Calculus 2 (for Physics)

Resit Exam

April 14, 2023 (8:30 – 10:30)

Please turn over and read the instructions!

1) Consider the hyperbolic paraboloid H given by the equation

$$x^2 - y^2 - z = 0$$

$\boxed{6}$ a) Treating H as a level surface of a function of three variables, find an equation of the tangent plane to H at the point $P(4, 3, 7)$.

$\boxed{6}$ b) Use the Implicit Function Theorem to show that near the point P in part a), H can be considered to be the graph of a function f of x and y. Compute the partial derivatives f_x and f_y and show that the tangent plane found in a) coincides with the graph of the linearization $L(x, y)$ of $f(x, y)$ at $(4, 3)$.

$\boxed{9}$ c) Use the method of Lagrange multipliers to find the point(s) closest to the origin along the intersection of H with the plane $z = x + y$.

2) Consider the vector field

$$\vec{G}(x, y, z) = (y^2 z + 2xz^2)\,\hat{\imath} + (2xyz)\,\hat{\jmath} + (xy^2 + 2x^2 z)\,\hat{k}$$

$\boxed{9}$ a) Show that $\operatorname{curl} \vec{G} = \vec{0}$.

$\boxed{8}$ b) Determine a scalar potential for $\vec{G}$.

$\boxed{4}$ c) Evaluate the line integral of $\vec{G}$ along the curve of intersection of the sphere $x^2 + y^2 + z^2 = 5$ and the plane $y = 1$.

3) Consider the velocity field given by

$$\vec{V}(x, y, z) = (-2yz)\,\hat{\imath} + (y)\,\hat{\jmath} + (3x)\,\hat{k}$$

and the surface $S = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z = 5,\ 1 \le z \le 4\}$ with upward normal vectors and positively-oriented boundary ∂S.

$\boxed{3}$ a) Describe and sketch the surface S and its boundary ∂S. Draw the orientation of ∂S.

Verify Stokes' Theorem by directly computing

$\boxed{12}$ b) the circulation of $\vec{V}$ along ∂S, i.e. $\displaystyle\int_{\partial S} \vec{V} \cdot d\vec{r}$ and

$\boxed{10}$ c) the flux of $\operatorname{curl} \vec{V}$ across S, that is $\displaystyle\iint_S \operatorname{curl} \vec{V} \cdot d\vec{S}$.

4) Consider the force field given by

$$\vec{F}(x, y, z) = (2x)\,\hat{\imath} + (2y)\,\hat{\jmath} + (z^2)\,\hat{k}$$

over the solid region $E = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \le 1,\ x \ge 0,\ y \ge 0\}$ and its outward-oriented boundary surface ∂E.

$\boxed{3}$ a) Describe and sketch the solid region E and the boundary surface ∂E. Draw the orientation of ∂E.

Verify the Divergence Theorem by directly calculating

$\boxed{12}$ b) the flux of $\vec{F}$ across ∂E, that is $\displaystyle\oiint_{\partial E} \vec{F} \cdot d\vec{S}$ and

$\boxed{8}$ c) the triple integral of $\operatorname{div} \vec{F}$ over E, i.e. $\displaystyle\iiint_E \operatorname{div} \vec{F}\, dV$.

Instructions

Please read and follow these instructions!

- **write your name and S-number on the top of each sheet of writing paper!**
- use the writing paper (lined) and scratch paper (blank) provided, raise your hand if you need more paper
- start each question on a new page
- use a pen with black or blue ink
- do not use any kind of correcting fluid or tape
- any rough work should be crossed through neatly so it can be seen
- this is a closed-book exam, you are not allowed to use the textbook or your notes or any other written material
- you are allowed to use the formula sheet provided or a simple pocket calculator
- programmable/graphing calculators are not allowed, nor the use of electronic devices (tablet, laptop, phone, etc.) to solve the exercises
- your work should be clearly and logically structured
- **explain your reasoning using words!!!**
- show all your calculations, an answer without any computation will not be rewarded
- you have 2 hours to solve all 4 problems
- you get 10 free points
- upon completion[1] submit your worksheets at the front desk

[1] At the end of the exam or after you finished whichever is sooner.

Formula sheet

Gradient of $g(x,y,z)$: $\operatorname{grad} g(x,y,z) = \nabla g(x,y,z) = \langle g_x, g_y, g_z\rangle$

Linearization of $f(x,y)$: $L(x,y) = f_x(a,b)(x-a) + f_y(a,b)(y-b) + f(a,b)$

Implicit Differentiation: Suppose $F(x,y,z)=0$ is used to define z implicitly as a function $f(x,y)$ in a neighbourhood of the point $P(a,b,c)$ such that $F(a,b,c)=0$ and $F_z(a,b,c)\neq 0$. Then $f_x = -\frac{F_x}{F_z}$, $\quad f_y = -\frac{F_y}{F_z}$.

Method of Lagrange Multipliers: To find the max./min. of f along the level surface $g(x,y,z)=k$ solve $\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ for (x,y,z,λ) while making sure that $g(x,y,z)=k$ is satisfied.

Cylindrical $\to$ Cartesian: $x = r\cos\theta$, $\quad y = r\sin\theta$, $\quad z = z$.

Spherical $\to$ Cartesian: $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$.

Change of Variables in Triple Integrals (T: $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$):

$$\iiint_{T(S)} f(x,y,z)\,dV = \iiint_{S} \tilde{f}(u,v,w)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Jacobians: $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$ (cylindrical), $\dfrac{\partial(x,y,z)}{\partial(\rho,\phi,\theta)} = \rho^2\sin\phi$ (spherical)

Line Integral of $\vec{V}$: $\displaystyle\int_C \vec{V}(x,y,z)\cdot d\vec{r} = \int_a^b \vec{V}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$

Surface Integral of $\vec{F}$: $\displaystyle\iint_S \vec{F}(x,y,z)\cdot d\vec{S} = \iint_D \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv$

Curl of $\vec{V}(x,y,z) = P(x,y,z)\,\hat{\imath} + Q(x,y,z)\,\hat{\jmath} + R(x,y,z)\,\hat{k}$:

$$\operatorname{curl}\vec{V} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

Divergence of $\vec{F}(x,y,z) = F_1(x,y,z)\,\hat{\imath} + F_2(x,y,z)\,\hat{\jmath} + F_3(x,y,z)\,\hat{k}$:

$$\operatorname{div}\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

Some basic trigonometric identities:

$\sin^2 x + \cos^2 x = 1$, $\quad \sin(2x) = 2\sin x\cos x$, $\quad \cos(2x) = \cos^2 x - \sin^2 x$