INDEFINITE INTEGRALS

ELECTRONIC VERSION OF LECTURE

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OUTLINE

1. ANTIDERIVATIVES AND INDEFINITE INTEGRALS
2. TECHNIQUES OF INTEGRATION
3. INTEGRATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS
4. INTEGRATION OF NONRATIONAL FUNCTIONS
5. TRIGONOMETRIC INTEGRALS

CENSUS

Data on the growth of world population provided by the U.S. Census Bureau can be used to create a model of Earth's population growth. According to this model, **the rate of change** of the world's population since 1950 is given by $p(t) = -0,012.t^2 + 48.t - 47925$, where t is the calendar year and $p(t)$ is in millions of people per year.

1. Given that the population in 2000 was about 6000 million people, find an equation for $P(t)$, the total population as a function of the calendar year.
2. Use the equation $P(t)$ to predict the world population in 2050.

1. $P(t)$ is the **antiderivative** of $p(t)$

$$p(t) = -0,012.t^2 + 48.t - 47925$$

$$\Rightarrow P(t) = -0,012.\frac{t^3}{3} + 48.\frac{t^2}{2} - 47925.t + C$$

To find C, substitute 2000 for t and 6000 for $P(t)$. We receive $C = 31856000$ and

$$P(t) = -0,004.t^3 + 24.t^2 - 47925.t + 31856000$$

2. Substitute 2050 for t, according to the model the world population in 2050 should be about $P(2050) = 9250$ million people.

Definition 1.1

A function F is called an ***antiderivative of** f* *on an interval X, if $F(x)$ is continuous and differentiable on X and $F'(x) = f(x)$, or $dF(x) = f(x)dx$ for all $x \in X$.*

Theorem 1.1

If F is an antiderivative of f on an interval $X \subset \mathbb{R}$ then the most general antiderivative of f on X is $\Phi(x) = F(x) + C$, where C is an arbitrary constant

Example 1.1

The general antiderivative of $f(x) = x^2$ is $\frac{1}{3}x^3 + C$.

Definition 1.2

*Let F be any antiderivative of f on an interval $X \subset \mathbb{R}$. The **indefinite integral of $f(x)$** is defined by*

$$\Phi(x) = F(x) + C,$$

where C is an arbitrary constant.

- Indefinite integral is denoted by $\int f(x)\,dx$.
- The process of computing an integral is called **integration**.
- Here, $f(x)$ is called the **integrand** and the term dx identifies x as the **variable of integration**.

Some basic formulas of indefinite integrals I

1. $\int 0.dx = C.$
2. $\int 1.dx = x + C.$
3. $\int x^\alpha dx = \dfrac{x^{\alpha+1}}{\alpha+1} + C, \alpha \neq -1.$
4. $\int \dfrac{dx}{x} = \ln|x| + C.$
5. $\int a^x dx = \dfrac{a^x}{\ln a} + C, a > 0, a \neq 1.$
6. $\int e^x dx = e^x + C.$

Some basic formulas of indefinite integrals II

7. $$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C, x \neq \pm 1.$$
8. $$\int \frac{dx}{1+x^2} = \arctan x + C.$$
9. $$\int \sin x dx = -\cos x + C.$$
10. $$\int \cos x dx = \sin x + C.$$
11. $$\int \frac{dx}{\cos^2 x} = \tan x + C.$$
12. $$\int \frac{dx}{\sin^2 x} = -\cot x + C.$$

SOME BASIC FORMULAS OF INDEFINITE INTEGRALS III

13. $$\int \sinh x dx = \cosh x + C.$$
14. $$\int \cosh x dx = \sinh x + C.$$
15. $$\int \frac{dx}{\cosh^2 x} = \tanh x + C.$$
16. $$\int \frac{dx}{\sinh^2 x} = -\coth x + C.$$
17. $$\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln \left| \frac{x-a}{x+a} \right| + C.$$
18. $$\int \frac{dx}{\sqrt{x^2 + a}} = \ln \left| x + \sqrt{x^2 + a} \right| + C.$$

PROPERTIES

Rule I. If $a \neq 0$ then

$$\int af(x)\,dx = a\int f(x)\,dx.$$

Rule II.

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx.$$

Rule III. If $\int f(t)\,dt = F(t) + C$ then

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C,\ (a \neq 0).$$

EXAMPLE 1.2

Find $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}}$.

SOLUTION

Substitute $t = \dfrac{x}{a}$ from the formula

$$\int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + C \Rightarrow \int \frac{dx}{\sqrt{1-\left(\frac{x}{a}\right)^2}} = \frac{1}{\frac{1}{a}} \arcsin \frac{x}{a} + C$$

$$\Rightarrow \int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{dx}{a.\sqrt{1-\left(\frac{x}{a}\right)^2}} =$$

$$= \frac{1}{a} \cdot a. \arcsin \frac{x}{a} + C = \arcsin \frac{x}{a} + C.$$

Example 1.3

Find $\displaystyle\int \frac{dx}{x^2+a^2}$.

