Interest Rates Modeling Report: Lognormal spot rate models

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February 2025

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1 Lognormal spot rate models

1.1 The importance of spot rate models

Spot rate models give a specific dynamics to the spot rate r_t to price bonds with any maturity T. If we assume to know the market price of risk λ_t, we can find the price of any bond $p(t,T)$ using a Feynman-Kaç approach to the term structure equation.

Theorem 1.1 (Term Structure Equation) *If the market is arbitrage free, the family of processes $F^T(t,r(t)) = p(t,T)$ satisfies the following equation*

$$\begin{cases} F_t^T + (\mu - \lambda\sigma)F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T = 0 \\ F^T(T,r) = 1 = p(T,T) \end{cases}$$

where we assume that the spot rate has the $\mathbb{Q}$-dynamics

$$dr_t = (\mu(t,r(t)) - \lambda_t\sigma(t,r(t)))dt + \sigma(t,r(t))dW_t$$

$$r(t) = r$$

Theorem 1.1 allows us to give a Feynman-Kaç representation to the price function $F(t,r,T)$.

$$F(t,r,T) = \mathbb{E}_{t,r}^{\mathbb{Q}}[e^{-\int_t^T r_s ds}] \tag{1}$$

The price does not require the dynamics of $r(t)$ under the objective probability measure $\mathbb{P}$, but under the martingale measure $\mathbb{Q}$. This suggests that we should develop short-rate models directly under the measure $\mathbb{Q}$, since in this way the modelization of the market price for risk λ_t is not needed.

This idea paved the way for short-rate martingale models. The dynamics of r_t can have different forms, and each of them can be calibrated using market data. Specifically, every model for r_t has particular properties that can be useful depending on the scenario considered.

1.2 Log-normal spot rate models: properties

A specific model class for the dynamics of r_t is the class of log-normal spot rate models. As we can deduce, the dynamics of r_t in these models is such that r_t is a log-normal random variable for every fixed t. As a result, we have various desirable properties for a risk-free rate.

A first realistic property is the positivity of the rates: the lognormality of r_t guarantees the impossibility for the rates to be negative in the short and long term, which is always expected in real-life market data. Moreover, in these models the volatility of r_t is always proportional to r_t itself, which is again a realistic assumption, as higher rates usually imply a higher volatility. In addition, log-normal models are often preferred for pricing interest rate options and bonds with embedded options because their behavior better reflects the dynamics observed in financial markets.

On the other hand, log-normal models sometimes lack in simplicity and interpretability, and rarely yield a closed formula for option and bond pricing. In that case, calibration is performed through Monte Carlo simulations or lattice methods (binomial trees), as will be the case for the Black-Derman-Toy model. Another flaw of log-normal models is the instability of the rates when shifting to instantaneous rates. In fact, log-normality leads to "double exponential" expressions, i.e. expressions where the exponential function is itself an argument of an exponential, in the pricing formula 1.

A further limit of these models is that, as mentioned before, they struggle to accommodate negative interest rates, which, although historically rare, have occurred in certain economic conditions, for example, during the 2008 financial crisis and the subsequent monetary policies implemented by central banks, such as the European Central Bank (ECB) and the Bank of Japan (BoJ). This limitation makes log-normal models less suitable for accurately modeling interest rate dynamics in environments where negative rates are a possibility.

Some examples of log-normal short-rate models include CIR, extended Vasiçek (Hull-White), Dothan, and Black-Derman-Toy, which will be thoroughly discussed in the rest of the paper.

2 The Black-Derman-Toy model

The Black-Derman-Toy model is a simple and versatile model of interest rates, according to which all security prices and rates depend on a single factor: the short rate itself. The dynamics of r_t is, in fact, the following.

$$dr_t = \theta(t)r_t dt + \sigma(t)r_t dW_t \tag{2}$$

It can also be defined through the dynamics of $log(r_t)$ by formally applying Ito formula to the logarithmic function.

$$d(log(r_t)) = \frac{1}{r_t}dr_t + \frac{1}{2}\frac{1}{r_t^2}d\langle r_t\rangle$$

By substituting 2 the above dynamics becomes

$$d(log(r_t)) = (\theta(t) - \frac{1}{2}\sigma^2(t))dt + \sigma(t)dW_t$$

And the model can be seen as a specific case of the Black and Karasinski (1991) model below, without mean reversion.

$$d(log(r_t)) = \kappa(\theta(t) - log(r_t))dt + \sigma(t)dW_t \quad \text{with } \kappa, \theta(t) \geq 0 \tag{3}$$

After analyzing the lattice method for the calibration of the BDT model, the last part of the report will focus on the problem of Eurodollar future pricing, and introduce a solution to it.

