Calculus 1 (for Physics), Block 1A, Midterm exam

BLOCK 1A 2022/2023 MIDTERM EXAM

WBPH057-05 (Calculus 1 (for Physics))

TIME ALLOWED: 2 HOURS

INSTRUCTIONS TO CANDIDATES

1、 Attempt all 8 questions in this test. The total number of points available is 100. You will get 10 points for free. Your grade will be your point total divided by 10.

2、 The number of points you can get for each question is shown next to it.

3、 In answering the questions in this paper it is particularly important to show your argumentation. The total number of points will only be given for full and detailed answers.

4、 Simple pocket calculators are allowed at this exam. Other electronic devices such as graphical/programmable calculators, tablets, laptops and mobile phones are not.

5、 Books, notes and formula sheets are all not allowed.

6、 Candidates are reminded of the need to use clear and accurate English.

DO NOT REMOVE THIS DOCUMENT FROM THE EXAMINATION ROOM

QUESTIONS

1) Find the following quantities:
   a) $\lim\limits_{x\to 3^-} \frac{x^2-x-6}{|x-3|}$.
   b) $\lim\limits_{x\to 0^+} \frac{1-\cos x}{x(x+1)}$.

Solution:

a) $$\lim_{x\to 3^-} \frac{x^2-x-6}{|x-3|} = \lim_{x\to 3^-} \frac{(x-3)(x+2)}{|x-3|} = \lim_{x\to 3} \frac{(x-3)(x+2)}{-(x-3)} = \lim_{x\to 3} -(x+2) = -5.$$

[7 points in total]

b) $$\lim_{x\to 0^+} \frac{1-\cos x}{x(x+1)} = \lim_{x\to 0^+} \frac{1}{x+1} \frac{(1-\cos x)'}{(x)'} = \lim_{x\to 0^+} \frac{1}{1+x} \frac{(0+\sin x)}{1} = 0.$$

[7 points in total]

2) Find all $z \in \mathbb{C}$ for which the following equation holds true:
   $z^4 = 2i$

Solution: $z^4 = 2i = 2e^{\frac{\pi}{2}i + 2\pi k i}, k \in \mathbb{Z}$, so
$z = \sqrt[9]{2} e^{\frac{\pi}{18}i + \frac{2\pi k}{9}i}, k \in \mathbb{Z}$.
So
$z = \sqrt[4]{2} e^{\frac{\pi}{8}i}$, $z = \sqrt[4]{2} e^{\frac{5\pi}{8}i}$, $z = \sqrt[4]{2} e^{\frac{9\pi}{8}i}$, or $z = \sqrt[4]{2} e^{\frac{13\pi}{8}i}$. [7 points in total]

3)
   a) Find $\lim\limits_{x\to\infty} \left(\frac{x+4}{x+2}\right)^{3x+5}$.
   b) If f and g are differentiable functions,
   $F(x) = f(e^x)g(x), f(1) = -2, f'(1) = 5, g(0) = 2$ and $g'(0) = 3$, then find $F'(0)$.

   Solution:

   a)
$$\lim_{x\to\infty} \left(\frac{x+4}{x+2}\right)^{3x+5} = \lim_{x\to\infty} \left(1 + \frac{2}{x+2}\right)^{3(x+2-2)+5} = \lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^{6x-1}$$
$$= \lim_{x\to\infty} \left(\left(1+\frac{1}{x}\right)^x\right)^6 \left(1+\frac{1}{x}\right)^{-1} = e^6 \cdot 1 = e^6.$$

[7 points]

   b)
$F'(x) = (f(e^x)g(x))' = f'(e^x)e^x g(x) + f(e^x)g'(x)$,
so $F'(0) = f'(e^0)e^0 g(0) + f(e^0)g'(0) = f'(1)g(0) + f(1)g'(0)$
$= 5 \cdot 2 + -2 \cdot 3 = 4$. [7 points]

4) Find the derivative of the following functions:
   a) $g(x) = \sin x \cos x + \tan x$.
   b) $h(x) = \ln(x^4 + x^2 + 9)$.
   c) $q(x) = \frac{x^3+5}{x^3-2}$.
   d) $p(x) = x^x$.

