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1) Consider the cone K given by the equation

$$9x^2 + y^2 - z^2 = 0$$

a) Treating K as a level surface of a function of three variables, find an equation of the tangent plane to K at the point $P(1, 4, 5)$.

b) Use the Implicit Function Theorem to show that near the point P in part a), K can be considered to be the graph of a function f of y and z. Compute the partial derivatives f_y and f_z and show that the tangent plane found in a) coincides with the graph of the linearization $L(y, z)$ of $f(y, z)$ at $(4, 5)$.

c) Use the method of Lagrange multipliers to find the point(s) closest to the z-axis along the intersection of K with the plane $z = 2$.

Solution. a) The cone K can be viewed as a level surface for the function $F(x, y, z) = 9x^2 + y^2 - z^2$. To obtain a normal vector to the tangent plane we may use the fact that the gradient vector is perpendicular to level surfaces. The gradient vector of $F(x, y, z)$ is

$$\nabla F(x, y, z) = F_x\,\hat{\imath} + F_y\,\hat{\jmath} + F_z\,\hat{k} = 18x\,\hat{\imath} + 2y\,\hat{\jmath} - 2z\,\hat{k}$$

which at the point $(1, 4, 5)$ becomes

$$\vec{n} = \nabla F(1, 4, 5) = 18(1)\,\hat{\imath} + 2(4)\,\hat{\jmath} - 2(5)\,\hat{k} = 18\,\hat{\imath} + 8\,\hat{\jmath} - 10\,\hat{k}.$$

For any point $Q(x, y, z)$ in the tangent plane, the vector $\overrightarrow{PQ} = (x-1)\,\hat{\imath} + (y-4)\,\hat{\jmath} + (z-5)\,\hat{k}$ lies in the plane and as such it is perpendicular to $\vec{n}$, i.e. we have

$$\vec{n} \cdot \overrightarrow{PQ} = 0 \quad \Leftrightarrow \quad 18(x-1) + 8(y-4) - 10(z-5) = 0 \quad \Leftrightarrow \quad 9x + 4y - 5z = 0.$$

Therefore $9x + 4y - 5z = 0$ is an equation for the tangent plane to K at $(1, 4, 5)$.

b) Since $F(x, y, z)$ is a polynomial function of x, y, z, its partial derivatives are continuous. Furthermore, we have $F(1, 4, 5) = 0$ and $F_x(1, 4, 5) = 18x|_{x=1} = 18 \neq 0$. By the Implicit Function Theorem, there is a neighbourhood of $(1, 4, 5)$ in which a unique function $x = f(y, z)$ exists that satisfies $F(f(y, z), y, z) = 0$. The partial derivatives of f are found via implicit differentiation

$$f_y = -\frac{F_y}{F_x} = -\frac{2y}{18x} = -\frac{y}{9x}, \qquad f_z = -\frac{F_z}{F_x} = -\frac{-2z}{18x} = \frac{z}{9x}$$

taking the following values at $(1, 4, 5)$:

$$f_y(4, 5) = -\frac{4}{9(1)} = -\frac{4}{9}, \qquad f_z(4, 5) = \frac{5}{9(1)} = \frac{5}{9}.$$

Hence the linearization of $x = f(y, z)$ at $(4, 5)$ is

$$\begin{aligned} L(y, z) &= f_y(4, 5)(y-4) + f_z(4, 5)(z-5) + f(4, 5) \\ &= -\frac{4}{9}(y-4) + \frac{5}{9}(z-5) + 1 \\ &= \frac{-4y + 5z}{9}. \end{aligned}$$

The graph of the linearization is given by the equation $x = \frac{-4y+5z}{9}$ which is equivalent to the equation $9x + 4y - 5z = 0$ of the tangent plane found in part a).

c) Finding the point(s) on the cone with a minimal distance from the z-axis is equivalent to minimizing the function $d(x,y) = x^2 + y^2$ (i.e. distance from the z-axis squared) subject to the constraint $g(x,y) = 9x^2 + y^2 - 2^2 = 0$ (i.e. the equation of the cone with the equation of the plane $z = 2$ taken into account). We use the method of Lagrange multipliers and solve

$$\begin{cases} \nabla d(x,y) = \lambda \nabla g(x,y) \\ \quad g(x,y) = 0 \end{cases} \qquad \Leftrightarrow \qquad \begin{cases} 2x = 18\lambda x \\ 2y = 2\lambda y \\ 9x^2 + y^2 - 4 = 0 \end{cases}$$

