Discrete Mathematics for Computer Science

Department of Computer Science

Lecturer: Nazeef Ul Haq

Reference Book: Discrete Mathematics and its applications BY Kenneth H. Rosen – 8^{th} edition

Lecture 3

Chapter 1. The Foundations

2. Propositional Equivalences
3. Predicates and Quantifiers

Truth Tables

- Truth table for ~p ∧ (q ∨ ~r)

p	q	r	~ r	q ∨ ~ r	~ p	~ p ∧ (q ∨ ~ r)
T	T	T	F	T	F	F
T	T	F	T	T	F	F
T	F	T	F	F	F	F
T	F	F	T	T	F	F
F	T	T	F	T	T	T
F	T	F	T	T	T	T
F	F	T	F	F	T	F
F	F	F	T	T	T	T

Double Negation

- Double Negative Property $\sim(\sim p) \equiv p$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

- **Example: “It is not true that I am not happy”**
- Solution:
- Let p = “I am happy”
- then ~ **p** = “I am not happy”
- and ~**(~ p)** = “It is not true that I am not happy”
- Since ~ (~**p**) ≡ **p**
- Hence the given statement is equivalent to:“**I am happy**”

De Morgan's Laws

- The negation of an **and** statement is logically equivalent to the **or** statement in which each component is negated.
  Symbolically $\sim(p \wedge q) \equiv \sim p \vee \sim q$.
- The negation of an **or** statement is logically equivalent to the **and** statement in which each component is negated.
  Symbolically: $\sim(p \vee q) \equiv \sim p \wedge \sim q$.

De Morgan's Laws

- How we can prove this?
  $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

Same truth values

De Morgan’s Laws

- Give negations for each of the following statements:
  a. The fan is slow **or** it is very hot.
  b. Akram is unfit **and** Saleem is injured.
- **Solution:**
  a. The fan is **not** slow **and** it is **not** very hot.
  b. Akram is **not** unfit **or** Saleem is **not** injured.
- **INEQUALITIES AND DEMORGAN’S LAWS:**
- Use DeMorgan’s Laws to write the negation of
  $-1 < x \leq 4$
- $-1 < x \leq 4$ means $x > -1$ **and** $x \leq 4$

By DeMorgan’s Law, the negation is:

$x > -1$ or $x \leq 4$ Which is equivalent to: $x \leq -1$ **or** $x > 4$

1.2 Propositional Equivalence

- A ***tautology*** is a compound proposition that is **true** *no matter what* the truth values of its atomic propositions are! And represented by a symbol “t”
  - *e.g.* $p \vee \neg p$ (“Today the sun will shine or today the sun will not shine.”) [What is its truth table?]
- A ***contradiction*** is a compound proposition that is **false** no matter what! And represented by a symbol “c”
  - *e.g.* $p \wedge \neg p$ (““Today is Wednesday and today is not Wednesday.”) [Truth table?]
- A ***contingency*** is a compound proposition that is neither a tautology nor a contradiction.
  - *e.g.* $(p \vee q) \to \neg r$

Logical Equivalence

- Compound proposition p is ***logically equivalent*** to compound proposition q, written $p \equiv q$ or $p \Leftrightarrow q$, **iff** the compound proposition $p \leftrightarrow q$ is a tautology.

- Compound propositions p and q are logically equivalent to each other **iff** p and q contain the same truth values as each other in <u>all</u> corresponding rows of their truth tables.

Proving Equivalence via Truth Tables

- Prove that $\neg(p \wedge q) \equiv \neg p \vee \neg q$. (De Morgan's law)

p	q	$p \wedge q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$\neg(p \wedge q)$
T	T	T	F	F	F	F
T	F	F	F	T	T	T
F	T	F	T	F	T	T
F	F	F	T	T	T	T

- Show that

  Check out the solution in the textbook!