SOLUTION

Substitute $t = \dfrac{x}{a}$ from the formula

$$\int \frac{dt}{t^2+1} = \arctan t + C \Rightarrow \int \frac{dx}{\left(\frac{x}{a}\right)^2+1} = \frac{1}{\frac{1}{a}}\arctan\frac{x}{a} + C$$

$$\Rightarrow \int \frac{dx}{x^2+a^2} = \int \frac{dx}{a^2\left[\left(\frac{x}{a}\right)^2+1\right]} =$$

$$= \frac{1}{a^2}\cdot a.\arctan\frac{x}{a} + C = \frac{1}{a}\arctan\frac{x}{a} + C.$$

The substitution rule

Theorem 2.1

Let composite function $f(u(x))$ define on interval X, and let function $t = u(x)$ be differentiable on interval X. If $f(t)$ has antiderivative $F(t)$ on an interval $T \supseteq u(X)$ then

$$\boxed{\int f(u(x))du(x) = F(u(x)) + C.} \tag{1}$$

CASE I (THE SUBSTITUTION RULE)

If we can not compute the integral $\int g(x)\,dx$ directly, we often look for a new variable u and function $f(u)$ for which

$$\int g(x)\,dx = \int f(u(x)).u'(x)\,dx = \int f(u(x))\,du(x) =$$

$$= \left(\int f(t)\,dt\right)\Bigg|_{t=u(x)}$$

where the integral $\int f(t)\,dt$ is easier to evaluate than $\int g(x)\,dx$.

EXAMPLE 2.1

Find $\int \sin^3 x \cos x dx.$

SOLUTION

Let $t = \sin x, dt = \cos x dx$. This gives us

$$\int \sin^3 x \cos x dx = \int t^3 dt = \frac{t^4}{4} + C = \frac{\sin^4(x)}{4} + C.$$

CASE II (THE INVERSE SUBSTITUTION RULE)

In some cases, the integral $\int f(x)\,dx$ will be easier to evaluate if we change x by a new function $x=\varphi(t)$ with a new variable t. At this time, we have

$$f(x)\,dx = f(\varphi(t)).\varphi'(t)\,dt.$$

So

$$\int f(x)\,dx = \int f(\varphi(t))\varphi'(t)\,dt$$

In the result, we substitue $t=\varphi^{-1}(x)$, where φ^{-1} is the **inverse function** of φ.

Example 2.2

Evaluate $I = \int \sqrt{a^2 - x^2}\,dx$, $\quad a > 0$ *is a constant.*

SOLUTION Let $x = a\sin t$, where $-\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}\cdot$ Then $t = \arcsin\frac{x}{a}$, $dx = a\cos t\,dt$.

$$\sqrt{a^2 - x^2} = \sqrt{a^2 - a^2\sin^2 t} = \sqrt{a^2\cos^2 t} = a|\cos t| =$$

$= a\cos t$. Note that $\cos t \geqslant 0$ because $-\frac{\pi}{2} \leqslant t \leqslant \frac{\pi}{2}\cdot$ Thus the Substitution Rule gives

$$\int \sqrt{a^2 - x^2}\,dx = \int a^2\cos^2 t\,dt =$$

$$= \frac{a^2}{2}\int (1 + \cos 2t)\,dt = \frac{a^2}{2}\left(t + \frac{1}{2}\sin 2t\right) + C =$$

Substituting $t = \arcsin \frac{x}{a}$, we have

$$I = \frac{a^2}{2}\left[\arcsin\frac{x}{a} + \sin\left(\arcsin\frac{x}{a}\right)\cos\left(\arcsin\frac{x}{a}\right)\right] + C =$$

$$= \frac{a^2}{2}\left[\arcsin\frac{x}{a} + \sin\left(\arcsin\frac{x}{a}\right)\sqrt{1-\sin^2\left(\arcsin\frac{x}{a}\right)}\right] + C$$

$$= \frac{a^2}{2}\left(\arcsin\frac{x}{a} + \frac{x}{a}\sqrt{1-\left(\frac{x}{a}\right)^2}\right) + C.$$

$$\boxed{\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}x\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{a} + C} \qquad (2)$$

If u and v are differentiable functions, then

$$\frac{d}{dx}[u(x).v(x)] = u(x).v'(x) + u'(x).v(x)$$

$$\Rightarrow \int [u(x).v'(x) + u'(x).v(x)]\,dx = u(x).v(x)$$

$$\Rightarrow \int u(x).v'(x)\,dx + \int u'(x).v(x)\,dx = u(x).v(x)$$

THEOREM 2.2

If functions $u = u(x)$ and $v = v(x)$ are differentiable on interval $X \subset \mathbb{R}$, then

