Linear Algebra - Midterm Exam

December 14, 2020 17.30-19.30 Online exam

Read carefully the exam rules on the Nestor page of the exam
Remember that you are not allowed to use the internet during the exam. The only exception is if you use an online text editor (e.g., overleaf) to write the exam. But notice that you are not allowed to use an online editor if it has tools to solve math problems.

QUESTIONS:

1. (a) $\boxed{1.5}$ You are given two 3×2 matrices, B and C, with, respectively, elements b_{ik} and c_{ik}, and two column vectors, $\mathbf{x}$ and $\mathbf{y}$ with, respectively, components x_i and y_i, $i = 1, 2, 3$, $k = 1, 2$.
   Find the elements, a_{ij}, of a 3×3 matrix, A, such that $AB = C$, and $A\mathbf{x} = \mathbf{y}$. Remember, B, C, $\mathbf{x}$ and $\mathbf{y}$ are known, and A is unknown. Finally, if necessary, assume that all square matrices in this exercise are invertible.
   Hint: First write each of the components of the matrix/matrix product $AB = C$ and the matrix/vector product $A\mathbf{x} = \mathbf{y}$ as a system of linear equations. By looking at those equations construct two new matrices, one called B', using the elements of B and $\mathbf{x}$, and the other one called C', using the elements of C and $\mathbf{y}$, such that a single matrix product, $AB' = C'$, gives exactly the same system of linear equations. Once you have the matrix product, explain with words (no need to do any complicated calculation) how you would get A.

   (b) $\boxed{0.5}$ What are the elements of A when $B = \begin{bmatrix} 1 & 0 \\ -5 & 1 \\ 0 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 3 & 1 \\ 5 & 5 \\ 1 & -1 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 4 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 7 \\ 4 \\ 3 \end{bmatrix}$?

2. Check whether in each case the given vectors form a basis in the corresponding vector space. Justify your answer.

   (a) $\boxed{1}$ $\mathbf{v}_1 = \begin{bmatrix} 4 \\ 5 \\ -9 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ 2 \\ -7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 5 \\ 8 \end{bmatrix}$.

   If the answer is yes, give the transition matrices from, respectively, the basis $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to the basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and from the basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ to the basis $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, and write the coordinate vector $[\mathbf{x}]_E$ given in the basis E as a coordinate vector $[\mathbf{x}]_V$ in the basis V:

   $$[\mathbf{x}]_E = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}_E$$

   (b) $\boxed{1}$ $\mathbf{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ -6 \\ 5 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

   If the answer is yes, give the transition matrices from, respectively, the basis $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ to the basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$, and from the basis $E = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ to the basis $V = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$, and write the coordinate vector $[\mathbf{x}]_E$ given in the basis E as a coordinate vector $[\mathbf{x}]_V$ in the basis V:

   $$[\mathbf{x}]_E = \begin{bmatrix} 2 \\ -1 \\ 4 \\ 0 \end{bmatrix}_E$$

Please turn over

3. (a) $\boxed{0.5}$ Show that the following two vectors are linearly independent:

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \ \mathbf{u}_2 = \begin{bmatrix} -5/2 \\ 0 \\ 1 \end{bmatrix}.$$

   (b) $\boxed{0.5}$ Find the general form of any vector that is a linear combination of these two vectors. This general new vector gives the equation of a plane. (The head of each particular vector represented by this general vector is a point in that plane.).

   (c) $\boxed{1}$ Write a general vector in the direction perpendicular to this plane. Write that vector in parametric form. Write also the unit vector in the direction perpendicular to this plane.
   Hint: You may want to select 2 out of the infinite vectors that define the plane.

4. (a) $\boxed{1}$ Show that the vectors $f_1(x) = x$, $f_2(x) = \sin x$ and $f_3(x) = \cos x$ in the vector space $C^2[0, \pi]$ (the set of all the functions that have a continuous second derivatives on the interval $[0, \pi]$) are linearly independent.

   (b) $\boxed{1}$ Given the vectors $p(x) = a$, $q(x) = 2x + 4$ and $r(x) = (a-1)x^2$ in the vector space P_3 of all polynomials with degree < 3, find for which values of a the three vectors are linearly independent.

5. Consider 3 libraries, one in Groningen, one in Assen and one in Haren. Books are borrowed in the morning, and returned in the evening. A book may be returned at each of the 3 libraries. The books are borrowed and returned in the following way:

   From the books borrowed in Groningen 60% are returned in Groningen, 20% are returned in Assen and 20% are returned in Haren.
   From the books borrowed in Assen 70% are returned in Assen, none is returned in Groningen and 30% are returned in Haren.
   All the books borrowed in Haren are returned in Haren.
   The fraction of books borrowed and returned at each library each day is the same, and the total number of books is constant.

