Advanced Asset Management

A.Y. 2024-2025

Asset Management & Sustainable Finance

Final Examination

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1 Exercise 1

1.1 Point 1

1.1.1 Question 1a

We consider the CAPM model:

$$R_i - r = \beta_i(R_m - r) + \varepsilon_i$$

Taking expectations on both sides and noting that the idiosyncratic risk ε_i has zero mean, we obtain:

$$\mathbb{E}[R_i] - r = \beta_i(\mathbb{E}[R_m] - r)$$

Rearranging:

$$\mu = r\mathbf{1}_6 + \boldsymbol{\beta}(\mu_m - r\mathbf{1}_6)$$

where:

- μ is the vector of expected returns
- r is the risk-free rate
- $\mathbf{1}_6$ is a 6×1 column vector of ones
- $\boldsymbol{\beta}$ is the vector of asset betas
- μ_m is the expected market return

Using this formula, we obtain:

$$\mu = \begin{bmatrix} 0.002 \\ 0.038 \\ 0.062 \\ 0.080 \\ 0.110 \\ 0.170 \end{bmatrix}$$

1.1.2 Question 1b

To compute the covariance matrix, we once again rely on the CAPM formulation. Given the model:

$$R_i - r = \beta_i(R_m - r) + \varepsilon_i$$

Taking variances on both sides and using the assumption that the market factor and the idiosyncratic risk are uncorrelated, we obtain the expression for the covariance matrix:

$$\Sigma = \boldsymbol{\beta}\boldsymbol{\beta}^T \sigma_m^2 + D$$

where:

- $\boldsymbol{\beta}$ is the vector of factor betas,
- σ_m^2 is the variance of the market factor,
- D is the diagonal covariance matrix associated with the idiosyncratic risk factors, where the diagonal elements correspond to the variances $\tilde{\sigma}_i^2$ of the idiosyncratic noise for each asset.

Using this formulation, we compute the covariance matrix:

$$\Sigma = \begin{bmatrix} 0.0261 & -0.0036 & -0.0084 & -0.0120 & -0.0180 & -0.0300 \\ -0.0036 & 0.0292 & 0.0084 & 0.0120 & 0.0180 & 0.0300 \\ -0.0084 & 0.0084 & 0.0296 & 0.0280 & 0.0420 & 0.0700 \\ -0.0120 & 0.0120 & 0.0280 & 0.0521 & 0.0600 & 0.1000 \\ -0.0180 & 0.0180 & 0.0420 & 0.0600 & 0.1044 & 0.1500 \\ -0.0300 & 0.0300 & 0.0700 & 0.1000 & 0.1500 & 0.2696 \end{bmatrix}$$

1.1.3 Question 1c

To obtain the vector of volatilities, we take the square root of the diagonal elements of the covariance matrix $\mathbf{\Sigma}$, which gives:

$$\sigma = \sqrt{\text{diag}(\Sigma)} = \begin{bmatrix} 0.1616 \\ 0.1709 \\ 0.1720 \\ 0.2283 \\ 0.3231 \\ 0.5192 \end{bmatrix}$$

Using this, we compute the correlation matrix using the formula:

$$\rho = \Sigma \oslash (\sigma\sigma^T)$$

where:

- $\oslash$ denotes the element-wise division operator
- $\sigma\sigma^T$ is the outer product of the volatility vector, forming a matrix where each element $\sigma_i\sigma_j$ scales the respective covariance

Thus, the resulting correlation matrix is:

$$
\rho = \begin{bmatrix}
1.0000 & -0.1304 & -0.3022 & -0.3254 & -0.3448 & -0.3576 \\
-0.1304 & 1.0000 & 0.2857 & 0.3077 & 0.3260 & 0.3381 \\
-0.3022 & 0.2857 & 1.0000 & 0.7130 & 0.7555 & 0.7836 \\
-0.3254 & 0.3077 & 0.7130 & 1.0000 & 0.8135 & 0.8438 \\
-0.3448 & 0.3260 & 0.7555 & 0.8135 & 1.0000 & 0.8941 \\
-0.3576 & 0.3381 & 0.7836 & 0.8438 & 0.8941 & 1.0000
\end{bmatrix}
$$

1.1.4 Question 1d

To compute the Sharpe Ratio of each asset, we use the formula:

$$SR = (\mu - r\mathbf{1}_6) \oslash \sigma$$

where:

- μ is the vector of expected returns
- r is the risk-free rate
- $\mathbf{1}_6$ is a column vector of ones
- σ is the vector of volatilities
- $\oslash$ denotes element-wise division

Applying this formula, we obtain the Sharpe Ratio for each asset:

$$
SR = \begin{bmatrix}
-0.1114 \\
0.1053 \\
0.2441 \\
0.2629 \\
0.2785 \\
0.2889
\end{bmatrix}
$$

1.2 Point 2

1.2.1 Question 2a

We aim to formulate the mean-variance optimization problem as a quadratic program (QP), given by:

$$x^\star = \arg\min \frac{1}{2} x^\top \Sigma x - \gamma x^\top (\mu - r\mathbf{1_6})$$

subject to the constraints:

$$\sum_{i=1}^{n} x_i = 1, \quad -10 \leq x_i \leq 10, \quad \forall i = 1, \ldots, n$$

We rewrite the problem in a form compatible with the `cvxopt` Python library, which requires the standard quadratic programming formulation:

$$x^* = \arg\min \frac{1}{2} x^T Q x - x^T R$$

subject to:

$$Ax = B, \quad Cx \leq D$$

where:

- $Q = \Sigma$, the covariance matrix of asset returns
- $R = \gamma(\mu - r\mathbf{1_6})$, where γ is the risk aversion parameter
- A and B define the equality constraint ensuring that the portfolio is fully invested:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \end{bmatrix}$$

- C and D define the inequality constraints enforcing the position limits:

$$C = \begin{bmatrix} -I_6 \\ I_6 \end{bmatrix}, \quad D = \begin{bmatrix} 10 \times \mathbf{1}_6 \\ 10 \times \mathbf{1}_6 \end{bmatrix}$$

where I_6 is the 6×6 identity matrix.

1.2.2 Question 2b

Solving the QP Problem formulated above for the given values of γ we obtain the following results:

γ	0.00	0.10	0.20	0.50	1.00
x_1	0.3330	0.3000	0.2669	0.1678	0.0027
x_2	0.2035	0.1880	0.1725	0.1259	0.0482
x_3	0.3689	0.3522	0.3355	0.2854	0.2019
x_4	0.2106	0.2111	0.2116	0.2130	0.2154
x_5	0.0450	0.0654	0.0858	0.1470	0.2490
x_6	−0.1610	−0.1166	−0.0722	0.0610	0.2829
$\mu(x^\star(\gamma))$	0.0257	0.0338	0.0420	0.0664	0.1071
$\sigma(x^\star(\gamma))$	0.0797	0.0846	0.0980	0.1634	0.2962
$\text{SR}(x^\star(\gamma)\|r)$	0.0716	0.1635	0.2242	0.2839	0.2940

Table 1: Optimal portfolio weights for different values of γ

1.2.3 Question 2c

To construct the efficient frontier, we solved the quadratic programming (QP) problem for a range of risk-aversion parameters, γ, using a finely discretized grid. The grid is divided into three segments:

$$\begin{aligned}
\gamma_1 &= \{-2.0, -1.9, -1.8, \ldots, -0.1\}, & \gamma_1 &\in [-2, 0), & \text{step size: } 0.1,\\
\gamma_2 &= \{0.00, 0.05, 0.10, \ldots, 1.00\}, & \gamma_2 &\in [0, 1.05], & \text{step size: } 0.05,\\
\gamma_3 &= \{2, 3, 4, \ldots, 10\}, & \gamma_3 &\in [2, 10], & \text{step size: } 1.
\end{aligned}$$

The resulting efficient frontier is illustrated in Figure 1. This frontier represents the set of optimal risk-return trade-offs available when investing in risky assets without access to a risk-free asset.

