Bayesian Methods for Cosmology: Measuring Dark Energy

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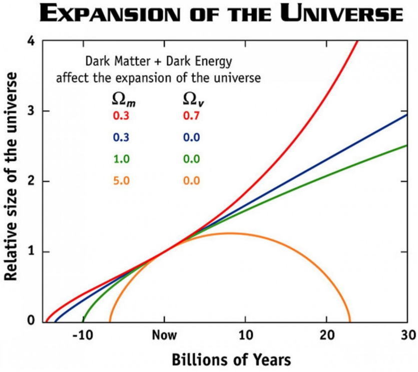
Topics

1. Introduction to the physical model
2. MCMC (Monte Carlo Markov Chain)
3. ABC (Approximate Bayesian Computation)
4. Comparison and conclusions

Why dark energy?

- Universe expansion is accelerating
- Dark energy: scales as $\rho \propto a^0$
- What is it?
  - Cosmological constant
  - Quintessence
  - ...
- From experiments (PDG 2018): $\Omega_\Lambda \approx 0.69$, $\Omega_m \approx 0.31$

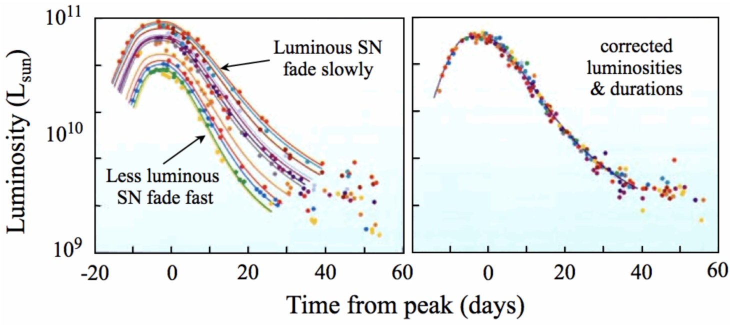


Type Ia Supernovae and Standard Candles

$$m_b = 5\log_{10} D_L + M \Rightarrow m_b = 5\log_{10} D_L - \alpha(s-1) + \beta c + M$$

where m_b: magnitude in the b-band, α: stretch parameter

β: color parameter, s: stretch, c: color



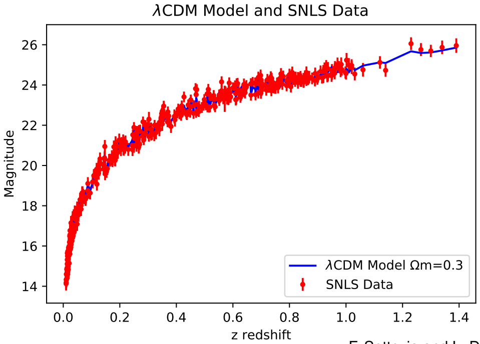
The Model

$$D_L = (1+z)\frac{c}{H_0}\int_0^z \frac{dz'}{\sqrt{\Omega_m(1+z')^3 + \Omega_\Lambda}}$$

Gaussian noise

Parameters: (Ω_m, α, β, M)

(for ABC (Ω_m, σ))



Why Bayesian methods in Cosmology?

- Incorporate previous knowledge about the Universe
- Methods useful for dealing with dimensionality curse:
  - MCMC: likelihood needed
  - ABC: no likelihood needed
- Tradition!

Monte Carlo Markov Chain (MCMC)

Bayes theorem: the posterior distribution of the parameters $\vec{\theta}$, given the prior $p(\vec{\theta})$ and the likelihood of the data $P(\text{data}|\vec{\theta})$, is:

$$P(\vec{\theta}|\text{data}) \propto P(\text{data}|\vec{\theta})p(\vec{\theta})$$

- So, given two parameter sets $\vec{\theta_1}$ and $\vec{\theta_2}$, we can use the ratio:

$$\frac{p(\vec{\theta_1}|\text{data})p(\vec{\theta_1})}{p(\vec{\theta_2}|\text{data})p(\vec{\theta_2})}$$

- If the prior is uniform, this simplifies to the likelihood ratio:

$$\frac{p(\vec{\theta_1}|\text{data})}{p(\vec{\theta_2}|\text{data})}$$

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   If $\frac{p(\vec{\theta}_1|\text{data})}{p(\vec{\theta}_0|\text{data})} < 1$: accept $\vec{\theta}_1$ with probability $\frac{p(\vec{\theta}_1|\text{data})}{p(\vec{\theta}_0|\text{data})}$.
4. If $\vec{\theta}_1$ is not accepted, store $\vec{\theta}_0$ a second time.

