2. Gyro-expansion

p.2 small m/q expansion

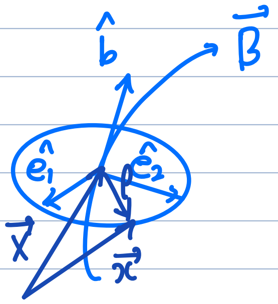
$$\frac{m}{q}\frac{d\vec{v}}{dt} = \vec{E} + \vec{v}\times\vec{B}$$

small "$\frac{m}{q}$" expansion

$$\begin{cases} ① \ \dfrac{m}{q} = \epsilon \quad (\epsilon \ll 1) \\ ② \ \dfrac{m}{q} \rightarrow \epsilon \dfrac{m}{q} \quad (\epsilon = 1) \quad \left(\text{eg. } \begin{matrix} \omega = \epsilon^{-1}\omega \\ \rho = \epsilon\rho \end{matrix}\right) \end{cases}$$

$$\Rightarrow \boxed{\epsilon \frac{m}{q}\frac{d\vec{v}}{dt} = \vec{E} + \vec{v}\times\vec{B}}$$

p.3 Expansion from guiding center



$$\vec{x} = \vec{X} + \epsilon\vec{\rho}$$

$$\vec{B}(\vec{X}+\vec{\rho}) = \vec{B}(\vec{X}) + \epsilon(\vec{\rho}\cdot\vec{\nabla})\vec{B} + O(\epsilon^2)$$

$$\vec{E_\perp}(\vec{X}+\vec{\rho}) = \vec{E_\perp}(\vec{X}) + \epsilon(\vec{\rho}\cdot\vec{\nabla})\vec{E_\perp} + O(\epsilon^2)$$

$$\vec{E_\parallel}(\vec{X}+\vec{\rho}) = \epsilon\vec{E_\parallel}(\vec{X}) + \epsilon^2(\vec{\rho}\cdot\vec{\nabla})\vec{E_\parallel} + O(\epsilon^3)$$

p.4 Multiple time-scale in expansion

$$\vec{\rho_0} = \rho_0(\hat{e}_1 \sin(\omega_0 t) + \hat{e}_2 \cos(\omega_0 t))$$

$$= \rho_0(\hat{e}_1 \sin\gamma + \hat{e}_2\cos\gamma)$$: gyrophase $\gamma = \gamma(t)$ $(\gamma_0 = \omega_0 t)$

$$\vec{\rho}(\vec{X},t) \longrightarrow \vec{\rho}(\vec{X},t,\gamma)$$

p.5 Time derivatives in gyroexpansion

$$\vec{v} = \frac{d\vec{X}}{dt} + \frac{d\vec{\rho}}{dt} \ :$$

$$\frac{d\vec{X}}{dt} = \underline{\vec{V_0} + \epsilon\vec{V_1}} + O(\epsilon^2) \leftarrow \text{we want to find } \vec{V_i} \text{ (various drift velocity)}$$

$$\frac{d\vec{\rho}}{dt} = \left(\frac{d}{dt} + \dot{\gamma}\frac{d}{d\gamma}\right)\vec{\rho} = \left(\frac{d}{dt} + \omega\frac{d}{d\gamma}\right)\vec{\rho} \quad (\omega(\vec{x},t) \equiv d\gamma/dt)$$

$$= \left(\frac{d}{dt} + \epsilon^{-1}\omega\frac{d}{d\gamma}\right)\epsilon\vec{\rho} = \underline{\epsilon\frac{d\vec{\rho}}{dt} + \omega\frac{d\vec{\rho}}{d\gamma}}$$

Also. $\vec{\rho} = \vec{\rho_0} + \epsilon\vec{\rho_1} + O(\epsilon^2)$, $\omega = \omega_0 + \epsilon\omega_1 + O(\epsilon^2)$

$$\rightarrow \vec{v} = \vec{V_0} + \epsilon\vec{V_1} + \epsilon\frac{d\vec{\rho_0}}{dt} + (\omega_0 + \epsilon\omega_1)\frac{d(\vec{\rho_0} + \epsilon\vec{\rho_1})}{d\gamma}$$

$$= \vec{V_0} + \epsilon\vec{V_1} + \omega_0\frac{d\vec{\rho_0}}{d\gamma} + \epsilon\left(\frac{d\vec{\rho_0}}{dt} + (\vec{V_0}\cdot\vec{\nabla})\vec{\rho_0} + \omega_0\frac{d\vec{\rho_1}}{d\gamma} + \omega_1\frac{d\vec{\rho_0}}{d\gamma}\right) + O(\epsilon^2)$$

