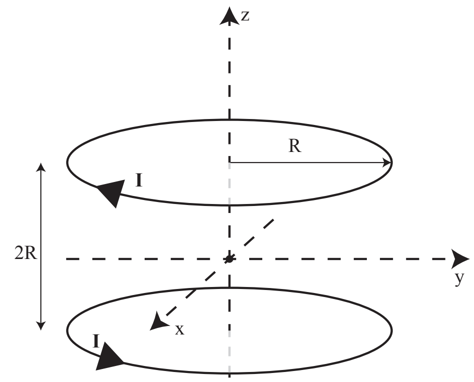
Electricity and Magnetism Part 1, resit, 03/07/2023

4 questions, 77 points

Write your name and student number on each answer sheet. Use of a calculator is allowed. In your handwritten answers, remember to indicate vectors (unit vectors) with an arrow (hat) above the symbol. Before handing in: check that you read all questions carefully, and that you explained your answers where needed

1. Consider two circular loops, each carrying a steady current **I**, but in opposite directions.

The situation is indicated in the figure on the right. The loops have a radius R, and the distance between the loops is equal to their diameter.



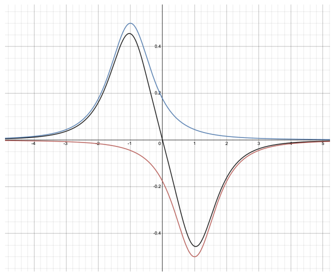
(a) (5 points) Derive an equation for the magnetic field strength along the z-axis.
Answer: like Example 5.6, but with two loops. Pay attention to the direction!

For one loop:
* *Single line element:* $B^{(1)}(z) = \frac{\mu_0 I}{4\pi r^2}$.
* *Take only the vertical component:* $B^{(1)}(z) = \frac{\mu_0 I}{4\pi r^2}\cos\theta$.
* *Integrate along the circle:* $B^{(1)}(z) = \frac{\mu_o I}{4\pi r^2}\cos\theta \int dl = \frac{\mu_0 I}{4\pi r^2} \cdot \frac{R}{r} \cdot 2\pi R$.
* *Picking the appropriate distance:* $r^2 = R^2 + (z \pm R)^2 \implies B^{(1)}(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2+(z\pm R)^2)^{3/2}}$.
* *Adding the two loops:* $B(z) = \frac{\mu_0 I R^2}{2}\left(\frac{1}{(R^2+(z+R)^2)^{3/2}} - \frac{1}{(R^2+(z-R)^2)^{3/2}}\right)$

(+1 for integrating elements dl, +1 for taking the vertical component, +1 for taking the appropriate distance r, +2 for final answer)

(b) (5 points) Make a graph showing the magnetic field on the z-axis due to each loop, and of the combined field of the two loops. Do this for z ranging from $-3R$ to $+3R$. For the vertical axis of this graph, you should use units of $\mu_0 I$. If the magnetic field points in the negative z-direction, it is negative; in the positive z-direction, it is positive.

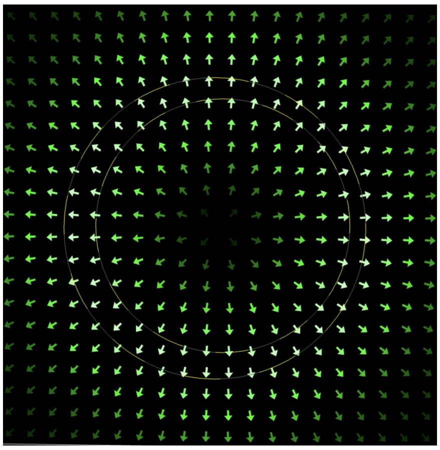
Answer: The two coils have opposite currents - so opposite magnetic fields (red and blue in the figure on the right). Maximum in the center of each coil. For the sum of the two coils, in between them - at $z = 0$ - the magnetic field is zero, linearly increasing in both directions. Desmos link



(+5 for correct figure, -1 not passing origin, -1 if not zero at boundaries, -1 if peak/graph is not smooth, -1 if not anti-symmetric, -1 for forgetting the axis)

(c) (5 points) Next we are interested in the magnetic field in the plane between the two coils, the (x, y) plane. Draw a sketch with magnetic field vectors to give a qualitative indication of the strength and the direction of the magnetic field in this plane, for a region that extends beyond the size of the current loops in the (z=0) plane between the coils. To draw these field vectors, you draw vectors starting at a number of sample points in the 2D plane to represent the magnitude and direction of the field.

