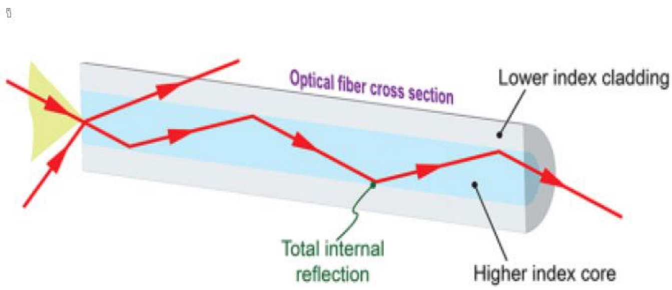
Electricity and Magnetism, Test 6 June 21 2023 8:30-10:30

Write your name and student number on each answer sheet. Use of a calculator is allowed. You may make use of 1 A4 (double sided) with handwritten notes and of the provided formula sheet. The same notation is used as in the book, i.e. a bold-face $\vec{\mathbf{A}}$ is a vector, $\hat{\mathbf{x}}$ is the unit vector in the x-direction. **In your handwritten answers, remember to indicate vectors (unit vectors) with an arrow (hat) above the symbol**.

4 questions, 43 points

Question 1 (8 points)



Optical fibers are used as a means to transmit light between the two ends of the fiber and find wide usage in fiber-optic communications. Optical fibers include a core surrounded by a transparent cladding material; light is kept in the core by the phenomenon of total internal reflection. The cladding (typically thicker than the core) protects the core from the environment, and also makes the fiber mechanically-stable.

An optical fiber with circular cross-section has refractive index of $n_1 = 1.5$ at the communication wavelength. It's surrounded by a thick cladding with refractive index of $n_2 = 1.4$.

A. Find the maximum angle (in degrees) *relative to the fiber axis* at which light can propagate down the fiber by undergoing successive internal reflections (2 points)

<u>Tip</u>: don't forget to set your calculator in the "degree" mode

B. Find the speed of light in the fiber (1 point)

C. For the angle you found in (A), find the speed at which light actually makes its way along the fiber – that is, the length of the fiber that the light traverses per unit time. (2 points)

<u>Tip</u>: this is not what you calculated in (B) as the light bounces back and forth rather than following a straight path along the fiber

D. Because of production failures, the cladding may include some defect regions with the refractive index different from that designed. Explain qualitatively the physics of what happens if the refractive index of the defect is higher than the designed value of $n_2 = 1.4$. And if the refractive index of the cladding defect is lower than $n_2 = 1.4$? (3 points)

<u>Model answers</u> (Waves in materials)

A. $\dfrac{sin\ \theta_T}{sin\ \theta_I} = \dfrac{n_1}{n_2}; \max(sin\ \theta_T) = 1 \ \Rightarrow sin\ \theta_I = \dfrac{n_2}{n_1}$ (0.5 point)

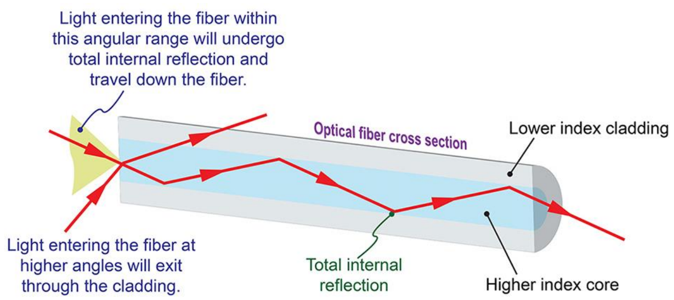
$arcsin\left(\dfrac{n_2}{n_1}\right) = arcsin\left(\dfrac{1.4}{1.5}\right) \cong 69^{\circ}$ (0.5 point)

The angle wrt fiber's axis is 21° (1 point)

B. $\mathrm{v} = \frac{c}{n_1} = \frac{3 \cdot 10^8}{1.5} = 2 \cdot 10^8 \, \frac{\mathrm{m}}{\mathrm{s}}$ (1 point)

C. The actual pathway is $l/\cos(21^\circ) = 1.07l$ (1 point)

The speed is $2 \cdot 10^8/1.07 = 1.87 \cdot 10^8$ m/s (1 point)



D. If the refractive index of the defect is larger than $n_2 = 1.4$ but smaller than $n_1 = 1.5$, the acceptance angle becomes smaller which would lead to some losses of the transmitted light. If the refractive index of the defect becomes larger than $n_1 = 1.5$, there will not be any internal reflection and hence the fiber will not work (=no light transmitted because of many leakages from the core into the cladding) (2 points)

If the refractive index of the defect is smaller than $n_2 = 1.4$, nothing really bad happen as the acceptance angle increases so that the already existing rays will continue to experience total internal reflection. (1 point)

However, there will be light scattering at the interfaces with the defects.