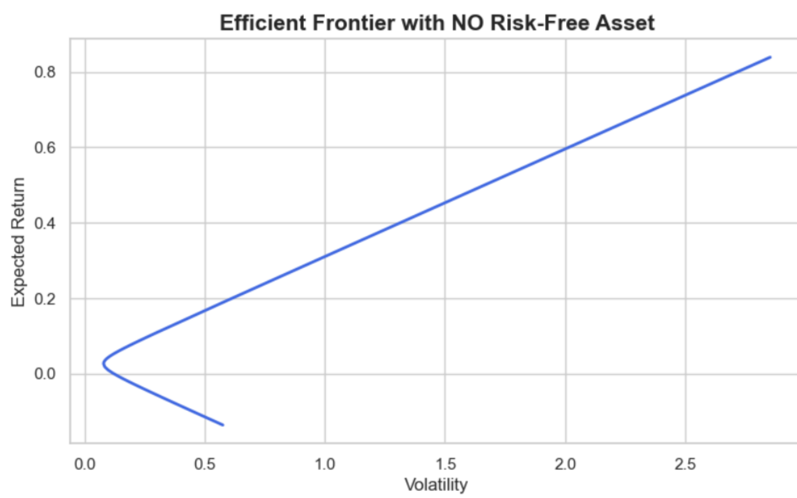


Figure 1: Efficient frontier without risk-free asset

1.2.4 Question 2d

Using the same Mean-Variance Optimization (MVO) solver, a bisection algorithm was implemented to determine the optimal portfolio allocations for target volatilities of 10% and 15%.

The search was conducted over the range $\gamma \in [-0.5, 10]$, refining the interval until its length was reduced to 10^{-11}, ensuring high precision in the risk-return trade-off. The optimal portfolio allocations and associated performance metrics are presented in Table 2.

1.2.5 Question 2e

From modern portfolio theory, the analytical formula for the unconstrained tangency portfolio is given by:

γ	0.21	0.45
x_1	0.2630	0.1860
x_2	0.1710	0.1340
x_3	0.3340	0.2940
x_4	0.2120	0.2130
x_5	0.0880	0.1360
x_6	-0.0670	0.0370
$\mu(x^\star(\gamma))$	0.0429	0.0620
$\sigma(x^\star(\gamma))$	0.1000	0.1500
$\mathrm{SR}(x^\star(\gamma)\|r)$	0.2293	0.2797

Table 2: Optimal portfolio weights for target volatilities of 10% and 15%

$$x^\star = \frac{\Sigma^{-1}(\mu - r\mathbf{1}_6)}{\mathbf{1}_6^\top \Sigma^{-1}(\mu - r\mathbf{1}_6)}$$

Using this formula we get the results summed in Table 3.

We observed that the obtained portfolio satisfies the weight constraints.

Later we will also need the value of gamma relative to this portfolio so we compute it here with the following formula:

$$\gamma = \frac{1}{\mathbf{1}_6^\top \Sigma^{-1}(\mu - r\mathbf{1}_6)}$$

and we obtain a value of 1.1135.

1.2.6 Question 2f

We employ a two step brute-force algorithm first with a step size of 0.001 over the range $\gamma \in [0, 5]$ and then on a narrower interval with a step size of 0.00001. The optimal γ^* that maximizes the Sharpe Ratio is identified, yielding the results reported in Table 4.

1.2.7 Question 2g

We aim to extend the Mean-Variance Optimization (MVO) problem by introducing a risk-free asset. This requires modifying both the covariance matrix and the return vector accordingly.

The optimization problem is given by:

$$x^\star = \arg\min \frac{1}{2} x^\top \tilde{\Sigma} x - \gamma x^\top (\tilde{\mu} - r_f \mathbf{1}_7)$$

subject to the constraints:

Table 3: Tangency portfolio weights

γ^*	1.1135
x_1	-0.0348
x_2	0.0306
x_3	0.1829
x_4	0.2159
x_5	0.2722
x_6	0.3333
$\mu(x^\star)$	0.1163
$\sigma(x^\star)$	0.3275
$\mathrm{SR}(x^\star \mid r)$	0.2941

Table 4: Brute-Force portfolio weights

γ^*	1.1135
x_1	-0.0348
x_2	0.0306
x_3	0.1829
x_4	0.2159
x_5	0.2722
x_6	0.3333
$\mu(x^\star(\gamma^*))$	0.1163
$\sigma(x^\star(\gamma^*))$	0.3275
$\mathrm{SR}(x^\star(\gamma^*) \mid r)$	0.2941

Table 5: Tangency portfolio with analytical formula and brute-force algorithm.

$$\sum_{i=1}^{n+1} x_i = 1, \quad -10 \leq x_i \leq 10, \quad \forall i = 1, \ldots, n+1.$$

We rewrite the problem in a form compatible with `cvxopt` which requires the standard quadratic programming formulation:

$$x^\star = \arg\min \frac{1}{2} x^\top Q x - x^\top R$$

subject to:

$$Ax = B, \quad Cx \leq D$$

The matrices in this formulation are defined as:

- $Q = \tilde{\Sigma}$, the modified covariance matrix
- $R = \gamma(\tilde{\mu} - r_f \mathbf{1}_7)$, where γ is the risk aversion parameter

The modified covariance matrix and return vector are given by:

$$\tilde{\Sigma} = \begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix}, \quad \tilde{\mu} = \begin{bmatrix} \mu \\ r_f \end{bmatrix}$$

The equality constraint ensures that the total portfolio allocation sums to 1:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \end{bmatrix}$$

The inequality constraint matrices enforce portfolio bounds:

$$C = \begin{bmatrix} -I_7 \\ I_7 \end{bmatrix}, \quad D = \begin{bmatrix} 10 \times \mathbf{1}_7 \\ 10 \times \mathbf{1}_7 \end{bmatrix}$$

where I_7 is the 7×7 identity matrix.
Solving for γ in the range $[0.5, 1.5]$, we obtain the following efficient frontier:

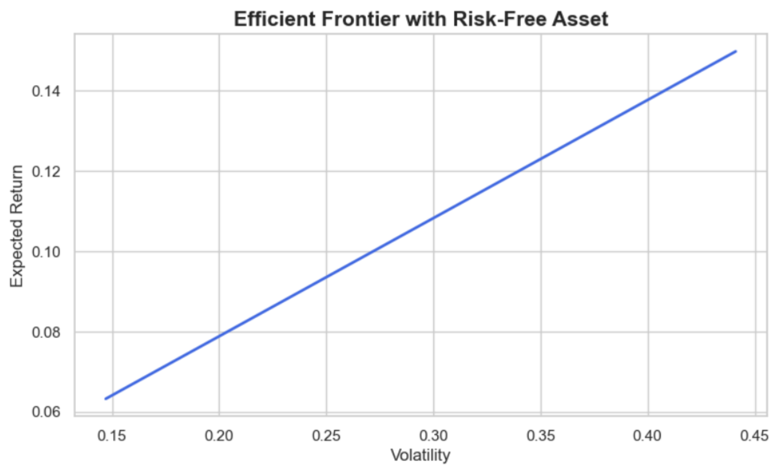


Figure 2: Efficient Frontier with a risk-free asset

As expected by the theory, the efficient frontier forms a straight line.

Next, we compute the Sharpe Ratio for different portfolios along the positive γ region of the efficient frontier. The results confirm that the Sharpe Ratio remains constant at 0.2941, which matches the value of the tangency portfolio in the case without a risk-free asset.

This result is in full agreement with Modern Portfolio Theory. According to the Two-Fund Separation Theorem, the positive segment of the efficient frontier forms a straight line connecting the risk-free rate with the tangency portfolio. Since all optimal portfolios lie on this line, they must have the same Sharpe Ratio, as the ratio itself represents the slope of the line connecting each portfolio to the risk-free rate:

$$SR(x^\star) = \frac{\mu(x^\star) - r_f}{\sigma(x^\star)}$$

Since all points along the positive efficient frontier lie on the same straight line, the slope, and therefore the Sharpe Ratio, remains constant across all portfolios. Thus, regardless of the specific portfolio composition, any efficient portfolio in this region maintains an identical risk-adjusted return, as dictated by the linear relationship between risk and return in the presence of a risk-free asset.

To determine the tangency portfolio using a brute-force approach with the new efficient frontier, we solve the Quadratic Programming (QP) problem over a finely spaced γ-grid and identify the first γ value where the asset allocation includes a negative weight in the last component, which is the one relative to the risk-free asset.

This idea follows directly from the Two-Fund Separation Theorem. According to the theorem, the tangency portfolio corresponds to the portfolio on the positive efficient frontier where the mixing coefficient α equals 1. For values of $\alpha > 1$, portfolios are obtained by shorting the risk-free asset, meaning the allocation in the risk-free asset (x_7) becomes negative.

Based on this insight, we employ the same two-level brute force algorithm as before to find the first portfolio where x_7 takes a negative value, marking the transition point where the risk-free asset is being shorted. This portfolio corresponds to the tangency portfolio.

