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(Linear Algebra)

1 Matrix Algebra

1.1 Definition

- A $m \times n$ matrix is a table of numbers that contains in m rows and n columns

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{1n} \\ a_{21} & a_{22} & a_{23} & a_{2n} \\ a_{n1} & a_{n2} & a_{n3} & a_{nn} \end{pmatrix} = (a_{ij})_{m \times n}$$

1.2 Definition

- If $a_{ij} = 0$, $\forall i, j$, then A is called a zero matrix and is denoted by 0

1.3 Definition

Let A = $(a_{ij})_{m \times n}$:
- A matrix A that is formed by columns is called a transpose of A
Ex: $\begin{pmatrix} 2 & -1 & 3 \\ 4 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ -1 & 7 \\ 3 & 8 \end{pmatrix}$

1.4 Definition

- If the number of rows and the number of columns equal, then the matrix is called a square matric of order
- Let A = $(a_{ij})_{m \times n}$ be a square matrix of order n

- All the elements $a_{11}, a_{22}, \ldots, a_{nn}$ is diagonal of A

- A sum of $a_{11}, a_{22}, \ldots, a_{nn}$ is called a trace of A
  $A = \begin{pmatrix} \mathbf{1} & 1 & 3 \\ 4 & \mathbf{2} & 7 \\ 2 & 5 & \mathbf{6} \end{pmatrix}$

  Trace(A) = 1 + 2 + 6 = 9.

- If $a_{ij} = 0, \forall i \neq j$, then A is called a diagonal matrix
  $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ is a diagonal matrix

- $$\text{If } a_{ij} = \begin{cases} 0, if \text{ i} \neq j \\ 1, if \text{ i} = j \end{cases}$$

  then A is called an identity matrix

2 Matrix Operation

2.1 Equality

$A = B \leftrightarrow$

$$\begin{cases} \text{Same size} \\ \text{Corresponding elements are equal} \end{cases}$$

Ex: $A = \begin{pmatrix} 2 & -1 & 3 \\ 4 & 7 & 8 \end{pmatrix}$

$B = \begin{pmatrix} 2 & x & 3 \\ 4 & 7 & 8 \end{pmatrix}$

A = B when

$$\begin{cases} x = -1 \\ y = 4 \end{cases}$$

2.2 Sum:

$A = (a_{ij})_{m\times n}$
$B = (b_{ij})_{m\times n}$
$\rightarrow A + B = (a_{ij} + b_{ij})_{m\times n}$
Ex:
A = $\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$

B = $\begin{pmatrix} 1 & -3 \\ 0 & 7 \end{pmatrix}$

$\rightarrow A + B = \begin{pmatrix} 3 & -4 \\ 3 & 11 \end{pmatrix}$

2.3 Multiplication by a number

$A = (a_{ij})_{m\times n}$
$\rightarrow \alpha.A = (\alpha a_{ij})_{m\times n}$
Ex:
A = $\begin{pmatrix} 1 & 2 & 1 \\ 3 & 5 & 4 \end{pmatrix}$

$\rightarrow 3A = \begin{pmatrix} 3 & 6 & 3 \\ 8 & 5 & 12 \end{pmatrix}$

2.4 Matrix Multiplication

$A = (a_{ij})_{m\times n}$; $B = (b_{ij})_{n\times p}$
$A \times B = C = (c_{ij})_{m\times p}$
c_{ij} = (i^{th} rows of A) × (j^{th} columns of B)
$c_{ij} = (a_{i1} \times a_{i2} \times a_{i3} \ldots \times a_{in}) \begin{pmatrix} b_{1j} \\ b_{2j} \\ \ldots \\ b_{nj} \end{pmatrix}$

$c_{ij} = a_{i1}b_{1j} + a_2b_{2j} + \ldots + a_{in}b_{nj}$

$c_{ij} = \sum_{n}^{k=1} a_{ik}b_{kj}$

Ex:

A $= \begin{pmatrix} 1 & -1 & 3 \\ 2 & 4 & -2 \end{pmatrix}_{2\times3}$; B $= \begin{pmatrix} 2 & 1 & -1 \\ 5 & 2 & 6 \\ 0 & 1 & -1 \end{pmatrix}_{3\times3}$

$\rightarrow A \times B = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{pmatrix}_{2\times3}$

With:

$c_{11} =$ (1 -1 3) $\begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} = 1 \times 2 + (-1) \times 5 + 3 \times 0 = -3$

2.5 Power of Square Matrix

Let A = is a square matrix:

$A^0 = 1$

$A^1 =$ A

$A^2 =$ A.A

$A^3 =$ A.A.A

...

