Linear Algebra - Midterm Exam

March 22, 2023   18.30-20.30

You **can** have a "cheat sheet": This is an A4 paper written on **one** side. You are **NOT** allowed to have books, course notes, homework assignments, etc., laptops, e-readers, tablets, telephones, etc., during the exam!). You can also use a normal calculator (not a programmable/graphic one).

And remember the advice I gave in class: Explain what you have to do, why and how and then start doing it because, if time is short and you cannot finish a calculation, at least you showed you knew what you had to do.

QUESTIONS:

1. A system of equations with 3 unknowns (x, y, z), with parameters a and b, is given by

$$\begin{aligned} -x_1 \qquad\quad + x_3 &= a \\ 2\,x_1 + 3\,x_2 + a\,x_3 &= -2 \\ -2\,x_1 + 4\,x_2 + 5\,x_3 &= 3\,b \end{aligned}$$

   (a) $\boxed{0.6}$ Solve the system for $(a, b) = (2, 1)$. Clearly indicate the applied steps. (You can use any of the methods given in the course.)

   (b) $\boxed{0.4}$ Use **the determinant** to find for which value(s) of a and/or b the system has exactly one solution.

   (c) $\boxed{0.6}$ Use **row reduction** to determine for which value(s) of a and/or b the system has no solutions.

   (d) $\boxed{0.4}$ Given $B = \begin{bmatrix} -4 & 0 & 4 \\ 8 & 12 & 1 \\ -8 & 16 & 20 \end{bmatrix}$, what is the null space of B, $\mathcal{N}(B)$?

2. Are these 5 vectors in $\mathbb{R}^5$ linearly independent?

   (a) $\boxed{1}$ $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \\ 7 \end{bmatrix}$ $\mathbf{u}_2 = \begin{bmatrix} 3 \\ 5 \\ -1 \\ -3 \\ 4 \end{bmatrix}$ $\mathbf{u}_3 = \begin{bmatrix} 7 \\ 8 \\ 10 \\ 5 \\ -6 \end{bmatrix}$ $\mathbf{u}_4 = \begin{bmatrix} -3 \\ 6 \\ 0 \\ -15 \\ -21 \end{bmatrix}$ $\mathbf{u}_5 = \begin{bmatrix} -9 \\ 0 \\ -9 \\ 3 \\ 13 \end{bmatrix}$

   (b) $\boxed{1}$ $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -6 \\ -3 \end{bmatrix}$ $\mathbf{v}_2 = \begin{bmatrix} 7 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 5 \\ 5 \end{bmatrix}$ $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 0 \\ -11 \\ 0 \\ 0 \end{bmatrix}$ $\mathbf{v}_5 = \begin{bmatrix} -9 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

   (Hint: Although this may appear to be a daunting task, there is a very simple way to find out.)

3. Given $A = \begin{bmatrix} 1 & -2 & -4 \\ -2 & 5 & 8 \\ 2 & 5 & 12 \end{bmatrix}$

   (a) $\boxed{0.5}$ Compute the adjoint matrix of A, $adj(A)$.

   (b) $\boxed{0.3}$ Is A non-singular?

   (c) $\boxed{0.3}$ If possible, depending on your answer to point (b), use $adj(A)$ to compute A^{-1}.

Please turn over

(d) $\boxed{0.3}$ Is $B = \begin{bmatrix} 1 & -2 & 3 & 0 & 0 & 0 \\ 2 & -5 & 10 & 0 & 0 & 0 \\ 2 & -5 & 11 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 0 & -6 & 2 \end{bmatrix}$

invertible? Explain. (Notice that, if it is, you do not need to calculate the inverse yet.)

(e) $\boxed{0.3}$ Is $C = \begin{bmatrix} 3 & 4 & -5 & 0 & 0 & 0 \\ 10 & 12 & -15 & 0 & 0 & 0 \\ -2 & -3 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 & -3 & 1 \end{bmatrix}$

invertible? Explain. (Notice that, if it is, you do not need to calculate the inverse yet.)

(f) $\boxed{0.3}$ For B and C, use any valid method to calculate the inverse where possible.