   On Tuesday Dec. 8 2020 there were 600 books in Groningen, 200 in Assen and 500 in Haren.

   (a) $\boxed{1}$ Determine the numbers of books at each library two days later, on Dec. 10 2020.

   (b) $\boxed{1}$ Determine the numbers of books at each library one day earlier, on Dec. 7 2020.

Solutions

1. (a) The given matrices and vectors are:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix},\ B=\begin{bmatrix} b_{11} & b_{12}\\ b_{21} & b_{22}\\ b_{31} & b_{32}\end{bmatrix} \text{ and } C=\begin{bmatrix} c_{11} & c_{12}\\ c_{21} & c_{22}\\ c_{31} & c_{32}\end{bmatrix},\ \mathbf{x}=\begin{bmatrix} x_1\\ x_2\\ x_3\end{bmatrix} \text{ and } \mathbf{y}=\begin{bmatrix} y_1\\ y_2\\ y_3\end{bmatrix},$$

Let's write $AB=C$ and $A\mathbf{x}=\mathbf{y}$ explicitly. Since C is a (3×2) matrix, it will have 6 components, and since $\mathbf{y}$ is a (3×1) vector, it will have 3 components. Therefore, I will have 9 equations. I will write first the 2 components of the first row of C, followed by the first component of $\mathbf{y}$, followed by the 2 components of the second row of C, followed by the second component $\mathbf{y}$, followed by the 2 components of the third row of C, followed by the last component of $\mathbf{y}$:

$$\begin{aligned}
&a_{11}\,b_{11}+a_{12}\,b_{21}+a_{13}\,b_{31}=c_{11} \leftarrow \text{(this and the next equation are the first row of } C)\\
&a_{11}\,b_{12}+a_{12}\,b_{22}+a_{13}\,b_{32}=c_{12}\\
&a_{11}\,x_1\ \ +a_{12}\,x_2\ \ +a_{13}\,x_3=y_1 \ \leftarrow \text{(this is the first element of } \mathbf{y})
\end{aligned}$$

$$\begin{aligned}
&a_{21}\,b_{11}+a_{22}\,b_{21}+a_{23}\,b_{31}=c_{21} \leftarrow \text{(this and the next equation are the second row of } C)\\
&a_{21}\,b_{12}+a_{22}\,b_{22}+a_{23}\,b_{32}=c_{22}\\
&a_{21}\,x_1\ \ +a_{22}\,x_2\ \ +a_{23}\,x_3=y_2 \ \leftarrow \text{(this is the second element of } \mathbf{y})
\end{aligned}$$

$$\begin{aligned}
&a_{31}\,b_{11}+a_{32}\,b_{21}+a_{33}\,b_{31}=c_{31} \leftarrow \text{(this and the next equation are the third row of } C)\\
&a_{31}\,b_{12}+a_{32}\,b_{22}+a_{33}\,b_{32}=c_{32}\\
&a_{31}\,x_1\ \ +a_{32}\,x_2\ \ +a_{33}\,x_3=y_1 \ \leftarrow \text{(this is the third element of } \mathbf{y})
\end{aligned}$$

0.5 points to write all the components correctly

These 9 equations are equivalent to the matrix product:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & x_1\\ b_{21} & b_{22} & x_2\\ b_{31} & b_{32} & x_3\end{bmatrix}=\begin{bmatrix} c_{11} & c_{12} & y_1\\ c_{21} & c_{22} & y_2\\ c_{31} & c_{32} & y_3\end{bmatrix}.$$

I can write this in a a more compact way if I call $B'=[B|\mathbf{x}]$, the original matrix B with a third column equal to $\mathbf{x}$, and $C'=[C|\mathbf{y}]$, the original matrix C with a third column equal to $\mathbf{y}$.

0.3 points to realise this and write back the system as a matrix product

Then:

$$AB'=C'.$$

Since I know B, C, $\mathbf{x}$, and $\mathbf{y}$, I also know B' and C', and I need to find A. I therefore multiply both sides by $(B')^{-1}$ **from the right**, and the answer is:

$$A=C'\,(B')^{-1}.$$

0.7 to write this correctly, namely by multiplying by B^{-1} from the **right**

They do not get the 0.7 points if they multiply by B^{-1} from the **left**

(b) In this case $B'=[B|\mathbf{x}]=\begin{bmatrix} 1 & 0 & 4\\ -5 & 1 & 1\\ 0 & 0 & -1\end{bmatrix}$ and $C'=[C|\mathbf{y}]=\begin{bmatrix} 3 & 1 & 7\\ 5 & 5 & 4\\ 1 & -1 & 3\end{bmatrix}$

I need $(B')^{-1}$. For that I write $[B'|I]$ and apply row operations to get $\left[I|\,(B')^{-1}\right]$.