$A^n =$ A.A...A

2.6 Itentity MAtrix

- In linear algebra, the otentity matrix, or sometimes amboguously called **a unit matrix** of size n is the $n \times n$ square matrix with ones on **the main diagonal** and zeroes elsewhere, denoted by I_n

$I_1 = \begin{bmatrix} 1 \end{bmatrix}$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

2.7 Echelon Form

- The first non-zero element of a row from the left is called a leading entry of the row

2.8 Definition of Echelon Form

- All non-zero rows are above any rows of all zeros
- Each leading entry of row is in a column to the right of the row above it

2.9 Elementary row operations:

1. Scaling: Multiplying all entries in a row by a non-zero constant

$$r_i \to ar_i; a \neq 0$$

2. Replacement: Replace one row by the sum of itself and a multiple of another row

$$r_i \to r_i + \beta r; \forall \beta$$

3. Intercharging: Intercharge two rows $r_i \leftrightarrow r_j$

2.10 Subtitution and Elimination

Example:

$$\begin{cases} x + 2y - z = 1 (eq1) \\ 2x + 5y - 3z = 3 (eq2) \\ 5x + 12y - 7z = 7 (eq3) \end{cases}$$

We have:

$$eq2 = 2eq1$$
$$eq3 = 5eq1$$

$\leftrightarrow$

$$\begin{cases} x + & 2y - z = 1 \\ & y - z = 1 \\ & 2y - 2z = 2 \end{cases}$$

2.11 Argumented Matrix

- In linear algebra, an argumeted matrix is a matrix obtained by appending the columns of two given matrices, usually for the purpose of performing the same **elementary row operation** on each of the given matrices.

- Given matrix A = $\begin{pmatrix} 1 & 1 & -1 & 2 \\ 2 & 3 & -1 & 4 \\ 3 & 2 & -3 & 7 \\ -1 & 1 & 2 & -3 \end{pmatrix}$ and B = $\begin{pmatrix} 1 \\ 5 \\ 4 \\ 1 \end{pmatrix}$

The argumented matrix is C = $\left(\begin{array}{rrrr|r} 1 & 1 & -1 & 2 & 1 \\ 2 & 3 & -1 & 4 & 5 \\ 3 & 2 & -3 & 7 & 4 \\ -1 & 1 & 2 & -3 & 1 \end{array}\right)$

2.12 Transform matrix in Echelon form

1. Begin with the left most non-zero column. Select a non-zero entry in the colmn as a pivot, then use elimination $\left(\begin{array}{cccc|c} 1 & 1 & -1 & 2 & 1 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 1 & 2 \end{array}\right)$
2. Then start with the second row and do the same until the last row

3 Determinant of the square matrix

3.1 Definition 1:

- A determinant of the a square matrix is a number that is denoted by **det(A)** = |A|.
Ex:A = $\begin{pmatrix} 2 & -1 \\ 3 & 6 \end{pmatrix}$
$\rightarrow det(A) = \begin{vmatrix} 2 & -1 \\ 3 & 6 \end{vmatrix} = 2 \times (-6) - 3 \times (-1) = 15$

3.2 Cofactor Matrix

- Let A be a square matrix and A_{ij} be an element of A:
Cofactor matrix of A is defined by A_{ij}, which:
$A_{ij} = (-1)^{i+j} \begin{vmatrix} \text{element A by deleting } i^{th} \text{ rows and} \\ j^{th} \text{ columns} \end{vmatrix}$

3.3 Definition 2:

- Let A be a square matrix of order n:

- Order 1: A = $(a_{11}) \rightarrow det(A) = a_{11}$
- Order 2: A = $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$\rightarrow \det(\mathrm{A}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \times a_{22} - a_{21} \times a_{12}$
$= a_{11} \times A_{11} + a_{12} \times A_{12}$

- Order 3:
  $\det(\mathrm{A}) = a_{11} \times A_{11} + a_{12} \times A_{12} + a_{13} \times A_{13}$
- For order n:
  $\det(\mathrm{A}) = a_{11} \times A_{11} + a_{12} \times A_{12} + \ldots + a_{1n} \times A_{1n}$

4 Complex number