4. Consider the subspace spanned by the set of vectors $U = \{-1 + 2x + 3x^2, x - x^2, x - 2x^2\}$ in P_3 (remember that P_3 is the vector space of polynomials of degree $n < 3$).

   (a) $\boxed{0.3}$ What is the dimension of this subspace? Explain.

   (b) $\boxed{0.3}$ It is stated that the set of vectors $W = \{x, x^2, 1 + x + x^2\}$ is a basis of that subspace. Explain why this is true.

   (c) $\boxed{1}$ Compute the transition matrix from the canonical basis, $E = \{1, x, x^2\}$ to the basis W.

   (b) $\boxed{0.4}$ Given a polynomial in the canonical basis, $p(x) = a + bx + cx^2$, write this polynomial in the basis W.

5. Which of the given sets of vectors are linearly independent? Explain in each case why.

   (a) $\boxed{0.8}$ $\{1, \sinh x, \cosh x\} \in C^2[0-1]$. (*Hint: In case you need it,* $\sinh x = \dfrac{e^x - e^{-x}}{2}$ *and* $\cosh x = \dfrac{e^x + e^{-x}}{2}$)

   (b) $\boxed{0.6}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$.

   (c) $\boxed{0.6}$ $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

$\boxed{\text{NOTE: Maximum points possible, } p = 10. \text{ Grade, } g = 0.9p + 1}$

Solutions

General consideration: In each problem the students should get some points if they explain correctly what they have to do (the method) to solve the problem even if either the final result is incorrect because they make algebraic mistakes or they did not manage to solve the problem.
I therefore give separately the points for the method as "X points for the method" and the points for the result as "Y points for the result".
If a student does not state explicitly the method, but they use it in a way that is clear what they do, you should add the X points of the method at the end.
They do not get points for a solution if they use a different method than the one asked for, even if the result is correct. I may not write this in all cases, but keep this in mind.

1. (a) I can write the augmented matrix if I take $a = 2$ and $b = 1$, row reduce it to find the solution. I do not give the steps but the students have to. They can use any other valid method to find the solutions:

$$\begin{bmatrix} -1 & 0 & 1 & 2 \\ 2 & 3 & 2 & -2 \\ -2 & 4 & 5 & 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 0 & -3/7 \\ 0 & 1 & 0 & -10/7 \\ 0 & 0 & 1 & 11/7 \end{bmatrix}$$

0.4 points for the row reduction. They get the points also if they use any other method taught in the course.

Subtract 0.1 points for each new algebraic mistake; the minimum grade should be 0, and not negative.

So the solution is $\mathbf{x} = \begin{bmatrix} -3/7 \\ -10/7 \\ 11/7 \end{bmatrix}$.

0.2 points for the solution. They get the points also if they do not write the solution explicitly as a vector, but it is clear from the previous step what the solution is

(b) Using the determinant:

$det(A) = \begin{vmatrix} -1 & 0 & 1 \\ 2 & 3 & a \\ -2 & 4 & 5 \end{vmatrix} = 4a - 1$. $\det(A) \neq 0 \Leftrightarrow 4a - 1 \neq 0 \Leftrightarrow a \neq \frac{1}{4}$. So the system will have only 1 solution as long as $a \neq \frac{1}{4}$, regardless the value of b.

0.3 points for the determinant

0.1 points for the statement about a, even if they say nothing about b.

They get no points for the whole item if they use a different method (see question)

(c) Applying row operations $\left[\begin{array}{ccc|c} -1 & 0 & 1 & a \\ 2 & 3 & a & -2 \\ -2 & 4 & 5 & 3b \end{array}\right] \rightsquigarrow \begin{bmatrix} 1-4a & 0 & 0 & 4a^2 - 15a + 9b + 8 \\ 0 & 1-4a & 0 & 2a^2 + 10a - 3ab - 6b - 6 \\ 0 & 0 & 1-4a & 9b + 8 - 14a \end{bmatrix}$.

