Final exam Calculus 1 (for Physics)

Block 1A, 2022–2023

university of groningen

INSTRUCTIONS TO CANDIDATES

1. Attempt all 8 questions in this test. The total number of points available is 100. You will get 10 points for free.
2. The number of points you can get for each question is shown next to it.
3. In answering the questions in this paper it is particularly important to show your argumentation. The total number of points will only be given for full and detailed answers.
4. Simple pocket calculators are allowed at this exam. Other electronic devices such as graphical/programmable calculators, tablets, laptops and mobile phones are not.
5. Books, notes and formula sheets are all not allowed.
6. Please write your solutions on the accompanying pages and make sure that all pages have your name and student ID on them.

DO NOT REMOVE THIS DOCUMENT FROM THE EXAMINATION ROOM.

$\boxed{\mathbf{1}}$ Find the following quantities:

a. $\lim_{x \to -2^+} \frac{x^2-x-6}{|x+2|}$. [**6 points**]

b. $\lim_{x \to 0^-} \frac{\sin 2x}{x(x+1)}$. [**6 points**]

c. $\lim_{x \to 1} (x-1) \cos\left(\frac{1}{x-1}\right)$. [**6 points**]

Solution:

a. We have that

$$\begin{aligned}\lim_{x \to -2^+} \frac{x^2-x-6}{|x+2|} &= \lim_{x \to -2^+} \frac{(x-3)(x+2)}{|x+2|} = \lim_{x \to -2^+} \frac{(x-3)(x+2)}{x+2} \\ &= \lim_{x \to -2^+} (x-3) = -5.\end{aligned}$$

b. We have that

$$\lim_{x \to 0^-} \frac{\sin 2x}{x(x+1)} = \lim_{x \to 0} \frac{1}{x+1} \cdot 2 \cdot \frac{\sin 2x}{2x} = \frac{1}{1+0} \cdot 2 \cdot 1 = 2.$$

c. We have that $-1 \le \cos\left(\frac{1}{x-1}\right) \le 1$ for $x \ne 1$, so

$$0 \le \left| \lim_{x \to 1} (x-1) \cos\left(\frac{1}{x-1}\right) \right| \le \left| \lim_{x \to 1} (x-1) \right| = 0,$$

so $\lim_{x \to 1} (x-1) \cos\left(\frac{1}{x-1}\right) = 0$.

$\boxed{\mathbf{2}}$ Find all $z \in \mathbb{C}$ for which $z^3 = -8$. [**8 points**]

Solution:
We have that $z^3 = -8$ if and only if $z^3 = 8e^{\pi i + 2\pi k i}$, where $k \in \mathbb{Z}$, so

$$z = 2e^{\frac{\pi}{3} i + \frac{2}{3}\pi k i}, \ k \in \mathbb{Z}.$$

So $z = 2e^{\frac{\pi}{3} i}$, $z = e^{\pi i}$, or $z = -e^{\frac{4\pi}{3} i}$, or equivalently

$$z = 1 + i\sqrt{3}, \ z = -2, \ \text{or } z = 1 - i\sqrt{3}.$$

$\boxed{\mathbf{3}}$ Find the derivative of the following functions:

a. $g(x) = \sin\left(e^{x^2}\right) + \tan^3 x$. [**5 points**]

b. $h(x) = \ln\left(x^2 + 11x + 2022\right)$. [**5 points**]

c. $q(x) = \frac{x^4+2}{x^4+3}$. [**5 points**]

d. $p(x) = x^{3x+14}$. [**8 points**]

Solution:

a. $g'(x) = 2xe^{x^2}\cos\left(e^{x^2}\right) + 3\frac{\tan^2 x}{\cos^2 x}$.

b. $h'(x) = \frac{1}{(x^2+11x+2022)} \cdot (2x+11+0) = \frac{2x+11}{x^2+11x+2022}$.

c. $q'(x) = \left(1 - \frac{1}{x^4+3}\right)' = \frac{4x^3}{(x^4+3)^2}$.

d. $p'(x) = \left(e^{(3x+14)\ln x}\right)' = e^{(3x+14)\ln x}\left(3\ln x + (3x+14)\cdot\frac{1}{x}\right) = x^{3x+14}\left(3\ln x + \frac{3x+14}{x}\right)$.

