Name: Student Number:

Mechanics and Relativity: R2

October 14th, 2022, Aletta Jacobshal

Duration: 90 mins

Before you start, read the following:

- There are 3 problems, for a total of 58 points.
- Write your name and student number on all sheets.
- Make clear arguments and derivations and use correct notation. *Derive* means to start from first principles, and show all intermediate (mathematical) steps you used to get to your answer!
- Support your arguments by clear drawings where appropriate. Draw your spacetime diagrams on the provided hyperbolic paper.
- Write your answers in the boxes provided. If you need more space, use the lined drafting paper.
- Generally use drafting paper for scratch work. Don't hand this in unless you ran out of space in the answer boxes.
- Write in a readable manner, illegible handwriting will not be graded.

	Points
Problem 1:	19
Problem 2:	21
Problem 3:	18
Total:	58
GRADE (1 + # Total/(58/9))	

Useful equations:

$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$
$$\Delta t \geq \Delta s \geq \Delta \tau$$

The Lorentz transformation equations with $\gamma \equiv (1-\beta^2)^{-1/2}$:

$$t' = \gamma(t - \beta\, x)\,,$$
$$x' = \gamma(x - \beta t)\,,$$
$$y' = y\,,$$
$$z' = z\,.$$

The relativistic Doppler shift formula:

$$\frac{\lambda_R}{\lambda_E} = \sqrt{\frac{1+v_x}{1-v_x}}\,.$$

Possibly relevant equations:

$$F = G\frac{Mm}{r^2}; \qquad F = ma; \qquad PV \propto k_b T; \qquad F = \frac{dp}{dt}$$

Possibly relevant numbers:

$$c = 299792458 \text{ m/s} \tag{1}$$

Question 1: Proper time and space-time in SR (19 pts)
During the lectures we derived the space-time interval as a frame independent measure of 'distance' in special relativity and proper time as 'path length'. Both the space-time and proper time interval are frame independent.

(a) **(4 pts)** Show that the space-time interval is not invariant under Galilean transformations (you are allowed to consider the 2D (t, x) space-time interval with the OTHER frame moving in the $+x$ direction.).

The Galilean transformation rules only affect the spatial components, i.e. $x' = x - \beta t$ where β is the speed (in the $+x$ direction) **(1 pt)**. Since time is not affected **(1 pt)**, and is absolute in Galilean relativity, the space time interval Δs will not be invariant. We write $\Delta s^2 = \Delta t'^2 - \Delta x'^2 = \Delta t^2 - \Delta(x+vt)^2$**(1 pt)** $= (1-v^2)\Delta t^2 - 2v\Delta t\Delta x - \Delta x^2 = \Delta s^2 - v\Delta t(v\Delta t + 2\Delta x) \neq \Delta s^2$ **(1 pt)**

(b) **(5 pts)** Show that the space-time interval is invariant under Lorentz transformations.

We can read of the Lorentz transformations from the cover page. We find $\Delta s'^2 = \Delta t'^2 - \Delta x'^2 = \gamma^2\Delta(t-\beta x)^2 - \gamma^2\Delta(x-\beta t)^2$**(1 pt)** $= \gamma^2(\Delta t^2 - 2\beta\Delta x\Delta t + \beta^2\Delta x^2) - \gamma^2(\Delta x^2 - 2\beta\Delta t\Delta x + \beta^2\Delta t^2)$**(1 pt)**. The cross term cancels: $\Delta s'^2 = \gamma^2(1-\beta^2)\Delta t^2 - \gamma^2(1-\beta^2)\Delta x^2$ **(1 pt)**. We note that $\gamma^2(1-\beta^2) = 1$ **(1 pt)**. We thus find $\Delta s'^2 = \Delta t^2 - \Delta x^2 = \Delta s^2$. **(1 pt)**

(c) **(5 pts)** Rederive the expression for v' in terms of v and β (i.e. the velocity transformation in 2 dimensions as we did in class, so $v = v_x$ and $v' = v'_x$). Use the Lorentz transformations provided on the cover page.

