Report

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Geometry and Kinematics

Exercise 1

1. The end-effector of the robot lies on the surface of a spherical shell, defined by the radius of q_3.

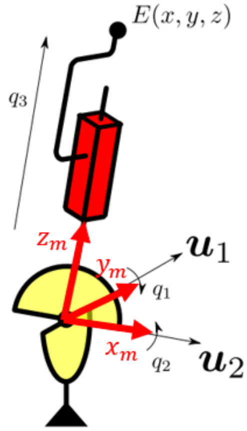

Figure 1: A 3 DOF robotic leg with 2 revolute and 1 prismatic joints

2. The problem can be solved by using Screw theory. Let us consider that the base frame (F_0) is located at the intersection of u_1 and u_2. The x_M axis is along the direction vector u_2 and y_M axis is along direction vector u_1. The initial base position can be defined as a point, since there isn't actually any link length mentioned in the problem.
   The screw **S** is defined by a unit direction $\hat{s}$, the position of a point s_0 on this axis with respect to a reference frame and pitch h.

$$S = \begin{bmatrix} \hat{s} \\ \mathbf{s_0} \times \hat{s} + h\hat{s} \end{bmatrix} \tag{1}$$

   Screw 1 (along u_1 axis):

$$\hat{s_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \; s_{1o} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; S_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2}$$

   Screw 2 (along u_2 axis):

$$\hat{s_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \; s_{1o} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \; S_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{3}$$

   Screw 3 (along the direction of prismatic axis):

$$\hat{s_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \; S_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \tag{4}$$

Now, the end-effector frame is consider to be aligned with the base frame. Since there is no translation of the zero configuration of the robot, so the transformation matrix from the base to end-effector at zer configuration can be expressed as:

$$ {}^{0}T_{ee}||_{0} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{5} $$

The screw axes in matrix screw representation $[\mathbf{S}] \in \mathrm{se}(3)$ are given by:

$$ [S_1] = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} [S_2] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} [S_3] = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6} $$

Now the forward kinematics of the mechanism can be obtained by using:

$$ {}^{\mathbf{0}}\mathbf{T}_{ee}(\mathbf{q}) = \exp([\mathbf{S}_1]\theta_1)\exp([\mathbf{S}_2]\theta_2)\exp([\mathbf{S}_3]\theta_3)\,{}^{\mathbf{0}}\mathbf{T}_4(0) \tag{7} $$

by using 7,

$$ {}^{\mathbf{0}}\mathbf{T}_{ee}(\mathbf{q}) = \begin{bmatrix} \cos(q_1) & \sin(q_1)\sin(q_2) & \sin(q_1)\cos(q_2) & q_3\sin(q_1)\cos(q_2) \\ 0 & \cos(q_2) & -\sin(q_2) & -q_3\sin(q_2) \\ -\sin(q_1) & \sin(q_2)\cos(q_1) & \cos(q_1)\cos(q_2) & q_3\cos(q_1)\cos(q_2) \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8} $$

So, we have to use the last column of the first 3 rows of 8,

$$ \begin{aligned} \mathbf{x} &= q_3\sin(q_1)\cos(q_2) \\ \mathbf{y} &= -q_3\sin(q_2) \\ \mathbf{z} &= q_3\cos(q_1)\cos(q_2) \end{aligned} \tag{9} $$

3. By exploiting equation 9, it can be formulated that,

$$\mathbf{q_1} = \tan^{-1}\left(\frac{x}{z}\right) \quad \mathbf{q_2} = \tan^{-1}\left(\frac{-y\sin q_1}{x}\right) \quad \mathbf{q_3} = \left(\frac{-y}{\sin q_2}\right) \tag{10}$$

Since for any fixed q_1, there are 2 possible q_2 and q_3 (without considering the joint limits), so the total number of inverse kinematics solutions are 2^2 or 4.