for x, y, λ. The second equation can be written as $2y(1-\lambda) = 0$ which implies that $y = 0$ or $\lambda = 1$. If $y = 0$, then the third equation yields $x = \pm 2/3$. If $\lambda = 1$, then the first equation becomes $2x = 18x$ whose only solution is $x = 0$ which according to the third equation means that $y = \pm 2$. Therefore we found four solutions

$$\left(\pm\frac{2}{3}, 0\right), \quad (0, \pm 2).$$

Evaluating the distance-squared function $d(x,y)$ at these points shows that

$$d\left(\pm\frac{2}{3}, 0\right) = \frac{4}{9}, \quad d(0, \pm 2) = 4$$

therefore there are two points, namely $(\frac{2}{3}, 0)$ and $(-\frac{2}{3}, 0)$, with a minimal distance of $\frac{2}{3}$ from the z-axis.

2) Consider the vector field

$$\vec{G}(x,y,z) = (2xy + e^{3z})\,\hat{\imath} + (x^2)\,\hat{\jmath} + (3xe^{3z})\,\hat{k}$$

a) Show that $\operatorname{curl} \vec{G} = \vec{0}$.

b) Determine a scalar potential for $\vec{G}$.

c) Evaluate the line integral of $\vec{G}$ along the curve of intersection of the hyperbolic paraboloid $z = x^2 - y^2$ and the plane $x = 1$ from the point $P_0(1, -1, 0)$ to the point $P_1(1, 1, 0)$.

Solution. a) We have

$$\begin{aligned} \operatorname{curl} \vec{G} = \nabla \times \vec{G} = & \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (2xy + e^{3z}) & (x^2) & (3xe^{3z}) \end{vmatrix} \\ = & \left[\frac{\partial}{\partial y}(3xe^{3z}) - \frac{\partial}{\partial z}(x^2)\right] \hat{\imath} \\ & + \left[\frac{\partial}{\partial z}(2xy + e^{3z}) - \frac{\partial}{\partial x}(3xe^{3z})\right] \hat{\jmath} \\ & + \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(2xy + e^{3z})\right] \hat{k} \\ = & [0 - 0]\,\hat{\imath} + \left[3e^{3z} - 3e^{3z}\right] \hat{\jmath} + [2x - 2x]\,\hat{k} \\ = & 0\,\hat{\imath} + 0\,\hat{\jmath} + 0\,\hat{k} = \vec{0}. \end{aligned}$$

so in fact $\operatorname{curl} \vec{G} = \vec{0}$ holds everywhere.

b) We need to solve the equation $\nabla g(x,y,z) = \vec{G}(x,y,z)$ for the unknown scalar potential $g(x,y,z)$. Written in component form, the equation reads

$$g_x = 2xy + e^{3z}, \tag{1}$$

$$g_y = x^2, \tag{2}$$

$$g_z = 3xe^{3z}. \tag{3}$$

Integrating both sides of equation (1) with respect to x, we obtain

$$g(x,y,z) = x^2y + xe^{3z} + h(y,z), \tag{4}$$

where $h(y,z)$ is a constant of integration depending on y and z (but not x). Differentiating (4) with respect to y, we get

$$g_y = x^2 + h_y(y,z) \tag{5}$$

Comparing equations (2) and (5) gives

$$h_y(y,z) = 0 \tag{6}$$

which integrated with respect to y yields

$$h(y,z) = k(z). \tag{7}$$

Again, we have constant of integration $k(z)$ that may depend on z (but not y). Plugging this into equation (4) gives us

$$g(x,y,z) = x^2y + xe^{3z} + k(z). \tag{8}$$

Differentiating (8) with respect to z and comparing the result to (3) yields

$$k'(z) = 0 \quad \Rightarrow \quad k(z) = K \text{ (constant)}. \tag{9}$$

Therefore we find that

$$g(x,y,z) = x^2y + xe^{3z} + K \tag{10}$$

is a scalar potential of the vector field $\vec{G}$.

c) Since $\vec{G}$ is conservative, i.e. we have $\vec{G}(x,y,z) = \nabla g(x,y,z)$ with the potential $g(x,y,z) = x^2y + xe^{3z} + K$ we have

$$\int\limits_{P_0 \to P_1} \vec{G} \cdot d\vec{r} = g(P_1) - g(P_0) = g(1,1,0) - g(1,-1,0) = (1^2(1) + 1e^0) - (1^2(-1) + 1e^0) = 2$$

by the Fundamental Theorem of Line Integrals.