<u>Note</u>: this subquestion tests students' creativity so that there might be deviations from the model answers

Question 2 (11 points)

Let us consider a gedanken ("thought") experiment in which a charge q suddenly appears at position $\vec{\mathbf{r}} = 0$ at time $t = 0$, persists for a while, and then disappears at time $t = \tau$. What is the electric field $\vec{\mathbf{E}}$ generated by such a charge?

A. Calculate the retarded potentials for such a situation and show that

$\vec{\mathbf{A}} = 0$

$$V = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r} \text{ for } 0 \le t - \dfrac{r}{c} \le \tau \\ 0 \text{ otherwise} \end{cases}$$

(2 points)

B. From the potentials, calculate the electric field $\vec{\mathbf{E}}$ and draw schematically the lines of the electric field at $t < 0$, $t = \tau/2$, and $t = 2\tau$ (5 points)

C. What is wrong with the formulation of this thought experiment? What fundamental law(s) of classical electrodynamics does it violate? Explain your point of view and suggest a fix for the formulation of the problem. What would your fix change in the solution? (4 points; 1 point the correct answer to each sentence above)

Model answers (Potentials and Fields; Radiation)

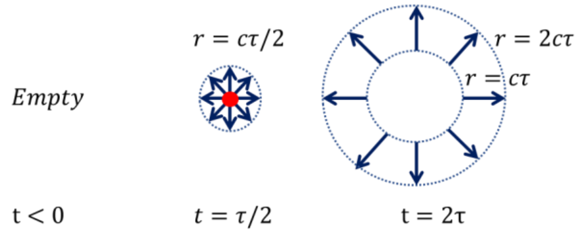
A. $\vec{\mathbf{A}} = 0$ as no currents in sight (1 point; given only if the explanation is presented)

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{\mathbf{r}}', t_r)}{\mathscr{r}} d\tau'; \; t_r \equiv t - \frac{\mathscr{r}}{c}$$

$$V = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r} \text{ for } 0 \le t - \dfrac{r}{c} \le \tau \\ 0 \text{ otherwise} \end{cases} \quad \text{(1 point)}$$

Note: as the separation vector $\vec{\boldsymbol{\mathscr{r}}} \equiv \vec{\mathbf{r}} - \vec{\mathbf{r}}'$ (Eq.1.23) and $\vec{\mathbf{r}}' = 0$ (the charge is at the origin), $\vec{\boldsymbol{\mathscr{r}}} = \vec{\mathbf{r}}$ so it doesn't matter which of the two to use.

$$\textbf{B. } \vec{\mathbf{E}} = -\vec{\nabla} V = \begin{cases} \dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^2}\hat{\mathbf{r}} \text{ for } 0 \le t - \dfrac{r}{c} \le \tau \\ 0 \text{ otherwise} \end{cases} \quad \text{(1 point)}$$



1 point 3 points (awarded only for the entirely correct answer; partial answers result in lowering the number of points)

C. The charge appears from nowhere and disappears to nowhere (1 point)

This violates the conservation-of-charge law. (1 point)

To fix it, there must be a current (density) added which changes in time (1 point)

This would result in a non-zero $\vec{\mathbf{A}}$ and therefore the appearance of the magnetic field. The electric field will be modified, too. (1 point)

Question 3 (14 points)

A. The energy (per unit time per unit area) radiated by an oscillating dipole $\vec{\mathbf{p}}(t) = p_0 cos(\omega t)\hat{\mathbf{z}}$ is determined by the Poynting vector

$$\vec{\mathbf{S}}(\vec{\mathbf{r}}, t) = \frac{\mu_0}{c}\left\{\frac{p_0\omega^2}{4\pi}\left(\frac{sin\theta}{r}\right)cos[\omega(t - r/c)]\right\}^2 \hat{\mathbf{r}}$$

Calculate the total power (averaged over many oscillation periods) radiated by an oscillating electric dipole and show that

$$\langle P \rangle = \frac{\mu_0 p_0^2 \omega^4}{12\pi c} \quad \text{(4 points)}$$