$$\rightarrow \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{X}}{dt} + \epsilon\frac{d\vec{\rho}}{dt}\right) = \frac{d}{dt}\left(\vec{V_0} + \epsilon\vec{V_1}\right) + \frac{d}{dt}\left(\epsilon\frac{d\vec{\rho}}{dt} + \omega\frac{d\vec{\rho}}{d\gamma}\right)$$

$$= \frac{d\vec{V_0}}{dt} + \epsilon\frac{d\vec{V_1}}{dt} + \left(\frac{d}{dt} + (\vec{V_0} + \epsilon\vec{V_1} + \cdots)\cdot\vec{\nabla} + \epsilon^{-1}\omega\frac{d}{d\gamma}\right)\left(\epsilon\frac{d\vec{\rho}}{dt} + \omega\frac{d\vec{\rho}}{d\gamma}\right)$$

$$= \frac{d\vec{V_0}}{dt} + \epsilon\frac{d\vec{V_1}}{dt} + \left[\underline{\epsilon^{-1}\omega_0\frac{d}{d\gamma}} + \underline{\omega_1\frac{d}{d\gamma} + \frac{d}{dt} + \vec{V_0}\cdot\vec{\nabla}}\right] \times$$

$$\left[\underline{\omega_0\frac{d\vec{\rho_0}}{d\gamma}} + \epsilon\underline{\left(\frac{d\vec{\rho_0}}{dt} + (\vec{V_0}\cdot\vec{\nabla})\vec{\rho_0} + \omega_0\frac{d\vec{\rho_1}}{d\gamma} + \omega_1\frac{d\vec{\rho_0}}{d\gamma}\right)}\right] + \cdots$$

$$= \epsilon^{-1}\omega_0^2\frac{d^2\vec{\rho_0}}{d\gamma^2} + \frac{d\vec{V_0}}{dt} + \underline{\omega_0\left(\frac{d}{dt} + (\vec{V_0}\cdot\vec{\nabla})\right)\frac{d\vec{\rho_0}}{d\gamma} + \omega_0^2\frac{d^2\vec{\rho_1}}{d\gamma^2} + \omega_0\omega_1\frac{d^2\vec{\rho_0}}{d\gamma^2}}$$

$$+ \underline{\omega_0\omega_1\frac{d^2\vec{\rho_0}}{d\gamma^2} + \omega_0\left(\frac{d}{dt} + \vec{V_0}\cdot\vec{\nabla}\right)\frac{d\vec{\rho_0}}{d\gamma}}$$

$$= \underbrace{\epsilon^{-1}\omega_0^2\frac{d^2\vec{\rho_0}}{d\gamma^2}}_{O(\varepsilon^{-1})} + \underbrace{\frac{d\vec{V_0}}{dt} + 2\omega_0\left(\frac{d}{dt} + \vec{V_0}\cdot\vec{\nabla}\right)\frac{d\vec{\rho_0}}{d\gamma} + \omega_0^2\frac{d^2\vec{\rho_1}}{d\gamma^2} + 2\omega_0\omega_1\frac{d^2\vec{\rho_0}}{d\gamma^2}}_{O(\varepsilon^0)} + O(\varepsilon)$$

$$\therefore \epsilon\frac{m}{q}\frac{d\vec{v}}{dt} = \frac{m}{q}\left[\omega_0^2\frac{d^2\vec{\rho_0}}{d\gamma^2} + \epsilon\left(\frac{d\vec{V_0}}{dt} + 2\omega_0\left(\frac{d}{dt} + \vec{V_0}\cdot\vec{\nabla}\right)\frac{d\vec{\rho_0}}{d\gamma} + \omega_0^2\frac{d^2\vec{\rho_1}}{d\gamma^2} + 2\omega_0\omega_1\frac{d^2\vec{\rho_0}}{d\gamma^2}\right)\right] \quad \cdots ①$$

p.8 Expansion for electromagnetic fields

$$\vec{E}+\vec{v}\times\vec{B} = \vec{E}_\perp + \varepsilon(\vec{\rho}\cdot\vec{\nabla})\vec{E}_\perp + \varepsilon\vec{E}_\parallel$$

(all 1st order)

$$+\left(\vec{v}_0+\epsilon\vec{v}_1+\epsilon\frac{d\vec{\rho}}{dt}+\omega\frac{d\vec{\rho}}{d\gamma}\right)\times\left(\vec{B}+\epsilon(\vec{\rho}\cdot\vec{\nabla})\vec{B}\right)$$

$$= \vec{E}_\perp + \vec{v}_0\times\vec{B} + \omega_0\frac{d\vec{\rho}_0}{d\gamma}\times\vec{B}$$

$$+\epsilon\Big[(\vec{\rho}_0\cdot\vec{\nabla})\vec{E}_\perp + \vec{E}_\parallel + \vec{v}_0\times(\vec{\rho}_0\cdot\vec{\nabla})\vec{B} + \vec{v}_1\times\vec{B}$$

$$+\left(\frac{d}{dt}+\vec{v}_0\cdot\vec{\nabla}\right)\vec{\rho}_0\times\vec{B} + \omega_0\frac{d\vec{\rho}_0}{d\gamma}\times(\vec{\rho}_0\cdot\vec{\nabla})\vec{B} + \omega_0\frac{d\vec{\rho}_1}{dt}\times\vec{B} + \omega_1\frac{d\vec{\rho}_0}{dt}\times\vec{B}\Big]$$