  - $\neg(p \vee q) \equiv \neg p \wedge \neg q$ (De Morgan's law)
  - $p \to q \equiv \neg p \vee q$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ (distributive law)

Equivalence Laws

- These are similar to the arithmetic identities you may have learned in algebra, but for propositional equivalences instead.

- They provide a pattern or template that can be used to match part of a much more complicated proposition and to find an equivalence for it and possibly simplify it.

Equivalence Laws

- *Identity*: $p \wedge \mathbf{T} \equiv p \qquad p \vee \mathbf{F} \equiv p$
- *Domination*: $p \vee \mathbf{T} \equiv \mathbf{T} \qquad p \wedge \mathbf{F} \equiv \mathbf{F}$
- *Idempotent*: $p \vee p \equiv p \qquad p \wedge p \equiv p$
- *Double negation:* $\neg\neg p \equiv p$
- *Commutative:* $p \vee q \equiv q \vee p \qquad p \wedge q \equiv q \wedge p$
- *Associative:* $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

More Equivalence Laws

- *Distributive*: $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
  $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- *De Morgan's*:
  $\neg(p \wedge q) \equiv \neg p \vee \neg q$
  $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- *Absorption*
  $p \vee (p \wedge q) \equiv p$ $\quad$ $p \wedge (p \vee q) \equiv p$
- *Trivial tautology/contradiction*:
  $p \vee \neg p \equiv \mathbf{T}$ $\quad$ $p \wedge \neg p \equiv \mathbf{F}$

See Table 6, 7, and 8 of Section 1.2

Defining Operators via Equivalences

Using equivalences, we can *define* operators in terms of other operators.

- Exclusive or: $p \oplus q \equiv (p \wedge \neg q) \vee (\neg p \wedge q)$

  $p \oplus q \equiv (p \vee q) \wedge \neg(p \wedge q)$

- Implies: $p \rightarrow q \equiv \neg p \vee q$

- Biconditional: $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

  $p \leftrightarrow q \equiv \neg(p \oplus q)$

This way we can "normalize" propositions

An Example Problem

- Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent.

$\neg(p \to q)$

$\equiv \neg(\neg p \vee q)$ [Expand definition of $\to$]

$\equiv \neg(\neg p) \wedge \neg q$ [DeMorgan's Law]

$\equiv p \wedge \neg q$ [Double negation Law]

EXERCISE

- Use Logical Equivalence to rewrite each of the following sentences more simply.
- **It is not true that I am tired and you are smart.**
  {I am not tired or you are not smart.}
- **It is not true that I am tired or you are smart.**
  {I am not tired and you are not smart.}
- **I forgot my pen or my bag and I forgot my pen or my glasses.**
  {I forgot my pen or I forgot my bag and glasses.}
- **It is raining and I have forgotten my umbrella, or it is raining and I have forgotten my hat.**
  {It is raining and I have forgotten my umbrella or my hat.}

Negation of Implication

- Since $p \rightarrow q \equiv \sim p \vee q$ therefore

$$\sim (p \rightarrow q) \equiv \sim (\sim p \vee q)$$
$$\equiv \sim (\sim p) \wedge (\sim q) \text{ by De Morgan's law}$$
$$\equiv p \wedge \sim q \text{ by the Double Negative law}$$

Thus the negation of "**if p then q**" is logically equivalent to "**p and not q**".

Example:

1. If Ali lives in Pakistan then he lives in Lahore.

   Ali lives in Pakistan and he does not live in Lahore.

