Examples of Covariance Rings

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1 Introduction

A bicategory is a 'weakened' 2-category. It can be obtained from a category by replacing the sets of arrows with categories of arrows and weakening the composition law of arrows so that it is only associative and unital up to natural isomorphisms. Thus, any 2-category is a bicategory and any category is a bicategory by taking only the identity 2-arrows. A 'functor' between bicategories is called a morphism. The concept of bicategory was introduced by J. Bénabou in [2]. He also introduced in [2] the bicategory $\mathfrak{Rings}$ of rings and bimodules. $\mathfrak{Rings}$ has rings with identity as objects, bimodules as arrows, and bimodule homomorphisms as 2-arrows.

View a category $\mathcal{C}$ with finitely many objects as a bicategory. For a morphism $F : \mathcal{C} \to \mathfrak{Rings}$, we can define a lax (or strong) covariant representation like for a group action. A lax covariant representation consists of a family of orthogonal idempotent elements and a transformation of morphisms. It is called a strong covariant representation if the transformation is strong. A universal ring describing the lax (or strong) covariant representations of F is called a lax (or strong) covariance ring for F. Lax and strong covariant representations and covariance rings are introduced by Ralf Meyer in [11].

This thesis is exploring covariance rings of various group actions. We show that the twisted crossed product is a lax and strong covariance ring for a twisted group action. For partial actions, strong covariance rings need not exist. We show that the crossed product is a lax covariance ring for a partial action and there exists a strong covariance ring for a partial action with an enveloping action.

In Section 2 we start by introducing the basics of bicategory theory and define the bicategory $\mathfrak{Rings}$ of rings and bimodules.

Then we see in Section 3 that a morphism $\mathcal{C} \to \mathfrak{Rings}$ from a category $\mathcal{C}$ can be described through a $\mathcal{C}^0$-graded ring $R = \bigoplus_{x\in\mathcal{C}^0} R_x$ and a $\mathcal{C}$-graded ring $S = \bigoplus_{g\in\mathcal{C}} S_g$ with a $\mathcal{C}^0$-graded, nondegenerate ring homomorphism $\lambda : R \to S|_{\mathcal{C}^0}$. This section plays an important role because a group G can be viewed as a groupoid with a single object. Then a group action defines a homomorphism $G \to \mathfrak{Rings}$ and this homomorphism can be described through a G-graded ring.

In Section 4 we finally define lax and strong covariance rings. It is known that lax covariance rings always exist, but the existence of strong covariance rings is only known for morphisms to the subbicategory $\mathfrak{Rings}_{\mathrm{fp}}$ of $\mathfrak{Rings}$ whose arrows are only those bimodules that are finitely generated and projective as right modules. The *Cohn localisation* is used here to obtain strong covariance rings. Since a Leavitt path algebra can be realized as a Cohn localisation, Leavitt path algebras appear as strong covariance rings of some homomorphisms $\mathbb{N} \to \mathfrak{Rings}_{\mathrm{fp}}$.

After Section 4, we consider various group actions. We start with twisted group actions. A twisted group action always defines a strictly unital homomorphism $G \to \mathfrak{Rings}$ and the crossed product is a strong covariance ring. Then we study partial actions. Unlike usual group actions, a partial action does not always define a morphism $G \to \mathfrak{Rings}$. It turns out that the sufficient and necessary condition to define a morphism is the associativity of its crossed product. But crossed products by partial actions are not always associative. The conditions for the associativity have been studied in [8] by R. Exel and M. Dokuchaev. We review these conditions in Section 6. If a partial action defines a strictly unital morphism, then its crossed product is a lax covariance ring. Leavitt path algebras appear again as lax covariance rings because they can be described as crossed products (see [9]). Furthermore, it has been shown in [8] that a partial action $\alpha = (\{\alpha_g\}, \{\mathcal{D}_g\})$ admits an enveloping action if and only if the ideals $\{\mathcal{D}_g : g \in G\}$ are unital algebras. Then α defines a morphism to $\mathfrak{Rings}_{\mathrm{fp}}$. Therefore, a strong covariance ring always exists for a partial action with an enveloping action. As an example, we consider the dual action of a topological partial action. The dual action always defines a strictly unital morphism to $\mathfrak{Rings}$, and its lax covariance ring is the *Steinberg algebra* of the transformation groupoid. Moreover, there is a strong covariance ring for the dual action, because it admits an enveloping action. In the last section, we show that a twisted partial action always defines a morphism and its lax covariance ring is again the crossed product.

2 Basics of bicategories

Definition 2.1. A bicategory is given by the following data:

(i) a set $\mathcal{C}^0$ of objects;

(ii) for any $x, y \in \mathcal{C}^0$, a category $\mathcal{C}(x, y)$ whose objects are arrows $x \to y$ of the bicategory and whose arrows are the 2-arrows between these arrows; the category structure provides an associative *vertical product* $\cdot$ on 2-arrows and a *unit* 2-arrow 1_f on each arrow f;

(iii) for all $x, y, z \in \mathcal{C}^0$, a bifunctor $\circ : \mathcal{C}(y, z) \times \mathcal{C}(x, y) \to \mathcal{C}(x, z)$. It contains a *product* $\circ$ on arrows and *horizontal product* $\bullet$ on 2-arrows; bifunctoriality says that $1_f \bullet 1_g = 1_{f \circ g}$ for composable arrows f and g and that $\bullet$ commutes with vertical products;

(iv) for each object $x \in \mathcal{C}^0$, a unit arrow $1_x : x \to x$;

(v) invertible natural transformations

$$l_f : 1_y \circ f \Rightarrow f \quad \text{and} \quad r_f : f \circ 1_x \Rightarrow f$$

for all arrows $f \in \mathcal{C}(x, y)$; they are called the left and right *uniters*, respectively;

(vi) invertible natural transformations

$$\mathrm{ass} : (f_1 \circ f_2) \circ f_3 \Rightarrow f_1 \circ (f_2 \circ f_3)$$

for all composable arrows f_i, $i = 1, 2, 3$; they are called the *associators*.

The naturality of the last two conditions means the following. We view $f \mapsto f$, $f \mapsto 1_y \circ f$ and $f \mapsto f \circ 1_x$ as functors $\mathcal{C}(x, y) \to \mathcal{C}(x, y)$ and $(f_1, f_2, f_3) \mapsto (f_1 \circ f_2) \circ f_3$ and $(f_1, f_2, f_3) \mapsto f_1 \circ (f_2 \circ f_3)$ as functors $\mathcal{C}(x_2, x_3) \times \mathcal{C}(x_1, x_2) \times \mathcal{C}(x_0, x_1) \to \mathcal{C}(x_0, x_3)$. The naturality of l_f and r_f says that for any 2-arrow $c : f_1 \Rightarrow f_2$ in $\mathcal{C}$ for arrows $f_1, f_2 : x \rightrightarrows y$, the following diagrams of 2-arrows commute:

$$\begin{tikzcd} 1_y \circ f_1 \arrow[r, Rightarrow, "l_{f_1}"] \arrow[d, Rightarrow, "1_y \bullet c"'] & f_1 \arrow[d, Rightarrow, "c"] \\ 1_y \circ f_2 \arrow[r, Rightarrow, "l_{f_2}"] & f_2 \end{tikzcd} \qquad \begin{tikzcd} f_1 \circ 1_x \arrow[r, Rightarrow, "r_{f_1}"] \arrow[d, Rightarrow, "c \bullet 1_x"'] & f_1 \arrow[d, Rightarrow, "c"] \\ f_2 \circ 1_y \arrow[r, Rightarrow, "r_{f_2}"] & f_2 \end{tikzcd} \tag{2.2}$$

The naturality of the associators says that if $f_1, f_1' \in \mathcal{C}(x_2, x_3)$, $f_2, f_2' \in \mathcal{C}(x_1, x_2)$ and $f_3, f_3' \in \mathcal{C}(x_0, x_1)$ and $c_j : f_j \Rightarrow f_j'$ are 2-arrows for $j = 0, 1, 2$, then the following diagram of 2-arrows commutes:

$$\begin{tikzcd} (f_1 \circ f_2) \circ f_3 \arrow[r, Rightarrow, "\mathrm{ass}_{f_1, f_2, f_3}"] \arrow[d, Rightarrow, "(c_1 \bullet c_2) \bullet c_3"'] & f_1 \circ (f_2 \circ f_3) \arrow[d, Rightarrow, "c_1 \bullet (c_2 \bullet c_3)"] \\ (f_1' \circ f_2') \circ f_3' \arrow[r, Rightarrow, "\mathrm{ass}_{f_1', f_2', f_3'}"] & f_1' \circ (f_2 \circ f_3') \end{tikzcd} \tag{2.3}$$

We also require the following diagrams of 2-arrows to commute:

$$\begin{tikzcd}
(f_1 \circ 1) \circ f_2 \arrow[r, Rightarrow, "r_{f_1} \bullet l_{f_2}"] \arrow[d, Rightarrow, "\mathrm{ass}"'] & f_1 \circ f_2 \\
f_1 \circ (1 \circ f_2) \arrow[ur, Rightarrow, "1_{f_1} \bullet l_{f_2}"'] &
\end{tikzcd} \tag{2.4}$$

for all pairs of composable arrows f_1, f_2 in $\mathcal{C}$ and

$$\begin{tikzcd}
 & (f_1 \circ f_2) \circ (f_3 \circ f_4) \arrow[dr, Rightarrow, "\mathrm{ass}"] & \\
((f_1 \circ f_2) \circ f_3) \circ f_4 \arrow[ur, Rightarrow, "\mathrm{ass}"] \arrow[d, Rightarrow, "\mathrm{ass}"'] & & f_1 \circ (f_2 \circ (f_3 \circ f_4)) \\
(f_1 \circ (f_2 \circ f_3)) \circ f_4 \arrow[dr, Rightarrow, "\mathrm{ass}"'] & & \\
 & f_1 \circ ((f_2 \circ f_3) \circ f_4) \arrow[uur, Rightarrow, "\mathrm{ass}"'] &
\end{tikzcd} \tag{2.5}$$

for all quadruples of composable arrows.

A bicategory is called *strictly unital* if all uniters l_f and r_f are identity maps, and *strict* if all uniters and associators are identities.

Example 2.6. The coherence conditions (2.4) and (2.5) are trivial if the bicategory is strict. Thus a strict bicategory is the same as a 2-category. Since any category is a 2-category by taking only the unit 2-arrows, it is also a strict bicategory.

Definition 2.7. Let $\mathcal{C}$ be a bicategory. Two arrows f, g in $\mathcal{C}$ are called *isomorphic* if there is an invertible 2-arrow $\alpha : f \Rightarrow g$. Then we write $f \simeq g$. An *equivalence* between $x, y \in \mathcal{C}^0$ is an arrow $\alpha : x \to y$ for which there is an arrow $\beta : y \to x$ such that $\beta \circ \alpha \simeq 1_x$ and $\alpha \circ \beta \simeq 1_y$. In this case, we say that they are *equivalent* and write $x \simeq y$.

Definition 2.8 ([10], Definition 2.1.15)**.** Let $\mathcal{A}$ and $\mathcal{B}$ be bicategories. Then $\mathcal{A}$ is a *subbicategory* of $\mathcal{B}$ if the following statements hold:

(i) $\mathcal{A}^0$ is a subset of $\mathcal{B}^0$.

(ii) For objects $X, Y \in \mathcal{A}$, the category $\mathcal{A}(X,Y)$ is a subcategory of $\mathcal{B}(X,Y)$.

(iii) For objects $X, Y, Z \in \mathcal{A}$, the product $\circ$ of arrows in $\mathcal{A}$ makes the following diagram

$$\begin{tikzcd}
\mathcal{A}(Y,Z) \times \mathcal{A}(X,Y) \arrow[r, "\circ"] \arrow[d] & \mathcal{A}(X,Z) \arrow[d] \\
\mathcal{B}(Y,Z) \times \mathcal{B}(X,Y) \arrow[r, "\circ"] & \mathcal{B}(X,Z)
\end{tikzcd} \tag{2.9}$$

commutative, in which the unnamed arrows are subcategory inclusions.

(iv) Every component of the associator in $\mathcal{A}$ is equal to the corresponding component of the associator in $\mathcal{B}$, and similarly for the left uniters and the right uniters.

Now we define one of the main objects of this thesis.

Definition 2.10. The bicategory $\mathfrak{Rings}$ consists of the following data:

1. rings with identity as objects;
2. A, B-bimodules as arrows $B \to A$;
3. bimodule homomorphisms as 2-arrows.