2.1 The Black-Derman-Toy model: link with its discrete version

For many pricing applications, it is convenient to have a binomial model that fits the current term structure of bond prices and the current term structure of volatilities. The main assumption for the construction of the binomial tree is that at each node interest rates evolve randomly in two possible scenarios, with equal probability. We will indicate the upward scenario with "u" and the downward with "d". The tree is recombining, meaning that an upward-downward sequence of movements leads to the same scenario of a downward-upward one. Therefore the number of scenarios starts from one at present time, as the present rate is certain, and increases by one at each period. In particular, after i periods we have $i+1$ scenarios. This assumption will be crucial for the calibration since it limits the number of unknowns.

2.1.1 Calibration of the BDT

As stated in the previous section, the BDT is a no-arbitrage model, meaning that the term structure of interest rates is an input of the model, not the desired output. In particular, from an implementation point of view, we consider as inputs the yield curve and the volatility curve. The first one is the array of the Zero Coupon interest rates with maturities $i\Delta t$ $\forall i = 1, \dots, n$, denoted by R_n, while the second one indicates the array of the volatilities at time zero of Zero Coupon Bonds maturing at $i\Delta t$, indicated with β_i $\forall i = 2, \dots, n$. The calibration process is based on finding the interest rate values at each time step $i = 0, ..., n-1$, in each scenario $j = 1, ..., i+1$, that is, $r_{i,j}$, where $j = 1$ corresponds to a sequence of all downward movements and $j = i+1$ to a sequence of all upward movements. These values should allow us to match both the values of ZCBs for different maturities, $\frac{1}{(1+R_i)^{i\Delta t}}$, and the volatilities σ_i within each period. Mathematically, the following equations must be satisfied:

Step 0 The initial rate r for the period between 0 and Δt, is known

$$r = R_1$$

Step 1 The initial value of a ZCB with maturity $2\Delta t$ can be computed not only using the yield rate R_2, but also by evaluating the bond's value in the two scenarios at time 1 and discounting with rate r. In the second case, we recall that each scenario occurs with 50% probability.

$$\frac{1}{(1+r)^{\Delta t}} \left[0.5 \frac{1}{(1+r_d)^{\Delta t}} + 0.5 \frac{1}{(1+r_u)^{\Delta t}} \right] = \frac{1}{(1+R_2)^{2\Delta t}}$$

Clearly, this equation is not sufficient to determine r_u and r_d. We compute the variance of the logarithm of interest rates as follows

$$\begin{aligned} \mathrm{VAR}\left(log(r_1)\right) &= 0.5log^2(r_u) + 0.5log^2(r_d) - \left[0.5(log(r_u) + log(r_d))\right]^2 \\ &= \left[0.5\left(log\left(r_u\right) - log(r_d)\right)\right]^2 = \left[0.5\left(log\left(r_u/r_d\right)\right)\right]^2 \end{aligned}$$

Considering the dynamics 3, we can write the following equation for its volatility.

$$0.5log\left(\frac{r_u}{r_d}\right) = \sigma_1\sqrt{\Delta t}$$

If we know the value of σ_1 we can compute r_u and r_d.

Step 2 Similarly, we write the following three equations to determine r_{uu}, r_{ud} and r_{dd}

$$\begin{aligned} &\frac{1}{(1+r)^{\Delta t}} \left(0.5 \frac{1}{(1+r_d)^{\Delta t}} \left[0.5 \frac{1}{(1+r_{dd})^{\Delta t}} + 0.5 \frac{1}{(1+r_{ud})^{\Delta t}} \right] + \right. \\ &\left. +0.5 \frac{1}{(1+r_u)^{\Delta t}} \left[0.5 \frac{1}{(1+r_{ud})^{\Delta t}} + 0.5 \frac{1}{(1+r_{uu})^{\Delta t}} \right] \right) = \frac{1}{(1+R_3)^{3\Delta t}} \end{aligned}$$

