Linear Algebra - Mock Exam

QUESTIONS:

1. $\boxed{\mathbf{2}}$ A system of equations with 3 unknowns (x, y, z), with parameters α and β, is given by

$$\begin{aligned} 2x + y - 3z &= \beta \\ x + \alpha z &= -2 \\ x + y &= 3\beta \end{aligned}$$

   (a) $\boxed{0.6}$ Solve the system for $(\alpha, \beta) = (2, 1)$. Clearly indicate the applied steps. (You can use any correct method.)

   (b) $\boxed{0.4}$ Use the determinant to determine for which value(s) of α and/or β the system has exactly one solution.

   (c) $\boxed{1}$ Use row reduction to determine for which value(s) of α and/or β the system has no solutions.

2. $\boxed{\mathbf{2}}$ A linear transformation $L_1(\mathbf{x})$ from $\mathbb{R}^3$ to $\mathbb{R}^3$ rotates a vector $\mathbf{x} \in \mathbb{R}^3$ by an angle ϕ around the x_3 axis, where ϕ is positive when the rotation is counterclockwise, that is when you rotate the vector in the direction from the x_1 axis towards the x_2 axis. A second transformation $L_2(\mathbf{x})$ from $\mathbb{R}^3$ to $\mathbb{R}^2$ projects a vector $\mathbf{x} \in \mathbb{R}^3$ onto the (x_1, x_2) plane.

   (a) $\boxed{0.4}$ Calculate the matrix A_1 of the transformation $L_1(\mathbf{x})$.

   (b) $\boxed{0.4}$ Calculate the matrix A_2 of the transformation $L_2(\mathbf{x})$.

   Both for (a) and (b) you need to explain how you got the matrix. (It is not enough to just write the correct matrix.)

   (c) $\boxed{0.4}$ Calculate the single matrix A of the combined transformation in which one first applies L_1 to $\mathbf{x}$ and then L_2 to the result of $L_1(\mathbf{x})$.

   (d) $\boxed{0.8}$ Find the eigenvalues and eigenvectors of A_1.

Please turn over

3. $\boxed{2}$ Given a generic matrix $A = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

   (a) $\boxed{0.6}$ Find its eigenvalues in terms of the elements of A, a_{ij}.

   Hint: If you choose things carefully, the characteristic polynomial will be naturally factored into the product of a first- and a second-order polynomial. Write the roots of those 2 polynomials in terms of the a_{ij}, noticing that $a_{13} = a_{23} = 0$, as given in the matrix.

   (b) $\boxed{0.6}$ Suppose that one of the eigenvalues of the previous matrix is 6, the trace of the matrix is $\mathrm{Tr}(A) = 16$, and the determinant of the 2×2 block of the matrix A given by $A_1 = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is $\det(A_1) = 21$. Find the other 2 eigenvalues of A.

   (c) $\boxed{0.8}$ Find all the elements of the matrix A if a set of three eigenvectors of A are:

   $$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -3 \\ 7 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

   where the order of the eigenvectors corresponds to the eigenvalues you found in (b) sorted such that $\lambda_1 < \lambda_2 < \lambda_3$.

4. $\boxed{2}$ A coupled system of differential equations is given by

$$\begin{aligned} x_1'(t) &= 2\,x_1(t) + 3\,x_2(t) \\ x_2'(t) &= -2\,x_1(t) + 6\,x_2(t) \end{aligned}$$

   with boundary conditions $x_1(0) = 4,\ x_2(0) = 1$.

   (a) $\boxed{1}$ Compute the eigenvalues and eigenvectors of the matrix of this system.

   (b) $\boxed{1}$ Use the eigenvalues and eigenvectors to compute the (real-valued) solution of the initial-value problem.

   Note: The eigenvalues/eigenvectors have the square roots of some number, so don't be discouraged if you get that.

5. $\boxed{2}$ Given the vector $\mathbf{v}_1 = \begin{bmatrix} 2 \\ a \\ b \end{bmatrix}$ in $\mathbb{R}^3$, with a and b being any two real numbers.

   (a) $\boxed{0.3}$ Write the unit vector in the direction of the given vector.

   (b) $\boxed{0.3}$ Write the vector $\mathbf{v}_1$ in (a), when $a = 1$ and $b = 2$, and find the unit vector in $\mathbb{R}^3$ in the direction of this vector.

