Calculus 2

Midterm Exam

March 25, 2022 (9:00 – 11:00)

university of groningen

Please turn over and read the instructions!

1) Consider the space curve C given by the vector function $\vec{r}\colon [0, 2\pi] \to \mathbb{R}^3$,

$$\vec{r}(t) = e^{-t}\,\vec{\imath} + e^{-t}\sin t\,\vec{\jmath} + e^{-t}\cos t\,\vec{k}, \qquad 0 \le t \le 2\pi.$$

$\boxed{9}$ a) Determine the first-, second- and third-order derivatives of $\vec{r}(t)$, i.e. calculate $\vec{r}\,'(t)$, $\vec{r}\,''(t)$ and $\vec{r}\,'''(t)$. Simplify as much as possible.

$\boxed{8}$ b) Find the length L of C and its parametrization by arc length s.

$\boxed{9}$ c) Determine the unit tangent vector $\vec{T}(t)$, principal normal vector $\vec{N}(t)$ and binormal vector $\vec{B}(t)$ to C.

$\boxed{6}$ d) Compute the curvature $\kappa(t)$ and the torsion $\tau(t)$ of C.

2) Consider the function $f\colon \mathbb{R}^2 \to \mathbb{R}$, $f(x, y) = 10 - 2x + 8y + x^2 + 2y^2$.

$\boxed{6}$ a) Compute all first- and second-order partial derivatives of $f(x, y)$.

$\boxed{4}$ b) Determine the maximum rate of change of $f(x, y)$ at $(x, y) = (0, 0)$.

$\boxed{2}$ c) Find the equation $z = L(x, y)$ for the tangent plane to the graph of $f(x, y)$ at the point $(0, 0, 10)$.

$\boxed{12}$ d) Find the absolute maximum and minimum of $f(x, y)$ on the closed region $E = \{(x, y) \mid x^2 - 2x + 2y^2 \le 7\}$.

$\boxed{16}$ 3) Evaluate the double integral of the function seen in Problem 2 over the region $R = \{(x, y) \mid (x-1)^2 + 2(y+2)^2 \le 1\}$. (Hint: Change variables via the transformation T: $x = 1 + u\cos v$, $y = -2 + \frac{1}{\sqrt{2}}u\sin v$.)

4) Evaluate the following line integrals along the curve C in Problem 1:

$\boxed{8}$ a) $\displaystyle\int_C g(x, y, z)\,ds$ with the function $g(x, y, z) = \dfrac{yz}{x^3}$.

$\boxed{10}$ b) $\displaystyle\int_C \vec{F}\cdot d\vec{r}$ with the vector field $\vec{F}(x, y, z) = z\,\vec{\jmath} - y\,\vec{k}$.

Formula sheet

Arc Length: $L = \int_a^b |\vec{r}\,'(t)|\,dt$, $\qquad s(t) = \int_a^t |\vec{r}\,'(u)|\,du$

$\vec{T}\vec{N}\vec{B}$ frame: $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$, $\quad \vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$, $\quad \vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

Curvature: $\kappa = \left|\dfrac{d\vec{T}}{ds}\right| = \dfrac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \dfrac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$

Torsion: $\tau = -\dfrac{d\vec{B}}{ds} \cdot \vec{N} = -\dfrac{\vec{B}\,'(t) \cdot \vec{N}(t)}{|\vec{r}\,'(t)|} = \dfrac{[\vec{r}\,'(t) \times \vec{r}\,''(t)] \cdot \vec{r}\,'''(t)}{|\vec{r}\,'(t) \times \vec{r}\,''(t)|^2}$

Gradient of f: $\operatorname{grad} f(x,y) = \nabla f(x,y) = \langle f_x, f_y \rangle$

Linearization at (a,b): $L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$

Method of Lagrange Multipliers: To find the max./min. of f along the level curve $g(x,y) = k$ solve $\nabla f(x,y) = \lambda \nabla g(x,y)$ for (x,y,λ) while making sure that $g(x,y) = k$ is satisfied.

Second Derivative Test: Let (a,b) be a stationary point of $f(x,y)$ and compute $D = D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$.
(a) If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.
(b) If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.
(c) If $D < 0$, then (a,b) is a saddle point of f.

Change of Variables in a Double Integral ($T\colon x = x(u,v),\ y = y(u,v)$)

$$\iint\limits_R f(x,y)\,dA = \iint\limits_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv,$$

where $R = T(S)$ and $\dfrac{\partial(x,y)}{\partial(u,v)} = \det \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix}$ is the Jacobian.

Line Integral of a Scalar Function: $\displaystyle\int\limits_C g(x,y,z)\,ds = \int_a^b g(\vec{r}(t))|\vec{r}\,'(t)|\,dt$

Line Integral of a Vector Field: $\displaystyle\int\limits_C \vec{F}(x,y,z) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$

Instructions

- **write your name and student number on the top of each page!**
- this is a closed-book exam, it is not allowed to use the textbook or the lecture notes
- you are allowed to use the formula sheet or a simple pocket calculator
- programmable calculators are not allowed, nor the use of electronic devices (tablet, laptop, mobile phone, etc.) to solve the exercises
- your work should be clearly and logically structured
- you should **explain your reasoning using words**
- include your computations, an answer without any computation will not be rewarded so also copy the computations from your scratch paper(s)
- you can achieve 100 points (including the 10 bonus points)