$$0.5log\left(\frac{r_{uu}}{r_{ud}}\right) = \sigma_2\sqrt{\Delta t}$$

$$0.5log\left(\frac{r_{ud}}{r_{dd}}\right) = \sigma_2\sqrt{\Delta t}$$

Again, the system of equations is determined provided that σ_2 is known. σ_i values are not given directly, so we need to recover them from the initial data, eventually through an approximation. Recalling that we have data on the volatilities β_i of bonds maturing at $i\Delta t$, we approximate β_i as the volatility of the bond yield between time zero and $i\Delta t$. For a bond with maturity $2\Delta t$ we obtain the following:

$$0.5log\left(\frac{y_{u2}}{y_{d2}}\right) = \beta_2\sqrt{\Delta t}$$

Since $y_{u2} = r_u$ and $y_{d2} = r_d$, this can be rewritten as

$$0.5log\left(\frac{r_u}{r_d}\right) = \beta_2\sqrt{\Delta t}$$

Hence $\sigma_1 = \beta_2$.

Unfortunately, there is no closed-form expression for the next σ_i values in function of β_i, so we must perform an iterative search. In particular, we start from an initial value of σ_i and calculate the interest rates at $i\Delta t$ as shown above. Then we compute the yields y_u and y_d from the obtained tree and we calculate the volatility of a bond maturing at $i\Delta t$ as $0.5log\left(\frac{y_u}{y_d}\right)$. We search σ_i iteratively until this quantity matches the corresponding bond yield volatility. The whole procedure can be iterated to calibrate the tree up to a desired depth.

2.1.2 Numerical Results

In this section we illustrate the results of the calibration of the BDT model starting from the following initial data: $r = 0.10$, $\Delta t = 1$, yield curve and the volatility curve are given in the tables below.

Maturity	**Yield Rate**
1 year	10.0
2 years	11.0
3 years	12.0
4 years	12.5
5 years	13.0

Maturity	Yield Volatility
2 years	19.0%
3 years	18.0%
4 years	17.5%
5 years	16.0%

Following the procedure explained in the previous section and recalling that $\sigma_1 = \beta_2$, we initially find r_u and r_d thanks to the equations:

$$\frac{1}{1.10}\left[0.5\frac{1}{1+r_d} + 0.5\frac{1}{1+r_u}\right] = \frac{1}{1.11^2}$$

$$0.5log\left(\frac{r_u}{r_d}\right) = 0.19$$

We obtain $r_u = 0.1432$ and $r_d = 0.0979$.
For the second step, we would need to solve the equations below:

$$\frac{1}{1.1}\left(0.5\frac{1}{1.0979}\left[0.5\frac{1}{1+r_{dd}} + 0.5\frac{1}{1+r_{ud}}\right] + \right.$$
$$\left. +0.5\frac{1}{1.1432}\left[0.5\frac{1}{1+r_{ud}} + 0.5\frac{1}{1+r_{uu}}\right]\right) = \frac{1}{1.12^3}$$

$$0.5log\left(\frac{r_{uu}}{r_{ud}}\right) = \sigma_2$$

$$0.5log\left(\frac{r_{ud}}{r_{dd}}\right) = \sigma_2$$

However, since the value of σ_2 is unknown, we try to solve them for different values of σ_2, obtaining several trees, and we calculate the price of a three-year bond in the two scenarios of the first time step, for each tree. The correct value of σ_2 is the one that allows for a perfect match of the three-year yield volatility. From an implementation point of view, we stop the iterative search when the error is smaller than a certain tolerance. The three-year bond prices calculated in the downward and upward scenarios are respectively

$$B_d = \frac{1}{1.0979}\left[0.5\frac{1}{1+r_{dd}} + 0.5\frac{1}{1+r_{ud}}\right]$$

$$B_u = \frac{1}{1.1432}\left[0.5\frac{1}{1+r_{ud}} + 0.5\frac{1}{1+r_{uu}}\right]$$

and the associated yields are

$$y_d = \sqrt{\frac{1}{B_d}} - 1$$

$$y_u = \sqrt{\frac{1}{B_u}} - 1$$

We obtain $\sigma_2 = 0.172$, and the corresponding level of the tree is given by

$$r_{uu} = 0.1942 \qquad r_{ud} = 0.1377 \qquad r_{dd} = 0.0976$$

These values gives the following

$$B_d = 0.8152 \qquad B_u = 0.7507$$

with the associated yields

$$y_d = 0.1076 \qquad y_u = 0.1542$$

Finally the yield volatility is

$$0.5log\left(\frac{0.1542}{0.1076}\right) = 0.18$$

and this value coincides with the given 3-year volatility β_3. These computational steps can be iterated to calibrate deeper levels of the tree.