$\cdots$ ②

p.9 Zeroth-order drift motion

① + ② for $\sim O(1)$

$$\omega_0^2\frac{d^2\vec{\rho}_0}{d\gamma^2} + \frac{q}{m}\vec{B}\times\frac{d\vec{\rho}_0}{d\gamma} = \frac{q}{m}\left(\vec{E}_\perp + \vec{v}_0\times\vec{B}\right)$$

- homogeneous solution : $\vec{\rho}_0 = \rho_0(\hat{e}_1\sin\gamma + \hat{e}_2\cos\gamma)$
- particular solution : $\vec{v}_{\perp 0} = \dfrac{\vec{E}\times\vec{B}}{B^2} \equiv \vec{v}_E$: leading-order drift

(taking gyro-average $\rightarrow$ LHS $=0$)

p.10 First-order (correction) equation

$$\frac{d\vec{v}_0}{dt} + 2\omega_0\left(\frac{d}{dt}+\vec{v}_0\cdot\vec{\nabla}\right)\frac{d\vec{\rho}_0}{d\gamma} + \omega_0^2\frac{d^2\vec{\rho}_1}{d\gamma^2} + 2\omega_0\omega_1\frac{d^2\vec{\rho}_0}{d\gamma^2}$$

$$= \frac{q}{m}\Big[(\vec{\rho}_0\cdot\vec{\nabla})\vec{E}_\perp + \vec{E}_\parallel + \vec{v}_0\times(\vec{\rho}_0\cdot\vec{\nabla})\vec{B} + \vec{v}_1\times\vec{B}$$

$$+\left(\frac{d}{dt}+\vec{v}_0\cdot\vec{\nabla}\right)\vec{\rho}_0\times\vec{B} + \omega_0\frac{d\vec{\rho}_0}{d\gamma}\times(\vec{\rho}_0\cdot\vec{\nabla})\vec{B} + \omega_0\frac{d\vec{\rho}_1}{dt}\times\vec{B} + \omega_1\frac{d\vec{\rho}_0}{dt}\times\vec{B}\Big]$$

Take gyro-average : $$\frac{d\vec{v}_0}{dt} = \frac{q}{m}\Big[\vec{E}_\parallel + \vec{v}_1\times\vec{B} + \omega_0\underbrace{\left\langle\frac{d\vec{\rho}_0}{d\gamma}\times(\vec{\rho}_0\cdot\vec{\nabla})\vec{B}\right\rangle}_{(*)}\Big]$$

$$\left\langle \frac{d\vec{\rho}_0}{d\nu} \times (\vec{\rho}_0 \cdot \vec{\nabla})\vec{B} \right\rangle \qquad (\vec{\rho}_0 = \hat{x}\sin\nu + \hat{y}\cos\nu)$$

$$= \rho_0^2 \left\langle (\hat{x}\cos\nu - \hat{y}\sin\nu) \times \left(\sin\nu \frac{d}{dx} + \cos\nu \frac{d}{dy}\right)(B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) \right\rangle$$

$$= \frac{1}{2}\rho_0^2 \left(\hat{x}\frac{d}{dy} - \hat{y}\frac{d}{dx}\right) \times (B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$$

$$= \frac{1}{2}\rho_0^2 \left(\frac{dB_y}{dy}\hat{z} - \frac{dB_z}{dy}\hat{y} + \frac{dB_x}{dx}\hat{z} - \frac{dB_z}{dx}\hat{x}\right)$$

$$= \frac{1}{2}\rho_0^2 \left(\underbrace{\left(\frac{dB_x}{dx} + \frac{dB_y}{dy} + \frac{dB_z}{dz}\right)}_{\vec{\nabla}\cdot\vec{B}=0}\hat{z} - \left(\frac{dB_z}{dx} + \frac{dB_z}{dy} + \frac{dB_z}{dz}\right)\hat{z}\right) = -\frac{1}{2}\rho_0^2 \vec{\nabla}B$$