Alternatively, we could parametrize the curve in question by the vector function

$$\vec{r}(y) = 1\,\hat{\imath} + y\,\hat{\jmath} + (1-y^2)\,\hat{k}, \qquad -1 \le y \le 1$$

whose derivative is

$$\vec{r}\,'(y) = \hat{\jmath} - 2y\,\hat{k}, \qquad -1 \le y \le 1$$

and along which the vector fields takes the values

$$\vec{G}(\vec{r}(y)) = (2y + e^{3(1-y^2)})\,\hat{\imath} + \hat{\jmath} + (3e^{3(1-y^2)})\,\hat{k}.$$

Hence the line integral can also be directly evaluated as follows

$$\int\limits_{P_0 \to P_1} \vec{G} \cdot d\vec{r} = \int_{-1}^{1} \vec{G}(\vec{r}(y)) \cdot \vec{r}\,'(y)\,dy = \int_{-1}^{1} \left(1 - 6ye^{3(1-y^2)}\right)\,dy = \left[y + e^{3(1-y^2)}\right]_{y=-1}^{y=1} = 1 - (-1) = 2.$$

3) Consider the velocity field given by

$$\vec{V}(x,y,z) = (ye^z)\,\hat{\imath} + (-xe^z)\,\hat{\jmath} + (e^z)\,\hat{k}$$

and the surface $S = \{(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = 4,\ y \ge 0,\ 0 \le z \le 2\}$ with outward normal vectors and positively-oriented boundary curve ∂S.

a) Describe and sketch the surface S and its boundary curve ∂S.
Draw the orientation of ∂S.

Verify Stokes' Theorem by directly computing

b) the circulation of $\vec{V}$ along ∂S, i.e. $\oint\limits_{\partial S} \vec{V} \cdot d\vec{r}$ and

c) the flux of $\operatorname{curl} \vec{V}$ across S, that is $\iint\limits_{S} \operatorname{curl} \vec{V} \cdot d\vec{S}$.

Solution. a) The surface S is the part of the cylinder of radius 2 with the z-axis as axis that is between the horizontal planes at heights $z = 0$ (i.e. the xy-plane) and $z = 2$, and the inequality $y \geq 0$ means considering only the half cylinder on the right-hand side of the xz-plane. This cylindrical "fence" can be seen in Figure 1. The boundary ∂S consists of the two semicircles (C_1 and C_3) and two line segments (C_2 and C_4):

$$C_1 = \{(x, y, 0) \mid x^2 + y^2 = 4,\ y \geq 0\}, \quad C_3 = \{(x, y, 2) \mid x^2 + y^2 = 4,\ y \geq 0\},$$

$$C_2 = \{(-2, 0, z) \mid 0 \leq z \leq 2\}, \quad C_4 = \{(2, 0, z) \mid 0 \leq z \leq 2\}.$$

Since the normal vectors point outward (meaning away from the z-axis), the boundary becomes positively-oriented if the lower semicircle C_1 is traversed counter-clockwise and the upper semicircle C_3 is traversed clockwise when viewed from above, the line segment C_2 needs to be traversed moving up (i.e. increasing z values) and C_4 is traversed moving down (i.e. decreasing z values).

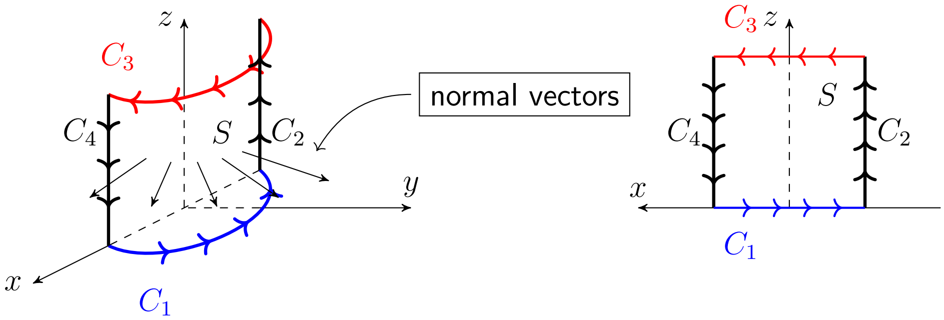

Figure 1: The surface S and its boundary $\partial S = C_1 \cup C_2$ (with positive orientation).

