MATFUS VÕRRANDID

KODUTÖÖ 4
TAAVI TAMMARU

4.1

avaldame lahendi $u(x,t)$ Greeni funktsiooni abil $G(\xi, \tau, x, t)$

Selleks korrutame algset võrrandit suvalise funktsiooni $v(x,t)$-ga ning integreerime mõlemad pool aja ning ruumi järgi.

defineerime $g(x) = A\sum_{j=1}^{3} \frac{3aj\pi}{L} \sin(\frac{3j\pi}{L}x)$

$$\int_0^T\int_0^L v \frac{1}{a^2}\frac{\partial^2 u}{\partial t^2}\,dx\,dt - \int_0^T\int_0^L v \frac{\partial^2 u}{\partial x^2}\,dx\,dt = \int_0^T\int_0^L v\, g(x)\,dx\,dt$$

integraalide manipulatsioonide tulemusel jõuame kujule.

$$u(x,t) = \frac{1}{a^2}\int_0^L \psi(\xi)\, G(\xi,0,x,t)\,d\xi - \frac{1}{a^2}\int_0^L \varphi(\xi) \frac{\partial G(\xi,\tau,x,t)}{\partial \tau}\Big|_{\tau=0} d\xi - \int_0^t \mu_1(L,t) \frac{\partial G(\xi,\tau,x,t)}{\partial \xi}\Big|_{\xi=L}$$
$$+ \int_0^t \mu_1(0,\tau)\frac{\partial G(\xi,\tau,x,t)}{\partial \xi}\Big|_{\xi=0} d\tau + \int_0^t\int_0^L g(\xi,\tau)\, G(\xi,\tau,x,t)\,d\xi\,d\tau$$

ning antud ülesande puhul

$\mu_1(x,t) = 0$, $\psi(x) = 0$, $\varphi(\xi) = B\sin(\frac{2\pi}{L}\xi)$, $g(\xi,\tau) = A\sum_{j=1}^{3} \frac{3aj\pi}{L}\sin(\frac{3j\pi}{L}\xi)$

arendame need saadud avaldisse

$$u(x,t) = -\frac{1}{a^2}\int_0^L B\sin(\frac{2\pi}{L}\xi)\frac{\partial G(\xi,\tau,x,t)}{\partial\tau}\Big|_{\tau=0} d\xi + \int_0^t\int_0^L A\sum_{j=1}^{3}\frac{3aj\pi}{L}\sin(\frac{3j\pi}{L}\xi)\, G(\xi,\tau,x,t)\,d\xi\,dt$$

Oleme avaldanud antud ülesande lahendi Greeni funktsiooni abil.

Ühemõõtmelise laineoperaatori Greeni funktsioon erinevat liiki rajatingimuste puhul on:

$$G(\xi,\tau,x,t) = H(t-\tau)\,a\sum X_n(x)X_n(\xi)\,\frac{\sin(\sqrt{\lambda_n}\,a(t-\tau))}{\sqrt{\lambda_n}}$$

leiame omaväärtused ja omafunktsioonid

$$G(\xi,\tau,x,t) = H(t-\tau)\frac{2a}{\pi}\sum_1^\infty \frac{1}{n}\sin(\frac{n\pi}{L}\xi)\sin(\frac{n\pi}{L}x)\sin(\frac{n\pi}{L}a(t-\tau))$$

nüüd arendame Greeni funktsiooni $u(x,t)$ avaldise

$$\frac{\partial G(\varepsilon,\tau,x,t)}{\partial \tau}\Big|_{\tau=0} \Rightarrow \quad \frac{d}{d\tau} H(t-\tau) = -\delta(t-\tau)$$

$$\frac{d}{d\tau} \sin\left(\frac{n\pi}{L} a(t-\tau)\right) = -\frac{n\pi}{L} a \cos\left(\frac{n\pi}{L} a(t-\tau)\right)$$