4. See Jupyter Notebook.
5. See Jupyter Notebook.

Exercise 2

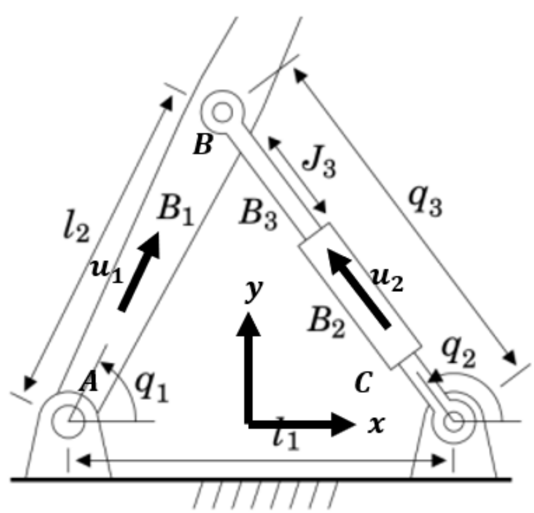

Figure 2: Slider crank linkage (1-RRPR)

1. The origin is taken at O. The co-ordinates of the points are listed below :

$$\mathbf{O} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \mathbf{A} = \begin{bmatrix} -\frac{l_1}{2} \\ 0 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} l_2 \cos q_1 \\ l_2 \sin q_1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} \frac{l_1}{2} \\ 0 \end{bmatrix} \tag{11}$$

The constraint equation can be written as :

$$\left\| \vec{BC} \right\|_2^2 = {q_3}^2 \tag{12}$$

Equation 12 can be expanded as :

$$(l_2 \cos q_1 - l_1)^2 + (l_2 \sin q_1)^2 = {q_3}^2$$
$${l_2}^2 \cos^2 q_1 - 2 l_1 l_2 \cos q_1 + {l_1}^2 + {l_2}^2 \sin q_1 = {q_3}^2$$

Ans finally the equation becomes:

$${l_1}^2 + {l_2}^2 - 2 l_1 l_2 \cos q_1 = {q_3}^2 \tag{13}$$

Using equation 13, the forward kinematics can be written as :

$$q_1 = \cos^{-1}(\frac{{l_1}^2 + {l_2}^2 - {q_3}^2}{2 l_1 l_2}) \tag{14}$$

2. Using equation 13, the inverse kinematics can be written as,

$$q_3 = \sqrt{{l_1}^2 + {l_2}^2 - 2 l_1 l_2 \cos q_1} \tag{15}$$

Due to architectural limitation, just the positive length is considered.

3. See Jupyter Notebook.

4. By using vector notations, it can be written

$$\vec{CA} + \vec{AB} = \vec{CB} \tag{16}$$

Equation 16 can be expanded by using the co-ordinates mentioned in equation 11,

$$l_1 \hat{i} + l_2 \hat{u}_1 = q_3 \hat{u}_2 \tag{17}$$

here we have considered that $\hat{u_1}$ is an unit vector along $\vec{AB}$ and $\hat{u_2}$ is an unit vector along $\vec{CB}$.

Taking derivative of equation 17, we get,

$$l_2\dot{\hat{u_1}} = \dot{q_3}\hat{u_2} + q_3\dot{\hat{u_2}} \tag{18}$$

The derivation of an unit vector $\hat{u} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ is expanded below :

$$\dot{\hat{u}} = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \dot{\theta} = \dot{\theta} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \hat{u} = \dot{\theta} E \hat{u} \tag{19}$$

By implementing equation 19 in equation 18, we get,

$$l_2\dot{q_1}E\hat{u_1} = \dot{q_3}\hat{u_2} + q_3\dot{q_2}E\hat{u_2} \tag{20}$$

Pre-multiplying $\hat{u_2}^T$ in equation 20 and replacing q_3 with d,

$$\dot{d} = l_2\dot{q_1}\hat{u_2}^T E\hat{u_1} \tag{21}$$

So the maximum velocity expression on the output joint side becomes,

$$\dot{q}_{1max} = \frac{\dot{d}_{max}}{l_2(\hat{u_2}^T E\hat{u_1})_{min}} \tag{22}$$

5. Considering the force components at B on body be F. Now to ensure that this force is maximum, it should act on perpendicular to direction vector **AB**. So, the torque equation can be written as,

$$\tau_{max} = \mathbf{r_{AB}} f_{max}$$

6. From equation 22, it is clear that if $\hat{u_1}$ and $\hat{u_2}$ become aligned along a single straight line (colinear), then

$$\hat{u_2}^T E\hat{u_1} = \hat{u_1}^T E\hat{u_1} = 0$$

Then from equation 22, $\dot{q_1}$ will be infinity, leading to a singular position.

