1 FUNCTION OF TWO VARIABLES

A function f of two variables is a formula z = f(x,y),(x,y) ∈ D.The set D = (x,y/f(x,y) is defined) ⊂ R^2 is the domain of f

Ex:

Find f(x,y) $= \dfrac{\sqrt{x+y+1}}{x-1}$, find the f(0,3),f(3,0) and domain D:

Sketch D in Oxy plane

Solution:

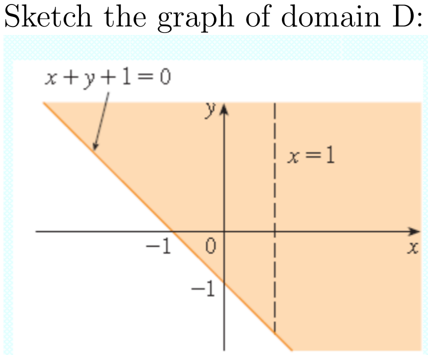
- f(0,3) $= \dfrac{\sqrt{0+3+1}}{0-1} = -4$

  Do same for f(3,0)

- Domain D of function f:

$$\begin{cases} x+y+1 \geq 0 \\ x-1 \neq 0 \end{cases}$$

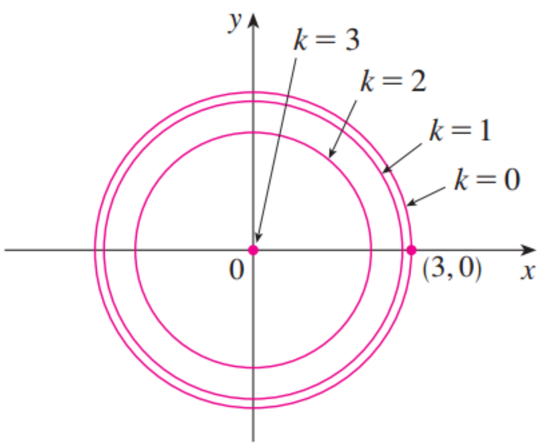
- Sketch the graph of domain D:



2 Level Curves

2.1 Definition

- A level curves of the function f of two variables are the curves with equation f(x,y) = k, where k is a constant (in the range of f).
Ex:
Sketch the curves of the function:g(x,y) = $\sqrt{9-x^2-y^2}$ for k = 0,1,2,3
Solution:
We consider: $\sqrt{9-x^2-y^2}$ = k or $x^2+y^2=9-k^2$
For each k = 0,1,2,3 we have function f represents a circle.



3 Partial Derivative

3.1 1st Order Partial Derivative

Notations for Partial Derivative: If z = f(x,y), we have:

$$f_x(x,y) = f_x = \frac{\delta f}{\delta x} = \frac{\delta}{\delta x} f(x,y) = \frac{\delta z}{\delta x} = f_1 = D_1 f = D_x f$$
$$f_y(x,y) = f_y = \frac{\delta f}{\delta y} = \frac{\delta}{\delta y} f(x,y) = \frac{\delta z}{\delta y} = f_2 = D_2 f = D_y f$$

Rule for Finding Partial Derivative of z = f(x,y)

1. To find f_x, regard y as constant and differentiate f(x,y) with respect to x.
2. To find f_y, regard x as constant and differentiate f(x,y) with respect to y.

3.2 Chain Rule

- Suppose that z = f(x,y) is a differential function of x and y, where x = g(t) and y = h(t) are both differential functions of t. Then z is a differential function of t and:

$$\frac{dz}{dt} = \frac{\delta f}{\delta x} . \frac{dx}{dt} + \frac{\delta f}{\delta y} . \frac{dy}{dx}$$

4 Constant Extrema

4.1 Local Max

Extrema

$$f(a,b) \geq f(x,y); \forall x, y \text{ NEAR } (a,b)$$

- To find Extrema z(x,y):

- Step 1: Solve:

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$$

$\rightarrow$

$$\begin{cases} M_1(x_1, y_1) \\ M_2(x_2, y_2) \end{cases}$$

- Step 2:

$$\begin{cases} f_{xx} \rightarrow A \\ f_{xy} \rightarrow B \\ f_{yy} \rightarrow C \end{cases}$$

$\rightarrow D = AC - B^2$ If:
- D > 0:
+ A > 0: Local min
+ A < 0: Local max
- D < 0: No extrema => Saddle point
- D = 0: No conclusion