$$\boxed{\int u\,dv = uv - \int v\,du} \tag{3}$$

EXAMPLE 2.3

Find $I = \int x \sin x dx$

SOLUTION

Let

$$u = x, \quad dv = \sin x dx.$$

Then

$$du = dx, \quad v = -\cos x.$$

Thus, using formula for integration by parts, we have

$$I = \int x \sin x dx = x(-\cos x) - \int (-\cos x) dx =$$

$$= -x \cos x + \int \cos x dx = -x \cos x + \sin x + C.$$

Example 2.4

Find $I = \int \ln x dx$

SOLUTION Let

$$u = \ln x, \quad dv = dx.$$

Then

$$du = \frac{1}{x} dx, \quad v = x.$$

Integrating by parts, we get

$$I = \int \ln x dx = x \ln x - \int x \cdot \frac{dx}{x} =$$

$$= -x \ln x - \int dx = x \ln x - x + C.$$

EXAMPLE 2.5

$$Find\ I = \int \sqrt{x^2+a}.dx, \quad a = constant$$

SOLUTION

Suppose that we choose $u = \sqrt{x^2+a}$, $\quad dv = dx$. Then

$$du = \frac{xdx}{\sqrt{x^2+a}}, \quad v = x.$$

Thus, using formula for integration by parts, we have

$$I = \int \sqrt{x^2+a}.dx = x\sqrt{x^2+a} - \int x \cdot \frac{xdx}{\sqrt{x^2+a}} =$$

$$= x\sqrt{x^2+a} - \int \frac{x^2+a}{\sqrt{x^2+a}}dx + \int \frac{adx}{\sqrt{x^2+a}}$$

$$I = x\sqrt{x^2+a} - \int \sqrt{x^2+a}.dx + a\int \frac{dx}{\sqrt{x^2+a}} =$$

$$= x\sqrt{x^2+a} - I + a\int \frac{dx}{\sqrt{x^2+a}}$$

$$\Rightarrow 2I = x\sqrt{x^2+a} + a\int \frac{dx}{\sqrt{x^2+a}}$$

So

$$\boxed{\int \sqrt{x^2+a}.dx = \frac{x\sqrt{x^2+a}}{2} + \frac{a}{2}\ln\left|x+\sqrt{x^2+a}\right| + C} \tag{4}$$

Note. We use **method of integration by parts** in the following cases:

1. $\displaystyle\int x^k \ln^m x dx \quad (k, m \in \mathbb{Z}_0)$
2. $\displaystyle\int x^k e^{ax} dx \quad (k \in \mathbb{Z}_0)$
3. $\displaystyle\int x^k \sin ax dx, \quad \int x^k \cos bx dx \quad (k \in \mathbb{Z}_0)$
4. $\displaystyle\int e^{ax} \sin bx dx, \quad \int e^{ax} \cos bx dx.$

REDUCTION FORMULA

EXAMPLE 2.6

Prove the reduction formula

$$\boxed{I_{n+1} = \frac{1}{2na^2} \cdot \frac{x}{(x^2+a^2)^n} + \frac{2n-1}{2na^2} I_n} \tag{5}$$

where

$$\boxed{I_n = \int \frac{dx}{(x^2+a^2)^n} \, (n \in \mathbb{N})} \tag{6}$$

SOLUTION

Let

$$u = \frac{1}{(x^2+a^2)^n}, dv = dx.$$

Then

$$du = \frac{-2nxdx}{(x^2+a^2)^{n+1}}, v = x.$$

So integration by parts gives

$$I_n = \frac{x}{(x^2+a^2)^n} + 2n\int \frac{x^2 dx}{(x^2+a^2)^{n+1}} =$$

$$= \frac{x}{(x^2+a^2)^n} + 2n\int \frac{(x^2+a^2)-a^2}{(x^2+a^2)^{n+1}} dx =$$

$$= \frac{x}{(x^2+a^2)^n} + 2nI_n - 2na^2 I_{n+1}$$

Therefore

$$I_{n+1} = \frac{1}{2na^2}\left[\frac{x}{(x^2+a^2)^n} + (2n-1)I_n\right].$$

Since

$$I_1 = \int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\frac{x}{a} + C,$$

so when $n = 1$ we can calculate I_2, and then I_3, etc.

EXAMPLE 2.7

Find $\displaystyle\int \frac{dx}{(x^2+4)^2}$

SOLUTION

By formula (5) we have $a=2, n=1$. Therefore

$$\int \frac{dx}{(x^2+4)^2} = \frac{1}{8}\cdot\frac{x}{x^2+4} + \frac{1}{8}\int\frac{dx}{x^2+4} =$$

$$= \frac{1}{8}\cdot\frac{x}{x^2+4} + \frac{1}{16}\cdot\arctan\frac{x}{2} + C.$$

Definition 3.1

The fractions of the forms

$$1)\ \frac{A}{x-a}; \qquad 2)\ \frac{A}{(x-a)^k}\ (k=2,3,\ldots);$$

$$3)\ \frac{Mx+N}{x^2+px+q}; \qquad 4)\ \frac{Mx+N}{(x^2+px+q)^m}(m=2,3,\ldots)$$

where A, a, M, N, p, q are constants and $q-\frac{p^2}{4}>0$, are called ***partial fractions***.