Ultimately, we obtain the following result:

γ^*	1.1135
x_1	-0.0348
x_2	0.0306
x_3	0.1829
x_4	0.2159
x_5	0.2722
x_6	0.3333
x_7	0.0000
$\mu(x^\star(\gamma^*))$	0.1163
$\sigma(x^\star(\gamma^*))$	0.3275
$\mathrm{SR}(x^\star(\gamma^*)\vert r)$	0.2941

Table 6: Brute-Force portfolio weights

1.2.8 Question 2h

Method	γ	μ	σ	SR
Direct Formula	1.1135	0.1163	0.3275	0.2941
Brute-Force I	1.1135	0.1163	0.3275	0.2941
Brute-Force II	1.1135	0.1163	0.3275	0.2941

Table 7: Comparison of tangency portfolio across methods

Both brute-force methods perform exceptionally well due to the use of a very fine grid and a two-step grid reduction method. This approach ensures highly accurate results without impacting computation time. As a result, the errors we obtain, when compared to the analytical formula, are smaller than the fourth decimal place relative to the exact values. For this reason we avoid showcasing also the weights for the three portfolios since they are the same up to the 4th decimal place.

The convergence of both methods confirms that the adopted optimization strategy is effective and allows for highly accurate identification of the desired tangency portfolio.

1.3 Point 3

1.3.1 Question 3a

We revert to the no risk-free asset case but modify the constraints to enforce long-only investments. The optimization problem can be formulated as a Quadratic Program (QP):

$$x^* = \arg\min \frac{1}{2} x^\top Q x - x^\top R$$

subject to:

$$Ax = B, \quad Cx \leq D$$

where:

- $Q = \Sigma$, the covariance matrix of asset returns
- $R = \gamma(\mu - r\mathbf{1_6})$, where γ is the risk aversion parameter
- A and B define the equality constraint ensuring that the portfolio is fully invested:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \end{bmatrix}$$

- C and D define the inequality constraints enforcing long-only condition:

$$C = \begin{bmatrix} -I_6 \end{bmatrix}, \quad D = \begin{bmatrix} \mathbf{0} \end{bmatrix}$$

where I_6 is the 6×6 identity matrix.

Solving the QP Problem formulated above for the given values of γ we obtain the following results:

γ	0.00	0.10	0.20	0.50	1.00
x_1	0.4458	0.3721	0.3042	0.1678	0.0027
x_2	0.2223	0.2024	0.1819	0.1258	0.0482
x_3	0.2799	0.3052	0.3200	0.2854	0.2019
x_4	0.0520	0.1203	0.1741	0.2130	0.2154
x_5	0.0000	0.0000	0.0200	0.1470	0.2490
x_6	0.0000	0.0000	0.0000	0.0610	0.2829
$\mu(x^\star(\gamma))$	0.0308	0.0370	0.0435	0.0664	0.1071
$\sigma(x^\star(\gamma))$	0.0887	0.0921	0.1022	0.1634	0.2961
$\text{SR}(x^\star(\gamma)\|r)$	0.1224	0.1845	0.2296	0.2839	0.2940

Table 8: Optimal portfolio weights for different values of γ

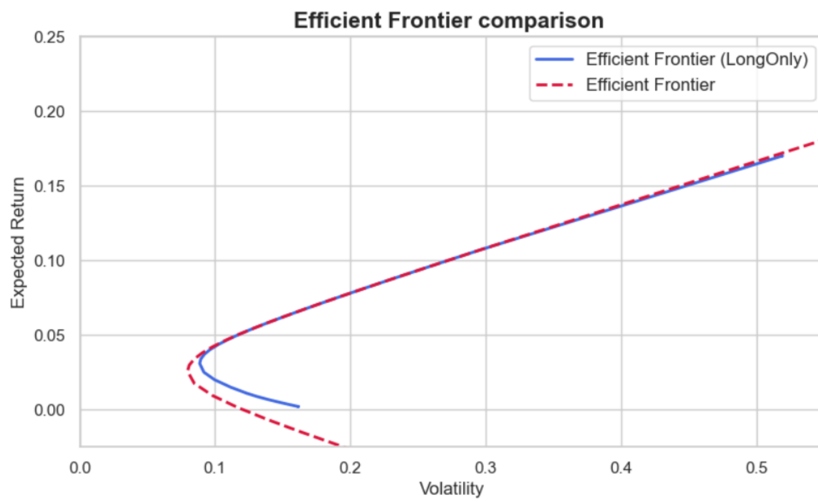


Figure 3: Comparison of efficient frontiers

1.3.2 Question 3b

The plot compares the long-short efficient frontier (dashed red) and the long-only frontier (solid blue). The long-only constraint restricts short-selling, reducing diversification and shifting the frontier to the right.

Short-selling in the long-short case enhances efficiency by combining negatively correlated assets, resulting in lower volatility (σ) for the same expected return (μ). Imposing constraints limits this effect, leading to a less efficient risk-return trade-off.

1.3.3 Question 3c

We apply the same bisection algorithm as in point 2d to determine the optimal portfolio allocations for target volatilities of 10% and 15%. The algorithm searches for the value of γ that yields the desired portfolio standard deviation, stopping when the interval length reaches 10^{-11}. The results are presented in the Table 9.

1.3.4 Question 3d

Using a brute-force algorithm we can find numerically the long-only portfolio. The portoflio is obtained by maximizing the Sharpe Ratio and the results are summed up in the Table 10.

As expected, the Sharpe ratio of the long-only portfolio is slightly lower than that of the long-short portfolio, as short-selling improves capital allocation.

The long-only portfolio has a lower expected return since it cannot leverage short positions to increase exposure to high-return assets. However, its volatility remains nearly unchanged, keeping the Sharpe ratio competitive.

Notably, the asset previously shorted in the long-short portfolio is entirely absent in the long-only allocation. This suggests that shorting was beneficial, and when restricted, the model excludes the asset rather than assigning it a positive weight.

γ	0.18	0.45
x_1	0.3132	0.1858
x_2	0.1854	0.1343
x_3	0.3211	0.2945
x_4	0.1703	0.2127
x_5	0.0101	0.1359
x_6	0.0000	0.0368
$\mu(x^\star(\gamma))$	0.0423	0.0620
$\sigma(x^\star(\gamma))$	0.1000	0.1500
$\mathrm{SR}(x^\star(\gamma) \mid r)$	0.2231	0.2797

Table 9: Optimal portfolio weights for target volatilities of 10% and 15%

γ	1.0690	1.1135
x_1	0.0000	-0.0348
x_2	0.0297	0.0306
x_3	0.1768	0.1829
x_4	0.2087	0.2159
x_5	0.2629	0.2722
x_6	0.3219	0.3333
$\mu(x^\star(\gamma))$	0.1124	0.1163
$\sigma(x^\star(\gamma))$	0.3143	0.3275
$\mathrm{SR}(x^\star(\gamma) \mid r)$	0.2940	0.2941

Table 10: Tangency portfolio weights for long-only (left) and long-short portfolio (right)

1.3.5 Question 3e

We compute the beta coefficient β_i as:

$$\beta_i = \frac{(\Sigma x)_i}{x^T \Sigma x}$$

where x represents the composition of the long-only tangency portfolio x_{MSR}.

Next, we determine the implied expected returns priced by the market using the Capital Asset Pricing Model (CAPM):

$$\tilde{\mu}_i = r + \beta_i(\mu_{MSR} - r)$$

where μ_{MSR} is the expected return of the market portfolio (given by the tangency portfolio), and r is the risk-free rate.

Finally, the alpha coefficient α_i, which measures the excess return beyond what is explained by market exposure, is computed as:

$$\alpha_i = \mu_i - \tilde{\mu}_i$$

where μ_i represents the realized expected return of asset i.

We obtain the following results:

Asset	β_i	$\tilde{\mu}_i$	α_i
Asset 1	-0.1871	0.0027	-0.0007
Asset 2	0.1948	0.0380	~ 0
Asset 3	0.4544	0.0620	~ 0
Asset 4	0.6492	0.0800	~ 0
Asset 5	0.9737	0.1100	~ 0
Asset 6	1.6229	0.1700	~ 0

Table 11: Computed values for β_i, $\tilde{\mu}_i$, and α_i for different assets

The only significant alpha (α_i) is for Asset 1, with a value of -0.0007. As expected, this negative alpha compensates for the distortion introduced by the long-only constraint, which creates artificial demand for an asset that was shorted in the unconstrained tangency portfolio. This leads to a CAPM mispricing, corrected by the negative α. The other alphas are approximately zero, suggesting that CAPM holds well for those assets.

