ORDINARY DIFFERENTIAL EQUATIONS

1. Separable Equations

- A **separable equation** is a first-order differential equation, which has a form:

$$P(x)dx + Q(y)dy = 0$$

Solution:

Integrate both sides, we receive:

$$\int P(x)dx + \int Q(y)dy = C$$

2. The differential Equation of the form

$$f_1(x).g_1(y)dx + f_2(x).g_2(y)dy = 0$$

Solution:

If $f_1(x), f_2(x), g_1(y), g_2(y) \neq 0$, then we divide both sides by $f_2(x).g_1(y)$. So: (1)

$$\frac{f_1(x)}{f_2(x)}dx + \frac{g_2(x)}{g_1(x)} = 0 \to \int \frac{f_1(x)}{f_2(x)}dx + \int \frac{g_2(y)}{g_1(y)}dy$$

3. COOL PROBLEM

(Liên quan đến the rate of change cần chú ý)

- The rate of change of the temperature T(t) is the derivative of $\frac{dT}{dt}$
- Acording to the Newton's law of cooling:

$$\frac{dT}{dt} = k(T - T_s)$$

When k is a constant of proportionality.

Separating the variables we have:

$$\frac{dT}{T - T_s} = kdt \leftrightarrow \frac{dT}{T - T_s} = kdt \tag{2}$$

$$\rightarrow \int \frac{dT}{T - T_s} = k \int dt \rightarrow \ln|T - T_s| = kt + \ln C \tag{3}$$

$$\rightarrow T - T_s = e^{kt + lnC} = e^{kt} \cdot e^{lnC} = Ce^{kt} \tag{4}$$

4. The first-order linear differential equation

- The first-order **linear** differential equation is one that can be put into the form:

$$\frac{dy}{dx} + P(x).y = Q(x) \tag{5}$$

- To solve the linear differential equation, multply both side by the **integrate factor** $e^{P(x)dx}$ and integrate both sides:

$$y = e^{-\int P(x)dx} \cdot \int Q(x) \cdot e^{\int P(x)dx} dx + C$$
 (6)

SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

- 1. Homogeneous and Non-Homogeneous Second Differential Equations.
 - Homogeneous Second Differential Equation has a form:

$$Ay'' + By' + Cy = 0(A, B, C \in \mathbb{R}; A \neq 0)$$
 (7)

- Non-homogeneous Second Differential Equation has a form:

$$Ay'' + By' + Cy = f(x)(A, B, C \in \mathbb{R}; A \neq 0)$$
 (8)

2. Let y_1, y_2 be any 2 solutions of the homogeneous linear differential equation (1), then:

$$C_1y_1 + C_2y_2$$

is also a **solution** of equation (1) where C_1, C_2 are arbitrary constants. (hàm tùy ý)

• The 2 function y_1, y_2 are **linear dependent** on $a \le x \le b$ if there exists constants C_1, C_2 not zero, such that:

$$C_1 y_1 + C_2 y_2 = 0$$

• The 2 function y_1, y_2 are linear independent on $a \le x \le b$ if:

$$C_1 y_1 + C_2 y_2 = 0$$

For all x such that $a \le x \le b$ implies $C_1 = C_2 = 0$.

3. Let y_1, y_2 be 2 real function each of which has the first derivative on the interval $a \leq x \leq b$. The determinant (định thức):

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x) \times y_2'(x) - y_1'(x) \times y_2(x)$$

4. If y_1, y_2 are 2 linearly independent solutions of equation (7), then every **homogeneous solution** y_h

$$y_h = C_1 y_1 + C_2 y_2$$

We have equation (7) as:

$$Ay'' + By' + Cy = 0$$

Consider that:

$$y = e^{kx}$$
$$y' = ke^{kx}$$
$$y" = k^{2}e^{kx}$$

We receive:

$$Ak^2e^{kx} + Bke^{kx} + Ce^{kx} = 0 (9)$$

$$\leftrightarrow Ak^2 + Bk + C = 0 \tag{10}$$

From equation (10), we have evaluate Δ of the equation:

• $\Delta > 0$, the equation has **real and different roots** $k_1, k_2 \neq 0$. Then:

$$y_1(x) = e^{k_1 x}$$
$$y_2(x) = e^{k_2 x}$$

The homogeneous solution is:

$$y_h = C_1 e^{k_1 x} + C_2 e^{k_2 x}$$

• $\Delta = 0$ then the characteristic equation has **double real root**: k = k0The the homogeneous solution is:

$$y_h = C_1 e^{k_0 x} + C_2 x e^{k_0 x}$$

• $\Delta < 0$, then characteristic equation has **complex conjugate roots** $k_1 = a + bi$ **and** $k_2 = a - bi$ Then the homogeneous solution is:

$$y_h = e^{ax}(C_1 cos(bx) + C_2 sin(bx))$$

- 5. Non-homogeneous Equation with constant coefficients
 - Case I: $f(x) = e^{\alpha x}.P_n(x)$

The particular solution of non-homogeneous equation has a form $y_p = x^s.e^{\alpha x}.Q_n(x)$

- If α is not the root of the characteristic equation then s =0 and the particular equation has a form $y_p = e^{\alpha x}.Q_n(x)$
- If α is one the 2 different roots of the characteristic equation then s =1 and the particular solution has a form $y_p x.e^{\alpha x}.Q_n(x)$
- If α is the double root of the characteristic equation then s=2 and the particular equation has a form $y_p=x^2.e^{\alpha x}.Q_n(x)$
- Case 2: $f(x) = e^{\alpha x} \cdot [P_n(x) \cdot cos(\beta x) \cdot Q_m(x) \cdot sin(\beta x)]$

The particular solution of non-homogeneous equation has a form:

$$y_p = x^s.e^{\alpha x}.[H_k(x).cos(\beta x) + T_m(x).sin(\beta x)]$$

- If $\alpha + i\beta$ is not the root of the characteristic equation then $\mathbf{s} = \mathbf{0}$ and the particular solution has a form $y_p = e^{\alpha x} . [H_k(x).cos(\beta x) + T_k(x).sin(\beta x)]$
- If $\alpha + i\beta$ is one of the complex conjugate roots of the characteristic equation then $\mathbf{s} = \mathbf{1}$ and the particular solution has a form $y_p = x.e^{\alpha x}.[H_k(x).cos(\beta x) + T_k(x).sin(\beta x)]$
- 6. Find the non-homogeneous equation:
 - Step 1:Solve the homogeneous equation, use characteristic equation to find k_1, k_2 .
 - Step 2:Evaluate the homogeneous equation.
 - Step 3:Find a particular solution of non-homogeneous equation.