We calibrated the model as we have just explained in Python data and developed a class to price options and compute Greeks. We then replicated all the results of the original paper.

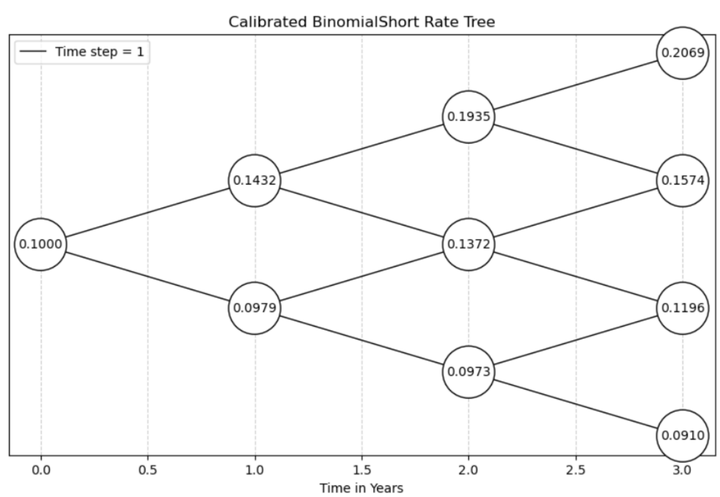

Figure 1: Black-Derman-Toy binomial short rate tree calibrated with the original paper's data

2.2 Zero Black-Derman-Toy model

The classical version of the Black-Derman-Toy model was originally developed in 1990 and was an industry standard for years. Following the 2007–2008 financial crisis in the United States, the Federal Reserve reduced the Fed Funds Rate by 425 basis points to practically zero. This decision was preserved for nine years and was called the Zero Interest Rate Policy (ZIRP policy). This led to the development of several new approaches to fixed income theory that would model the ZIRP.

In 2020 Grzegorz Krzyzanowsky, Ernesto Mordecki, and Andrés Sosa developed an extension of the Black-Derman-Toy model that would incorporate in its dynamics the possibility of a downward jump to a practically zero interest rate value and a high probability that this null value persists in time.

In the new model, the nodes labeled (i, j) with $j \geq 2$ have the same characteristics as in the Black-Derman-Toy model. Specifically, they follow a binomial structure with up-and-down probabilities of $\frac{1}{2}$, and jump values that need to be calibrated. Furthermore, nodes of the form $(1, j)$ introduce a third possible downward jump with a small probability p, while the other two possible jumps occur with probability $\hat{p} = \frac{1-p}{2}$. If this downward jump occurs, the process enters the so-called zero interest rate policy (ZIRP) zone, which means that the interest rate falls to a small value x_0, which is close to the central bank's policy target. When the process is in the ZIRP zone, it remains there with a high probability $(1 - q)$ and exits with probability q.

To calibrate the tree, following the same methodology as in the classical BDT model, the authors impose that the variance at each node for the same time period remains constant, to be determined by calibration and denoted as $\sigma(n)$ for period n. In addition to the standard interest rates $r_{i,j}$ and the bond prices $B_{i,j}$ of the BDT model, the ZBDT model introduces the (unknown) bond prices $B_{i,0}$ for $i = 1, \ldots, n-1$ and sets $B_{0,n} = 100$. The corresponding interest rates $r_{i,0}$ for $i = 1, \ldots, i+1$ are fixed at x_0.

Figure 2 illustrates the tree structure for pricing a zero-coupon bond with expiration in $n = 3$ years.

The parameters p, q, and x_0 have clear economic interpretations:

- p represents the probability of entering a financial crisis in a basic time unit (one year in this context),
- q is the conditional probability of economic recovery from the financial crisis,
- x_0 is the assumed interest rate in the ZIRP zone.