b) Based on part a), the following vector functions can be used to represent the (oriented) boundary $\partial S = C_1 \cup C_2 \cup C_3 \cup C_4$:

$$\begin{aligned}
C_1: &\quad \vec{r}_1(t) = (2\cos t)\,\hat{\imath} + (2\sin t)\,\hat{\jmath}, \quad 0 \leq t \leq \pi, \\
C_2: &\quad \vec{r}_2(t) = -2\,\hat{\imath} + t\,\hat{k}, \quad 0 \leq t \leq 2, \\
C_3: &\quad \vec{r}_3(t) = (-2\cos t)\,\hat{\imath} + (2\sin t)\,\hat{\jmath} + 2\,\hat{k}, \quad 0 \leq t \leq \pi, \\
C_4: &\quad \vec{r}_4(t) = 2\,\hat{\imath} + (2-t)\,\hat{k}, \quad 0 \leq t \leq 2.
\end{aligned}$$

The circulation of $\vec{V}$ along the boundary $\partial S = C_1 \cup C_2 \cup C_3 \cup C_4$ is the sum of the line integrals of $\vec{V}$ along each of the curves:

$$\oint\limits_{\partial S} \vec{V} \cdot d\vec{r} = \int\limits_{C_1} \vec{V} \cdot d\vec{r} + \int\limits_{C_2} \vec{V} \cdot d\vec{r} + \int\limits_{C_3} \vec{V} \cdot d\vec{r} + \int\limits_{C_4} \vec{V} \cdot d\vec{r}.$$

To compute these line integrals, we need the tangent vectors as well as the values $\vec{V}$ takes along the curves. Along C_1, we obtain

$$\vec{r}_1{}'(t) = (-2\sin t)\,\hat{\imath} + (2\cos t)\,\hat{\jmath}$$

and

$$\vec{V}(\vec{r}_1(t)) = (2\sin t)e^0\,\hat{\imath} + (-2\cos t)e^0\,\hat{\jmath} + (e^0)\,\hat{k} = (2\sin t)\,\hat{\imath} - (2\cos t)\,\hat{\jmath} + \hat{k}$$

hence the line integral is

$$\begin{aligned}\int_{C_1} \vec{V}\cdot d\vec{r} &= \int_0^{\pi} \vec{V}(\vec{r}_1(t))\cdot \vec{r}_1{}'(t)\,dt = \int_0^{\pi} [(2\sin t)\,\hat{\imath} - (2\cos t)\,\hat{\jmath} + \hat{k}]\cdot[(-2\sin t)\,\hat{\imath} + (2\cos t)\,\hat{\jmath}]\,dt \\ &= \int_0^{\pi} -4(\sin^2 t + \cos^2 t)\,dt = \int_0^{\pi} -4\,dt = [-4t]_{t=0}^{t=\pi} = -4\pi.\end{aligned}$$

Along C_2 we get

$$\vec{r}_2{}'(t) = \hat{k}$$

and

$$\vec{V}(\vec{r}_2(t)) = (0e^t)\,\hat{\imath} + (2e^t)\,\hat{\jmath} + (e^t)\,\hat{k} = (2e^t)\,\hat{\jmath} + (e^t)\,\hat{k}.$$

hence the line integral equals

$$\begin{aligned}\int_{C_2} \vec{V}\cdot d\vec{r} &= \int_0^{2} \vec{V}(\vec{r}_2(t))\cdot \vec{r}_2{}'(t)\,dt = \int_0^{2} [(2e^t)\,\hat{\jmath} + (e^t)\,\hat{k}]\cdot\hat{k}\,dt \\ &= \int_0^{2} e^t\,dt = [e^t]_{t=0}^{t=2} = e^2 - e^0 = e^2 - 1.\end{aligned}$$