$v' = \frac{dx'}{dt'} = \frac{\gamma d(x-\beta t))}{\gamma(t-\beta x)} = \frac{dx-\beta dt}{dt-\beta dx}$ **(2 pts)**. Pull out a factor of $dt \rightarrow \frac{dt}{dt}\frac{dx/dt-\beta}{1-\beta dx/dt}$ **(2 pts)**. Noting that $v = dx/dt$ we recover $v' = \frac{v-\beta}{1-\beta v}$.**(1 pt)**

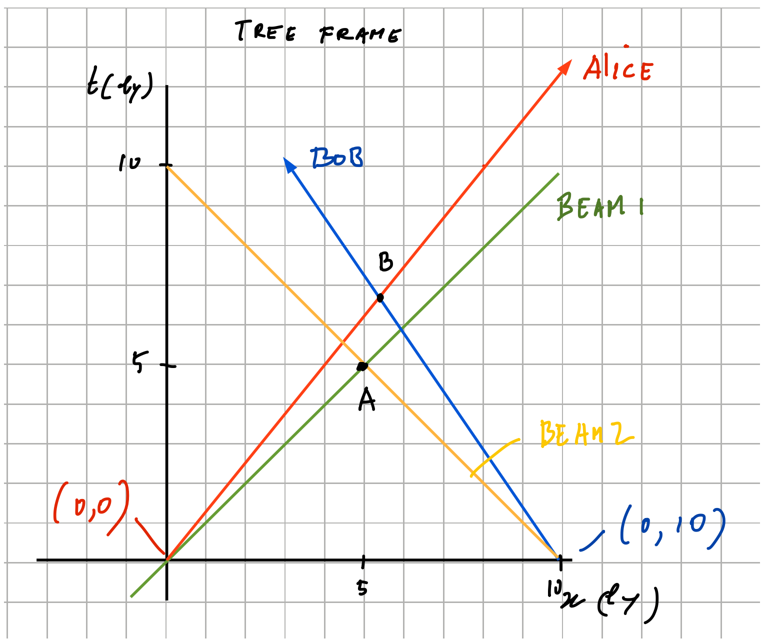
(d) **(9 pts)** Finally show that $\Delta\tau$ is frame independent. Hint: use both Lorentz and Einstein velocity transformations (rederived in the previous question). Again you can assume 2D.

$\Delta\tau = \int(1-v^2)^{1/2}dt$ **(1 pt)**. We can drop the vector bar on top of v since this is a 2D example. Let us move to a new frame, i.e. $\Delta\tau' = \int(1-v'^2)^{1/2}dt'$ **(1 pt)**. We have $dt' = \gamma(dt - \beta dx) = \gamma dt(1-\beta v)$ **(1 pt)**, where we used that $dx/dt = v$. The velocity transformation is given by $v' = (v-\beta)(1-\beta v)$ (as derived in the previous question). We insert this back into the integral to obtain $\Delta\tau' = \int(1-(v-\beta)^2/(1-\beta v))^{1/2}\gamma(1-\beta v)dt$ **(2 pts)**. We pull the factor of $(1-\beta v)$ through the square root: $\int((1-\beta v)^2-(v-\beta)^2)^{1/2}\gamma dt = \int(1-2\beta v+\beta^2v^2-v^2+2\beta v+\beta^2)^{1/2}\gamma dt = \int((1-\beta^2)v^2)^{1/2}\gamma dt = \int(1-v^2)^{1/2}dt$ **(3 pts)**. Here we used that $(1-\beta^2)^{1/2} = \gamma^{-1}$. Thus we conclude $\Delta\tau' = \Delta\tau$ **(1 pt)**, and proper time is indeed frame independent as expected.

Question 2: Twins reunion (21 pts)

Alice and Bob, both in their own spaceship travel in opposite directions. Alice is traveling to the right at 0.8c and Bob is traveling to the left at 0.7c. A tree on Earth, Alice in a spaceship, and a light beam (1) go through the origin at $(t, x) = (0, 0)$). Bob in his spaceship and a second light beam (2) pass through $(t, x) = (0, 10\text{ly})$. Light beam (1) is moving to the right with speed c and light beam (2) is moving to the left with speed c as observed in the tree frame on Earth.

(a) **(3 pts)** Draw a spacetime diagram in the tree frame. Draw the worldlines of the two spaceships and the two beams. Make sure that the diagram can be read unambiguously by adding the necessary markings to the axes. **Use the provided diagram paper at end of the exam.**



(1/2 pt): drawing a diagram
(1/2 pt): drawing labels and ticks
(1/2 pt): worldline BOB
(1/2 pt): worldline ALICE
(1/2 pt): worldline BEAM 1
(1/2 pt): worldline BEAM 2

(b) **(1 pt)** When do the two light beams cross each other as observed in the tree frame? Call this event A.