Example

- Show that **~(p→q) → p** is a tautology without using truth tables.
- ~(p→q) → p Given statement form

$\equiv \sim[\sim(p \wedge \sim q)] \rightarrow p$	Implication law $p \rightarrow q \equiv \sim(p \wedge \sim q)$
$\equiv (p \wedge \sim q) \rightarrow p$	Double negation law
$\equiv \sim(p \wedge \sim q) \vee p$	Implication law $p \rightarrow q \equiv \sim p \vee q$
$\equiv (\sim p \vee q) \vee p$	De Morgan's law
$\equiv (q \vee \sim p) \vee p$	Commutative law
$\equiv q \vee (\sim p \vee p)$	Associative law
$\equiv q \vee t$	Negation law
$\equiv t$	

Another Example Problem

- Check using a symbolic derivation whether

$$\boxed{(p \wedge \neg q) \rightarrow (p \oplus r) \equiv \neg p \vee q \vee \neg r}$$

$(p \wedge \neg q) \rightarrow (p \oplus r)$ [Expand definition of $\rightarrow$]

$\equiv \neg(p \wedge \neg q) \vee (p \oplus r)$ [Expand definition of $\oplus$]

$\equiv \neg(p \wedge \neg q) \vee (\,(p \vee r) \wedge \neg(p \wedge r)\,)$

[DeMorgan's Law]

$\equiv (\neg p \vee q) \vee (\,(p \vee r) \wedge \neg(p \wedge r)\,)$

cont.

Example Continued...

$$\boxed{(p \wedge \neg q) \rightarrow (p \oplus r) \equiv \neg p \vee q \vee \neg r}$$

$(\neg p \vee q) \vee ((p \vee r) \wedge \neg(p \wedge r))$ [$\vee$ Commutative]

$\equiv (q \vee \neg p) \vee ((p \vee r) \wedge \neg(p \wedge r))$ [$\vee$ Associative]

$\equiv q \vee (\neg p \vee ((p \vee r) \wedge \neg(p \wedge r)))$ [Distribute $\vee$ over $\wedge$]

$\equiv q \vee (((\neg p \vee (p \vee r)) \wedge (\neg p \vee \neg(p \wedge r)))$ [$\vee$ Assoc.]

$\equiv q \vee (((\neg p \vee p) \vee r) \wedge (\neg p \vee \neg(p \wedge r)))$ [Trivial taut.]

$\equiv q \vee ((\mathbf{T} \vee r) \wedge (\neg p \vee \neg(p \wedge r)))$ [Domination]

$\equiv q \vee (\mathbf{T} \wedge (\neg p \vee \neg(p \wedge r)))$ [Identity]

$\equiv q \vee (\neg p \vee \neg(p \wedge r))$

cont.

End of Long Example

$$\boxed{(p \wedge \neg q) \rightarrow (p \oplus r) \equiv \neg p \vee q \vee \neg r}$$

$q \vee (\neg p \vee \neg(p \wedge r))$ [DeMorgan's Law]

$\equiv q \vee (\neg p \vee (\neg p \vee \neg r))$ [$\vee$ Associative]

$\equiv q \vee ((\neg p \vee \neg p) \vee \neg r)$ [Idempotent]

$\equiv q \vee (\neg p \vee \neg r)$ [Associative]

$\equiv (q \vee \neg p) \vee \neg r$ [$\vee$ Commutative]

$\equiv \neg p \vee q \vee \neg r$ ■

Review: Propositional Logic

- Atomic propositions: $p, q, r, \ldots$
- Boolean operators: $\neg \wedge \vee \oplus \rightarrow \leftrightarrow$
- Compound propositions: $(p \wedge \neg q) \vee r$
- Equivalences: $p \wedge \neg q \Leftrightarrow \equiv \neg(p \rightarrow q)$
- Proving equivalences using:
  - Truth tables
  - Symbolic derivations (series of logical equivalences) $p \equiv q \equiv r \equiv \cdots$

1.3 Predicate Logic

- Consider the sentence

  "For every x, $x > 0$"

  If this were a true statement about the positive integers, it could not be adequately symbolized using only statement letters, parentheses and logical connectives.

