Calculus 2
Midterm Exam – Solutions
Exam Date: March 25, 2022 (9:00 – 11:00)

university of groningen

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1) Consider the space curve C given by the vector function $\vec{r}\colon [0, 2\pi] \to \mathbb{R}^3$,

$$\vec{r}(t) = e^{-t}\,\vec{\imath} + e^{-t}\sin t\,\vec{\jmath} + e^{-t}\cos t\,\vec{k}, \qquad 0 \le t \le 2\pi.$$

$\boxed{9}$ a) Determine the first-, second- and third-order derivatives of $\vec{r}(t)$, i.e. calculate $\vec{r}\,'(t)$, $\vec{r}\,''(t)$ and $\vec{r}\,'''(t)$. Simplify as much as possible.

$\boxed{8}$ b) Find the length L of C and its parametrization by arc length s.

$\boxed{9}$ c) Determine the unit tangent vector $\vec{T}(t)$, principal normal vector $\vec{N}(t)$ and binormal vector $\vec{B}(t)$ to C.

$\boxed{6}$ d) Compute the curvature $\kappa(t)$ and the torsion $\tau(t)$ of C.

Solution. a) Since e^{-t} is a common factor in all three components of $\vec{r}(t)$ we may write

$$\vec{r}(t) = f(t)\vec{u}(t) \quad \text{with} \quad f(t) = e^{-t} \text{ and } \vec{u}(t) = \vec{\imath} + \sin t\,\vec{\jmath} + \cos t\,\vec{k}$$

and compute the derivative of $\vec{r}(t)$ by applying the Product Rule:

$$\begin{aligned}
\vec{r}\,'(t) &= f'(t)\vec{u}(t) + f(t)\vec{u}\,'(t) \\
&= \big(e^{-t}\big)'\big[\vec{\imath} + \sin t\,\vec{\jmath} + \cos t\,\vec{k}\,\big] + e^{-t}\big[(1)'\vec{\imath} + (\sin t)'\,\vec{\jmath} + (\cos t)'\,\vec{k}\,\big] \\
&= -e^{-t}\big[\vec{\imath} + \sin t\,\vec{\jmath} + \cos t\,\vec{k}\,\big] + e^{-t}\big[0\vec{\imath} + \cos t\,\vec{\jmath} - \sin t\,\vec{k}\,\big] \\
&= -e^{-t}\big[\vec{\imath} + (\sin t - \cos t)\,\vec{\jmath} + (\sin t + \cos t)\,\vec{k}\,\big]
\end{aligned}$$

(Note that we also needed to use $\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$ and the Chain Rule.) Thus we have

$$\vec{r}\,'(t) = -e^{-t}\big[\vec{\imath} + (\sin t - \cos t)\,\vec{\jmath} + (\sin t + \cos t)\,\vec{k}\,\big] \tag{1}$$

By differentiating (1) and applying the Product Rule, we compute the second derivative of $\vec{r}(t)$:

$$\begin{aligned}
\vec{r}\,''(t) &= \big(-e^{-t}\big)'\big[\vec{\imath} + (\sin t - \cos t)\,\vec{\jmath} + (\sin t + \cos t)\,\vec{k}\,\big] - e^{-t}\big[(1)'\vec{\imath} + (\sin t - \cos t)'\,\vec{\jmath} + (\sin t + \cos t)'\,\vec{k}\,\big] \\
&= e^{-t}\big[\vec{\imath} + (\sin t - \cos t)\,\vec{\jmath} + (\sin t + \cos t)\,\vec{k}\,\big] - e^{-t}\big[0\vec{\imath} + (\cos t + \sin t)\,\vec{\jmath} + (\cos t - \sin t)\,\vec{k}\,\big] \\
&= e^{-t}\big[\vec{\imath} - 2\cos t\,\vec{\jmath} + 2\sin t\,\vec{k}\,\big]
\end{aligned}$$

(Again, we used $\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$, the Chain Rule as well as the Sum and Constant Multiple Rules.) We have

$$\vec{r}\,''(t) = e^{-t}\big[\vec{\imath} - 2\cos t\,\vec{\jmath} + 2\sin t\,\vec{k}\,\big] \tag{2}$$