   (c) $\boxed{0.4}$ Using the dot product, find the general vector, $\mathbf{v}_2$ in $\mathbb{R}^3$, different from the null vector, that is orthogonal to the vector in part (b). (*Note: There are infinite vectors, each with a different modulus, that are orthogonal to a given vector. Here we ask you to write the general form of that vector; start with components x_1, x_2 and x_3, to find the form of that general vector.*)

   (d) $\boxed{0.3}$ Find the unit vector in $\mathbb{R}^3$ in the direction of $\mathbf{v}_2$. (*Hint: For this you can choose one, any, of the infinite vectors that you found in the previous point.*)

   (e) $\boxed{0.3}$ Write down any vector in $\mathbb{R}^3$ in the direction of $\mathbf{v}_2$.

   (f) $\boxed{0.4}$ Use the cross product to find a vector perpendicular to both the vector from part (a) and (c)

Solutions:

1. Here I solve the problem using the augmented matrix and row reduction, but the students can choose other methods since the question did not specify which method to use.

   (a) We can write the system of equations as $A\mathbf{x} = \mathbf{b}$, where:

   $$A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & \alpha \\ 1 & 1 & 0 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ and } \mathbf{b} = \begin{bmatrix} \beta \\ -2 \\ 3\beta \end{bmatrix}.$$

   For $\alpha = 2$ and $\beta = 1$, the matrix A and vector $\mathbf{b}$ are:

   $$A = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}.$$

   Use for instance row reduction on the augmented matrix:

   $$\begin{bmatrix} 2 & 1 & -3 & | & 1 \\ 1 & 0 & 2 & | & -2 \\ 1 & 1 & 0 & | & 3 \end{bmatrix},$$

   to get:

   $$\begin{bmatrix} 1 & 0 & 0 & | & -2 \\ 0 & 1 & 0 & | & 5 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

   0.5 points for solving the system; I used row reduction, but any other (correct) method is acceptable. They have to explain which method they applied.

   And the solution is:

   $$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2 \\ 5 \\ 0 \end{bmatrix}.$$

   0.1 points for writing the solution in this or any other form

   They loose this extra 0.1 points nit they do not give the solution

   (b) The system has only 1 solution $\Leftrightarrow \det(A) \neq 0$.

   0.2 points for this or similar statement; they get the points also if they use the statement without making it explicitly

   $\det(A) = -\alpha - 3$. The system will have exactly one solution whenever $-\alpha - 3 \neq 0 \Rightarrow \alpha \neq -3$.

   0.2 points for the correct answer.

   They get 0 points if they use other methods, even if the result is correct, since the question asked them to use the determinant.

(c) Apply row reduction to the general matrix:

$$\begin{bmatrix} 2 & 1 & -3 & | & \beta \\ 1 & 0 & \alpha & | & -2 \\ 1 & 1 & 0 & | & 3\beta \end{bmatrix},$$

to get

$$\begin{bmatrix} 2 & 1 & -3 & \beta \\ 0 & -\frac{1}{2} & \frac{2\alpha+3}{2} & \frac{-4-\beta}{2} \\ 0 & 0 & \alpha+3 & 2(\beta-1) \end{bmatrix}$$

0.5 points for doing row reduction and getting to this, or a similar, form that allows them to answer the question

It is apparent that the last row would be inconsistent if $\alpha = -3$ and $\beta \neq 1$.

0.5 for the result

If they use a different method, they do not get points for this exercise, even if the result is correct, because the question was to use row reduction

(Notice that in the current form we cannot state anything from rows 1 or 2.)

2. (a) You need to check what the effect of the transformation is on the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, e.g. sketching the transformation of the unit vectors. I do not show that part but they must do something to show how they got the matrix.

The matrix is: $A_1 = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

0.4 points for the matrix

They get no points here if they do not explain how they found the matrix

(b) You need to check what the effect of the transformation is on the unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$, e.g. sketching the transformation of the unit vectors. I do not show that part but they must do something to show how they got the matrix.

The matrix is:

$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

0.4 points for the matrix

They get no points here if they do not explain how they found the matrix

(c) $A = A_2 A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \end{bmatrix}$.