1.4 Point 4

1.4.1 Question 4a

The results for the long-only tangency portfolio are summed up in table below:

Asset	x_i	MR_i	RC_i	RC_i^*
1	0.0000	-0.0588	0.0000	0.0000
2	0.0297	0.0612	0.0018	0.0058
3	0.1768	0.1428	0.0253	0.0804
4	0.2087	0.2041	0.0426	0.1355
5	0.2629	0.3061	0.0805	0.2560
6	0.3219	0.5101	0.1642	0.5224
Volatility	0.3143			

Table 12: Long-only tangency portfolio risk decomposition

1.4.2 Question 4b

The equally weighted portfolio allocates the same weight to each asset, serving as a simple yet effective benchmark for evaluating more sophisticated risk-budgeting strategies.

The results are as follows:

Asset	x_i	MR_i	RC_i	RC_i^*
1	0.1667	−0.0387	−0.0065	−0.0327
2	0.1667	0.0793	0.0132	0.0670
3	0.1667	0.1431	0.0239	0.1208
4	0.1667	0.2026	0.0338	0.1710
5	0.1667	0.3008	0.0501	0.2539
6	0.1667	0.4976	0.0829	0.4200
Volatility	0.1975			

Table 13: Equally weighted portfolio risk decomposition

1.4.3 Question 4c

We aim to determine the minimum variance portfolio by removing the return component from the optimization and focusing solely on risk minimization. The problem is formulated as a quadratic program (QP), given by:

$$x^\star = \arg\min \frac{1}{2} x^\top Q x$$

subject to the constraints:

$$\sum_{i=1}^{6} x_i = 1, \quad 0 \leq x_i \leq 1, \quad \forall i = 1, \dots, 6.$$

We rewrite the problem in a form compatible with the `cvxopt` Python library, which requires the standard quadratic programming formulation:

$$x^\star = \arg\min \frac{1}{2} x^\top Q x$$

subject to:

$$Ax = B, \quad Cx \leq D$$

where:

- $Q = \Sigma$, the covariance matrix of asset returns

- A and B define the equality constraint ensuring full investment:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \end{bmatrix}$$

- C and D define the inequality constraints enforcing the long-only condition:

$$C = \begin{bmatrix} -I_6 \end{bmatrix}, \quad D = \begin{bmatrix} \mathbf{0} \end{bmatrix}$$

where I_6 is the 6×6 identity matrix.

The results of the optimization are presented below:

Asset	x_i	MR_i	RC_i	RC_i^*
1	0.4458	0.0887	0.0395	0.4458
2	0.2223	0.0887	0.0197	0.2223
3	0.2799	0.0887	0.0248	0.2799
4	0.0520	0.0887	0.0046	0.0520
5	0.0000	0.1224	0.0000	0.0000
6	0.0000	0.2039	0.0000	0.0000
Volatility	0.0886			

Table 14: Minimum variance portfolio risk decomposition

1.4.4 Question 4d

The Diversification Ratio is defined as:

$$\mathcal{DR}(x) = \frac{\sum_{i=1}^{n} \mathcal{R}(L_i)}{\mathcal{R}\left(\sum_{i=1}^{n} L_i\right)} = \frac{\sum_{i=1}^{n} x_i \mathcal{R}(e_i)}{\mathcal{R}(x)}$$

where $\mathcal{R}(x)$ represents the chosen risk measure. In our case, using volatility as the risk measure, the expression simplifies to:

$$\mathcal{D}(x) = \frac{\sum_{i=1}^{n} x_i \sigma_i}{\sqrt{x^\top \Sigma x}}$$

The Most Diversified Portfolio is obtained by solving the following optimization problem:

$$\max \quad \frac{x^\top \sigma}{\sqrt{x^\top \Sigma x}}$$

subject to:

$$x^\top \mathbf{e} = 1, \quad x \geq 0$$

where $\mathbf{e} = (1, \ldots, 1)^T$ is the vector of ones.

This problem is equivalent to the following minimization formulation:

$$\min \quad \frac{\sqrt{x^\top \Sigma x}}{x^\top \sigma}$$

subject to:

$$x \geq 0$$

Since the objective function remains invariant under scalar multiplication, if x^* is an optimal solution, then for any $\lambda \in \mathbb{R}$, the scaled portfolio λx^* is also optimal.

Thus, there exists a scalar λ such that:

$$\lambda (x^*)^\top \sigma = 1$$

which allows us to rewrite the optimization problem as:

$$\min \quad \sqrt{x^\top \Sigma x}$$

subject to:

$$x^\top \sigma = 1, \quad x \geq 0$$

Once we have found a solution to the previous problem, we can rescale it to obtain a portfolio whose weights sum to 1.

The last formulation we derived is equivalent to the following quadratic programming (QP) formulation:

$$\min \quad \frac{1}{2} x^\top \Sigma x$$

subject to:

$$x^\top \sigma = 1, \quad x \geq 0$$

We rewrite the problem in a form compatible with the `cvxopt` Python library, which requires the standard quadratic programming formulation:

$$x^* = \arg\min \frac{1}{2} x^\top Q x$$

subject to:

$$Ax = B, \quad Cx \leq D$$

where:

- $Q = \Sigma$, the covariance matrix of asset returns.
- A and B define the equality constraint ensuring the portfolio is properly scaled:

$$A = \sigma^T, \quad B = [1]$$

- C and D are defined as follows:

$$C = \begin{bmatrix} -I_6 \end{bmatrix}, \quad D = \begin{bmatrix} \mathbf{0} \end{bmatrix}$$

Asset	x_i	MR_i	RC_i	RC_i^*
1	0.4799	0.0821	0.0394	0.4130
2	0.2074	0.0868	0.0180	0.1888
3	0.1405	0.0874	0.0123	0.1288
4	0.0957	0.1160	0.0111	0.1164
5	0.0558	0.1642	0.0092	0.0960
6	0.0206	0.2638	0.0054	0.0571
Total Volatility	0.0954			

Table 15: Most diversified portfolio risk decomposition

1.4.5 Question 4e

Employing the CCD algorithm we obtain:

Asset	x_i	MR_i	RC_i	RC_i^*
1	0.4131	0.0431	0.0178	0.1667
2	0.1963	0.0908	0.0178	0.1667
3	0.1530	0.1165	0.0178	0.1667
4	0.1123	0.1587	0.0178	0.1667
5	0.0777	0.2294	0.0178	0.1667
6	0.0477	0.3736	0.0178	0.1667
Total Volatility	0.1069			

Table 16: ERC portfolio risk decomposition

Portfolio	β wrt MSR	β wrt EW
Long-Only Tangency Portfolio	1.0000	1.5659
Equally Weighted Portfolio	0.6179	1.0000
Minimum Variance Portfolio	0.1208	0.2581
Most Diversified Portfolio	0.1643	0.3262
Equal Risk Contribution Portfolio	0.2564	0.4625

Table 17: Beta values of different portfolios

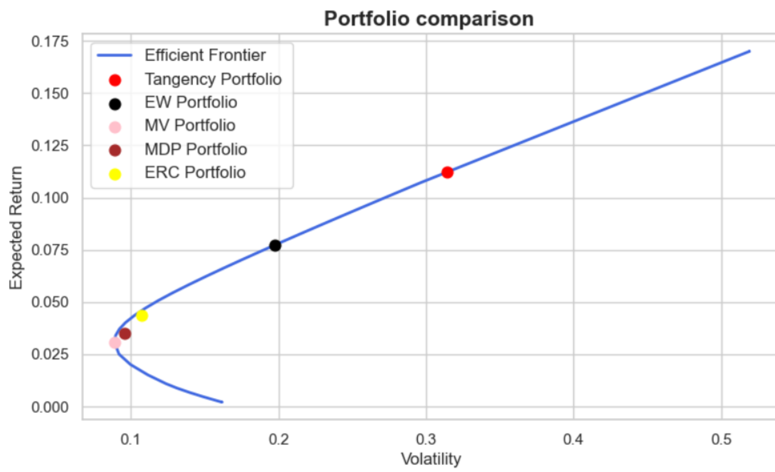


Figure 4: Portfolios visualization

1.4.6 Question 4f

The values of beta are summed up in the Table 17 and in Figure 4 we plot the various long-only portfolios in the mean-variance graph.

By examining the figure, it is clear that the risk-budgeting portfolios exhibit lower variance than the tangency portfolio, albeit at the cost of lower returns.

