Caculus 2 – SUMMARY – Chapter 2: Double Integral. Book: Page 973 – 1004.

Topic 0: Iterated integral $\int_a^b \int_c^d f(x,y)dydx$. More general: $\int_a^b \int_{g(x)}^{h(x)} f(x,y)dydx$.

Example: Evaluate a/ I = $\iint_R x^2ydydx$ where R = $\{(x,y) \mid 0 \le x \le 3, 1 \le y \le 2\}$

b/ I = $\iint_D (1+2y)dydx$, D = $\{(x,y) \mid 0 \le x \le 1, x^2 \le y \le x\}$

Answer: a/ I = $\int_0^3 \left[\int_1^2 x^2ydy\right]dx = \frac{27}{2}$

b/ I = $\int_0^1 \left[\int_{x^2}^x (1+2y)dy\right]dx = \int_0^1 \left[y+y^2\right]_{y=x^2}^{y=x} dx = \int_0^1 \left(x-x^4\right)dx = \frac{3}{10}$.

Topic 1: Double Integral over General Domain D: $\iint_D f(x,y)dA$.

Easiest: D – rectangle [a, b] x [c, d] or D: $\begin{cases} a \le x \le b \\ c \le y \le d \end{cases}$

$\Rightarrow \iint_D f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx$.

Separable case: $\begin{cases} D = [a,b]\times[c,d] \\ f(x,y) = g(x)h(y) \end{cases} \Rightarrow \iint_D f(x,y)dA = \int_a^b g(x)dx \cdot \int_c^d h(y)dy$.

Example: Evaluate numerically I = $\iint_D \frac{e^{2x+y}}{\sqrt{xy}}dA$ where D = [1, 2] x [1,3].

Answer: I = $\int_1^2 \frac{e^{2x}}{\sqrt{x}}dx \cdot \int_1^3 \frac{e^y}{\sqrt{y}}dy$. Calculator $\Rightarrow$ I $\approx$ 216.92

Basic case: D is $\begin{cases} a \le x \le b \\ g(x) \le y \le h(x) \end{cases}$

Example: Evaluate I = $\iint_D y^2dA$, D = $\{(x,y) \mid -1 \le y \le 1, -y-2 \le x \le y\}$.

Answer: I = $\int_{-1}^1 \int_{-y-2}^y y^2dxdy = \int_{-1}^1 y^2[x]_{x=-y-2}^{x=y} dy = \int_{-1}^1 y^2(2y+2)dy = \frac{4}{3}$.

Remark: The case D: $\begin{cases} c \le y \le d \\ g(y) \le x \le h(y) \end{cases}$ is solved similarly.

Special: D is bounded by $\begin{vmatrix} y = g(x) \\ y = h(x) \end{vmatrix}$ $\Rightarrow$ Technique: Solve Intercept

$g(x) = h(x)$

Example: Evaluate I = $\iint_D x\cos y dA$, D is bounded by $y=0$, $y=x^2$, $x=1$.

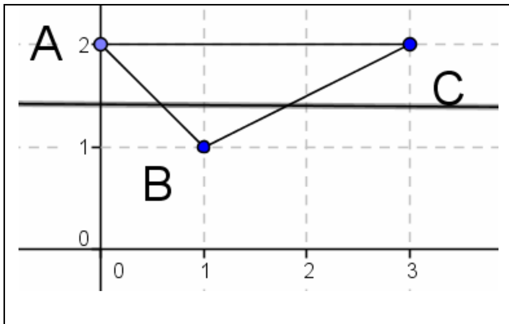
Answer: D: $\begin{vmatrix} y=0,\ y=x^2 \\ x=1 \end{vmatrix} \Rightarrow$ Miss one x. Intercept equation: $x^2=0 \Rightarrow x=0 \Rightarrow$

D: $\begin{vmatrix} 0\le x\le 1 \\ 0\le y\le x^2 \end{vmatrix} \Rightarrow \mathrm{I} = \int_0^1\int_0^{x^2} x\cos y\,dy\,dx = \int_0^1 x[\sin y]_{y=0}^{y=x^2}\,dx = \int_0^1 x\sin\left(x^2\right)dx$.

Calculator or Change $t=x^2 \Rightarrow \mathrm{I} = \frac{1}{2}\int_0^1 \sin t\,dt = \frac{1}{2}(1-\cos 1)$.

3rd case: D is described by geometry, for example triangle with 3 vertices.

Example: Evaluate I $= \iint_D y^3 dA$, D: triangular region with vertices (0;2), (1;1), (3;2).



D: $\begin{vmatrix} 1\le y\le 2 \\ \mathrm{x}_{AB}\le x\le \mathrm{x}_{BC} \end{vmatrix}$. AB: $y=ax+b$. Through A(0; 2), B(1; 1) $\Rightarrow \begin{cases} 0\cdot a+b=2 \\ 1\cdot a+b=1 \end{cases} \Rightarrow \begin{cases} a=-1 \\ b=2 \end{cases} \Rightarrow$ $y=-x+2 \Rightarrow x=2-y$. Same for BC: $y=\frac{x+1}{2}$ $\Rightarrow x=2y-1$

$\Rightarrow$ D: $\begin{vmatrix} 1\le y\le 2 \\ 2-y\le x\le 2y-1 \end{vmatrix} \Rightarrow \mathrm{I} = \int_1^2\int_{2-y}^{2y-1} y^3\,dx\,dy = \int_1^2 y^3[x]_{x=2-y}^{x=2y-1}\,dy = \frac{147}{20}$.

Topic 2: Application – Mass & Charge: Formula (1) – (2) page 1004.