  The sentence contains two new features: a ***predicate*** *and a* ***quantifier***

Subjects and Predicates

- In the sentence "The dog is sleeping":
  - The phrase "the dog" denotes the ***subject*** – the *object* or *entity* that the sentence is about.
  - The phrase "is sleeping" denotes the ***predicate*** – a property that the subject of the statement can have.
- In predicate logic, a ***predicate*** is modeled as a ***proposional function P(·)*** from subjects to propositions.
  - $P(x)$ = "x is sleeping" (where x is any subject).
  - P(The cat) = "*The cat* is sleeping" (proposition!)

More About Predicates

- Convention: Lowercase variables *x*, *y*, *z*... denote subjects; uppercase variables *P*, *Q*, *R*… denote propositional functions (or predicates).

- Keep in mind that the *result of applying* a predicate *P* to a value of subject *x* is the *proposition*. But the predicate *P*, or the statement *P*(*x*) **itself** (*e.g.* *P* = “is sleeping” or *P*(*x*) = “*x* is sleeping”) is **not** a proposition.
  - *e.g.* if *P*(*x*) = “*x* is a prime number”,

    P(3) is the *proposition* “3 is a prime number.”

Propositional Functions

- Predicate logic *generalizes* the grammatical notion of a predicate to also include propositional functions of **any** number of arguments, each of which may take **any** grammatical role that a noun can take.
  - *e.g.*:

    let $P(x,y,z)$ = "x gave y the grade z"

    then if

    x = "Mike", y = "Mary", z = "A",

    then

    $P(x,y,z)$ = "**Mike** gave **Mary** the grade **A**."

Examples

- Let $P(x)$: $x > 3$. Then
  - $P(4)$ is TRUE/FALSE $4 > 3$
  - $P(2)$ is TRUE/FALSE $\boxed{2 > 3}$
- Let $Q(x, y)$: x is the capital of y. Then
  - Q(Washington D.C., U.S.A.) is TRUE
  - Q(Hilo, Hawaii) is FALSE
  - Q(Massachusetts, Boston) is FALSE
  - Q(Denver, Colorado) is TRUE
  - Q(New York, New York) is FALSE
- Read EXAMPLE 6 (pp.33)
  - If **x > 0** then x:= x + 1 (in a computer program)

Universe of Discourse (U.D.)

- The power of distinguishing subjects from predicates is that it lets you state things about *many* objects at once.
- e.g., let $P(x)$ = "$x + 1 > x$". We can then say, "For ***any*** number x, $P(x)$ is true" instead of $(\mathbf{0} + 1 > \mathbf{0}) \land (\mathbf{1} + 1 > \mathbf{1}) \land (\mathbf{2} + 1 > \mathbf{2}) \land \ldots$

- The collection of values that a variable x can take is called x's ***universe of discourse*** or the ***domain of discourse*** (often just referred to as the ***domain***).

Quantifier Expressions

- ***Quantifiers*** provide a notation that allows us to *quantify* (count) *how many* objects in the universe of discourse satisfy the given predicate.
- "∀" is the FOR∀LL or ***universal*** quantifier.
$\forall x\, P(x)$ means *for all* x in the domain, $P(x)$.
- "∃" is the ∃XISTS or ***existential*** quantifier.
$\exists x\, P(x)$ means there *exists* an x in the domain (that is, 1 or more) such that $P(x)$.

The Universal Quantifier ∀

- ***∀x P(x): For all x in the domain, P(x).***
- $\forall x\, P(x)$ is
  - *true* if $P(x)$ is true <u>for every</u> x in D (*D: domain of discourse*)
  - *false* if $P(x)$ is false for <u>at least one</u> x in D
    - For every real number x, $x^2 \geq 0$ TRUE
    - For every real number x, $x^2 - 1 > 0$ FALSE
- A ***counterexample*** to the statement $\forall x\, P(x)$ is a value x in the domain D that makes $P(x)$ false
- What is the truth value of $\forall x\, P(x)$ when the domain is empty? TRUE

The Universal Quantifier $\forall$

- If all the elements in the domain can be listed as x_1, $x_2, \ldots, x_n$ then, $\forall x\, P(x)$ is the same as the conjunction:

$$P(x_1) \wedge P(x_2) \wedge \cdots \wedge P(x_n)$$

- Example: Let the domain of x be parking spaces at UH. Let $P(x)$ be the statement "x is full." Then the ***universal quantification*** of $P(x)$, $\forall x\, P(x)$, is the *proposition:*
  - "All parking spaces at UH are full."
  - or "Every parking space at UH is full."
  - or "For each parking space at UH, that space is full."