The indefinite integrals of these partial fractions are:

1. $\displaystyle\int \frac{A}{x-a}dx = A\ln|x-a| + C.$

2. $\displaystyle\int \frac{A}{(x-a)^n}dx = \frac{A}{(1-n)(x-a)^{n-1}} + C,\ (n \neq 1).$

3. $\displaystyle\int \frac{Mx+N}{x^2+px+q}dx = \frac{M}{2}\int \frac{(2x+p)dx}{x^2+px+q} +$

$$\left(N - \frac{Mp}{2}\right)\int \frac{dx}{(x+p/2)^2+q-p^2/4} = \frac{M}{2}\ln|x^2+px+q| +$$

$$\left(N - \frac{Mp}{2}\right)\cdot \frac{1}{\sqrt{q-p^2/4}}\arctan\frac{x+p/2}{\sqrt{q-p^2/4}} + C$$

4. $$\int \frac{Mx+N}{(x^2+px+q)^m}dx =$$

$$\frac{M}{2}\int \frac{2x+p}{(x^2+px+q)^m}dx + \left(N-\frac{Mp}{2}\right)\int \frac{dx}{(x^2+px+q)^m} =$$

$$\frac{M}{2}\cdot\frac{(x^2+px+q)^{-m+1}}{-m+1} + \left(N-\frac{Mp}{2}\right)\int \frac{dx}{(x^2+px+q)^m},$$

where

$$\int \frac{dx}{(x^2+px+q)^m} = \int \frac{dx}{\left[\left(x+\frac{p}{2}\right)^2+\left(q-\frac{p^2}{4}\right)\right]^m}$$

can be found by putting $t = x+\frac{p}{2}, q-\frac{p^2}{4} = a^2$ and applying reduction formula.

Integration of rational functions $I = \int \frac{P_n(x)}{Q_m(x)} dx$,

where $P_n(x), Q_m(x)$ are polynomials with degrees n and m respectively.

Method of partial fractions

1. If $n \geqslant m$ then we divide $P_n(x)$ into $Q_m(x)$

$$\frac{P_n(x)}{Q_m(x)} = S(x) + \frac{R(x)}{Q_m(x)}$$

2. If $n < m$ then we factorize the denominator as

$$Q_m(x) = (x-a)^k \dots (x^2 + px + q)^\ell,$$

where $k + \dots + 2\ell = m, \frac{p^2}{4} - q < 0$

The partial fraction decomposition of the integrand $\dfrac{P_n(x)}{Q_m(x)}$ has the form

$$\frac{P_n(x)}{Q_m(x)} = \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \ldots + \frac{A_k}{(x-a)^k} +$$

$$+ \ldots + \frac{M_1 x + N_1}{x^2+px+q} + \ldots + \frac{M_\ell x + N_\ell}{(x^2+px+q)^\ell}$$

In order to find the coefficients $A_1, A_2, \ldots, A_k$, $M_1, \ldots, M_\ell, N_1, \ldots, N_\ell$, we can choose values x that simplify the equation.

Example 3.1

Find $\int \frac{x^3+x}{x-1}dx.$

SOLUTION Since the degree of the numerator is greater than the degree of the denominator, we perform the division.

$$\int \frac{x^3+x}{x-1}dx = \int \left(x^2+x+2+\frac{2}{x-1}\right)dx =$$

$$= \frac{x^3}{3}+\frac{x^2}{2}+2x+2\ln|x-1|+C.$$

Example 3.2

Evaluate $I = \int \frac{x^2+2x+6}{(x-1)(x-2)(x-4)}dx.$

SOLUTION The method of partial fractions gives

$$\frac{x^2+2x+6}{(x-1)(x-2)(x-4)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-4}$$

$$\Rightarrow x^2+2x+6 \equiv A(x-2)(x-4) + B(x-1)(x-4)+$$

$$+C(x-1)(x-2), \forall x \in \mathbb{R}$$

We put $x = 1$ and get: $9 = A(1-2)(1-4) \Rightarrow A = 3$
We put $x = 2$, we get: $14 = B(2-1)(2-4) \Rightarrow B = -7$
We put $x = 4$, we get: $30 = C(4-1)(4-2) \Rightarrow C = 5$

$$I = 3\int \frac{dx}{x-1} - 7\int \frac{dx}{x-2} + 5\int \frac{dx}{x-4} =$$

$$= 3\ln|x-1| - 7\ln|x-2| + 5\ln|x-4| + C =$$

$$= \ln\left|\frac{(x-1)^3(x-4)^5}{(x-2)^7}\right| + C$$

EXAMPLE 3.3

Evaluate $I = \int \frac{x^2+1}{(x-1)^3(x+3)}dx.$

SOLUTION The partial fraction decompostion is

$$\frac{x^2+1}{(x-1)^3(x+3)} = \frac{A}{(x-1)^3} + \frac{B}{(x-1)^2} + \frac{C}{x-1} + \frac{D}{x+3}$$

$$\Rightarrow x^2+1 \equiv A(x+3) + B(x-1)(x+3) + C(x-1)^2(x+3) +$$

$$+D(x-1)^3, \forall x \in \mathbb{R}$$

We put $x = 1$, we get: $2 = 4A \Rightarrow A = \frac{1}{2}$

We put $x = -3$, we get: $10 = -64D \Rightarrow D = -\frac{5}{32}$

Now we equate coefficients of x^3 in both sides, we get $C + D = 0 \Rightarrow C = \frac{5}{32}$