The beta values in the first column are all below one because the tangency portfolio has the highest volatility, while all other portfolios have lower volatility. Since beta measures a portfolio's return sensitivity relative to the market (here represented by the tangency portfolio), lower-volatility portfolios exhibit proportionally smaller returns, resulting in beta values below one.

This behavior arises because all portfolios, except the tangency portfolio, are constructed using a risk-budgeting approach aimed at reducing total risk in some sense. As a result, they tend to have lower volatility, especially when the tangency portfolio exhibits very high volatility, as is the case here.

In the second column, beta values generally increase. The only beta greater than one is that of the tangency portfolio itself, as it is the only portfolio with higher volatility than the equally weighted portfolio. In general, beta values increase because the volatility of the equally weighted portfolio is lower than that of the tangency portfolio, making the other portfolios appear more volatile relative to it.

Finally by the figure we can observe that the variance of the ERC portfolio is in between that of the MV and the EW portfolios which is in agreement with the theoretical results.

2 Exercise 2

2.1 Point 1

2.1.1 Question 1a

To compute the covariance matrix we use the formula

$$\mathbf{\Sigma} = \mathbf{D}_\sigma \cdot \mathbb{C} \cdot \mathbf{D}_\sigma$$

where:

- $\mathbb{C}$ is the given correlation matrix.
- $\mathbf{D}_\sigma$ is the diagonal matrix of individual asset volatilities σ_i

$$\mathbf{D}_\sigma = \begin{bmatrix} 0.2 & 0 & 0 & \cdots & 0 \\ 0 & 0.22 & 0 & \cdots & 0 \\ 0 & 0 & 0.25 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0.405 \end{bmatrix}$$

We obtain:

$$\mathbf{\Sigma} = \begin{bmatrix} 0.0400 & 0.0220 & 0.0150 & 0.0216 & 0.0360 & 0.0480 & 0.0280 & 0.0243 \\ 0.0220 & 0.0484 & 0.0165 & 0.0238 & 0.0297 & 0.0352 & 0.0462 & 0.0267 \\ 0.0150 & 0.0165 & 0.0625 & 0.0270 & 0.0563 & 0.0800 & 0.0438 & 0.0506 \\ 0.0216 & 0.0238 & 0.0270 & 0.0324 & 0.0243 & 0.1008 & 0.0378 & 0.0219 \\ 0.0360 & 0.0297 & 0.0563 & 0.0243 & 0.2025 & 0.1800 & 0.0788 & 0.0547 \\ 0.0480 & 0.0352 & 0.0800 & 0.1008 & 0.1800 & 0.6400 & 0.1680 & 0.0972 \\ 0.0280 & 0.0462 & 0.0438 & 0.0378 & 0.0788 & 0.1680 & 0.1225 & 0.0851 \\ 0.0243 & 0.0267 & 0.0506 & 0.0219 & 0.0547 & 0.0972 & 0.0851 & 0.1640 \end{bmatrix}$$

2.1.2 Question 1b

To compute the volatility $\sigma(b)$ of the benchmark, we use the following formula:

$$\sigma(b) = \sqrt{b^\top \Sigma b} = 0.3263$$

where b is the vector of benchmark weights, given by:

$$b = (0.095, 0.155, 0.055, \ldots, 0.09)$$

2.1.3 Question 1c

The results are summarized in the following Table 18, where:

Metric	**Formula**	**Result**
Carbon Intensity $\mathcal{CI}(b)$	$\mathcal{CI}(b) = b^\top \mathcal{CI}$	239.2000
Carbon Momentum $\mathcal{CM}(b)$	$\mathcal{CM}(b) = b^\top \mathcal{CM}$	-0.0310
Green Intensity $\mathcal{GI}(b)$	$\mathcal{GI}(b) = b^\top \mathcal{GI}$	0.2598
ESG Score $\mathcal{S}(b)$	$\mathcal{S}(b) = b^\top \mathcal{S}$	0.1850

Table 18: Summary of benchmark metrics

- $b = (0.095, 0.155, 0.055, \ldots, 0.09)$ is the vector of benchmark weights
- $\mathcal{CI} = (80, 200, 390, \ldots, 580)$ is the vector of carbon intensities
- $\mathcal{CM} = (-0.05, -0.075, -0.015, \ldots, 0.02)$ is the vector of Carbon Momentums
- $\mathcal{GI} = (0.05, 0.805, 0.15, \ldots, 0.20)$ is the vector of green intensities
- $\mathcal{S} = (-2, 2.5, 1.5, \ldots, 0.5)$ is the vector of ESG Scores

2.1.4 Question 1d

We define two portfolios, each corresponding to a different sector, by starting from the benchmark allocation and selecting only the weights associated with each sector, setting the weights of the other sector to zero:

$$\text{Portfolio Sector 1} = (0.095, 0, 0.055, 0.085, 0, 0, 0, 0.090)$$

$$\text{Portfolio Sector 2} = (0, 0.155, 0, 0, 0.100, 0.250, 0.170, 0)$$

This sectoral decomposition allows us to determine the benchmark's exposure to each sector and subsequently compute the relevant metrics for each.

The exposure to each sector is given by the sum of the weights of the respective portfolios. While we do not need this result immediately, it will be useful later in our analysis. For reference, the exposure is 0.325 for the first sector and 0.675 for the second sector.

To compute the metrics for each sector, we normalize the sectoral portfolios so that the sum of their weights equals 1. This normalization results in the following adjusted portfolios:

$$\text{Portfolio Sector 1} = (0.2923, 0, 0.1692, 0.2615, 0, 0, 0, 0.2769)$$

$$\text{Portfolio Sector 2} = (0, 0.2296, 0, 0, 0.1481, 0.3704, 0.2519, 0)$$

These adjusted portfolios allow us to properly compute sector-specific metrics while maintaining consistency with the original benchmark allocation. The results for the metrics are summarized in the following table:

Metric	**Sector 1**	**Sector 2**
Carbon Intensity $\mathcal{CI}$	459.23	133.26
Carbon Momentum $\mathcal{CM}$	-0.0168	-0.0378
Green Intensity $\mathcal{GI}$	0.0954	0.3389
ESG Score $\mathcal{S}$	0.3308	0.1148

Table 19: Sector-wise breakdown of benchmark metrics

2.2 Point 2

2.2.1 Question 2a

We aim to formulate the portfolio optimization problem as a quadratic program (QP), given by:

$$x^* = \arg\min \left((x-b)^\top \Sigma (x-b)\right)$$

subject to the constraints:

$$x \geq \mathbf{0}, \quad \sum_{i=1}^{n} x_i = 1, \quad \mathcal{CI}(t,x) \leq (1-20\%) \cdot (1-7\%)^t \cdot \mathcal{CI}(b)$$

where:

- x is the vector of portfolio weights
- b is the benchmark portfolio weights
- Σ is the covariance matrix of asset returns
- $\mathcal{CI}$ is the vector of carbon intensities
- $\mathcal{CI}(b)$ is the Carbon Intensity of the benchmark
- t is a given time parameter

We rewrite the problem in a form compatible with the `cvxpy` Python library, which requires the standard quadratic programming formulation:

$$x^* = \arg\min \frac{1}{2} x^\top Q x - x^\top R$$

subject to:

$$Ax = B, \quad Cx \leq D$$

where:

- $Q = \Sigma$, the covariance matrix of asset returns
- $R = \Sigma b$, ensuring proximity to the benchmark
- A and B define the equality constraint ensuring full investment:

$$A = [1, 1, \ldots, 1], \quad B = [1]$$

- C and D define the inequality constraints enforcing the long-only constraint and the Carbon Intensity limit:

$$C = \begin{bmatrix} -I_6 \\ \mathcal{CI}^\top \end{bmatrix}, \quad D = \begin{bmatrix} \mathbf{0} \\ (1-0.2) \cdot (1-0.07)^t \cdot \mathcal{CI}(b) \end{bmatrix}$$

This formulation allows the problem to be solved efficiently using quadratic programming techniques.