In the tree frame this moment can be read of from the space time diagram, i.e. at $(t_A, x_A) = (5\text{ly}, 5\text{ly})$**(1 pt)**

(c) **(7 pts)** Event B is when Alice and Bob cross. Determine the coordinates of this event in the tree frame using algebra. Are events A and B causally connected? Compute Δs. Explain how we decide whether two events can be causally connected. Label these two events in the diagram.

From the tree frame $x_{\text{BOB}} = -0.7t + c_1$ **(1/2 pt)** and $x_{\text{ALICE}} = 0.8t + c_2$ **(1/2 pt)** . Bob passes through $(t, x) = (0, 10) \rightarrow x_{\text{BOB}} = -0.7t + 10$ **(1/2 pt), (**$c_1 = 10$**)** while Alice passes the origin, i.e. $x_{\text{ALICE}} = 0.8t$ **(1/2 pt)**,($c_2 = 0$) . We equate these two $-0.7t + 10 = 0.8t \rightarrow 10 = 1.5t \rightarrow t = 20/3\text{ly} \sim 6.66\text{ly}$ **(1 pt)** . To obtain the x coordinate we just use one of the equations (they should give the same answer!), $x = 0.8(20/3) = 16/3\text{ly} \sim 5.33\text{ly}$ **(1 pt)** . This looks about right from the diagram. Events A and B are causally connected since the spacetime interval $\Delta s^2 > 0$ **(1 pt)** . To be precise, $\Delta s = \sqrt{(5/3)^2 - (1/3)^2} = 2\sqrt{6}/3$ **(1 pt)** . This is a timelike interval. Labelling event A **(1/2 pt)**, B **(1/2 pt)** in diagram above.

(d) **(5 pts)** Calculate the velocity of Bob in Alice frame and vice versa. Does your answer make sense? Explain.

To go from the tree frame to Alice's frame we need to boost $\beta = 0.8c$ **(1/2 pt)** . In the tree frame, $v_{\text{BOB}} = -0.7c$ **(1/2 pt)** . To get the velocity as observed in Alice's frame we thus need to compute $v'_{\text{BOB}} = \frac{v_{\text{BOB}}-\beta}{1-\beta v_{\text{BOB}}} = \frac{-0.7-0.8}{1-(-0.7*0.8))} = -\frac{15}{10}\frac{100}{156} = -\frac{150}{156}$c **(2 pts)** .
To get the Alice her velocity in Bob's frame we do very similar steps to find $v'_{\text{ALICE}} = \frac{0.8+0.7}{1+0.7*0.8))} = \frac{150}{156}$c **(1 pt)**. As expected, since they are both moving in inertial frames, from their respective frames they move at the same speed but opposite directions. The net speed is very close to the speed of light, but does not exceed it! **(1 pt)**

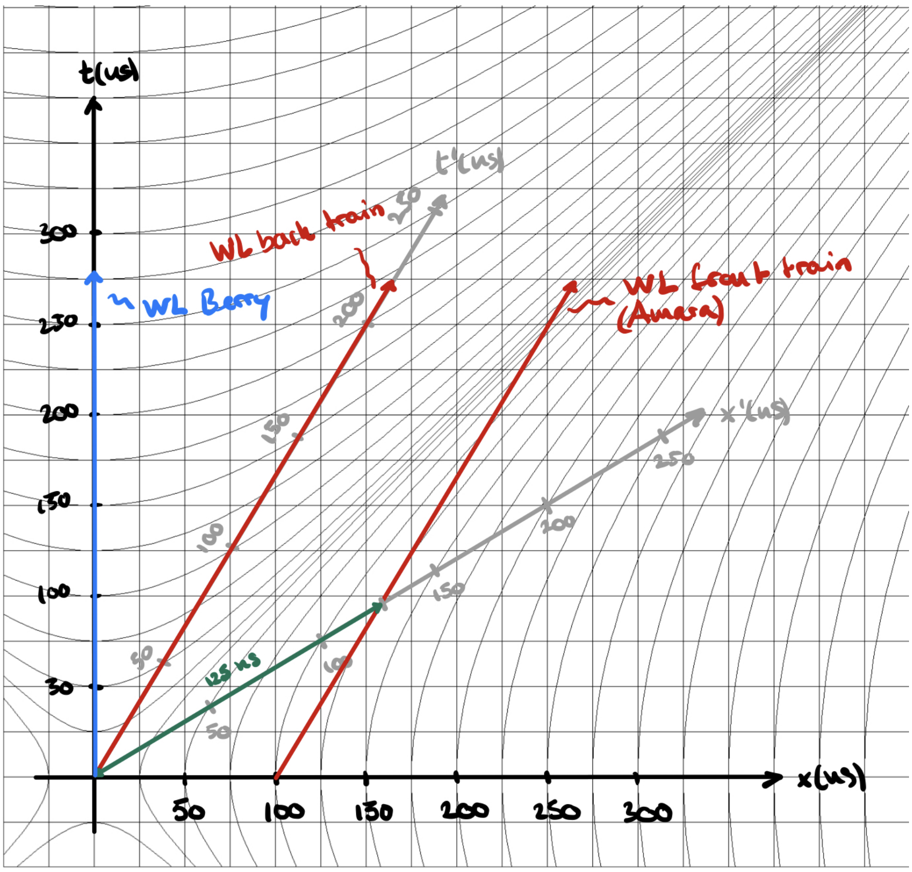
(e) **(5 pts)**Calculate the velocity of Bob in Alice frame and vice versa using Galilean relativity and compare it with (d). Is the result you get consistent with special relativity? Explain your answer.