We put $x = 0$, we get: $1 = 3A - 3B + 3C - D \Rightarrow B = \frac{3}{8}$. So $I =$

$$= \frac{1}{2}\int \frac{dx}{(x-1)^3} + \frac{3}{8}\int \frac{dx}{(x-1)^2} + \frac{5}{32}\int \frac{dx}{x-1} - \frac{5}{32}\int \frac{dx}{x+3} =$$

$$= -\frac{1}{4(x-1)^2} - \frac{3}{8(x-1)} + \frac{5}{32}\ln\left|\frac{x-1}{x+3}\right| + C$$

Example 3.4

Evaluate $I = \int \frac{dx}{x^5 - x^2}$

SOLUTION We factorize the denominator

$$x^5 - x^2 = x^2(x-1)(x^2+x+1).$$

Then the partial fraction decompostion is

$$\frac{1}{x^5 - x^2} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-1} + \frac{Dx+E}{x^2+x+1}$$

$$\Rightarrow 1 \equiv A(x-1)(x^2+x+1) + Bx(x-1)(x^2+x+1) +$$

$$+ Cx^2(x^2+x+1) + (Dx+E)x^2(x-1), \forall x \in \mathbb{R}.$$

We put $x = 0$, we get: $1 = -A \Rightarrow A = -1$

We put $x = 1$, we get: $1 = 3C \Rightarrow C = \dfrac{1}{3}$

Now we equate coefficients of x^4, x^3, x^2 in both sides

$$1 \equiv A(x^3-1)+B(x^4-x)+C(x^4+x^3+x^2)+Dx^4+Ex^3-Dx^3-Ex^2,$$

we have

$$\begin{cases} B+C+D &= 0 \\ A+C+E-D &= 0 \\ C-E &= 0 \end{cases} \Leftrightarrow \begin{cases} B = 0 \\ D = -\dfrac{1}{3} \\ E = \dfrac{1}{3} \end{cases}$$

Therefore

$$I=-\int\frac{dx}{x^2}+\frac{1}{3}\int\frac{dx}{x-1}-\frac{1}{3}\int\frac{x-1}{x^2+x+1}dx=$$

$$=\frac{1}{x}+\frac{1}{3}\ln|x-1|-\frac{1}{6}\int\frac{2x+1-3}{x^2+x+1}dx=$$

$$=\frac{1}{x}+\frac{1}{3}\ln|x-1|-\frac{1}{6}\ln(x^2+x+1)+\frac{1}{2}\int\frac{dx}{x^2+x+1}=$$

$$=\frac{1}{x}+\frac{1}{3}\ln|x-1|-\frac{1}{6}\ln(x^2+x+1)+\frac{1}{2}\int\frac{d\left(x+\frac{1}{2}\right)}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}=$$

$$=\frac{1}{x}+\frac{1}{6}\ln\frac{(x-1)^2}{x^2+x+1}+\frac{1}{\sqrt{3}}\arctan\frac{2x+1}{\sqrt{3}}+C.$$

$$\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{p_1}, \left(\frac{ax+b}{cx+d}\right)^{p_2}, \dots, \left(\frac{ax+b}{cx+d}\right)^{p_n}\right) dx$$

where

- $p_1, p_2, \dots, p_n$ are rational numbers,
- a, b, c, d are real numbers.

SOLUTION Let

$$\frac{ax+b}{cx+d} = t^m,$$

where m is the **lowest common multiple** of **denominators of rational numbers** $p_1, p_2, \dots, p_n$

Example 4.1

Evaluate $I = \int \frac{dx}{\sqrt[3]{(2x+1)^2} - \sqrt{2x+1}}$

SOLUTION Let

$$2x+1 = t^6 \Rightarrow x = \frac{t^6-1}{2}, dx = 3t^5 dt.$$

$$I = \int \frac{3t^5 dt}{t^4 - t^3} = 3\int \frac{t^2 dt}{t-1} = 3\int \left(t+1+\frac{1}{t-1}\right) dt =$$

$$= \frac{3}{2}t^2 + 3t + 3\ln|t-1| + C.$$

Substitute $t = (2x+1)^{1/6}$, we have

$$I = \frac{3}{2}(2x+1)^{1/3} + 3(2x+1)^{1/6} + 3\ln|(2x+1)^{1/6} - 1| + C.$$

$$\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

Example 4.2

Evaluate $I = \displaystyle\int \frac{dx}{\sqrt{x^2+2x+5}}$

SOLUTION

$$I = \int \frac{dx}{\sqrt{x^2+2x+1+4}} = \int \frac{d(x+1)}{\sqrt{(x+1)^2+4}} =$$

$$= \ln\left|x+1+\sqrt{x^2+2x+5}\right| + C.$$

EXAMPLE 4.3

Evaluate $I = \displaystyle\int \frac{dx}{\sqrt{-3x^2+4x-1}}$

SOLUTION

$$I = \int \frac{dx}{\sqrt{3\left[\frac{1}{9} - \left(x^2 - 2\cdot\frac{2}{3}\cdot x + \frac{4}{9}\right)\right]}} = \frac{1}{\sqrt{3}}\int \frac{d\left(x-\frac{2}{3}\right)}{\sqrt{\frac{1}{9} - \left(x-\frac{2}{3}\right)^2}} =$$

