**Final Exam Mathematical Physics, Prof. G. Palasantzas**

- Date: 19-06-2023
- Total number of points 100
- 10 points for taking the exam
- For all problems justify briefly your answer

**<u>Problem 1</u> (20 points)**

(a: 10 points) Find the value of c if $\sum_{n=1}^{\infty} e^{nc} = 30$

(b: 10 points) Prove that $\frac{1+x}{1-x} = 1 + 2\sum_{n=1}^{\infty} x^n$ if |x|<1.

**<u>Problem 2</u> (25 points)**

Consider a spring with spring constant $k$ and a mass $m$ attached to it. The mass m is in motion under the influence of the external periodic force $F(t) = F_1\cos(n_1\omega_o t)\cos(n_2\omega_o t) + F_2\cos(n_3\omega_o t)$ with $\omega_o \neq \sqrt{k/m}$ and $n_{1,2,3}$ being positive integers. The equation of motion of the mass $m$ is given by the equation

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$$

Find a particular periodic solution of the form $X_p(t) = \sum_{-\infty}^{+\infty} X_n e^{in\omega_o t}$ and transform it into the real final form.

**<u>Problem 3</u> (20 points)**

Using the delta function $\delta(x) = \int_{-\infty}^{+\infty} e^{-i2\pi kx} dk$ calculate the double integral

$$\Gamma = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-|x|}[\cos(2\pi kx) - i\sin(2\pi kx)]\cos(2\pi\varepsilon k)\cos(4\pi\varepsilon k)dxdk$$

with ε a real finite number.

**<u>Problem 4</u> (25 points)**

Consider the definition of the Fourier transform $F(k) = \int_{-\infty}^{+\infty} F(x)e^{-i2\pi kx}dx$ and the definition of the delta function $\delta(k) = \int_{-\infty}^{+\infty} e^{-i2\pi kx}dx$.

Calculate the Fourier transform of the function $F(x) = cos^n(2\pi k_o x)$ with n a positive integer.

*Tip: Consider the Binomial identity* $(x+y)^n = \sum_{m=0}^{n}\frac{n!}{m!(n-m)!}x^{n-m}y^m$

**Problem 1**

(a) In general we have:

$$\sum_{n=1}^{\infty} e^{nc} = \frac{e^c}{1-e^c} = 30$$

Geometric series

$$e^c(1+30) = 30$$

$$c = \ln(\tfrac{30}{31}) \quad <0$$

Consistent to apply geometric series since $|e^c| < 1 \; or \; c < 0$

(b)

$$f(x) = \frac{1+x}{1-x} = (1+x)\left(\frac{1}{1-x}\right) = (1+x)\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} x^{n+1}$$

$$= 1 + \sum_{n=1}^{\infty} x^n + \sum_{n=1}^{\infty} x^n = 1 + 2\sum_{n=1}^{\infty} x^n$$

*A second approach:* $f(x) = \dfrac{1+x}{1-x} = \dfrac{-(1-x)+2}{1-x} = -1 + 2\left(\dfrac{1}{1-x}\right) = -1 + 2\sum_{n=0}^{\infty} x^n = 1 + 2\sum_{n=1}^{\infty} x^n$

*A third approach:*

$$f(x) = \frac{1+x}{1-x} = (1+x)\left(\frac{1}{1-x}\right) = (1+x)(1+x+x^2+x^3+\cdots)$$

$$= (1+x+x^2+x^3+\cdots) + (x+x^2+x^3+x^4+\cdots) = 1+2x+2x^2+2x^3+\cdots = 1+2\sum_{n=1}^{\infty} x^n.$$

