Calculus I (MTH.263) Notes

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Contents

Chapter 2

2.3 Solving Limits Algebraically

Definition 2.3.1: Direct Substitution

$$\lim_{x\to 5}\left(2x^2 - 3x + 4\right)$$

$$2(5)^2 - 3(5) + 4 = 39$$

Definition 2.3.2: Factoring (goal of cancelling denominator)

$$\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{(x+1)(x-1)}{x-1} = \lim_{x\to 1}(x+1) = 2$$

Definition 2.3.3: Multiply by the conjugate

$$\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2}\cdot\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}$$

$$\lim_{t\to 0}\frac{t^2+9-9}{t^2\left(\sqrt{t^2+9}+3\right)} = \lim_{t\to 0}\frac{t^2}{t^2\left(\sqrt{t^2+9}+3\right)}$$

$$\lim_{t\to 0}\frac{1}{\sqrt{t^2+9}+3}$$

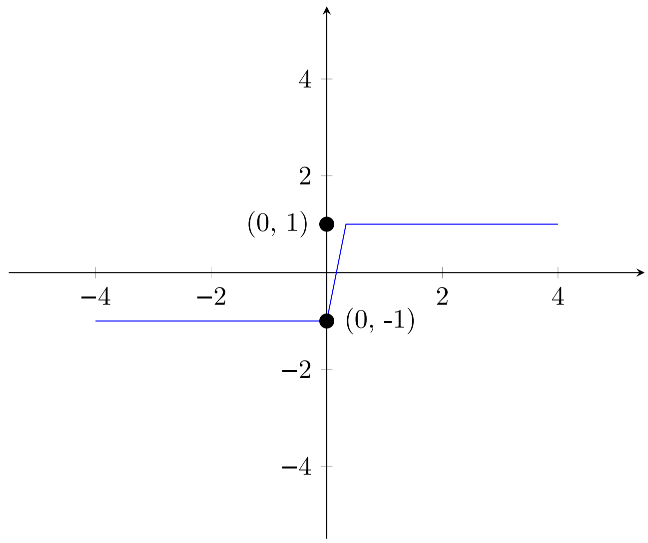
$$\lim_{t\to 0}\frac{1}{\sqrt{(0)^2+9}+3} = \frac{1}{6}$$

Example 2.3.1

$$\lim_{x\to 0}|x| = 0$$

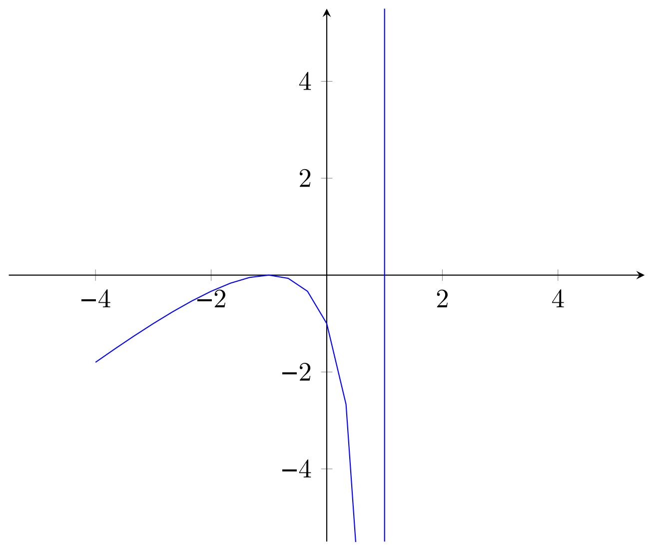
Example 2.3.2

$$\lim_{x \to 0} \frac{|x|}{x} = \text{DNE}$$



Example 2.3.3

$$\lim_{x \to 1} \frac{(x+1)^2}{x-1} = \text{DNE}$$



Note:-

- The vertical asymptote is at $x = 1$
- Highest power of x is in the numerator: no horizontal asymptote
- Highest pwoer in denominator: $x = 0$
- Same powers: highest power coefficient
- Find diagonal asymptote with long or synthetic division

2.4 Formal Definition of Limits

Definition 2.4.1: Formal Definition of Limits

$$\lim_{x \to a} f(x) = L$$

if for every number $\epsilon > 0$ there is a number $\delta > 0$ such that

if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$

Example 2.4.1

Given that $\lim_{x \to 3}(4x - 5) = 7$, find the value of δ that corresponds to $\epsilon = 0.01$

First, we define our variables

$$a = 3$$

$$f(x) = 4x - 5$$

$$L = 7$$

Now, we set up the equations for delta and epsilon

$$|x - 3| < \delta$$

$$|4x - 5 - 7| < \epsilon$$

From here, we'll simplify epsilon to be equal to delta ($|x - 3|$), and then substitute our given value (0.01) for epsilon to solve for delta

$$|4x - 12| < \epsilon$$

$$4|x - 3| < \epsilon$$

$$|x - 3| < \frac{\epsilon}{4}$$

$$\delta = \frac{0.01}{4}$$

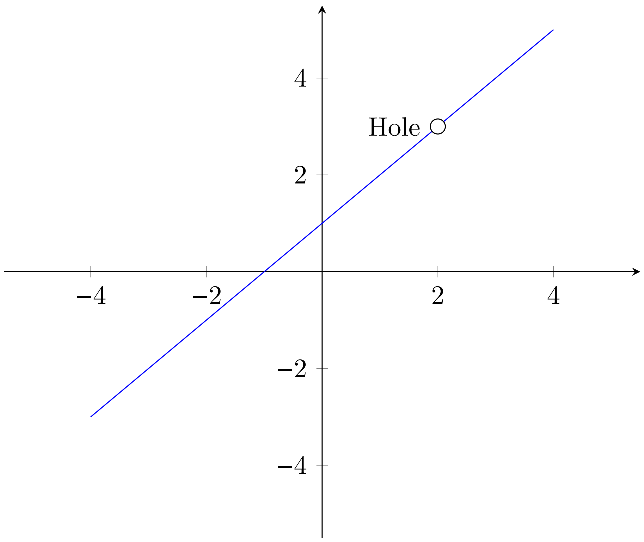
2.5 Continuity

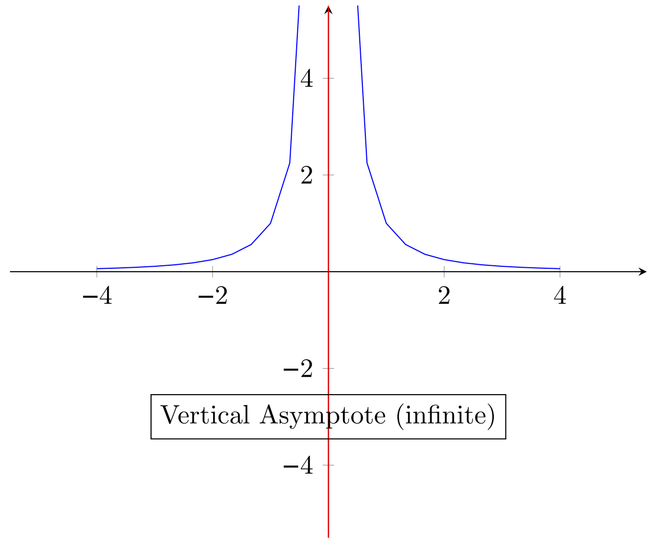
Definition 2.5.1: Continuity

A function is continuous everywhere **EXCEPT** where it isn't:

- Hole (point discontinuity or removable discontinuity)
  - You could write a piecewise function that fills in the hole, hence removable
- Vertical asymptote (infinite discontinutity)
- Jump discontinutity

Continuity is described using interval notation





Note:-

Continuous from the right if $\lim_{x \to a+} f(x) = f(a)$ (goes into a closed point)
Continuous from the left if $\lim_{x \to a-} f(x) = f(a)$ (goes into a closed point)

Definition 2.5.2: Formal definition of continuity

A function $f(x)$ is continuous at a point a if and only if the following three conditions are satisied:

- $f(a)$ is defined
- $\lim_{x \to a} f(x)$ exists
- $\lim_{x \to a} f(x) = f(a)$

Theorem 2.5.1 Intermediate Value Theorem (IVT)

If a function is continuous on an interval $[a, b]$, then there exists a number c between a and b such that $f(c)$ is between $f(a)$ and $f(b)$
You must say four things:

- Show that function is continuous
- Find $f(a)$
- Find $f(b)$
- If it changes from negative to positive (or vice versa), the graph crosses the x-axis, so there is a root between $x = a$ and $x = b$

Note:-

The IVT tells you something exists, but not what it is.

Chapter 3

3.1 Formal Definition of Limits

Definition 3.1.1: Limit Definition of the Derivative

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Example 3.1.1

$$\lim_{h \to 0} \frac{(1+h)^{10} - 1}{h}$$

$$a = 1$$

$$f(x) = x^{10}$$

Example 3.1.2 (Finding the equation of the tangent line)

$$f(x) = \frac{x-1}{x-2}, (3, 2)$$

$$f'(x) = \frac{\frac{x+h-1}{x+h-2} - \frac{x-1}{x-2}}{h}$$

$$f'(x) = \frac{\frac{(x+h-1)(x-2)-((x-1)(x+h-2))}{(x+h-2)(x-2)}}{h}$$

$$f'(x) = \frac{\frac{x^2-2x+hx-2h-x+2-x^2+x-hx+h+2x-2}{(x+h-2)(x-2)}}{h}$$

$$f'(x) = \frac{\frac{-h}{(x+h-2)(x-2)}}{h}$$

$$f'(x) = \frac{-1}{(x+h-2)(x-2)}$$

Solve the limit

$$\lim_{h \to 0} \frac{-1}{(x+0-2)(x-2)} = \frac{-1}{(x-2)^2}$$

Plug in x to find the slope of the tangent line

$$m = \frac{-1}{(3-2)^2} = -1$$

Use point-slope form to construct a formula

$$y - 2 = -(x - 3)$$

Note:-

A derivative is equal to

- Slope of the tangent line
- Instantaneous rate of change
- Instantaneous velocity (velocity is a change in position)

Note:-

If $s(t)$ (the position function) represents the position of a particle at time t, $s'(t) = v(t)$ represents change in the position (velocity).
Velocity is a vector; it has both direction and magnitude Speed is a scalar; it has magnitude but not direction
Speed $= |v(t)|$

Question 1

A manufacturer produces bolts of a fabric with a fixed width. The cost of producing x yards of this fabric is $C = f(x)$ dollars.

1. In practical terms, what does it mean to say that $f'(1000) = 9$?
2. Which do you think is greater, $f'(50)$ or $f'(500)$? What about $f'(5000)$?

Solution:

1. The rate of change at $x = 1000$ is 9
2. $f'(50)$ will be lower because the cost of making the 500th yard is less than the cost of the 50th yard because of economies of scale. But as production expands, $f'(5000)$ may be higher due to increased maintanance and material costs

3.2 Derivative Notition

There are many different ways to notate derivatives. Here are a few examples:

Example 3.2.1 (First Derivatives)

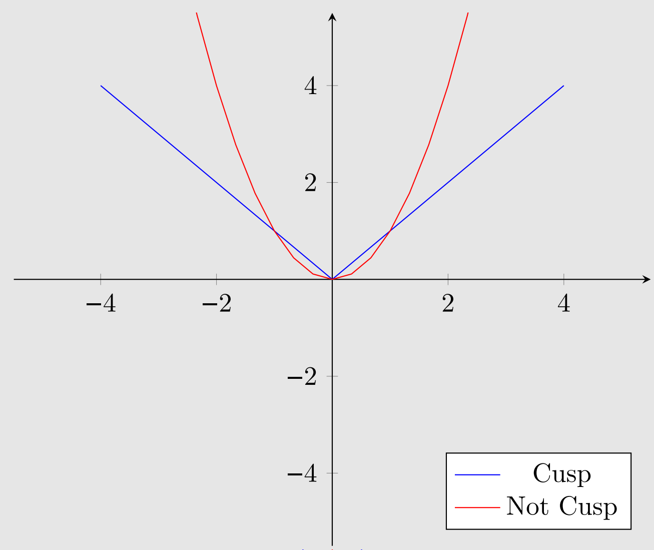
- $f'(x)$
- y'
- $\frac{dy}{dx}$
- $\frac{d}{dx}(f(x))$
- $Df(x)$
- $D_x f(x)$
- $\frac{dy}{dt}$

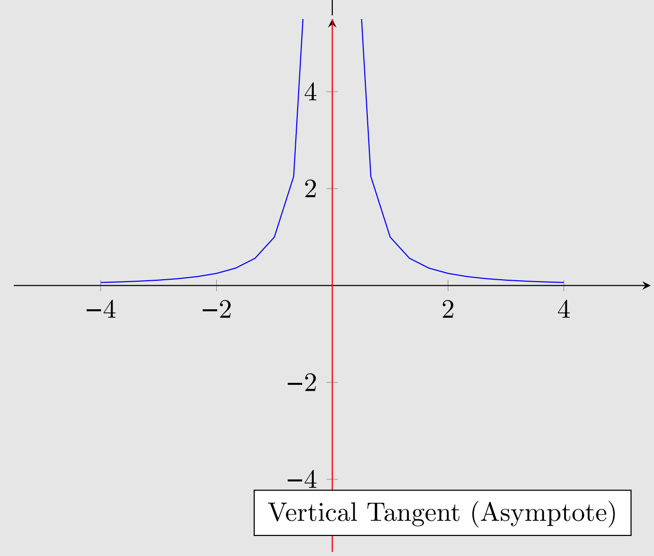
Example 3.2.2 (Higher Derivatives)

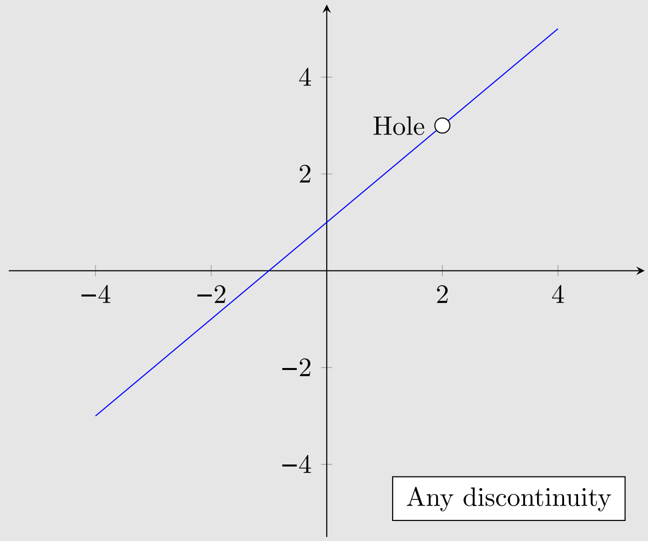
- $f''(x)$
- $f'''(x)$
- $f^4(x)$
- $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$