$$= \frac{1}{\sqrt{3}}\arcsin\frac{x-2/3}{1/3} + C = \frac{1}{\sqrt{3}}\arcsin(3x-2) + C.$$

$$\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$$

SOLUTION

$$\int \frac{Ax+B}{\sqrt{ax^2+bx+c}}dx = \int \frac{\frac{A}{2a}(2ax+b)+B-\frac{Ab}{2a}}{\sqrt{ax^2+bx+c}}dx$$

$$= \frac{A}{2a}\int \frac{d(ax^2+bx+c)}{\sqrt{ax^2+bx+c}} + \left(B-\frac{Ab}{2a}\right)\int \frac{dx}{\sqrt{ax^2+bx+c}}$$

EXAMPLE 4.4

Evaluate $I = \displaystyle\int \frac{5x-3}{\sqrt{2x^2+8x+1}}dx$

SOLUTION

$$I = \int \frac{5x-3}{\sqrt{2x^2+8x+1}}dx = \int \frac{\frac{5}{4}(4x+8)-13}{\sqrt{2x^2+8x+1}} =$$

$$= \frac{5}{4}\int \frac{4x+8}{\sqrt{2x^2+8x+1}}dx - 13\int \frac{dx}{\sqrt{2x^2+8x+1}} =$$

$$= \frac{5}{2}\sqrt{2x^2+8x+1} - \frac{13}{\sqrt{2}}\int \frac{dx}{\sqrt{(x+2)^2-\frac{7}{2}}} =$$

$$= \frac{5}{2}\sqrt{2x^2+8x+1} - \frac{13}{\sqrt{2}}\ln\left|x+2+\sqrt{x^2+4x+\frac{1}{2}}\right| + C.$$

$$\int \frac{dx}{(x-\alpha)\sqrt{ax^2+bx+c}}$$

SOLUTION Let $x-\alpha = \dfrac{1}{t}$

EXAMPLE 4.5

Evaluate $I = \displaystyle\int \frac{dx}{x\sqrt{5x^2-2x+1}}$

SOLUTION Let $x = \dfrac{1}{t} \Rightarrow dx = -\dfrac{1}{t^2}dt$. Then

$$I = -\int \frac{dt}{t^2\cdot\frac{1}{t}\sqrt{\frac{5}{t^2}-\frac{2}{t}+1}} = -\int \frac{dt}{\sqrt{t^2-2t+5}} =$$

$$I = -\ln\left|t - 1 + \sqrt{t^2 - 2t + 5}\right| + C =$$

$$= -\ln\left|\frac{1}{x} - 1 + \sqrt{\frac{1}{x^2} - \frac{2}{x} + 5}\right| + C =$$

$$= -\ln\left|\frac{1 - x + \sqrt{5x^2 - 2x + 1}}{x}\right| + C.$$

$$\int \frac{dx}{(x^2+\alpha)\sqrt{ax^2+c}}$$

SOLUTION Let $t = \sqrt{a + \dfrac{c}{x^2}}$

EXAMPLE 4.6

Evaluate $I = \displaystyle\int \frac{dx}{(x^2+2)\sqrt{x^2-1}}$

SOLUTION $I = \displaystyle\int \frac{xdx}{x^2(x^2+2)\sqrt{1-\dfrac{1}{x^2}}}.$

Let $t = \sqrt{1 - \dfrac{1}{x^2}} \Rightarrow x^2 = \dfrac{1}{1 - t^2} \Rightarrow xdx = \dfrac{tdt}{(1 - t^2)^2}$

$$I = \int \frac{tdt}{(1-t^2)^2 \cdot \dfrac{1}{1-t^2} \cdot \left(\dfrac{1}{1-t^2} + 2\right).t} =$$

$$= \int \frac{dt}{2(3/2 - t^2)} = \frac{1}{2\sqrt{6}} \left[\ln \left| \frac{\sqrt{3/2} + t}{\sqrt{3/2} - t} \right| \right] + C$$

So

$$I = \frac{1}{2\sqrt{6}} \left[\ln \left| \frac{\sqrt{3/2} + \sqrt{1 - \dfrac{1}{x^2}}}{\sqrt{3/2} - \sqrt{1 - \dfrac{1}{x^2}}} \right| \right] + C$$

$\int R(\sin x, \cos x)\, dx$

Where $R(\sin x, \cos x)$ is a rational function with respect to variables $\sin x, \cos x$.