To go from the tree frame to Alice's frame we need to boost $\beta = 0.8c$ **(1/2 pt)** . In the tree frame, $v_{\text{BOB}} = -0.7c$ **(1/2 pt)** . To get the velocity as observed in Alice's frame we thus need to compute $v'_{\text{BOB}} = v_{\text{BOB}} - \beta$ **(1 pt)** as we do in Galilean relativity. $v'_{\text{BOB}} = -1.5$c **(1 pt)** . To get the Alice her velocity in Bob's frame we do very similar steps to find $v'_{\text{ALICE}} = 0.8 + 0.7 = 1.5$c **(1 pt)** . From their respective frames they move at the same speed but opposite directions but it is not consistent with special relativity since their speeds exceeds the speed of light **(1 pt)** .

Question 3: Length Contraction and the Doppler Shift (18 pts)

Amara is a high speed train driver driving her train at speed 3/5 with respect to the track. Berry is next to the track as the train passes by. Berry is located at the origin of his frame. The back end of the train passes him at $t = 0$. At that time, he measures the length of the train and finds it to be 100 ns. Take Berry to be in the home frame and Amara in the other frame. Take the origin of both frames to coincide.

(a) **(6 pts)** Draw a two observer diagram corresponding to the above scenario. Draw the worldlines of the back and front of the train and label them. Graphically, determine the length of the train in Amara's frame. **Use the provided 2-observer diagram paper at the end of the exam.**



(1/2 pt): drawing diagram
(1/2 pt): drawing labels and ticks
(1 pt): drawing x' axis
(1 pt): drawing y' axis
(1 pt): drawing worldline back of train
(1 pt): drawing worldline front of train
(1 pt): Reading of from the diagram the length is ~ 125 ns.

(b) **(2 pts)** Confirm that your result in part (a) is consistent with the Lorentz contraction formula.

The length as observed in the HOME frame should be contracted (since the train is moving). We therefore expect the length in the moving frame to be larger than 100 ns. The length contraction is determined by the the Lorentz factor $\gamma = (1-\beta^2)^{-1/2} = (1-(3/5)^2)^{-1/2} = 5/4$ **(1 pt)**. The length in the moving frame $L_{\text{AMARA}} = \gamma L_{\text{BERRY}} = \frac{5}{4} \times 100$ ns $= 125$ ns **(1 pt)**.

(c) **(6 pts)** In the rest frame of the train, the nose of the train makes an angle of 60° with the horizontal. What is this angle as observed by Berry?

Length is only contracted in the direction of motion. This means the height of the train is unaffected: $y = y'$ **(1 pt)**, $x = \gamma^{-1}x'$ **(1 pt)**. The angle is determined by $\tan\Theta' = \frac{y'}{x'}$, or $y = x'\tan\Theta' = \gamma^{-1}x\tan\Theta$ **(2 pts)**. This last equality is determined by the fact that y is unaffected (is the same in both frames). We thus find that in Berry's frame $\Theta = \tan^{-1}(\gamma^{-1}\tan\Theta')$ **(1 pt)**. Inserting 60° we obtain $\Theta = 65.21°$ **(1 pt)**.

At a different part of the journey, the Amara and her train travel at a speed of 0.16 towards a signal, located next to the track. In the signal's rest frame, the signal is red ($\lambda_{\text{red}} = 650$ nm).

(d) **(4 pts)** Will Amara pass the signal or stop? *Hint:* The wavelength of green light is about 15% smaller than that of red light.