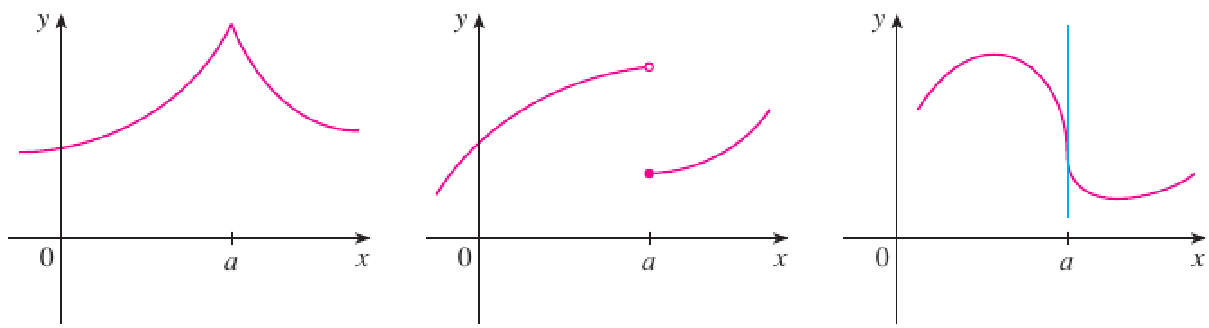
Note:-

A function is not differentiable (the derivative does not exist) when

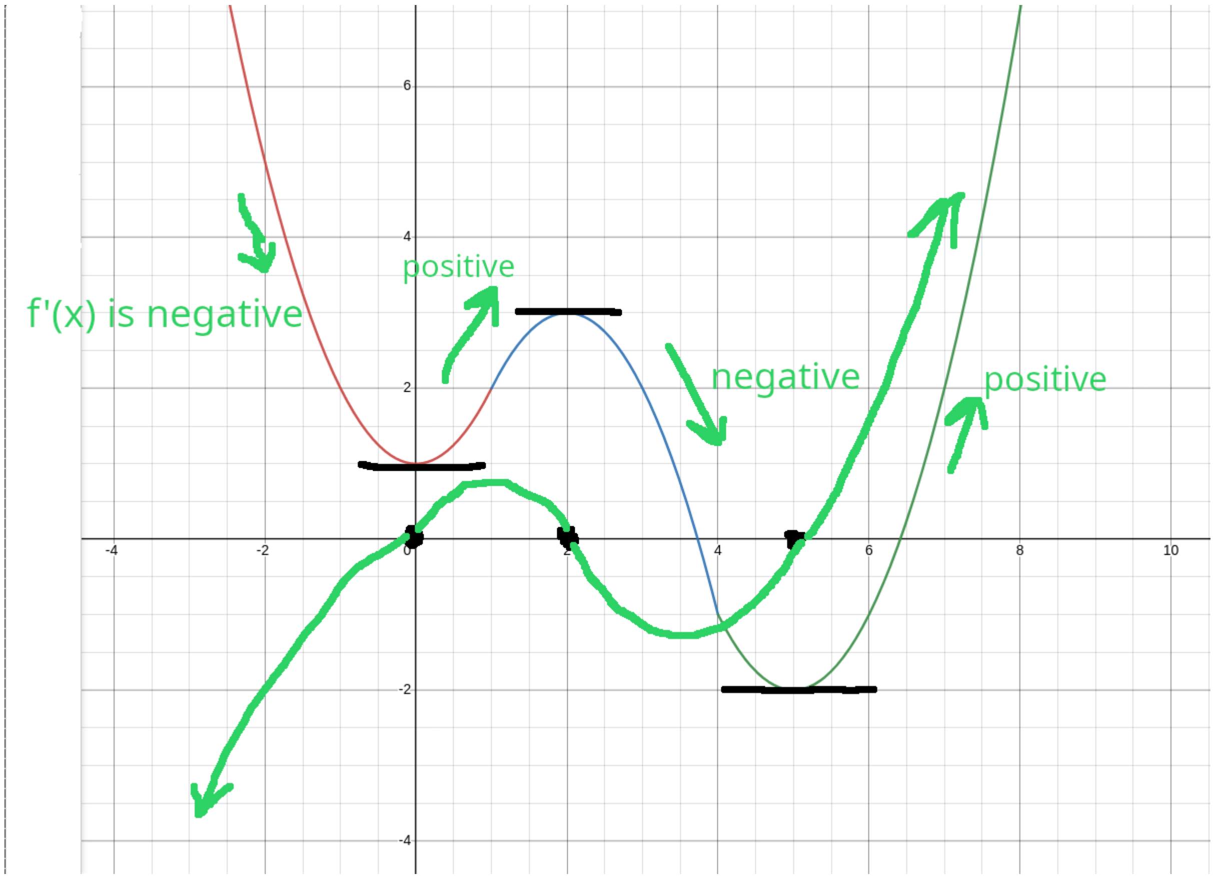








(a) A corner (b) A discontinuity (c) A vertical tangent



3.3 Rules of Derivatives

Definition 3.3.1: Constant Rule

The derivative of a constant is 0, as constants don't change, and derivates are the rate of change (slope)

Definition 3.3.2: Power Rule

Multiply coefficient by the exponent and subtract 1 from the exponent

$$f(x) = x^2 - x$$

Doing it with the definition of a derivative

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^2 - (x+h) - (x^2 - x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{x^2 + 2hx + h^2 - x - h - x^2 + x}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{h(2x + h - 1)}{h}$$

$$f'(x) = \lim_{h \to 0} 2x + h - 1 = 2x - 1$$

Doing it with the power rule

$$x^2 - x^1 \to 2x^1 - 1x^0 \to 2x - 1$$

Example 3.3.1

$$f(x) = x^3 - x$$
$$f'(x) = 3x^2 - 1$$

Example 3.3.2

$$f(x) = 2x^3 - 6x^2 + 3x - 4$$
$$f'(x) = 6x^2 - 12x + 3 + 0$$

Example 3.3.3

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$
$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

Example 3.3.4

$$f(x) = \frac{1}{x^2} = x^{-2}$$
$$f'(x) = -2x^{-3}$$

Example 3.3.5

$$f(x) = x^{\pi}$$

$$f'(x) = \pi x^{\pi - 1}$$

Definition 3.3.3: Product Rule

1st times derivative of 2nd plus 2nd times derivative of 1st

$$f(x) = (6x^3 + 2x + 1)(7x^4 - 4x^3 - 3)$$

DO NOT SIMPLIFY

$$f'(x) = \big((6x^3 + 2x + 1)(28x^3 - 12x^2)\big) + \big((7x^4 - 4x^3 - 3)(18x^2 + 2)\big)$$

Example 3.3.6

$$f(t) = \sqrt{t}(a + bt)$$

$$f'(t) = \left(t^{\frac{1}{2}}(b)\right) + \left((a + bt)\left(\frac{1}{2}t^{-\frac{1}{2}}\right)\right)$$

Definition 3.3.4: Quotient Rule

Bottom times derivative of top **minus** top times derivative of bottom all over bottom squared

$$f(x) = \frac{x^2 + x - 2}{x^3 + 6}$$

$$f'(x) = \frac{\big((x^3+6)(2x+1)\big) - \big((x^2+x-2)(3x^2)\big)}{(x^3+6)^2}$$

Definition 3.3.5: Normal Line

The normal line is perpendicular to the tangent line

3.4 Triginometry Limits

Proposition 3.4.1 MEMORIZE

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$$

Proposition 3.4.2 MEMORIZE

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$$

Example 3.4.1

$$\lim_{x \to 0} \frac{\sin 7x}{x} \cdot 7$$

$$\lim_{x \to 0} \frac{7\sin 7x}{7x}$$

$$\frac{\sin 7x}{7x} = 1$$

$$\frac{7}{1}$$

Example 3.4.2

$$\lim_{x \to 0} \frac{\sin 7x}{4x}$$

$$\lim_{x \to 0} \frac{\sin 7x}{4x} \cdot \frac{7}{4}$$

$$\lim_{x \to 0} \frac{\frac{7}{4}\sin 7x}{7x}$$

$$\frac{7}{4}$$

Example 3.4.3

$$\lim_{x\to 0} x \cot x$$

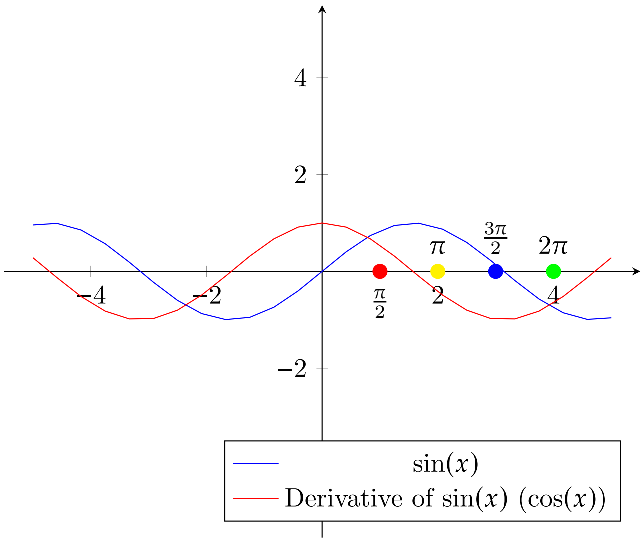
$$\lim_{x\to 0} x \frac{\cos x}{\sin x} \cdot \frac{1}{x}$$

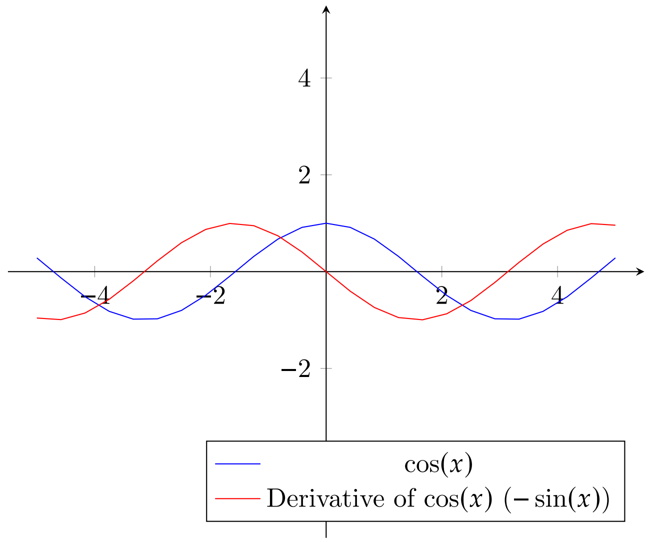
$$\lim_{x\to 0} \frac{\cos x}{\frac{sin x}{x}}$$

$$\lim_{x\to 0} \frac{\cos x}{1}$$

$$\cos 0 = 1$$

Finding trig derivatives:





Example 3.4.4 (sin(x))

$$f(x) = \sin(x)$$

$$f'(x) = \lim_{h\to 0} = \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} = \frac{\sin(x+h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)\cos(h) - \sin(x)}{h} + \lim_{h\to 0} \frac{\cos(x)\sin(h)}{h}$$

$$= \lim_{h\to 0} \frac{\sin(x)(\cos(h-1))}{h} + \lim_{h\to 0} \frac{\cos(x)}{1}$$

$$= \sin(x)(0) + \cos(x)$$

$$= \cos(x)$$

Example 3.4.5 ($\tan(x)$)

$$
\begin{aligned}
y &= \tan(x) = \frac{\sin(x)}{\cos(x)} \\
y' &= \frac{\cos(x)(\cos(x)) - \sin(x)(-\sin(x))}{\cos^2(x)} \\
&= \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} \\
&= \frac{1}{\cos^2(x)} \\
y' &= \sec^2(x)
\end{aligned}
$$

Example 3.4.6 ($\cot(x)$)

$$
\begin{aligned}
y &= \cot(x) = \frac{\cos(x)}{\sin(x)} \\
y' &= \frac{(\sin(x)(-\sin(x))) - (\cos(x)\cos(x))}{\sin^2(x)} \\
&= \frac{-\sin^2(x) - \cos^2(x)}{\sin^2(x)} \\
&= \frac{-1(\sin^2(x) + \cos^2(x))}{\sin^2(x)} \\
&= \frac{-1}{\sin^2(x)} = -\csc^2(x)
\end{aligned}
$$

Example 3.4.7 ($\csc(x)$)

$$
\begin{aligned}
y &= \csc(x) = \frac{1}{\sin(x)} \\
y' &= \frac{\sin(x)0 - 1(\cos(x))}{\sin^2(x)} \\
&= \frac{-\cos(x)}{\sin^2(x)} = \frac{-1}{\sin(x)} \cdot \frac{\cos(x)}{\sin(x)} \\
&= -\cot(x)\csc(x)
\end{aligned}
$$

Example 3.4.8 ($\sec(x)$)

$$
\begin{aligned}
y &= \sec(x) = \frac{1}{\cos(x)} \\
y' &= \frac{\cos(x)0 - -\sin(x)(1)}{\cos^2(x)} \\
&= \frac{\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos(x)} \cdot \frac{1}{\cos(x)} \\
&= \sec(x)\tan(x)
\end{aligned}
$$

Definition 3.4.1: Trig Derivatives

y	y'
$\sin(x)$	$\cos(x)$
$\cos(x)$	$-\sin(x)$
$\tan(x)$	$\sec^2(x)$
$\cot(x)$	$-\csc^2(x)$
$\csc(x)$	$-\cot(x)\csc(x)$
$\sec(x)$	$\sec(x)\tan(x)$