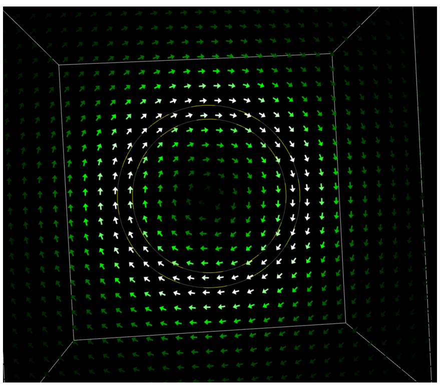
Answer: In the centre (x,y)=(0,0) the magnetic field is zero. From that point the magnitude increases in all directions. For the given current directions the magnetic field points outwards everywhere. Typical magnetic field lines and their directions are sketched in the figure on the right. The highest magnetic field strength is in between the loops; outside it decreases again in strength. You can also explore an interactive version of the situation via this link: Falstad link.



(+2 for qualitative correct (zero in the center, increasing towards rings), +2 for direction outwards everywhere, +1 for decreasing in strength outside the loops)

(d) (5 points) Make a qualitative drawing using vectors to depict the magnetic vector potential $\mathbf{A}$ in the (x,y) plane, at the location of the top coil, so at the location $z = R$.

The magnetic vector potential is directed along the current, and is strongest close to the current loop. This can be visualized using the same Falstad link as above. +2 for qualitative correct (zero in the center, same direction as the current), +2 for same curl everywhere, +1 for maximum strength along the loop. In the graph on the right the strenght is indicated by the brightness; on paper, the best way is to have longer vectors where the field is strong. Note: for a realistic wire with non-zero cross-section the current density is finite, and the vector potential therefore also.



(e) (5 points) Consider an atom, which has a magnetic dipole moment associated with the spin of an unpaired electron. From a quantum mechanical description of this atom, we know that its magnetic dipole moment can be oriented either parallel or anti-parallel to the external magnetic field, and once it is in a certain orientation, it will maintain that orientation. It is possible for this atom to be levitated at the position $(x, y, z) = (0, 0, 0)$, but only for one of the two orientations. Which one? Explain!

Answer: The atom will experience a force towards the center of the trap if the magnetic dipole moment is (and remains) oriented anti-parallel to the magnetic field. In this configuration, the force on the dipole is proportional to the gradient of the magnetic field (equation 6.3). For the current loop in figures 6.3 and 6.4 (see below), the dipole moment is parallel to the magnetic field, and the force is directed towards stronger magnetic fields. In the case of this question, we have the reverse situation. The result is analogous to the Stark deceleration example shown in the lectures: the dipoles are pushed to the minimum of the field strength. Equation 6.3: $\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B})$. *Staying in place means* $\vec{F} = \vec{0} \implies \vec{\nabla}(\vec{m} \cdot \vec{B}) = 0$. *Having a small derivative in B needs to give a restoring force, hence* $\vec{m} \cdot \vec{B} < 0 \implies \vec{m}$ *is anti-parallel to* $\vec{B}$. *(+1 for discussing electron orientation, +1 for realizing that the force is proportional to the derivative of B, +1 for noting that F = 0, +1 for realizing we need to minimize the derivative of B, +1 for correct answer. Or +5 for an overall convincing answer.)*

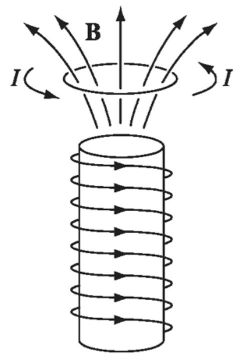


FIGURE 6.3

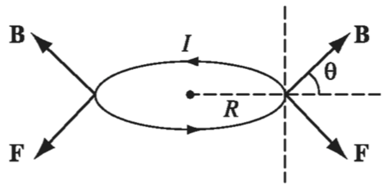
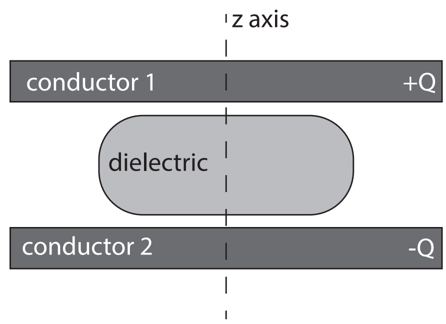


FIGURE 6.4

2. Consider two parallel conducting plates, each with thickness a. Their separation is d. Conductor 1 and 2 carry a charge of +Q and -Q respectively. In between the two conducting plates a neutral and solid block of a linear dielectric material is placed. There is a distance (gap) between the conductor and the dielectric material. The dielectric material has rounded edges. A cross-section of the situation is depicted in the figure on the right.