Along C_3 we have

$$\vec{r}_3{}'(t) = (2\sin t)\,\hat{\imath} + (2\cos t)\,\hat{\jmath}$$

and

$$\vec{V}(\vec{r}_3(t)) = (2\sin t)e^2\,\hat{\imath} + (2\cos t)e^2\,\hat{\jmath} + (e^2)\,\hat{k} = (2e^2\sin t)\,\hat{\imath} + (2e^2\cos t)\,\hat{\jmath} + (e^2)\hat{k}.$$

hence the line integral equals

$$\begin{aligned}\int_{C_3} \vec{V}\cdot d\vec{r} &= \int_0^{\pi} \vec{V}(\vec{r}_3(t))\cdot \vec{r}_3{}'(t)\,dt = \int_0^{\pi} [(2e^2\sin t)\,\hat{\imath} + (2e^2\cos t)\,\hat{\jmath} + (e^2)\hat{k}]\cdot[(2\sin t)\,\hat{\imath} + (2\cos t)\,\hat{\jmath}]\,dt \\ &= \int_0^{\pi} 4e^2(\sin^2 t + \cos^2 t)\,dt = \int_0^{\pi} 4e^2\,dt = [4e^2 t]_{t=0}^{t=\pi} = 4e^2\pi.\end{aligned}$$

Along C_4 we get

$$\vec{r}_4{}'(t) = -\hat{k}$$

and

$$\vec{V}(\vec{r}_4(t)) = (0e^{2-t})\,\hat{\imath} + (-2e^{2-t})\,\hat{\jmath} + (e^{2-t})\,\hat{k} = (-2e^{2-t})\,\hat{\jmath} + (e^{2-t})\,\hat{k}.$$

hence the line integral equals

$$\begin{aligned}\int_{C_4} \vec{V}\cdot d\vec{r} &= \int_0^{2} \vec{V}(\vec{r}_4(t))\cdot \vec{r}_4{}'(t)\,dt = \int_0^{2} [(-2e^{2-t})\,\hat{\jmath} + (e^{2-t})\,\hat{k}]\cdot[-\hat{k}]\,dt \\ &= \int_0^{2} (-e^{2-t})\,dt = [e^{2-t}]_{t=0}^{t=2} = e^0 - e^2 = 1 - e^2.\end{aligned}$$

When we add these four integrals the second and fourth terms cancel leaving us with

$$\oint_{\partial S} \vec{V}\cdot d\vec{r} = 4\pi(e^2 - 1).$$

Thus we see that the circulation of $\vec{V}$ along the boundary of S equals $4\pi(e^2-1)$.

c) Let us first compute the curl of $\vec{V}$:

$$\operatorname{curl}\vec{V} = \nabla\times\vec{V} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ (ye^z) & (-xe^z) & (e^z) \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(e^z) - \frac{\partial}{\partial z}(-xe^z)\right]\hat{\imath} + \left[\frac{\partial}{\partial z}(ye^z) - \frac{\partial}{\partial x}(e^z)\right]\hat{\jmath} + \left[\frac{\partial}{\partial x}(-xe^z) - \frac{\partial}{\partial y}(ye^z)\right]\hat{k}$$

$$= [(0)-(-xe^z)]\,\hat{\imath} + [(ye^z)-(0)]\,\hat{\jmath} + [(-e^z)-(e^z)]\,\hat{k}$$

$$= (xe^z)\,\hat{\imath} + (ye^z)\,\hat{\jmath} - (2e^z)\,\hat{k}.$$

The surface S is a piece of a cylinder so let us parametrize it using cylindrical coordinates via the vector function

$$\vec{r}(\theta,z) = (2\cos\theta)\,\hat{\imath} + (2\sin\theta)\,\hat{\jmath} + (z)\,\hat{k}, \qquad 0\le\theta\le\pi,\ 0\le z\le 2$$

The derivatives of $\vec{r}(\theta,z)$ with respect to θ and z are

$$\vec{r}_\theta = (-2\sin\theta)\,\hat{\imath} + (2\cos\theta)\,\hat{\jmath}, \qquad \vec{r}_z = \hat{k}$$

and therefore we have

$$\vec{r}_\theta\times\vec{r}_z = [(-2\sin\theta)\,\hat{\imath} + (2\cos\theta)\,\hat{\jmath}]\times\hat{k} = -2\sin\theta\,(\hat{\imath}\times\hat{k}) + 2\cos\theta\,(\hat{\jmath}\times\hat{k}) = -2\sin\theta\,(-\hat{\jmath}) + 2\cos\theta\,\hat{\imath}$$

that is

$$\vec{r}_\theta\times\vec{r}_z = (2\cos\theta)\,\hat{\imath} + (2\sin\theta)\,\hat{\jmath}.$$

The vector field $\operatorname{curl}\vec{V}$ takes the following values on S:

$$\operatorname{curl}\vec{V}(\vec{r}(\theta,z)) = (2\cos\theta)e^z\,\hat{\imath} + (2\sin\theta)e^z\,\hat{\jmath} - (2e^z)\,\hat{k}.$$