0.3 points for the row reduction

They get no points for the whole item if they use a different method (see question)

From the last row:
$(1 - 4a)x_3 = 9b + 8 - 14a$. If take $a = \frac{1}{4} \Rightarrow 0 = 9b + 8 - \frac{7}{2} = 9b + \frac{9}{2}$, which will only be consistent if $b = -\frac{1}{2}$.
So the system will have no solution (will be inconsistent) if $a = \frac{1}{4}$ **and** $b \neq -\frac{1}{2}$. (Notice that you would get the same result if you used any of the rows, it is only that the last one is the easiest.)

0.3 points for this condition

They get 0 points here if they do not state this condition correctly; in particular it is important that they say that **both** conditions must be satisfied

Subtract 0.1 points for each new algebraic mistake; the minimum grade should be 0, and not negative.

If they make algebraic mistakes, and reach a different conclusion, they should get 0.2 points for the second part if they write the condition correctly and self consistently with what they did, including their mistakes

(d) The null space of $B \in \mathbb{R}^{3\times 3}$ is defined as all the vectors $\mathbf{x}$ that are solutions of the homogeneous system $B\mathbf{x} = \mathbf{0}$:
$\mathcal{N}(B) = \{\mathbf{x} \in \mathbb{R}^3 \mid B\mathbf{x} = \mathbf{0}\}$.

Apply row operations to the augmented matrix: $\begin{bmatrix} -4 & 0 & 4 & 0 \\ 8 & 12 & 1 & 0 \\ 8 & 16 & 20 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 3/4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

0.2 points for the row operations

Subtract 0.1 points for each new algebraic mistake; the minimum grade should be 0, and not negative.

The last row says that there is a free variable. If I take, for instance, x_3 free, the first two rows give $x_1 = x_3$ and $x_2 = -\frac{3}{4}x_3$, and I can write the solution in parametric form: $\mathbf{x} = x_3 \begin{bmatrix} 1 \\ -3/4 \\ 1 \end{bmatrix}$. Hence $\mathcal{N}(B)$ is the set of vectors $\mathbf{x} = t \begin{bmatrix} 1 \\ -3/4 \\ 1 \end{bmatrix}$ with $t \in \mathbb{R}$.

0.2 points for this or a similar statement; in particular, they must have somewhere that there is a free variable and the vector has to be in parametric form (not necessarily written as I did, but it is okay if it is in some equivalent way)

If they make algebraic mistakes, and reach a different conclusion, they should get 0.2 points for the second part if they write the condition correctly and self consistently with what they did, including their mistakes

They get 0 points for this part if they do not have a parameter (t in my case, but name is not relevant) indicating that there are infinite vectors

2. Notice that both in (a) and (b) I used the determinant, but they can use other valid methods if they want and still get full points

(a) A very simple way to find out is to calculate the determinant of the matrix that has columns given by the column vectors. The vectors are linearly independent if and only if the determinant is different from 0. In this case:

0.4 points for this or similar statement about the determinant

$$A = \begin{bmatrix} 1 & 3 & 7 & -3 & -9 \\ -2 & 5 & 8 & 6 & 0 \\ 0 & -1 & 10 & 0 & -9 \\ 5 & -3 & 5 & -15 & 3 \\ 7 & 4 & -6 & -21 & 13 \end{bmatrix}.$$

But I noticed that column 4 is column 1 multiplied by -3, so the determinant will be 0, and hence the vectors are not linearly independent.

0.4 points for noticing that because the columns are multiple of each other the determinant is 0

They get the points also if they do not notice this and calculate the full determinant

0.2 points for stating that the vectors are not linearly independent

(b) Proceed as in part (a), but sort the vectors such that the matrix is a block matrix:

0.4 points for this or similar statement about the determinant

$$A = \begin{bmatrix} 0 & 0 & 0 & 7 & -9 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & -11 & 0 & 0 \\ -6 & 5 & 0 & 0 & 0 \\ -3 & 5 & 0 & 0 & 0 \end{bmatrix},$$

which makes it easier to calculate the determinant as the product of the determinant of each block (I do not show the steps). The result is $\det(A) = 4125 \neq 0$, so the vectors are linearly independent.