Tip 1: integrate the average energy over a sphere of radius r

Tip 2: you might find useful the following integral $\int sin^3\theta\, d\theta = \frac{cos^3\theta}{3} - cos\theta$

B. The oscillating dipole emits power into the outer space so that the power should be externally supplied to the oscillating dipole. The resistance that would give the same average loss-to-heat power as the oscillating dipole in *fact* puts out in the form of radiation, is called the “radiation resistance”. Find the “radiation resistance” R_{rad} of the wire joining the two ends of the radiative dipole and show that

$$R_{rad} = \frac{\mu_0 d^2 \omega^2}{6\pi c} \quad \text{(5 points)}$$

Tip: use Joule’s heat law and the energy conservation law

C. Express your result in the size of the dipole d and the emitted wavelength λ; re-calculate all constants into a single number. (3 points)

D. For the wires in an ordinary radio (say, $d = 1\ cm$ and $\lambda = 0.1\ m$), calculate and compare the values of the radiative resistance R_{rad} and the total resistance R. For the latter, consider a copper (resistivity $\rho = 2 \cdot 10^{-8}\ \Omega\, m$) wire of diameter $D = 1\ mm$ (2 points)

Answer to Question 3 (Griffiths, p.471 and Problem 11.3) 14 points

A. (4 points)

$$I = \langle \mathrm{S} \rangle_t = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\left(\frac{sin^2\theta}{r^2}\right) \qquad \text{(1 point)}$$

The total power radiated:

$$\langle P \rangle = \int \langle \mathbf{S} \rangle \cdot d\mathbf{a} = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c}\int \frac{sin^2\theta}{r^2} r^2 sin\theta\, d\theta d\varphi \qquad \text{(1 point)}$$

$$= \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} \int_0^{2\pi} d\varphi \int_0^{\pi} sin^3\theta \, d\theta = \frac{\mu_0 p_0^2 \omega^4}{32\pi^2 c} 2\pi \left(\frac{cos^3\theta}{3} - cos\theta \right) \Bigg|_0^{\pi} = \frac{\mu_0 p_0^2 \omega^4}{16\pi c} \frac{4}{3}$$

$$= \frac{\mu_0 p_0^2 \omega^4}{12\pi c}$$ (2 points)

B. (5 points)

$$I = \frac{dq}{dt} = -q_0 \omega sin(\omega t)$$ (1 point)

$$P = I^2 R = q_0^2 \omega^2 sin^2(\omega t) R$$ (1 point)

Averaged power $\langle P \rangle = \frac{1}{2} q_0^2 \omega^2 R$ (1 point)

$$\frac{1}{2} q_0^2 \omega^2 R = \frac{\mu_0 d^2 q_0^2 \omega^4}{12\pi c}$$

$$R_{rad} = \frac{\mu_0 d^2 \omega^2}{6\pi c}$$ (2 points)

C. (3 points)

$$\omega = \frac{2\pi c}{\lambda}$$ (1 point)

$$R_{rad} = \frac{\mu_0 d^2 4\pi^2 c^2}{6\pi c \lambda^2} = \frac{2}{3}\pi \mu_0 c \left(\frac{d}{\lambda}\right)^2 = \frac{2}{3}\pi \, (4\pi \times 10^{-7})(3 \times 10^8) \left(\frac{d}{\lambda}\right)^2 \cong 790 \left(\frac{d}{\lambda}\right)^2 \; [\Omega]$$

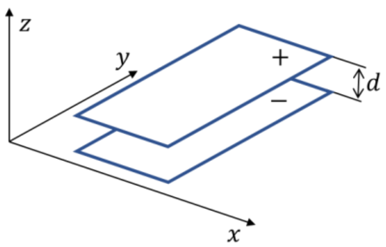
(2 points)

D. (2 points)

$$R_{rad} = 790 \left(\frac{0.01}{0.1}\right)^2 = 7.9 \, \Omega; \; R = \rho \frac{d}{(\pi D^2/4)} = 2 \; 10^{-8} \frac{0.01}{(\pi 10^{-6}/4)} = 2.5 \; 10^{-4} \, \Omega$$

R is negligible compared to R_{rad}. (2 points)

Question 4 (10 points)



An ideal capacitor consists of two parallel rectangular plates with a vertical separation of d (see the figure above). The capacitor has been charged by connecting it temporarily to a battery of the voltage V, with the positive and negative charges at the upper and lower plate, respectively. The capacitor is at rest in the laboratory system of coordinates. You need to find how the fields inside the capacitor look like from the viewpoint of a moving observer. As usual, neglect the fringing fields.