So the flux of $\operatorname{curl}\vec{V}$ across S is

$$\iint_S \operatorname{curl}\vec{V}\cdot d\vec{S} = \int_0^\pi\int_0^2 \operatorname{curl}\vec{V}(\vec{r}(\theta,z))\cdot(\vec{r}_\theta\times\vec{r}_z)\,dz\,d\theta$$

$$= \int_0^\pi\int_0^2 [(2\cos\theta)e^z\,\hat{\imath} + (2\sin\theta)e^z\,\hat{\jmath} - (2e^z)\,\hat{k}]\cdot[(2\cos\theta)\,\hat{\imath} + (2\sin\theta)\,\hat{\jmath}]\,dz\,d\theta$$

$$= \int_0^\pi\int_0^2 4e^z(\cos^2\theta+\sin^2\theta)\,dz\,d\theta = \int_0^\pi\int_0^2 4e^z\,dz\,d\theta = 4\pi[e^z]_{z=0}^{z=2} = 4\pi(e^2-1).$$

Thus we see that the flux of $\operatorname{curl}\vec{V}$ across S is $4\pi(e^2-1)$.

4) Consider the force field given by

$$\vec{F}(x,y,z) = (-x^2y)\,\hat{\imath} + (xy^2)\,\hat{\jmath} + (z^2)\,\hat{k}$$

over the solid region $E = \{(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2+z^2\le 4,\ z\ge 1\}$ and its outward-oriented boundary surface ∂E.

a) Describe and sketch the solid region E and the boundary surface ∂E. Draw the orientation of ∂E.

Verify the Divergence Theorem by directly calculating

b) the flux of $\vec{F}$ across ∂E, that is $\displaystyle\oiint_{\partial E} \vec{F} \cdot d\vec{S}$ and

c) the triple integral of $\operatorname{div} \vec{F}$ over E, i.e. $\displaystyle\iiint_E \operatorname{div} \vec{F}\, dV$.

Solution. a) The inequality $x^2 + y^2 + z^2 \leq 4$ yields the closed solid ball of radius 2 centred at the origin. The other inequality $z \geq 1$ corresponds to the region on and above the horizontal plane at height 1. The solid region E is the intersection of these two regions , i.e. a solid spherical cap (see Figure 2). Accordingly, the boundary of E is the union of a spherical cap and a disc. More precisely, let S_1 denote of the portion of the sphere of radius 2 centred at the origin for which we have the polar angle $0 \leq \phi \leq \pi/3$. (The upper limit for ϕ is extracted from the right triangle with vertices $(0,0,1)$, $(0,\sqrt{3},1)$, $(0,0,0)$). And let S_2 denote the horizontal disc of radius $\sqrt{3}$ centred at $(0,0,1)$. The radius of the disc was obtained from $(x^2+y^2)+1^2 \leq 4 \quad \Rightarrow \quad \sqrt{x^2+y^2} \leq \sqrt{3}$. In summary, we have $\partial E = S_1 \cup S_2$.

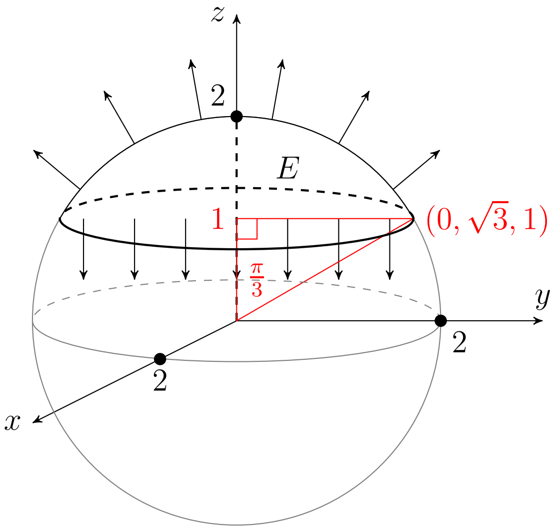

Figure 2: The solid region E (with outward-pointing normal vectors).