SOLUTION Let $t = \tan \frac{x}{2} \cdot$ Then

$$x = 2\arctan t, dx = \frac{2dt}{1+t^2},$$

$$\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}$$

Example 5.1

Evaluate $I = \int \frac{dx}{\sin x}$

SOLUTION

Let $t = \tan\frac{x}{2}\cdot$ Then

$$\sin x = \frac{2t}{1+t^2}, dx = \frac{2dt}{1+t^2}.$$

So

$$I = \int \frac{\frac{2dt}{1+t^2}}{\frac{2t}{1+t^2}} = \int \frac{dt}{t} = \ln|t| + C = \ln\left|\tan\frac{x}{2}\right| + C.$$

Example 5.2

Evaluate $I = \int \frac{dx}{\cos x}\cdot$

SOLUTION Let $t = \tan\frac{x}{2}\cdot$ Then

$$\cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2dt}{1+t^2}\cdot$$

So

$$I = \int \frac{\frac{2dt}{1+t^2}}{\frac{1-t^2}{1+t^2}} = \int \frac{2dt}{1-t^2} = \int \frac{(1+t)+(1-t)dt}{(1-t)(1+t)} =$$

$$= -\ln|1-t| + \ln|1+t| + C = \ln\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right| + C.$$

Example 5.3

Evaluate $I = \displaystyle\int \frac{dx}{4\sin x + 3\cos x + 5}$

SOLUTION Let $t = \tan\dfrac{x}{2}\cdot$ Then

$$\sin x = \frac{2t}{1+t^2}, \cos x = \frac{1-t^2}{1+t^2}, dx = \frac{2dt}{1+t^2}\cdot$$

So

$$I = \int \frac{\frac{2dt}{1+t^2}}{4.\frac{2t}{1+t^2} + 3.\frac{1-t^2}{1+t^2} + 5} = 2\int \frac{dt}{2t^2 + 8t + 8} =$$

$$= \int \frac{dt}{(t+2)^2} = -\frac{1}{t+2} + C = -\frac{1}{\tan\frac{x}{2} + 2} + C.$$

SOME SPECIAL CASES

1. If $R(\sin x, \cos x)$ is odd function with respect to $\sin x$, that means $R(-\sin x, \cos x) = -R(\sin x, \cos x)$, then we let $t = \cos x$.
2. If $R(\sin x, \cos x)$ is odd function with respect to $\cos x$, that means $R(\sin x, -\cos x) = -R(\sin x, \cos x)$, then we let $t = \sin x$.
3. If $R(\sin x, \cos x)$ is even function with respect to $\sin x, \cos x$, that means

$$R(-\sin x, -\cos x) = R(\sin x, \cos x),$$

then we let $t = \tan x$.

EXAMPLE 5.4

Evaluate $I = \displaystyle\int \frac{(\sin x + \sin^3 x)\,dx}{\cos 2x}$

SOLUTION Since the integrand is odd function with respect to $\sin x$, then we let

$t = \cos x \Rightarrow dt = -\sin x dx, \sin^2 x = 1 - t^2, \cos 2x = 2t^2 - 1.$

Therefore

$$I = \int \frac{(2-t^2)(-dt)}{2t^2-1} = \int \frac{(t^2-2)\,dt}{2t^2-1} = \frac{1}{2}\int dt - \frac{3}{2}\int \frac{dt}{2t^2-1} =$$

$$= \frac{t}{2} - \frac{3}{2\sqrt{2}} \ln\left|\frac{t\sqrt{2}-1}{t\sqrt{2}+1}\right| + C = \frac{1}{2}\cos x - \frac{3}{2\sqrt{2}} \ln\left|\frac{\sqrt{2}\cos x - 1}{\sqrt{2}\cos x + 1}\right| + C.$$

EXAMPLE 5.5

Evaluate $I = \displaystyle\int \frac{(\cos^3 x + \cos^5 x)\,dx}{\sin^2 x + \sin^4 x}$

SOLUTION Since the integrand is odd function with respect to $\cos x$, then we let $t = \sin x \Rightarrow dt = \cos x dx$, $\cos^2 x = 1 - t^2$. Therefore

$$I = \int \frac{\cos^2 x(1+\cos^2 x)\cos x dx}{\sin^2 x + \sin^4 x} = \int \frac{(1-t^2)(2-t^2)\,dt}{t^2 + t^4} =$$

$$= \int \left(1 + \frac{2}{t^2} - \frac{6}{1+t^2}\right) dt = t - \frac{2}{t} - 6\arctan t + C =$$

$$= \sin x - \frac{2}{\sin x} - 6\arctan(\sin x) + C.$$

EXAMPLE 5.6

Evaluate $I = \displaystyle\int \frac{dx}{\sin^2 x + 2\sin x \cos x - \cos^2 x}$

SOLUTION

Since the integrand is even function with respect to $\sin x, \cos x$, then we let

$$t = \tan x \Rightarrow \sin x = \frac{t}{\sqrt{1+t^2}}, \cos x = \frac{1}{\sqrt{1+t^2}}$$

$$\Rightarrow x = \arctan t, dx = \frac{dt}{1+t^2}.$$

Therefore

$$I=\int \frac{\frac{dt}{1+t^2}}{\frac{t^2}{1+t^2}+\frac{2t}{\sqrt{1+t^2}}\cdot\frac{1}{\sqrt{1+t^2}}-\frac{1}{1+t^2}}=$$

$$=\int\frac{dt}{t^2+2t-1}=\int\frac{d(t+1)}{(t+1)^2-2}=$$

$$=\frac{1}{2\sqrt{2}}\ln\left|\frac{t+1-\sqrt{2}}{t+1+\sqrt{2}}\right|+C=$$

$$=\frac{1}{2\sqrt{2}}\ln\left|\frac{\tan x+1-\sqrt{2}}{\tan x+1+\sqrt{2}}\right|+C$$