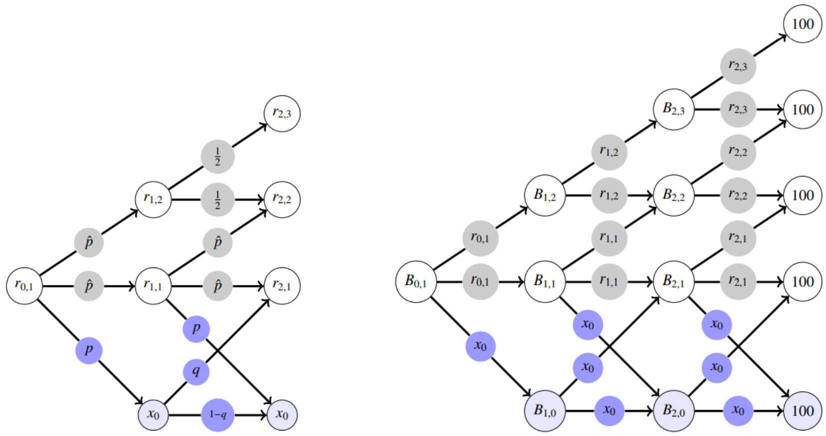

Figure 2: The ZBDT interest rate tree and the corresponding ZC-bond tree

2.2.1 Calibration of the model

As in the Black-Derman-Toy model the inputs are the yield curve and the volatility curve.

The first step uses only $y(1)$ and we conclude that $r_{0,1} = y(1)$.

The bond prices at time 1 are initialized as:

$$B_{1,0} = B_{1,1} = B_{1,2} = 100$$

The price of a bond at time 0 is computed by incorporating the probability of a downward jump p:

$$B_{0,1} = \frac{1}{1 + r_{0,1}} \left[\frac{1-p}{2} (B_{1,2} + B_{1,1}) + pB_{1,0} \right]$$

For the next step, we introduce the information regarding the volatility.

In the ZBDT model, a node can now have three possible outcomes:

$$Y = \begin{cases} y_u, & \text{with probability } \hat{p} = \frac{1-p}{2} \\ y_d, & \text{with probability } \hat{p} = \frac{1-p}{2} \\ y_0, & \text{with probability } p \end{cases}$$

where $y_0 = x_0$, representing the ZIRP interest rate.

To maintain the variance consistency, we define:

$$\log\left(\frac{Y}{y_0}\right) = \begin{cases} \log\left(\frac{y_u}{y_0}\right), & \text{with probability } \hat{p} \\ \log\left(\frac{y_d}{y_0}\right), & \text{with probability } \hat{p} \\ 0, & \text{with probability } p \end{cases}$$

which leads to the following variance expression:

$$\operatorname{Var}\left(\log\frac{Y}{y_0}\right) = \frac{1-p}{2}\left(\log^2\frac{y_u}{y_0} + \log^2\frac{y_d}{y_0}\right) - \frac{(1-p)^2}{2}\log\frac{y_u}{y_0}\log\frac{y_d}{y_0}$$

Introducing the notation:

$$\ell_u = \log\frac{y_u}{y_0}, \quad \ell_d = \log\frac{y_d}{y_0}$$

The variance simplifies to

$$\beta(2)^2 = \frac{1-p^2}{4}(\ell_u^2 + \ell_d^2) - \frac{(1-p)^2}{2}\ell_u\ell_d$$

The short rate values are then calibrated using

$$y_d = r_{1,1}, \quad y_u = r_{1,2}, \quad y_0 = x_0$$

and the bond price equation

$$B_{0,1} = \frac{1}{(1+y(2))^2} = \frac{1}{1+r_{0,1}}\left[\frac{1-p}{2}(B_u + B_d) + pB_0\right]$$

where

$$B_u = \frac{100}{1+y_u}, \quad B_d = \frac{100}{1+y_d}, \quad B_0 = \frac{100}{1+y_0}$$

We obviously set the following.