The vertical product of 2-arrows is the composition of bimodule homomorphisms. The product of arrows and the horizontal product of 2-arrows are $\otimes_R$. The uniters and the associator are the canonical isomorphisms

$$\begin{aligned} S \otimes_S Q \simeq Q, &\qquad s \otimes q \mapsto s \cdot q, \\ Q \otimes_R R \simeq Q, &\qquad q \otimes r \mapsto q \cdot r, \\ (O \otimes_T P) \otimes_S Q \simeq O \otimes_T (P \otimes_s Q), &\qquad (o \otimes p) \otimes q \mapsto o \otimes (p \otimes q). \end{aligned}$$

Definition 2.11. Let $\mathfrak{Rings}_{\mathrm{fp}}$ be the data obtained from $\mathfrak{Rings}$ by taking only those S, R-bimodules that are finitely generated and projective as right R-modules as arrows $R \to S$. Then $\mathfrak{Rings}_{\mathrm{fp}}$ is a subbicategory of $\mathfrak{Rings}$.

2.1 Morphisms, Transformations, and Modifications.

2.1.1 Morphisms and homomorphisms.

Definition 2.12. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories. A morphism $F : \mathcal{C} \to \mathcal{D}$ consists of the following data:

(i) a function $F^0 : \mathcal{C}^0 \to \mathcal{D}^0$;

(ii) functors $F : \mathcal{C}(x, y) \to \mathcal{D}(F^0(x), F^0(y))$ for all $x, y \in \mathcal{C}^0$;

(iii) natural 2-arrows $\mu_{f,g} : F(f) \circ F(g) \Rightarrow F(f \circ g)$ for all composable $f, g \in \mathcal{C}$;

(iv) 2-arrows $\lambda_x : 1_{F^0(x)} \Rightarrow F(1_x)$ for all objects $x \in \mathcal{C}^0$;

such that the following diagrams commute:

$$\begin{tikzcd}
(Ff \circ Fg) \circ Fh \arrow[r, Rightarrow, "\mu_{f,g} \bullet 1_h"] \arrow[d, Rightarrow, "\mathrm{ass}_{Ff,Fg,Fh}"', "\cong"] & F(f \circ g) \circ Fh \arrow[r, Rightarrow, "\mu_{f \circ g,h}"] & F((f \circ g) \circ h) \arrow[d, Rightarrow, "\cong"', "F(\mathrm{ass}_{f,g,h})"] \\
Ff \circ (Fg \circ Fh) \arrow[r, Rightarrow, "1_f \bullet \mu_{g,h}"] & Ff \circ F(g \circ h) \arrow[r, Rightarrow, "\mu_{f,g \circ h}"] & F(f \circ (g \circ h))
\end{tikzcd} \tag{2.13}$$

for three composable arrows f, g, h in $\mathcal{C}$ and

$$\begin{tikzcd}
Ff \circ 1_{F^0 x} \arrow[r, Rightarrow, "1_{Ff} \bullet \lambda_x"] \arrow[d, Rightarrow, "r_{Ff}"', "\cong"] & Ff \circ F(1_x) \arrow[d, Rightarrow, "\mu_{f,1_x}"] \\
Ff & F(f \circ 1_x) \arrow[l, Rightarrow, "F(r_f)"', "\cong"]
\end{tikzcd}
\qquad
\begin{tikzcd}
1_{F^0 y} \circ Ff \arrow[r, Rightarrow, "\lambda_y \bullet 1_{Ff}"] \arrow[d, Rightarrow, "l_{Ff}"', "\cong"] & F(1_y) \circ Ff \arrow[d, Rightarrow, "\mu_{1_y,f}"] \\
Ff & F(1_y \circ f) \arrow[l, Rightarrow, "F(l_f)"', "\cong"]
\end{tikzcd} \tag{2.14}$$

for an arrow $f : x \to y$ in $\mathcal{C}$.

A morphism is a *homomorphism* if the 2-arrows $\mu_{f,g}$ and λ_x are invertible. It is a *strict homomorphism* if the 2-arrows $\mu_{g,h}$ and λ_x are identities. It is *strictly unital* if the 2-arrows λ_x are identities.

Example 2.15. We will see that twisted group actions on unital algebras define strictly unital homomorphisms $G \to \mathfrak{Rings}$. But partial actions do not always define morphisms $G \to \mathfrak{Rings}$, although twisted partial actions always do.
Note that a partial action is not always a special case of a twisted partial action. Twisted partial actions are defined by R. Exel in [7]. His definition of twisted partial action requires a 2-cocycle σ and the ideals to be idempotent. In general, the ideals need not be idempotent and the trivial 2-cocycle need not exist, because the ideals need not be unital.

2.1.2 Transformations and modifications.

Definition 2.16. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories and $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ morphisms. A *transformation* $F \Rightarrow G$ consists of the following data:

(i) arrows $\sigma_x : F^0(x) \to G^0(x)$ for all $x \in \mathcal{C}^0$;

(ii) natural 2-arrow $\sigma_f : Gf \circ \sigma_x \Rightarrow \sigma_y \circ Ff$ for all arrows $f : x \to y$ in $\mathcal{C}$ such that, for $f : x \to y$, $g : y \to z$ and all $x \in \mathcal{C}^0$ the following diagrams commute:

$$\begin{tikzcd}
(Gg \circ Gf) \circ \sigma_x \arrow[r, Rightarrow, "\text{ass}", "\cong"'] \arrow[d, Rightarrow, "\mu^G_{g,f} \bullet 1"'] & Gg \circ (Gf \circ \sigma_x) \arrow[r, Rightarrow, "1_{Gg} \bullet \sigma_f"] & Gg \circ (\sigma_y \circ Ff) \arrow[r, Rightarrow, "\text{ass}^{-1}", "\cong"'] & (Gg \circ \sigma_y) \circ Ff \arrow[d, Rightarrow, "\sigma_g \bullet 1_{Ff}"] \\
G(g \circ f) \circ \sigma_x \arrow[r, Rightarrow, "\sigma_{g \circ f}"] & \sigma_x \circ F(g \circ f) & \sigma_x \circ (Fg \circ Ff) \arrow[l, Rightarrow, "1_x \bullet \mu^F_{g,f}"'] & (\sigma_x \circ Fg) \circ Ff \arrow[l, Rightarrow, "\text{ass}"', "\cong"]
\end{tikzcd} \tag{2.17}$$

$$\begin{tikzcd}
1_{G^0 x} \circ \sigma_x \arrow[r, Rightarrow, "l_{\sigma_x}", "\cong"'] \arrow[d, Rightarrow, "\lambda_x \bullet 1"'] & \sigma_x \arrow[r, Rightarrow, "r^{-1}_{\sigma_x}", "\cong"'] & \sigma_x \circ 1_{F^0 x} \arrow[d, Rightarrow, "1 \bullet \lambda_x"] \\
G(1_x) \circ \sigma_x \arrow[rr, Rightarrow, "\sigma_{1_x}"] & & \sigma_x \circ F(1_x)
\end{tikzcd} \tag{2.18}$$

A transformation is called *strong* if all σ_f are invertible, and *strict* if they are all identities.

The following proposition says that if all arrows of $\mathcal{C}$ are equivalences, then all transformations between homomorphisms are strong. This will be used in Section 4.

Proposition 2.19 ([11], Proposition 3.3.9)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories. Assume that any arrow in $\mathcal{C}$ is an equivalence. Let $F, G : \mathcal{C} \rightrightarrows \mathcal{D}$ be homomorphisms. Then any transformation $F \Rightarrow G$ is strong.*

Proof. Step 1: σ_{1_x} is invertible.
Since F and G are homomorphisms, all arrows in the following diagram except σ_{1_x} are already known to be invertible. This implies the claim.

$$\begin{tikzcd}
1_{G^0 x} \circ \sigma_x \arrow[r, Rightarrow, "l_{\sigma_x}", "\cong"'] \arrow[d, Rightarrow, "\lambda_x \bullet 1"', "\cong"] & \sigma_x \arrow[r, Rightarrow, "r^{-1}_{\sigma_x}", "\cong"'] & \sigma_x \circ 1_{F^0 x} \arrow[d, Rightarrow, "1 \bullet \lambda_x", "\cong"'] \\
G(1_x) \circ \sigma_x \arrow[rr, Rightarrow, "\sigma_{1_x}"] & & \sigma_x \circ F(1_x)
\end{tikzcd}$$

Step 2: $\sigma_{g \circ f}$ is invertible.
Since f is an equivalence, there is a $g : y \to x$ and an invertible $\alpha : g \circ f \Rightarrow 1_x$. Then $F(\alpha)$ and $G(\alpha)$ are invertible. So are $1_x \bullet F(\alpha)$ and $G(\alpha) \bullet 1_x$.

Since $f \mapsto \sigma_f$ is natural, the following diagram commutes:

$$\begin{tikzcd}
G(g \circ f) \circ \sigma_x \arrow[r, Rightarrow, "\sigma_{g \circ f}"] \arrow[d, Rightarrow, "G(\alpha) \bullet 1_x"', "\cong"] & \sigma_x \circ F(g \circ f) \arrow[d, Rightarrow, "1_x \bullet F(\alpha)", "\cong"'] \\
G(1_x) \circ \sigma_x \arrow[r, Rightarrow, "\sigma_{1_x}", "\cong"'] & \sigma_x \circ F(1_x)
\end{tikzcd}$$

Since all arrows in this diagram except $\sigma_{g \circ f}$ are known to be invertible, $\sigma_{g \circ f}$ is invertible.

Step 3: $1_{Gg} \bullet \sigma_f$ is left invertible.
Consider the diagram (2.17) from the definition of a transformation. Note that the associators are invertible. Since F and G are homomorphisms, $\mu_{g,f}$ is invertible. By Step 2, $\sigma_{g \circ f}$ is invertible. Then we obtain that $1_{Gg} \bullet \sigma_f$ is left invertible.
Notice that the following holds:

$$1_{Gf \circ Gg} \bullet \sigma_f = (1_{Gf} \bullet 1_{Gg}) \bullet \sigma_f = \text{ass}^{-1} \circ (1_{Gf} \bullet (1_{Gg} \bullet \sigma_f)) \circ \text{ass}$$

If $1_{Gg} \bullet \sigma_f$ is left invertible, $1_{Gf \circ Gg} \bullet \sigma_f$ is left invertible.

Step 4: $1_{G^0(y)} \bullet \sigma_f$ is left invertible.

Since G is a homomorphism, there is an invertible 2-arrow

$$\Gamma : Gf \circ Gg \xRightarrow[\cong]{\mu^G_{f,g}} G(f \circ g) \xRightarrow[\cong]{G(\alpha)} G(1_y) \xRightarrow[\cong]{(\lambda^G_y)^{-1}} 1_{G^0(y)}.$$

Then we obtain the claim by the following diagram:

$$\begin{tikzcd}
1_{G^0(y)} \circ (Gf \circ \sigma_x) \arrow[r, Rightarrow, "1_{G^0(y)} \bullet \sigma_f", "\cong"'] & 1_{G^0(y)} \circ (\sigma_y \circ Ff) \\
(Gf \circ Gg) \circ (Gf \circ \sigma_f) \arrow[u, Rightarrow, "\Gamma \bullet 1", "\cong"'] \arrow[r, Rightarrow, "1_{Gf \circ Gg} \bullet \sigma_f"'] & (Gf \circ Gg) \circ (\sigma_y \circ Ff) \arrow[u, Rightarrow, "\cong", "\Gamma \bullet 1"']
\end{tikzcd}$$

Then σ_f is left invertible by the naturality of the left uniters of $\mathcal{D}$ in (2.14). A similar argument shows that σ_f is right invertible. So σ_f is invertible. $\square$

Definition 2.20. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories, let $F, G : \mathcal{C} \to \mathcal{D}$ be morphisms and let $\sigma, \sigma' : F \Rightarrow G$ be transformations. A *modification* $\Gamma : \sigma \Rrightarrow \sigma'$ is a collection of 2-arrows $\Gamma_x : \sigma_x \Rightarrow \sigma'_x$ for all $x \in \mathcal{C}^0$ making the following diagrams commute for all arrows $f : x \to y$ in $\mathcal{C}$:

$$\begin{tikzcd}
Gf \circ \sigma_x \arrow[r, Rightarrow, "1 \bullet \Gamma_x"] \arrow[d, Rightarrow, "\sigma_f"] & Gf \circ \sigma'_x \arrow[d, Rightarrow, "\sigma'_f"] \\
\sigma_y \circ Ff \arrow[r, Rightarrow, "\Gamma_y \bullet 1"] & \sigma'_y \circ Ff
\end{tikzcd}$$

3 Weakened dynamical systems on rings

In this section, we describe a morphism $\mathcal{C} \to \mathfrak{Rings}$ from a category $\mathcal{C}$ through graded rings. Since group actions can be generalized to such morphisms, Proposition 3.2 plays an important role in this thesis.

Definition 3.1. Let $\mathcal{C}$ be a category. A *$\mathcal{C}$-graded ring* is a possibly non-unital ring S with a direct sum decomposition $S = \bigoplus_{\gamma \in \mathcal{C}} S_\gamma$ as an abelian group, such that $S_\gamma \cdot S_\eta \subseteq S_{\gamma \cdot \eta}$ if γ and η are composable, $S_\gamma \cdot S_\eta = 0$ if $\gamma, \eta \in \mathcal{C}$ are not composable, and there are elements $1_x \in S_x := S_{1_x}$ for all $x \in \mathcal{C}^0$ with $1_y \cdot a = a = a \cdot 1_x$ for all $\gamma \in \mathcal{C}(x, y)$ and $a \in S_\gamma$.