$\int \sin^m x \cos^n x dx$

1. If $n > 0$ is odd, then substitute $t = \sin x$.
2. If $m > 0$ is odd, then substitute $t = \cos x$.
3. If both $m, n > 0$ are even then we use the trigonometric identities

$$\sin x \cos x = \frac{1}{2}\sin 2x,$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x),$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x).$$

EXAMPLE 5.7

Find $I = \int \sin^4 x \cos^5 x dx$

SOLUTION Let $t = \sin x$, then $dt = \cos x dx$. So

$$I = \int t^4(1-t^2)^2 dt = \int (t^4 - 2t^6 + t^8) dt =$$

$$= \frac{1}{5}t^5 - \frac{2}{7}t^7 + \frac{1}{9}t^9 + C =$$

$$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C.$$

EXAMPLE 5.8

Find $I = \int \dfrac{\sin^3 x dx}{\cos x.\sqrt[3]{\cos x}}$

SOLUTION Let $t = \cos x$, then $dt = -\sin x dx$. So

$$I = \int (1 - \cos^2 x) \cos^{-4/3} x \sin x dx =$$

$$= -\int (1 - t^2) t^{-4/3} dt = -\int t^{-4/3} dt + \int t^{2/3} dt =$$

$$= 3t^{-1/3} + \frac{3}{5} t^{5/3} + C = \frac{3}{\sqrt[3]{\cos x}} + \frac{3}{5} \cos x.\sqrt[3]{\cos^2 x} + C.$$

EXAMPLE 5.9

Find $I = \int \sin^2 x \cos^2 x dx$

SOLUTION

$$I = \int \frac{1}{4}\sin^2 2x = \frac{1}{8}\int (1-\cos 4x)dx =$$

$$= \frac{1}{8}\int dx - \frac{1}{8}\int \cos 4x dx = \frac{1}{8}x - \frac{1}{32}\sin 4x + C.$$

$\int \sin mx \cos nx dx, \int \cos mx \cos nx dx, \int \sin mx \sin nx dx$

We can make use of set of the trigonometric identities

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

EXAMPLE 5.10

Find $I = \int \cos x \cos \frac{x}{2} \cos \frac{x}{4} dx$

SOLUTION

$$I = \frac{1}{2}\int \left(\cos\frac{3x}{2} + \cos\frac{x}{2}\right)\cos\frac{x}{4}dx =$$

$$= \frac{1}{2}\int \cos\frac{3x}{2}\cos\frac{x}{4}dx + \frac{1}{2}\int \cos\frac{x}{2}\cos\frac{x}{4}dx =$$

$$= \frac{1}{4}\int \left(\cos\frac{7x}{4} + \cos\frac{5x}{4}\right)dx + \frac{1}{4}\int \left(\cos\frac{3x}{4} + \cos\frac{x}{4}\right)dx =$$

$$= \frac{1}{7}\sin\frac{7x}{4} + \frac{1}{5}\sin\frac{5x}{4} + \frac{1}{3}\sin\frac{3x}{4} + \sin\frac{x}{4} + C.$$

$$\int \frac{a_1 \sin x + b_1 \cos x}{a_2 \sin x + b_2 \cos x} dx$$

SOLUTION We express numerator as

$$a_1 \sin x + b_1 \cos x = A(a_2 \sin x + b_2 \cos x)' + B(a_2 \sin x + b_2 \cos x)$$

$$\Leftrightarrow a_1 \sin x + b_1 \cos x = A(a_2 \cos x - b_2 \sin x) +$$

$$+ B(a_2 \sin x + b_2 \cos x)$$

$$\Leftrightarrow a_1 \sin x + b_1 \cos x = (Ba_2 - Ab_2) \sin x + (Aa_2 + Bb_2) \cos x$$

$$\Rightarrow \begin{cases} a_1 = Ba_2 - Ab_2 \\ b_1 = Aa_2 + Bb_2 \end{cases}$$

$\Rightarrow$ Solve this system of equations to find out A, B.

Therefore $I = A \ln |a_2 \sin x + b_2 \cos x| + Bx + C.$

EXAMPLE 5.11

Find $I = \displaystyle\int \frac{2\sin x + 3\cos x}{\sin x + 3\cos x} dx$

SOLUTION

$$2\sin x + 3\cos x = A(\sin x + 3\cos x)' + B(\sin x + 3\cos x)$$

$$\Leftrightarrow 2\sin x + 3\cos x = A(\cos x - 3\sin x) + B(\sin x + 3\cos x)$$

$$\Leftrightarrow 2\sin x + 3\cos x = (B - 3A)\sin x + (A + 3B)\cos x$$

$$\Rightarrow \begin{cases} 2 = B - 3A \\ 3 = A + 3B \end{cases} \Leftrightarrow \begin{cases} A = -\frac{3}{10} \\ B = \frac{11}{10} \end{cases}$$

Therefore $I = -\dfrac{3}{10}\ln|\sin x + 3\cos x| + \dfrac{11}{10}x + C.$

THANK YOU FOR YOUR ATTENTION