nüüd kasutame korrutise tuletise valemit ja saame

$$\frac{d}{d\tau} G = -\delta(t-\tau) \frac{2a}{\pi} \sum_1^\infty \frac{1}{n} \sin\left(\frac{n\pi}{L}\varepsilon\right) \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi}{L} a(t-\tau)\right)$$
$$- H(t-\tau) \frac{2a}{\pi} \sum^\infty \frac{1}{n} \sin\left(\frac{n\pi}{L}\varepsilon\right) \sin\left(\frac{n\pi}{L}x\right) \left(-\frac{n\pi}{L} a \cos\left(\frac{n\pi}{L} a(t-\tau)\right)\right)$$

$$\frac{\partial G(\varepsilon,\tau,x,t)}{\partial \tau}\Big|_{\tau=0} = -\delta(t) \frac{2a}{\pi} \sum_1^\infty \frac{1}{n} \sin\left(\frac{n\pi}{L}\varepsilon\right) \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi}{L} a\, t\right)$$
$$- H(t) \frac{2a}{\pi} \sum^\infty \frac{1}{n} \sin\left(\frac{n\pi}{L}\varepsilon\right) \sin\left(\frac{n\pi}{L}x\right) \left(-\frac{n\pi}{L} a \cos\left(\frac{n\pi}{L} a(t)\right)\right)$$

nüüd läheme tagasi algse avaldise juurde

$$u(x,t) = -\frac{1}{a^2} \int_0^L B \sin\left(\frac{7\pi}{L}\varepsilon\right) \frac{\partial G(\varepsilon,\tau,x,t)}{\partial \tau}\Big|_{\tau=0} d\varepsilon + \int_0^t \int_0^L A \sum_{j=1}^3 \frac{3aj\pi}{L} \sin\left(\frac{3j\pi}{L}\varepsilon\right) G(\varepsilon,\tau,x,t)\, d\varepsilon\, dt$$

arendame $\frac{\partial G(\varepsilon,\tau,x,t)}{\partial \tau}\Big|_{\tau=0}$ ja $G(\varepsilon,\tau,x,t)$ tagasi sisse

$$u(x,t) = -\frac{1}{a^2} \int_0^L B \sin\left(\frac{7\pi}{L}\varepsilon\right) \Bigg(-\delta(t) \frac{2a}{\pi} \sum_1^\infty \frac{1}{n} \sin\left(\frac{n\pi}{L}\varepsilon\right) \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi}{L} a\, t\right)$$
$$- H(t) \frac{2a}{\pi} \sum^\infty \frac{1}{n} \sin\left(\frac{n\pi}{L}\varepsilon\right) \sin\left(\frac{n\pi}{L}x\right) \left(-\frac{n\pi}{L} a \cos\left(\frac{n\pi}{L} a(t)\right)\right)\Bigg) d\varepsilon,$$

$$+ \int_0^t \int_0^L A \sum_{j=1}^3 \frac{3aj\pi}{L} \sin\left(\frac{3j\pi}{L}\varepsilon\right) H(t-\tau) \frac{2a}{\pi} \sum_1^\infty \frac{1}{n} \sin\left(\frac{n\pi}{L}\varepsilon\right) \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi}{L} a(t-\tau)\right) d\varepsilon\, dt$$

nüüd on võimalik tuua konstandid integraalide ette ja avaldis lihtsustub

$$u(x,t) = -\frac{1}{a^2}\left(-\delta(t) \frac{2a}{\pi} \sum_1^\infty \frac{1}{n} \sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{n\pi}{L} a\, t\right) - H(t) \frac{2a}{\pi} \sum^\infty \frac{1}{n} \sin\left(\frac{n\pi}{L}x\right) \left(-\frac{n\pi}{L} a \cos\left(\frac{n\pi}{L} a(t)\right)\right)\right)$$
$$\cdot \int_0^L B \sin\left(\frac{7\pi}{L}\varepsilon\right) \left(\sum_1^\infty \sin\left(\frac{n\pi}{L}\varepsilon\right)\right) d\varepsilon$$