A $\mathcal{C}$-graded ring S restricts to a $\mathcal{C}^0$-graded ring by $S|_{\mathcal{C}^0} := \bigoplus_{x \in \mathcal{C}^0} S_x$.

Proposition 3.2 ([11], Proposition 3.4.3)**.** *Let $\mathcal{C}$ be a category and let $\mathcal{C}^0$ be its set of objects. Now we view $\mathcal{C}$ as a 2-category with only unit 2-arrows. A morphism $F : \mathcal{C} \to \mathfrak{Rings}$ is "equivalent" to a $\mathcal{C}^0$-graded ring $R = \bigoplus_{x \in \mathcal{C}^0} R_x$ and a $\mathcal{C}$-graded ring $S = \bigoplus_{\gamma \in \mathcal{C}} S_\gamma$ with a $\mathcal{C}^0$-graded, nondegenerate ring homomorphism $\lambda : R \to S|_{\mathcal{C}^0}$, that is, $\lambda|_{R_x} : R_x \to S_x$ is a unital homomorphism for each $x \in \mathcal{C}^0$.*
The morphism is strictly unital if and only if λ is the identity map; then R and λ may be left out, making a strictly unital morphism $\mathcal{C} \to \mathfrak{Rings}$ "equivalent" to a $\mathcal{C}$-graded ring S. The data above corresponds to a homomorphism if and only if λ is an isomorphism and the multiplication maps induce isomorphisms $S_\gamma \otimes_{S_{s(\gamma)}} S_\eta \xrightarrow{\simeq} S_{\gamma\eta}$ for composable $\gamma, \eta \in \mathcal{C}$.

Proof. A morphism $\mathcal{C} \to \mathfrak{Rings}$ consists of unital rings R_x for $x \in \mathcal{C}^0$, $R_{r(\gamma)}, R_{s(\gamma)}$-bimodules S_γ for $\gamma \in \mathcal{C}$, R_x, R_x-bimodule maps $\lambda_x : R_x \to S_x$ for $x \in \mathcal{C}^0$, and $R_{r(\gamma)}, R_{s(\gamma)}$-bimodule homomorphisms $\mu_{\gamma,\eta} : S_\gamma \otimes_{R_{s(\gamma)}} S_\eta \to S_{\gamma\eta}$ for composable $\gamma, \eta \in \mathcal{C}$, such that the diagrams (2.13) and (2.14) commute.
Now we define a multiplication on the abelian group $S := \bigoplus_{\gamma \in \mathcal{C}} S_\gamma$ by

$$x \cdot y = \begin{cases} \mu_{\gamma,\eta}(x \otimes y) & \text{if } x \in S_\gamma,\ y \in S_\eta \text{ and } \gamma, \eta \text{ are composable,} \\ 0 & \text{if } x \in S_\gamma,\ y \in S_\eta \text{ and } \gamma, \eta \text{ are not composable.} \end{cases}$$

We check that S is a $\mathcal{C}$-graded ring. The above multiplication is associative if and only if the diagram (2.13) commutes. And the two diagrams in (2.14) commute if and only if the bimodule structure of S_γ is of the form $a \cdot b \cdot c = \lambda_{r(\gamma)}(a) \cdot b \cdot \lambda_{s(\gamma)}(c)$ for all $a \in R_{r(\gamma)}$, $b \in S_\gamma$, $c \in R_{s(\gamma)}$; here the product is taken in the ring S. Then $\lambda_{r(\gamma)}(1_{r(\gamma)})$ and $\lambda_{s(\gamma)}(1_{s(\gamma)})$ act like left and right units on S_γ, respectively. So $S = \bigoplus_{\gamma \in \mathcal{C}} S_\gamma$ is a $\mathcal{C}$-graded ring and λ_x for $x \in \mathcal{C}^0$ is a unital R_x-bimodule map. This is equivalent to being a nondegenerate $\mathcal{C}^0$-graded ring homomorphism $\lambda : R \to S|_{\mathcal{C}^0}$. Note that λ determines the $R_{r(\gamma)}, R_{s(\gamma)}$-bimodule structure of S_γ. Then the description of morphisms in the first paragraph of the proposition follows.

A morphism is strictly unital if and only if λ_x is an identity map for each $x \in \mathcal{C}^0$. Then $R = S|_{\mathcal{C}^0}$. So R and λ can be left out.
A morphism is a homomorphism if and only if the maps $\mu_{\gamma,\eta}$ and λ_x are isomorphisms for each composable $\gamma, \eta \in \mathcal{C}$ and $x \in \mathcal{C}^0$. If $\lambda_{s(\gamma)}$ is an isomorphism, then $S_\gamma \otimes_{S_{s(\gamma)}} S_\eta \simeq S_\gamma \otimes_{R_{s(\gamma)}} S_\eta$. The map $\mu_{\gamma,\eta}$ induces the same map $S_\gamma \otimes_{S_{s(\gamma)}} S_\eta \xrightarrow{\simeq} S_{\gamma\eta}$ as the multiplication map in S. Then the claims about homomorphisms follow. $\square$

Remark 3.3. The meaning of "*equivalent*" in the proposition is unclear because we did not introduce equivalences of bicategories. But the explanation is out of the scope of this thesis. You can find the details in Remark 3.5.7 of [11].

Remark 3.4. Proposition 3.2 says that strictly unital morphisms $\mathcal{C} \to \mathfrak{Rings}$ are equivalent to $\mathcal{C}$-graded rings. Furthermore, strictly unital homomorphisms are equivalent to strictly $\mathcal{C}$-graded rings, where the grading is called *strict* if

$$S_\gamma \otimes_{s(\gamma)} S_\eta \cong S_{\gamma\eta}$$

for all composable $\gamma, \eta \in \mathcal{C}$.

Example 3.5. Commonly, we mean by a 'graded ring' R that it is graded by $\mathbb{N}$, that is, $R = \bigoplus_{n\in\mathbb{N}} R_n$ and $R_n \cdot R_m \subseteq R_{n+m}$. View $\mathbb{N}$ as a 2-category with a single object, the natural numbers as arrows, and only unit 2-arrows. Proposition 3.2 says that strictly unital morphisms $\mathbb{N} \to \mathfrak{Rings}$ are equivalent to graded rings and homomorphisms $\mathbb{N} \to \mathfrak{Rings}$ are equivalent to strictly graded rings. For strictly graded rings $R = \bigoplus_{n\in\mathbb{N}} R_n$, we have isomorphisms $R_n \otimes_{R_0} R_m \cong R_{n+m}$ for $n, m \in \mathbb{N}$. Then the nth tensor power of R_1 is isomorphic to R_n by the n-fold multiplication map ${R_1^{\otimes_{R_0}}}^n \to R_n$. Thus, R_0 and R_1 determine the strictly graded ring $R = \bigoplus_{n\in\mathbb{N}} R_n \simeq \bigoplus_{n\in\mathbb{N}} {R_1^{\otimes_{R_0}}}^n$ uniquely up to isomorphism where $R_1^0 := R_0$. So any strictly unital homomorphism $\mathbb{N} \to \mathfrak{Rings}$ is completely determined by R_0 and R_1 and equivalent to $R = \bigoplus_{n\in\mathbb{N}} {R_1^{\otimes}}^n$. In this case, we call R the *tensor algebra* of the bimodule R_1 over R_0.

4 Lax and strong covariance rings

In this section, we introduce covariant representations and covariance rings for morphisms to $\mathfrak{Rings}$. Roughly speaking, a covariance ring is a ring that describes the covariant representations of the morphism.

Definition 4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories and let x be an object of $\mathcal{D}$. The *constant homomorphism* $\mathrm{const}_x : \mathcal{C} \to \mathcal{D}$ is the homomorphism where each object, arrow, and 2-arrow in the definition of a homomorphism are x, the identity arrow 1_x, or uniter $1_x \circ 1_x \Rightarrow 1_x$ as $\mu_{f,g}$, or the unit 2-arrow on 1_x, respectively.

Example 4.2 ([11], Example 3.6.2)**.** Let D be a ring and let $\mathcal{C}$ be a category. The homomorphism $\mathrm{const}_D : \mathcal{C} \to \mathfrak{Rings}$ is equivalent to a $\mathcal{C}$-graded ring $S = \bigoplus_{\gamma \in \mathcal{C}} S_\gamma$ by Proposition 3.2. This ring is the category ring of $\mathcal{C}$ with coefficients in D. It has $S_\gamma = D$ for all $\gamma \in \mathcal{C}$. The multiplication of S is defined as follows:

$$(u_1\delta_\gamma) \cdot (u_2\delta_\eta) = \begin{cases} (u_1 \cdot u_2)\delta_{\gamma\eta} & \text{if } s(\gamma) = r(\eta), \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4.3. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories. Let $A : \mathcal{C} \to \mathfrak{Rings}$ be a morphism and let $D \in \mathcal{C}^0$. A *lax cone over A with summit D* is a transformation $\mathrm{const}_D \Rightarrow A$. Let $\mathrm{Cone}_{\mathrm{lax}}(D, A)$ be the category that has these lax cones as objects and modifications between them as arrows. A *cone over A with summit D* is a strong transformation $\mathrm{const}_D \Rightarrow A$. The category $\mathrm{Cone}(D, A)$ is a full subcategory of $\mathrm{Cone}_{\mathrm{lax}}(D, A)$.

Dually, a *lax cone under A with nadir D* is transformation $A \Rightarrow \mathrm{const}_D$, and a *cone under A with nadir D* is a strong transformation $A \Rightarrow \mathrm{const}_D$. They are the objects of categories $\mathrm{Cone}(A, D) \subseteq \mathrm{Cone}_{\mathrm{lax}}(A, D)$.

Remark 4.4. The composition of arrows in $\mathrm{Cone}_{\mathrm{lax}}(D, A)$ is defined by the composition of modifications. It is well-defined. You can find more details in Lemma 3.5.1 of [11].

Proposition 4.5 ([11], Proposition 3.6.4)**.** *Let $\mathcal{C}$ be a category. Describe a morphism $A : \mathcal{C} \to \mathfrak{Rings}$ by a $\mathcal{C}^0$-graded ring $R = \bigoplus_{x \in \mathcal{C}^0} R_x$ and a $\mathcal{C}$-graded ring $S = \bigoplus_{\gamma \in \mathcal{C}} S_\gamma$ with a nondegenerate $\mathcal{C}^0$-graded ring homomorphism $\lambda : R \to S|_{\mathcal{C}^0}$. Let D be a ring. The category $\mathrm{Cone}_{\mathrm{lax}}(D, A)$ is equivalent to the category of nondegenerate S, D-bimodules and bimodule homomorphisms. Here an S, D-bimodule is nondegenerate if the right D-module structure is unital and $S \cdot M = M$.*

Proof. A transformation $\mathrm{const}_D \Rightarrow A$ consists of the following data:

- R_x, D bimodules-M_x for $x \in \mathcal{C}^0$,
- R_x, D-bimodule maps $\sigma_\gamma : S_\gamma \otimes_{R_x} M_x \to M_y \otimes_D D$ for $\gamma : x \to y$ in $\mathcal{C}$,

such that the diagrams (2.17) and (2.18) commute. Since $M_y \otimes_D D \cong M_y$ as an R_y, D-bimodule, we may replace σ_γ with an R_y, D-bimodule map $S_\gamma \otimes_{R_x} M_x \to M_y$. Let $M := \bigoplus_{x \in \mathcal{C}^0} M_x$. Extend the R_y, R_x-bimodule S_γ to an R-bimodule defining $R_{y'} \cdot S_\gamma = 0 = S_\gamma \cdot R_{x'}$ for $y \neq y'$ and $x \neq x'$. Then $S_\gamma \otimes_R M = S_\gamma \otimes_{R_x} M_x$, and we may turn σ_γ into an R, D-bimodule map $S_\gamma \otimes_R M \to M$ – the projection onto R_x in the multiplier ring of R is used to see that the range of such a bimodule map is contained in M_y. Now we combine all σ_γ to an R, D-bimodule map $\sigma : S \otimes_R M \to M$. Since a lax cone is a transformation, it makes the diagrams (2.17) and (2.18) commute. We can check that σ defines a left S-module structure of M by chasing the diagrams (2.17). Thus, M becomes an S, D-bimodule. It is nondegenerate by construction. The diagram (2.18) says that the left S-module structure restricted to R gives the original R-module structure on M. Thus the R-module structure on M is redundant.