The Existential Quantifier $\exists$

- ***$\exists x\, P(x)$: There exists an x in the domain (that is, 1 or more) such that $P(x)$.***
- $\exists x\, P(x)$ is
  - *true* if $P(x)$ is true for at least one x in the domain
  - *false* if $P(x)$ is false for every x in the domain
- What is the truth value of $\exists x\, P(x)$ when the domain is empty? FALSE
- If all the elements in the domain can be listed as $x_1, x_2, \ldots, x_n$ then, $\exists x\, P(x)$ is the same as the disjunction:

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n)$$

The Existential Quantifier ∃

- Example:

  Let the domain of *x* be <u>parking spaces at UH</u>.

  Let *P*(*x*) be the statement "*x* is full."

  Then the ***existential quantification*** of *P*(*x*), $\exists x\, P(x)$, is the *proposition*:

  - "Some parking spaces at UH are full."
  - or "There is a parking space at UH that is full."
  - or "At least one parking space at UH is full."

Free and Bound Variables

- An expression like $P(x)$ is said to have a ***free variable x*** (meaning, x is undefined).

- A quantifier (either $\forall$ or $\exists$) *operates* on an expression having one or more free variables, and ***binds*** one or more of those variables, to produce an expression having one or more ***bound variables***.

Example of Binding

- $P(x,y)$ has 2 free variables, x and y.
- $\forall x\, P(x,y)$ has 1 free variable *y*, and one bound variable *x*. [Which is which?]
- "$P(x)$, where $x = 3$" is another way to bind x.
- An expression with <u>zero</u> free variables is a bona-fide (actual) proposition.
- An expression with <u>one or more</u> free variables is not a proposition:

e.g. $\square x\, P(x,y) = Q(y)$

Quantifiers with Restricted Domain

- Sometimes the universe of discourse is restricted within the quantification, *e.g.*,
  - $\forall x{>}0\ P(x)$ is shorthand for
    "For all x that are greater than zero, $P(x)$."
    $= \forall x\ (x > 0 \rightarrow P(x))$
  - $\exists x{>}0\ P(x)$ is shorthand for
    "There is an x greater than zero such that $P(x)$."
    $= \exists x\ (x > 0 \wedge P(x))$

Translating from English

- Express the statement "*Every student in this class has studied calculus*" using predicates and quantifiers.
  - Let $C(x)$ be the statement: "x has studied calculus."
  - If <u>domain for x consists of the students in this class</u>, then
  - it can be translated as $\forall x\, C(x)$

or

  - If <u>domain for x consists of all people</u>
  - Let $S(x)$ be the predicate: "x is in this class"
  - Translation: $\forall x\, (S(x) \rightarrow C(x))$

Translating from English

- Express the statement "*Some students in this class has visited Mexico*" using predicates and quantifiers.
  - Let $M(x)$ be the statement: "x has visited Mexico"
  - If <u>domain for *x* consists of the students in this class</u>, then
  - it can be translated as $\exists x\, M(x)$

  or
  - If <u>domain for *x* consists of all people</u>
  - Let $S(x)$ be the statement: "x is in this class"
  - Then, the translation is $\exists x\, (S(x) \land M(x))$

Translating from English

- Express the statement "*Every student in this class has visited either Canada or Mexico*" using predicates and quantifiers.
  - Let $C(x)$ be the statement: "x has visited Canada" and $M(x)$ be the statement: "x has visited Mexico"
  - If <u>domain for *x* consists of the students in this class</u>, then
  - it can be translated as $\forall x\, (C(x) \vee M(x))$

