

CALCULUS II

1 FUNCTION OF TWO VARIABLES

A function f of two variables is a formula $z = f(x,y), (x,y) \in D$. The set $D = \{(x,y) | f(x,y) \text{ is defined}\} \subset \mathbb{R}^2$ is the domain of f

Ex:

Find $f(x,y) = \frac{\sqrt{x+y+1}}{x-1}$, find the $f(0,3), f(3,0)$ and domain D :

Sketch D in Oxy plane

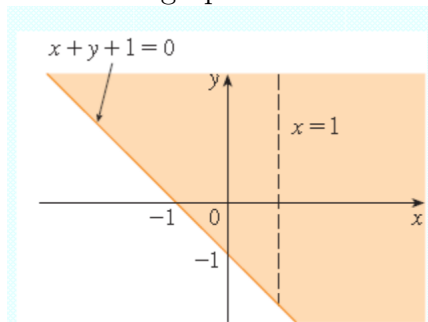
Solution:

- $f(0,3) = \frac{\sqrt{0+3+1}}{0-1} = -2$
Do same for $f(3,0)$

- Domain D of function f :

$$\begin{cases} x + y + 1 \geq 0 \\ x - 1 \neq 0 \end{cases}$$

- Sketch the graph of domain D :



2 Level Curves

2.1 Definition

- A level curves of the function f of two variables are the curves with equation $f(x,y) = k$, where k is a constant (in the range of f).

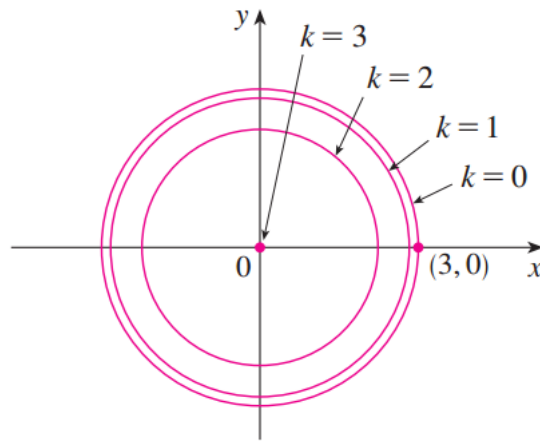
Ex:

Sketch the curves of the function: $g(x,y) = \sqrt{9 - x^2 - y^2}$ for $k = 0,1,2,3$

Solution:

We consider: $\sqrt{9 - x^2 - y^2} = k$ or $x^2 + y^2 = 9 - k^2$

For each $k = 0,1,2,3$ we have function f represents a circle.



3 Partial Derivative

3.1 1st Order Partial Derivative

Notations for Partial Derivative: If $z = f(x,y)$, we have:

$$f_x(x, y) = f_x = \frac{\delta f}{\delta x} = \frac{\delta}{\delta x} f(x, y) = \frac{\delta z}{\delta x} = f_1 = D_1 f = D_x f$$

$$f_y(x, y) = f_y = \frac{\delta f}{\delta y} = \frac{\delta}{\delta y} f(x, y) = \frac{\delta z}{\delta y} = f_2 = D_2 f = D_y f$$

Rule for Finding Partial Derivative of $z = f(x, y)$

1. To find f_x , regard y as constant and differentiate $f(x, y)$ with respect to x .
2. To find f_y , regard x as constant and differentiate $f(x, y)$ with respect to y .

3.2 Chain Rule

- Suppose that $z = f(x, y)$ is a differential function of x and y , where $x = g(t)$ and $y = h(t)$ are both differential functions of t . Then z is a differential function of t and:

$$\frac{dz}{dt} = \frac{\delta f}{\delta x} \cdot \frac{dx}{dt} + \frac{\delta f}{\delta y} \cdot \frac{dy}{dt}$$

4 Constant Extrema

4.1 Local Max

Extrema

$$f(a, b) \geq f(x, y); \forall x, y \text{ NEAR } (a, b)$$

- To find Extrema $z(x, y)$:

- Step 1: Solve:

$$\begin{cases} f_x = 0 \\ f_y = 0 \end{cases}$$

→

$$\begin{cases} M_1(x_1, y_1) \\ M_2(x_2, y_2) \end{cases}$$

- Step 2:

$$\begin{cases} f_{xx} \rightarrow A \\ f_{xy} \rightarrow B \\ f_{yy} \rightarrow C \end{cases}$$

→ $D = AC - B^2$ If:

- $D > 0$:
- + $A > 0$: Local min
- + $A < 0$: Local max
- $D < 0$: No extrema => Saddle point
- $D = 0$: No conclusion

4.2 Global Max

(Max and Min point)

$$f(a, b) \geq f(x, y); \forall x, y \in D$$

- To find Global Max and Min at $g(x, y) = 0$, we use Lagrange Function:

$$L(x, y, \lambda) = f(x, y) + \lambda g(x, y)$$

Then solve:

$$\begin{cases} \frac{\delta L}{\delta x} = 0 \\ \frac{\delta L}{\delta y} = 0 \\ g(x, y) = 0 \end{cases}$$

→

$$\begin{cases} M_1(x_1, y_1, \lambda_1) \\ M_2(x_2, y_2, \lambda_2) \end{cases}$$

- Finally, evaluate $f(x,y)$ at M_1 and M_2 to find max and min point.