Conversely, let M be a nondegenerate S, D-bimodule. Let $M_x := S_x \cdot M \subseteq M$. Then the map $\bigoplus_{x\in\mathcal{C}^0} M_x \to M$ is injective because the subrings $S_x \subseteq S$ are orthogonal. It is also surjective because

$$M = S \cdot M = \sum_{\gamma\in\mathcal{C}} S_\gamma \cdot M = \sum_{\gamma\in\mathcal{C}} S_{r(\gamma)} \cdot S_\gamma \cdot M = \sum_{x\in\mathcal{C}^0} S_x \cdot M = \sum_{x\in\mathcal{C}^0} M_x.$$

Now we reverse the above proof. The left S-module structure induces an R_x, D -bimodule structure on M_x and gives bimodule homomorphisms $\sigma_\gamma : S_\gamma \otimes_{R_x} M_x \to M_y \simeq M_y \otimes_D D$ for arrows $\gamma \in \mathcal{C}$ such that the diagrams (2.17) and (2.18) commute. The S-module structure on $M \cong \bigoplus_{x\in\mathcal{C}^0} M_x$ built by combining the maps σ_γ is the given one. Thus a nondegenerate S, D-bimodule gives a transformation and vice versa.
The rest of the proof about modifications and bimodule maps can be found in the proof of Proposition 3.6.4 of [11]. □

Remark 4.6. If $\mathcal{C}^0$ is finite, the ring S is unital. Let $p_x \in S_x \subseteq S$ be the unit element of S_x. Then the unit of S is given by $\sum_{x\in\mathcal{C}^0} p_x$. In this case, a nondegenerate S, D-bimodule is the same as a 'unital' S, D-bimodule or, in other words, an arrow $D \to S$ in the bicategory $\mathfrak{Rings}$.

Definition 4.7. Let $\mathcal{C}$ be a bicategory with finitely many objects. A *lax covariant representation* of a morphism $F : \mathcal{C} \to \mathfrak{Rings}$ on a ring D consists of a family of orthogonal idempotent elements $(p_x)_{x\in\mathcal{C}^0}$ in D with $\sum_{x\in\mathcal{C}^0} p_x = 1$ and a transformation $\mathrm{const}_D \Rightarrow F$ where $M_x = p_x \cdot D \subseteq D$ for all $x \in \mathcal{C}^0$. A *lax covariance ring* for F is a ring U such that ring homomorphisms $U \to D$ are naturally in bijection with lax covariant representations of F on D. A *strong covariant representation* is a lax covariant representation with a strong transformation $\mathrm{const}_D \Rightarrow F$. The strong covariance ring is defined like the lax one, but with strong covariant representations.

Remark 4.8. In this thesis, categories with a single object are important, since a group can be viewed as a category with a single object. In this case, the projection p_x must be the unit of D. Then a lax covariant representation is a transformation $\mathrm{const}_D \Rightarrow F$ where the underlying $F(x), D$-bimodule is just D with some left $F(x)$-module structure.

The following proposition describes lax covariance rings of morphisms from categories.

Corollary 4.9 ([11], Proposition 3.6.8)**.** *Let $\mathcal{C}$ be a category with finitely many objects. Describe a morphism $F : \mathcal{C} \to \mathfrak{Rings}$ by graded rings $R = \bigoplus_{x\in\mathcal{C}^0} R_x$ and $S = \bigoplus_{\gamma\in\mathcal{C}} S_\gamma$ with a $\mathcal{C}^0$-graded unital ring homomorphism $\lambda : R \to S|_{\mathcal{C}^0}$. Then S is a lax covariance ring for F.*

Proof. The proof of Proposition 4.5 and Remark 4.6 give us a bijection between lax covariant representations on D and S, D-bimodules. Now we check that these bijections for different rings D are natural. Let $\Sigma = (\{p_x\}, \sigma : \mathrm{const}_D \Rightarrow F)$ be a lax covariant representation for F on a ring D. Let E be a ring and let $f : D \to E$ be a ring homomorphism. f induces a left D-module structure on E by $d \cdot e = f(d)e$. Then $M'_x := p_x \cdot D \otimes_D E \simeq f(p_x) \cdot E$ is an S_x, E-bimodule. Identify the bimodule maps $\sigma_\gamma \otimes E : S_\gamma \otimes_{S_{s(\gamma)}} (p_{s(\gamma)} \cdot D) \otimes_D E \to p_{r(\gamma)} \cdot D \otimes_D E$ with $\theta_\gamma : S_\gamma \otimes M'_{s(\gamma)} \to M'_{r(\gamma)}$. Since σ is a transformation, the S_x, E-bimodules M'_x and the bimodule maps θ_γ define a transformation $\theta : \mathrm{const}_E \Rightarrow F$. So the family of orthogonal idempotent elements $f(p_x) \in E$ and the transformation θ define a lax covariant representation Θ on E. Let $M' = \bigoplus_{x \in \mathcal{C}^0} M'_x$. By the proof of Proposition 4.5, M' becomes a S, E-bimodule. □

4.1 Strong covariance rings.

Let $\mathfrak{Rings}_{\mathrm{fp}}$ be the subbicategory of $\mathfrak{Rings}$ whose arrows are finitely generated projective as right modules. The existence of strong covariance rings is only known for morphisms to $\mathfrak{Rings}_{\mathrm{fp}}$. The idea is to localise the lax covariance ring at certain bimodule 'maps' between finitely generated projective modules. This localisation is called the *Cohn localisation*.

Remark 4.10 ([11], Example 3.6.10). Recall that if all arrows of a category $\mathcal{C}$ are equivalences, then any transformations between homomorphisms $F, G : \mathcal{C} \to \mathfrak{Rings}$ are strong by Proposition 2.19. Then for a homomorphism $F : \mathcal{C} \to \mathfrak{Rings}$, all transformations $\mathrm{const}_D \Rightarrow F$ are strong. Thus the lax covariance ring for F is also a strong covariance ring.

Definition 4.11. Let R be a ring. Let $u_i : P_i \to Q_i$ for $i \in I$ be a set of right R-module maps between finitely generated projective right R-modules P_i and Q_i. The *Cohn localisation* of R at $\{u_i : i \in I\}$ is the universal ring R' with a homomorphism $R \to R'$ such that the maps

$$u_i \otimes_R \mathrm{id}_{R'} : P_i \otimes_R R' \to Q_i \otimes_R R'$$

are invertible for all $i \in I$. Here we mean by 'universal' that if D is another ring and $f : R \to D$ is a homomorphism, then f factors through R' if and only if $u_i \otimes_R \mathrm{id}_D : P_i \otimes_R D \to Q_i \otimes_R D$ is invertible for all $i \in I$, and this factorisation is unique if it exists.

The following lemma says that the Cohn localisation always exists.

Lemma 4.12. *In the above setting, the Cohn localisation of R at $\{u_i\}_{i \in I}$ exists.*

Proof. See Lemma 3.6.12 in [11]. □

Let $\mathcal{C}$ be a bicategory with finitely many objects and let $F : \mathcal{C} \to \mathfrak{Rings}_{\mathrm{fp}}$ be a morphism. Describe F through a $\mathcal{C}^0$-graded ring R and a $\mathcal{C}$-graded ring S with a $\mathcal{C}^0$-graded ring homomorphism $\lambda : R \to S|_{\mathcal{C}^0}$ by Proposition 3.2. By definition, a strong covariant representation on D is also a lax one. Thus, it gives us an S, D-bimodule M. Furthermore, the strong covariance ring also has the property that its category of left modules is contained in the category of left S-modules. Since each S_g is finitely generated and projective as a right $R_{s(g)}$-module, both $S_g \otimes_{R_{s(g)}} S$ and $S_{r(g)} \otimes_{R_{r(g)}} S$ are finitely generated and projective right S-modules. Now we identify

$$S_g \otimes_{R_{s(g)}} S \cong \bigoplus_{h \in r^{-1}(s(g))} S_g \otimes_{R_{s(g)}} S_h.$$

The multiplication maps $\mu_{g,h} : S_g \otimes_{R_{s(g)}} S_h \to S_{gh}$ give a right S-module map

$$\psi_g : S_g \otimes_{R_{s(g)}} S \to S_{r(g)} \otimes_{R_{r(g)}} S.$$

Let M be an S, D-bimodule. The left S-action on M implies a direct sum decomposition $M = \bigoplus_{x \in \mathcal{C}^0} M_x$. The map ψ_g induces a right D-module map

$$\psi_g \otimes_S \mathrm{id}_M : S_g \otimes_{R_{s(g)}} M_{s(g)} \cong S_g \otimes_{R_{s(g)}} M \to S_{r(g)} \otimes_{R_{r(g)}} M. \tag{4.13}$$

The transformation which corresponds to M by (4.5) is strong if and only if the maps ψ_g are invertible for all $g \in \mathcal{C}$.

Proposition 4.14. *Let $\mathcal{C}$ be a bicategory with finitely many objects. Let $F : \mathcal{C} \to \mathfrak{Rings}_{\mathrm{fp}}$ be a morphism. The Cohn localisation of the lax covariance ring for F at the set of homomorphisms $\{\psi_g : g \in \mathcal{C}\}$ is a strong covariance ring for F. In particular, the strong covariance ring exists.*

Proof. The argument above proves this proposition. This original argument is on page 117 and 118 of [11]. □

4.2 Leavitt path algebras as strong covariance rings

Leavitt path algebras appear as strong covariance rings of some homomorphisms $\mathbb{N} \to \mathfrak{Rings}$. The idea is that a Leavitt path algebra may be realized as a Cohn localization of an algebra.

Definition 4.15. Let $E = (E^0, E^1, r, s)$ be a directed graph. Then a vertex v is called:

1. a *sink* if $s^{-1}(v) = \emptyset$.
2. an *infinite emitter* if $|s^{-1}(v)| = \infty$.
3. *regular* if it is neither a sink nor an infinite emitter.

Definition 4.16. Let $E = (E^0, E^1, r, s)$ be a directed graph and let K be any field.
We define a set $(E^1)^*$ consisting of symbols of the form $\{e^* \mid e \in E^1\}$. The Leavitt path algebra of E with coefficients in K, denoted $L_K(E)$, is the free associative K-algebra generated by the set $E^0 \cup E^1 \cup (E^1)^*$ subject to the following relations:

(V) $vv' = \delta_{vv'} v$ for $v \in E^0$

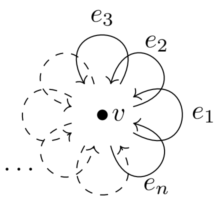
(E1) $s(e)e = er(e) = e$ for all $e \in e \in E^1$

(E2) $r(e)e^* = e^* s(e) = e^*$ for all $e \in E^1$

(CK1) $e^* e' = \delta_{e,e'} r(e)$ for all $e, e' \in E^1$ and

(CK2) $v = \sum_{\{e \in E^1 | s(e) = v\}} ee^*$ for all regular vertices $v \in E^0$.

Example 4.17. Consider the rose E_n with n petals

$E_n =$ 

Its Leavitt path algebra is the so called Leavitt algebra $L_K(1,n)$.
For $n = 1$,

$E_1 =$ 

$L_K(E_1)$ is the Laurent polynomial algebra $K[x, x^{-1}]$ over K.

Constructing a directed graph from a homomorphism $\mathbb{N} \to \mathfrak{Rings}$.

Recall that a strictly unital homomorphism $\mathbb{N} \to \mathfrak{Rings}$ is determined by a ring R_0 and an R_0-bimodule R_1. Let K be a field and let V be a finite set. Consider a homomorphism $\mathbb{N} \to \mathfrak{Rings}_{\mathrm{fp}}$, such that $R_0 = K^V$. Then R_1 is a finite-dimensional R_0-vector space. Note that $K \otimes K \simeq K$. Thus R_1 is a finite-dimensional module over $R_0 \otimes R_0 = K^{V^2}$, which is equivalent to a family of vector spaces $(R_1)_{(v,w)}$ for $v, w \in V$. For each $(v,w) \in V^2$, we choose a basis $\mathcal{B}_{(v,w)}$ of $(R_1)_{(v,w)}$. Set $E = \bigcup_{(v,w) \in V^2} \mathcal{B}_{(v,w)}$ and define two maps:

$$s, r : E \to V, \quad s(\gamma) = v\,,\; r(\gamma) = w \text{ for } \gamma \in \mathcal{B}_{(v,w)}. \tag{4.18}$$

Then (E, V, s, r) defines a directed graph Γ. By Proposition 3.2, our strictly unital homomorphism can be described by the graded ring $S = \bigoplus_{n \in \mathbb{N}} {R_1}^{\otimes_{R_0} n}$. Here the R_0-bimodule $R_n \simeq R_1^{\otimes_{R_0} n}$ as a K-vector space has the basis

$$\mathcal{P}^n(E) := \{(e_1, \ldots, e_n) \in E^n \,|\, s(e_{i+1}) = r(e_i) \text{ for } i = 1, 2, \ldots, n-1\}, \tag{4.19}$$

the set of paths of length n in Γ. The graded ring S as a K-algebra is called the *path algebra* of Γ.