   (a) (3 points) Copy the sketch to your answer sheet, and indicate where the free and bound (surface and/or volume) charges in this situation are located. Indicate their sign and indicate their density if it is not uniform.

   (b) (5 points) Make a qualitative graph of the potential along the z-axis, for the region in between the two conductors, so including the dielectric material. Clearly indicate for all relevant sections whether the potential is constant, or changes linearly, or has a different behaviour ($1/r$ or $1/r^2$ for example).

   (c) (5 points) Make a qualitative graph of the magnitude of the z-component of the electric field along the z-axis, for the same region. Clearly indicate for all relevant sections whether the magnitude of the field is constant, or changes linearly, or has a different behaviour ($1/r$ or $1/r^2$ for example). Represent the direction of the field in the sign of the magnitude.

(d) (3 points) Next, while we keep the conductors connected to the power supply that was used to charge them, we remove the dielectric material. Compare the electric field (direction and magnitude) between the dielectric material and the conductor in the old situation to the electric field between the conductors in the new situation.

(a):The free charges are on the surfaces of the conductors - positive on conductor 1, negative on conductor 2. Most of the bound charges are on the top and the bottom of the dielectric, and their density tapers of around the rounded edges. In the central part of the capacitor the field is uniform - so there are no volume bound charges. The density of the free charges on the conductor is a bit higher near the dielectric.
(b):The potential is constant inside the conductors: positive for conductor 1, negative for conductor 2. There is a steep linear decline between the conductors and the dielectric, and a steady (but less steep) decline of the potential inside the dielectric. Outside the conductors, the potential drops off first linearly then as $1/r$ then to $1/r^2$.
(c):The electric field is zero inside the conductors. It is constant high between the conductors and the dielectric, and constant but lower inside the dielectric. Outside the conductors the field is first constant then drops of as $1/r^2$ then $1/r^3$ (Take derivative of Potential to see).
(d):The capacitance is reduced when the dielectric material is removed. This happens because the free charge is no longer partially cancelled by the bound charges. However, the voltage difference between the conductors is kept constant by the power supply. As a result, the gradient of the potential is reduced, and the electric field between 1 point mentioning the field increases, full points for complete explanation

3. A solenoid

(a) (5 points) Calculate the magnetic field (magnitude and direction) inside and outside an infinitely long solenoid with n turns per unit length, radius R and a current I. Explain the direction of the field using symmetries, and use Ampere's law to derive the expressions.

(b) (5 points) Inside the solenoid from subquestion (a) we place another, with the inner solenoid having $2n$ turns per unit length, and radius $\frac{1}{2}R$. The two solenoids are co-axial, and each has the same current I. Give the magnetic field inside, between, and outside the solenoids when the current flows in the same and opposite directions. Explain your answer.

(c) (5 points) Consider a cylindrical paramagnetic core inside a single infinitely long solenoid, which has a current I flowing through its windings. Sketch the direction of the magnetisation of this material and the direction of any volume bound current and/or surface bound current. In your sketch also include the direction of the free current in the coil.

(d) (5 points) Let's assume the material has a linear susceptibility χ_m. Find the general expression for $\mathbf{B}$ inside the material (still inside the solenoid).

Answer:
(a): The solenoid is chosen to be oriented with its main axis in the z-direction. From symmetry arguments, the field outside is zero, and the field inside is pointing either in +z or -z, depending on the direction of the current chosen. For an infinite solenoid without material inside, the equation found from Ampere's law is $\mathbf{B} = \mu_0 nI\hat{\mathbf{z}}$. Outside the field is zero. The radius is not relevant.
(b): The magnetic fields add inside the inner solenoid. In between the solenoids there is just the field of the outer solenoid, and outside the field is still zero. Depending on the direction of the current, the direction of the field of the two coils is in the same direction or not. The radius of the solenoids is not relevant. The inner solenoid has twice the magnetic field strength compared to the outer solenoid.

(c): For paramagnetism, the magnetisation M is parallel to B. A surface bound current runs in the same direction as the free current, circumferential to the core material. There is no volume bound current since the magnetisation is uniform, and therefore it has no curl.
(d): The magnetic field is now given by $\mathbf{B} = \mu\mathbf{H} = \mu n I\hat{\mathbf{z}} = \mu_0(1+\chi_m)nI\hat{\mathbf{z}}$

4. Right or wrong? For each subquestion there are four statements. Identify **all wrong** statements (for each subquestion there is at least one, but there could be more), and explain why they are **not correct**. If you find that a statement is not generally true (i.e. in some cases it would be wrong), then also classify it as wrong, and explain why.

   (a) (4 points) Consider a solid sphere, made of a linear dielectric, at the center of which a charge $+Q$ is placed. Outside the sphere there are no free charges. The electric field ...