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5. Repeat the iteration N times

Toy model: One-dimensional Gaussian distribution

- Simulate a sample of N independent events according to $\mathcal{N}(\mu = 0, \sigma^2 = 1)$.
  The estimator of their mean will be distributed according to $\mathcal{N}(\mu = 0, \sigma^2 = 1/N)$ (CLT). (Here $\theta_{j+1} \sim \mathcal{N}(\theta_j, 1)$)
  Results: Mean and variance as expected!
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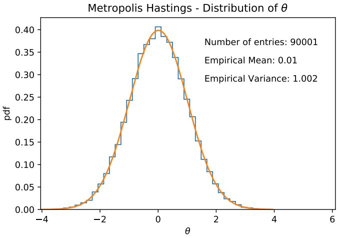

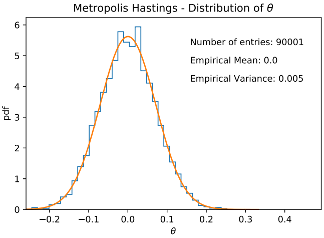

Back to our model (MCMC)

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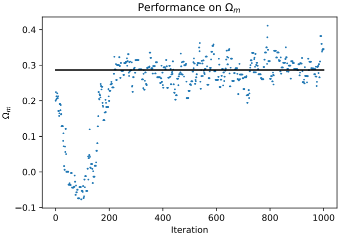
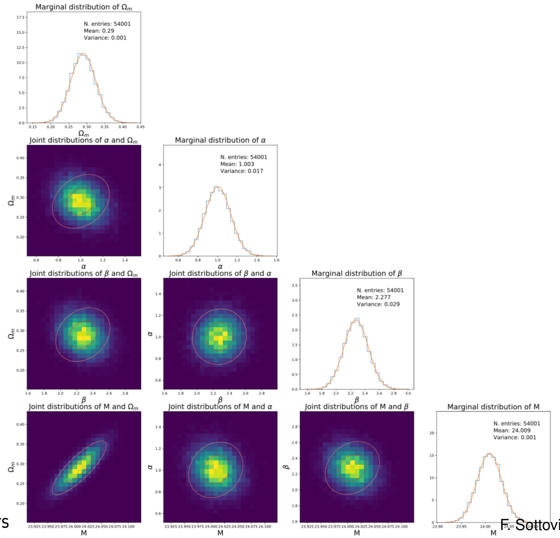
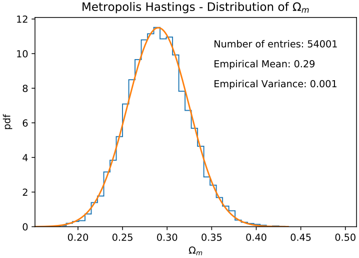
- How fast is the convergence?

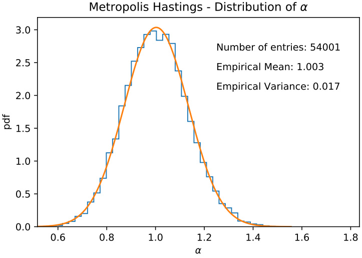


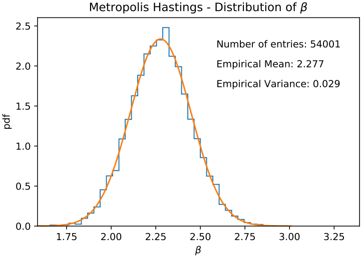
Estimated distr., N = 60000, discarding first 10%