Example 3.4.9 (Finding the equation of the tangent line)

$$y = 2x\sin(x); \left(\frac{\pi}{2}, \pi\right)$$
$$m = y' = 2x\cos(x) + 2\sin(x)$$
$$= 2\left(\frac{\pi}{2}\right)\cos\frac{\pi}{2} + 2\sin\frac{\pi}{2}$$
$$= \pi(0) + 2(1)$$
$$= 2$$
$$y - \pi = 2\left(x - \frac{\pi}{2}\right)$$

3.5 Chain Rule

Example 3.5.1 (Using the chain rule on a simple function)

$$f \circ g = \sqrt{x^2+1}$$
$$f(x) = \sqrt{x}$$
$$u^{\frac{1}{2}} = \frac{1}{2}u^{-\frac{1}{2}}$$
$$g(x) = x^2 + 1$$
$$y = x^2 + 1$$
$$du = 2x$$

$$F'(x) = \frac{1}{2}u^{-\frac{1}{2}}(2x)$$
$$= \frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)$$

Definition 3.5.1: Chain Rule

- Take the derivatives of all the embedded functions
- Chain them together through multiplication
- Name the inside function u

Example 3.5.2

$$y = (x^3 - 1)^{100} \longrightarrow u^{100}$$
$$u = x^3 - 1$$
$$du = 3x^2$$
$$y' = 100u^{99}(3x^2)$$
$$y' = 100(x^3 - 1)^{99}(3x^2)$$

Example 3.5.3

$$y = \frac{1}{\sqrt[3]{x^2 + x + 1}} = u^{-\frac{1}{3}}$$
$$u = x^2 + x + 1$$
$$du = 2x + 1$$
$$f'(x) = -\frac{1}{3}u^{-\frac{4}{3}}du$$
$$-\frac{1}{3}(x^2 + x + 1)(2x + 1)$$

Example 3.5.4

$$g(t) = \left(\frac{t-2}{2t+1}\right)^9 = u^9$$
$$u = \frac{t-2}{2t+1}$$
$$du = \frac{(2t+1)(1) - (t-2)(2)}{(2t+1)^2}$$
$$g'(t) = 9\left(\frac{t-2}{2t+1}\right)^8 \left(\frac{(2t+1)(1) - (t-2)(2)}{(2t+1)^2}\right)$$

Example 3.5.5

$$y = \cos 3x = \cos u$$
$$u = 3x$$
$$du = 3$$
$$y' = -3\sin 3x$$

Example 3.5.6

$$\cos(\tan x) = \cos u$$
$$u = \tan x$$
$$du = \sec^2 x$$
$$y' = (-\sin(\tan x))(\sec^2 x)$$

Example 3.5.7

$$y = \sin(\cos(\tan x)) = \sin u$$
$$y' = \cos(\cos(\tan x))du$$
$$y' = \cos(\cos(\tan x))(-\sin(\tan x))$$

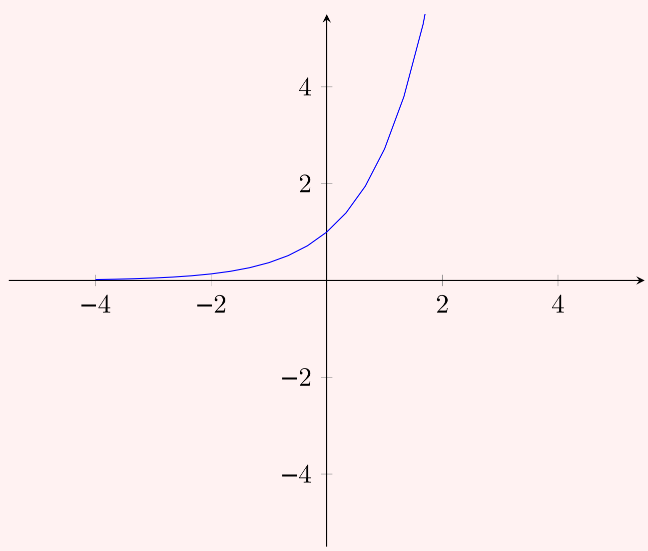
Example 3.5.8

$$y = \sin^2 x$$
$$u = \sin x$$
$$du = \cos x$$
$$y' = 2u du$$
$$y' = 2\sin x \cos x$$

Definition 3.5.2: Derivative of $y = e^x$

The tangent line at any point will grow exponentially with the graph

$$y = e^u$$
$$y' = e^u du$$
$$y' = e^x$$



Example 3.5.9

$$y = e^{2x}$$
$$u = 2x, du = 2$$
$$y' = 2e^{2x}$$

Example 3.5.10

$$y = e^{x^2}$$
$$u = x^2, du = 2x$$
$$y' = 2xe^{x^2}$$

Example 3.5.11

$$y = e^{\cos x}$$
$$u = \cos x, du = -\sin x$$
$$y' = -\sin(x)e^{\cos x}$$

Example 3.5.12

$$y = \frac{e^{2x}}{e^{2x}+1}$$
$$y' = \frac{(e^{2x}+1)(2e^{2x}) - (e^{2x})(2e^{2x})}{(e^{2x}+1)^2}$$

Definition 3.5.3: Derivative of $\ln x$

The derivative of $y = \ln x$ is $\frac{1}{x}$

$$y = \ln u \longrightarrow y' = \frac{1}{u}du = \frac{du}{u}$$

Example 3.5.13

$$y = \ln(x^2)$$
$$u = x^2, du = 2x$$
$$y' = \frac{1}{x^2}(2x) = \frac{2x}{x^2} = \frac{2}{x}$$

Example 3.5.14

$$y = \ln(|\tan x|)$$
$$u = \tan x, du = \sec^2 x$$
$$\frac{1}{\tan x}\sec^2 x = \frac{\sec^2 x}{\tan x}$$

Example 3.5.15

$$y = \ln\left|\frac{x+1}{\sqrt{x+2}}\right|$$

$$y' = \frac{1}{\frac{x+1}{\sqrt{x+2}}} \left(\frac{\sqrt{x+2}(1) - (x+1)\frac{1}{2}(x+2)^{-\frac{1}{2}}}{x+2} \right)$$

$$\frac{1}{\frac{x+1}{\sqrt{x+2}}} = \frac{\sqrt{x+2}}{x+1}$$

3.6 Implicit Differentiation

Example 3.6.1 (Solving for y)

$$x^2 + y^2 = 25$$
$$2x + 2y\frac{dy}{dx} = 0$$
$$2y\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = \frac{-2x}{2y} = -\frac{x}{y}$$

Example 3.6.2 (Example with product rule)

$$x^3 + y^3 = 6xy$$

($6xy$ is product rule, $\frac{dy}{dx} = y'$)

$$3x^2 + 3y^2\frac{dy}{dx} = 6x\frac{dy}{dx} + 6y$$
$$3y^2\frac{dy}{dx} - 6x\frac{dy}{dx} = 6y - 3x^2$$
$$\frac{dy}{dx}(3y^2 - 6x) = 6y - 3x^2$$
$$\frac{dy}{dx} = \frac{6y - 3x^2}{3y^2 - 6x}$$

Example 3.6.3 (Example with trig)

$$\sin(x+y) = y^2 \cos x$$
$$\cos(x+y)\left(1 + \frac{dy}{dx}\right) = y^2(-\sin(x)) + 2y\frac{dy}{dx}\cos(x)$$
$$\cos(x+y) + \cos(x+y)\frac{dy}{dx} = -y^2 \sin x + 2y\cos(x)\frac{dy}{dx}$$
$$\cos(x+y) + y^2 \sin x = -\cos(x+y)\frac{dy}{dx} + 2y\cos(x)\frac{dy}{dx}$$
$$\cos(x+y) + y^2 \sin x = \frac{dy}{dx}(2y\cos x - \cos(x+y))$$
$$\frac{\cos(x+y) + y^2 \sin x}{2y\cos x - \cos(x+y)} = \frac{dy}{dx}$$

Example 3.6.4 (Finding the second derivative)

Find the first derivative

$$x^4 + y^4 = 16$$

$$4x^3 + 4y^3 \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{4x^3}{4y^3} = -\frac{x^3}{y^3}$$

Start evaluating the second derivative

$$y'' = \frac{d^2y}{dx^2} = \frac{y^3(-3x^2) - (-x^3)(3y^2 \frac{dy}{dx})}{y^6}$$

$$y'' = \frac{-3x^2y^3 + 3x^3y^2 \frac{dy}{dx}}{y^6}$$

Substitute out $\frac{dy}{dx}$ for the first derivative that we calculated

$$= \frac{-3x^2y^3 + 3x^3y^2 \left(-\frac{x^3}{y^3}\right)}{y^6}$$

$$= \frac{-3x^2y^3 - \frac{3x^6}{y}}{y^6}$$

$$= \frac{\frac{-3x^2y^4 - x^6}{y}}{y^6} = \frac{-3x^2(y^4 - x^4)}{y^7}$$

$y^4 - x^4$ can be simplified to 16 since this was the original equation

$$x^4 + y^4 = 16, y'' = \frac{(-3x^2)(16)}{y^7}$$

3.7 Rates of Change - Particle Motion

Example 3.7.1

$$s(t) = t^3 - 6t^2 + 9t$$

1. Find the velocity at time t

   Velocity is rate of change in position - derivative

$$v(t) = s'(t) = 3t^2 - 12t + 9$$

2. What is the velocity after 2 seconds?

$$v(2) = 3(2)^2 - 12(2) + 9 = -3\text{m/s}$$

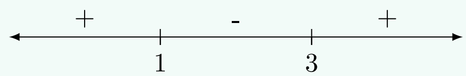
3. When is the particle at rest?

$$0 = 3t^2 - 12t + 9$$

$$0 = 3(t^2 - 4t + 3)$$
$$0 = 3(t-1)(t-3)$$
$$t = 3, 1$$

4. When is the particle moving forward?

$$(t-1)(t-3) > 0$$



$$(-\infty, 1) \cup (3, \infty)$$

5. Find the total distance travelled by the particle during the first 5 seconds

$$s(1) - s(0) = 4 - 0 = 4\text{m}$$
$$s(3) - s(1) = 0 - 4 = |-4| = 4\text{m}$$
$$s(5) - s(3) = 20 - 0 = 20\text{m}$$
$$28\text{m}$$

6. Find the displacement of the particle in the first 5 seconds

$$s(5) - s(0) = 20 - 0 = 20\text{m}$$

7. Find the acceleration at time t

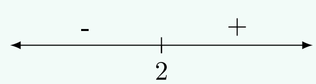
$$a(t) = s''(t) = v'(t) = 6t - 12$$

8. What is the acceleration after 4 seconds?

$$a(4) = 6(4) - 12 = 12\text{m/s}^2$$

9. When is acceleration positive?

$$6t - 12 = 0$$
$$t = 2$$



$$(2, \infty)$$

10. When is the particle speeding up? (NOT THE SAME QUESTION)
    When velocity and acceleration have the same sign (look at cut point graphs)

$$(1, 2) \cup (3, \infty)$$

11. When is the particle slowing down?
    When velocity and acceleration have different signs

$$(-\infty, 1) \cup (2, 3)$$

Note:-

Applications of the derivative (Page 174)

- Linear density
  - The linear density is the rate of change of mass with respect to length
- Electric Current
  - The rate of change in which charge flows through a surface
- Power (rate at which work is done)
- Rate of heat flow
- Temperature gradient (change of temperature with respect to position)
- Rate of decay of radioactive substances
- Instantaneous rate of reaction
- Isothermal compressibility
- Instantaneous rate of growth
- Blood flow in veins
- Marginal cost

3.8 Related Rates (of change)

Question 2

Air is being pumped into a spherical balloon so that its volume increases at a rate of 100 cm^3/s. How fast is the radius of the balloon increasing when the diameter is 50 cm?

Solution:

Note:-

We will need to formula for the volume of a sphere

$$V = \frac{4}{3}\pi r^3$$

We know the derivative of the volume with respect to time

$$\frac{dV}{dt} = 100\text{cm}^3/\text{s}$$

We are looking for the derivative of the radius with respect to time
$\frac{dr}{dt}$ =? at the instant that $d = 50$cm

Take the derivative of the formula for the volume of a sphere

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

Substitute the rate of volume increasing ($\frac{dV}{dt}$) and the radius

$$100 = 4\pi(25)^2 \frac{dr}{dt}$$

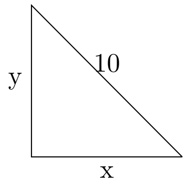
$$\frac{1}{25\pi}\text{cm/s} = \frac{dr}{dt}$$

Note:-

IMPORTANT: Units are required, not optional

Question 3

A ladder 10 ft long rests against a vertical wall. If the bottom of the ladder slides away from the wall at a rate of 1 ft/s, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 6 ft from the wall?