The following proposition says that the Leavitt path algebra $L_K(\Gamma)$ is a Cohn localization of the path algebra S of Γ.

Proposition 4.20 ([11], Proposition 3.6.15)**.** *Let Γ be a finite directed graph with surjective source map $s : E \to V$. The Cohn localisation of S at the map $\psi : R_1 \otimes_{R_0} S \to S$ is the Leavitt path algebra of Γ.*

Proof. See Proposition 3.6.15 of [11]. □

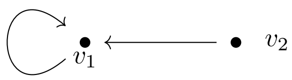
Thus, Leavitt path algebras are strong covariance rings for homomorphisms $\mathbb{N} \to \mathfrak{Rings}_{\mathrm{fp}}$.

Corollary 4.21. *Let $F : \mathbb{N} \to \mathfrak{Rings}_{\mathrm{fp}}$ be a homomorphism such that the directed graph Γ constructed as above is finite and the source map is $s : E \to V$ surjective. Then the Leavitt path algebra of Γ is a strong covariance ring for F.*

Example 4.22. Let $F_1 : \mathbb{N} \to \mathfrak{Rings}$ be the homomorphism determined by $R_0 = K^2$ and R_1 given by the following vector spaces:

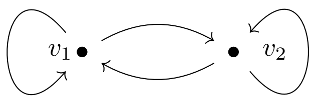
- $(R_1)_{(v_1,v_1)} = K$ and $(R_1)_{(v_2,v_1)} = K$,
- $(R_1)_{(v_1,v_2)\in V^2} = 0$ and $(R_1)_{(v_2,v_2)} = 0$.

F_1 corresponds to the following graph Γ_1:



For a field K, the Leavitt path algebra $L_K(\Gamma_1)$ is the 2×2-matrix algebra $\mathcal{M}_2(K[x, x^{-1}])$ over K-Laurent polynomials. So $\mathcal{M}_2(K[x, x^{-1}])$ is a strong covariance ring for F_1.

Example 4.23. Let $F_2 : \mathbb{N} \to \mathfrak{Rings}$ be the homomorphism determined by $R_0 = K^2$ and $(R_1)_{(v_i,v_j)} = K$ for $i, j = 1, 2$. Then F_2 corresponds to the following graph Γ_2:

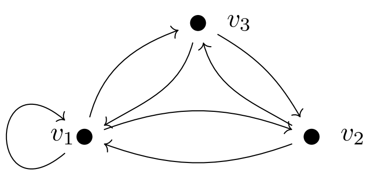


For a field K, the Leavitt path algebra $L_K(\Gamma_2)$ is isomorphic to the Leavitt algebra $L(1, 2)$. See the table on page 7 of [4].
Thus, $L(1, 2)$ is a strong covariance ring for F_2.

Example 4.24. Let $F_3 : \mathbb{N} \to \mathfrak{Rings}$ be the homomorphism determined by $R_0 = K^3$ and R_1 given by the following vector spaces:

$$(R_1)_{i,j} = \begin{cases} 0 & \text{if } i = j \text{ and } i.j = 2, 3, \\ K & \text{otherwise.} \end{cases}$$

Then F_3 corresponds to the following graph Γ_3:



For a field K, the Leavitt path algebra $L_K(\Gamma_2)$ is isomorphic to the 2×2-matrix algebra $\mathcal{M}_2(L(1, 5))$ over the Leavitt algebra $L(1, 5)$. See the page 15 of [4]. Thus, $\mathcal{M}_2(L(1, 5))$ is a strong covariance ring for F_3.

Many other examples of Leavitt path algebras can be found in [4].

5 Twisted group actions as morphisms

In this section, we show that a twisted group action defines a strictly unital homomorphism and that the strong covariance ring for a morphism induced by a twisted group action is its crossed product. As an example of a strong covariance ring, we will consider crossed products (often also called skew group rings).

Definition 5.1. Let G be a group and let R be a ring and $R^\times$ its group of units. Then a *twisted action* of G on R is defined as a pair (ρ, u) of a homomorphism $\rho : G \to \mathrm{Aut}(R)$ and a 2-cocycle $u : G \times G \to R^\times$ such that:

1. $u(f,g) \cdot u(fg,h) = \rho_f(u(g,h)) \cdot u(f,gh)$ for all $f,g,h \in G$,
2. $u(g,h) \cdot \rho_{gh}(c) = \rho_g \rho_h(c) \cdot u(g,h)$ for $g,h \in G$ and $c \in R$,
3. $\rho_1 = \mathrm{id}_R$ and $u(g,1) = 1$, $u(1,g)$ for all $g \in G$.

Note that all usual group actions are twisted with the trivial 2-cocycle. Now we generalize a twisted action to a homomorphism $F : G \to \mathfrak{Rings}$ considering G as a groupoid with a single object and elements of G as arrows.

Proposition 5.2. *Let G be a group and $A : G \to \mathfrak{Rings}$ a strictly unital homomorphism. Describe this homomorphism through a G-graded ring $S = \bigoplus_{g \in G} A_g$. This homomorphism comes from a twisted action if and only if $A_g \simeq A_1$ as a left module for all $g \in G$.*

Proof. If $A_g \simeq A_1$ as a left module, the right module structure of A_g comes from a map $\rho_g : A_1 \to A_1$, that is, $x \cdot y = x\rho_g(y)$. Set $u(g,h) := \mu_{g,h}(1 \otimes 1)$. Then the multiplication map $\mu_{g,h}$ is given by

$$\mu_{g,h}(x \otimes y) = \mu_{g,h}(x \cdot y \otimes 1) = (x \cdot y)\mu_{g,h}(1 \otimes 1) = x\rho_g(y)u(g,h).$$

Since $\mathcal{A}$ is a homomorphism, $\mu_{g,h}$ is invertible, which implies that $u(g,h)$ is also invertible. Taking $h = 1$ gives that ρ_g is an automorphism. It remains to show the conditions in Definition 5.1 .
By the definition of morphism, we get the following diagram:

$$\begin{tikzcd}
(A_f \otimes_{A_1} A_g) \otimes_{A_1} A_h \arrow[r, Rightarrow, "\mu_{f,g} \bullet 1_{A_h}"] \arrow[d, Rightarrow, "\mathrm{ass}"'] & A_{fg} \otimes_{A_1} A_h \arrow[r, Rightarrow, "\mu_{fg,h}"] & A_{fgh} \arrow[d, equal] \\
A_f \otimes_{A_1} (A_g \otimes_{A_1} A_h) \arrow[r, Rightarrow, "1_{A_f} \bullet \mu_{g,h}"] & Af \otimes_{A_1} A_{gh} \arrow[r, Rightarrow, "\mu_{f,gh}"] & A_{fgh}
\end{tikzcd}$$

Evaluating on $x \otimes y \otimes z$ gives us the following equation:

$$x\rho_f(y)u(f,g)\rho_{fg}(z)u(fg,h) = x\rho_f(y\rho_g(z)u(g,h))u(f,gh)$$

Since x, y and z are arbitrary,

- $x = 1$, $y = 1$, $z = 1$ gives

$$u(f,g)u(fg,h) = \rho_f(u(g,h))u(f,gh)$$

- $x = 1$, $y = 1$, $h = 1$ gives

$$u(f,g) \cdot \rho_{fg}(z) = \rho_f \rho_g(z) \cdot u(f,g).$$

Conversely, let (ρ, u) be a twisted action. We get a strictly unital morphism $A : G \to \mathfrak{Rings}$ by defining $\mathcal{A}$ as above. By construction, we get that $A_g \simeq A_1$ as left modules. □

Corollary 5.3. *Let $A : G \to \mathfrak{Rings}$ be a strictly unital homomorphism such that $A_1 \simeq A_g$ for all $g \in G$ as left modules. Then the twisted crossed product $A_1 \ltimes_{\alpha,u} G$ is a strong covariance ring.*

Proof. By the above proposition, there is a twisted action (α, u) of G on A_1 that the homomorphism A is coming from. Describe A through the crossed product $A_1 \rtimes_{\alpha,u} G$ by Proposition 3.2. Then the crossed product $A_1 \rtimes_{\alpha,u} G$ is a lax covariance ring for A by Proposition 4.9. Since all arrows of G are equivalences, all lax covariant representations are strong by Proposition 2.19. Therefore, $A_1 \ltimes_{\alpha,u} G$ is the strong covariance ring. □

5.1 Crossed products as strong covariance rings

By the following proposition, group rings of semidirect products are crossed products. Since semidirect products appear not only in group theory but also in physics, it gives us many interesting examples of strong covariance rings.

Proposition 5.4. *Let N and H be groups, K a ring, and $\varphi : H \to \mathrm{Aut}(N)$ a homomorphism. It induces a homomorphism $\eta : H \to \mathrm{Aut}(K[N])$. Then there is an isomorphism $K[N] \rtimes_\eta H \to K[N \rtimes_\varphi H]$.*

Proof. Define a map $\psi : K[N] \rtimes_\eta H \to K[N \rtimes_\varphi H]$ by $\psi((rn)h) = r(n,h)$ for $r \in K$. It is a bijective K-linear map. Note that $(sn)h \cdot (tm)h' = [(st)\{n\varphi_h(m)\}] \cdot hh'$. Therefore,

$$\psi((sn)h \cdot (tm)h') = \psi([(st)\{n\varphi_h(m)\}]) \cdot hh') = (st)(n\varphi_h(m), hh') = \psi((sn)h) \cdot \psi((tm)h')$$

So it is an algebra isomorphism. □

Example 5.5 (Dihedral group D_n)**.** Consider the inversion action φ of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}/n\mathbb{Z}$. It induces an action η on $K[\mathbb{Z}/n\mathbb{Z}]$ for a ring K with 1. Since $D_n \simeq \mathbb{Z}/n\mathbb{Z} \rtimes_\varphi \mathbb{Z}/2\mathbb{Z}$, $K[D_n]$ is a strong covariance ring for η.

Example 5.6 (Euclidean group $E(n)$)**.** Let $E(n)$ be the isometry group of $\mathbb{R}^n$. It is a semidirect product of the canonical action $\varphi : O(n) \to \mathrm{Aut}(\mathbb{R}^n)$. As we have seen above, φ induces an action η on $K[\mathbb{R}^n]$. Therefore, $K[\mathbb{R}^n] \rtimes_\eta O(n) \simeq K[E(n)]$ is a strong covariance ring for η.

6 Partial actions as morphisms

The concept of partial actions on algebras was introduced by R. Exel. Unlike usual group actions, partial actions do not always behave well. A partial action defines a morphism to $\mathfrak{Rings}$ if and only if its crossed product is associative. In [8], the authors provide an example of a non-associative crossed product and study the conditions for associativity of crossed products. The crossed product of a partial action $(\{\alpha_g\}, \{\mathcal{D}_g\})$ is associative if each $\mathcal{D}_g$ is non-degenerate or idempotent. The definition of a partial action on a C^*-algebra requires that each $\mathcal{D}_g$ is closed, and it is known that each closed ideal in a C^*-algebra is an idempotent ideal. Thus, all crossed products of partial actions on C^*-algebras are associative. In this section, we review the associativity question and explore covariance rings of partial actions.

Definition 6.1. A *partial action* of a group G on a K-algebra $\mathcal{A}$ is a family $\alpha = (\{\alpha_g\}, \{\mathcal{D}_g\})_{g\in G}$, where $\mathcal{D}_g$ is a two-sided ideal of $\mathcal{A}$ and $\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g$ is an isomorphism of algebras such that:

(i) $\mathcal{D}_1 = \mathcal{A}$ and α_1 is the identity map of $\mathcal{A}$

(ii) $\alpha_g(\mathcal{D}_{g^{-1}} \cap \mathcal{D}_h) = \mathcal{D}_g \cap \mathcal{D}_{gh}$

(iii) $\alpha_g(\alpha_h(x)) = \alpha_g\alpha_h(x)$ for $x \in \mathcal{D}_{h^{-1}} \cap \mathcal{D}_{(gh)^{-1}}$

Definition 6.2. A *topological partial action* on a topological space X can be defined by setting all $\mathcal{D}_g$ as open sets and $\alpha_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g$ homeomorphisms.