0.2 points for writing the matrix product in the right order

0.2 points for the correct end result

They get no points if they write $A_1 A_2$

During the exam some students mentioned that if they could not solve part (b) they would not be able to solve (c) either, which might be unfair. So I said that they should at least explain how they would solve part (c) for some other matrix instead of the one in part (b). If they did that, they should get the full points for part (c), but not for (b) if they did not solve it

(d) $A_1 = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

To find the eigenvectors solve $\det(A_1 - \lambda I) = 0$, $\det \begin{bmatrix} \cos\phi - \lambda & -\sin\phi & 0 \\ \sin\phi & \cos\phi - \lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix} = 0 \implies$

$\implies (1-\lambda)\left[(\cos\phi - \lambda)^2 + \sin^2\phi\right] = 0$

$(1-\lambda) = 0 \implies \lambda = 1$

or

$\left[(\cos\phi - \lambda)^2 + \sin^2\phi\right] = 0 \implies \cos^2\phi - 2\lambda\cos\phi + \lambda^2 + \sin^2\phi = 0 \implies \lambda^2 - 2\lambda\cos\phi + \sin^2\phi + \cos^2\phi = 0 \implies$

$\lambda^2 - 2\lambda\cos\phi + 1 = 0 \implies \lambda = \dfrac{2\cos\phi \pm \sqrt{4\cos^2\phi - 4}}{2} = \cos\phi \pm i\sqrt{1-\cos^2\phi} = \cos\phi \pm i\sin\phi.$

The eigenvalues are then:

$\lambda_1 = 1$, $\lambda_2 = \cos\phi + i\sin\phi$, $\lambda_3 = \cos\phi - i\sin\phi$.

0.3 points maximum, 0.1 points for each correct eigenvalue

They get 0.1(instead of 0.3) points if they say how they would find the eigenvaluess, but they don't do it

To find the eigenvectors use row reduction to solve $(A - \lambda I)\mathbf{x} = \mathbf{0} \implies \left[\begin{array}{ccc|c} \cos\phi - \lambda & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi - \lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \end{array}\right]$

(i) $\lambda_1 = 1$:

$\left[\begin{array}{ccc|c} \cos\phi - 1 & -\sin\phi & 0 & 0 \\ \sin\phi & \cos\phi - 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_3 =$free

$x_1(\cos\phi - 1) = x_2\sin\phi$

$x_1\sin\phi = -x_2(\cos\phi - 1) \implies x_2\sin^2\phi = -x_2(\cos\phi - 1)^2 \implies$

$x_2(\sin^2\phi + \cos^2\phi + 2\cos\phi - 1) = 0 \implies 2x_2\cos\phi = 0$ for any ϕ, hence $x_2 = 0$.

And from the previous one $x_1 = 0$. Take $x_3 = 1 \implies \mathbf{v}_1 = [\ 0\ 0\ 1\ ]^T$.

0.2 points for this eigenvector

(ii) $\lambda_2 = \cos\phi + i\sin\phi$:

$\left[\begin{array}{ccc|c} -i\sin\phi & -\sin\phi & 0 & 0 \\ \sin\phi & -i\sin\phi & 0 & 0 \\ 0 & 0 & 1-\cos\phi - i\sin\phi & 0 \end{array}\right] \implies \left[\begin{array}{ccc|c} \sin\phi & -i\sin\phi & 0 & 0 \\ \sin\phi & -i\sin\phi & 0 & 0 \\ 0 & 0 & 1-\cos\phi - i\sin\phi & 0 \end{array}\right] \implies$

$\implies \left[\begin{array}{ccc|c} \sin\phi & -i\sin\phi & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1-\cos\phi - i\sin\phi & 0 \end{array}\right]$

$x_1\sin\phi - ix_2\sin\phi = 0 \implies x1 = ix_2$

$x_2 =$free

$x_3(1 - \cos\phi - i\sin\phi) = 0$ for any ϕ so $x_3 = 0$. Take $x_2 = 1 \implies \mathbf{v}_2 = [\ i\ 1\ 0\ ]^T$.

0.2 points for this eigenvector

(iii) $\lambda_2 = \cos\phi - i\sin\phi$:

This is the complex conjugate of λ_2; the corresponding eigenvector is the complex conjugate of $\mathbf{v}_2$, $\mathbf{v}_3 = [\ -i\ 1\ 0\ ]^T$.

0.1 points for this eigenvector

They get 0.2 (instead of 0.5) points if they say how they would find the eigenvectors, but they don't do it

3. (a) To find the eigenvalues I need to solve:

$\det(A-\lambda I)=0 \Rightarrow \det\begin{bmatrix} a_{11}-\lambda & a_{12} & 0 \\ a_{21} & a_{22}-\lambda & 0 \\ a_{31} & a_{32} & a_{33}-\lambda \end{bmatrix}$. If I expand through the las column:

$(a_{33}-\lambda)\left[\lambda^2-(a_{11}+a_{22})\lambda+(a_{11}a_{22}-a_{12}a_{21})\right]=0.$

0.2 points for writing the way to find the eigenvalues and the characteristic equation

From this I know that one eigenvalue is $\lambda_1 = a_{33}$.