Differentiating (2) via the Product Rule yields the third derivative of $\vec{r}(t)$:

$$\begin{aligned}
\vec{r}\,'''(t) &= \big(e^{-t}\big)'\big[\vec{\imath} - 2\cos t\,\vec{\jmath} + 2\sin t\,\vec{k}\,\big] + e^{-t}\big[(1)'\vec{\imath} - 2(\cos t)'\,\vec{\jmath} + 2(\sin t)'\,\vec{k}\,\big] \\
&= -e^{-t}\big[\vec{\imath} - 2\cos t\,\vec{\jmath} + 2\sin t\,\vec{k}\,\big] + e^{-t}\big[0\vec{\imath} + 2\sin t\,\vec{\jmath} + 2\cos t\,\vec{k}\,\big] \\
&= -e^{-t}\big[\vec{\imath} - 2(\sin t + \cos t)\,\vec{\jmath} + 2(\sin t - \cos t)\,\vec{k}\,\big]
\end{aligned}$$

($\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$, the Chain, the Sum and the Constant Multiple Rules.) We find

$$\vec{r}\,'''(t) = -e^{-t}\big[\vec{\imath} - 2(\sin t + \cos t)\,\vec{\jmath} + 2(\sin t - \cos t)\,\vec{k}\,\big] \tag{3}$$

b) We need to compute the length of the derivative $\vec{r}\,'(t)$ (cf. (1)). Since $|c\vec{v}| = |c||\vec{v}|$ for any scalar c and vector $\vec{v}$, we may write

$$\begin{aligned}
|\vec{r}\,'(t)| &= e^{-t}\sqrt{1^2 + (\sin t - \cos t)^2 + (\sin t + \cos t)^2} \\
&= e^{-t}\sqrt{1 + \sin^2 t - 2\sin t\cos t + \cos^2 t + \sin^2 t + 2\sin t\cos t + \cos^2 t} \\
&= e^{-t}\sqrt{1 + 2(\sin^2 t + \cos^2 t)} = e^{-t}\sqrt{3}
\end{aligned}$$

(where we used the formula $|\langle a, b, c\rangle| = \sqrt{a^2 + b^2 + c^2}$ and the trigonometric identity $\sin^2 t + \cos^2 t = 1$). Hence we have

$$|\vec{r}\,'(t)| = e^{-t}\sqrt{3} \tag{4}$$

and the length of the curve is

$$L = \int_a^b |\vec{r}\,'(t)|\,dt = \int_0^{2\pi} e^{-t}\sqrt{3}\,dt = \sqrt{3}\left[-e^{-t}\right]_{t=0}^{t=2\pi} = \sqrt{3}\left(1 - e^{-2\pi}\right)$$

that is

$$L = \sqrt{3}\left(1 - e^{-2\pi}\right) \tag{5}$$

Similarly, the arc length function is

$$s(t) = \int_a^t |\vec{r}\,'(u)|\,du = \int_0^t e^{-u}\sqrt{3}\,du = \sqrt{3}\left[-e^{-u}\right]_{u=0}^{u=t} = \sqrt{3}\left(1 - e^{-t}\right)$$

which can be inverted (solved for t) as follows

$$s(t) = \sqrt{3}\left(1 - e^{-t}\right) \quad \Rightarrow \quad e^{-t} = 1 - \frac{s}{\sqrt{3}} \quad \Rightarrow \quad t(s) = \ln\left(\frac{\sqrt{3}}{\sqrt{3} - s}\right) \tag{6}$$

The parametrization of C by arc length is $\vec{\tilde{r}}(s) = \vec{r}(t(s))$, that is

$$\vec{\tilde{r}}(s) = \left(1 - \frac{s}{\sqrt{3}}\right)\left[\vec{\imath} + \sin\ln\left(\frac{\sqrt{3}}{\sqrt{3} - s}\right)\vec{\jmath} + \cos\ln\left(\frac{\sqrt{3}}{\sqrt{3} - s}\right)\vec{k}\right], \quad 0 \le s \le \sqrt{3}\left(1 - e^{-2\pi}\right) \tag{7}$$

c) The results of parts a) and b) combined give us the unit tangent vector

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = -\frac{1}{\sqrt{3}}\left[\vec{\imath} + (\sin t - \cos t)\,\vec{\jmath} + (\sin t + \cos t)\,\vec{k}\,\right] \tag{8}$$