Definition 6.3. Let $\alpha = (\{\alpha_g\}, \{\mathcal{D}_g\})$ be a partial action of G. The *crossed product* (also often called *partial skew group ring*) $\mathcal{A} \rtimes_\alpha G$ is the set of all finite formal sums $\{\sum\limits_{g\in G} a_g\delta_g : a_g \in \mathcal{D}_g\}$, where δ_g are symbols. The addition is defined in the obvious way, and the multiplication is determined by

$$(a_g\delta_g) \cdot (b_h\delta_h) = \alpha_g(\alpha_{g^{-1}}(a_g)b_h)\delta_{gh}. \tag{6.4}$$

Proposition 6.5. *Let $(\{\alpha_g\}, \{\mathcal{D}_g\})$ be a partial action of G on a unital algebra $\mathcal{A}$. Then α defines a strictly unital morphism $G \to \mathfrak{Rings}$ if and only if the crossed product $\mathcal{A} \rtimes_\alpha G$ is associative.*

Proof. Note that a G-graded ring gives us a strictly unital morphism $A : G \to \mathfrak{Rings}$ by Proposition 3.2. The crossed product $\mathcal{A} \rtimes_\alpha G$ is associative if and only if it is a G-graded ring. So the claim follows. Explicitly, we can define a strictly unital morphism as follows:

A partial action $(\{\alpha_g\}, \{\mathcal{D}_g\})$ gives us the following data:

- an algebra $\mathcal{A}$
- the ideals $\mathcal{D}_g$ as bimodules over $\mathcal{A}$ with left module structure by multiplication, and the right module structure given by

$$a \cdot b = \alpha_g(\alpha_{g^{-1}}(a)b);$$

- multiplication maps

$$\mu_{g,h} : \mathcal{D}_g \otimes \mathcal{D}_h \to \mathcal{D}_{gh},$$
$$\mu_{g,h}(a \otimes b) = \alpha_g(\alpha_{g^{-1}}(a)b).$$

This satisfies the diagram condition of a morphism if and only if $\mathcal{A} \rtimes_\alpha G$ is associative. □

However, the associativity of $\mathcal{A} \rtimes_\alpha G$ is not a trivial question. The following example shows that crossed products are not always associative.

Example 6.6 ([8] Example 3.5)**.** Let $\mathcal{A} = \operatorname{span}_K\{1, t, u, v\}$. We define the multiplication as follows:

$$u^2 = v^2 = uv = vu = tu = ut = t^2 = 0, \qquad tv = vt = u$$

and 1 as the identity. Let $G = \langle g : g^2 = 1 \rangle$ and $I = \langle v \rangle = \operatorname{span}_K\{u, v\}$. Consider the partial action given by $\mathcal{D}_g = I$, $\alpha_g : u \mapsto v, v \mapsto u$. Let $x = t\delta_1 + u\delta_g$. Then $x(xx) = u\delta_g$ and $(xx)x = 0$.

6.1 Associativity of crossed products

The associativity question of (twisted) crossed products has been already studied in [8] by R. Exel. He proved that $\mathcal{A} \rtimes_\alpha G$ is associative if each $\mathcal{D}_g$ is (L, R)-associative and that all ideals in a semiprime ring are (L, R)-associative. As a consequence, all crossed products of partial actions on semiprime rings are associative. So any partial action on a semiprime algebra defines a morphism $G \to \mathfrak{Rings}$.

Definition 6.7. The *multiplier algebra* $\mathcal{M}(\mathcal{I})$ of an algebra $\mathcal{I}$ is the set of all ordered pairs (L, R) of linear transformations of $\mathcal{I}$ satisfying the following:

$$L(ab) = L(a)b, \qquad R(ab) = aR(b), \qquad R(a)b = aL(b)$$

The algebra structure is defined by componentwise addition, scalar multiplication and $(L, R) \cdot (L', R') = (L \circ L', R' \circ R)$.

Note that the multiplier algebra $\mathcal{M}(\mathcal{I})$ is associative. Define a map $\varphi : \mathcal{I} \to \mathcal{M}(\mathcal{I})$ by $\varphi(x) = (L_x, R_x)$ where L_x and R_x are the left and right multiplication of x.

Definition 6.8. A K-algebra $\mathcal{I}$ is called *non-degenerate* if φ is injective.

Lemma 6.9 ([8], Lemma 2.3)**.** *The following statements hold:*

(i) $\varphi(\mathcal{I})$ is an ideal of $\mathcal{M}(\mathcal{I})$.

(ii) φ is an isomorphism if and only if $\mathcal{I}$ is unital.

Proof. (i) Let $x \in \mathcal{I}$ and $(L, R) \in \mathcal{M}(\mathcal{I})$ be arbitrary.

Then $(L_x, R_x) \cdot (L, R) = (L_x \circ L, R \circ R_x)$. For $a \in \mathcal{I}$, we have the following equations:

$$L_x(L(a)) = xL(a) = R(x)a = L_{R(x)}(a)$$
$$R(R_x(a)) = R(ax) = aR(x) = R_{R(x)}(a)$$

Thus, $(L_x \circ L, R \circ R_x) \in \varphi(\mathcal{I})$. The condition $(L \circ L_x, R_x \circ R) \in \varphi(\mathcal{I})$ is shown in exactly the same way.

(ii) $\Rightarrow$ Since $\mathcal{M}(\mathcal{I})$ is unital, it is trivial.

$\Leftarrow$ If $\mathcal{I}$ is unital, $\varphi(1)$ is the identity in $\mathcal{M}(\mathcal{I})$. Since $\varphi(\mathcal{I})$ is an ideal, it holds that $\varphi(\mathcal{I}) = \mathcal{M}(\mathcal{I})$. In this case, φ is injective. Thus φ is an isomorphism. □

Definition 6.10. An algebra $\mathcal{I}$ is said to be (L,R)-*associative* if given any two multipliers (L,R) and (L',R') in $\mathcal{M}(\mathcal{I})$, one has that

$$R' \circ L = L \circ R'.$$

Proposition 6.11 ([8] Proposition 2.5)**.** *An algebra $\mathcal{I}$ is (L,R)-associative if one of the following holds:*

(i) $\mathcal{I}$ is non-degenerate or

(ii) $\mathcal{I}$ is idempotent, that is, $\mathcal{I}^2 = \mathcal{I}$.

Proof. (i) We observe that for $(L,R),(L',R') \in \mathcal{M}(\mathcal{I})$ and $a,b \in \mathcal{I}$ the following holds:

$$R(L'(a))b = L'(a)L(b) = L'(aL(b)) = L'(R(a)b) = L'(R(a))b$$

Therefore, $R(L'(a)) - L'(R(a))$ lies in the left annihilator of $\mathcal{I}$. A similar calculation shows that it also lies in the right annihilator. Let φ as in Definition 6.8. Then $\varphi(R(L'(a)) - L'(R(a)))$ vanishes because $R(L'(a)) - L'(R(a))$ lies in the left (and right) annihilator. If $\mathcal{I}$ is non-degenerate, that is, the map φ is injective. Thus, we have that $R(L'(a)) = L'(R(a))$.

(ii) We make another observation. Let $a = a_1 a_2 \in \mathcal{I}$. Then we have

$$\begin{aligned} R(L'(a)) &= R(L'(a_1a_2)) = R(L'(a_1)a_2) = L'(a_1)R(a_2) \\ &= L'(a_1R(a_2)) = L'(R(a_1a_2)) = L'(R(a)) \end{aligned}$$

If $\mathcal{I}$ is idempotent, every element is a sum of terms of the form a_1a_2. Therefore, $\mathcal{I}$ is (L,R)-associative. $\square$

Now we show that every non-zero ideal of a unital semiprime algebra $\mathcal{A}$ is (L,R)-associative.

Definition 6.12. A unital algebra $\mathcal{A}$ is called semiprime if $\mathcal{A}$ has no nonzero nilpotent ideal.

Proposition 6.13 ([8], Proposition 2.6)**.** *The following statements are equivalent:*

(i) Every nonzero ideal of $\mathcal{A}$ is non-degenerate.

(ii) Every nonzero ideal of $\mathcal{A}$ is non-degenerate or idempotent.

(iii) Every nonzero ideal of $\mathcal{A}$ is right-non-degenerate.

(iv) Every nonzero ideal of $\mathcal{A}$ is left-non-degenerate.

(v) $\mathcal{A}$ is semiprime.

In this case, every ideal of $\mathcal{A}$ is (L,R)-associative.

Proposition 6.14. *Let $\pi : \mathcal{I} \to \mathcal{J}$ be an isomorphism of algebras. Then it induces an isomorphism $\tilde{\pi} : \mathcal{M}(\mathcal{I}) \to \mathcal{M}(\mathcal{J})$, defined by, $(L,R) \mapsto (\pi \circ L \circ \pi^{-1}, \pi \circ R \circ \pi^{-1})$.*

Now we can answer the associativity question.

Theorem 6.15 ([8], Theorem 3.1)**.** *If $\alpha = (\{\alpha_g\},\{\mathcal{D}_g\})$ is a partial action of G on an algebra $\mathcal{A}$ such that each $\mathcal{D}_g$ is (L,R)-associative, then $\mathcal{A} \rtimes_\alpha G$ is associative.*

Proof. $\mathcal{A} \rtimes_\alpha G$ is associative if and only if

$$(a\delta_h b\delta_g)c\delta_f = a\delta_h(b\delta_g c\delta_f) \tag{6.16}$$

We compute the left hand side:

$$(a\delta_h b\delta_g)c\delta_f = \alpha_h\{\alpha_h^{-1}(a)b\}\delta_{hg}c\delta_f = \alpha_{hg}[\alpha_{hg}^{-1}\{\alpha_h(\alpha_h^{-1}(a)b)\}c]\delta_{hgf}$$

Since $\alpha_h(\alpha_h^{-1}(a)b) \in \mathcal{D}_h \cap \mathcal{D}_{hg}$, we have $\alpha_{hg}^{-1}\{\alpha_h(\alpha_h^{-1}(a)b)\} = \alpha_g^{-1}(\alpha_h^{-1}(a)b)$. This belongs again to $\mathcal{D}_{g^{-1}} \cap \mathcal{D}_{g^{-1}h^{-1}}$. Therefore, we obtain the following by splitting α_{hg}:

$$(a\delta_h b\delta_g)c\delta_f = \alpha_{hg}[\alpha_{hg}^{-1}\{\alpha_h(\alpha_h^{-1}(a)b)\}c]\delta_{hgf} = \alpha_h[\alpha_g\{\alpha_g^{-1}(\alpha_h^{-1}(a)b)c\}]\delta_{hgf}$$

The right hand side is:

$$a\delta_h(b\delta_g c\delta_f) = a\delta_h(\alpha_g(\alpha_g^{-1}(b)c))\delta_{gf} = \alpha_h\{\alpha_h^{-1}(a)\alpha_g(\alpha_g^{-1}(b)c)\}\delta_{hgf}$$

Applying α_h^{-1}, (6.16) holds if and only if

$$\alpha_g\{\alpha_g^{-1}(\alpha_h^{-1}(a)b)c\} = \alpha_h^{-1}(a)\alpha_g(\alpha_g^{-1}(b)c) \tag{6.17}$$

Since α_h^{-1} is an isomorphism, we can just replace $\alpha_{h^{-1}}(a)$ by $a \in \mathcal{D}_{h^{-1}}$. Moreover, if $h = f = 1$, then $\mathcal{D}_h = \mathcal{D}_f = \mathcal{A}$. Thus the associativity holds for $h = f = 1$ if and only if

$$\alpha_g(\alpha_g^{-1}(ab)c) = a\alpha_g(\alpha_g^{-1}(b)c) \tag{6.18}$$

for arbitrary $g \in G$, $b \in \mathcal{D}_g$ and $a, c \in \mathcal{A}$. This is again equivalent to

$$(\alpha_g \circ R_c \circ \alpha_g^{-1}) \circ L_a = L_a \circ (\alpha_g \circ R_c \circ \alpha_g^{-1}) \tag{6.19}$$

for any $\mathcal{D}_g$ and $a, c \in \mathcal{A}$. Consider L_a as a left multiplier of $\mathcal{D}_g$ and R_c as a right multiplier of $\mathcal{D}_{g^{-1}}$. By Proposition 6.14, $\alpha_g \circ R_c \circ \alpha_g^{-1}$ is a right multiplier of $\mathcal{D}_g$. Hence if $\mathcal{D}_g$ is (L, R)-associative, then (6.19) holds. □

Corollary 6.20. *Let $\alpha = (\{\alpha_g\}, \{\mathcal{D}_g\})$ be a partial action of G on an algebra $\mathcal{A}$ such that each $\mathcal{D}_g$ is non-degenerate or idempotent. Then $\mathcal{A} \rtimes_\alpha G$ is associative. Furthermore, $\mathcal{A} \rtimes_\alpha G$ is always associative if $\mathcal{A}$ is semiprime.*

Corollary 6.21. *Let $\alpha = (\{\alpha_g\}, \{\mathcal{D}_g\})$ be a partial action such that $\mathcal{A} \rtimes_\alpha G$ is associative. Then it defines a strictly unital morphism $F : G \to \mathfrak{Rings}$. By Corollary 4.9, $\mathcal{A} \rtimes_\alpha G$ is a lax covariance ring for F.*

Proof. By Proposition 6.5, α defines a strictly unital morphism $F : G \to \mathfrak{Rings}$. Then, Proposition 4.9 implies the claim. □

Leavitt path algebras can be described as partial skew group rings (see [9], Thm 3.3). Thus, Leavitt path algebras are also lax covariance rings for partial actions.