b) From part a), we deduce that the spherical cap S_1 is given by the vector function

$$S_1 : \qquad \vec{r}_1(\phi,\theta) = (2\sin\phi\cos\theta)\,\hat{\imath} + (2\sin\phi\sin\theta)\,\hat{\jmath} + (2\cos\phi)\,\hat{k}, \quad 0 \leq \phi \leq \pi/3,\ 0 \leq \theta \leq 2\pi.$$

and the disc S_2 is given by the vector function

$$S_2 : \qquad \vec{r}_2(r,\theta) = (r\cos\theta)\,\hat{\imath} + (r\sin\theta)\,\hat{\jmath} + (1)\,\hat{k}, \quad 0 \leq r \leq \sqrt{3},\ 0 \leq \theta \leq 2\pi.$$

The flux of $\vec{F}$ across $\partial E = S_1 \cup S_2$ is the sum of the surface integrals of $\vec{F}$ across S_1 and S_2:

$$\oiint_{\partial E} \vec{F} \cdot d\vec{S} = \iint_{S_1} \vec{F} \cdot d\vec{S} + \iint_{S_2} \vec{F} \cdot d\vec{S}.$$

To evaluate these surface integrals, we need to find (outward) normal vectors to the surfaces as well as the values of $\vec{F}$ on the surfaces. On S_1, we have

$$(\vec{r}_1)_\phi = (2\cos\phi\cos\theta)\,\hat{\imath} + (2\cos\phi\sin\theta)\,\hat{\jmath} - (2\sin\phi)\,\hat{k}$$

and

$$(\vec{r}_1)_\theta = (-2\sin\phi\sin\theta)\,\hat{\imath} + (2\sin\phi\cos\theta)\,\hat{\jmath}.$$

The cross product of these derivatives yields normal vectors to S_1:

$$\begin{aligned}(\vec{r}_1)_\phi \times (\vec{r}_1)_\theta =&(2\cos\phi\cos\theta\,\hat{\imath} + 2\cos\phi\sin\theta\,\hat{\jmath} - 2\sin\phi\,\hat{k}) \times (-2\sin\phi\sin\theta\,\hat{\imath} + 2\sin\phi\cos\theta\,\hat{\jmath})\\
=&(2\cos\phi\cos\theta)(2\sin\phi\cos\theta)(\hat{\imath}\times\hat{\jmath}) - (2\cos\phi\sin\theta)(2\sin\phi\sin\theta)(\hat{\jmath}\times\hat{\imath})\\
&+ (2\sin\phi)(2\sin\phi\sin\theta)(\hat{k}\times\hat{\imath}) - (2\sin\phi)(2\sin\phi\cos\theta)(\hat{k}\times\hat{\jmath})\\
=&(4\sin\phi\cos\phi\cos^2\theta)(\hat{k}) - (4\sin\phi\cos\phi\sin^2\theta)(-\hat{k})\\
&+ (4\sin^2\phi\sin\theta)(\hat{\jmath}) - (4\sin^2\phi\cos\theta)(-\hat{\imath})\\
=&(4\sin^2\phi\cos\theta)\,\hat{\imath} + (4\sin^2\phi\sin\theta)\,\hat{\jmath} + [4\sin\phi\cos\phi(\cos^2\theta+\sin^2\theta)]\,\hat{k}\\
=&(4\sin^2\phi\cos\theta)\,\hat{\imath} + (4\sin^2\phi\sin\theta)\,\hat{\jmath} + (4\sin\phi\cos\phi)\,\hat{k}.\end{aligned}$$

As for the values of $\vec{F}$ on S_1, we get

$$\vec{F}(\vec{r}_1(\phi,\theta)) = (-8\sin^3\phi\cos^2\theta\sin\theta)\,\hat{\imath} + (8\sin^3\phi\cos\theta\sin^2\theta)\,\hat{\jmath} + (4\cos^2\phi)\,\hat{k}.$$

Hence the normal component of $\vec{F}$ is

$$\begin{aligned}&\vec{F}(\vec{r}_1(\phi,\theta)) \cdot \big((\vec{r}_1)_\phi \times (\vec{r}_1)_\theta\big)\\
&= (-8\sin^3\phi\cos^2\theta\sin\theta)(4\sin^2\phi\cos\theta) + (8\sin^3\phi\cos\theta\sin^2\theta)(4\sin^2\phi\sin\theta) + (4\cos^2\phi)(4\sin\phi\cos\phi)\\
&= -32\sin^5\phi\sin\theta\cos^3\theta + 32\sin^5\phi\sin^3\theta\cos\theta + 16\cos^3\phi\sin\phi.\end{aligned}$$