0.1 points for this eigenvalue

For the other 2 eigenvalues I need to solve $\lambda^2-(a_{11}+a_{22})\lambda+(a_{11}a_{22}-a_{12}a_{21})=0$:

$$\lambda = \frac{a_{11}+a_{22}}{2} \pm \sqrt{\left(\frac{a_{11}+a_{22}}{2}\right)^2-(a_{11}a_{22}-a_{12}a_{21})}.$$

0.3 points for writing this to find the other 2 eigenvalues

If someone had a different (correct) way to find the eigenvalues they should get the full 0.3 points

(b) $\mathrm{Tr}(A) = a_{11}+a_{22}+a_{33} = \lambda_1+\lambda_2+\lambda_3 = 16$ and $\lambda_1 = 6$. I will assume that this eigenvalue is the one from $(a_{33}-\lambda)=0$ in point (a). (I know that, since the eigenvalues are unique, if this works this must be the correct solution.) So $a_{33}=6$ and $a_{11}+a_{22}=10$.

0.1 points for saying or using that $\mathrm{Tr}(A)$ is equal to the sum of the eigenvalues

At the same time, $\det(A_1) = a_{11}a_{22}-a_{12}a_{21} = 21$, so I can find the other 2 eigenvalues from the solution of the quadratic equation in point (a):

0.1 points for writing or using $\det(A)$ to solve the quadratic equation of the previous point

$\lambda = \frac{10}{2} \pm \sqrt{\left(\frac{10}{2}\right)^2-21} = 5 \pm 2$. So the other eigenvalues are $\lambda_2 = 3$ and $\lambda_3 = 7$.

0.4 points for these other 2 eigenvalues, 0.2 points for each of them

After the exam, a student contacted me and told me that he did not assume that $\lambda_1 = a_{33}$, and nevertheless found the 3 eigenvalues in a consistent way, but the values were different than the ones I give here. He also did it my way, and he could not decide which ones were right. At this point, if there is another way to do this that is different than how I did it, and that way is correct, the students should get the full points (again, as long as the procedure is consistent and correct). In the next point, however, since I give the eigenvectors, the other solution that this student found would not work. He noticed that (he told me) and went on with the solution that was the same as mine. This guy did more than I had expected, so he should not loose points (if anything, he should get extra points!) for being so thorough

(c) Since the given eigenvectors correspond the eigenvalues sorted in increasing order, I can make 2 matrices $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ and $P = [\mathbf{v}_1\ \ \mathbf{v}_2\ \ \mathbf{v}_3]$ such that $A = PDP^{-1}$.

0.3 points for this or similar statement, also if they do not make this statement explicitly but they use this to solve the problem

Then $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix}$, $P = \begin{bmatrix} 3 & 0 & 1 \\ -3 & 0 & 1 \\ 7 & 1 & 1 \end{bmatrix}$ and $P^{-1} = \frac{1}{6}\begin{bmatrix} 1 & -1 & 0 \\ -10 & 4 & 6 \\ 3 & 3 & 0 \end{bmatrix}$

0.4 points (0.1 points for D, 0.1 points for P and 0.2 points for P^{-1})

Such that $A = PDP^{-1} = \begin{bmatrix} 5 & 2 & 0 \\ 2 & 5 & 0 \\ -3 & 4 & 6 \end{bmatrix}$

$\boxed{\text{0.1 points for } A\text{, or its elements}}$

4. A coupled system of differential equations is given by

$$\begin{aligned} x_1'(t) &= 2\,x_1(t) + 3\,x_2(t) \\ x_2'(t) &= -2\,x_1(t) + 6\,x_2(t) \end{aligned}$$

with boundary conditions $x_1(0) = 4,\ x_1(0) = 1$.

(a) To find the eigenvalues solve $\det(A - \lambda I) = 0 \implies \det \begin{bmatrix} 2-\lambda & 3 \\ -2 & 6-\lambda \end{bmatrix} = 0 \implies$

0.2 points for writing this or similar statement. They get the points also if they use this even if they do not say it explicitly

$(2-\lambda)(6-\lambda) + 6 = 0 \implies \lambda^2 - 8\lambda + 18 = 0 \implies \lambda_1 = 4 + \sqrt{2}i,\ \lambda_2 = 4 - \sqrt{2}i.$

0.2 points for the eigenvalues (0.1 points for each eigenvalue)

To calculate the eigenvectors I need to solve $(A - \lambda_i I)\mathbf{v_i} = \mathbf{0}$ for each eigenvalue.