Negations of Quantifiers

- $\forall x\, P(x)$: "Every student in the class has taken a course in calculus" ($P(x)$: "x has taken a course in calculus")
  - "<u>Not</u> every student in the class … calculus"

    $\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$

- Consider $\exists x\, P(x)$: "There is a student in the class who has taken a course in calculus"
  - "There is <u>no</u> student in the class who has taken a course in calculus"

    $\neg\exists x\, P(x) \equiv \forall x\, \neg P(x)$

Negations of Quantifiers

- Definitions of quantifiers: If the domain = {a, b, c,…}
  - $\forall x\, P(x) \equiv P(a) \wedge P(b) \wedge P(c) \wedge \cdots$
  - $\exists x\, P(x) \equiv P(a) \vee P(b) \vee P(c) \vee \cdots$
- From those, we can prove the laws:
  - $\neg\forall x\, P(x) \equiv \neg(P(a) \wedge P(b) \wedge P(c) \wedge \cdots)$
    $\equiv \neg P(a) \vee \neg P(b) \vee \neg P(c) \vee \cdots$
    $\equiv \exists x\, \neg P(x)$
  - $\neg\exists x\, P(x) \equiv \neg(P(a) \vee P(b) \vee P(c) \vee \cdots)$
    $\equiv \neg P(a) \wedge \neg P(b) \wedge \neg P(c) \wedge \cdots$
    $\equiv \forall x\, \neg P(x)$
- Which *propositional* equivalence law was used to prove this?

DeMorgan's

Negations of Quantifiers

Theorem:

- **Generalized De Morgan's laws for logic**

1. $\neg\forall x\, P(x) \equiv \exists x\, \neg P(x)$

2. $\neg\, \exists x\, P(x) \equiv \forall x\, \neg P(x)$

Negations: Examples

- What are the negations of the statements $\forall x\ (x^2 > x)$ and $\exists x\ (x^2 = 2)$?
  - $\neg\forall x\ (x^2 > x) \equiv \exists x\ \neg(x^2 > x) \equiv \exists x\ (x^2 \leq x)$
  - $\neg\exists x\ (x^2 = 2) \equiv \forall x\ \neg(x^2 = 2) \equiv \forall x\ (x^2 \neq 2)$

- Show that $\neg\forall x(P(x) \rightarrow Q(x))$ and $\exists x(P(x) \wedge \neg Q(x))$ are logically equivalent.
  - $$\begin{aligned} \neg\forall x(P(x) \rightarrow Q(x)) &\equiv \exists x\ \neg(P(x) \rightarrow Q(x)) \\ &\equiv \exists x\ \neg(\neg P(x) \vee Q(x)) \\ &\equiv \exists x\ (P(x) \wedge \neg Q(x)) \end{aligned}$$

Summary

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Nesting of Quantifiers

- Example:

  Let the domain of x and y be people.

  Let $L(x,y)$ = "x likes y" (A statement with 2 free variables – not a proposition)

  - Then $\exists y\, L(x,y)$ = "There is someone whom x likes." (A statement with 1 free variable x – not a proposition)

  - Then $\forall x\, (\exists y\, L(x,y))$ =

    "Everyone has someone whom they like."

    (A __Proposition__ with __0__ free variables.)

Nested Quantifiers

- Nested quantifiers are quantifiers that occur within the scope of other quantifiers.
- The order of the quantifiers is important, unless all the quantifiers are universal quantifiers or all are existential quantifiers.

TABLE 1 Quantifications of Two Variables.