4.2 Global Max

(Max and Min point)

$$\boxed{f(a, b) \geq f(x, y); \forall x, y \in D}$$

- To find Global Max and Min at g(x,y) = 0, we use Lagrange Function:

$$\boxed{L(x, y, \lambda) = f(x, y) + \lambda g(x, y)}$$

Then solve:

$$\begin{cases} \dfrac{\delta L}{\delta x} = 0 \\ \dfrac{\delta L}{\delta y} = 0 \\ g(x,y) = 0 \end{cases}$$

$\rightarrow$

$$\begin{cases} M_1(x_1, y_1, \lambda_1) \\ M_2(x_2, y_2, \lambda_2) \end{cases}$$

- Finally, evaluate f(x,y) at M_1 and M_2 to find max and min point.

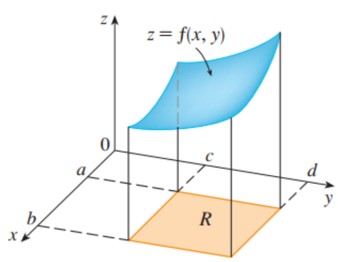
5 Double Integral

5.1 Volume and Double Integrals

- We consider the function f of two variables defined on the closed rectangle:

$$R = [a,b] \times [c,d] = (x,y) \in \mathbb{R}^2 | a \leq x \leq b; c \leq y \leq d$$

- Suppose that f(x,y) $>$ 0. The graph of f:

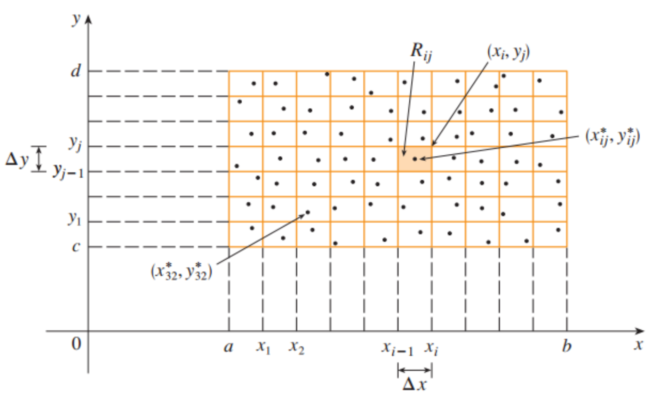


- The volume V of the funtion f is denoted by:

$$S = (X, y, z) \in \mathbb{R}^3 | 0 \leq z \leq f(x, y), (x, y) \in \mathbb{R}$$

- To find the volume V:

- First we should devide the rectangle R into sub-rectangles
- Divide the integral [a,b] into m sub-integrals $[x_{i-1}, x_i]$ and the $\delta x = \dfrac{b-a}{m}$ and devide the integral [c,d] into n subintegrals $[y_{i-1}, y_i]$ and $\delta y = \dfrac{d-c}{m}$

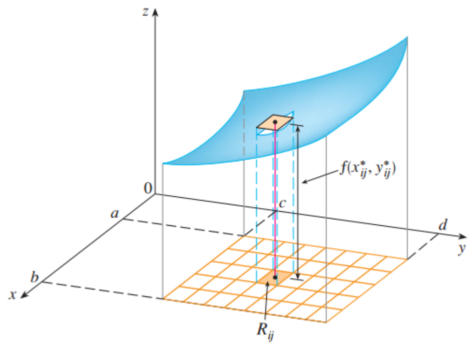


- The volume of sub-rectangles equal to height times area:

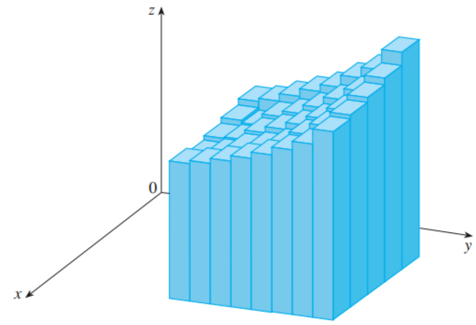
$$\boxed{V_{ij} = f(x_{ij}, y_{ij})\delta A}$$

$\rightarrow$

$$\boxed{V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}, y_{ij})\delta A}$$



z
0
a
c
d
y
b
x
$f(x_{ij}^*, y_{ij}^*)$
R_{ij}



z
0
y
x