6.2 Enveloping actions

A natural way to make a partial action is by restricting a global action. A (twisted) group action always defines a homomorphism of bicategories and always has a strong covariance ring, namely, the (twisted) crossed product. Thus, we can expect that partial actions that extend to global actions have similar properties. Indeed, such partial actions arising from global actions have also strong covariance rings.

Definition 6.22. Let G be a group and $\mathcal{A}$ be an ideal of an unital algebra $\mathcal{B}$. Given an action $\beta : G \to \mathrm{Aut}(\mathcal{B})$, define $\mathcal{D}_g = \mathcal{A} \cap \beta_g(\mathcal{B})$ and $\alpha_g := \beta_g|_{\mathcal{D}_{g^{-1}}}$. Then α defines a partial action and it is called a *restriction of* β. If $\bigcup_{g\in G} \beta_g(\mathcal{A})$ spans $\mathcal{B}$, α is called an *admissible restriction of* β.

Definition 6.23 ([8], Definition 4.1)**.** We say that two partial actions $\alpha = (\{\alpha_g\}, \{\mathcal{D}_g\}, \mathcal{A})$ and $\alpha' = (\{\alpha'_g\}, \{D'_g\}, \mathcal{A})$ of G are *equivalent* if there is an algebra isomorphism $\varphi : \mathcal{A} \to \mathcal{A}'$ such that

- $\varphi(\mathcal{D}_g) = D'_g$
- $\alpha'_g \circ \varphi(x) = \varphi \circ \alpha_g(x)$ for all $x \in \mathcal{D}_{g^{-1}}$.

Definition 6.24. An action β of G on an algebra $\mathcal{B}$ is called *an enveloping action* of α if α is equivalent to an admissible restriction of β. In other words, there is an algebra monomorphism $\varphi : \mathcal{A} \to \mathcal{B}$ such that $\varphi(\mathcal{A})$ is an ideal of $\mathcal{B}$ satisfying the following conditions for each $g \in G$:

(i) $\varphi(\mathcal{D}_g) = \varphi(\mathcal{A}) \cap \beta_g(\varphi(\mathcal{A}))$

(ii) $\varphi \circ \alpha_g(x) = \beta_g \circ \varphi(x)$ for each $x \in \mathcal{D}_{g^{-1}}$

(iii) $\mathcal{B}$ is generated by $\bigcup_{g\in G} \beta_g(\varphi(\mathcal{A}))$

Proposition 6.25 ([8], Proposition 4.3)**.** *If α is a partial action of G on an algebra $\mathcal{A}$ with an enveloping algebra β on $\mathcal{B}$, there is an embedding $\iota : \mathcal{A} \rtimes_\alpha G \hookrightarrow \mathcal{B} \rtimes_\beta G$. In particular, $\mathcal{A} \rtimes_\alpha G$ is associative.*

It is then clear that partial actions with enveloping actions define strictly unital morphisms $F : G \to \mathfrak{Rings}$. In the next section, we talk about their strong covariance rings.

6.3 Strong covariance ring for a partial action with an enveloping action

The following theorem states that if a partial action $\alpha = (\{\alpha_g\}, \{\mathcal{D}_g\})_{g\in G}$ admits an enveloping action, then the ideals $\mathcal{D}_g$ are unital algebras. That implies that they are finitely generated projective bimodules. Thus, α defines a strictly unital morphism to $\mathfrak{Rings}_{\mathrm{fp}}$. Since the strong covariance ring of such a morphism is the Cohn localization of the lax covariance ring, it follows that the Cohn localization of the crossed product is the strong covariance ring of a partial action with an enveloping action.

Proposition 6.26 ([8], Proposition 4.5)**.** *Let $\mathcal{A}$ be a unital algebra. Then a partial action $\alpha = (\{\alpha_g\}, \{\mathcal{D}_g\})_{g\in G}$ of G on $\mathcal{A}$ admits an enveloping algebra if and only if each $\mathcal{D}_g$ is a unital algebra. Moreover, if the enveloping action exists, this is unique up to equivalence.*

Proof. Suppose that α admits an enveloping action β. Let φ be a monomorphism as in Definition 6.23. Then each $\mathcal{D}_g$ is unital because $\varphi(\mathcal{D}_g) = \varphi(\mathcal{A}) \cap \beta_g(\varphi(\mathcal{A}))$.

Now assume that each $\mathcal{D}_g$ is a unital algebra. This means that $\mathcal{D}_g$ is generated by a central idempotent 1_g, i.e., $\mathcal{D}_g = 1_g\mathcal{A}$. We will construct an action β of G on some larger algebra $\mathcal{F}$ and a monomorphism $\varphi : \mathcal{A} \to \mathcal{F}$. Then we show that the restriction of β on $\mathcal{B} = \bigcup_{g\in G} \beta_g(\varphi(\mathcal{A}))$ is an enveloping action of α.

Let $\mathcal{F} := \mathcal{F}(G, \mathcal{A})$ be the algebra of functions from G into $\mathcal{A}$. For $f \in \mathcal{F}$ and $g \in G$, we denote $f(g)$ by $f|_g$. Then define $\beta_g(f) \in \mathcal{F}$ by

$$\beta_g(f)|_h = f(g^{-1}h)$$

It is easy to see that $\beta_g : f \mapsto \beta_g(f)$ is an automorphism whose inverse is obviously $\beta_{g^{-1}}$. Thus, the map $g \mapsto \beta_g$ is an action of G on $\mathcal{F}$. Now we define the monomorphism $\varphi : \mathcal{A} \to \mathcal{F}$ by the formula:

$$\varphi(a)|_g = \alpha_{g^{-1}}(a1_g)$$

Let $\mathcal{B} \subseteq \mathcal{F}$ be the subalgebra generated by $\bigcup_{g\in G} \beta_g(\varphi(\mathcal{A}))$ for $g \in G$. We denote the restriction of β on $\mathcal{B}$ by the same letter β. Note that for $g, h \in G$, 1_g1_h is the unit of $\mathcal{D}_g \cap \mathcal{D}_h$. Thus, the condition (ii) in Definition 6.1 gives us the following equation

$$\alpha_g(\alpha_{g^{-1}}(1_g)1_h) = \alpha_g(1_{g^{-1}}1_h) = 1_g1_{gh} \tag{6.27}$$

Claim: The restriction β is an enveloping action of α.

We start by checking the condition (ii). For $g, h \in G$ and $a \in \mathcal{D}_{g^{-1}}$ we have that $\beta_g(\varphi(a))|_h = \varphi(a)|_{g^{-1}h} = \alpha_{h^{-1}g}(a1_{g^{-1}h})$ and $\varphi(\alpha_g(a))|_h = \alpha_{h^{-1}}(\alpha_g(a)1_h)$. Thus, (ii) is satisfied if and only if the following holds for all $g, h \in G$ and $a \in \mathcal{D}_{g^{-1}}$:

$$\alpha_{h^{-1}g}(a1_{g^{-1}h}) = \alpha_{h^{-1}}(\alpha_g(a)1_h) \tag{6.28}$$

Since $a1_{g^{-1}h} \in \mathcal{D}_{g^{-1}} \cap \mathcal{D}_{g^{-1}h}$, we can split $\alpha_{h^{-1}g}$ on the left hand side of (6.28) into $\alpha_{h^{-1}}\alpha_g$ and using (6.27) we have

$$\begin{aligned}\alpha_{h^{-1}g}(a1_{g^{-1}h}) &= \alpha_{h^{-1}}(\alpha_g(a1_{g^{-1}h}))\\ &= \alpha_{h^{-1}}(\alpha_g(a)\alpha_g(1_{g^{-1}}1_{g^{-1}h}))\\ &= \alpha_{h^{-1}}(\alpha_g(a)1_g1_h)\\ &= \alpha_{h^{-1}}(\alpha_g(a)1_h).\end{aligned} \tag{6.29}$$

Now it only remains to check the condition (i). The elements in $\varphi(\mathcal{A}) \cap \beta_g(\varphi(\mathcal{A}))$ are of the form $\varphi(a) = \beta_g(\varphi(b))$ for $a, b \in \mathcal{A}$.

For each $h \in G$ the equality $\varphi(a)|_h = \beta_g(\varphi(b))|_h$ is equivalent to

$$\alpha_{h^{-1}}(a1_h) = \alpha_{h^{-1}g}(b1_{g^{-1}h}) \tag{6.30}$$

Taking $h = 1$ this gives $a = \alpha_g(b1_{g^{-1}}) \in \mathcal{D}_g$. So $\varphi(\mathcal{D}_g) \supseteq \varphi(\mathcal{A}) \cap \beta_g(\varphi(\mathcal{A}))$. For the other inclusion, we show that for each $a \in \mathcal{D}_g$ there is $b \in \mathcal{A}$ such that (6.30) holds. For $b = \alpha_{g^{-1}}(a)$ the right hand side of (6.30) is $\alpha_{h^{-1}g}(\alpha_{g^{-1}}(a)1_{g^{-1}h}) = \alpha_{h^{-1}}(a1_h)$. Thus (i) is also satisfied. In conclusion, β is an enveloping action of α. □

This proposition implies that partial actions with an enveloping action admit strong covariance rings:

Theorem 6.31. *Let α be a partial action of G on a unital algebra $\mathcal{A}$. Suppose that α admits an enveloping action. Then the Cohn localisation of $\mathcal{A} \rtimes_\alpha G$ at the maps $\{\psi_g : \mathcal{D}_g \otimes_{\mathcal{A}} \mathcal{A} \rtimes G \to \mathcal{A} \rtimes G \mid g \in G\}$ is a strong covariance ring of α.*

Proof. Since α admits an enveloping action, each $\mathcal{D}_g$ is unital by (6.26). If $\mathcal{D}_g$ is an unital algebra, it is isomorphic to $p \cdot \mathcal{A}$ where p is a central idempotent of $\mathcal{A}$. Thus, it is a finitely generated projective module. Then the rest of the proof follows from Proposition 4.14. □

6.4 Partial actions on a topological space.

Let X be a compact, Hausdorff, and totally disconnected space and let $\theta = (\{\theta_g\}, \{X_g\})_{g\in G}$ be a topological partial action of a discrete group G, such that each X_g is clopen. Then X_g is also compact. For a commutative ring K with unit equipped with the discrete topology, define

$$A_K(X_g) := \{f : X \to K \mid \mathrm{supp}(f) \subseteq X_g,\ f \text{ continuous }\}.$$

Since X_g is compact, the algebra $A_K(X_g)$ has the unit $1_g := 1_{X_g}$. Now define

$$\alpha_g : A_K(X_{g^{-1}}) \to A_K(X_g), \quad f \mapsto f \circ \theta_{g^{-1}}. \tag{6.32}$$

It is easy to check that $\alpha = (\{\alpha_g\}, \{A_K(X_g)\})_{g\in G}$ defines a partial action on the algebra $A_K(X)$. The partial action α is called the *dual action* of θ.

Lemma 6.33. *(i) The partial action α defines a strictly unital morphism $G \to \mathfrak{Rings}_{fp}$.*

(ii) α admits an enveloping action.

Proof. It follows directly that each $A_K(X_g)$ is unital. So there is a central idempotent $1_g \in A_K(X)$ such that $A_K(X_g) = 1_g A_K(X)$. The right module structure of $A_K(X_g)$ is given by $a \cdot x = \alpha_g(\alpha_g^{-1}(a)x) = \alpha_g(\alpha_g^{-1}(1_g)ax) = 1_g ax = ax$ for $a \in A_K(X_g)$ and $x \in A_K(X)$. Therefore, $A_K(X_g)$ is a finitely generated projective right $A_K(X)$-module. □

Proposition 6.34. *(i) The crossed product $A_K(X) \rtimes_\alpha G$ is a lax covariance ring for α.*

(ii) The Cohn localisation of $A_K(X) \rtimes_\alpha G$ at the maps $\{\psi_g : A_K(X_g) \otimes_{A_K(X)} A_K(X) \mid g \in G\}$ is a strong covariance ring for α. In particular, the strong covariance ring exists.

In a certain setting, we have that $A_K(X \rtimes G) \simeq A_K(X) \rtimes G$. We did not yet define $X \rtimes G$ and $A_K(X \rtimes G)$. They can be explained in terms of *Steinberg algebras*. The isomorphism is proved in [3].

6.4.1 Steinberg algebra and groupoid of germs.

A *groupoid* $\mathcal{G}$ is a small category with only invertible arrows. We identify $\mathcal{G}$ with the set of arrows and the set of objects with the set of unit arrows and write $\mathcal{G}^0$ for the set of unit arrows. For an arrow $g \in \mathcal{G}$, the source and range maps are given by $s(g) = g^{-1}g$ and $r(g) = gg^{-1}$. A *topological* groupoid is a groupoid whose underlying set is equipped with a topology such that the composition and inversion maps of arrows are continuous.