<magnetic moment>

$$\mu \equiv \frac{1}{2} q \omega_0 \rho_0^2 \to \frac{m v_\perp^2}{2B} + O(\epsilon)$$

$$\therefore \frac{d\vec{v}_0}{dt} = \frac{q}{m}\left[\vec{E}_\parallel + \vec{v}_1 \times \vec{B} - \frac{1}{2}\omega_0 \rho_0^2 \vec{\nabla}B\right]$$

$$m\frac{d\vec{v}_0}{dt} = q\left[\vec{E}_\parallel + \vec{v}_1 \times \vec{B} - \frac{1}{2}\omega_0\rho_0^2\vec{\nabla}B\right] = q\left[\vec{E}_\parallel + \vec{v}_1 \times \vec{B}\right] - \mu\vec{\nabla}B \quad \cdots ③$$

$\vec{B}\cdot③$

i) parallel : $m\dfrac{d\vec{v}_{\parallel 0}}{dt} = q\vec{E}_\parallel - \mu(\hat{b}\cdot\vec{\nabla})B$

$\vec{B}\times③$

ii) perpendicular : $\vec{B} \times m\dfrac{d\vec{v}_0}{dt} = q\vec{B}\times(\vec{v}_1 \times \vec{B}) - \mu\vec{B}\times\vec{\nabla}B$

$$= q\vec{v}_{\perp}B^2 - \mu\vec{B}\times\vec{\nabla}B$$

$$\therefore \vec{v}_\perp = \underbrace{\frac{m}{qB^2}\left(\vec{B}\times\frac{d\vec{v}_0}{dt}\right)}_{\text{inertial drift.}} + \mu\frac{\vec{B}\times\vec{\nabla}B}{qB^2} \quad \cdots ④$$

← curvature drift가 안보이네?

⇒ First-order correction for drift motion

p.12 First-order correction for drift motion

$$\vec{V}_0 = \vec{V}_{\parallel 0}\hat{b} + \frac{\vec{E}\times\vec{B}}{B^2} = \vec{V}_{\parallel 0}\hat{b} + \vec{V}_E$$

$$\frac{d\vec{v}_0}{dt} = \left(\frac{d}{dt} + v_{\parallel 0}\hat{b}\cdot\vec{\nabla} + \vec{v}_E\cdot\vec{\nabla}\right)(v_{\parallel 0}\hat{b} + \vec{v}_E) = \frac{d}{dt}(v_{\parallel 0}\hat{b}) + \underbrace{(v_{\parallel 0}\hat{b} + \vec{v}_E)\cdot\vec{\nabla}(v_{\parallel 0}\hat{b})}_{\text{curvature drift}} + \underbrace{\frac{d\vec{v}_E}{dt}}_{\text{polarization drift}} \quad \cdots ⑤$$

($\vec{v}_E \cdot \vec{\nabla}\vec{v}_E$ are new and can be potentially important as Northrop discussed.)

combine ⑤ to ④, then

$$\vec{V}_{d\perp} = \frac{m}{qB^2}\left(\vec{B}\times\left(\frac{d}{dt}(v_{\parallel 0}\hat{b}) + (v_{\parallel 0}\hat{b}+\vec{v}_E)\cdot\vec{\nabla}(v_{\parallel 0}\hat{b}) + \frac{d\vec{v}_E}{dt}\right)\right) + \mu\frac{\vec{B}\times\vec{\nabla}B}{qB^2}$$

$$= \frac{mv_{\parallel 0}^2}{qB^2}(\vec{B}\times\vec{\kappa}) + \mu\frac{\vec{B}\times\vec{\nabla}B}{qB^2} + \underline{\frac{m}{qB^2}\frac{d\vec{E}_\perp}{dt}}$$

$$+ \frac{mv_{\parallel 0}}{qB^2}\vec{B}\times\left(\frac{d}{dt}+\vec{v}_E\cdot\vec{\nabla}\right)\hat{b} - \underline{\frac{m\vec{E}_\perp}{qB^3}\frac{d\vec{B}}{dt}}$$

(x)

$$\frac{m}{qB^2}\vec{B}\times\frac{d}{dt}\left(\frac{\vec{E}\times\vec{B}}{B^2}\right) = \frac{m}{qB^2}\vec{B}\times\left(\frac{(\dot{\vec{E}}_\perp\times\vec{B})+(\vec{E}_\perp\times\dot{\vec{B}})}{B^2} - \frac{2}{B^3}(\vec{E}\times\vec{B})\right)$$

$$= \frac{m}{qB^2}\dot{\vec{E}}_\perp + \frac{m\vec{E}_\perp}{qB^2}\frac{(\vec{B}\cdot\dot{\vec{B}})}{B^2} - \frac{m\vec{E}}{qB^2}\frac{2}{B^{\cancel{3}}}B^{\cancel{2}}$$