$$+ A \sum_{j=1}^3 \frac{3aj\pi}{L} \frac{2a}{\pi} \sum_1^\infty \frac{1}{n} \sin\left(\frac{n\pi}{L}x\right) \int_0^t H(t-\tau) \sum_1^\infty \sin\left(\frac{n\pi}{L} a(t-\tau)\right) dt \int_0^L \sum_{j=1}^3 \sin\left(\frac{3j\pi}{L}\varepsilon\right) \sum_1^\infty \sin\left(\frac{n\pi}{L}\varepsilon\right) d\varepsilon$$

Vaatame igat integraali eraldi ja kasutame ortogonaalsus printsiipi.

$$\int_0^L B \sin\left(\frac{7\pi}{L}\varepsilon\right)\left(\sum_1^{\infty} \sin\left(\frac{n\pi}{L}\varepsilon\right)\right) d\varepsilon = 0$$

$$\int_0^t H(t-\tau) \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L} a(t-\tau)\right) dt = ?$$

$$\int_0^L \sum_{j=1}^{3} \sin\left(\frac{3j\pi}{L}\varepsilon\right) \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}\varepsilon\right) d\varepsilon = \frac{3L}{2}$$

Seega

$$u(x,t) = A \sum_{j=1}^{3} \frac{3aj\pi}{L} \frac{2a}{\pi} \sum_1^{\infty} \frac{1}{n} \sin\left(\frac{n\pi}{L}x\right) \int_0^t H(t-\tau) \sum_1^{\infty} \sin\left(\frac{n\pi}{L} a(t-\tau)\right) dt \; \frac{3L}{2}$$

Ülesanne 4.2

Üldlahendi kuju on $u(r,\phi) = C_{10} + C_{20}\ln(r) + \sum_{n=1}^{\infty}\Big(\big(B_{1n}r^{n} + B_{2n}r^{-n}\big)\cos(n\phi)$
$+ \big(B_{3n}r^{n} + B_{4n}r^{-n}\big)\sin(n\phi)\Big)$

vorrandisüsteem konstantide leidmiseks

$$C_{10} + C_{20}\ln(r) = I_1$$
$$C_{10} + C_{20}\ln(r) = I_2$$

$$I_1 = \frac{1}{2\pi}\int_0^{2\pi} \mu_1(r_1,\phi)\,d\phi$$
$$= \frac{1}{2\pi}\int_0^{2\pi} 8a\sin(3\phi)\cos(3\phi)\,d\phi + \frac{1}{2\pi}\int_0^{2\pi}\beta\left(\cos^2(\phi)-\sin^2(\phi)\right)d\phi$$
$$= 0 + 0 = 0$$

$$I_2 = \frac{1}{2\pi}\int_0^{2\pi}\beta\cos(3\phi)\,d\phi - \frac{1}{2\pi}\int_0^{2\pi} a\sin(4\phi)\,d\phi$$
$$= 0 + 0 = 0$$

Seega

$$C_{10} = C_{20} = 0$$

vorrandisüsteem konstantide B_{1n} ja B_{2n} leidmiseks.

$$B_{1n}r_1^{n} + B_{2n}r_1^{-n} = I_{c1}$$
$$B_{1n}r_2^{n} + B_{2n}r_2^{-n} = I_{c2}$$

$$I_{c1} = \frac{1}{\pi}\int_0^{2\pi}\mu_1(r_1,\phi)\cos(n\phi)\,d\phi$$

$$I_{c1} = \frac{1}{\pi}\int_0^{2\pi} 8a\sin(3\phi)\cos(3\phi)\cos(n\phi)\,d\phi + \frac{1}{\pi}\int_0^{2\pi}\beta\left(\cos^2(\phi)-\sin^2(\phi)\right)\cos(n\phi)\,d\phi$$
$$= \frac{1}{\pi}\left(-\frac{48a\sin^2(n\pi)}{9-n^2} + \frac{n\beta\sin(2n\pi)}{-4+n^2}\right)$$

