
- Total number of points 100
- 10 points for taking the exam
- For all problems justify briefly your answer

<u>Problem 1</u> (25 points)

Find the value of c if $\sum_{n=2}^{\infty} (1 + c)^{-n} = 2$

<u>Problem 2</u> (20 points)

Consider the boundary value problem for the one-dimensional heat equation for a bar with zero-temperature ends:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad u = u(x,t), \quad t > 0, \quad 0 < x < L,$$
$$u(x,0) = f(x),$$
$$u(0,t) = 0, \quad u(L,t) = 0, \quad t \geq 0.$$

u(x,t): Temperature

The general solution for *u(x,t)* is given by:

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-\lambda_n^2 t} \sin \frac{n\pi}{L} x \ , \ \lambda_n = \frac{cn\pi}{L}$$

Derive the solution u(x,t)

(**a: 10 points**) for $f(x) = 20\sin(\pi x/L)$

(**b: 10 ponts**) for $f(x) = 20\sin(\pi x/L) + 50\sin 3\pi x/L)$

<u>Problem 3</u> (20 points)

Consider a spring with spring constant k and a mass m attached to it ($k=m\omega^2$). The mass m is assumed to perform motion under the influence of an external periodic force $F(t) = F_o[cos^2(9\omega_o t) - 0.5] + 2F_o \cos(7\omega_o t)$ ($F(t) = F(t+T)$ with $T=2\pi/\omega_o$). The equation of motion of the mass m reads of the form

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

Find a particular periodic solution for the motion of the mass m by considering as a trial solution a Fourier series of the form $X_p(t) = \sum_{-\infty}^{+\infty} X_n e^{in\omega_o t}$.

<u>Problem 4</u> (25 points)

Consider the definition of the Fourier transform $F(k) = \int_{-\infty}^{+\infty} F(x)e^{-i2\pi kx}dx$ and the definition of the delta function $\delta(k) = \int_{-\infty}^{+\infty} e^{-i2\pi kx}dx$.

Calculate the Fourier transform of the function $F(x) = sin^n(2\pi k_o x)$ with n a positive integer.

Tip: Consider the Binomial identity $(x+y)^n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} x^{n-m}y^m$

Problem 1

$\sum_{n=2}^{\infty}(1+c)^{-n}$ is a geometric series with $a=(1+c)^{-2}$ and $r=(1+c)^{-1}$, so the series converges when

$\left|(1+c)^{-1}\right|<1 \;\Leftrightarrow\; |1+c|>1 \;\Leftrightarrow\; 1+c>1$ or $1+c<-1 \;\Leftrightarrow\; c>0$ or $c<-2$. We calculate the sum of the

series and set it equal to 2: $\dfrac{(1+c)^{-2}}{1-(1+c)^{-1}}=2 \;\Leftrightarrow\; \left(\dfrac{1}{1+c}\right)^2=2-2\left(\dfrac{1}{1+c}\right) \;\Leftrightarrow\; 1=2(1+c)^2-2(1+c) \;\Leftrightarrow$

$2c^2+2c-1=0 \;\Leftrightarrow\; c=\frac{-2\pm\sqrt{12}}{4}=\frac{\pm\sqrt{3}-1}{2}$. However, the negative root is inadmissible because $-2<\frac{-\sqrt{3}-1}{2}<0$.

So $c=\frac{\sqrt{3}-1}{2}$.

Problem 2

(a) Here you satisfy the intial condition u(x,0)= *f*(x) =20sin(πx/L) for

$$\lambda_1 = c\pi/L$$

Thus, the solution is

$$u(x,t) = 20\exp[-\lambda_1{}^2 t]\sin(\frac{c\pi x}{L})$$

(b) Satisfy the initial conditions:

U(x,0)=20sin(πx/l)+50sin(3πx/L)

Only the terms λ_1=cπ/L and λ_3=3cπ/L give non-zero contribution to the series solution with B_1=20 and B_3=50

So the solution is:

U(x,t)=20exp[- λ_1^2 t]sin(πx/L)+50exp[- λ_3^2 t]sin(3πx/L)

Problem 3

$$F(t) = F_0\left[\cos^2(9\omega_0 t) - 0.5\right] + 2F_0\cos(7\omega_0 t)$$

$$\cos^2(9\omega_0 t) = \frac{1}{4}\left(e^{i\,18\omega_0 t} + e^{-i\,18\omega_0 t} + 2\right) =$$

$= \frac{1}{2}\cos(18\omega_0 t) + 0.5$. Thus, we have

$F(t) = \frac{F_0}{2}\cos(18\omega_0 t) + 2F_0\cos(7\omega_0 t)$ or

$$F(t) = \frac{F_0}{4}e^{i\,18\omega_0 t} + \frac{F_0}{4}e^{-i\,18\omega_0 t} + F_0 e^{i7\omega_0 t} + F_0 e^{-i7\omega_0 t}$$