2.2.2 Question 2b

Solving the problem for time values of 0, 1, 2, 5, and 10 we obtain the following values:

Metric	$t = 0$	$t = 1$	$t = 2$	$t = 5$	$t = 10$
Tracking Error Volatility	0.0133	0.0196	0.0255	0.0412	0.1336
Carbon Intensity	167.4393	155.7187	144.8188	116.4861	81.0380
Carbon Momentum	-0.0373	-0.0383	-0.0393	-0.0414	-0.0032
Green Intensity	0.2822	0.2788	0.2756	0.2661	0.0297
Reduction Rate	0.3000	0.3490	0.3946	0.5130	0.6612

Table 20: Evolution of portfolio metrics over time

The table above illustrates the evolution of key portfolio metrics over time as the Carbon Intensity constraint becomes more stringent. This leads to four main visible effects:

- **Increase in Tracking Error Volatility:** As the constraint tightens over time, the portfolio is less able to replicate the benchmark, leading to a progressive increase in tracking error volatility
- **Decrease in Carbon Intensity:** Since the constraint explicitly forces a reduction in Carbon Intensity, the portfolio's carbon footprint naturally declines as t increases
- **Decrease in Green Intensity:** As the portfolio becomes more aligned with sustainability objectives, we observe a progressive decline in Green Intensity, which is an expected result since Green Intensity and Carbon Intensity are correlated

t	0	1	2	5	10
w_1	0.1662	0.1868	0.2059	0.2585	0.4564
w_2	0.1628	0.1496	0.1374	0.1052	0.0000
w_3	0.0645	0.0519	0.0402	0.0029	0.0000
w_4	0.0000	0.0000	0.0000	0.0000	0.0000
w_5	0.0921	0.0956	0.0989	0.1088	0.3451
w_6	0.2577	0.2555	0.2534	0.2482	0.1985
w_7	0.1998	0.2178	0.2346	0.2764	0.0000
w_8	0.0569	0.0428	0.0296	0.0000	0.0000

Table 21: Evolution of portfolio weights over time

- **Increase in the Reduction Rate:** The reduction rate is inversely proportional to the portfolio's Carbon Intensity. As Carbon Intensity decreases due to the constraint, the reduction rate must increase over time

As a footnote, we might also expect Carbon Momentum to increase for similar reasons as Green Intensity. However, this does not happen. The main reason is that companies with low Carbon Intensity do not necessarily exhibit high Carbon Momentum. Once a company has significantly reduced its Carbon Intensity, achieving further reductions becomes increasingly difficult.
Overall, the tightening of the carbon constraint directly impacts the portfolio's composition, making it increasingly different from the benchmark and leading to a natural trade-off between sustainability goals and benchmark tracking performance.

2.2.3 Question 2c

In the following table, we compare the sector allocation of the optimized portfolios over time with the sector allocation of the benchmark:

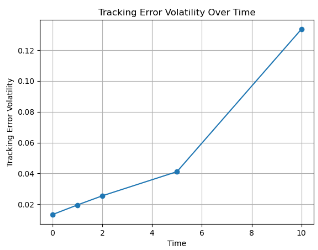
Time	**Benchmark**	$t = 0$	$t = 1$	$t = 2$	$t = 5$	$t = 10$
Sector 1 Exposure	0.3250	0.2876	0.2815	0.2758	0.2614	0.4564
Sector 2 Exposure	0.6750	0.7124	0.7185	0.7242	0.7386	0.5436

Table 22: Sector exposures over time, including benchmark values

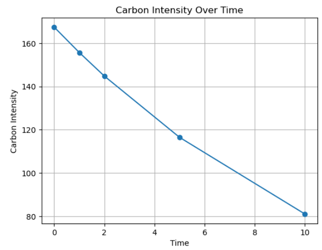
From previous computations, we know that Sector 2 has a significantly lower Carbon Intensity than Sector 1. As the Carbon Intensity constraint tightens over time, we would expect the portfolio to progressively reduce its exposure to high-carbon assets, leading to a shift in allocation toward Sector 2.

However, the table clearly shows that this is not the case. The primary reason lies in Asset 1, which belongs to Sector 1 but has a much lower Carbon Intensity compared to other assets within the same sector. As a result, the portfolio's allocation to Asset 1 increases over time, which, in turn, drives an overall increase in Sector 1 exposure.

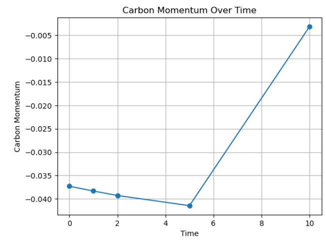
In Figure 5 we can see the evolution of the metrics over time.



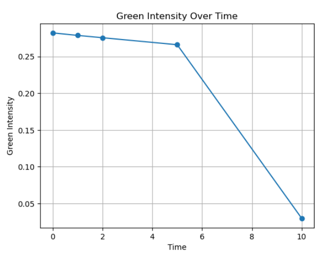
(a) Tracking Error Volatility



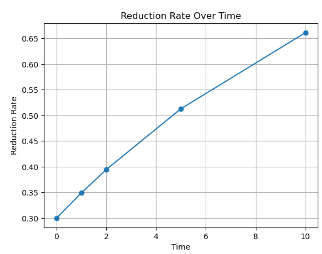
(b) Carbon Intensity



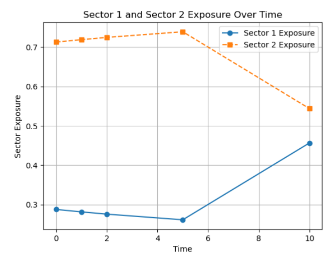
(c) Carbon Momentum



(d) Green Intensity



(e) Reduction Rate



(f) Sector Exposures

Figure 5: Evolution of Portfolio Metrics Over Time

2.2.4 Question 2d

We aim to formulate the portfolio optimization problem as a quadratic program (QP), given by:

$$x^* = \arg\min \left((x-b)^\top \Sigma (x-b) \right)$$

subject to the constraints:

$$x \geq \mathbf{0}, \quad \sum_{i=1}^{n} x_i = 1, \quad \mathcal{CI}(t,x) \leq (1-0.3)\cdot(1-0.07)^t \cdot \mathcal{CI}(b)$$

$$x^\top X_1 = 0.325, \quad x^\top X_2 = 0.675$$

where:

- x is the vector of portfolio weights
- b is the benchmark portfolio weights
- Σ is the covariance matrix of asset returns
- $\mathcal{CI}$ is the vector of carbon intensities

- $\mathcal{CI}(b)$ is the Carbon Intensity of the benchmark
- t is a given time parameter

and

$$X_1 = (1, 0, 1, 1, 0, 0, 0, 1)$$
$$X_2 = (0, 1, 0, 0, 1, 1, 1, 0)$$

We rewrite the problem in a form compatible with the `cvxpy` Python library, which requires the standard quadratic programming formulation:

$$x^* = \arg\min \frac{1}{2} x^\top Q x - x^\top R$$

subject to:

$$Ax = B, \quad Cx \leq D$$

where:

- $Q = \Sigma$, the covariance matrix of asset returns
- $R = \Sigma b$, ensuring proximity to the benchmark
- A and B define the equality constraint ensuring full investment and sector neutrality:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.325 \end{bmatrix}$$

- C and D define the inequality constraints enforcing the long-only condition and the Carbon Intensity limit:

$$C = \begin{bmatrix} -I_6 \\ \mathcal{CI}^\top \end{bmatrix}, \quad D = \begin{bmatrix} \mathbf{0} \\ (1 - 0.2) \cdot (1 - 0.07)^t \cdot \mathcal{CI}(b) \end{bmatrix}$$

Solving the problem for the same time values as before we obtain:

Metric	$t = 0$	$t = 1$	$t = 2$	$t = 5$	$t = 10$
Tracking Error Volatility	0.0148	0.0210	0.0269	0.0426	0.1453
Carbon Intensity	167.4400	155.7191	144.8189	116.4860	81.0380
Carbon Momentum	-0.0369	-0.0379	-0.0388	-0.0409	0.0087
Green Intensity	0.2610	0.2541	0.2477	0.2303	0.0355
Reduction Rate	0.3000	0.3490	0.3946	0.5130	0.6612

Table 23: Evolution of portfolio metrics over time

t	0	1	2	5	10
w_1	0.1928	0.2178	0.2410	0.3031	0.3250
w_2	0.1334	0.1154	0.0986	0.0551	0.0000
w_3	0.0745	0.0635	0.0532	0.0218	0.0000
w_4	0.0000	0.0000	0.0000	0.0000	0.0000
w_5	0.0859	0.0884	0.0907	0.0978	0.4371
w_6	0.2565	0.2541	0.2518	0.2461	0.2205
w_7	0.1993	0.2172	0.2338	0.2759	0.0174
w_8	0.0577	0.0437	0.0307	0.0001	0.0000