Its derivative is the vector function

$$\begin{aligned}
\vec{T}\,'(t) &= -\frac{1}{\sqrt{3}}\left[(1)'\vec{\imath} + (\sin t - \cos t)'\,\vec{\jmath} + (\sin t + \cos t)'\,\vec{k}\,\right] \\
&= -\frac{1}{\sqrt{3}}\left[0\vec{\imath} + (\cos t + \sin t)\,\vec{\jmath} + (\cos t - \sin t)\,\vec{k}\,\right] \\
&= -\frac{1}{\sqrt{3}}\left[(\cos t + \sin t)\,\vec{\jmath} + (\cos t - \sin t)\,\vec{k}\,\right]
\end{aligned}$$

which has the length

$$\begin{aligned}
|\vec{T}\,'(t)| &= \frac{1}{\sqrt{3}}\sqrt{(\cos t + \sin t)^2 + (\cos t - \sin t)^2} \\
&= \frac{1}{\sqrt{3}}\sqrt{\cos^2 t + 2\cos t\sin t + \sin^2 t + \cos^2 t - 2\cos t\sin t + \sin^2 t} \\
&= \frac{1}{\sqrt{3}}\sqrt{2(\cos^2 t + \sin^2 t)} = \frac{1}{\sqrt{3}}\sqrt{2}
\end{aligned}$$

Therefore the unit normal vector is

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = -\frac{1}{\sqrt{2}}\big[(\cos t + \sin t)\,\vec{\jmath} + (\cos t - \sin t)\,\vec{k}\,\big] \tag{9}$$

and the binormal vector is

$$\begin{aligned}
\vec{B}(t) &= \vec{T}(t) \times \vec{N}(t) \\
&= -\frac{1}{\sqrt{3}}\big[\vec{\imath} + (\sin t - \cos t)\,\vec{\jmath} + (\sin t + \cos t)\,\vec{k}\,\big] \times \left(-\frac{1}{\sqrt{2}}\right)\big[(\cos t + \sin t)\,\vec{\jmath} + (\cos t - \sin t)\,\vec{k}\,\big] \\
&= \frac{1}{\sqrt{6}}\big[\vec{\imath} + (\sin t - \cos t)\,\vec{\jmath} + (\sin t + \cos t)\,\vec{k}\,\big] \times \big[(\cos t + \sin t)\,\vec{\jmath} + (\cos t - \sin t)\,\vec{k}\,\big] \\
&= \frac{1}{\sqrt{6}}\begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 1 & (\sin t - \cos t) & (\sin t + \cos t) \\ 0 & (\cos t + \sin t) & (\cos t - \sin t) \end{vmatrix} \\
&= \frac{1}{\sqrt{6}}\big[\{-(\sin t - \cos t)^2 - (\sin t + \cos t)^2\}\,\vec{\imath} - (\cos t - \sin t)\,\vec{\jmath} + (\cos t + \sin t)\,\vec{k}\,\big] \\
&= \frac{1}{\sqrt{6}}\big[-2(\sin^2 t + \cos^2 t)\,\vec{\imath} + (\sin t - \cos t)\,\vec{\jmath} + (\sin t + \cos t)\,\vec{k}\,\big] \\
&= \frac{1}{\sqrt{6}}\big[-2\,\vec{\imath} + (\sin t - \cos t)\,\vec{\jmath} + (\sin t + \cos t)\,\vec{k}\,\big]
\end{aligned}$$

So we have

$$\vec{B}(t) = \frac{1}{\sqrt{6}}\big[-2\,\vec{\imath} + (\sin t - \cos t)\,\vec{\jmath} + (\sin t + \cos t)\,\vec{k}\,\big] \tag{10}$$

d) The curvature can be computed using the results of parts b) and c):

$$\kappa(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \frac{\sqrt{2}/\sqrt{3}}{e^{-t}\sqrt{3}} = \frac{\sqrt{2}}{3}e^t \tag{11}$$

To determine the torsion we compute the derivative of the binormal vector:

$$\begin{aligned}
\vec{B}\,'(t) &= \frac{1}{\sqrt{6}}\big[(-2)'\,\vec{\imath} + (\sin t - \cos t)'\,\vec{\jmath} + (\sin t + \cos t)'\,\vec{k}\,\big] \\
&= \frac{1}{\sqrt{6}}\big[(\cos t + \sin t)\,\vec{\jmath} + (\cos t - \sin t)\,\vec{k}\,\big]
\end{aligned}$$