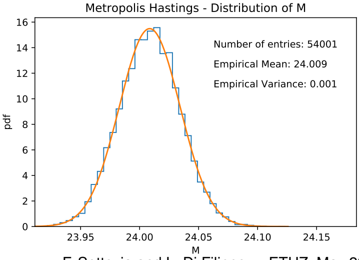


MCMC: Results

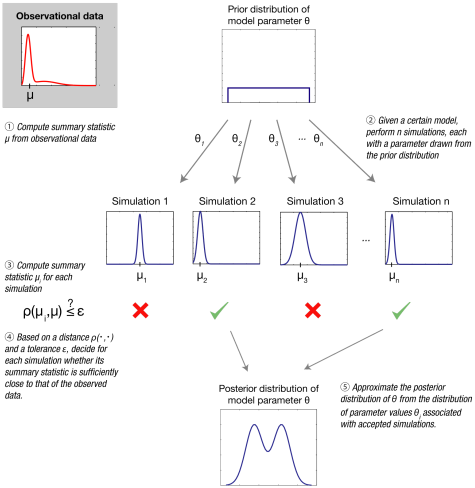








ABC (Approximate Bayesian Computation)



ABC Toy Model

- Gaussian model: 100 samples, $\sim \mathcal{N}(\mu, \sigma = 10)$
  $\Rightarrow$ estimate μ
- Expected variance on μ: $\sigma_\mu^2 = \frac{10^2}{100} = 1$ (CLT)
- From theory [2]: $\sigma_\mu^2(\epsilon) = \frac{\sigma_{\text{data}}^2}{N} + \frac{\epsilon^2}{3}$ (for flat prior)

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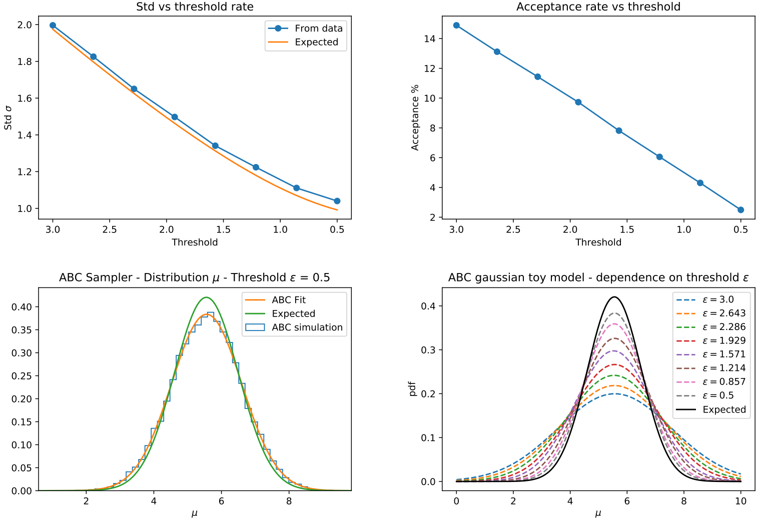
- **Results**: $\mu \approx 5.55$ (uncertainty depends on threshold)

ABC Toy Model - Results



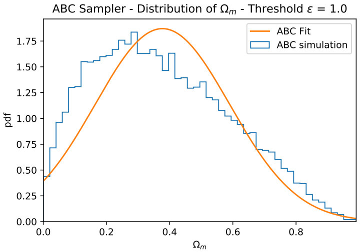
Back to our model (ABC)

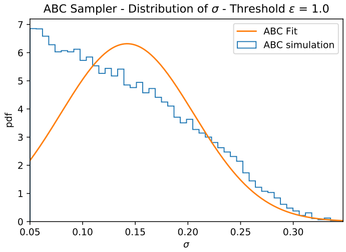
- Our model: gaussian noise $\sim \mathcal{N}(0, \sigma)$ on each (z, D_L)
- 2-dimensional parameter space, $\vec{\theta} = (\Omega_m, \sigma)$
- Distance measure
  a) $d = \frac{1}{N} \sum_z \left| D_{L,\text{obs.}}(z) - D_{L,\text{sim.}}(z) \right|$
  b) $d = \frac{1}{N} \sum_z \left(D_{L,\text{obs.}}(z) - D_{L,\text{sim.}}(z) \right)^2$
  c) $d = \max_z \left| D_{L,\text{obs.}}(z) - D_{L,\text{sim.}}(z) \right|$
- Flat prior on both parameters:

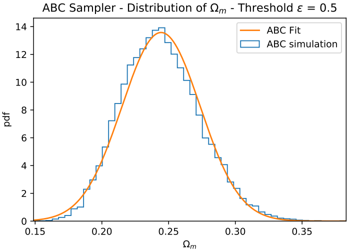
  $\Omega_m \in [0, 1]$

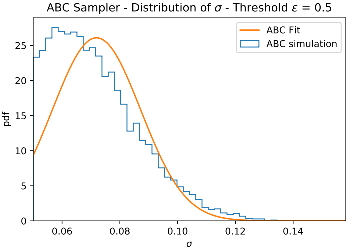
  $\sigma \in [0.05, 0.35]$: est. noise from data with $\sigma = 0.15$

ABC: Results

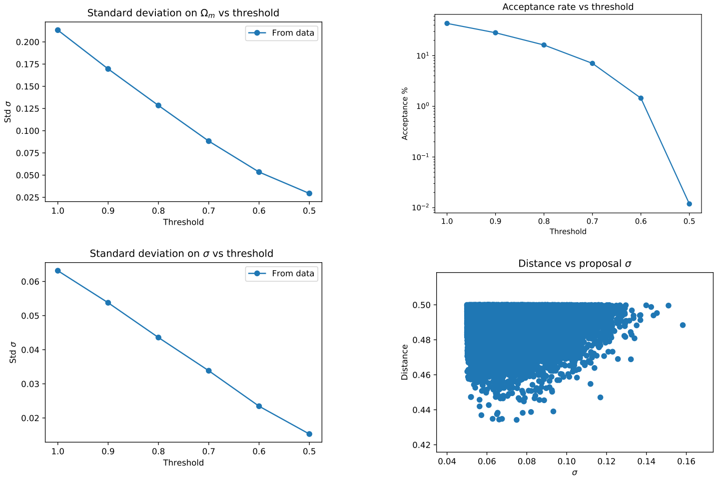

ABC Sampler - Distribution of Ω_m - Threshold ϵ = 1.0
ABC Fit
ABC simulation
pdf
Ω_m


ABC Sampler - Distribution of σ - Threshold ϵ = 1.0
ABC Fit
ABC simulation
pdf
σ


ABC Sampler - Distribution of Ω_m - Threshold ϵ = 0.5
ABC Fit
ABC simulation
pdf
Ω_m


ABC Sampler - Distribution of σ - Threshold ϵ = 0.5
ABC Fit
ABC simulation
pdf
σ

ABC: Performance



Hints of convergence for $\epsilon \lesssim 0.5$...

ABC: Results and Observations

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- Why?
  1. Choice of the distance
  2. Threshold - computation time
  3. Noise model
  4. Should we really get a gaussian distribution?

Conclusions

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Conclusions

- MCMC more efficient (smaller variance), more “robust”
- ABC: no likelihood needed
- Convergence: better in MCMC, but depends on various factors!
- Important: in MCMC high acceptance ratio not at expense of distribution
  ABC: high acceptance ratio for high threshold $\Rightarrow$ “bad” posterior
- Results:
  - MCMC: $\Omega_m = 0.29 \pm 0.01$
  - ABC: $\Omega_m = 0.24 \pm 0.03$
  - Literature [3]: $\Omega_m = 0.308 \pm 0.012$

References

- ABC page on Wikipedia: https://en.wikipedia.org/wiki/Approximate_Bayesian_computation
- Joel Akeret, Alexandre Refregier, Adam Amara, Sebastian Seehars, Caspar Hasner
  Approximate Bayesian Computation for Forward Modeling in Cosmology
  https://en.wikipedia.org/wiki/Approximate_Bayesian_computation
- Particle Data Group 2018
  http://pdg.lbl.gov/2018/html/computer_read.html

Thank you for your attention!