Solution:

Use the pythagorean theorem to set up what we know

$$x^2 + y^2 = 10^2$$

$$\frac{dx}{dt} = 1\text{ft/s}$$

We are looking for how fast the *top* (y) of the ladder is sliding away from the wall at $x = 6$.

$$\frac{dy}{dt} = ?$$

Take the derivative of the formula we know to find a substitute. Since the independent variable is time (t) and not x, we will need to take $\frac{dx}{dt}$ as well as $\frac{dy}{dt}$.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$2(6)(1) + 2y\frac{dy}{dt} = 0$$

We cannot solve for $\frac{dy}{dt}$ because of the unknown y. We can solve the initial equation for y at $x = 6$.

$$6^2 + y^2 = 100$$

$$y^2 = 64$$

$$y = 8$$

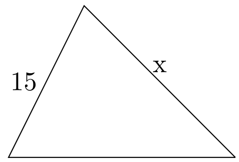
Now, we can plug in y

$$12 + 16\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = -\frac{3}{4}\text{ft/s}$$

Question 4

Two sides of a triangle have lengths 12m and 15m. The angle between them is increasing at a rate of $2°$/min. How fast is the length of the third side increasing when the angle between the sides of fixed length is $60°$

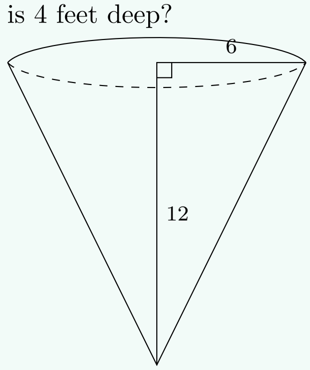


Solution:

$$\frac{d\theta}{dt} = 2^\circ/\text{min}$$
$$\theta = 60^\circ$$
$$\frac{dc}{dt} = ?$$
$$c^2 = a^2 + b^2 - 2ab\cos\theta$$
$$c^2 = 12^2 + 15^2 - 2(12)(15)\cos\theta$$
$$c^2 = 369 - 360\cos\theta$$
$$2c\frac{dc}{dt} = 360\sin\theta\frac{d\theta}{dt}$$
$$c\frac{dc}{dt} = 180\sin\theta(2)$$
$$\frac{dc}{dt} = \frac{360\left(\frac{\sqrt{3}}{2}\right)}{\sqrt{189}} = \frac{180\sqrt{3}}{\sqrt{189}}$$

Example 3.8.1

Water is pouring into a conical cistern at the rate of 8 cubic feet per minute. If the height of the cistern is 12 feet and the radius of its circular opening is 6 feet, how fast is the water level rising when the water is 4 feet deep?



$$\frac{dV}{dt} = 8\text{ft}^2/\text{min}$$
$$\frac{dh}{dt} = ?@h = 4$$
$$V = \frac{1}{3}\pi r^2 h$$

Solve for r

$$\frac{6}{12} = \frac{r}{h}$$
$$\frac{12r}{12} = \frac{6h}{12}$$
$$r = \frac{1}{2}h = \frac{h}{2}$$

Plug in r

$$V = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$$

$$V = \frac{1}{12}\pi h^3$$

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt}$$

$$8 = \frac{\pi}{4}(16)\frac{dh}{dt}$$

$$\frac{8}{4\pi} = \frac{4\pi}{4\pi}\frac{dh}{dt}$$

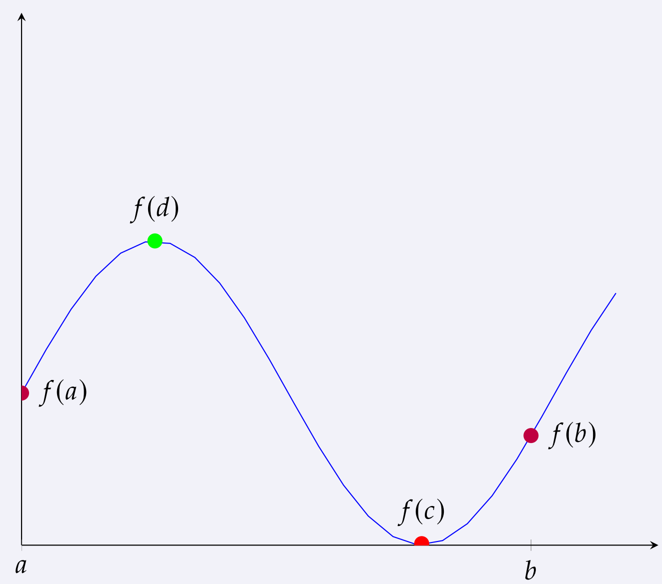
$$\frac{dh}{dt} = \frac{2}{\pi}\text{ft/min}$$

Chapter 4

4.1 Mix/max

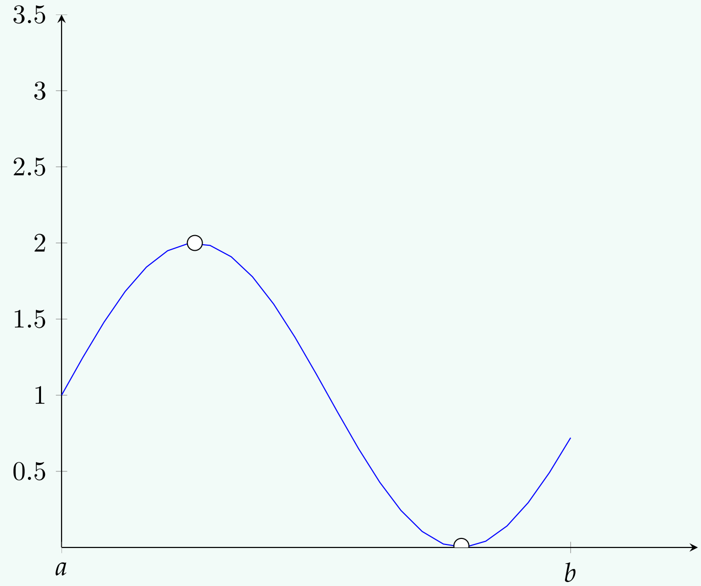
Theorem 4.1.1 Extreme Value Theorem

If f is continuous in $[a, b]$, then
$\exists$ absolute maximum value of f over $[a, b]$
$\exists$ absolute minimum value of f over $[a, b]$
$\exists c, d \in [a, b] : f(c) \leqslant f(x) \leqslant f(d)$ for all $x \in [a, b]$

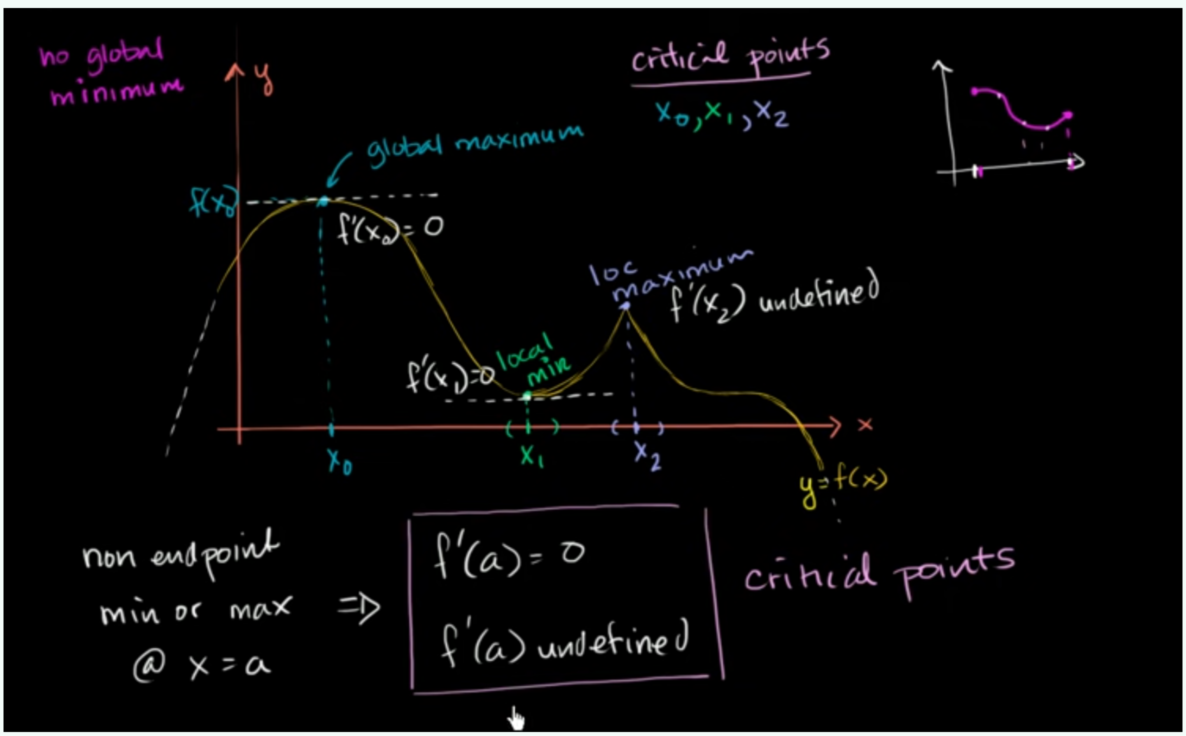


Example 4.1.1 (Why the function must be continuous)

As $x \longrightarrow$ max, you will get closer and closer but never actually hit the max



Example 4.1.2 (Critical points)



Example 4.1.3 (Finding critical points and mins/maxs)

Since critical points have a tangent line that has a slope equal to 0 or undefined, set the derivative of a function equal to zero to find the minima and maxima. Then, set up a cut points number line and plug the interval points into the derivative to see if it's positive or negative. When it changes from a positive to a negative, there is a max, and when it changes from a negative to a positive, there's a min.

Example 4.1.4 (Finding absolute min/max)

You are given an interval with endpoints. Take the derivative and set it equal to zero and solve to find the critical numbers. Instead of setting up intervals, set up an x/y table that include both the endpoints and the critical numbers, and plug the x values into the ORIGINAL equation to find the y values.

4.3 How derivatives affect the shape of a graph

Definition 4.3.1

1. $f(x)$ is increasing if $f'(x)$ is in $y > 0$, and decreasing if $f'(x)$ is in $y < 0$
2. $f(x)$ has local maxima if $f'(x)$ is decreasing through $x = 0$, and local minima if $f'(x)$ is increasing through $x = 0$
3. Find $f''(x)$ to determine when $f(x)$ is concave up and concave down. Set $f''(x)$ equal to zero and solve to find cutpoints. Find when concave upwards and downwards by plugging in numbers. Given a graph, it is CU up $f'(x)$ is increasing on a curve, and CD of $f'(x)$ is decreasing on a curve.

4.4 Limits at Infinity

$$\lim_{x\to\infty} f(x)$$

gives horizontal asymptotes

Definition 4.4.1: Finding horizontal asymptotes

Must be RATIONAL (fraction)

1. Greatest power in top: no HA
2. Greatest power in bottom: HA: $y = 0$
3. Greatest power matches in top and bottom: HA: $y =$ coefficients of highest powers

Example 4.4.1

$$\lim_{x\to\infty} \frac{3x^2 - x - 2}{5x^2 + 4x + 1} = \frac{3}{5}$$

This means that there is a horizontal asymptote at $y = \frac{3}{5}$

Showing work

$$\begin{aligned}
&= \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} - \frac{x}{x^2} - \frac{2}{x^2}}{\frac{5x^2}{x^2} + \frac{4x}{x^2} + \frac{1}{x^2}} \\
&= \lim_{x\to\infty} \frac{3 - \frac{1}{x} - \frac{2}{x^2}}{5 + \frac{4}{x} + \frac{1}{x^2}} \\
&= \frac{3 - 0 - 0}{5 + 0 + 0} \\
&= \frac{3}{5}
\end{aligned}$$

Note:-

In the formal proof of a limit, you'd need to put the limit notation in front of each of those 6 parts

Example 4.4.2

$$\lim_{x\to\infty} \frac{\sqrt{2x^2+1}}{3x-5} = \frac{\sqrt{2}}{3}$$

Example 4.4.3

$$\lim_{x\to\infty} \sqrt{x^2+1} - x$$

Multiply by the conjugate

$$\lim_{x\to\infty} \sqrt{x^2+1} - x \cdot \frac{\sqrt{x^2+1}+x}{\sqrt{x^2+1}+x}$$
$$= \lim_{x\to\infty} \frac{x^2+1-x^2}{\sqrt{x^2+1}+x}$$
$$= \lim_{x\to\infty} \frac{1}{\sqrt{x^2+1}+x} = 0$$

Limit is zero because highest power is in denominator

Example 4.4.4

$$\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}+x}$$
$$\lim_{x\to\infty} \frac{x}{x+1+x} = \lim_{x\to\infty} \frac{x}{2x+1} = \frac{1}{2}$$

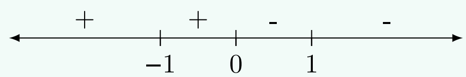
4.5 Sketching Curves

Example 4.5.1

$$y = \frac{2x^2}{x^2-1}$$

- H.A.: $y = 2$
- V.A.: $x = 1, x = -1$
- S.A.: none (there is a horizontal asymptote)
- y-intercept: $(0, 0)$
- x-intercept: $(0, 0)$
-

$$y' = \frac{(x^2-1)4x - 2x^2(2x)}{(x^2-1)^2}$$
$$\frac{\cancel{4x^2} - 4x - \cancel{4x^3}}{(x^2-1)^2} = 0$$



There is a max going between $(-1, 0)$ and $(0, 1)$

- $$y'' = \frac{(x^2-1)^2(-4) - (-4x)2(x^2-1)(2x)}{(x^2-1)^4}$$

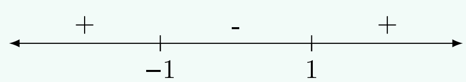
$$\frac{-4(x^2-1)\left[(x^2-1)\cancel{(-4)} - (\cancel{-4}x)2\cancel{(x^2-1)}(2x)\right]}{(x^2-1)^4}$$

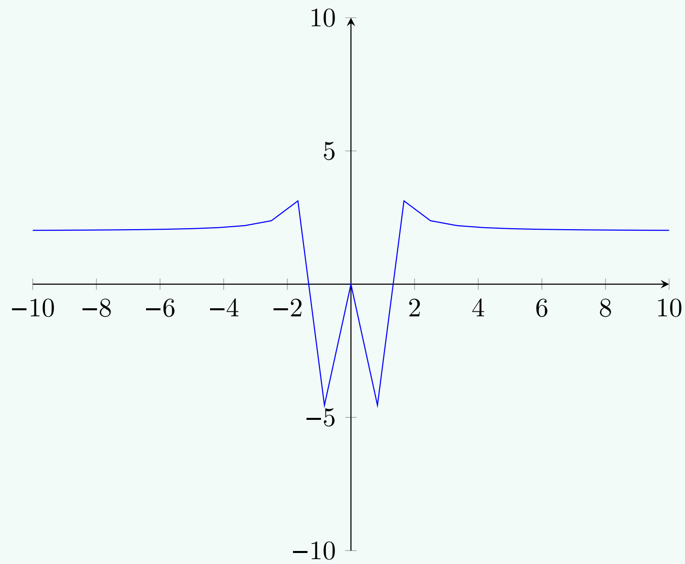
$$\frac{-4\left[x^2-1-(4x^2)\right]}{(x^2-1)^3}$$

$$\frac{(-4(-1))(3x^2+1)}{(x^2-1)^3}$$

$$\frac{4(3x^2+1)}{(x^2-1)^3} = 0$$

(no critical numbers because we have imaginary; only use asymptotes)