Table 24: Optimal portfolio weights for different values of t

2.3 Point 3

2.3.1 Question 3a

We aim to formulate the portfolio optimization problem as a quadratic program (QP), given by:

$$x^* = \arg\min \left((x-b)^\top \Sigma (x-b)\right)$$

subject to the constraints:

$$x \geq \mathbf{0}, \quad \sum_{i=1}^{n} x_i = 1, \quad \mathcal{CI}(x) \leq (1-30\%) \cdot \mathcal{CI}(b)$$

where:

- x is the vector of portfolio weights
- b is the benchmark portfolio weights
- Σ is the covariance matrix of asset returns
- $\mathcal{CI}(x)$ is the Carbon Intensity of the portfolio
- $\mathcal{CI}(b)$ is the Carbon Intensity of the benchmark

We rewrite the problem in a form compatible with the `cvxpy` Python library, which requires the standard quadratic programming formulation:

$$x^* = \arg\min \frac{1}{2} x^\top Q x - x^\top R$$

subject to:

$$Ax = B, \quad Cx \leq D$$

where:

- $Q = \Sigma$, the covariance matrix of asset returns
- $R = \Sigma b$, ensuring proximity to the benchmark
- A and B define the equality constraint ensuring full investment:

$$A = [1, 1, \ldots, 1], \quad B = [1]$$

- C and D define the inequality constraints enforcing the long-only condition and the Carbon Intensity limit:

$$C = \begin{bmatrix} -I_6 \\ \mathcal{CI}^\top \end{bmatrix}, \quad D = \begin{bmatrix} \mathbf{0} \\ (1 - 0.3) \cdot \mathcal{CI}(b) \end{bmatrix}$$

This formulation allows the problem to be solved efficiently using quadratic programming techniques and we obtain:

$$x^* = \begin{bmatrix} 0.1662 \\ 0.1628 \\ 0.0645 \\ 0.0000 \\ 0.0092 \\ 0.2577 \\ 0.1998 \\ 0.0569 \end{bmatrix}$$

2.3.2 Question 3b

We introduce an additional constraint:

$$\mathcal{GI}(x) \geq (1 + 50\%) \cdot \mathcal{GI}(b)$$

where:

- x represents the portfolio weights
- b represents the benchmark portfolio weights
- $\mathcal{GI}(x)$ is the Green Intensity of the portfolio
- $\mathcal{GI}(b)$ is the Green Intensity of the benchmark

This constraint ensures that the Green Intensity of the portfolio is at least 50% higher than that of the benchmark. The Green Intensity, $\mathcal{GI}$, measures the proportion of capital expenditures (CapEx) dedicated to environmentally friendly projects.

Implications:

- The optimizer will favor companies with a higher share of investments in green projects

- Companies with low green CapEx will be penalized or excluded
- This constraint encourages a more environmentally sustainable investment portfolio

To enforce this constraint, we redefine the general inequality form:

$$Cx \leq D$$

where the constraint matrix C and vector D are:

$$C = \begin{bmatrix} -I_6 \\ \mathcal{CI}^\top \\ -\mathcal{GI}^\top \end{bmatrix}$$

$$D = \begin{bmatrix} \mathbf{0} \\ (1 - 0.3) \cdot \mathcal{CI}(b) \\ -(1 + 0.5) \cdot \mathcal{GI}(b) \end{bmatrix}$$

where:

- The first row of $Cx \leq D$ ensures the long-only condition
- The second row of $Cx \leq D$ ensures the Carbon Intensity limit
- The third row of $Cx \leq D$ enforces the Green Intensity constraint

Solving the optimization problem under these constraints yields the following optimal portfolio:

$$x^* = \begin{bmatrix} 0.0638 \\ 0.2981 \\ 0.0320 \\ 0.0000 \\ 0.0875 \\ 0.2572 \\ 0.2190 \\ 0.0423 \end{bmatrix}$$

2.3.3 Question 3c

We introduce the following additional constraints to our optimization problem:

$$\mathcal{CM}(x) \leq (1 + 50\%) \cdot \mathcal{CM}(b)$$

where:

- x represents the portfolio weights
- b represents the benchmark portfolio weights
- $\mathcal{CM}(x)$ is the Carbon Momentum of the portfolio

- $\mathcal{CM}(b)$ is the Carbon Momentum of the benchmark

This constraint ensures that the Carbon Momentum of the portfolio does not exceed 150% of the benchmark's Carbon Momentum. Carbon Momentum, $\mathcal{CM}$, measures how much a company's Carbon Intensity has changed over time.

Implications:

- Companies that are actively reducing their carbon emissions will be preferred
- The portfolio limits exposure to firms increasing their carbon footprint
- It enforces a preference for transitioning companies, encouraging investment in firms that are making sustainability improvements

To enforce these constraints, we redefine the general inequality form:

$$Cx \leq D$$

where the constraint matrix C and vector D are now:

$$C = \begin{bmatrix} -I_6 \\ \mathcal{CI}^\top \\ -\mathcal{GI}^\top \\ \mathcal{CM}^\top \end{bmatrix}$$

$$D = \begin{bmatrix} \mathbf{0} \\ (1 - 0.3) \cdot \mathcal{CI}(b) \\ -(1 + 0.5) \cdot \mathcal{GI}(b) \\ (1 + 0.5) \cdot \mathcal{CM}(b) \end{bmatrix}$$

where:

- The first row of $Cx \leq D$ ensures the long-only condition
- The second row of $Cx \leq D$ ensures the Carbon Intensity limit
- The third row of $Cx \leq D$ enforces the Green Intensity constraint
- The fourth row of $Cx \leq D$ enforces the Carbon Momentum constraint

Solving the optimization problem under these constraints yields the following optimal portfolio:

$$x^* = \begin{bmatrix} 0.0763 \\ 0.2908 \\ 0.0418 \\ 0.0000 \\ 0.0674 \\ 0.2594 \\ 0.2282 \\ 0.0360 \end{bmatrix}$$

2.3.4 Question 3d

We introduce the following additional constraints to our optimization problem:

$$\mathcal{S}(x) \geq \mathcal{S}(b) + 0.5$$

where:

- x represents the portfolio weights
- b represents the benchmark portfolio weights
- $\mathcal{S}(x)$ is the ESG Score of the portfolio
- $\mathcal{S}(b)$ is the ESG Score of the benchmark

This constraint requires the ESG Score of the portfolio to be at least 0.5 points higher than that of the benchmark.

Implications:

- The optimizer prioritizes companies with strong ESG ratings
- Firms with poor governance, weak social policies, or high emissions may be excluded
- This ensures alignment with sustainable and ethical investment principles

To enforce these constraints, we redefine the general inequality form:

$$Cx \leq D$$

where the constraint matrix C and vector D are now:

$$C = \begin{bmatrix} -I_6 \\ \mathcal{CI}^\top \\ -\mathcal{GI}^\top \\ \mathcal{CM}^\top \\ -\mathcal{S}^\top \end{bmatrix}$$

$$D = \begin{bmatrix} \mathbf{0} \\ (1-0.3) \cdot \mathcal{CI}(b) \\ -(1+0.5) \cdot \mathcal{GI}(b) \\ (1+0.5) \cdot \mathcal{CM}(b) \\ -(\mathcal{S}(b) + 0.5) \end{bmatrix}$$

where:

- The first row of $Cx \leq D$ ensures the long-only condition
- The second row of $Cx \leq D$ ensures the Carbon Intensity limit
- The third row of $Cx \leq D$ enforces the Green Intensity constraint
- The fourth row of $Cx \leq D$ enforces the Carbon Momentum constraint

- The fifth row of $Cx \leq D$ enforces the ESG Score constraint

Solving the optimization problem under these constraints yields the following optimal portfolio:

$$x^* = \begin{bmatrix} 0.0265 \\ 0.3756 \\ 0.0665 \\ 0.0000 \\ 0.0826 \\ 0.2740 \\ 0.1684 \\ 0.0064 \end{bmatrix}$$

2.3.5 Question 3e

The results for the previous problem are summarized in the following table:

QP Problem	**TE**	$\mathcal{CI}$	$\mathcal{CM}$	$\mathcal{GI}$	$\mathcal{S}$	**Sector 1**	**Sector 2**
3a	0.0133	167.44	-0.0373	0.2822	-0.1209	0.2876	0.7124
3b	0.0321	167.44	-0.0438	0.3897	0.3614	0.1382	0.8618
3c	0.0329	167.44	-0.0465	0.3897	0.3440	0.1541	0.8459
3d	0.0441	167.44	-0.0465	0.4176	0.6850	0.0994	0.9006

Table 25: Summary of QP problem results

2.3.6 Question 3f

Given the four constraints:

$$\mathcal{CI}(x) \leq (1-\mathcal{R})\mathcal{CI}(x), \quad \mathcal{GI}(x) \geq (1+50\%)\mathcal{GI}(x), \quad \mathcal{CM}(x) \leq (1+50\%)\mathcal{CM}(x), \quad \mathcal{S}(x) \geq \mathcal{S}(b)+0.5$$

we can iterate over possible values of $\mathcal{R}$ and we can find the first value that makes the constraints incompatibles. In particular we obtain a value of 39.27%.