**Problem 2**

$$F(t) = F_1 \cos(m_1 w_0 t)\cos(m_2 w_0 t) + F_2 \cos(m_3 w_0 t)$$

$$F(t) = \frac{F_1}{4}\left[e^{i(m_1 w_0)t} + e^{-i m_1 w_0 t}\right]\left[e^{i m_2 w_0 t} + e^{-i m_2 w_0 t}\right] + \frac{F_2}{2}\left[e^{i m_3 w_0 t} + e^{-i m_3 w_0 t}\right]$$

$$F(t) = \frac{F_1}{4}\left[e^{i(m_1+m_2)w_0 t} + e^{-i(m_1+m_2)w_0 t} + e^{i(m_1-m_2)w_0 t} + e^{-i(m_1-m_2)w_0 t}\right]$$

$$+ \frac{F_2}{2}\left[e^{i m_3 w_0 t} + e^{-i m_3 w_0 t}\right] \quad (1)$$

Try $\quad X_p(t) = \sum_{-\infty}^{+\infty} X_n e^{i n w_0 t}$

$$m\sum_{-\infty}^{+\infty}\left[-(n w_0)^2\right]X_n e^{i n w_0 t} + c\sum_{-\infty}^{+\infty} i(n w_0) X_n e^{i n w_0 t} + k\sum_{-\infty}^{+\infty} X_n e^{i n w_0 t} = \sum_{-\infty}^{+\infty} f_n e^{i n w_0 t}$$

$$\Rightarrow \sum_{-\infty}^{+\infty}\left\{\left[(k - m(n w_0)^2) + i c n w_0\right]X_n - f_n\right\} e^{i n w_0 t} = 0 \Rightarrow$$

$$\left[(k - m(n w_0)^2) + i c n w_0\right]X_n - f_n = 0 \Rightarrow X_n = \frac{f_n}{(k - m(n w_0)^2) + i c n w_0} \quad (2)$$

$$(2) \Rightarrow X_n = f_n \frac{e^{i\varphi_n}}{D_n} \quad (3)$$

$$D_n = \sqrt{(k - m(n w_0)^2)^2 + (c n w_0)^2}$$

$$\tan\varphi_n = \frac{c n w_0}{m(n w_0)^2 - k}, \quad D_{-n} = D_n, \quad \varphi_{-n} = -\varphi_n$$

$$(1) \Rightarrow \begin{cases} f_n = \dfrac{F_1}{4} & \text{for } n = \pm(m_1 \pm m_2) \\ f_n = \dfrac{F_2}{2} & \text{for } n = \pm m_3 \end{cases} \Leftrightarrow F(t) = \sum_{-\infty}^{+\infty} f_n e^{i n w_0 t} \quad (4)$$

$$X_p(t) = \frac{F_1}{4}\frac{1}{D_{m_1+m_2}}\left[e^{i[(m_1+m_2)w_0 t + \varphi_{m_1+m_2}]} + e^{-i[(m_1+m_2)w_0 t + \varphi_{m_1+m_2}]}\right] +$$

$$+ \frac{F_1}{4}\frac{1}{D_{m_1-m_2}}\left[e^{i[(m_1-m_2)w_0 t + \varphi_{m_1-m_2}]} + e^{-i[(m_1-m_2)w_0 t + \varphi_{m_1-m_2}]}\right]$$

$$+ \frac{F_2}{2}\frac{1}{D_{m_3}}\left[e^{i[m_3 w_0 t + \varphi_{m_3}]} + e^{-i[m_3 w_0 t + \varphi_{m_3}]}\right]$$

$$X_p(t) = \frac{F_1}{2}\frac{1}{D_{m_1+m_2}}\cos\left((m_1+m_2)w_0 t + \varphi_{m_1+m_2}\right) + \frac{F_1}{2}\frac{1}{D_{m_1-m_2}}\cos\left[(m_1-m_2)w_0 t + \varphi_{m_1-m_2}\right]$$

$$+ F_2\frac{1}{D_{m_3}}\cos\left(m_3 w_0 t + \varphi_{m_3}\right)$$

Replace above $m_{1,2,3}$ with $n_{1,2,3}$

**Problem 3**

$$\Gamma = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-|x|}\, e^{-i2\pi kx} \cos(2\pi\varepsilon k)\cos(4\pi\varepsilon k)\, dx\, dk$$