Definition 6.35. A topological groupoid $\mathcal{G}$ is called *étale* if the source and range maps $s, r : \mathcal{G} \to \mathcal{G}^0$ of arrows are local homeomorphisms. Furthermore, an *étale* groupoid is called *ample* if $\mathcal{G}^0$ is Hausdorff and has a basis of compact open subsets.

Definition 6.36. Let θ be a partial action as above. The *transformation groupoid* $X \rtimes_\theta G$ is defined as follows:

(i) The underlying set is $\{(x, g) | g \in G,\, x \in X_g\}$

(ii) If $(y, h), (x, g) \in X \rtimes_\theta G$, then they are composable if, and only if, $\theta_{h^{-1}}(y) = x$. In this case, we have $(y, h) \cdot (x, g) = (y, hg)$.

Definition 6.37 ([13],Definition 3.1.)**.** If $\mathcal{G}$ is an ample groupoid and K is a commutative ring with unit equipped with the discrete topology, then $A_K(\mathcal{G})$ (also written as $K\mathcal{G}$ by Steinberg) is the space of all K-valued functions on $\mathcal{G}$ spanned by functions $f : \mathcal{G} \to K$ such that:

(i) there is an open Hausdorff subspace V in $\mathcal{G}$ so that f vanishes outside V;

(ii) $f|_V$ is continuous with compact support.

The algebra $A_K(\mathcal{G})$ is called the *Steinberg algebra.*

The following proposition gives us the desired isomorphism between algebras.

Proposition 6.38 ([3], Theorem 3.2)**.** *Let θ be a partial action of a discrete group G on a compact, Hausdorff and totally disconnected space X, such that each X_g is clopen. Let α be the dual action of θ and let $X \rtimes_\theta G$ be the transformation groupoid associated with θ. Then there is an isomorphism of algebras $A_K(X \rtimes_\theta G) \simeq A_K(X) \rtimes_\alpha G$.*

So the Steinberg algebra $A_K(X \rtimes G)$ is a lax covariance ring for a partial action.

7 Twisted partial group actions as morphisms

Now we twist a partial action and generalize it to a strictly unital morphism to $\mathfrak{Rings}$. The associativity of the crossed product plays a crucial role to define a strictly unital morphism $G \to \mathfrak{Rings}$. This has been also studied in [7] by R. Exel. He has shown that all crossed products by twisted partial actions are associative. Thus, a twisted partial action always defines a strictly unital morphism to $\mathfrak{Rings}$. Furthermore, we can easily check that the twisted crossed product is its lax covariance ring.

Definition 7.1 ([7], Def. 2.1)**.** A twisted partial action of a group G on a K-algebra A is a triple

$$\theta = (\{\mathcal{D}_g\}_{g\in G}, \{\theta_g\}_{g\in G}, \{w_{g,h}\}_{(g,h)\in G\times G})$$

where $\mathcal{D}_g$ is a two-sided ideal of A, $\theta_g : \mathcal{D}_{g^{-1}} \to \mathcal{D}_g$ is an isomorphism and $w_{g,h}$ is an invertible element of $\mathcal{M}(\mathcal{D}_g \cdot \mathcal{D}_{gh})$ satisfying the following properties for all $g, h, t \in G$:

(i) $\mathcal{D}_g^2 = \mathcal{D}_g$ (idempotent), $\mathcal{D}_g\mathcal{D}_h = \mathcal{D}_h\mathcal{D}_g$ for all $g, h \in G$;

(ii) $\mathcal{D}_1 = A$ and θ_1 is the identity map of A;

(iii) $\theta_g(\mathcal{D}_{g^{-1}} \cdot \mathcal{D}_h) = \mathcal{D}_g \cdot \mathcal{D}_{gh}$;

(iv) $\theta_g \circ \theta_h(a) = w_{g,h}\theta_{gh}(a)w_{g,h}^{-1}$;

(v) $w_{1,g} = w_{g,1} = 1$;

(vi) $\theta_g(aw_{h,t})w_{g,ht} = \theta_g(a)w_{g,h}w_{gh,t}$.

Remark 7.2. To clarify some ambiguities in the definition, it is useful to remark the following:

- $\prod_{i=1}^{n} \mathcal{D}_{g_i}$ is also idempotent.
- Any algebra $\mathcal{I}$ with $\mathcal{I}^2 = \mathcal{I}$ is (L, R)-associative. Thus, $(w_{g,h}\theta_{gh}(a))w_{g,h}^{-1} = w_{g,h}(\theta_{gh}(a)w_{g,h}^{-1})$. That allows us to leave out the brackets in (iv).
- $\theta_g^{-1}(\mathcal{D}_g\mathcal{D}_h) = \mathcal{D}_{g^{-1}}\mathcal{D}_{g^{-1}h}$ by (iii).

Definition 7.3. Let $\Theta = (\{\alpha_g\}, \{\mathcal{D}_g\}, \{w_{g,h}\})$ be a twisted partial action of G on an algebra A. Then the crossed product $A \rtimes_\Theta G$ is defined as

$$\bigoplus_{g\in G} \mathcal{D}_g\delta_g$$

where δ_g are symbols. The multiplication is defined by

$$(a_g\delta_g)(b_h\delta_h) = \theta_g(\theta_{g^{-1}}(a)b)w_{g,h}$$

Here $w_{g,h}$ acts as a right multiplier on $\theta_g(\mathcal{D}_{g^{-1}}\mathcal{D}_h) = \mathcal{D}_g\mathcal{D}_{gh}$.

Similar to partial actions, a twisted partial action gives us the following data:

1. an algebra A;

2. $\mathcal{D}_g$ as bimodules over A with the left module structure by multiplication and the right module structure by $a \cdot b = \theta_g(\theta_{g^{-1}}(a)b)$;
3. multiplication maps

$$\mu_{g,h} : \mathcal{D}_g \otimes \mathcal{D}_h \to \mathcal{D}_{gh},$$
$$\mu_{g,h}(a \otimes b) = \theta_g(\theta_{g^{-1}}(a)b)w_{g,h}.$$

The diagram conditions of a morphism hold if and only if $A \rtimes_\Theta G$ is associative. The following proposition proves the associativity. Thus, twisted partial actions always define strictly unital morphisms to $\mathfrak{Rings}$.
Before we prove the associativity of $\mathcal{A} \rtimes_\Theta G$, we need to show some technical equations.

Lemma 7.4 ([7], Lemma 2.3)**.** *(i) If $a, c \in \mathcal{A}, b \in \mathcal{D}_h$ and $h \in G$, then*

$$a\theta_h(\theta_{h^{-1}}(b)c) = \theta_h(\theta_{h^{-1}}(ab)c). \tag{7.5}$$

(ii) If $x \in \mathcal{D}_{h^{-1}}\mathcal{D}_{h^{-1}g-1}$, $g, h \in G$ and $c \in \mathcal{A}$, then

$$\theta_{gh}^{-1}(w_{g,h}\theta_{gh}(x))c = \theta_{gh}^{-1}(w_{g,h}\theta_{gh}(xc)). \tag{7.6}$$

Proof. (i) Since $\theta_h : \mathcal{D}_{h^{-1}} \to \mathcal{D}_h$ is an isomorphism, $(\theta_h L_c \theta_h^{-1}, \theta_h^{-1} R_c \theta_h)$ is a multiplier of $\mathcal{D}_h$. Using the (L, R)-associativity on (L_a, R_a) and $(\theta_h L_c \theta_h^{-1}, \theta_h^{-1} R_c \theta_h)$, we see that

$$L_a \cdot (b \cdot \theta_h^{-1} R_c \theta_h) = L_a(b) \cdot (\theta_h^{-1} R_c \theta_h) = \theta_h(\theta_{h^{-1}}(ab)c).$$

This is precisely what we want to show.

(ii) By (iii), $\theta_{gh}|_{\mathcal{D}_{h^{-1}}\mathcal{D}_{h^{-1}g^{-1}}} : \mathcal{D}_{h^{-1}}\mathcal{D}_{h^{-1}g^{-1}} \to \mathcal{D}_g\mathcal{D}_{gh}$ is an isomorphism. Hence $(\theta_{gh}^{-1} w_{g,h} \theta_{gh})$ is a multiplier of $\mathcal{D}_{h^{-1}}\mathcal{D}_{h^{-1}g^{-1}}$. The (L, R)-associativity gives the following equation, which proves the claim:

$$[(\theta_{gh}^{-1} w_{g,h} \theta_{gh}) \cdot x] \cdot R_c = (\theta_{gh}^{-1} w_{g,h} \theta_{gh}) \cdot (x \cdot R_c)$$

□

Proposition 7.7 ([7], Theorem 2.4)**.** *$A \rtimes_\Theta G$ is associative.*

Proof. $\mathcal{A} \rtimes_\Theta G$ is associative if and only if the following equation holds for all $a \in \mathcal{D}_g$, $b \in \mathcal{D}_h$, $c \in \mathcal{D}_t$ and $g, h, t \in G$:

$$(a\delta_g b\delta_h)c\delta_t = a\delta_g(b\delta_h c\delta_t) \tag{7.8}$$

We compute the left hand side:

$$(a\delta_g b\delta_h)c\delta_t = (\theta_g(\theta_{g^{-1}}(a)b)w_{g,h}\delta_{gh})c\delta_t = \theta_{gh}(\theta_{gh}^{-1}(\theta_g(\theta_{g^{-1}}(a)b)w_{g,h})c)w_{gh,t}\delta_{ght}$$

The right hand side is :

$$\begin{aligned} a\delta_g(b\delta_h c\delta_t) &= a\delta_g(\theta_h(\theta_{h^{-1}}(b)c)w_{h,t}\delta_{ht}) \\ &= \theta_g[\theta_{g^{-1}}(a)\theta_h(\theta_{h^{-1}}(b)c)w_{h,t}]w_{g,ht}\delta_{ght} \\ &= \theta_g[\theta_{g^{-1}}(a)\theta_h(\theta_{h^{-1}}(b)c)]w_{g,h}w_{gh,t}\delta_{ght} \end{aligned}$$

The last equation holds because $\theta_{g^{-1}}(a)\theta_h(\theta_{h^{-1}}(b)c)$ lies in $\mathcal{D}_{g^{-1}}\mathcal{D}_h\mathcal{D}_{ht}$, so we can apply (vi). Comparing both terms, we can cancel $w_{gh,t}$ because it is invertible. Since $\theta_{g^{-1}}(a)$ runs over $\mathcal{D}_{g^{-1}}$, (7.8) is equivalent to

$$\theta_{gh}[\theta_{gh}^{-1}(\theta_g(ab)w_{g,h})c] = \theta_g[a\theta_h(\theta_{h^{-1}}(b)c)]w_{g,h} \tag{7.9}$$

for all $g, h \in G$, $a \in \mathcal{D}_{g^{-1}}, b \in \mathcal{D}_h$ and $c \in \mathcal{A}$. The right hand side is equal to $\theta_g(\theta_h(\theta_{h^{-1}}(ab)c))w_{g,h}$ by (7.6). Now $y = ab$ lies in $\mathcal{D}_{g^{-1}}\mathcal{D}_h$. So it is enough to show that

$$\theta_{gh}(\theta_{gh}^{-1}(\theta_g(y)w_{g,h})c) = \theta_g(\theta_h(\theta_{h^{-1}}(y)c))w_{g,h} \tag{7.10}$$

for all $g, h \in G$ and $y \in \mathcal{D}_{g^{-1}}\mathcal{D}_h$. Set $x = \theta_h^{-1}(y) \in \theta_h^{-1}(\mathcal{D}_{g^{-1}}\mathcal{D}_h) = \mathcal{D}_{h^{-1}}\mathcal{D}_{h^{-1}g^{-1}}$. Then applying (iv) on the left hand side and (7.6) gives us

$$\theta_{gh}(\theta_{gh}^{-1}(\theta_g \circ \theta_h(x)w_{g,h})c) = \theta_{gh}(\theta_{gh}^{-1}(w_{g,h}\theta_{gh}(x))c) = \theta_{gh}(\theta_{gh}^{-1}(w_{g,h}\theta_{gh}(xc))) = w_{g,h}(\theta_{gh}(xc)) \tag{7.11}$$

Now we set again $z = xc$. Then (7.8) is finally equivalent to

$$w_{g,h}\theta_{gh}(z) = \theta_g(\theta_h(z))w_{g,h} \tag{7.12}$$

with arbitrary $g, h \in G$ and $z \in \mathcal{D}_{h^{-1}}\mathcal{D}_{h^{-1}g^{-1}}$, which is (iv). $\square$

Corollary 7.13. *Let $\Theta = (\{\theta_g\}_{g\in G}, \{\mathcal{D}_g\}_{g\in G}, \{w_{g,h}\}_{g,h\in G^2})$ be a twisted partial action of G on a unital algebra $\mathcal{A}$. It defines a strictly unital morphism $\Theta : G \to \mathfrak{Rings}$. Moreover, the twisted partial crossed product $\mathcal{A} \rtimes_\Theta G$ is a lax covariance ring for Θ.*

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