A. Write down the electric field strength $\vec{\mathbf{E}}_0$ and the magnetic field strength $\vec{\mathbf{B}}_0$ between the plates in the laboratory system of coordinates (3 points)

<u>Tip</u>: don't forget to express the fields via the parameters given (i.e. V and d)

Now consider an observer in a frame that is moving to the +*x*-direction with respect to the laboratory frame, with speed v. Provide the following quantities as they would be measured by the moving observer:

B. Change in the number of excess electrons on the negative plate (1 points)

C. The electric field $\vec{\mathbf{E}}$ and the magnetic field $\vec{\mathbf{B}}$ between the plates expressing the fields via the parameters given (i.e. V and d) (4 points)

D. Explain why the moving observer observes the magnetic field (2 points)

<u>Tip</u>: answers like "this is due to relativistic electrodynamics" do not count; you need to explain the physics behind

<u>Model Answers (How fields transform) (10 points)</u>

Similar to Problem 12.43

A. $\vec{\mathbf{E}}_0 = -\frac{V}{d}\hat{\mathbf{z}}$ (2 points)

$\vec{\mathbf{B}}_0 = 0$ as there are no currents (1 point)

B. None – charge is invariant. (1 point)

C. $\vec{\mathbf{E}} = \gamma\vec{\mathbf{E}}_0 = \frac{\vec{\mathbf{E}}_0}{\sqrt{1-(\mathrm{v}/c)^2}} = -\frac{V}{d\sqrt{1-(\mathrm{v}/c)^2}}$ (1 point)

The direction is the same as before because other components of $\vec{\mathbf{E}}_0$ are zero (1 point)

As $\vec{\mathbf{B}}_0 = 0$ in the lab frame, in the moving frame

$\vec{\mathbf{B}} = \frac{1}{c^2}(\mathbf{v}\times\vec{\mathbf{E}})$ (1 point)

so the magnetic field is $\hat{\mathbf{x}}\times(-\hat{\mathbf{z}}) = \hat{\mathbf{y}}$ directed

and its magnitude is

$B = \frac{\mathrm{v}}{c^2}E = \frac{\mathrm{v}}{c^2}\frac{1}{\sqrt{1-(\mathrm{v}/c)^2}}E_0 = -\frac{\mathrm{v}}{c^2}\frac{1}{\sqrt{1-(\mathrm{v}/c)^2}}\frac{V}{d}$ (1 point)

Alternative solution: use Eq.12.109 directly

In the rest system, $E_z = -\frac{V}{d}; B = 0$

In the moving system:

$\bar{E}_z = \gamma(E_z + \mathrm{v}B_y) = \gamma E_z = -\frac{V}{d\sqrt{1-(\mathrm{v}/c)^2}}$ (2 points)

$$\bar{B}_y = \gamma\left(B_y + \frac{\mathrm{v}}{c^2}E_z\right) = \gamma\frac{\mathrm{v}}{c^2}E_z = -\frac{\mathrm{v}}{c^2}\frac{1}{\sqrt{1-(\mathrm{v}/c)^2}}\frac{V}{d} \text{ (2 points)}$$

Alternative solution: you forgot how the fields are transformed

$$\sigma = \frac{\sigma_0}{\sqrt{1-(\mathrm{v}/c)^2}};\ \bar{E}_z = \gamma E_z = -\frac{V}{d\sqrt{1-(\mathrm{v}/c)^2}} \quad \text{(2 points)}$$

The factor $\sqrt{1-(\mathrm{v}/c)^2}$ is because the plates Lorentz-contract in one direction but the charge does not change so that the charge density increases.

$$\vec{\mathbf{K}}_\pm = \mp\sigma\mathrm{v}\hat{\mathbf{x}};\, B = \mu_0\sigma\mathrm{v} = \mu_0\mathrm{v}\frac{\sigma_0}{\sqrt{1-(\mathrm{v}/c)^2}} = \frac{\mathrm{v}}{c^2}\frac{1}{\sqrt{1-(\mathrm{v}/c)^2}}\frac{\sigma_0}{\epsilon_0}$$

$$= -\frac{V}{d\sqrt{1-(\mathrm{v}/c)^2}} \quad \text{(2 points)}$$

D. Because the observer sees two surface current densities $\vec{\mathbf{K}}_\pm = \mp\sigma\mathrm{v}\hat{\mathbf{x}}$ in the opposite directions which create the magnetic field (2 points)