Exercise 3

1. If Angela wants to convert the coordinates from meters to inches, she can multiply each component of the 3D position by the conversion factor of 39.3701.

   $$[x_{in}, y_{in}, z_{in}] = 40 \times [x_m, y_m, z_m]$$

   This is a simple linear scaling and Angela can apply this factor to convert the 3D position given in meters to inches without modifying Donald's code.

2. Yes, he can find a linear scale of conversion, but it will be different for translation and rotation parts.

   he can convert the quaternions in roll, pitch & yaw angles:

   $$\begin{aligned} \text{Roll} &= \text{atan2}(2(q_w q_x + q_y q_z), 1 - 2(q_x^2 + q_y^2)) \\ \text{pitch} &= \text{asin}(2(q_w q_y - q_z q_x)) \\ \text{yaw} &= \text{atan2}(2(q_w q_z + q_x q_y), 1 - 2(q_y^2 + q_z^2)) \end{aligned}$$

   Then they can be converted by using:

   $$(\text{Roll}^{\text{degrees}}, \text{Pitch}^{\text{degrees}}, \text{Yaw}^{\text{degrees}}) = 57.3 \times (\text{Roll}^{\text{radians}}, \text{Pitch}^{\text{radians}}, \text{Yaw}^{\text{radians}})$$

3. Yes it is possible to work with each other's system. The only part that has to be changed is the translation part, with a suitable linear scale.

   For a given pose, the rotational matrix will be same, irrespective of the method of computation. Just to make sure that the euler angle convention is suitable for robotics terms. Post-computation, of the results one of the persons have to modify the translation part for other person to use.

Dynamics

Exercise 4

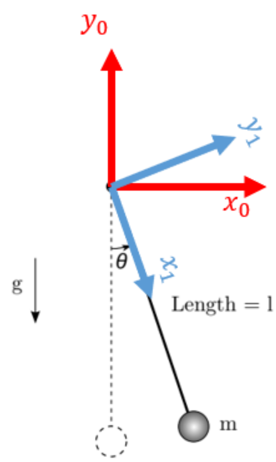

Figure 3: Simple pendulum

1. Frame F_0 (base frame) and F_1 are defined at the hinge joint of the system. Axis $\hat{x_1}$ is always directed along string of the pendulum. The position vector along $\vec{AB}$ can be written as,

$$x_{\vec{AB}} = l\hat{x_1} = l(\sin\theta\hat{x_0} - \cos\theta\hat{y_0}) \tag{23}$$

   Differentiating eq. 23, we get,

$$x_{\dot{\vec{AB}}} = \frac{d}{dt}[l\hat{x_1}]_{F_0} x_{\dot{\vec{AB}}} = \frac{d}{dt}[l\hat{x_1}]_{F_1} + \omega_1 \times [l\hat{x_1}] = 0 + [\dot{\theta} z_1] \times [l\hat{x_1}]$$

   Finally, it becomes,

$$x_{\dot{\vec{AB}}} = \dot{\theta} l \hat{y_1} \tag{24}$$

The kinetic energy of the system can be defined as,

$$KE = \frac{1}{2} m \dot{\vec{x_{AB}}}^T \dot{\vec{x_{AB}}} = \frac{1}{2} m l^2 \dot{\theta}^2 \tag{25}$$

And the potential energy of the system can be defined as,

$$PE = -m\vec{g}\vec{x}_{AB} = -m[-g\hat{y_0}]l[\sin\theta\hat{x_0} - \cos\theta\hat{y_0}]$$

Finally the equation becomes,

$$PE = -mgl\cos\theta \tag{26}$$

The Lagrangian of the system,

$$L = KE - PE = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\cos\theta \tag{27}$$

The expression for induced torque in the system is,

$$\tau = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} \tag{28}$$

Computing these terms,

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \qquad \frac{\partial L}{\partial \theta} = -mgl\sin\theta \qquad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$$

Replacing the above derivations in eq. 28, we get,

$$\tau = ml^2\ddot{\theta} + mgl\sin\theta \tag{29}$$

Now adding the damping term in eq. 29, the equation becomes,

$$\tau = ml^2\ddot{\theta} + b\dot{\theta} + mgl\sin\theta \tag{30}$$

2. The inverse dynamics equation should be,

$$\ddot{\theta} = \frac{1}{ml^2}(\tau - b\dot{\theta} - mgl\sin\theta) \tag{31}$$