0.4 points for noticing that they can use the properties of block matrices to compute the determinant

They get the points also if they do not notice this and calculate the full determinant

0.2 points for stating that the vectors are linearly independent

3. (a) The $adj(A)$ is the matrix with elements that are the cofactors of a_{ij} transposed.

0.3 points for saying this or using this to get the result.

I do not write intermediate steps. The students do not have to give every detail, but they have to explain something.

$$adj(A) = \begin{bmatrix} 20 & 4 & 4 \\ 40 & 20 & 0 \\ -20 & -9 & 1 \end{bmatrix}$$

0.2 points for the result.

(b) I use the determinant (students can use other valid methods) because if $\det(A) \neq 0 \Leftrightarrow \exists A^{-1}$:

$$\det(A) = \begin{vmatrix} 1 & -2 & -4 \\ -2 & 5 & 8 \\ 2 & 5 & 12 \end{vmatrix} = 20 \text{ so } A \text{ is non-singular.}$$

0.3 points for the result.

They can use any valid method. I used the determinant, but they can also calculate A^{-1} to prove that it exists.

(c) $A^{-1} = adj(A)/\det(A) = \begin{bmatrix} 1 & 1/5 & 1/5 \\ 2 & 1 & 0 \\ -1 & -9/20 & 1/20 \end{bmatrix}$.

0.3 points for the result.

(d) B is a block matrix, so the determinant is the product of the determinants of the blocks. But $\begin{vmatrix} 3 & -1 \\ -6 & 2 \end{vmatrix} = 0$ so no, B is not invertible.

0.3 points for this statement.

They can use any valid method. I used the determinant of the blocks, but they can also notice that column 5 is column 6 times -3, or calculate the determinant of the full matrix to prove that B^{-1} does not exists. It is also okay if they try to calculate B^{-1} and conclude it does not exist.

(e) C is a block matrix, so the determinant is again the product of the determinants of the blocks.

$\begin{vmatrix} 3 & 4 & -5 \\ 10 & 12 & -15 \\ -2 & -3 & 4 \end{vmatrix} = -1$, $\det(7) = 7$, and $\begin{vmatrix} 3 & 2 \\ -3 & 1 \end{vmatrix} = 9$, so yes, C is invertible.

0.3 points for this statement.

They can use any valid method. I used the determinant of the blocks, but they can also calculate the determinant of the full matrix to prove that C^{-1} exists. It is also okay if they calculate C^{-1} to prove it exists.

(f) Calculate the inverse of the blocks of C (B is not invertible); I do not give the steps, but the students have to show how they got the inverses of the blocks:

$$\begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 10 & -2 & 5 & 0 & 0 & 0 \\ 6 & -1 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/9 & -2/9 \\ 0 & 0 & 0 & 0 & 1/3 & 1/3 \end{bmatrix}.$$

0.3 points for B^{-1}. They can use any valid method to do this. It is also okay if they do not use the blocks (although it would be much more work).

4. (a) Calculate the Wronskian, $W\{-1+2x+3x^2, x-x^2, x-2x^2\} = 2 \neq 0$, so the 3 vectors are linearly independent, That means that the three vectors are a basis of the subspace that they themselves define and hence you can generate the whole subspace with only three vectors. Therefore the dimension is 3.

0.3 points for the result. If they express the idea in words but do not work it out they get 0.2 points.

I did not think of other methods of doing this, but if they use a valid method they get the full points.

(b) Again calculate the Wronskian, $W\{x, x^2, 1+x+x^2\} = 2 \neq 0$, so the three vectors are linearly independent. Since the the dimension of this subspace is 3 (see previous point) and these are 3 linearly independent vectors they generate a 3-dimension space and hence they are basis of the subspace.

0.3 points for the result. If they express the idea in words but do not work it out they get 0.2 points.

I did not think of other methods of doing this, but if they use a valid method they get the full points.