and write

$$\begin{aligned}
\tau(t) &= -\frac{\vec{B}\,'(t)\cdot\vec{N}(t)}{|\vec{r}\,'(t)|} \\
&= -\left(-\frac{1}{\sqrt{2}\sqrt{6}\,e^{-t}\sqrt{3}}\right)\big[(\cos t + \sin t)\,\vec{\jmath} + (\cos t - \sin t)\,\vec{k}\,\big]\cdot\big[(\cos t + \sin t)\,\vec{\jmath} + (\cos t - \sin t)\,\vec{k}\,\big] \\
&= \frac{1}{6}e^t\big[(\cos t + \sin t)^2 + (\cos t - \sin t)^2\big] \\
&= \frac{1}{6}e^t\big[\cos^2 t + 2\cos t\sin t + \sin^2 t + \cos^2 t - 2\cos t\sin t + \sin^2 t\big] \\
&= \frac{1}{6}e^t\big[2(\cos^2 t + \sin^2 t)\big] = \frac{1}{6}e^t\big[2\big] = \frac{1}{3}e^t
\end{aligned}$$

Thus we have found that

$$\tau(t) = \frac{1}{3}e^t \tag{12}$$

2) Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = 10 - 2x + 8y + x^2 + 2y^2$.

$\boxed{6}$ a) Compute all first- and second-order partial derivatives of $f(x,y)$.

$\boxed{4}$ b) Determine the maximum rate of change of $f(x,y)$ at $(x,y) = (0,0)$.

$\boxed{2}$ c) Find the equation $z = L(x,y)$ for the tangent plane to the graph of $f(x,y)$ at the point $(0,0,10)$.

$\boxed{12}$ d) Find the absolute maximum and minimum of $f(x,y)$ on the closed region $E = \{(x,y) \mid x^2 - 2x + 2y^2 \leq 7\}$.

Solution. a) The first-order partial derivatives are

$$f_x = \frac{\partial}{\partial x}(10 - 2x + 8y + x^2 + 2y^2) = -2 + 2x \qquad f_y = \frac{\partial}{\partial y}(10 - 2x + 8y + x^2 + 2y^2) = 8 + 4y \tag{13}$$

Differentiating these with respect to x and y, we obtain the second-order partial derivatives

$$f_{xx} = \frac{\partial}{\partial x}(-2 + 2x) = 2, \quad f_{yy} = \frac{\partial}{\partial y}(8 + 4y) = 4 \tag{14}$$

and

$$f_{xy} = \frac{\partial}{\partial y}(-2 + 2x) = 0, \qquad f_{yx} = \frac{\partial}{\partial x}(8 + 4y) = 0 \tag{15}$$

b) The maximum rate of change at any point is given by the length of the gradient vector. It follows from (13) that the gradient vector is

$$\nabla f(x,y) = \langle 2x - 2, 4y + 8 \rangle \tag{16}$$

At at the origin we have the vector $\nabla f(0,0) = \langle -2, 8 \rangle$ whose length is

$$|\nabla f(0,0)| = \sqrt{(-2)^2 + 8^2} = \sqrt{68} = 2\sqrt{17} \tag{17}$$

c) Evaluating at the origin, the function gives $f(0,0) = 10$ whereas the first-order partial derivatives (found in (13)) take the values

$$f_x(0,0) = -2 \quad f_y(0,0) = 8 \tag{18}$$

Therefore the linearization of f at $(0,0)$ is

$$L(x,y) = f_x(0,0)(x - 0) + f_y(0,0)(y - 0) + f(0,0) = -2x + 8y + 10 \tag{19}$$

and the tangent plane can be expressed via the equation $z = -2x + 8y + 10$.

d) First we locate the stationary points for the function by solving $f_x = 0$, $f_y = 0$ for x and y:

$$\begin{cases} f_x = -2 + 2x = 0 \\ f_y = 8 + 4y = 0 \end{cases} \quad \Rightarrow \quad x = 1, \; y = -2. \tag{20}$$

Hence f has a single stationary point at $(1,-2)$. To determine what type of stationary point it is, we apply the Second Derivative Test. For this we compute

$$D = D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - \left[f_{xy}(0,0)\right]^2 = (2)(4) - [0^2] = 8 \tag{21}$$

Since we have $D > 0$ and $f_{xx}(0,0) = 2 > 0$ (cf. (14)), the function f has a local minimum at $(1,-2)$ with a value of $f(1,-2) = 1$.