3. See Jupyter Notebook.

Exercise 5

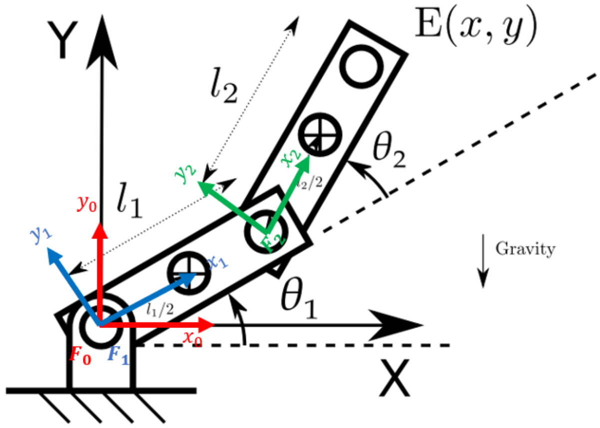

Figure 4: A planar 2R robot arm with two degrees of freedom

1. The notations of the solution are: I_j = Identity matrix of dimension j, $\hat{ms}_j$ = first moment of inertia of body j = $\begin{bmatrix} mx_j & my_j & mz_j \end{bmatrix}^T$,

   I_{Oj} = moment of Inertia of body j in the origin = $\begin{bmatrix} xx_i & xy_i & xz_i \\ yx_i & yy_i & yz_i \\ zx_i & zy_i & zx_i \end{bmatrix}$

   Assumptions taken for the solution:

   - The center of mass lies along O_iO_j, hence $my_i, mz_i = 0$.
   - Bodies are symmetrical, so $xy_i, xz_i, yz_i = 0$.

   The frames are mentioned in the figure. To Compute the forces using Lagrange's method, first we have to compute the velocities of the 2 bodies.

$$v_1 = 0$$

$$\omega_1 = \dot{\theta}_1 \hat{z}_0$$

$$v_2 = \frac{d}{dt}[O_0\vec{O}_2]_{F_0} = \frac{d}{dt}[l_{O_0O_2}\hat{x_1}]_{F_0} = \frac{d}{dt}[l_1\hat{x_1}]_{F_1} + \omega_1 \times [l_1\hat{x_1}] = l_1\dot{\theta}_1\hat{y_1}$$

$$\omega_2 = (\dot{\theta}_1 + \dot{\theta}_2)\hat{z_0}$$

The kinetic and potential energies of the body can be computed by using the following formulations,

$$E_j = \frac{1}{2}\begin{bmatrix} {v_j}^T & {\omega_j}^T \end{bmatrix} \begin{bmatrix} m_j\mathbf{I}_3 & \widehat{\mathbf{ms}}_j^T \\ \widehat{\mathbf{ms}}_j & \mathbf{I}_{O_j} \end{bmatrix} \begin{bmatrix} v_j \\ \omega_j \end{bmatrix} = \frac{1}{2}\mathbf{t}_j^T\mathbf{M}_j\mathbf{t}_j \tag{32}$$

$$U_j = -{}^0\mathbf{g}^T \left(m_j\,{}^0\mathbf{r}_{O_0O_j} +^0 R_j\,{}^j\mathbf{ms}_j \right) \tag{33}$$

Using eq. 32, kinetic energies of the two bodies can be computed,

$$\begin{aligned}
E_1 &= \frac{1}{2}\begin{bmatrix} \mathbf{0}^T & \dot{\theta}_1\hat{z_0} \end{bmatrix} \begin{bmatrix} m_1\mathbf{I_3} & \mathbf{\hat{ms}_1^T} \\ \mathbf{\hat{ms}_1} & \mathbf{I}_{O_1} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \dot{\theta}_1\hat{z_0} \end{bmatrix} \\
&= \frac{1}{2}[\dot{\theta}_1\hat{z_0}]^T\,\mathbf{I}_{O_1}\,[\dot{\theta}_1\hat{z_0}] \\
&= \frac{1}{2}\begin{bmatrix} 0 & 0 & \dot{\theta}_1 \end{bmatrix} \begin{bmatrix} xx_1 & 0 & 0 \\ 0 & yy_1 & 0 \\ 0 & 0 & zz_1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \\
&= \frac{1}{2}zz_1{\dot{\theta}_1}^2
\end{aligned}$$