Example 4.5.2

$$y = \sqrt{x^2+x} - x$$

1. No vertical asymptote (no denominator)
2. H.A. is found by taking the limit as $x \to \infty$. Multiply by conjugate to make fraction

$$\lim_{x\to\infty} \sqrt{x^2+x} - x \cdot \frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}$$

$$= \lim_{x\to\infty} \frac{\cancel{x^2} + 1x \cancel{- x^2}}{\sqrt{1x^2 + x} + 1x} = \frac{1}{2}$$

$$y = \frac{1}{2}$$

3. Y-Intercept is found by plugging in 0 for x

$$\sqrt{0^2 + 0} - 0 = 0$$

$$\text{Y-Intercept: } (0, 0)$$

4. X-Intercept is found by setting y equal to 0

$$0 = \sqrt{x^2 + x} - x$$

$$x = \sqrt{x^2 + x}$$

$$x^2 = x^2 + x$$

$$0 = x$$

$$\text{X-Intercept: } (0, 0)$$

5. Take the derivative to find mins/maxs

$$y' = \frac{1}{2}(x^2 + x)^{-\frac{1}{2}}(2x + 1) - 1$$

$$= \frac{2x + 1}{2\sqrt{x^2 + x}} - 1$$

$$= \frac{2x + 1}{2\sqrt{x^2 + x}} - \frac{2\sqrt{x^2 + x}}{2\sqrt{x^2 + x}}$$

6. Set derivative equal to zero to solve for cut points

$$\frac{2x + 1 - 2\sqrt{x^2 + x}}{2\sqrt{x^2 + x}} = 0$$

$$2x + 1 - 2\sqrt{x^2 + x} = 0$$

$$2x + 1 = 2\sqrt{x^2 + x}$$

Get rid of root by squaring both sides

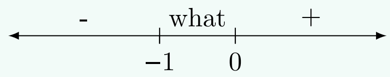
$$(2x + 1)^2 = (2\sqrt{x^2 + x})^2$$

$$4x^2 + 4x + 1 = 4(x^2 + x)$$

$$4x^2 + 4x + 1 = 4x^2 + 4x$$

$$1 = 0$$

No solution means that there are no critical numbers, so we cant have cut points. Our interval is $(-\infty, \infty)$ If we plug in zero, our denominator evaluates to zero. This means that the derivative doesn't exist at zero, and we can use that as a cut point Once we plug in -1, we find that the derivative also doesn't exist at -1 and we need to make that a cut point. Once we try to plug something in for $(-1, 0)$, we find that we get an imaginary number. We do not know what is happening in this interval

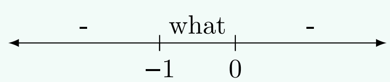


7. Now, take the second derivative

$$y' = (x^2+x)^{-\frac{1}{2}}\left(x+\frac{1}{2}\right)-1$$

$$\begin{aligned} y'' &= \left(x^2+x\right)^{-\frac{1}{2}}(1) + \left(x+\frac{1}{2}\right)\frac{-1}{2}\left(x^2+x\right)^{-\frac{3}{2}}(2x+1) = 0 \\ &= \left(x^2+x\right)^{-\frac{3}{2}}\left[x^2+x+\left(-\frac{1}{2}x-\frac{1}{4}\right)(2x+1)\right] \\ &= \left(x^2+x\right)^{-\frac{3}{2}}\left[x^2+x-x^2-\frac{1}{2}x-\frac{1}{2}x-\frac{1}{4}\right] \\ &= \left(x^2+x\right)^{-\frac{3}{2}}\left[x^2+x-x^2-x-\frac{1}{4}\right] \\ &= \left(x^2+x\right)^{-\frac{3}{2}}\left[\cancel{x^2}+\cancel{x}\cancel{-x^2}\cancel{-x}-\frac{1}{4}\right] \\ &= \frac{-1}{4\sqrt{(x^2+x)^3}} = 0 \\ -1 &= 0 \end{aligned}$$

Doesn't exist at 0. After plugging in -1, we find that it doesn't exist there either



8. We know that the graph has a horizontal asymptote at $y = \frac{1}{2}$, we know that theres a point at $(0, 0)$, we know that it's coming down from $-\infty$, and it's going up towards ∞

9. If we take the limit as $x \to -\infty$, we find that we get $\sqrt{\infty - \infty}$, which is undefined. This means that in $(-\infty, 0)$ we do not have a horizontal asymptote.

4.6 Formulas

Definition 4.6.1: Linear Approximation (linearization) (of f at a)

$$L(x) = f(a) + f'(a)(x-a)$$

Derivative is slope of the tangent line (m). This equation could be written as

$$y = b + mx$$

or

$$y = mx + b$$

Example 4.6.1

$$f(x) = \sqrt{x+3} \text{ at } a = 1$$
$$f'(x) = \frac{1}{2}(x+3)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+3}}$$
$$f(1) = \sqrt{1+3} = 2$$
$$f'(1) = \frac{1}{2\sqrt{1+3}} = \frac{1}{4}$$
$$L(x) = 2 + \frac{1}{4}(x-1)$$

Definition 4.6.2: Differentials

$$y = \sqrt{x+3}$$
$$\frac{dy}{dx} = \frac{1}{2}(x+3)^{-\frac{1}{2}} = \frac{1}{2\sqrt{x+3}}$$
$$dy = \frac{1}{2\sqrt{x+3}}dx$$

Definition 4.6.3: Mean Value Theorem (MVT)

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$f(x)$ must be continuous and differentiable on the interval $[a,b]$ This equation is similar to $\frac{y_2-y_1}{x_2-x_1}$

1. State $f(x)$ is continuous and differentiable on $[a,b]$
2. Find $f(b)$
3. Find $f(a)$
4. Find $f'(c)$
5. Substitute and solve for c

Example 4.6.2

$$f(x) = x^3 - x \text{ on the interval } [0,2]$$
$$f(x) \text{ is both continuous and differentiable on } [0,2]$$
$$f(b) = 2^3 - 2 = 6$$
$$f(a) = 0$$
$$f'(c) = 3c^2 - 1$$
$$3c^2 - 1 = \frac{6-0}{2-0}$$
$$3c^2 - 1 = 3$$
$$c^2 = \frac{4}{3} = \pm\sqrt{\frac{4}{3}}$$

The negative one isn't on the interval, so the answer is $\sqrt{\frac{4}{3}}$

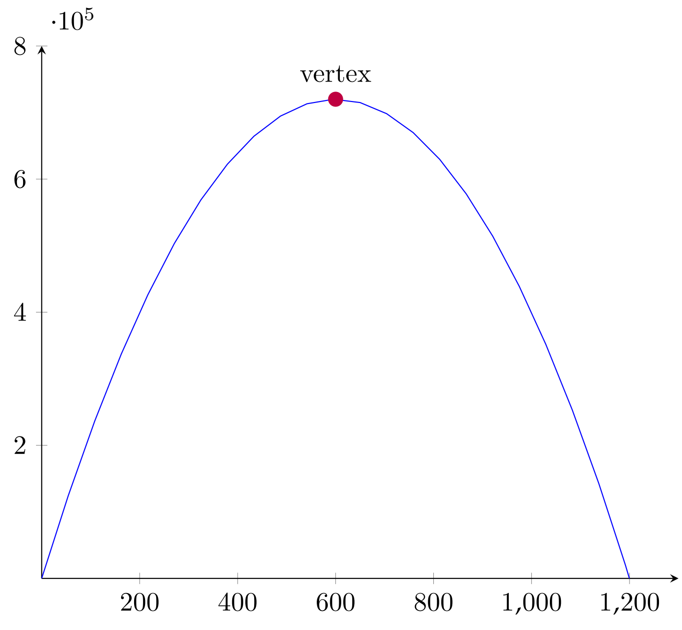
4.7 Optimization

Question 5

A farmer has 2400 ft of fencing and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?

- 2400ft fencing
- $A = LW$
- $P = L + 2W$
- $P = 2400$

$$2400 = L + 2W$$
$$L = 2400 - 2W$$
$$A(W) = (2400 - 2W)(W)$$
$$A(W) = 2400W - 2W^2$$



To find the max (vertex), find the derivative and set equal to zero (where the tangent line's slope is zero)

$$A'(W) = 2400 - 4W$$

$$2400 - 4W = 0$$
$$2400 = 4W$$
$$600 = W$$

Note:-

$$\text{Profit} = \text{Revenue} - \text{Cost}$$
$$P(x) = R(x) - C(x)$$

4.9 Integrals

4.9.1 Indefinite Integrals

$$y = 3x^2, y' = 6x$$
$$y = 3\sqrt{x}, y' = \frac{3}{2}x^{-\frac{1}{2}}$$
$$y = \frac{2}{x}, y' = \frac{-2}{x^2}$$

Example 4.9.1 (Undo the power rule)
Add one to power and divide by new power

$$y' = 3x^2, y = x^3 + c$$
$$y' = 3\sqrt{x} = 3x^{\frac{1}{2}}, y = 2x^{\frac{3}{2}} + c$$
$$y' = \frac{2}{x} = 2x^{-1}, y = 2\ln x + c$$

Example 4.9.2 (Constants)

$$y = 3x^2 + 1, y' = 6x$$
$$y = 3x^2 + 2, y' = 6x$$
$$y = 3x^2 - 3, y' = 6x$$
$$y = 3x^2 + \frac{1}{2}, y' = 6x$$

Definition 4.9.1: Indefinite Integrals

You cannot account for constants in the derivative without additional information since the derivative of a constant is 0. To solve this, we will always need to add a $+c$ to the end of the function.

Definition 4.9.2: Notation for integrals

$\int dx$ - This means integral with respect to x

$\int$	y	y'
$-\cos(x)$	$\sin(x)$	$\cos(x)$
$\sin(x)$	$\cos(x)$	$-\sin(x)$
$\ln(\lvert\cos(x)\rvert)$	$\tan(x)$	$\sec^2(x)$
$\ln(\lvert\sin(x)\rvert)$	$\cot(x)$	$-\csc^2(x)$
$-\ln(\lvert\csc(x) + \cot(x)\rvert)$	$\csc(x)$	$-\cot(x)\csc(x)$
$\ln(\lvert\tan(x) + \sec(x)\rvert)$	$\sec(x)$	$\sec(x)\tan(x)$

Example 4.9.3 (Particular Solutions 1)

$$f'(x) = x\sqrt{x} \text{ and } f(1) = 2$$

$f(1) = 2$ is our additional information
Integrate $f'(x) = x\sqrt{x} = x^{\frac{3}{2}}$

$$f(x) = \frac{2}{5}x^{\frac{5}{2}} + c$$

$$\frac{2}{5}(1)^{\frac{5}{2}} + c = 2$$

$$\frac{2}{5} + c = 2$$

$$c = \frac{8}{5}$$

$$f(x) = \frac{2}{5}x^{\frac{5}{2}} + \frac{8}{5}$$

Example 4.9.4 (Particular Solutions 2)

$$f''(x) = 12x^2 + 6x - 4, f(0) = 4, f(1) = 1$$

Integrate

$$f'(x) = 4x^3 + 3x^2 - 4x + c$$

Integrate and add another variable

$$f(x) = x^4 + x^3 - 2x^2 + cx + d$$

Plug in 0 and solve for d

$$f(0) = 0^4 + 0^3 - 2(0)^2 + c(0) + d = 4$$

$$d = 4$$

Plug in d and 1 and solve for c

$$f(x) = x^4 + x^3 - 2x^2 + cx + 4$$

$$f(1) = 1^4 + 1^3 - 2(1)^2 + c(1) + 4 = 1$$

$$1 + 1 - 2 + c + 4 = 1$$

$$c = -3$$

Plug in c and d

$$f(x) = x^4 + x^3 - 2x^2 + 3x + 4$$

4.9.2 Definite Integrals

Definition 4.9.3: Definite Integrals

The answer is a number

$$\begin{aligned}\int_0^1 \left(4+3x^2\right) dx &= \left[4x+x^3\right]_0^1 \\ &= 4(1)+(1)^3-\left(4(0)+0^3\right) \\ &= 5\end{aligned}$$

Example 4.9.5

$$\begin{aligned}\int_1^4 \sqrt{x}dx = \int_1^4 x^{\frac{1}{x}}dx &= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4 \\ &= \frac{2}{3}(4)^{\frac{3}{2}} - \frac{2}{3}(1)^{\frac{3}{2}} \\ &= \frac{2}{3}(8) - \frac{2}{3} = \frac{16}{3} - \frac{2}{3} = \frac{14}{3}\end{aligned}$$

Example 4.9.6

$$\int_0^4 \sqrt{2x+1}dx$$

$$u = 2x+1$$

$$du = 2dx, \frac{1}{2}du = 2x$$

Note:-

Some people change the limits of integration to match u, but it's typically easier to just back substitute and use the limits of x

$$\frac{1}{2}\int u^{\frac{1}{2}}du$$

$$\frac{1}{2}\cdot\frac{2}{3}u^{\frac{3}{2}} = \left[\frac{1}{3}(2x+1)^{\frac{3}{2}}\right]_0^4$$

$$\frac{1}{3}\left[9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right] = \frac{1}{3}(27-1) = \frac{26}{3}$$

Example 4.9.7

$$\int_1^2 \frac{dx}{(3-5x)^2}$$

$$u = 3-5x$$

$$du = -5dx$$

$$-\frac{1}{5}du = dx$$

$$= -\frac{1}{5}\int u^{-2}du$$

$$= -\frac{1}{5}(-1)u^{-1} = \left[\frac{1}{5} \cdot \frac{1}{3-5x}\right]_1^2$$

$$= \frac{1}{5}\left(\frac{1}{3-10} - \frac{1}{3-5}\right) = \frac{1}{5}\left(\frac{5}{14}\right) = \frac{1}{14}$$