$$B_{2,j} = 100, \quad j = 1,2,3$$

After the second time step, the calibration is based on the following calibration formulas:

$$B_{0,1} = \frac{1}{(1+y(n))^n} = \frac{1}{1+r_{0,1}}\left(\frac{1-p}{2}B_u + \frac{1-p}{2}B_d + pB_0\right),$$

$$B_{n,j} = 100, \quad j = 0, \ldots, (n+1),$$

$$B_{i,j} = \frac{1}{2(1+r_{i,j})}(B_{i+1,j+1} + B_{i+1,j}), \quad i = 1, \ldots, (n-1), \quad j = 2, \ldots, i+1,$$

$$B_{i,1} = \frac{1}{1+r_{i,j}} \left(\frac{1-p}{2} B_{i+1,2} + \frac{1-p}{2} B_{i+1,1} + pB_{i+1,0} \right), \quad i = 1, \ldots, (n-1),$$

$$B_{i,0} = \frac{1}{1+x_0} \left(qB_{i+1,1} + (1-q)B_{i+1,0} \right), \quad i = 1, \ldots, (n-1).$$

$$B_u = \frac{100}{(1+y_u)^{n-1}}, \quad B_d = \frac{100}{(1+y_d)^{n-1}}, \quad B_0 = \frac{100}{(1+y_0)^{n-1}},$$

$$\ell_u = \log \frac{y_u}{y_0}, \quad \ell_d = \log \frac{y_d}{y_0},$$

$$\beta(n)^2 = \frac{1-p^2}{4}(\ell_u^2 + \ell_d^2) - \frac{(1-p)^2}{2}\ell_u \ell_d,$$

$$\sigma(n) = \frac{1}{2} \log \frac{r_{n-1,j+1}}{r_{n-1,j}}, \quad j = 2, \ldots, n,$$

$$\ell_1 = \log \frac{r_{n-1,1}}{x_0}, \quad \ell_2 = \log \frac{r_{n-1,2}}{x_0},$$

$$\sigma(n)^2 = \frac{1-p^2}{4}(\ell_1^2 + \ell_2^2) - \frac{(1-p)^2}{2}\ell_1 \ell_2.$$

Using these equations, we implemented a Python class to calibrate the model and build the short-rate tree, as well as the associated ZCB Price Tree.

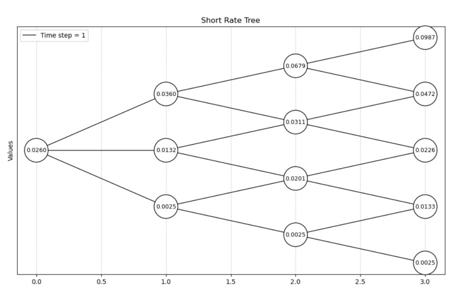

Figure 3: Calibrated Short-Rate tree

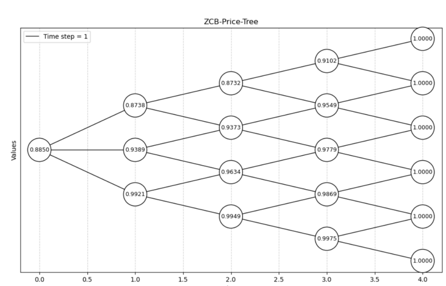

Figure 4: ZCB Price Tree

2.2.2 Pricing results from the literature

The study concludes with an application of the extended ZBDT model to option pricing. The researchers used data from the Federal Reserve Board of the United States, covering the period from August 6, 2002, to April 28, 2017. These data are used to calibrate the model and compute bond prices at each step, following the approach described above.

With these bond prices, Black's formula is applied to price vanilla call options on bonds with a face value of $100. The strike prices range from $80 to $100, the bond matures in five years, and the call option has an exercise time of two years. The key parameters of the model are set as follows: $x_0 = 0.25\%$, $p = 0.02$, and $q = 0.07$.

The analysis is carried out under six different market conditions: a normal yield curve, a flat yield curve, the beginning of the financial crisis, the US financial crisis, the European crisis, and the recovery from the US crisis. Across all scenarios, the results indicate that the ZBDT model effectively prices options with higher strike prices, particularly those close to the full value of the bond.