$$\begin{aligned}
E_2 &= \frac{1}{2}\begin{bmatrix} \mathbf{v_2}^T & \omega_\mathbf{2}^T \end{bmatrix} \begin{bmatrix} m_2\mathbf{I_3} & \mathbf{\hat{ms}_2^T} \\ \mathbf{\hat{ms}_2} & \mathbf{I}_{O_2} \end{bmatrix} \begin{bmatrix} \mathbf{v_2} \\ \omega_\mathbf{2} \end{bmatrix} \\
&= \frac{1}{2}[m_2{v_2}^Tv_2 + {\omega_2}^TI_{o2}\,\omega_2 + 2\hat{ms_2}^T(\vec{v_2} \times \omega 2)] \\
&= \frac{1}{2}[m_2{l_1}^2{\dot{\theta}_1}^2 + zz_2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2\hat{ms}_2\,[(l_1\dot{\theta}_1\hat{y_1}) \times ((\dot{\theta}_1 + \dot{\theta}_2)\hat{z_1})]] \\
&= \frac{1}{2}[m_2{l_1}^2{\dot{\theta}_1}^2 + zz_2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2\hat{ms}_2\,[l_1\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\hat{x_1}]] \\
&= \frac{1}{2}[m_2{l_1}^2{\dot{\theta}_1}^2 + zz_2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2mx_2\,l_1\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)[\hat{x_2}\hat{x_1}]] \\
&= \frac{1}{2}[m_2{l_1}^2{\dot{\theta}_1}^2 + zz_2(\dot{\theta}_1 + \dot{\theta}_2)^2 + 2mx_2\,l_1\dot{\theta}_1(\dot{\theta}_1 + \dot{\theta}_2)\cos\theta_2]
\end{aligned}$$

using eq. 33, Potential energies of the bodies can be computed,

$$
\begin{aligned}
U_1 &= -{}^0\mathbf{g}^T \left(m_1 \cancel{{}^0\mathbf{r}_{O_0O_1}} +^0 R_1\, {}^1\mathbf{ms}_1\right) \\
&= -[-g\hat{y_0}]\,[mx_1(\cos\theta_1\hat{x_0} + \sin\theta_1\hat{y_0})] \\
&= gmx_1\sin\theta_1
\end{aligned}
$$

$$
\begin{aligned}
U_2 &= -{}^0\mathbf{g}^T \left(m_2\, {}^0\mathbf{r}_{O_0O_2} +^0 R_2\, {}^1\mathbf{ms}_2\right) \\
&= -[-g\hat{y_0}]\,[m_2l_1(\cos\theta_1\hat{x_0} + \sin\theta_1\hat{y_0}) + mx_2(\cos(\theta_1+\theta_2)\hat{x_0} + \sin(\theta_1+\theta_2)\hat{y_0})] \\
&= gm_2l_1\sin\theta_1 + mx_2\sin(\theta_1+\theta_2)g
\end{aligned}
$$

The lagrangian of the system can be computed as,

$$
\begin{aligned}
L &= E_1 + E_2 - U_1 - U_2 \\
&= \frac{1}{2}zz_1\dot{\theta_1}^2 + \frac{1}{2}\left[m_2l_1^2\dot{\theta_1}^2 + zz_2(\dot{\theta_1}+\dot{\theta_2})^2 + 2mx_2l_1\dot{\theta_1}(\dot{\theta_1}+\dot{\theta_2})\cos\theta_2\right] \\
&\quad - gmx_1\sin\theta_1 - [gm_2l_1\sin\theta_1 + mx_2\sin(\theta_1+\theta_2)g] \\
&= \frac{1}{2}(zz_1 + m_2l_1^2)\dot{\theta_1}^2 + \frac{1}{2}zz_2(\dot{\theta_1}+\dot{\theta_2})^2 + mx_2l_1\dot{\theta_1}(\dot{\theta_1}+\dot{\theta_2})\cos\theta_2 \\
&\quad - gmx_1\sin\theta_1 - gm_2l_1\sin\theta_1 - gmx_2\sin(\theta_1+\theta_2)
\end{aligned}
$$

Now the expressions for τ_1 and τ_2 can be obtained,

$$\tau_1 = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta_1}}\right) - \frac{\partial L}{\partial\theta_1} \tag{34}$$

$$\tau_2 = \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta_2}}\right) - \frac{\partial L}{\partial\theta_2} \tag{35}$$