The flux of $\vec{F}$ across S_1 is

$$\begin{aligned}\iint_{S_1} \vec{F}\cdot d\vec{S} &= \int_0^{\pi/3}\int_0^{2\pi} \vec{F}(\vec{r}_1(\phi,\theta)) \cdot ((\vec{r}_1)_\phi \times (\vec{r}_1)_\theta)\, d\theta\, d\phi\\
&= \int_0^{\pi/3}\int_0^{2\pi} (-32\sin^5\phi\sin\theta\cos^3\theta + 32\sin^5\phi\sin^3\theta\cos\theta + 16\cos^3\phi\sin\phi)\, d\theta\, d\phi\\
&= \int_0^{\pi/3} ([8\sin^5\phi\cos^4\theta]_{\theta=0}^{\theta=2\pi} + [8\sin^5\phi\sin^4\theta]_{\theta=0}^{\theta=2\pi} + 32\pi\cos^3\phi\sin\phi)\, d\phi\\
&= \int_0^{\pi/3} (0 + 0 + 32\pi\cos^3\phi\sin\phi)\, d\phi = 32\pi\int_0^{\pi/3}\cos^3\phi\sin\phi\, d\phi\\
&= 8\pi[-\cos^4\phi]_{\phi=0}^{\phi=\pi/3} = 8\pi(-\left(\tfrac{1}{2}\right)^4 + 1^4) = \tfrac{15}{2}\pi.\end{aligned}$$

On the disc S_2, the outward-pointing normal vector is $-\hat{k}$. and the vector field $\vec{F}$ takes on the following values

$$\vec{F}(\vec{r}_2(\theta,z)) = (-r^3\cos^2\theta\sin\theta)\,\hat{\imath} + (r^3\cos\theta\sin^2\theta)\,\hat{\jmath} + (1^2)\,\hat{k}.$$

Its normal component is

$$\vec{F}(\vec{r}_2(\theta,z)) \cdot (-\hat{k}) = (-r^3\cos^2\theta\sin\theta)(0) + (r^3\cos\theta\sin^2\theta)(0) + (1)(-1) = -1$$

and hence the flux of $\vec{F}$ across S_2 is minus the area of the disc

$$\iint_{S_2} \vec{F}\cdot d\vec{S} = \int_0^{\sqrt{3}}\int_0^{2\pi} \vec{F}(\vec{r}_2(\theta,z)) \cdot (-\hat{k})\, d\theta\, dr = \int_0^{\sqrt{3}}\int_0^{2\pi} (-1)\, d\theta\, dr = -\pi(\sqrt{3})^2 = -3\pi.$$

Therefore the total flux of $\vec{F}$ across ∂E is $\frac{15}{2}\pi - 3\pi = \frac{9}{2}\pi$.

c) The divergence of $\vec{F}$ is

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{\partial}{\partial x}(-x^2y) + \frac{\partial}{\partial y}(xy^2) + \frac{\partial}{\partial z}(z^2) = -2xy + 2xy + 2z = 2z.$$

In terms of cylindrical coordinates (r, θ, z) the equation for the sphere reads $r^2 + z^2 = 4$, hence the solid E can be expressed as follows

$$E: \qquad 0 \le r \le \sqrt{4 - z^2}, \quad 0 \le \theta \le 2\pi, \quad 1 \le z \le 2.$$

Therefore the triple integral of $\operatorname{div} \vec{F}$ over E is

$$\begin{aligned}
\iiint\limits_E \operatorname{div} \vec{F}\, dV &= \iiint\limits_E 2z\, dV = \int\limits_0^{2\pi} \int\limits_1^2 \int\limits_0^{\sqrt{4-z^2}} (2z)(r)\, dr\, dz\, d\theta = \int\limits_0^{2\pi} d\theta \int\limits_1^2 z \int\limits_0^{\sqrt{4-z^2}} (2r)\, dr\, dz \\
&= 2\pi \int_1^2 z[r^2]_{r=0}^{r=\sqrt{4-z^2}}\, dz = 2\pi \int\limits_1^2 z(4 - z^2)\, dz = 2\pi[2z^2 - \tfrac{1}{4}z^4]_{z=1}^{z=2} \\
&= 2\pi[2(2^2) - \tfrac{1}{4}(2^4) - 2(1^2) + \tfrac{1}{4}(1^4)] = 2\pi(\tfrac{9}{4}) = \tfrac{9}{2}\pi.
\end{aligned}$$