$$I_{c_2} = \frac{1}{\pi}\int_0^{2\pi} \beta\cos(3\phi)\cos(n\phi)\,d\phi + \frac{1}{\pi}\int_0^{2\pi} a\sin(4\phi)\cos(n\phi)\,d\phi$$

$$= \frac{1}{\pi}\left(\frac{24a\sin(2n\pi)}{n(64-20n^2+n^4)} + \frac{\beta n(-7+n^2)\sin(2n\pi)}{9-10n^2+n^4}\right)$$

$$B_{1n} = S_1(n)\frac{1}{\pi}\left(-\frac{48a\sin^2(n\pi)}{9-n^2} + \frac{n\beta\sin(2n\pi)}{-4+n^2}\right)$$

$$B_{2n} = S_1(-n)\frac{1}{\pi}\left(\frac{24a\sin(2n\pi)}{n(64-20n^2+n^4)} + \frac{\beta n(-7+n^2)\sin(2n\pi)}{9-10n^2+n^4}\right)$$

kus $S_1(n) = \dfrac{r_1^n}{r_1^{2n} - r_2^{2n}}$

leiame kordajad B_{3n} ja B_{4n}

$$I_{s1} = \frac{1}{\pi}\int_0^{2\pi} 8a\sin(3\phi)\cos(3\phi)\sin(n\phi)\,d\phi + \frac{1}{\pi}\int_0^{2\pi}\beta\left(\cos^2(\phi) - \sin^2(\phi)\right)\sin(n\phi)\,d\phi$$

$$= \frac{1}{\pi}\left(\frac{2\beta n\sin^2(n\pi)}{-4+n^2} + \frac{24a\sin(2n\pi)}{-9+n^2}\right)$$

$$I_{s2} = \frac{1}{\pi}\int_0^{2\pi}\beta\cos(3\phi)\sin(n\phi)\,d\phi + \frac{1}{\pi}\int_0^{2\pi} a\sin(4\phi)\sin(n\phi)\,d\phi$$

$$= \frac{1}{\pi}\left(\frac{2\beta n\sin^2(n\pi)}{-9+n^2} + \frac{48a\sin^2(n\pi)}{64n-20n^3+n^5}\right)$$

$$B_{3n} = S_1(n)\frac{1}{\pi}\left(\frac{2\beta n\sin^2(n\pi)}{-4+n^2} + \frac{24a\sin(2n\pi)}{-9+n^2}\right)$$

$$B_{4n} = S_1(-n)\frac{1}{\pi}\left(\frac{2\beta n\sin^2(n\pi)}{-9+n^2} + \frac{48a\sin^2(n\pi)}{64n-20n^3+n^5}\right)$$

üldlahend avaldub kujul:

$$\begin{aligned} u(r,\phi) = \sum_{n=1}^{\infty}\Bigg(&\Bigg(S_1(n)\frac{1}{\pi}\left(-\frac{48a\sin^2(n\pi)}{9-n^2} + \frac{n\beta\sin(2n\pi)}{-4+n^2}\right) r^n + \\ &S_1(-n)\frac{1}{\pi}\left(\frac{24a\sin(2n\pi)}{n(64-20n^2+n^4)} + \frac{\beta n(-7+n^2)\sin(2n\pi)}{9-10n^2+n^4}\right) r^{-n}\Bigg)\cos(n\phi) \\ &+ S_1(n)\frac{1}{\pi}\left(\frac{2\beta n\sin^2(n\pi)}{-4+n^2} + \frac{24a\sin(2n\pi)}{-9+n^2}\right) r^n + \\ &S_1(-n)\frac{1}{\pi}\left(\frac{2\beta n\sin^2(n\pi)}{-9+n^2} + \frac{48a\sin^2(n\pi)}{64n-20n^3+n^5}\right) r^{-n}\Bigg)\sin(n\phi)\Bigg) \end{aligned}$$