0.2 points for writing this or similar statement. They get the points also if they use this even if they do not say it explicitly

But since the eigenvalues are complex conjugates, the eigenvectors will also be, so I only need to calculate one eigenvector. Take $\lambda_1 = 4 + \sqrt{2}i$. The augmented matrix will be:

$$\begin{bmatrix} 2-(4+\sqrt{2}i & 3 & | & 0 \\ 2 & 6-(4+\sqrt{2}i & | & 0 \end{bmatrix} = \begin{bmatrix} -2-\sqrt{2}i & 3 & | & 0 \\ 2 & 2-\sqrt{2}i & | & 0 \end{bmatrix}$$

Row operations (e.g., multiply row 2 by $2 + \sqrt{2}i$, which gives that the second row is 2 times row 1) give:

$$\begin{bmatrix} -2-\sqrt{2}i & 3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

So $\dfrac{(2+\sqrt{2}i)x_1}{3}$. If I take $x_1 = 2 - \sqrt{2}i$ then $x_2 = 2$. So one eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 2-\sqrt{2}i \\ 2 \end{bmatrix}$. The other one is the complex conjugate of this one, $\mathbf{v}_2 = \begin{bmatrix} 2+\sqrt{2}i \\ 2 \end{bmatrix}$.

0.4 points for the eigenvectors (0.2 points for each eigenvector)

(b) I can use the formula from the notes (which they have during the exam) using $\mathbf{v}_1$ and λ_1, where $a = \mathrm{Re}(\lambda_1)$ and $b = \mathrm{Im}(\lambda_1)$:

They can do the same using λ_2 and $\mathbf{v}_2$

$\mathbf{y}_1(t) = [\mathrm{Re}(\mathbf{v}_1)\cos bt - \mathrm{Im}(\mathbf{v}_1)\sin bt]\, e^{at}$
$\mathbf{y}_2(t) = [\mathrm{Re}(\mathbf{v}_1])\sin bt + \mathrm{Im}(\mathbf{v}_1)\cos bt]\, e^{at}$
and $\mathbf{x}(t) = c_1\mathbf{y}_1(t) + c_2\mathbf{y}_2(t)$.

0.2 points for writing these equations, or similar, or for using them without writing them explicitly

They should not use the formula for the case of real eigenvalues/eigenvectors, since the 2 eigenvectors are not linearly independent. If they do, they get no points for part (b) of the exercise

$$\mathbf{y}_1(t) = \left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} \cos\sqrt{2}t + \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} \sin\sqrt{2}t \right) e^{4t}$$
$$\mathbf{y}_2(t) = \left(\begin{bmatrix} 2 \\ 2 \end{bmatrix} \sin\sqrt{2}t - \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} \sin\sqrt{2}t \right) e^{4t} \implies$$

0.2 points for the 2 particular solutions (0.1 points per solution)

$$\mathbf{x}(t) = \begin{bmatrix} 2c_1\cos\sqrt{2}t + \sqrt{2}c_1\sin\sqrt{2}t + 2c_2\sin\sqrt{2}t - \sqrt{2}c_2\cos\sqrt{2}t \\ 2c_1\cos\sqrt{2}t + 2c_2\sin\sqrt{2}t \end{bmatrix} e^{4t}.$$

0.2 points for the general solution

The condition $x_1(0) = 4$ $x_2(0) = 1$ imply:

$$\mathbf{x}(0) = \begin{bmatrix} 2c_1 - \sqrt{2}c_2 \\ 2c_1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \implies c_1 = \frac{1}{2}, c_2 = -\frac{3}{\sqrt{2}}.$$

0.2 points for the values of c_1 and c_2

The solution of the initial-value problem is then:

$$\mathbf{x}(t) = \begin{bmatrix} \cos\sqrt{2}t + \frac{\sqrt{2}}{2}\sin\sqrt{2}t - 3\sqrt{2}\sin\sqrt{2}t + 3\cos\sqrt{2}t \\ \cos\sqrt{2}t - 3\sqrt{2}\sin\sqrt{2}t \end{bmatrix} e^{4t} = \begin{bmatrix} 4\cos\sqrt{2}t - \frac{5\sqrt{2}}{2}\sin\sqrt{2}t \\ \cos\sqrt{2}t - 3\sqrt{2}\sin\sqrt{2}t \end{bmatrix} e^{4t}.$$