Example: (a – last digit of student ID number, m = a + 1) Given one lamina of the shape D: $\begin{cases} 1\le x\le 2 \\ x\le y\le 2x \end{cases}$ with the limear density $\rho(x,y)=2x+my+3xy$, find the lamina mass.

Answer: (For example: m = 5) Mass = $\int_1^2\int_x^{2x}(2x+5y+3xy)\,dy\,dx$

$= \int_1^2 \left[2xy+(3x+5)\cdot\frac{y^2}{2}\right]_{y=x}^{y=2x} dx$. Calculator $\Rightarrow$ Mass $\approx$ 39.04

Numerical Answer:

m = 1:	Mass = 25.04
m = 2:	Mass = 28.54
m = 3:	Mass = 32.04
m = 4:	Mass = 35.54
m = 5:	Mass = 39.04
m = 6:	Mass = 42.54
m = 7:	Mass = 46.04
m = 8:	Mass = 49.54
m = 9:	Mass = 53.04
m = 10:	Mass = 56.54

Topic 3: Volume of the solid (S)

Simple case: Given $\begin{cases} f(x,\ y)\le z\le g(x,\ y) \\ (x,\ y)\in D \end{cases}$ (or Below $z=f(x,\ y)$ and Above Domain D $\subseteq$ Oxy $\Rightarrow$ $z=0 \Rightarrow 0\le z\le f(x,\ y)$)

Example: Find the volume of the solid (S) under the plane x + 2y – z = 0 and above the region bounded by y = x and y = x^4.

Answer: Region bounded by $y = x$, $y = x^4$ belongs to Oxy: $z = 0 \Rightarrow$

Solid (S): under the plane $z = x + 2y$ & above $z = 0 \Rightarrow 0 \le z \le x + 2y$.

Domain D: bounded by $\begin{vmatrix} y = x \\ y = x^4 \end{vmatrix}$. Intercept equation: $x^4 = x \Rightarrow \left[\begin{array}{l} x = 0 \\ x = 1 \end{array}\right.$.

Test with $x = \frac{1}{2} \in (0;1)$: $\frac{1}{2^4} < \frac{1}{2} \Rightarrow x^4 \le x \ \forall \text{ x} \in [0;1]$

$\Rightarrow$ D = $\begin{cases} 0 \le x \le 1 \\ x^4 \le y \le x \end{cases} \Rightarrow$ V = $\int\limits_0^1 \int\limits_{x^4}^{x} [(x + 2y) - 0] dydx$.

General case: Volume of solide (S), bounded by some equations $\Rightarrow$ Find one variables, for example z, that appears exactly only TWO times in equations $\Rightarrow \begin{vmatrix} z = f(x, y) \\ z = g(x, y) \end{vmatrix}$. Remaning equations (having only x, y) $\rightarrow$ Domain D. If D is not finite, add Intercept equation $f(x, y) = g(x, y)$ to D.

Example: Find the volumes of the solid bounded by the coordinates planes and the plane 3x + 2y + z = 6.

Answer: Coordinate planes: $\begin{vmatrix} \text{Oxy}: z = 0 \\ \text{Oyz}: x = 0 \\ \text{Oxz}: y = 0 \end{vmatrix}$. 2 equations with z $\Rightarrow$ Solid (S) is between $\begin{vmatrix} z = 0 \\ z = 6 - 3x - 2y \end{vmatrix}$ (*). 2 remaining equations: $\begin{vmatrix} x = 0 \\ y = 0 \end{vmatrix}$ &

Intercept equation (from (*)): $6 - 3x - 2y = 0 \Rightarrow y = \frac{6 - 3x}{2}$.

Draw picture $\Rightarrow$ Domain D: $\begin{cases} 0 \le x \le 2 \\ 0 \le y \le \frac{6 - 3x}{2} \end{cases}$.

Test by P(1, 1) inside D $\Rightarrow 0 \le 6 - 3x - 2y$ at D $\Rightarrow$

V = $\int\limits_0^2 \int\limits_0^{\frac{6-3x}{2}} [(6 - 3x - 2y) - 0] dydx = \int\limits_0^2 \left[(6 - 3x)y - y^2\right]_{y=0}^{y=\frac{6-3x}{2}} dx = 6$.

Topic 4: Domain D related to circle $\Rightarrow$ Polar: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$.

Example: Evaluate I = $\iint\limits_D e^{-x^2-y^2} dA$, where D is the disk $x^2 + y^2 \le 4$

Answer: Polar $\Rightarrow$ D: $\begin{cases} 0 \le r \le 2 \\ 0 \le \theta \le 2\pi \end{cases} \Rightarrow$ I = $\int\limits_0^2 \int\limits_0^{2\pi} re^{-r^2} d\theta dr$. $t = r^2 \Rightarrow$ I = $\pi\left(1 - \frac{1}{e^4}\right)$.

Example: Find the volume of the solid under $z = x^2 + y^2$, above Oxy – plane and insde the cylinder: $x^2 + y^2 = 2x$.

Answer: Solid (S): $\begin{cases} 0 \le z \le x^2 + y^2 \\ \text{D}: x^2 + y^2 \le 2x \end{cases}$. Polar for D: $x^2 + y^2 = 2x \Leftrightarrow r = 2\cos\theta \ge 0$

$\Rightarrow$ D: $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, $0 \le r \le 2\cos\theta \Rightarrow$ V = $\iint\limits_D \left[(x^2 + y^2) - 0\right] dA = \int\limits_{-\pi/2}^{\pi/2} \int\limits_0^{2\cos\theta} r^2 \cdot rdrd\theta = \frac{3\pi}{2}$.