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Nested Quantifiers

- Let the domain of x and y is **R**, and $P(x,y)$: $xy = 0$. Find the truth value of the following propositions.
  - $\forall x \, \forall y \, P(x, y)$ (F)
  - $\forall x \, \exists y \, P(x, y)$ (T)
  - $\exists x \, \forall y \, P(x, y)$ (T)
  - $\exists x \, \exists y \, P(x, y)$ (T)

R:set of real numbers

- $\forall x \, \exists y \, P(x,y) \not\equiv \exists y \, \forall x \, P(x,y)$
  - For every x, there exists y such that $x + y = 0$. (T)
  - There exists y such that, for every x, $x + y = 0$. (F)

Nested Quantifiers: Example

- Let the domain = {1, 2, 3}. Find an expression equivalent to $\forall x\, \exists y\, P(x,y)$ where the variables are bound by substitution instead:
  - Expand from inside out or outside in.
  - Outside in:

$$\forall x\, \exists y\, P(x,y)$$

$$\equiv \exists y\, P(1,y) \wedge \exists y\, P(2,y) \wedge \exists y\, P(3,y)$$

$$\equiv [P(1,1) \vee P(1,2) \vee P(1,3)] \wedge$$
$$[P(2,1) \vee P(2,2) \vee P(2,3)] \wedge$$
$$[P(3,1) \vee P(3,2) \vee P(3,3)]$$

Quantifier Exercise

- If $R(x,y)$="x relies upon y," express the following in unambiguous English when the domain is all people

$\forall x(\exists y\, R(x,y))$ = Everyone has *someone* to rely on.

$\exists y(\forall x\, R(x,y))$ = There's a poor overburdened soul whom *everyone* relies upon (including himself)!

$\exists x(\forall y\, R(x,y))$ = There's some needy person who relies upon *everybody* (including himself).

$\forall y(\exists x\, R(x,y))$ = Everyone has *someone* who relies upon them.

$\forall x(\forall y\, R(x,y))$ = *Everyone* relies upon *everybody*, (including themselves)!

Negating Nested Quantifiers

- Successively apply the rules for negating statements involving a single quantifier

- Example: Express the negation of the statement $\forall x\, \exists y\, (P(x,y) \wedge \exists z\, R(x,y,z))$ so that all negation symbols immediately precede predicates.
  - $\neg\forall x\, \boxed{\exists y\, (P(x,y) \wedge \exists z\, R(x,y,z))}$

    $\equiv \exists x\, \neg\boxed{\exists y\, \boxed{(P(x,y) \wedge \exists z\, R(x,y,z))}}$

    $\equiv \exists x\, \forall y\, \neg\boxed{(P(x,y) \wedge \exists z\, R(x,y,z))}$

    $\equiv \exists x\, \forall y\, (\neg P(x,y) \vee \boxed{\neg\exists z\, R(x,y,z)})$

    $\equiv \exists x\, \forall y\, (\neg P(x,y) \vee \forall z\, \neg R(x,y,z))$

Equivalence Laws

- $\forall x\ \forall y\ P(x,y) \equiv \forall y\ \forall x\ P(x,y)$

  $\exists x\ \exists y\ P(x,y) \equiv \exists y\ \exists x\ P(x,y)$

- $\forall x\ \mathbf{(}P(x) \land Q(x)\mathbf{)} \equiv \mathbf{(}\forall x\ P(x)\mathbf{)} \land \mathbf{(}\forall x\ Q(x)\mathbf{)}$

  $\exists x\ \mathbf{(}P(x) \lor Q(x)\mathbf{)} \equiv \mathbf{(}\exists x\ P(x)\mathbf{)} \lor \mathbf{(}\exists x\ Q(x)\mathbf{)}$

- Exercise:

  See if you can prove these yourself.

Notational Conventions

- Quantifiers have higher precedence than all logical operators from propositional logic:

$$\left(\forall x\, P(x)\right) \wedge Q(x)$$

- Consecutive quantifiers of the same type can be combined:

$\forall x\ \forall y\ \forall z\ P(x,y,z) \equiv \forall x,y,z\ P(x,y,z)$

or even $\forall xyz\ P(x,y,z)$