Thus $\underbrace{F(t) = \sum_{-\infty}^{+\infty} f_n e^{i n\omega_0 t}}_{\text{Fourier series fo } F(t)}$ $f_{18} = f_{-18} = \frac{F_0}{4}$

$f_7 = f_{-7} = F_0$

$f_n = 0,\ n \neq \pm 7, \pm 18$

Try solution

$$X_p = \sum_{n=-\infty}^{+\infty} X_n e^{i n\omega_0 t}, \qquad X_p' = \sum_{-\infty}^{+\infty} X_n (i n\omega_0) e^{i n\omega_0 t}$$

$$X_p'' = \sum_{-\infty}^{+\infty} X_n\left(-(n\omega_0)^2\right) e^{i n\omega_0 t}$$

then subtitute into the differential equation.

$$m\sum_{-\infty}^{+\infty}\left(-(n\omega_0)^2\right)X_n e^{i n\omega_0 t} + c\sum_{-\infty}^{+\infty}(i\,n\omega_0)X_n e^{i n\omega_0 t} + k\sum_{-\infty}^{+\infty} X_n e^{i n\omega_0 t}$$

$$= \sum_{-\infty}^{+\infty} F_n e^{i n\omega_0 t} \quad \Rightarrow$$

$$\sum_{-\infty}^{+\infty}\Big[\underbrace{\left\{\left[k - m(n\omega_0)^2\right] + i\,c(n\omega_0)\right\}X_n - F_n}_{=0}\Big] e^{i n\omega_0 t} = 0$$

$\forall t \Rightarrow$

$$X_n = \frac{F_n}{\left(k - m(n\omega_0)^2\right) + i\,c n\omega_0} = \frac{F_n}{D_n}e^{i\delta_n}$$

$$\boxed{\begin{aligned} D_n &= \sqrt{\left(k - m(n\omega_0)^2\right)^2 + (cn\omega_0)^2} \\ \tan\delta_n &= \frac{cn\omega_0}{m(n\omega_0)^2 - k} \\ \tan\delta_{-n} &= -\tan\delta_n \\ \text{or } \delta_{-n} &= -\delta_n \\ D_{-n} &= D_n \end{aligned}}$$

$$\underbrace{X_p(t) = \sum_{-\infty}^{+\infty}\frac{F_n}{D_n}e^{i\delta_n}e^{i n\omega_0 t}}_{\text{general solution in fact for any periodic force } F(t)}$$

For our particular case we have:

$$X_p(t) = \frac{F_0}{4D_{18}} e^{i(18\omega_0 t + \delta_{18})} + \frac{F_0}{4D_{18}} e^{-i(18\omega_0 t + \delta_{18})} +$$

$$+ \frac{F_0}{D_7} e^{i(7\omega_0 t + \delta_7)} + \frac{F_0}{D_7} e^{-i(7\omega_0 t + \delta_7)}$$

$$\Rightarrow X_p(t) = \frac{F_0}{2D_{18}} \cos(18\omega_0 t + \delta_{18}) + \frac{2F_0}{D_7} \cos(7\omega_0 t + \delta_7)$$

Problem 4

Consider $f(x) = \sin^n(2\pi k_0 x)$

Find the Fourier transform.

$$F(k) = \int_{-\infty}^{+\infty} \sin^n(2\pi k_0 x)\, e^{-i2\pi k x}\, dx$$

$$\sin^n(2\pi k_0 x) = \frac{1}{2^n i^n}\left(e^{i2\pi k_0 x} - e^{-i2\pi k_0 x}\right)^n =$$

$$= \underbrace{\frac{1}{2^n i^n}\sum_{m=0}^{n} \frac{n!}{m!\,(n-m)!}\, e^{i2\pi k_0 x(n-m)}\, e^{-i2\pi k_0 x m}\,(-1)^m}_{\text{Binomial Identity}} \Rightarrow$$

$$F(k) = \frac{1}{2^n i^n}\sum_{m=0}^{n} \frac{n!}{m!\,(n-m)!}(-1)^m \underbrace{\int_{-\infty}^{+\infty} e^{i2\pi k_0 (n-2m)x}\, e^{-i2\pi k x}\, dx}_{\int_{-\infty}^{+\infty} e^{-i2\pi[k-k_0(n-2m)]x}\,dx} =$$

$$\underbrace{\int_{-\infty}^{+\infty} e^{-i2\pi[k-k_0(n-2m)]x}\, dx}_{\delta(k-k_0(n-2m))}$$

$$\Rightarrow F(k) = \frac{1}{2^n i^n}\sum_{m=0}^{n} \frac{n!}{m!\,(n-m)!}(-1)^m\, \delta(k - k_0(n-2m))$$