3 The problem with the pricing of Euro-Dollar future contracts

3.1 The Euro-Dollar future contract

Before introducing the pricing problem, we dedicate this paragraph to a brief recall of the way a Euro-Dollar future contract works. This contract allows banks, hedge funds and firms to hedge variations in interest rates with a derivative on the 3-month LIBOR rate r_L. More specifically, the contract has a fixed maturity T and its value is linked to the market expectation of the LIBOR rate at maturity. In the formula below, we set $\delta = 0.25$ since the contract is always related to the three-month LIBOR rate, which is indicated with the notation $r_L(T) = L(T, T + \delta)$, and $\mathbb{E}_t^{\mathbb{Q}}$ is the risk-neutral $\mathcal{F}_t-$conditional expectation.

$$F(t,T) = 100 - \mathbb{E}_t^{\mathbb{Q}}[\delta r_L(T)]$$

In analogy with the relation between bond and rates, the future's price grows as the future value of the LIBOR rate decreases. It is important to note that the buyer of such a contract does not pay anything at the time the contract is stipulated. Instead, the buyer agrees to pay or receive the difference between the agreed contract price and the actual future value of the LIBOR rate at the time of settlement. In fact, in a future contract, settlement occurs daily through the *marking-to-market* process. Each day, the market value of the contract is calculated again and the positions are adjusted by transferring funds between the counterparties based on the price change from the previous day.

That being said, it is easy to understand that the pricing of a Eurodollar future contract requires a stable short-rate model, and in the next paragraph we will see how log-normal models are not suited for this task.

3.2 Pricing of Eurodollar future contracts with continuously compounded log-normal rates

In this chapter, we demonstrate how log-normal short-rate models exhibit instability in expected accumulation factors and can lead to negative infinite prices for Eurodollar futures contracts.

Firstly, Hogan and Weintraub (1993) proved that BDT and BK model are such that for any $0 \le \tau < t < T = t + \delta$

$$\mathbb{E}_\tau[B(t,T)^{-1}] = \infty \tag{4}$$

As a result, we can apply Jensen's inequality using the function $f(x) = \frac{1}{x}$ to get an inequality involving the accumulation factors $\beta(t,T) = e^{\int_t^T r_s ds}$.

$$B(t,T)^{-1} = \mathbb{E}_t[e^{-\int_t^T r_s ds}]^{-1} \le \mathbb{E}_t[e^{\int_t^T r_s ds}] = \mathbb{E}_t[\beta(t,T)]$$

It is now possible to combine 4, the monotonicity of conditional expectation and the tower property to prove the explosive behaviour of the expected accumulation factors.

$$\text{For any } \tau < t, \ \mathbb{E}_\tau[\beta(t,T)] \geq \mathbb{E}_\tau[B(t,T)^{-1}] = \infty$$

This proves that the model is ill defined, since explosive behaviour of expected accumulation factors in log-normal short-rate models leads to zero bond prices, unusable investments, and infinite arbitrage opportunities. A direct consequence of this is the incorrect pricing of the Eurodollar future contract. As mentioned before, the price of this contract is

$$F_\tau(t,T) = 100 - \mathbb{E}_t^{\mathbb{Q}}[\delta r_L(T)]$$

By definition of percentage LIBOR rate, we know that

$$B(t,T)^{-1} = 1 + \frac{\delta r_L(T)}{100}$$

We can combine the two equations to find an expression of the price in terms of $B(t,T)^{-1}$.

$$F_\tau(t,T) = 100(2 - \mathbb{E}_\tau[B(t,T)^{-1}]) = -\infty$$

This result proves the inadequacy of log-normal continuously compounded short-rate models, in particular the Black-Derman-Toy model. It is in fact impossible for such a contract to have an infinite and negative price, since it would lead to inevitable arbitrage opportunities. A solution to this was provided by Sandmann and Sondermann (1993), and it consisted in modelling short rates with a finite accrual period instead of instantaneous short rates.

3.3 The correction by Sandmann and Sondermann

The main idea of Sandmann and Sondermann is to assume log-normal dynamics for the effective rates $r_e(t)$ instead of the istantaneous rates $r_c(t)$. The two rates are related through the following formula.

$$r_c(t) = ln(1 + r_e(t)) \iff e^{r_c(t)} = 1 + r_e(t)$$

The difference between the continuously compounded and effective interest rate is best explained in terms of the cash flows at the end of one period. The left-hand side is equal to the one-period payoff at the rate $r_c(t)$ continuously compounded while the right-hand side assumes a single payment at the interest rate $r_e(t)$ at the end of the period.

The advantage of deriving $r_c(t)$ from the dynamics of $r_e(t)$ is that $r_c(t)$ can exhibit both normal and log-normal behavior, depending on its magnitude. The following theorem shows the result on the dynamics of $r_c(t)$ if we assume $r_e(t)$ to be log-normal.