The result is $(B')^{-1}=\begin{bmatrix} 1 & 0 & 4\\ 5 & 1 & 21\\ 0 & 0 & -1\end{bmatrix}$ (the students need to give the row operations),

0.4 points for B^{-1}

then

$$A = C'\left(B'\right)^{-1} = \begin{bmatrix} 8 & 1 & 26 \\ 30 & 5 & 121 \\ -4 & -1 & -20 \end{bmatrix}.$$

$\boxed{\text{0.1 points for } A}$

2. (a) In this case it is easy to see that the vectors are not linearly independent because, if I call $V = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3]$, $\det V = 0$. So they do not form a basis in $\mathbb{R}^3$.

$$\det V = \begin{bmatrix} 4 & 3 & -3 \\ 5 & 2 & 5 \\ -9 & -7 & 8 \end{bmatrix} = 0$$

$\boxed{\text{0.6 points for noticing that if } \det V = 0 \text{ is enough to show no independence}}$

$\boxed{\text{0.4 points for calculating the determinant. They get the full point also if they use a different, valid, method to prove that the vectors are not independent}}$

(b) In this case the vectors are linearly independent. To see this I call $V = [\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\ \mathbf{v}_4]$, which is a block diagonal matrix. It is easy to calculate the determinant of the two (2×2) blocks and multiply them by each other to get that $\det V = \det \begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix} = 6.$

$\boxed{\text{0.6 points for noticing that if } \det V = 0 \text{ is enough to show no independence}}$

$\boxed{\text{0.1 points for the determinant}}$

So the answer is yes, these vectors form a basis in $\mathbb{R}^4$.

To calculate the vector in the new basis, I need to find V^{-1}; since V is a block matrix, the inverse is straightforward (the invers of each block):

$$[\mathbf{x}]_V = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 5 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ -1 \\ 4 \\ 0 \end{bmatrix}_E = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & -1/6 & 0 \\ 0 & 0 & 5/6 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 4 \\ 0 \end{bmatrix}_E = \begin{bmatrix} 3 \\ 4 \\ -2/3 \\ 10/3 \end{bmatrix}_V.$$

$\boxed{\text{0.3 points}}$

3. (a) Two vectors are linearly independent if one is **not** a multiple of the other. Since $\mathbf{u}_1$ is not a multiple of $\mathbf{u}_2$, these two vectors are linearly independent.

$\boxed{\text{0.5 points for indicating that they are not independent by this or any other correct method. Notice, however, that in this case they cannot use the determinant of the matrix with these vectors as columns because the matrix is not square}}$

(b) That vector will be $\mathbf{v}_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1\mathbf{u}_1 + c_2\mathbf{u}_1 = c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3c_1 - \frac{5}{2}c_2 \\ c_1 \\ c_2 \end{bmatrix}.$

So $x_2 = c_1$ and $x_3 = c_2$ are free, and $x_1 = 3x_2 - \frac{5}{2}x_3$, which is the equation of the plane. (One can also write it as $2x_1 - 6x_2 + 5x_3 = 0$ or, in a more familiar way, $2x - 6y + 5z = 0$.) Explicitly:

$$\mathbf{v}_1 = \begin{bmatrix} 3x_2 - \frac{5}{2}x_3 \\ x_2 \\ x_3 \end{bmatrix}.$$

$\boxed{\text{0.5 points for writing the final vector}}$

(c) I first make specific vectors in the plane by choosing values for x_2 and x_3 in $\mathbf{v}_1$. Taking $x_2 = 1$ and $x_3 = 0$ and $x_2 = 0$ and $x_3 = 1$ I get these 2 vectors in the plane:

$$\mathbf{u}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \text{ and } \mathbf{u}_2 = \begin{bmatrix} -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix}.$$

A generic vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ will be perpendicular both to $\mathbf{u}_1$ and $\mathbf{u}_2 \iff \mathbf{u}_1\mathbf{x} = 0$ and $\mathbf{u}_2\mathbf{x} = 0$:

0.4 points for this or similar statement. They get the points also if they use this without saying it explicitly

$\mathbf{u}_1\mathbf{x} = 3x_1 + x_2 = 0 \Longrightarrow x_2 = -3x_1$
$\mathbf{u}_2\mathbf{x} = -\frac{5}{2}x_1 + x_3 = 0 \Longrightarrow x_3 = \frac{5}{2}x_1$

0.2 points for getting this solution

Then $\mathbf{x} = \begin{bmatrix} x_1 \\ -3x_1 \\ \frac{5}{2}x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -3 \\ \frac{5}{2} \end{bmatrix}$. The last one is the parametric form.