Computing the right hand side terms of equation 34,

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_1} &= \dot{\theta}_1 zz_1 + m_2 l_1^2 \dot{\theta}_1 + zz_2(\dot{\theta}_1 + \dot{\theta}_2) + mx_2 l_1 \cos\theta_2 (2\dot{\theta}_1 + \dot{\theta}_2) \\
\frac{\partial L}{\partial \theta_1} &= -gmx_1 \cos(\theta_1 + \theta_2) - g\cos\theta_1 (mx_1 + m_2 l_1) \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) &= \ddot{\theta}_1 zz_1 + m_2 l_1^2 \ddot{\theta}_1 + zz_2(\ddot{\theta}_1 + \ddot{\theta}_2) + m_2 l_1 \cos\theta_2 (2\ddot{\theta}_1 + \ddot{\theta}_2) \\
&\quad - m_2 l_1 \sin\theta_2 (2\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2
\end{aligned} \tag{36}
$$

Computing the right hand side terms of equation 35,

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_2} &= zz_2(\dot{\theta}_1 + \dot{\theta}_2) + mx_2 l_1 \dot{\theta}_1 \cos\theta_2 \\
\frac{\partial L}{\partial \theta_2} &= -gmx_2 \cos(\theta_1 + \theta_2) - mx_2 l_1 \dot{\theta}_1 (\dot{\theta}_1 + \dot{\theta}_2) \sin\theta_2 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) &= zz_2(\ddot{\theta}_1 + \ddot{\theta}_2) + mx_2 l_1 (\ddot{\theta}_1 \cos\theta_2 - \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2)
\end{aligned} \tag{37}
$$

Torque at Joint 1 can be expressed as,

$$
\begin{aligned}
\tau_1 = \ & \ddot{\theta}_1 zz_1 + m_2 l_1^2 \ddot{\theta}_1 + zz_2(\ddot{\theta}_1 + \ddot{\theta}_2) + mx_2 l_1 \cos\theta_2 (2\ddot{\theta}_1 + \ddot{\theta}_2) \\
& - m_2 l_1 \sin\theta_2 (2\dot{\theta}_1 + \dot{\theta}_2)\dot{\theta}_2 + gmx_1 \cos(\theta_1 + \theta_2) \\
& + g\cos\theta_1 (mx_1 + m_2 l_1)
\end{aligned} \tag{38}
$$

Torque at Joint 2 can be expressed as,

$$
\begin{aligned}
\tau_2 &= zz_2(\ddot{\theta}_1 + \ddot{\theta}_2) + gmx_2 \cos(\theta_1 + \theta_2) + mx_2 l_1 \dot{\theta}_1^{\,2} \sin\theta_2 \\
&\quad + mx_2 l_1 \ddot{\theta}_1 \cos\theta_2
\end{aligned} \tag{39}
$$

Expressing equations 38 and 39 in $\tau = M\ddot{q} + C\dot{q} + G$, we get,

$$
M = \begin{bmatrix} zz_1 + m_2 l_1^2 + zz_2 + 2m_2 l_1 \cos\theta_2 & zz_2 + mx_2 l_1 \cos\theta_2 \\ zz_2 + mx_2 l_1 \cos\theta_2 & zz_2 \end{bmatrix}
$$

$$C = \begin{bmatrix} 0 & -m_2 l_1 \sin\theta_2(2\dot{\theta}_1 + \dot{\theta}_2) \\ m x_2 l_1 \dot{\theta}_1 \sin\theta_2 & 0 \end{bmatrix}$$

$$G = \begin{bmatrix} g m x_1 \cos\theta 1 + \theta 2 + g\cos\theta_1(m x_1 + m_2 l_1) \\ g m x_2 \cos\theta_1 + \theta_2 \end{bmatrix}$$

$$\ddot{q} = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} \quad \dot{q} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

2. See Jupyter Notebook

3. I think there is some issue with the notations of the question. It shouldn't be torque τ in task space, instead it should be force notation F. Moreover M and C are replaced by M_x and C_x respectively. The task space coordinates of the end effector are related to the joint-space coordinates through the Jacobian $J(q)$:

$$\dot{x} = J(q)\dot{q}$$
$$\dot{q} = J^{-1}(q)\dot{x}$$

and

$$\ddot{x} = J(q)\ddot{q} + \dot{J}(q, \dot{q})\dot{q}$$
$$\ddot{q} = J^{-1}(q)(\ddot{x} - \dot{J}(q, \dot{q})\dot{q})$$
$$\ddot{q} = J^{-1}(q)(\ddot{x} - \dot{J}(q, \dot{q})J^{-1}(q)\dot{x})$$

where $x = \begin{bmatrix} x \\ y \end{bmatrix}$ is the task-space coordinates of the end-effector.