Theorem 3.1 (Dynamics of $\mathbf{r_c(t)}$) *Let $r_e(t)$ be such that*

$$\frac{dr_e(t)}{r_e(t)} = \mu(t)dt + \sigma(t)dW_t \tag{5}$$

Then the continuously compounded rate $r_c(t)$ follows the following diffusion process:

$$\frac{dr_c(t)}{r_c(t)} = \left\{1 - e^{-r_c(t)}\right\}\left\{\left(\mu(t) - \frac{1}{2}(1 - e^{-r_c(t)})\sigma^2(t)\right)dt + \sigma(t)dW_t\right\}$$

Proof. The connection between the continuously compounded rate $r_c(t)$ and the effective rate $r_e(t)$ is $r_c(t) = \ln x(t)$ with $x(t) = 1 + r_e(t)$. Hence, the dynamics 5 and Ito formula imply

$$dr_c = \frac{1}{x}dx - \frac{1}{2}\frac{1}{x^2}d\langle x\rangle = \frac{1}{1+r_e}(\mu r_e dt + \sigma r_e dW_t) - \frac{1}{2}\frac{1}{(1+r_e)^2}\sigma^2 r_e^2 dt.$$

Using the relation $\frac{r_e}{1+r_e} = 1 - e^{-r_c}$ yields 3.1. $\square$

The main consequence of this is that

(i) For $r_c(t) \to \infty$, the dynamics of $r_c(t)$ converges to the normal diffusion:

$$dr_c(t) = \left(\mu(t) - \frac{1}{2}\sigma^2(t)\right)dt + \sigma(t)dW_t$$

(ii) For $r_c(t) = o(dt)$, since $1 - e^{-r_c(t)} = r_c(t) + o(dt^2)$ and $r_c^2(t) = o(dt^2)$, the dynamics of $r_c(t)$ becomes the same as $r_e(t)$, i.e., a log-normal dynamics:

$$\frac{dr_e(t)}{r_e(t)} = \mu(t)dt + \sigma(t)dW_t$$

This suggests that the corresponding accumulation factor will not exhibit double-exponential growth, as $r_c(t)$ follows a Gaussian distribution for large values. At the same time, $r_c(t)$ retains all the desirable properties of the log-normal distribution discussed in Chapter 2 for small values, particularly its non-negativity. Moreover, in the rest of the chapter we will prove that such dynamics allows stable pricing of the Eurodollar future contracts.

Theorem 3.2 (Stability of log-normal short-rate models) *Let $r_e(t)$ follow the dynamics in 5, where $\mu(t)$ and $\sigma^2(t)$ are integrable and bounded functions on [0, T]. Then, for any $0 \le \tau < t < T$,*

$$\mathbb{E}_\tau[B(t,T)^{-1}] \quad \textit{and} \quad \mathbb{E}_\tau[\beta(t,T)]$$

are finite.

Proof. Since by Jensen's inequality

$$\mathbb{E}_\tau[B(t,T)^{-1}] < \mathbb{E}_\tau[\beta(t,T)]$$

It is sufficient to show that $\mathbb{E}_\tau[\beta(t,T)]$ is finite. Jensen's inequality and the relation $r_c(t) = ln(1 + r_e(t))$ imply that

$$\mathbb{E}_\tau[\beta(t,T)] = \mathbb{E}_\tau[e^{\int_t^T ln(1+r_e(s))ds}] = \mathbb{E}_\tau[e^{\int_t^T \frac{1}{T-t} ln(1+r_e(s))^{T-t} ds}] \leq$$

$$\leq \mathbb{E}_\tau[e^{ln(\int_t^T \frac{1}{T-t}(1+r_e(s))^{T-t} ds)}] = \frac{1}{T-t}\int_t^T \mathbb{E}_\tau[(1+r_e(s))^{T-t}]ds \leq$$

$$\leq \mathbb{E}_\tau[(1+r_e(s))^k]ds \quad \text{for} \quad k = min\{i \in \mathbb{N} \mid i \geq T-t\}$$

The initial assumptions on $\mu(t)$ and $\sigma^2(t)$ imply that this last term is finite.

$$\int_t^T e^{\{\int_\tau^s (\mu(u)+\frac{i}{2}\sigma^2(u))du\}^i} ds < \infty$$

□

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