ORDINARY DIFFERENTIAL EQUATIONS

1. **Separable Equations**
   - A **separable equation** is a first-order differential equation, which has a form:

$$\boxed{P(x)dx + Q(y)dy = 0}$$

Solution:
Integrate both sides, we receive:

$$\int P(x)dx + \int Q(y)dy = C$$

2. **The differential Equation of the form**

$$\boxed{f_1(x).g_1(y)dx + f_2(x).g_2(y)dy = 0}$$

Solution:

$$\text{If } f_1(x), f_2(x), g_1(y), g_2(y) \neq 0, \text{then we divide both sides by } f_2(x).g_1(y)\text{. So:} \tag{1}$$

$$\frac{f_1(x)}{f_2(x)}dx + \frac{g_2(x)}{g_1(x)} = 0 \rightarrow \int \frac{f_1(x)}{f_2(x)}dx + \int \frac{g_2(y)}{g_1(y)}dy$$

3. **COOL PROBLEM**
   (Liên quan đến the rate of change cần chú ý)
   - The rate of change of the temperature T(t) is the derivative of $\frac{dT}{dt}$
   - Acording to the Newton's law of cooling:

$$\boxed{\frac{dT}{dt} = k(T - T_s)}$$

When k is a constant of propotionality.
Separating the variables we have:

$$\frac{dT}{T - T_s} = kdt \leftrightarrow \frac{dT}{T - T_s} = kdt \tag{2}$$

$$\rightarrow \int \frac{dT}{T - T_s} = k\int dt \rightarrow ln|T - T_s| = kt + lnC \tag{3}$$

$$\rightarrow T - T_s = e^{kt+lnC} = e^{kt}.e^{lnC} = Ce^{kt} \tag{4}$$

4. **The first-order linear differential equation**
   - The first-order **linear** differential equation is one that can be put into the form:

$$\boxed{\frac{dy}{dx} + P(x).y = Q(x)} \tag{5}$$

- To solve the linear difernetial equation, multply both side by the **integrate factor** $e^{P(x)dx}$ and integrate both sides:

$$\boxed{y = e^{-\int P(x)dx}.\int Q(x).e^{\int P(x)dx}dx + C \qquad (6)}$$

SECOND ORDER LINEAR DIIFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

1. **Homogeneous and Non-Homogeneous Second Differential Equations**.
   - Homogeneous Second Differential Equation has a form:

$$\boxed{Ay" + By' + Cy = 0 (A, B, C \in \mathbb{R}; A \neq 0) \quad (7)}$$

   - Non-homogeneous Second Differential Equation has a form:

$$\boxed{Ay" + By' + Cy = f(x)(A, B, C \in \mathbb{R}; A \neq 0) \quad (8)}$$

2. Let y_1, y_2 be any 2 solutions of the homogeneous linear differential equation (1), then:

$$C_1y_1 + C_2y_2$$

   is also a **solution** of equation (1) where C_1, C_2 are arbitrary constants.(hàm tùy ý)

   - The 2 function y_1, y_2 are **linear dependent** on $a \leq x \leq b$ if *there exists constants C_1, C_2 not zero*, such that:

$$C_1y_1 + C_2y_2 = 0$$

   - The 2 function y_1, y_2 are **linear independent** on $a \leq x \leq b$ if:

$$C_1y_1 + C_2y_2 = 0$$

     For all x such that $a \leq x \leq b$ implies $C_1 = C_2 = 0$.

3. Let y_1, y_2 be 2 real function each of which has the first derivative on the interval $a \leq x \leq b$. The determinant (định thức):

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x) \times y_2'(x) - y_1'(x) \times y_2(x)$$

4. If y_1, y_2 are 2 linearly independent solutions of equation (7), then every **homogeneous solution** y_h is:

$$y_h = C_1y_1 + C_2y_2$$

   We have equation (7) as:

$$Ay" + By' + Cy = 0$$

   Consider that:

$$y = e^{kx}$$
$$y' = ke^{kx}$$
$$y" = k^2e^{kx}$$

We receive:

$$Ak^2e^{kx} + Bke^{kx} + Ce^{kx} = 0 \tag{9}$$

$$\leftrightarrow Ak^2 + Bk + C = 0 \tag{10}$$

From equation(10), we have evaluate Δ of the equation:

- $\Delta > 0$, the equation has **real and different roots** $k_1, k_2 \neq 0$.Then:

  $$y_1(x) = e^{k_1x}$$
  $$y_2(x) = e^{k_2x}$$

  The homogeneous solution is:

  $$\boxed{y_h = C_1e^{k_1x} + C_2e^{k_2x}}$$

- $\Delta = 0$ then the characteristic equation has **double real root**: $k = k0$
  The the homogeneous solution is:

  $$\boxed{y_h = C_1e^{k_0x} + C_2xe^{k_0x}}$$

- $\Delta < 0$, then characteristic equation has **complex conjugate roots** $k_1 = a + bi$ **and** $k_2 = a - bi$
  Then the homogeneous solution is:

  $$\boxed{y_h = e^{ax}(C_1cos(bx) + C_2sin(bx))}$$

5. **Non-homogeneous Equation with constant coefficients**

   - **Case I:**$f(x) = e^{\alpha x}.P_n(x)$
     The particular solution of non-homogeneous equation has a form $y_p = x^s.e^{\alpha x}.Q_n(x)$
     - If α is not the root of the characteristic equation then s =0 and the particular equation has a form $y_p = e^{\alpha x}.Q_n(x)$
     - If α is one the 2 different roots of the characteristic equation then s =1 and the particular solution has a form $y_px.e^{\alpha x}.Q_n(x)$
     - If α is the double root of the characteristic equation then s = 2 and the particular equation has a form $y_p = x^2.e^{\alpha x}.Q_n(x)$
   - **Case 2**: f(x) = $e^{\alpha x}.[P_n(x).cos(\beta x).Q_m(x).sin(\beta x)]$
     The particular solution of non-homogeneous equation has a form:

     $$y_p = x^s.e^{\alpha x}.[H_k(x).cos(\beta x) + T_m(x).sin(\beta x)]$$

     - If $\alpha + i\beta$ **is not the root** of the characteristic equation then **s = 0** and the particular solution has a form $y_p = e^{\alpha x}.[H_k(x).cos(\beta x) + T_k(x).sin(\beta x)]$
     - If $\alpha + i\beta$ **is one of the complex conjugate roots** of the characteristic equation then **s = 1** and the particular solution has a form $y_p = x.e^{\alpha x}.[H_k(x).cos(\beta x) + T_k(x).sin(\beta x)]$

6. **Find the non-homogeneous equation:**
   - Step 1:Solve the homogeneous equation, use characteristic equation to find k_1, k_2.
   - Step 2:Evaluate the homogeneous equation.
   - Step 3:Find a particular solution of non-homogeneous equation.