$$\cos(2\pi\varepsilon k)\cos(4\pi\varepsilon k) = \frac{1}{4}\left[e^{i2\pi\varepsilon k} + e^{-i2\pi\varepsilon k}\right]\left[e^{i4\pi\varepsilon k} + e^{-i4\pi\varepsilon k}\right] \Rightarrow$$

$$\cos(2\pi\varepsilon k)\cos(4\pi\varepsilon k) = \frac{1}{4}\left[e^{i6\pi\varepsilon k} + e^{-i6\pi\varepsilon k} + e^{i2\pi\varepsilon k} + e^{-i2\pi\varepsilon k}\right] \Rightarrow$$

$$\Gamma = \frac{1}{4}\int_{-\infty}^{+\infty} e^{-|x|}\, dx \left\{\int_{-\infty}^{+\infty}\left[e^{-i2\pi(x-3\varepsilon)k} + e^{-i2\pi(x+3\varepsilon)k} + e^{-i2\pi(x+\varepsilon)k} + e^{-i2\pi(x-\varepsilon)k}\right] dk\right\}$$

$$\Gamma = \frac{1}{4}\int_{-\infty}^{+\infty} e^{-|x|}\, dx \left\{ \underbrace{\int_{-\infty}^{+\infty} e^{-i2\pi(x-3\varepsilon)k}\, dk}_{\delta(x-3\varepsilon)} + \underbrace{\int_{-\infty}^{+\infty} e^{-i2\pi(x+3\varepsilon)k}\, dk}_{\delta(x+3\varepsilon)} + \underbrace{\int_{-\infty}^{+\infty} e^{-i2\pi(x-\varepsilon)k}\, dk}_{\delta(x-\varepsilon)} + \underbrace{\int_{-\infty}^{+\infty} e^{-i2\pi(x+\varepsilon)k}\, dk}_{\delta(x+\varepsilon)} \right\}$$

$$\Gamma = \frac{1}{4}\left[\int_{-\infty}^{+\infty} e^{-|x|}\delta(x-3\varepsilon)\, dx + \int_{-\infty}^{+\infty} e^{-|x|}\delta(x+3\varepsilon)\, dx + \int_{-\infty}^{+\infty} e^{-|x|}\delta(x-\varepsilon)\, dx + \int_{-\infty}^{+\infty} e^{-|x|}\delta(x+\varepsilon)\, dx\right] \Rightarrow$$

$$\Gamma = \frac{1}{4}\left[2e^{-3|\varepsilon|} + 2e^{-|\varepsilon|}\right] = \frac{1}{2}\left[e^{-3|\varepsilon|} + e^{-|\varepsilon|}\right]$$

**Problem 4**

$$F(k) = \int_{-\infty}^{+\infty} \cos^n(2\pi k_0 x)\, e^{-i2\pi k x}\, dx$$

$$F(k) = \frac{1}{2^n} \int_{-\infty}^{+\infty} \left(e^{i2\pi k_0 x} + e^{-i2\pi k_0 x}\right)^n e^{-i2\pi k x}\, dx$$

$$\left(e^{i2\pi k_0 x} + e^{-i2\pi k_0 x}\right)^n = \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} e^{i2\pi k_0 x(n-m)}\, e^{-i2\pi k_0 x m} \Rightarrow$$

$$\left(e^{i2\pi k_0 x} + e^{-i2\pi k_0 x}\right)^n = \sum_{m=0}^{\infty} \frac{n!}{m!(n-m)!} e^{i2\pi k_0 (n-2m)x} \Rightarrow$$

$$F(k) = \frac{1}{2^n} \int_{-\infty}^{+\infty} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} e^{-i2\pi x[k - k_0(n-2m)]}\, dx \Rightarrow$$

$$\Rightarrow F(k) = \frac{1}{2^n} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \underbrace{\int_{-\infty}^{+\infty} e^{-i2\pi x[k - k_0(n-2m)]}\, dx}_{\delta(k - k_0(n-2m))} \Rightarrow$$

$$\Rightarrow F(k) = \frac{1}{2^n} \sum_{m=0}^{n} \frac{n!}{m!(n-m)!} \delta(k - k_0(n-2m))$$