   1. ... outside the sphere is like an electric dipole at the origin
   2. ... outside the sphere is indistinguishable from the field created by a point charge $+Q$ a the origin
   3. ... inside the sphere is reduced due the polarization of the dielectric material by an amount that is proportional to charge Q
   4. ... outside the sphere is equal to ϵ times the electric displacement outside the sphere

   Answer: 1. is not correct - there is no dipole, since the charge is at the center. All others are correct.
   Points: +4 points for correct answer and reasoning (1 point if reasoning is missing), -1 points for incorrectly marking 2., 3. or 4. as incorrect).

   (b) (4 points) Consider 4 charges, with unspecified sign and magnitude, on the corners of a rectangle of size $2a$ by a, with its center at the origin of a coordinate system. The potential sufficiently far way ...

   1. ... is always dominated by the monopole term.
   2. ... is never exactly described by a monopole term.
   3. ... drops in magnitude faster or equal to $1/r^2$
   4. ... is well approximated by the electric field due to a single charge at the coordinate center, with a magnitude equal to the sum of the four charges.

   Answer: 1. is not correct: if there are two positive and two negative charges, there is no monopole term. Depending on the configuration, the field far away is best approximated by the monopole (++++, +++ -, - - - + or - - - -), dipole (++ - -) or quadrupole term (+ - + -). 2. is correct: never exactly the monopole, always some extra term. 3. is not correct, for example for four equal charges, when there are four monopole terms, the potential goes like 1/r (note: 1/r should be explicitly mentioned!). Also 4. is wrong; the electric field due to a dipole is not similar to that of charge zero.
   Points: +1 for marking either 1. or 3. as incorrect with reason, +3 for marking both 1. and 3. and +1 for marking 4. as incorrect with reason, -1 for naming 2..

   (c) (4 points) Consider two parallel large conducting plates. A power supply is connected to keep one plate at a potential of $+V$ and the other plate at $-V$. The distance between the plates is d. Now the plates, still connected to the power supply, are completely submerged in a liquid with a dielectric constant of 2.

   1. As a result the capacitance of the configuration described above is doubled.
   2. As a result the force between the two plates is doubled.
   3. The electric potential is zero at a point that is equidistant between the two plates.

4. If the liquid consists of polar molecules, a net force acts on the molecules at the edges of the plates, and its magnitude is proportional to the dielectric constant of the liquid.

Answer: About 1. $C = Q/V$; V is constant, but due to the dielectric constant more free charge is needed, which is supplied by the power supply. The capacitance is therefore increasing - 1. is correct. 2. is wrong - the force remains the same. 3. is correct - the potential varies linearly from one plate to the other. Statement 4. is not correct: since the electric field between the plates is proportional to the voltage, which is kept constant, the gradient of the electric field is also constant. Therefore the force on the molecules is also constant.
Points: +2 for naming 2. with (either) reason, +2 for naming 4. with reason, -1 for naming 1. or 3. or for giving an incorrect reason (when two reasons for given for the same point).

(d) (4 points) Consider an electron (a negatively charged particle), initially at rest in the center of a coordinate system. There is a positive, constant and uniform magnetic field in the $+\hat{\boldsymbol{x}}$ direction.

1. Following the application (in addition to the magnetic field) of an electric field that is positive, constant and uniform in the $+\hat{\boldsymbol{y}}$ direction, the electron's velocity will develop a positive component in the $+\hat{\boldsymbol{x}}$ direction.
2. When an electric field is applied that is under 45 degrees with respect to the magnetic field, the electron will obtain a non-zero velocity component in all three directions $\hat{\boldsymbol{x}}, \hat{\boldsymbol{y}}, \hat{\boldsymbol{z}}$.
3. When switching on an electric field, the electron will start to move; as soon as the electric field is switched off, the electron will come to rest again.
4. When an electric field is switched on that is in the same direction as the magnetic field, the electron will move in a straight line.

Answer: 1. is incorrect: the velocity will be perpendicular to $\boldsymbol{B}$, so the velocity points in $\hat{y}$ and $\hat{z}$. 2. is correct. 3. is incorrect: magnetic field does not do work, the electron will keep its speed (but the velocity will still change direction). 4. is true
Points: +2 points for naming 1. with reasoning, +2 points for naming 3. with reasoning. -1 point for incorrectly naming 2 or 4..
Remarks: many students argued that 4. is incorrect because the magnetic field keeps the electron moving. This is wrong, the electron will move because the electric field gave it some initial velocity (kinetic energy). The magnetic field can only redirect this velocity but does not alter the speed. Points were not awarded if this or a similar answer including the magnetic field was given.