0.2 points for the solution of the initial-value problem. They do not need to simplify the expressions as I did

5. (a) To find the unit vector divide the given vector by its modulus. The modulus of $\mathbf{v}_1$ is $|\mathbf{v}_1| = \sqrt{4 + a^2 + b^2}$. Hence the unit vector is:

0.1 points for the modulus

$$\frac{\mathbf{v}_1}{|\mathbf{v}_1|} = \frac{1}{\sqrt{4 + a^2 + b^2}} \begin{bmatrix} 2 \\ a \\ b \end{bmatrix}$$

0.2 points for the unit vector

They get the full 0.3 points if they write the unit vector directly, without first writing the modulus

(b) When $a = 1$ and $b = 2$, $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$, $|\mathbf{v}_1| = \sqrt{4+1+4} = 3$, and the unit vector is $\dfrac{\mathbf{v}_1}{|\mathbf{v}_1|} = \dfrac{1}{3}\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

0.3 points for replacing the values and getting the unit vector

(c) In general $\mathbf{v}_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and I need to find x_1, x_2 and x_3 such that $\mathbf{v}_2^T.\mathbf{v}_1 = 0$, with the condition that x_1, x_2 and x_3 cannot all be 0:

0.2 points for saying this or using this without saying it explicitly

$\mathbf{v}_2^T.\mathbf{v}_1 = 2x_1 + x_2 + 2x_3 = 0 \implies x_3 = -\left(x_1 + \dfrac{x_2}{2}\right)$. Therefore the general form of $\mathbf{v}_2$ is: $\mathbf{v}_2 = \begin{bmatrix} x_1 \\ x_2 \\ -\left(x_1 + \frac{x_2}{2}\right) \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1/2 \end{bmatrix}$, with x_1 and x_2 free.

0.2 points for the general vector. It doesn't have to be given in parametric form, but it must be general, i.e., inn terms of the free variables x_1 and x_2, and using numbers for those. I used x_1 and x_2, but they can use another set of two free variables

They loose the 0.2 points if they do not write one variable in terms of the other two

(d) There are infinite vectors perpendicular to a given vector, all of them contained in a plane that is perpendicular to the given vector. To use one of those infinite vectors I am free to choose x_2 and x_3. Take the simplest case, $x_2 = 0$ and $x_1 = 1$ (I cannot take both $x_1 = 0$ and $x_1 = 0$ cause then $x_3 = 0$ and it would be $\mathbf{v}_2 = \mathbf{0}$), so $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

0.1 points for fixing the values (any values) of the free variables and writing this vector

The modulus of $\mathbf{v}_2$ is $|\mathbf{v}_2| = \sqrt{2}$, and the unit vector in the direction of $\mathbf{v}_2$ is:

$$\frac{\mathbf{v}_2}{|\mathbf{v}_2|} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

0.2 points for writing the corresponding unit vector

(e) For any $c \in \mathbb{R}$, $c\mathbf{v}_2$ is in the direction of $\mathbf{v}_2$.

0.3 points for realising that it is a constant times the vector (or unit vector) that they had in the previous part

They get the 0.2 points if they solved this differently but correctly, as long as they wrote **any** vector in $\mathbb{R}^3$ along $\mathbf{v_2}$

.

(f) I calculate a new vector $\mathbf{v}_3 = \mathbf{v}_1 \times \mathbf{v}_2$ from parts (a) and (c):

$$\mathbf{v}_3 = \det \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ 2 & a & b \\ x_1 & x_2 & -\left(x1 + \frac{x_2}{2}\right) \end{bmatrix} = \left[-a\left(x_1 + \frac{x_2}{2}\right) - bx_2\right] \mathbf{e}_1 + \left[2\left(x_1 + \frac{x_2}{2}\right) + bx_1\right] \mathbf{e}_2 + \left[2x_2 - ax_1\right] \mathbf{e}_3.$$

0.4 points for any (correct) version of the cross product

They have to have variables in the components of the final vector, since the vectors in part (a) and (c) had variables in the components. If they replaced the variables with numbers, they do not get the points

Based on the questions in the chat during the exam, some may use the unit vector in (a). This is fine. Then they will have the same vector as in the solution above, only that bit will be divided by the modulus of $\mathbf{v}_1$.