2.4 Point 4

2.4.1 Question 4a

As we previously computed the carbon intensities of sector 1 and sector 2 are:

$$\mathcal{CI}(\text{Sector 1}) = 459.2308$$

$$\mathcal{CI}(\text{Sector 2}) = 133.2593$$

2.4.2 Question 4b

We aim to formulate the portfolio optimization problem as a quadratic program (QP), given by:

$$x^* = \arg\min \left((x-b)^\top \Sigma (x-b)\right)$$

subject to the constraints:

$$x \geq \mathbf{0}, \quad \sum_{i=1}^{n} x_i = 1, x^T \left(x_1^T CI - 0.7 \cdot \mathcal{CI}(\text{Sector 1})\right) \leq 0, x^T \left(x_2^T CI - 0.5 \cdot \mathcal{CI}(\text{Sector 2})\right) \leq 0$$

We rewrite the problem in a form compatible with the `cvxpy` Python library, which requires the standard quadratic programming formulation:

$$x^* = \arg\min \frac{1}{2} x^\top Q x - x^\top R$$

subject to:

$$Ax = B, \quad Cx \leq D$$

where:

- $Q = \Sigma$, the covariance matrix of asset returns
- $R = \Sigma b$, ensuring proximity to the benchmark
- A and B define the equality constraint ensuring full investment:

$$A = [1, 1, \ldots, 1], \quad B = [1]$$

- C and D define the inequality constraints enforcing the Carbon Intensity reduction for each sector:

$$C = \begin{bmatrix} \cdots & -I_8 & \cdots & \\ 80 - 0.7 \cdot \mathcal{CI}(\text{Sector 1}) & 0 & 390 - 0.7 \cdot \mathcal{CI}(\text{Sector 1}) & \cdots \\ 0 & 200 - 0.5 \cdot \mathcal{CI}(\text{Sector 2}) & 0 & \cdots \end{bmatrix}$$

$$D = \begin{bmatrix} \mathbf{0} \\ 0 \\ 0 \end{bmatrix}$$

  - The first row controls the ong-only condition
  - The second row controls Carbon Intensity for Sector 1

- The third row controls Carbon Intensity for Sector 2

Solving the previous problem, we obtain:

$$x^* = \begin{bmatrix} 0.3352 \\ 0.0000 \\ 0.0396 \\ 0.0921 \\ 0.3567 \\ 0.0443 \\ 0.0000 \\ 0.1321 \end{bmatrix}$$

2.4.3 Question 4c

To enforce sector neutrality, we introduce the following sector allocation constraints:

$$x^\top X_1 = 0.325, \quad x^\top X_2 = 0.675$$

The updated optimization problem is:

$$x^* = \arg\min \frac{1}{2} x^\top Q x - x^\top R$$

subject to:

$$Ax = B, \quad Cx \leq D$$

where:

- $Q = \Sigma$, the covariance matrix of asset returns
- $R = \Sigma b$, ensuring proximity to the benchmark
- A and B define the equality constraint ensuring full investment and sector neutrality:

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0.325 \end{bmatrix}$$

  - The first row ensures the portfolio is fully invested
  - The second row enforce the sector allocation to Sector 1
- C and D define the inequality constraints enforcing the Carbon Intensity reduction for each sector and they are the same as in the previous point

Solving the previous problem, we obtain:

$$x^* = \begin{bmatrix} 0.1898 \\ 0.0000 \\ 0.0000 \\ 0.0495 \\ 0.6004 \\ 0.0746 \\ 0.0000 \\ 0.0857 \end{bmatrix}$$

2.4.4 Question 4d

The results are summarized in the following table:

QP Problem	**TE**	$\mathcal{CI}$	$\mathcal{CM}$	$\mathcal{GI}$	$\mathcal{S}$	**Sector 1**	**Sector 2**
4b	0.1733	219.2644	0.0102	0.0563	-0.7396	0.5989	0.4011
4c	0.2031	149.4500	0.0363	0.0386	-0.8755	0.3250	0.6750

Table 26: Summary of QP problem results

2.4.5 Question 4e

From the first to the second problem, the tracking error volatility (TE) increases, which is expected given that an additional constraint has been imposed.

We observe that the new portfolio meets the imposed allocation constraints for both Sector 1 and Sector 2.

As the Sector 1 exposure decreases in the new allocation, the ESG score declines which is consistent with the fact that, as previously calculated, the average ESG score of Sector 1 assets is higher than that of Sector 2.

The Green-Intensity score decreased even though on average the Green-intensity of sector 1 assets is lower.

Finally, as mentioned earlier, Sector 1 has a significantly higher Carbon Intensity than Sector 2, so we would naturally expect the overall Carbon Intensity to decrease if we require an higher exposure to Sector 2. In this case it happens, however, as we saw in question 2c, it is not always the case.

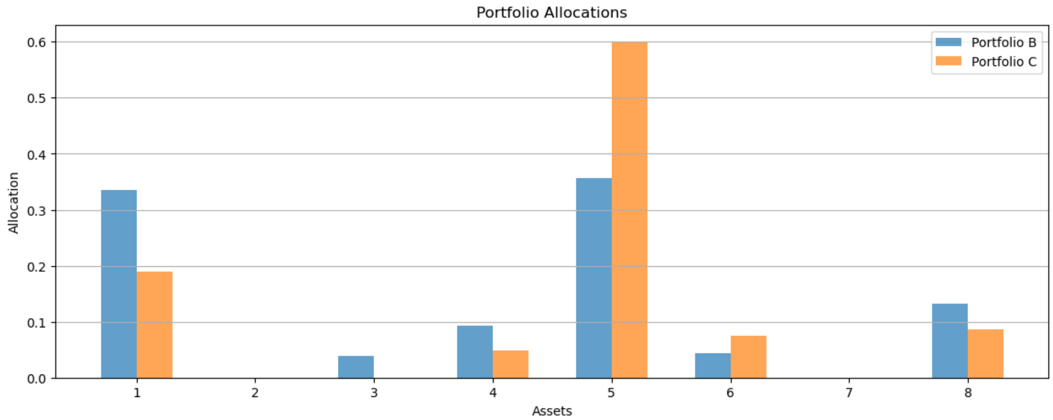
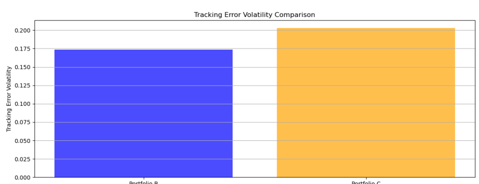
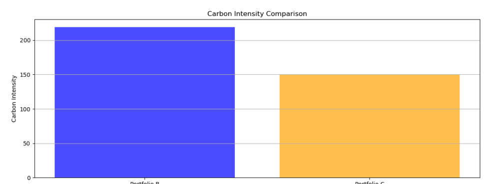


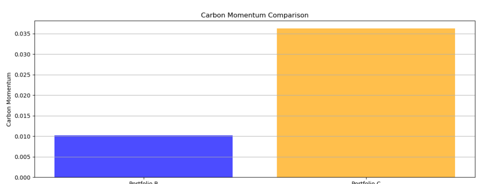
Figure 6: Portfolio Allocation Comparison



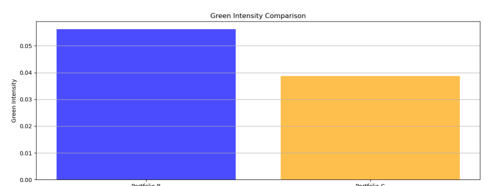
(a) Tracking Error Volatility



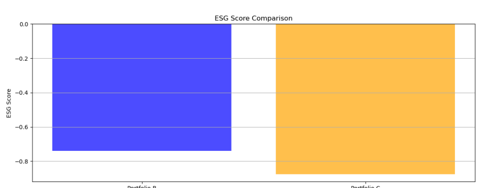
(b) Carbon Intensity



(c) Carbon Momentum



(d) Green Intensity



(e) ESG Score

(f) Sector Allocations

Figure 7: Metrics Comparison