

ORDINARY DIFFERENTIAL EQUATIONS

1. Separable Equations

- A **separable equation** is a first-order differential equation, which has a form:

$$P(x)dx + Q(y)dy = 0$$

Solution:

Integrate both sides, we receive:

$$\int P(x)dx + \int Q(y)dy = C$$

2. The differential Equation of the form

$$f_1(x).g_1(y)dx + f_2(x).g_2(y)dy = 0$$

Solution:

If $f_1(x), f_2(x), g_1(y), g_2(y) \neq 0$, then we divide both sides by $f_2(x).g_1(y)$. So: (1)

$$\frac{f_1(x)}{f_2(x)}dx + \frac{g_2(y)}{g_1(y)}dy = 0 \rightarrow \int \frac{f_1(x)}{f_2(x)}dx + \int \frac{g_2(y)}{g_1(y)}dy$$

3. COOL PROBLEM

(Liên quan đến the rate of change cần chú ý)

- The rate of change of the temperature $T(t)$ is the derivative of $\frac{dT}{dt}$
- According to the Newton's law of cooling:

$$\frac{dT}{dt} = k(T - T_s)$$

When k is a constant of proportionality.

Separating the variables we have:

$$\frac{dT}{T - T_s} = kdt \leftrightarrow \frac{dT}{T - T_s} = kdt \quad (2)$$

$$\rightarrow \int \frac{dT}{T - T_s} = k \int dt \rightarrow \ln|T - T_s| = kt + \ln C \quad (3)$$

$$\rightarrow T - T_s = e^{kt + \ln C} = e^{kt} \cdot e^{\ln C} = C e^{kt} \quad (4)$$

4. The first-order linear differential equation

- The first-order **linear** differential equation is one that can be put into the form:

$$\frac{dy}{dx} + P(x).y = Q(x) \quad (5)$$

- To solve the linear differential equation, multiply both side by the **integrate factor** $e^{P(x)dx}$ and integrate both sides:

$$y = e^{-\int P(x)dx} \cdot \int Q(x) \cdot e^{\int P(x)dx} dx + C \quad (6)$$

SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

1. Homogeneous and Non-Homogeneous Second Differential Equations.

- Homogeneous Second Differential Equation has a form:

$$Ay'' + By' + Cy = 0 (A, B, C \in \mathbb{R}; A \neq 0) \quad (7)$$

- Non-homogeneous Second Differential Equation has a form:

$$Ay'' + By' + Cy = f(x) (A, B, C \in \mathbb{R}; A \neq 0) \quad (8)$$

2. Let y_1, y_2 be any 2 solutions of the homogeneous linear differential equation (1), then:

$$C_1 y_1 + C_2 y_2$$

is also a **solution** of equation (1) where C_1, C_2 are arbitrary constants. (hàm tùy ý)

- The 2 function y_1, y_2 are **linear dependent** on $a \leq x \leq b$ if *there exists constants C_1, C_2 not zero*, such that:

$$C_1 y_1 + C_2 y_2 = 0$$

- The 2 function y_1, y_2 are **linear independent** on $a \leq x \leq b$ if:

$$C_1 y_1 + C_2 y_2 = 0$$

For all x such that $a \leq x \leq b$ implies $C_1 = C_2 = 0$.

3. Let y_1, y_2 be 2 real function each of which has the first derivative on the interval $a \leq x \leq b$. The determinant (định thức):

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x) \times y_2'(x) - y_1'(x) \times y_2(x)$$

4. If y_1, y_2 are 2 linearly independent solutions of equation (7), then every **homogeneous solution** y_h is:

$$y_h = C_1 y_1 + C_2 y_2$$

We have equation (7) as:

$$Ay'' + By' + Cy = 0$$

Consider that:

$$\begin{aligned} y &= e^{kx} \\ y' &= k e^{kx} \\ y'' &= k^2 e^{kx} \end{aligned}$$

We receive:

$$Ak^2e^{kx} + Bke^{kx} + Ce^{kx} = 0 \quad (9)$$

$$\Leftrightarrow Ak^2 + Bk + C = 0 \quad (10)$$

From equation(10), we have evaluate Δ of the equation:

- $\Delta > 0$, the equation has **real and different roots** $k_1, k_2 \neq 0$. Then:

$$\begin{aligned} y_1(x) &= e^{k_1x} \\ y_2(x) &= e^{k_2x} \end{aligned}$$

The homogeneous solution is:

$$y_h = C_1e^{k_1x} + C_2e^{k_2x}$$

- $\Delta = 0$ then the characteristic equation has **double real root**: $k = k_0$

The the homogeneous solution is:

$$y_h = C_1e^{k_0x} + C_2xe^{k_0x}$$

- $\Delta < 0$, then characteristic equation has **complex conjugate roots** $k_1 = a + bi$ **and** $k_2 = a - bi$

The the homogeneous solution is:

$$y_h = e^{ax}(C_1\cos(bx) + C_2\sin(bx))$$

5. Non-homogeneous Equation with constant coefficients

- **Case I:** $f(x) = e^{\alpha x} \cdot P_n(x)$

The particular solution of non-homogeneous equation has a form $y_p = x^s \cdot e^{\alpha x} \cdot Q_n(x)$

- If α is not the root of the characteristic equation then $s = 0$ and the particular equation has a form $y_p = e^{\alpha x} \cdot Q_n(x)$
- If α is one the 2 different roots of the characteristic equation then $s = 1$ and the particular solution has a form $y_p = x \cdot e^{\alpha x} \cdot Q_n(x)$
- If α is the double root of the characteristic equation then $s = 2$ and the particular equation has a form $y_p = x^2 \cdot e^{\alpha x} \cdot Q_n(x)$

- **Case 2:** $f(x) = e^{\alpha x} \cdot [P_n(x) \cdot \cos(\beta x) \cdot Q_m(x) \cdot \sin(\beta x)]$

The particular solution of non-homogeneous equation has a form:

$$y_p = x^s \cdot e^{\alpha x} \cdot [H_k(x) \cdot \cos(\beta x) + T_m(x) \cdot \sin(\beta x)]$$

- If $\alpha + i\beta$ **is not the root** of the characteristic equation then **$s = 0$** and the particular solution has a form $y_p = e^{\alpha x} \cdot [H_k(x) \cdot \cos(\beta x) + T_k(x) \cdot \sin(\beta x)]$
- If $\alpha + i\beta$ **is one of the complex conjugate roots** of the characteristic equation then **$s = 1$** and the particular solution has a form $y_p = x \cdot e^{\alpha x} \cdot [H_k(x) \cdot \cos(\beta x) + T_k(x) \cdot \sin(\beta x)]$

6. Find the non-homogeneous equation:

- Step 1: Solve the homogeneous equation, use characteristic equation to find k_1, k_2 .
- Step 2: Evaluate the homogeneous equation.
- Step 3: Find a particular solution of non-homogeneous equation.