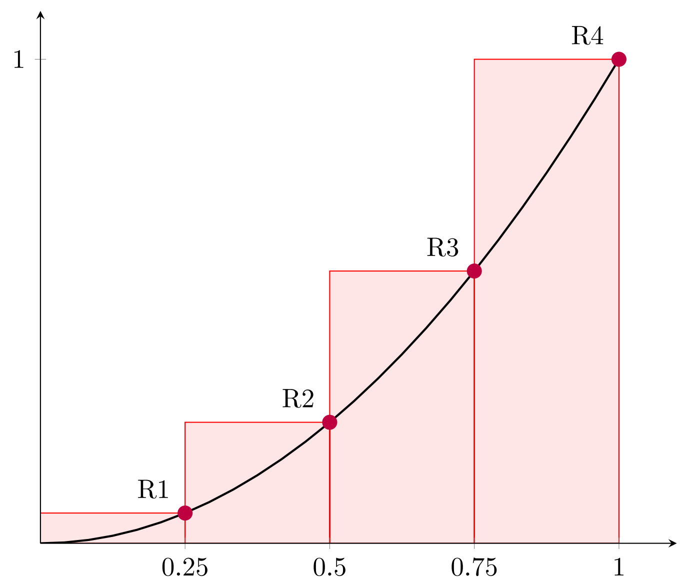
Chapter 5

5.1 Riemann Sums

$$y = x^2, [0, 1]$$

$$n = 4$$

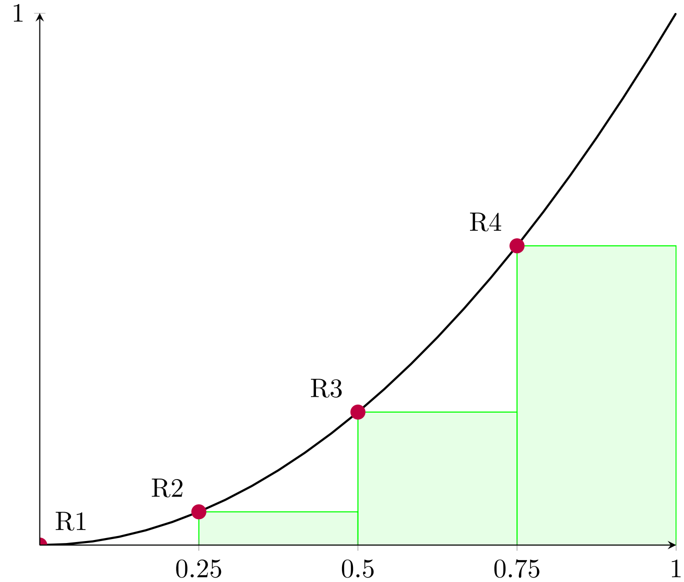
Width of each rectangle: $\frac{1}{4}$



Right endpoints

R1	$\frac{1}{4} \cdot \frac{1}{16}$
R2	$\frac{1}{4} \cdot \frac{4}{16}$
R3	$\frac{1}{4} \cdot \frac{9}{16}$
R4	$\frac{1}{4} \cdot \frac{16}{16}$

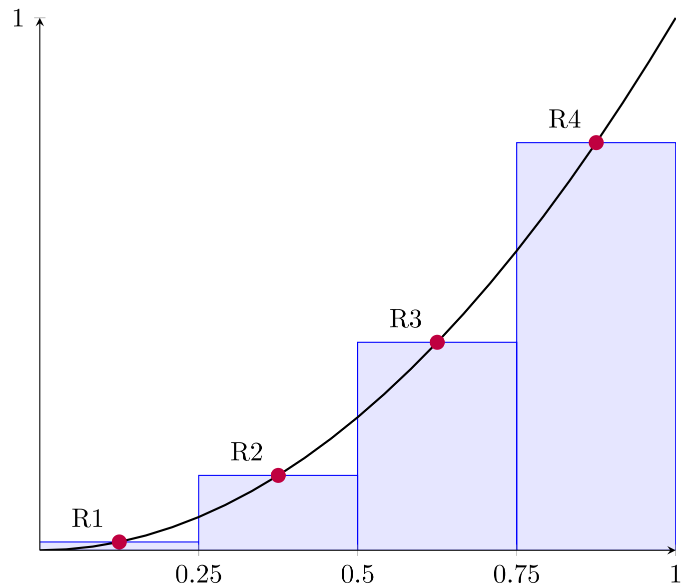
$\Sigma \frac{1}{4} \cdot \frac{30}{16} = \quad \frac{15}{32}$sq. units



Left Endpoints

R1	$\frac{1}{4} \cdot 0$
R2	$\frac{1}{4} \cdot \frac{1}{16}$
R3	$\frac{1}{4} \cdot \frac{4}{16}$
R4	$\frac{1}{4} \cdot \frac{9}{16}$

$\Sigma \frac{1}{4} \cdot \frac{14}{16} =$ $\frac{7}{32}$sq. units

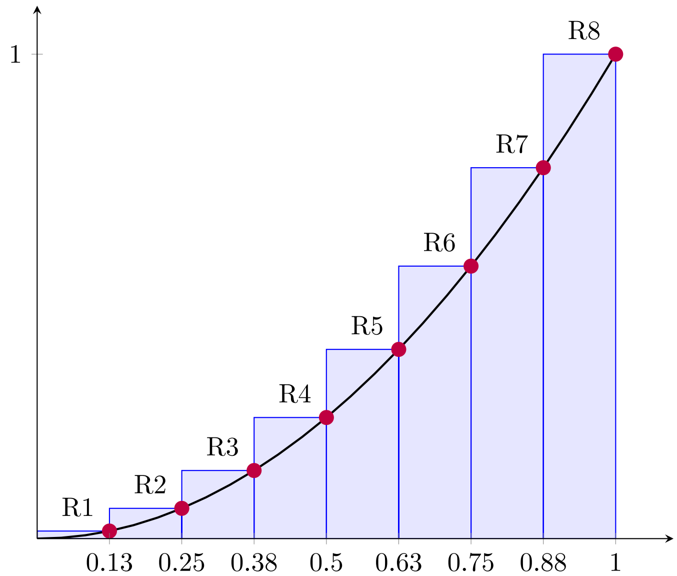


Midpoints

Note:-
Distance travelled is equal to the space under the graph of the velocity function

R1	$\frac{1}{4} \cdot \frac{1}{64}$
R2	$\frac{1}{4} \cdot \frac{9}{64}$
R3	$\frac{1}{4} \cdot \frac{25}{64}$
R4	$\frac{1}{4} \cdot \frac{49}{64}$

$\Sigma \frac{1}{4} \cdot \frac{84}{64} =$ $\frac{21}{16}$sq. units



R1	$\frac{1}{8} \cdot \frac{1}{64}$
R2	$\frac{1}{8} \cdot \frac{4}{64}$
R3	$\frac{1}{8} \cdot \frac{9}{64}$
R4	$\frac{1}{8} \cdot \frac{16}{64}$
R5	$\frac{1}{8} \cdot \frac{25}{64}$
R6	$\frac{1}{8} \cdot \frac{36}{64}$
R7	$\frac{1}{8} \cdot \frac{49}{64}$
R8	$\frac{1}{8} \cdot \frac{64}{64}$

$\Sigma \frac{1}{8} \cdot \frac{204}{64} = \frac{51}{128}$ $\frac{12\frac{3}{4}}{32}$sq. units

5.3 Fundamental Theorem of Calculus (FTC)

Theorem 5.3.1 FTC

Integrals and derivatives undo each other

Note:-

Calc II will need Page 314 Ex. 1

Conditions:

1. Continuous on $[a, b]$ - if there is a discontinuity on the interval, you can't just integrate
2. Differentiable on $[a, b]$

Example 5.3.1

Note:-

There must be a constant on the bottom, and a variable on the top of the integral

$$\frac{d}{dx} \int_0^x \sqrt{1+t^2}dt$$

Derivative of x is 1

$$= \sqrt{1 + x^2}$$

Example 5.3.2

$$\frac{d}{dx} \int_1^{x^4} \sec(t) dt$$

x^4 has its own derivative, $4x^3$

$$= 4x^3 \sec(x^4)$$

Example 5.3.3

Since we're not taking the derivative of an integral, it is not an FTC problem

$$\int_{-2}^{1} x^3 dx$$

$$= \left[\frac{1}{4}x^4\right]_{-2}^{1}$$

$$= \frac{1}{4}(1)^4 - \frac{1}{4}(-2)^4$$

$$= -\frac{15}{4}$$

Example 5.3.4

$$\int_{-1}^{3} \frac{1}{x^2} dx$$

Not continuous on the interval $[-1, 3]$ because when $x = 0$ it does not exist since x is in the denominator

5.4 More Particle Motion and Net Change Theorem

5.4.1 More Particle Motion

Definition 5.4.1

$f(t)$: position function
$f'(t)$: velocity function
$f''(t)$: acceleration function

Example 5.4.1

$$v(t) = t^2 - t - 6$$

1. Find displacement on $[1, 4]$

$$\int_1^4 (t^2 - t - 6) dt = \left[\frac{1}{3}t^3 - \frac{1}{2}t^2 - 6t\right]_1^4$$

$$= \left(\frac{1}{3}(4)^3 - \frac{1}{2}(4)^2 - 6(4)\right) - \left(\frac{1}{3}(1)^3 - \frac{1}{2}(1)^2 - 6(1)\right)$$

$$= \frac{1}{3}(64) - \frac{1}{2}(16) - 24 - \left(\frac{1}{3} - \frac{1}{2} - 6\right)$$

$$= \frac{64}{3} - 8 - 24 - \frac{1}{3} + \frac{1}{2} + 6$$

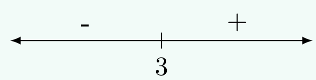
$$= -4\frac{1}{2}$$

2. Find the distance travelled on $[1, 4]$ We need to factor the velocity function to find the cut points

$$v(t) = t^2 - t - 6 = (t - 3)(t + 2) = 0$$

$$t = 3, \cancel{t = -2}$$

We dont use -2 since it's not in the interval of $[1, 4]$. Plug into the velocity function



Add the absolute value of the distance travelled negative to the distance travelled positive

$$\left|\int_1^3 v(t)dt\right| + \left|\int_3^4 v(t)dt\right|$$

$$= \left|\left[\frac{1}{3}t^3 - \frac{1}{2}t^2 - 6t\right]_1^3\right| + \left|\left[\frac{1}{3}t^3 - \frac{1}{2}t^2 - 6t\right]_3^4\right|$$

$$= \left|9 - \frac{9}{2} - 18 - \frac{1}{3} + \frac{1}{2} + 6\right| + \left|\frac{64}{3} - 8 - 24 - \left(9 - \frac{9}{2} - 18\right)\right|$$

$$= \left|-3 - 4 - \frac{1}{3}\right| + \left|-23 + \frac{64}{3} + \frac{9}{2}\right|$$

$$= \frac{83}{6}$$

Note:-

Except something is wrong, this isn't the correct answer

5.4.2 Net Change Theorem

Theorem 5.4.1 Net Change Theorem

Integral of a rate of change is the net change

$$\int_a^b F'(x)dx = F(b) - F(a)$$

5.5 U-Substitution (undoes chain rule)

Example 5.5.1

$$\int 2x\sqrt{1+x^2}dx$$

Indentify u

$$u = 1 + x^2$$

Find du

$$du = 2xdx$$

Substitute with u

$$\int u^{\frac{1}{2}} du$$

Undo power rule

$$\frac{2}{3}u^{\frac{3}{2}}$$

Backsubstitute

$$\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}} + c$$

Example 5.5.2

$$\int x^3 \cos(x^4+2)dx$$

$$u = x^4 + 2$$

$$du = 4x^3 dx$$

$$\frac{1}{4}du = x^3 dx$$

You can move constants out front when you substitute

$$\frac{1}{4}\int \cos u\, du$$

$$\frac{1}{4}\sin u$$

$$\frac{1}{4}\sin\left(1+x^2\right) + c$$

Example 5.5.3

$$\int \sqrt{2x+1}dx$$

$$u = 2x + 1$$

$$du = 2dx$$

$$\frac{1}{2}du = dx$$

$$\frac{1}{2}\int u^{\frac{1}{2}} du$$

$$\frac{1}{2}\frac{2}{3}u^{\frac{3}{2}}$$

$$\frac{1}{3}\left(2x+1\right)^{\frac{3}{2}}+c$$

Example 5.5.4

$$\int \frac{x}{\sqrt{1-4x^2}}dx$$

$$u = 1-4x^2$$

$$du = -8xdx$$

$$-\frac{1}{8}du = xdx$$

$$-\frac{1}{8}\int u^{-\frac{1}{2}}du$$

$$-\frac{1}{8}2u^{\frac{1}{2}}$$

$$-\frac{1}{4}\left(1-4x^2\right)^{\frac{1}{2}}+c$$

Example 5.5.5

$$\int \cos 5xdx$$

$$u = 5x$$

$$du = 5dx$$

$$\frac{1}{5}du = dx$$

$$\frac{1}{5}\int \cos(u)du$$

$$\frac{1}{5}\sin(u)$$

$$\frac{1}{5}\sin\left(5x\right)+c$$

Example 5.5.6

$$\int x^5\sqrt{1+x^2}dx$$

$$u = 1+x^2 \longrightarrow u-1 = x^2$$

$$x^4 = (u-1)^2$$

$$du = 2xdx$$

$$\frac{1}{2}du = xdx$$

We have $x \cdot x^4$ so that we can substitute, but we're left with x^4. We can rearrange the equation for u to solve for x^4

$$\frac{1}{2}\int u^{\frac{1}{2}} du x^4$$

$$\frac{1}{2}\int u^{\frac{1}{2}} du(u-1)^2$$

$$\frac{1}{2}\int u^{\frac{1}{2}}\left(u^2-2u+1\right) du$$

$$\frac{1}{2}\int \left(u^{\frac{5}{2}}-2u^{\frac{3}{2}}+u^{\frac{1}{2}}\right) du$$

$$\frac{1}{2}\left(\frac{2}{7}u^{\frac{7}{2}}-2\frac{2}{5}u^{\frac{5}{2}}+\frac{2}{3}u^{\frac{3}{2}}\right)$$

$$\frac{1}{7}\left(1+x^2\right)^{\frac{7}{2}}-\frac{2}{5}\left(1+x^2\right)^{\frac{5}{2}}+\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+c$$