Next, we set $g(x,y) = x^2 - 2x + 2y^2$ and solve $\nabla f(x,y) = \lambda \nabla g(x,y)$ and $g(x,y) = 7$ for x and y. The equations – written in component form – read

$$2x - 2 = \lambda(2x - 2) \tag{22}$$

$$4y + 8 = \lambda(4y) \tag{23}$$

$$x^2 - 2x + 2y^2 = 7 \tag{24}$$

Equation (22) can be rearranged to get $(2x-2)(\lambda-1)=0$ which implies that $x=1$ or $\lambda=1$. Assuming $\lambda=1$ would turn eq. (23) into $4y+8=4y$ which is a contradiction, therefore we must have $x=1$. Substituting this into eq. (24) yields $1^2-2(1)+2y^2=7$, i.e. $y^2=4$. This is solved by $y=\pm 2$. Hence the candidates for extrema along the boundary of the region E are $(1,2)$ and $(1,-2)$. The function f takes the following values at these places

$$f(1,2)=33, \quad f(1,-2)=1 \tag{25}$$

therefore f – taken over the region E – reaches its absolute maximum of 33 at $(1,2)$ and its absolute minimum of 1 at $(1,-2)$.

3) 16 Evaluate the double integral of the function seen in Problem 2 over the region $R=\{(x,y)\mid (x-1)^2+2(y+2)^2\leq 1\}$. (Hint: Change variables via the transformation T: $x=1+u\cos v$, $y=-2+\frac{1}{\sqrt{2}}u\sin v$.)

Solution. We start by noting that the region R is bounded the curve with equation $(x-1)^2+2(y+2)^2=1$ which is an ellipse centred at $(1,-2)$ with semi axes of lengths 1 and $1/\sqrt{2}$. This is the motivation for using the transformation T: $x=1+u\cos v$, $y=-2+\frac{1}{\sqrt{2}}u\sin v$. Let us express the function $f(x,y)=10-2x+8y+x^2+2y^2$ in terms of u and v. This can be done either by directly substituting $x=1+u\cos v$, $y=-2+\frac{1}{\sqrt{2}}u\sin v$ into f or by first rewriting f as

$$f(x,y)=10-2x+8y+x^2+2y^2=(x-1)^2+2(y+2)^2+1 \tag{26}$$

and then making the substitution. We obtain

$$\begin{aligned} f(x(u,v),y(u,v)) &= \big((1+u\cos v)-1\big)^2+2\big((-2+\tfrac{1}{\sqrt{2}}u\sin v)+2\big)^2+1 \\ &= \big(u^2\cos^2 v\big)+2\big(\tfrac{1}{2}u^2\sin^2 v\big)+1 \\ &= u^2(\cos^2 v+\sin^2 v)+1=u^2+1 \end{aligned}$$

that is

$$f(x(u,v),y(u,v))=u^2+1 \tag{27}$$

Next, we observe that the region R is the image of the rectangle $S=\{(u,v)\mid 0\leq u\leq 1,\ 0\leq v\leq 2\pi\}$ under the transformation T. This is because the defining inequality $(x-1)^2+2(y+2)^2\leq 1$ of R turns into $u^2\leq 1$ when written in terms of u and v. This is the only restriction we have on the u,v values. Taking $0\leq u\leq 1$ and $0\leq v\leq 2\pi$ ensures that T is one-to-one (except on the boundary of S). We also need to compute the Jacobian of T:

$$\frac{\partial(x,y)}{\partial(u,v)}=\det\begin{pmatrix}\dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v}\\[2mm] \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v}\end{pmatrix}=\det\begin{pmatrix}\cos v & -u\sin v\\ \frac{1}{\sqrt{2}}\sin v & \frac{1}{\sqrt{2}}u\cos v\end{pmatrix}=\frac{1}{\sqrt{2}}u(\cos^2 v+\sin^2 v)=\frac{1}{\sqrt{2}}u \tag{28}$$

Since u is non-negative over S, we have

$$\begin{aligned} \iint_R f(x,y)\,dA &= \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|du\,dv \\ &= \int_0^{2\pi}\int_0^1 (u^2+1)\frac{1}{\sqrt{2}}u\,du\,dv=\frac{1}{\sqrt{2}}\int_0^1(u^3+u)\,du\int_0^{2\pi}dv \\ &= \frac{1}{\sqrt{2}}\left[\frac{1}{4}u^4+\frac{1}{2}u^2\right]_{u=0}^{u=1}[v]_{v=0}^{v=2\pi}=\frac{1}{\sqrt{2}}\left(\frac{1}{4}+\frac{1}{2}\right)(2\pi)=\frac{1}{\sqrt{2}}\left(\frac{3}{4}\right)(2\pi)=\frac{3\pi}{2\sqrt{2}} \end{aligned}$$

That is

$$\iint_R f(x,y)\,dA=\frac{3\pi}{2\sqrt{2}} \tag{29}$$

4) Evaluate the following line integrals along the curve C in Problem 1:

$\boxed{8}$ a) $\displaystyle\int_C g(x,y,z)\,ds$ with the function $g(x,y,z) = \dfrac{yz}{x^3}$.