0.2 points for writing the vector in parametric form

To get the unit vector I take $x_1 = 1$ (this is not necessary, but it makes my life easier) and calculate the modulus of $\mathbf{x}$, $|\mathbf{x}| = \sqrt{1^2 + 3^2 + \left(\frac{5}{2}\right)^2} = \frac{\sqrt{65}}{2}$. The unit vector in this direction is then:

$$\frac{\mathbf{x}}{|\mathbf{x}|} = \frac{2}{\sqrt{65}} \begin{bmatrix} 1 \\ -3 \\ \frac{5}{2} \end{bmatrix}.$$

0.2 points for writing the unit vector

I did not do it myself, but it may be possible to do the same using the cross product. If they do this correctly, they should get the full points as well

4. (a) $f_1(x) = x,\ f_2(x) = \sin x,\ f_3(x) = \cos x$
$f_1'(x) = 1,\ f_2'(x) = \cos x,\ f_3'(x) = -\sin x$
$f_1''(x) = 0,\ f_2''(x) = -\sin x,\ f_3''(x) = -\cos x$

$$W[f_1, f_2, f_3] = \det \begin{bmatrix} x & \sin x & \cos x \\ 1 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{bmatrix} = x \det \begin{bmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{bmatrix} - 1 \det \begin{bmatrix} \sin x & \cos x \\ -\sin x & -\cos x \end{bmatrix} =$$
$$x(-\cos^2 x - \sin^2 x) + (-\sin x \cos x + \sin x \cos x) = -x.$$

0.4 points for writing the correct Wronskian

0.4 points for calculating the determinant correctly

Since there is an $x_0 \in [0, \pi]$ for which $W(f_1, f_2, f_3) \neq 0$ (namely, any $x_0 \neq 0$), the vectors (functions) are linearly independent

0.2 points for this or similar statement. They **have** to say that the given vectors are a basis of the vector space

They get the full points if they use a different method, and the method is correct

(b) $p(x) = a$, $q(x) = 2x + 4$, $r(x) = (a-1)x^2$
$p'(x) = 0$, $q'(x) = 2$, $r'(x) = 2(a-1)x$
$p''(x) = 0$, $q''(x) = 0$, $r''(x) = 2(a-1)$

$$W[p(x), q(x), r(x)] = \det \begin{bmatrix} a & 2x+4 & (a-1)x^2 \\ 0 & 2 & 2(a-1)x \\ 0 & 0 & 2(a-1) \end{bmatrix} = a.2.2.(a-1) = 4a(a-1).$$

(The last step makes use that the determinant of a triangular matrix is the product of the elements along the diagonal.)

0.4 points for writing the correct Wronskian

0.4 points for calculating the determinant correctly

So $W[p(x), q(x), r(x)] \neq 0 \Leftrightarrow a(a-1) \neq 0$, which is true as long as $a \neq 0$ and $a \neq 1$.

0.2 points for this or similar statement. They **have** to say that the given vectors are a basis of the vector space

They get the full points if they use a different method, and the method is correct

5. First write the matrix of this problem:

BOROWED FROM

RETURNED TO

	G	A	H
G	0.6	0	0
A	0.2	0.7	0
H	0.2	0.3	1

$$\rightarrow \quad A = \begin{bmatrix} 0.6 & 0 & 0 \\ 0.2 & 0.7 & 0 \\ 0.2 & 0.3 & 1 \end{bmatrix}.$$

(a) To find the number of books 2 days later I need to calculate $\mathbf{x}_2 = A^2\mathbf{x}_0$, where $\mathbf{x}_0 = \begin{bmatrix} 600 \\ 200 \\ 500 \end{bmatrix}$, which gives $\mathbf{x}_2 = \begin{bmatrix} 216 \\ 254 \\ 830 \end{bmatrix}$.

0.2 points for writing the matrix

Notice that they can write the matrix transposed, but then they have to use row vectors and must multiply them by the matrix from the left, otherwise the total number of books will not be conserved. In other words, the columns of A add up to 1 if they write it this way, but if they use the transposed matrix it is the rows of A that add up to 1

0.8 points for finding the solution

(b) To find the number of books 1 day earlier I need to calculate $\mathbf{x}_{-1} = A^{-1}\mathbf{x}_0$.

$$A =^{-1} \begin{bmatrix} 5/3 & 0 & 0 \\ -10/21 & 10/7 & 0 \\ -4/21 & -3/7 & 1 \end{bmatrix}$$

0.6 points for A^{-1}. They are free to choose how to find the inverse.

$$\mathbf{x}_{-1} = A^{-1}\mathbf{x}_0 = \begin{bmatrix} 1000 \\ 0 \\ 300 \end{bmatrix}.$$

0.4 points for the final result