Solution:

a) $g'(x) = (\sin x \cos x + \tan x)' = \left(\frac{1}{2}\sin 2x + \tan x\right)'$
$= \cos 2x + \frac{1}{\cos^2 x}$. [4 points]

b) $h'(x) = (\ln(x^4 + x^2 + 9))' = \frac{1}{x^4+x^2+9}(x^4 + x^2 + 9)' = \frac{4x^3+2x}{x^4+x^2+9}$. [4 points]

c) $q'(x) = \left(\frac{x^3+5}{x^3-2}\right)' = \left(1 + \frac{7}{x^3-2}\right)' = \frac{-21x^2}{(x^3-2)^2}$. [4 points]

d)
$p'(x) = (x^x)' = (e^{x\ln x})' = e^{x\ln x}(x\ln x)' = e^{x\ln x}\left(\ln x + x \cdot \frac{1}{x}\right)$
$= e^{x\ln x}(1 + \ln x) = x^x(1 + \ln x)$. [7 points in total]

5) Find the maximum of the function $f(x) = \ln(x(10 - x))$ on the interval $[1,9]$.

Solution: Possible candidates for the maximum of f on the interval $[1,9]$ are $f(1)$, $f(9)$ and $f(x)$ for any $x \in (1,9)$ for which $f'(x) = 0$.
$f'(x) = 0 \;\Leftrightarrow\; 0 = (\ln(x(10-x)))' \Leftrightarrow 0 = \frac{1}{x(10-x)} \cdot (10 - 2x) \;\Leftrightarrow\; x = 5.$
As $f(1) = f(9) = \ln 9$ and $f(5) = \ln 25$, we have that the maximum of f on the interval $[1,9]$ is $\ln 25$. [10 points]

6) A car leaves Groningen at 13:00 and arrives in Zwolle at 14:00, having traveled 100 km. Show that there was a time between 13:00 and 14:00 at which the car was traveling at a speed of 100 km per hour.

Solution: Let $f(t)$ be the distance traveled in kilometers at time t in with t expressed in hours, t=0 corresponding with 13:00 and t=1 corresponding with 14:00. Then $f(0) = 0$ and $f(1) = 100$, so by the mean value theorem, there is a $c \in (0,1)$ such that $f'(c) = \frac{f(1)-f(0)}{1-0} = \frac{100-0}{1}$=100. So there was indeed a time between 13:00 and 14:00 at which the car was traveling at a speed of 100 km per hour. *q.e.d.* [7 points]

7) Let $q(x) = sin^2 x + 2$. Find $\frac{d}{dx}\int_0^x e^{q(t)}dt$.
   Solution: $\frac{d}{dx}\int_0^x e^{q(t)}dt = e^{q(x)} = e^{sin^2 x+2}$. [7 points]

8) Let a, b, c be constants and let $f(x) = ax^5 + bx + c$. Let the graph of f touch the line with equation $y = -2x + 1$ at the point $(0,1)$. Additionally, assume that the line tangent to the graph of f for $x = 1$ is parallel to the line tangent to the graph with equation $y = x^4$ for $x = 1$. Find a, b and c.

Solution: For the graph of f to touch the line with equation $y = -2x + 1$ at the point $(0,1)$, we need that $f(0) = 1$ and $f'(0) = -2$, so because $f'(x) = 5ax^4 + b$, this means that $1 = f(0) = a \cdot 0 + b \cdot 0 + c = c$ and $-2 = f'(0) = 0 + b = b$. The line tangent to the graph of f for $x = 1$ is parallel to the line tangent to the graph with equation $y = x^4$ for $x = 1$, so we need that $(x^4)'|_{x=1} = f'(1)$, so $4 = 5a - 2$, giving $a = \frac{6}{5}$. So $a = \frac{6}{5}$, $b = -2$ and $c = 1$. [12 points]

The end of the paper