Question 6: Question 29

$$\int \frac{x}{\sqrt[4]{x+2}}dx$$

$$u=x+2$$

$$x=u-2$$

$$du=dx$$

$$\int u^{-\frac{1}{4}}(u-2)du=\int \left(u^{\frac{3}{4}}-2u^{-\frac{1}{4}}\right) du$$

$$\frac{4}{7}u^{\frac{7}{4}}-2\frac{4}{3}u^{\frac{3}{4}}$$

$$\frac{4}{7}(x+2)^{\frac{7}{4}}-\frac{8}{3}(x+2)^{\frac{3}{4}}+c$$

Question 7: Question 25

$$\int \sec^3 x \tan x dx$$

y	y'
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$

$$\int \sec^2 x \sec x \tan x dx$$

$$u=\sec x$$

$$du=\sec x \tan x dx$$

$$\int u^2 du=\frac{1}{3}u^3$$

$$\frac{1}{3}\sec^3 x+c$$

Question 8: Question 17

$$\int \frac{\cos\sqrt{t}}{\sqrt{t}}dt$$

$$u = \sqrt{t} = t^{\frac{1}{2}}$$

$$du = \frac{1}{2}t^{-\frac{1}{2}}dt$$

$$2du = t^{-\frac{1}{2}}dt$$

$$2\int \cos u du$$

$$2\sin u$$

$$2\sin\left(\sqrt{t}\right) + c$$

Question 9

y	y'
$\tan(x)$	$\sec^2(x)$
$\sec(x)$	$\sec(x)\tan(x)$

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} \tan^3\theta d\theta$$

$$= \int \tan\theta\tan^2\theta d\theta$$

Note:-

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x$$

$$\tan^2 x = \sec^2 x - 1$$

$$\int \tan\theta\left(\sec^2\theta - 1\right)d\theta$$

Distribute

$$= \int \tan\theta\sec^2\theta d\theta - \int \tan\theta d\theta$$

You can now either set u to $\tan\theta$ or $\sec\theta$

$$u = \tan\theta$$

$$du = \sec^2\theta d\theta$$

$$= \left[\frac{1}{2}\tan^2\theta - -\ln|\cos\theta|\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}$$

$$= \left(\frac{1}{2}\tan^2\left(\frac{\pi}{6}\right) + \ln\left|\cos\frac{\pi}{6}\right|\right) - \left(\frac{1}{2}\tan^2\left(-\frac{\pi}{6}\right) + \ln\left|\cos -\frac{\pi}{6}\right|\right)$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{3}}\right)^2 + \ln\frac{\sqrt{3}}{2} - \frac{1}{2}\left(-\frac{1}{\sqrt{3}}\right)^2 - \ln\frac{\sqrt{3}}{2}$$

$$= \frac{1}{6} + \ln\frac{\sqrt{3}}{2} - \frac{1}{6} - \ln\frac{\sqrt{3}}{2} = 0$$

Question 10

$$\int_0^a x\sqrt{x^2+a^2}dx$$

$$u = x^2 + a^2$$

$$du = 2xdx$$

$$\frac{1}{2}du = xdx$$

$$= \frac{1}{2}\int u^{\frac{1}{2}}du$$

$$= \frac{1}{2}\frac{2}{3}u^{\frac{3}{2}}$$

$$\left[\frac{1}{3}\left(x^2+a^2\right)^{\frac{3}{2}}\right]_0^a = \frac{1}{3}\left(\left(2a^2\right)^{\frac{3}{2}} - \left(a^2\right)^{\frac{3}{2}}\right)$$

$$= \frac{1}{3}\left(2\sqrt{2}a^3 - a^3\right)$$

Note:-

If the constant is at the top and the variable is at the bottom, you can just swap them and put a negative out front

Exponential integrals

Definition 5.5.1: Integrals of exponentials

$$e^u = e^u du \longrightarrow \int e^u du = e^u + c$$

$$\ln u = \frac{1}{u}du \longrightarrow \int \frac{1}{u}du = \ln u + c$$

Example 5.5.7

$$\int x^2 e^{x^3} dx$$

$$u = x^3$$

$$du = 3x^2 dx$$

$$\frac{1}{3}du = x^2 dx$$

$$\frac{1}{3}\int e^u du$$

$$\frac{1}{3}e^{x^3} + c$$

Example 5.5.8

$$\int \frac{x}{x^2+1}dx$$
$$u = x^2 + 1$$
$$du = 2xdx$$
$$\frac{1}{2}du = xdx$$
$$\frac{1}{2}\int u^{-1}du$$
$$\frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|x^2+1| + c$$

Example 5.5.9

$$\int \tan x dx = \int \frac{\sin x}{\cos x}dx$$
$$u = \cos x$$
$$du = -\sin x dx$$
$$-\int \frac{1}{u}du$$
$$-\ln|\cos x| + c$$
$$= \ln|\cos x|^{-1} + c$$
$$= \ln|\frac{1}{\cos x}| + c$$
$$= \ln|\sec x| + c$$

Question 11

$$\int (e^x + e^{-x})^2\,dx =$$
$$\int \left(e^{2x} + 2 + e^{-2x}\right)dx$$
$$u_1 = 2x, du_1 = 2dx, dx = \frac{1}{2}du_1$$
$$u_2 = -2x, du_2 = -2dx, dx = \frac{-1}{2}du_2$$
$$\frac{1}{2}\int e^u du + 2\int dx + \frac{-1}{2}\int e^u du$$
$$\frac{1}{2}e^{2x} + 2x - \frac{1}{2}e^{-2x} + c$$

Question 12

$$\int \frac{e^{\frac{1}{x}}}{x^2}dx$$

$$u = \frac{1}{x} = x^{-1}$$
$$du = -x^{-2}dx = \frac{-1}{x^2}dx$$
$$-du = \frac{1}{x^2}dx$$
$$-\int e^u du$$
$$-e^{\frac{1}{x}} + c$$

Question 13

$$\int \frac{e^{\sqrt{x}}}{\sqrt{x}}$$
$$u = \sqrt{x} = x^{\frac{1}{2}}$$
$$du = \frac{1}{2}x^{-\frac{1}{2}}dx = \frac{1}{2} \cdot \frac{1}{\sqrt{x}}dx$$
$$2du = \frac{1}{\sqrt{x}}dx$$
$$2\int e^u du$$
$$2e^{\sqrt{x}} + c$$

Question 14

$$\int e^x \sin(e^x)dx$$
$$u = e^x$$
$$du = e^x dx$$
$$\int \sin(u)du$$
$$-\cos(e^x) + c$$

Question 15

$$\int \frac{\sin 2x}{1 + \cos^2 x}dx$$
$$u = 1 + \cos^2 x$$
$$du = -2\cos(x)\sin(x)dx$$
$$-du = 2\sin(x)\cos(x) = \sin 2x dx$$
$$-\int \frac{\sin 2x}{u}du$$
$$-\int \frac{1}{u}du = -\ln|1 + \cos^2 x| + c$$

Chapter 6

6.1 Area Between Curves

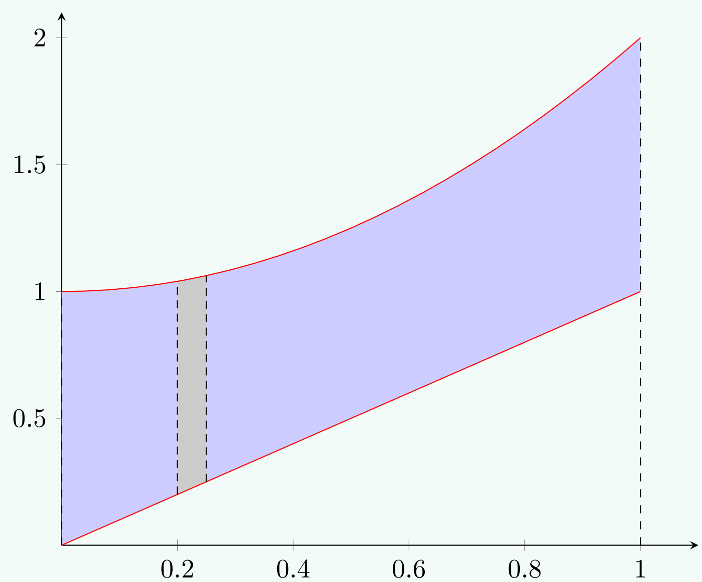
Definition 6.1.1

- Top curve minus bottom curve in $y =$ form

Example 6.1.1

Find the area of the region bounded above by $y = x^2 + 1$, bounded below by $y = x$, and bounded on the sides by $x = 0$ and $x = 1$

Since we're finding the area, we know that we need to integrate. Graph the functions to see what they look like.



Note:-

$$\frac{f(x+h) - f(x)}{h}$$

h represents change in x that's really close to 0 δ means change, so change in x is δx, or dx, which is the width of each arbitrary rectangle

Set up the integral

$$\int_0^1 x^2 + 1 - x dx$$

Top curve $(x^2 + 1)$ minus bottom curve (x) times the change in x (dx). The 0 and 1 represent how far the rectangles are stacking in the x direction (the left and right bounds)

Now we can solve

$$\int_0^1 x^2 + 1 - x dx$$

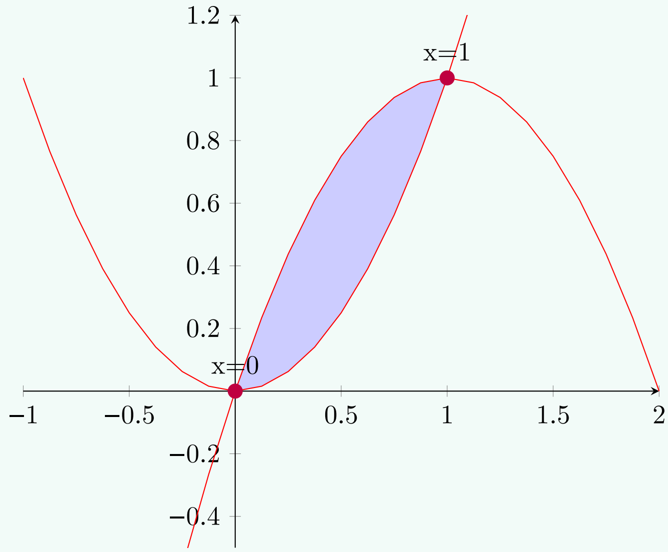
$$= \left[\frac{1}{3}x^3 + x - \frac{1}{2}x^2\right]_0^1$$

$$= \frac{1}{3} + 1 - \frac{1}{2} - 0$$

$$= \frac{5}{6}$$

Example 6.1.2

Find the area enclosed by $y = x^2$ and $y = 2x - x^2$
$2x - x^2 = x(2 - x)$ - intercepts are 2 and 0



Top curve is $y = 2x - x^2$ and bottom curve is $y = x^2$ We need to know the points of intersection (the bounds), so set the equations equal to each other $x^2 = 2x - x^2$

$$x^2 = 2x - x^2$$

$$2x - 2x = 0$$

$$2x(x - 1) = 0$$

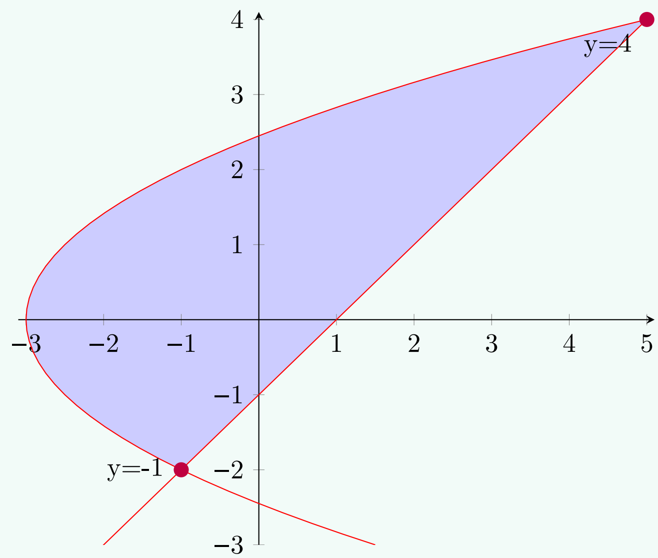
$$x = 0, x = 1$$

Construct the integral

$$\int_0^1 \left(2x - x^2 - x^2\right) dx$$

Example 6.1.3

Find the area bounded by $y = x - 1$ and $y^2 = 2x + 6$ $y^2 = 2x + 6$ is a sideways parabola



We need to draw the rectangles sideways so that the top and bottom curve doesn't change, there fore we have dy instead of dx, and we need to change the equations to $x =$ instead of $y =$. In this case, it's right curve minus left curve instead of top curve minus bottom curve

$$y = x - 1 \rightarrow x = y + 1$$
$$y^2 = 2x + 6 \rightarrow x = \frac{1}{2}y^2 - 3$$

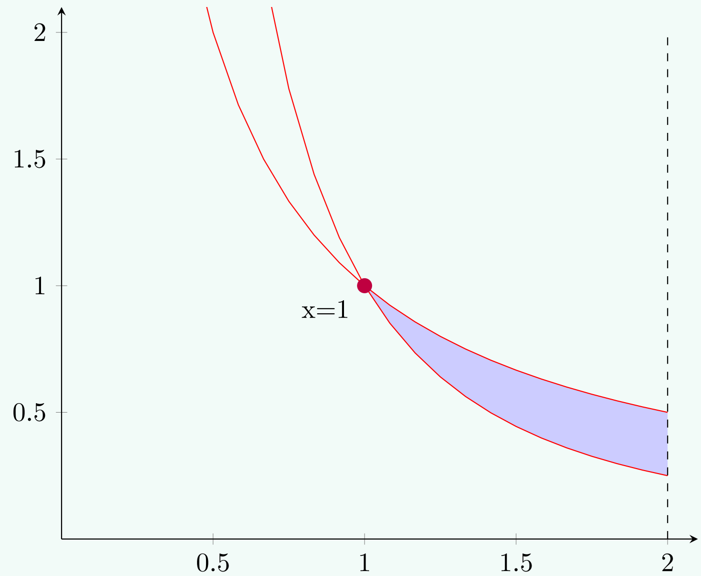
Find the bounds by setting them equal to each other

$$y + 1 = \frac{1}{2}y^2 - 3$$
$$0 = \frac{1}{2}y^2 - y - 4$$
$$0 = y^2 - 2y - 8$$
$$0 = (y + 2)(y - 4)$$
$$y = -2, y = 4$$

$$\int_{-2}^{4} y + 1 - \left(\frac{1}{2}y^2 - 3\right) dy$$
$$\int_{-2}^{4} \left(y - \frac{1}{2}y^2 + 4\right) dy$$