To express the dynamics in task space, we transform the joint-space dynamics into task space. First, the task-space inertia matrix M_x is given by:

$$M_x = MJ(q)^{-1} \tag{40}$$

The task-space Coriolis and centrifugal matrix C_x is:

$$C_x = CJ(q)^{-1} - MJ(q)^{-1}\dot{J}(q, \dot{q})J(q)^{-1} \tag{41}$$

The task-space gravitational vector G_x is:

$$G_x = G \tag{42}$$

Finally, the robot dynamics in operational space can be written as:

$$\tau = M_x \ddot{x} + C_x \dot{x} + G_x \tag{43}$$

The expression would have been different if we would have considered forces instead of torques.

Exercise 6

1. Minimum no of scalar parameters :
   n = 1 : 1 parameter (1 Translation)
   n = 2 : 3 parameter (2 Translations and 1 Rotation)
   n = 3 : 6 parameter (3 Translations and 3 Rotations)

2. Properties of mass-inertia matrix (M):
   1. M is a square matrix with dimension n, where n is the number of induced wrenches in the system.
   2. M is symmetric.
   3. M depends on joint position (q).

3. Yes, it is possible to convert imperial and metric system. There is a linear relationship between Joule and ft-lb.

   $$1 \quad \text{J} = 0.74 quad \text{ft-lb}$$

4. As seen in equation 32, the mass-inertia matrix depends on the body j. So the computed kinetic energies will be different if the bodies do not match.

   On the other hand if on of them has accounted for transformation in velocities, then the kinetic energy will come out to be same.

Control

Exercise 7

1. For first part, check jupyter notebook.
   Proof that [π, 0] is asymptotically stable from any initial state :

   Let V(x_1, x_2) be a lyapunov function. It can be defined as,

   $$V(\theta, \dot{\theta}) = \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1 - \cos\theta) \tag{44}$$

   The function is continuous, differentiable and always definite positive i.e. $V > 0$.

   Now, computing the derivative of eq. 44,

   $$\begin{aligned}\dot{V}(\theta, \dot{\theta}) &= \frac{d}{dt}V(\theta, \dot{\theta}) \\ &= \frac{\partial V}{\partial \theta}\dot{\theta} + \frac{\partial V}{\partial \dot{\theta}}\ddot{\theta} \\ &= mgl\sin\theta\dot{\theta} + ml^2\dot{\theta}\ddot{\theta} \\ &= \dot{\theta}(ml^2\ddot{\theta} + mgl\sin\theta) \\ &= -b\dot{\theta}^2\end{aligned}$$

   The above expression is obtained by using equation 30 for a damped simple pendulum by making the left hand term 0. $\dot{V}$ will be always negative except for $\dot{\theta} = 0$.
   From LaSalle's Invariance Theorem, if $V_x > 0$ and $V_x < 0$, then the state $[\theta, \dot{\theta}]$ will converge to the largest invariant set where $V_x = 0$. The largest invariant set will be sets of fixed points of the pendulum (the points where the plot crosses θ axis in the phase plot). So the pendulum will be asymptotically stable at fixed point $[\pi, 0]$.

Exercise 8

See Jupyter Notebook.

Exercise 9

1. The advantages are listed below :

   Direct Drive : Very compact design, no gear so no mechanical backlash, very accurate and fast response, Higher efficiency.
   Geared DC Actuator : Due to gears the torque output will be high, simple mechanical setup.
   Series Elastic Actuator : Can store and release energy whenever needed, absorbs shock and vibrations, the control will be improved because of elasticity (helpful for robots).

   The drawbacks are listed below :

   Direct Drive : Very limited torque, External cooling system required due to heat generation.
   Geared DC Actuator : Mechanical backlash exists reducing the energy efficiency, can be heavy, mechanical backlash can affect the accuracy and control.
   Series Elastic Actuator : Control might be complex, delays in response might be observed.

   Selection of an actuator for real world implementation of a torque limited simple pendulum : The simple pendulum has a simple design, hence the control shouldn't be that complicated.
   So we can ignore series elastic actuators. Mechanical backlash will affect the accuracy of the pendulum, causing issues in control task. So , we can avoid the geared DC actuators.