(c) I need to express the canonical basis in terms of W. I write the elements of W in between parentheses for clarity:

$1 \;\, = a_1(x) + a_2(x^2) + a_3(1+x+x^2)$
$x \;\, = b_1(x) + b_2(x^2) + b_3(1+x+x^2)$
$x^2 = c_1(x) + c_2(x^2) + c_3(1+x+x^2)$

0.2 points for writing the 3 canonical basis vectors in terms of the basis vectors given

and I need to find all the constants, $a_1, a_2, \ldots$, etc. For that I use that 2 polynomials are equal if and only if the coefficients in front of equal powers of x are equal, which yields:

$1 = a_3,\ 0 = a_1 + a_3,\ 0 = a_2 + a_3$
$0 = b_3,\ 1 = b_1 + b_3,\ 0 = b_2 + b_3$
$0 = c_3,\ 0 = c_1 + c_3,\ 1 = c_2 + c_3$

0.1 points for writing the equations that need to be solved to find the unknown coefficients

Each of these is a system of 3 equations with 3 unknowns. I can solve them (I do not give any step) to find:

$a_1 = -1,\ a_2 = -1,\ a_3 = 1$
$b_1 = 1,\ b_2 = 0,\ b_3 = 0$
$c_1 = 0,\ c_2 = 1,\ c_3 = 0$

0.6 points for solving all the 3 system of equations to find the coefficients; 0.2 points for each triplet of coefficients.

So the transition matrix from W to the canonical basis is $S = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

0.1 points for the transition matrix.

Another way of doing it is to find first the transition matrix from W to E, S^{-1} (I call this transition matrix S^{-1} given that it is the inverse of the transition matrix I need, which I called S above), which is very simple to calculate, and then invert that to get S, the transition matrix from E to W. (This is the way it is done in example 6 of §3.5 in the book.) Start by writing the vectors of the basis W in terms of those of the basis E (as before, I write the elements of E in parentheses for clarity):

$x = 0(1) + 1(x) + 0(x^2)$
$x^2 = 0(1) + 0(x) + 1(x^2)$
$1 + x + x^2 = 1(1) + 1(x) + 1(x^2)$,

Give 0.2 points for these 3 equations.

take the coefficients in each row as the elements of the columns of the transition matrix from W to E, $S^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Give 0.2 points for writing S^{-1}
and invert S^{-1} to get S, which gives the same I have above. Give full points if they do it this way.
Give 0.6 points for inverting S^{-1} to get S.

(d) Given $p(x) = a + bx + cx^2$ in the canonical basis, I can find the same polynomial in the basis W using:

$S \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b-a \\ c-a \\ a \end{bmatrix}$, such that in the new basis: $p(x) = (b-a)x + (c-a)x^2 + a(1+x+x^2)$.

0.4 points for transforming the polynomial from one basis to the other.

I can check this (they do not need to do this, but it helps to be sure you got it right) expanding the above : $p(x) = (b-a)x + (c-a)x^2 + a(1+x+x^2) = bx - ax + cx^2 - ax^2 + a + ax + ax^2 = a + bx + cx^2$.

I did it, but they do not need to check the result.

5. (a) Here I use the Wronskian, $W(1, \sinh x, \cosh x) = \begin{bmatrix} 1 & \sinh x & \cosh x \\ 0 & \cosh x & \sinh x \\ 0 & \sinh x & \cosh x \end{bmatrix} = \cosh^2 x - \sinh^2 x = 1 \neq 0$, so the answer is yes.

0.8 points for this. They can also write $c_1(1) + c_2(\sinh x) + c_3(\cosh x) = 0$, take 3 arbitrary values for x and get a system of 3 equations with 3 unknowns, the c_i, and see whether this system has only the trivial solution, etc., like was done in one of the HW. See also example 8 in §3.3 of the book.

(b) Given $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}$, I calculate $\det[\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3] = \begin{vmatrix} 1 & 1 & -1 \\ 0 & 3 & 2 \\ -3 & 1 & 2 \end{vmatrix} = -11 \neq 0$ so yes.

0.6 points for this. They get full points if they use any other valid method.

(c) Here I do not need to calculate anything (but it is okay if the students do) because in $\mathbb{R}^2$ more than 2 vectors cannot be are linearly independent, so the answer is no.

0.6 points for this. They get full points if they use any other valid method.