Example 6.1.4

$$y = \frac{1}{x}, y = \frac{1}{x^2}, x = 2$$



$$\int_1^2 \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$$

$$= \left[\ln x + \frac{1}{x}\right]_1^2$$

$$= \ln 2 + \frac{1}{2} - (\ln 1 + 1)$$

$$= \ln 2 - \frac{1}{2}$$

6.2 Volume of Solids of Revolution

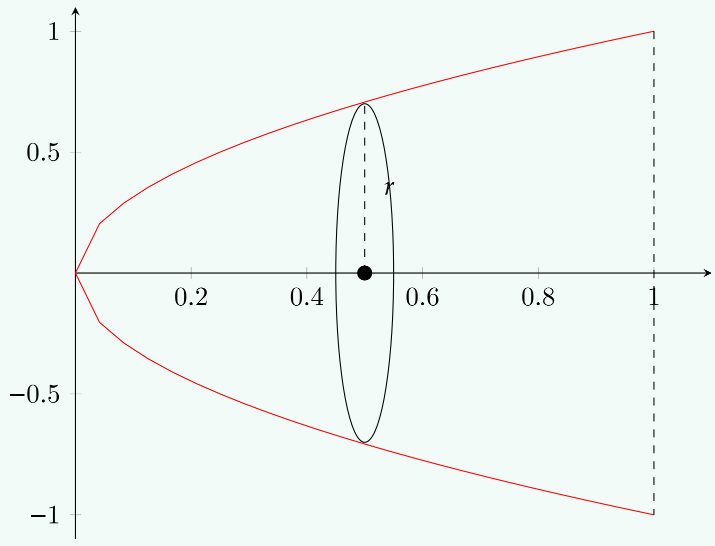
Definition 6.2.1

The volume for solid of revolution between curves is the volume of object between a curve $f(x)$ and a curve $g(x)$ rotated around $y = d$ on an interval $[a, b]$ given by

$$V = \int_a^b \pi \left|(f(x) - d)^2 - (g(x) - d)^2\right| dx$$

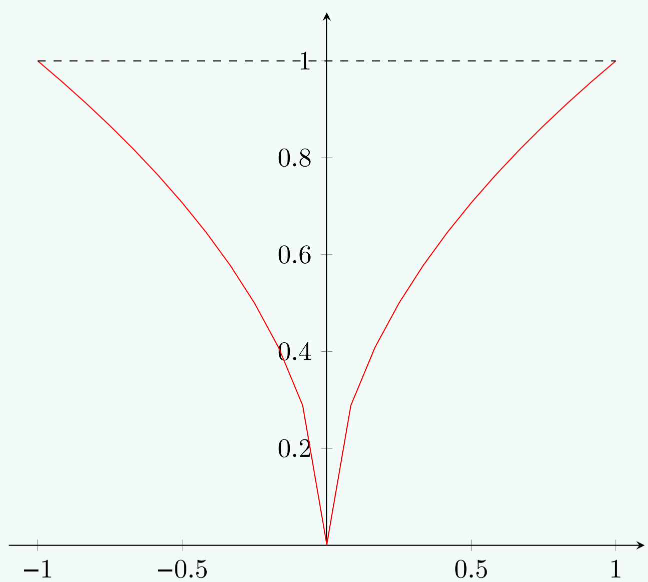
Example 6.2.1

$$y = \sqrt{x}, 0 \text{ to } 1, \text{ about x-axis}$$



$$r = \sqrt{x} - 0$$
$$r^2 = x$$
$$\pi \int_0^1 (x)dx$$

Example 6.2.2



$$y = \sqrt{x}$$
$$x = y^2$$
$$r = y^2$$
$$r^2 = y^4$$
$$\pi \int_0^1 (y^4)dy$$

6.5 Average Change

Definition 6.5.1

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)dx$$

Must be continuous on $[a, b]$

Example 6.5.1

Find average value of $f(x) = 1 + x^2$ on $[-1, 2]$

$$\frac{1}{2--1}\int_{-1}^{2} 1 + x^2 dx$$

$$= \frac{1}{3}\left[\left(x + \frac{1}{3}x^3\right)\right]_{-1}^{2}$$

$$= \frac{1}{3}\left(2 + \frac{8}{3} - \left(-1 - \frac{1}{3}\right)\right)$$

$$= \frac{1}{3}(3 + 3) = 2$$

Example 6.5.2

$$f(x) = (x-3)^2 \text{ on } [2, 5]$$

$$\frac{1}{5-2}\int_2^5 (x-3)^2\, dx$$

$$= \frac{1}{3}\left[\left(\frac{1}{3}(x-3)^3\right)\right]_2^5$$

$$= \frac{1}{3}\left(\frac{8}{3} - -\frac{1}{3}\right)$$

$$= \frac{1}{3}(3) = 1$$

Chapter 7

7.4 Logarithmic Differentiation

Example 7.4.1

$$y = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$$

Take natural log of both sides

$$\ln y = \ln\left(\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\right)$$

Expand using laws of logarithms

$$\ln y = \frac{3}{4}\ln x + \frac{1}{2}\ln\left(x^2+1\right) - 5\ln(3x+2)$$

Use implicit differentiation

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{2}\cdot\frac{1}{x^2+1}\cdot(2x) - 5\frac{1}{3x+2}(3)$$

$$\frac{dy}{dx} = \frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left(\frac{3}{4x} + \frac{x}{x^2+1} - \frac{15}{3x+2}\right)$$

Example 7.4.2

$$y = x^{\sqrt{x}}$$

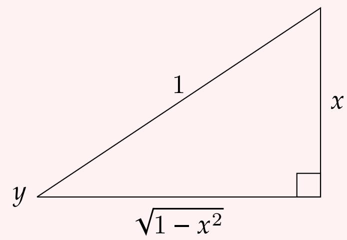
$$\ln y = \sqrt{x}\ln x$$

$$\frac{1}{y}y' = \sqrt{x}\frac{1}{x} + \ln x\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

7.6 Inverse Trig

Definition 7.6.1: Differential of inverse sine

$$y = \sin^{-1} x$$
$$x = \sin y$$
$$1 = \cos y \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{\cos y}$$



$$x^2 + ?^2 = 1$$
$$?^2 = 1 - x^2$$
$$? = \sqrt{1 - x^2}$$

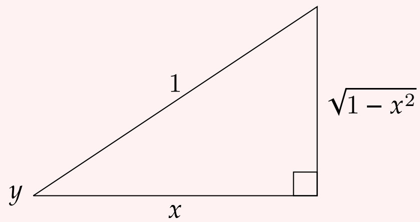
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$
or
$$\frac{1}{\sqrt{1-u^2}} du$$

Example 7.6.1

$$f(x) = \sin^{-1}(x^2 - 1)$$
$$u = x^2 - 1$$
$$dy = 2x$$
$$f'(x) = \frac{1}{\sqrt{1-(x^2-1)^2}} 2x$$

Definition 7.6.2: Differential of inverse cosine

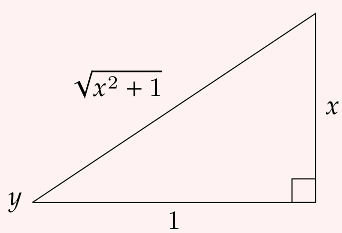
$$y = \cos^{-1} x$$
$$x = \cos y$$
$$1 = -\sin y \frac{dy}{dx}$$
$$\frac{dy}{dx} = -\frac{1}{\sin y}$$



$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-x^2}}$$

Definition 7.6.3: Differential of inverse tangent

$$y = \tan^{-1} x$$
$$x = \tan y$$
$$1 = \sec^2 y \frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{1}{\sec^2 y}$$



$$\frac{dy}{dx} = \cos^2 y = \left(\frac{1}{\sqrt{x^2+1}}\right)^2 = \frac{1}{x^2+1}$$

Example 7.6.2

$$f(x) = x \arctan \sqrt{x}$$
$$u = \sqrt{x}$$
$$du = \frac{1}{2} x^{-\frac{1}{2}}$$

$$f'(x) = x\left[\frac{1}{\left(\sqrt{x}\right)^2+1}\left(\frac{1}{2\sqrt{x}}\right)\right] + \arctan\sqrt{x}(1)$$

$$= x\left(\frac{1}{2\sqrt{x}\,(x+1)}\right) + \arctan\sqrt{x}$$

Example 7.6.3

$$y = \frac{1}{\sin^{-1}x} = \left(\sin^{-1}x\right)^{-1}$$

$$y' = -1\left(\sin^{-1}x\right)^{-2}\left(\frac{1}{\sqrt{1-x^2}}\right)$$

Example 7.6.4

$$\int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-4x^2}}dx = \int_0^{\frac{1}{4}} \frac{1}{\sqrt{1-(2x)^2}}dx$$

$$u = 2x$$

$$du = 2dx$$

$$\frac{1}{2}du = dx$$

$$\frac{1}{2}\int \frac{1}{\sqrt{1-u^2}}du$$

$$= \frac{1}{2}\sin^{-1}u$$

$$= \left[\frac{1}{2}\sin^{-1}(2x)\right]_0^{\frac{1}{4}}$$

$$= \frac{1}{2}\left(\sin^{-1}\frac{1}{2} - \sin^{-1}0\right)$$

$$= \frac{1}{2}\left(\frac{\pi}{6} - 0\right) = \frac{\pi}{12}$$

7.7 Hyperbolic Functions

Definition 7.7.1: Definition of the hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x}$$

Definition 7.7.2: Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh x) = \cosh x$$

$$\frac{d}{dx}(\cosh x) = \sinh x$$

$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x$$

$$\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x x$$

$$\frac{d}{dx}(\operatorname{sech} x) = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

Definition 7.7.3: Inverse hyperbolic functions

$$y = \sinh^{-1} x \iff \sinh y = x$$

$$y = \cosh^{-1} x \iff \cosh y = x \text{ and } y \geqslant 0$$

$$y = \tanh^{-1} x \iff \tanh y = x$$

Definition 7.7.4: Derivatives of inverse hyperbolic functions

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}}$$
$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}$$
$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}$$
$$\frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{x^2+1}}$$
$$\frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}$$
$$\frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}$$

7.8 L'Hopital's Rule

Question 16

$$\lim_{x\to 1} \frac{x^2-1}{x-1}$$
$$= \lim_{x\to 1} \frac{\cancel{(x-1)}(x+1)}{\cancel{x-1}}$$
$$= \lim_{x\to 1} x+1 = 1+1 = 2$$

Question 17

$$\lim_{h\to 0} \frac{(3+h)^2-9}{h}$$
$$= \lim_{h\to 0} \frac{h^2+6h\cancel{+9-9}}{h}$$
$$= \lim_{h\to 0} h+6 = 0+6 = 6$$

Question 18

$$\lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2}$$
$$= \lim_{x\to 0} \frac{\sqrt{x^2+9}-3}{x^2} \cdot \frac{\sqrt{x^2+9}+3}{\sqrt{x^2+9}+3}$$
$$= \lim_{x\to 0} \frac{x^2\cancel{+9-9}}{x^2\left(\sqrt{x^2+9}+3\right)}$$
$$= \lim_{x\to 0} \frac{1}{\sqrt{x^2+9}+3}$$
$$= \frac{1}{\pm 3+3} = \frac{1}{6}$$

Definition 7.8.1: L'Hopital's Rule

If and only if direct substitution results in an indeterminate form $\left[\frac{0}{0}, \frac{\infty}{\infty}, \text{etc.}\right]$, you can use L'Hopital's Rule (take the derivative of the top and bottom and try substitution again). Things like $\frac{1}{0}$ are not indeterminate, they just don't exist.

Example 7.8.1

$$\lim_{x\to 1} \frac{\ln x}{x-1} = \frac{0}{0}$$

Since our answer is indeterminate, we must use L'Hopital's Rule

$$\lim_{x\to 1} \frac{\frac{1}{x}}{1} = 1$$

Example 7.8.2

$$\lim_{x\to 1} \frac{x^2-1}{x-1} = \lim_{x\to 1} \frac{2x}{1} = 2$$

Example 7.8.3

$$\lim_{h\to 0} \frac{(3+h)^2-9}{h} = \lim_{h\to 0} \frac{2(3+h)}{1} = 6$$

Example 7.8.4

$$\begin{aligned}
&\lim_{x\to\infty} \frac{e^x}{x^2} \\
&\overset{\text{(H)}}{=} \lim_{x\to\infty} \frac{e^x}{2x} \\
&\overset{\text{(H)}}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty
\end{aligned}$$

Example 7.8.5

$$\begin{aligned}
&\lim_{x\to 0} \frac{\tan x - x}{x^3} \\
&\overset{\text{(H)}}{=} \lim_{x\to 0} \frac{\sec^2 x - 1}{3x^2} \\
&\overset{\text{(H)}}{=} \lim_{x\to 0} \frac{2\sec^2 x \tan x}{6x} \\
&\overset{\text{(H)}}{=} \lim_{x\to 0} \frac{2\sec^2 x \sec^2 x + \tan x \left(4\sec^2 x \tan x\right)}{6} \\
&= \frac{2+0}{6} = \frac{1}{3}
\end{aligned}$$

Example 7.8.6

$$\begin{aligned}\lim_{x\to 0+} x\ln x &= \lim_{x\to 0+} \frac{\ln x}{\frac{1}{x}} \\ &\overset{\text{(H)}}{=} \lim_{x\to 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} \\ &= \lim_{x\to 0+} \frac{1}{x}\cdot -\frac{x^2}{1} = \lim_{x\to 0+} -x \\ &= 0\end{aligned}$$

Example 7.8.7

$$\begin{aligned}&\lim_{x\to 0} \frac{\tan px}{\tan qx} \\ &\overset{\text{(H)}}{=} \lim_{x\to 0} \frac{p\sec^2 px}{q\sec^2 qx} \\ &= \frac{p\cancel{\sec^2 p(0)}}{q\cancel{\sec^2 q(0)}} \\ &= \frac{p}{q}\end{aligned}$$