   Since the torque is already limited, so i think a **direct drive** will be a suitable option for the selection.

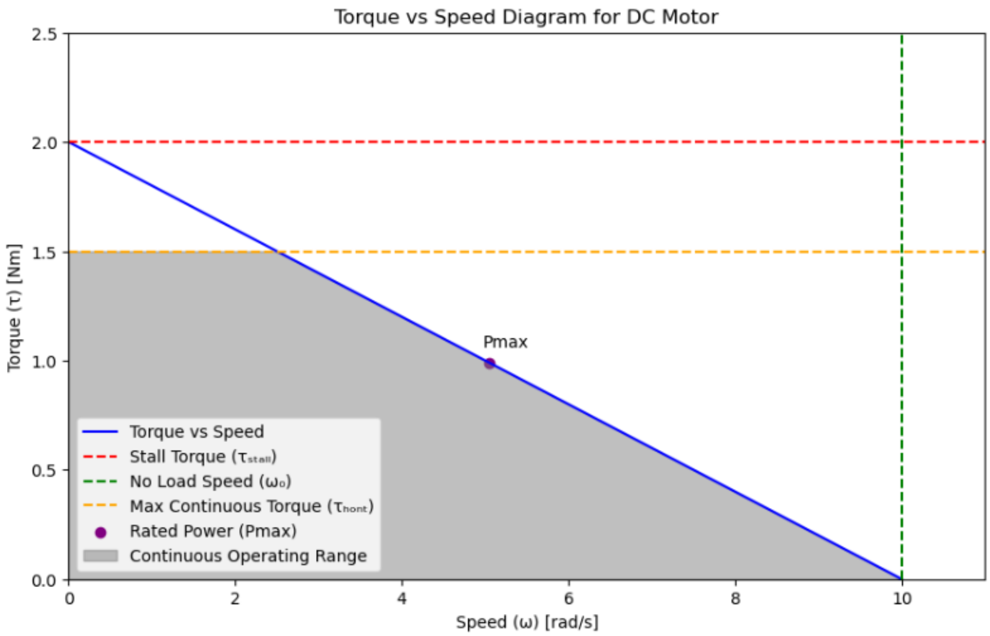

Figure 5: τ-ω curve of a DC motor

2. **Effects of Gear Ratio on the Operating Range :**
1. If the gear ratio is more than 1, it will increase the amount of exerted torque at the expense of speed. So, the stall torque remains the same, but the torque at lower speeds will be higher.
2. Since the output torque increases, the maximum speed decreases. The no-load speed (ω_0) will also reduce with a comparatively higher gear ratio.

3. **Adding friction effect :** Friction is a force that resists motion. Friction in robot joints can arise from several sources, such as mechanical components and the environment. It can manifest as static friction (resisting motion when the joint is stationary) or dynamic friction (resisting motion when the joint is rotating). It causes loss in energy and increases the control effect.

   Though there are many possible friction models, most simple and reasonable model will be the combination of Coulomb and the viscous friction model (without the Stribeck effect). The friction term hence becomes,

$$\tau_{fi} = f_{si} sign(\theta) + f_{vi}\dot{\theta} \tag{45}$$

$$\begin{aligned} \text{Here, } f_{si} &= \text{Constant due to static friction} \\ f_{vi} &= \text{Constant due to viscous friction} \\ \theta &= \text{Angle for joint rotation} \end{aligned}$$

Adding motor inertia effect : Rotor inertia refers to the resistance of the motor's rotor to change in rotational speed. The rotor inertia plays a significant role in how quickly the joint can move (accelerate or decelerate).

The motor provides impact on the kinetic energy part. Considering the input motor velocity as $\dot{q_{mi}}$, gear ratio be N_i and I_{mi} be the inertia of the motor. So the output motor velocity becomes $\dot{q_{mo}} = N_i \dot{q_{mi}}$. The kinetic energy due to motor i will be,

$$\begin{aligned} E_{mi} &= \frac{1}{2} I_{mi} \dot{q_{mo}} \\ &= \frac{1}{2} I_{mi} (N_i \dot{q}_{mo})^2 \\ &= \frac{1}{2} I_{mi} N_i^2 \dot{q}_{mo}^2 \end{aligned}$$

So, the Torque due to motor i becomes,

$$\tau_{mi} = I_{mi} N_i^2 \ddot{q}_{mo} \tag{46}$$