$\boxed{4}$ Determine the Taylor series of $f(x) = \ln(2x+1)$ near $x = 0$. **[9 points]**

Solution: $f'(x) = \frac{2}{2x+1} = 2\sum_{n=0}^{\infty}(-2x)^n$, so the Taylor series of $f(x) = \ln(2x+1)$ near $x = 0$ is

$$f(x) = \int_0^x 2\sum_{n=0}^{\infty}(-2t)^n\,dt = -\sum_{n=0}^{\infty}\frac{1}{n+1}(-2x)^{n+1}.$$

$\boxed{5}$ Solve the differential equation $\frac{1}{(x+1)y^2}\frac{dy}{dx} = \cos x$. **[9 points]**

Solution: We have that $\frac{1}{y^2}\frac{dy}{dx} = (x+1)\cos x$, so

$$-\frac{1}{y} = \int (x+1)\cos x dx = [(x+1)\sin x] - \int 1\cdot(\sin x)dx = (x+1)\sin x - \cos x - C$$

where C is a constant, so

$$y = \frac{1}{C - (x+1)\sin x + \cos x},$$

is the general solution.

$\boxed{6}$ Use mathematical induction to prove that for all $n \in \mathbb{N}\cup\{0\}$ we have that $\int_0^{\infty} x^n e^{-2x}dx = \frac{n!}{2^{n+1}}$. **[8 points]**

Proof: For $n = 0$ we get that

$$\int_0^{\infty} x^n e^{-2x}dx = \int_0^{\infty} e^{-2x}dx = \left[-\frac{1}{2}e^{-2x}\right]_0^{\infty} = \frac{1!}{2^{0+1}},$$

so for $n = 0$ we have that indeed $\int_0^{\infty} x^n e^{-2x}dx = \frac{n!}{2^{n+1}}$. Suppose that for an $n \geq 0$ we have that $\int_0^{\infty} x^n e^{-2x}dx = \frac{n!}{2^{n+1}}$. Then using integration by parts gives

$$\int_0^{\infty} x^{n+1}e^{-2x}dx = -\frac{1}{2}\left[x^{n+1}e^{-2x}\right]_0^{\infty} + \frac{1}{2}(n+1)\int_0^{\infty} x^n e^{-2x}dx = \frac{(n+1)!}{2^{n+2}},$$

which proves that if $\int\limits_0^\infty x^n e^{-2x}dx = \frac{n!}{2^{n+1}}$ for a value $n = k$, $k \in \mathbb{N} \cup \{0\}$, then $\int\limits_0^\infty x^n e^{-2x}dx = \frac{n!}{2^{n+1}}$ for $n = k+1$ as well. As we have already proven that $\int\limits_0^\infty x^n e^{-2x}dx = \frac{n!}{2^{n+1}}$ for $n = 0$, we have now proved using mathematical induction that $\int\limits_0^\infty x^n e^{-2x}dx = \frac{n!}{2^{n+1}}$ for all $n \in \mathbb{N} \cup \{0\}$. *q.e.d.*

$\boxed{7}$ A spaceship accelerates for 2 hours from rest to a certain maximum speed v, traversing 1000 kilometers in the process. Prove that v is at least 500 kilometers per hour. **[10 points]**

Proof: Let $x(t)$ be the distance traveled by the spaceship at time t, t measured in hours. By the mean value theorem, there is a $c \in (0, 2)$ such that $x'(c) = \frac{x(2)-x(0)}{2-0} = 500$, which means that there was a time between when the spaceship started accelerating and when the spaceship reached maximum speed when the speed was 500 kilometers per hour. As maximum speed was reached after that point, the maximum speed v has to be at least 500 kilometers per hour. *q.e.d.*

$\boxed{8}$ Find the maximum of $f(x) = x^3 e^{-x^2}$, $x \geq 0$. **[5 points]**

Solution: $f'(x) = 3x^2 e^{-x^2} - 2x^4 e^{-x^2}$. Solving $f'(x) = 0$ gives $x = 0$, or $x = \pm\sqrt{\frac{3}{2}}$. For $x > \sqrt{\frac{3}{2}}$ we have that $f'(x) < 0$ and for $0 < x < \sqrt{\frac{3}{2}}$ we have that $f'(x) > 0$, so f takes on a global maximum for $x = \sqrt{\frac{3}{2}}$. So the maximum of $f(x) = x^3 e^{-x^2}$, $x \in \mathbb{R}$ is $f\left(\sqrt{\frac{3}{2}}\right) = \frac{3}{2e}\sqrt{\frac{3}{2e}}$.