The received wavelength by Amara is given by (frome the cover page):

$$\lambda_R = \sqrt{\frac{1+v}{1-v}}\lambda_E, \tag{2}$$

where $v = -0.16$ **(1 pt)** as the train is moving towards the signal and $\lambda_E = 650$ nm . We find $\lambda_R = 553$ nm **(1 pt)** (the signal is blue shifted). The wavelength of green light is 85% of λ_{red} yielding 553 nm **(1 pt)**. Indeed, Amara will perceive the signal as being on green **(1 pt)**, so she will pass the signal.

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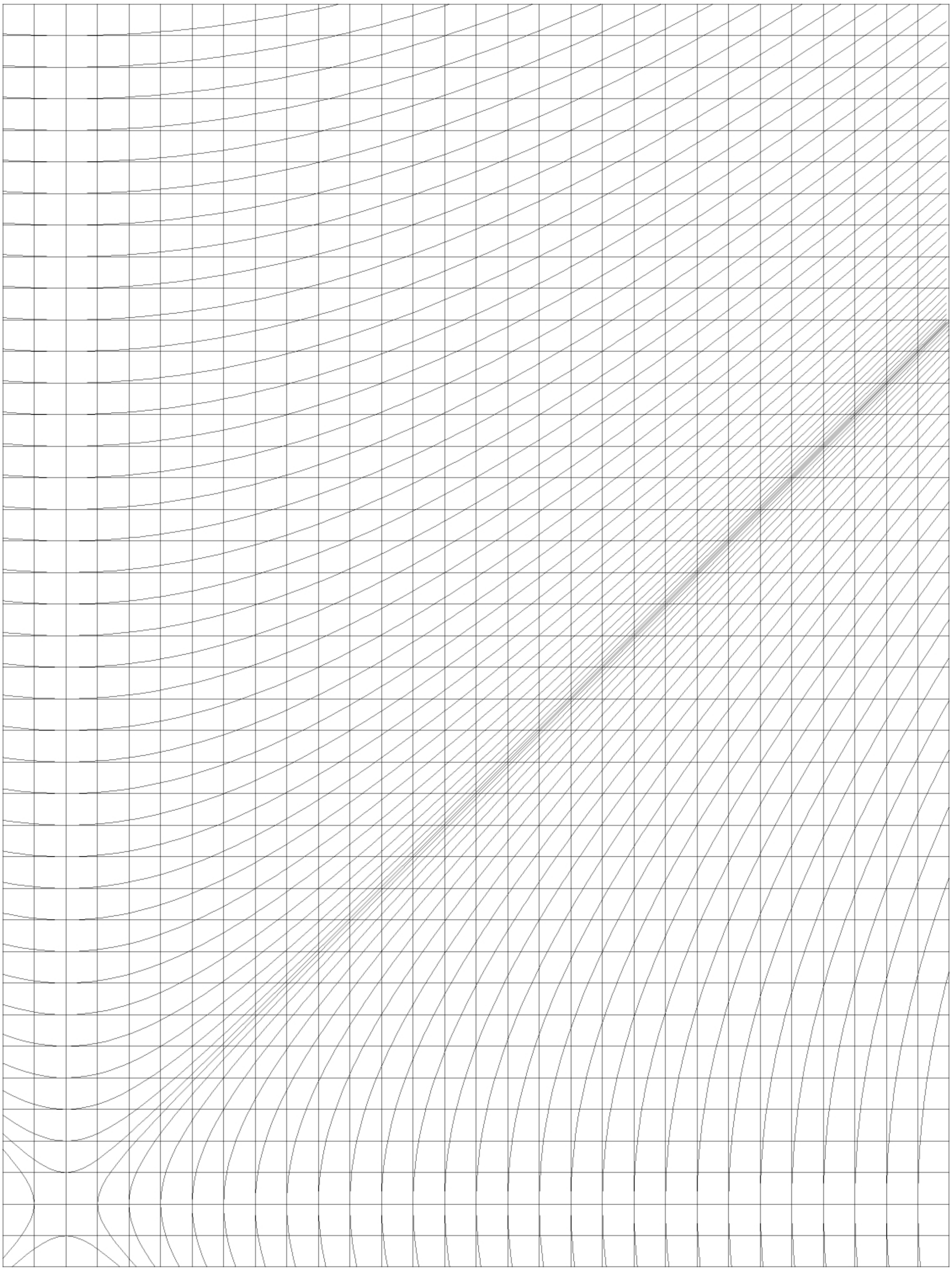

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