$\boxed{10}$ b) $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ with the vector field $\vec{F}(x,y,z) = z\,\vec{\jmath} - y\,\vec{k}$.

Solution. a) The curve C has the parametric equations $x(t) = e^{-t}$, $y(t) = e^{-t}\sin t$, $z(t) = e^{-t}\cos t$, $0 \le t \le 2\pi$. Therefore the function g takes the following values along C:

$$g(\vec{r}(t)) = g(x(t),y(t),z(t)) = \frac{y(t)z(t)}{x(t)^3} = \frac{(e^{-t}\sin t)(e^{-t}\cos t)}{e^{-3t}} = e^t \sin t\cos t$$

In part b) of Problem 1 we found that the length of the derivative of $\vec{r}(t)$ is $|\vec{r}\,'(t)| = e^{-t}\sqrt{3}$. Thus we have

$$\begin{aligned}\int_C g(x,y,z)\,ds &= \int_a^b g(\vec{r}(t))|\vec{r}\,'(t)|\,dt \\ &= \int_0^{2\pi} (e^t\sin t\cos t)(\sqrt{3}e^{-t})\,dt = \int_0^{2\pi} \sqrt{3}\sin t\cos t\,dt \\ &= \sqrt{3}\left[-\frac{1}{2}\cos^2 t\right]_{t=0}^{t=2\pi} = -\frac{\sqrt{3}}{2}\big(\cos^2 2\pi - \cos^2 0\big) = -\frac{\sqrt{3}}{2}\big(1-1\big) = 0\end{aligned}$$

That is

$$\int_C g(x,y,z)\,ds = 0 \tag{30}$$

b) The vector field along the C takes the following values

$$\vec{F}(\vec{r}(t)) = \vec{F}(x(t),y(t),z(t)) = e^{-t}\cos t\,\vec{\jmath} - e^{-t}\sin t\,\vec{k} = e^{-t}\big[\cos t\,\vec{\jmath} - \sin t\,\vec{k}\,\big]$$

In part a) of Problem 1 we computed the derivative of $\vec{r}(t)$:

$$\vec{r}\,'(t) = -e^{-t}\big[\vec{\imath} + (\sin t - \cos t)\,\vec{\jmath} + (\sin t + \cos t)\,\vec{k}\,\big]$$

Hence we have

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_a^b \vec{F}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt \\ &= \int_0^{2\pi} e^{-t}\big[\cos t\,\vec{\jmath} - \sin t\,\vec{k}\,\big]\cdot(-e^{-t})\big[\vec{\imath} + (\sin t - \cos t)\,\vec{\jmath} + (\sin t + \cos t)\,\vec{k}\,\big]\,dt \\ &= \int_0^{2\pi} (-e^{-2t})\big[\cos t\,\vec{\jmath} - \sin t\,\vec{k}\,\big]\cdot\big[\vec{\imath} + (\sin t - \cos t)\,\vec{\jmath} + (\sin t + \cos t)\,\vec{k}\,\big]\,dt \\ &= \int_0^{2\pi} (-e^{-2t})\big(0(1) + (\cos t)(\sin t - \cos t) + (-\sin t)(\sin t + \cos t)\big)\,dt \\ &= \int_0^{2\pi} (-e^{-2t})(-\cos^2 t - \sin^2 t)\,dt = \int_0^{2\pi} (-e^{-2t})(-1)\,dt = \int_0^{2\pi} e^{-2t}\,dt \\ &= \left[-\frac{1}{2}e^{-2t}\right]_{t=0}^{t=2\pi} = -\frac{1}{2}(e^{-2(2\pi)} - e^0) = \frac{1-e^{-4\pi}}{2}\end{aligned}$$

That is

$$\int_C \vec{F}\cdot d\vec{r} = \frac{1-e^{-4\pi}}{2} \tag{31}$$