Lecture 2 : Single Particle Motion

p.2 Lorentz force

$$m\frac{d\vec{v}}{dt} = q\left(\vec{E}(\vec{x},t) + \vec{v}\times\vec{B}(\vec{x},t)\right)$$

p.3,4 Gyro-motion

- $\vec{E}=0,\ \vec{B}=B\hat{b}=\text{const.}$ Break down $\vec{v}=v_{\parallel}\hat{b}+\vec{v_\perp}$

(i) $m\frac{d\vec{v}}{dt}\cdot\vec{B} = q(\vec{E}+\vec{v}\times\vec{B})\cdot\vec{B} = 0 \quad \therefore v_{\parallel}=\text{const}$

(ii) $m\frac{d\vec{v}}{dt}\times\vec{B} = q(\vec{E}+\vec{v}\times\vec{B})\times\vec{B}$

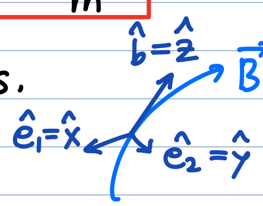
$$m\frac{d}{dt}(\underline{\vec{v}\times\vec{B}}) = q(\vec{v}\times\vec{B})\times\vec{B} = q\vec{B}\times(\vec{B}\times\vec{v}) = q\left[\vec{B}(\vec{B}\cdot\vec{v})-\vec{v}B^2\right]$$

$$\left(\vec{v}\times\vec{B} = \frac{m}{q}\dot{\vec{v}}\right)$$

$$\frac{m^2}{q}\ddot{\vec{v}} = q\left[\vec{B}(\vec{B}\cdot\vec{v})-\vec{v}B^2\right] \rightarrow \ddot{\vec{v_\perp}} + \frac{q^2B^2}{m^2}\vec{v_\perp} = 0$$

$$\rightarrow \ddot{\vec{v_\perp}} + \omega_c^2\vec{v_\perp} = 0 \quad \text{where } \boxed{\omega_c = \frac{|q|B}{m}} \rightarrow \vec{v_\perp} = \vec{v_{\perp 0}}e^{\pm i\omega_c t} \equiv \vec{v}_{gyro}$$

Consider local orthogonal coordinates.



$v_x = v_\perp\cos(\omega_c t)$

$v_y = \mp v_\perp\sin(\omega_c t)$

$$\Rightarrow \vec{v_\perp} = v_\perp\left(\hat{e}_1\cos(\omega_c t) \mp \hat{e}_2\sin(\omega_c t)\right)$$

$$\vec{x_\perp} = \left(\frac{v_\perp}{\omega_c}\right)\left(\hat{e}_1\sin(\omega_c t) \pm \hat{e}_2\cos(\omega_c t)\right)$$

적분

$$\boxed{\rho \equiv \frac{v_\perp}{\omega_c} = \frac{mv_\perp}{|q|B}}$$

gyro (Larmor) radius,

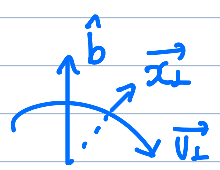
$$\left(\begin{aligned} &= \pm\frac{1}{\omega_c}\hat{b}\times\vec{v_\perp} \\ &= \frac{1}{\omega_c}\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{b} \\ 0 & 0 & 1 \\ v_\perp\cos\omega_c t & \mp v_\perp\sin\omega_c t & 0 \end{vmatrix} \end{aligned} \right)$$

참고

* rotation direction is always in a way to reduce magnetic field.

: plasma is <u>diamagnetic</u>

$$\vec{x_\perp} = \pm\frac{1}{\omega_c}\hat{b}\times\vec{v_\perp}$$



p.5 Diamagnetism and Feynmann's paradox

Lorentz force does not change particle energy : $\vec F\cdot\vec v = \frac{d}{dt}\left(\frac{1}{2}mv^2\right) = q(\vec v\times\vec B)\cdot\vec v = 0$

For this reason, Feynman claims that

“The diamagnetism or paramagnetism is a purely quantum mechanical effect.”

Since otherwise magnetic field energy will be changed without changing particle energy.

- Contradictory to energy conservation.

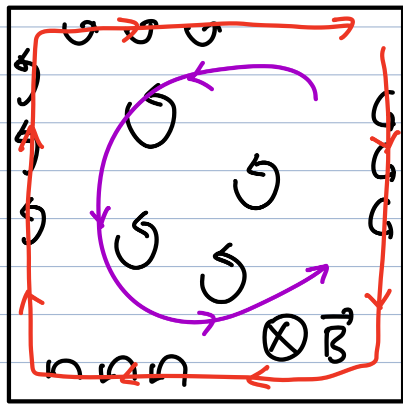
Bohr argues: Plasma needs wall for thermodynamic equilibrium.
Then, diamagnetism goes away.

J.-K. Park argues: Eddy currents on ideal wall -> compensate diamagnetism
However, in a reality, we have resistive wall -> wall current dissipation.

Lorentz force → E 변화불가.

But diamagnetism → B 에너지 ↓

사라진 에너지는 어디로 가는가?



벽에서 반사되는 것이 diamagnetism을 cancel 시킨다는 Bohr의 주장

p.6 $\vec E\times\vec B$ drift

$\vec E$ = const. $\vec B = B\hat b$ = const. Break down $\vec v = v_\parallel\hat b + \vec v_\perp$

(i) $m\dot{\vec v}\cdot\vec B = q(\vec E+\vec v\times\vec B)\cdot\vec B$

$m\dot v_\parallel B = qE_\parallel B$ $\quad\therefore \dot v_\parallel = \frac{q}{m}E_\parallel \quad\rightarrow v_\parallel = \frac{q}{m}E_\parallel t$

(ii) $m\dot{\vec v}\times\vec B = q(\vec E+\vec v\times\vec B)\times\vec B$

$m\frac{d}{dt}(\vec v\times\vec B) = m\cdot\frac{m}{q}\ddot{\vec v}_\perp = q(\vec E+\vec v\times\vec B)\times\vec B$

$\frac{m^2}{q}\frac{d^2\vec v_\perp}{dt^2} = q\vec E\times\vec B + \vec B(\vec B\cdot\vec v) - \vec v B^2 \rightarrow \frac{m}{q}\frac{d^2\vec v_\perp}{dt^2} = q\vec E\times\vec B - \vec v_\perp B^2$

$\therefore \frac{m}{q}\frac{d^2\vec v_\perp^{\,2}}{dt^2} - \vec v_\perp B^2 = -q\vec E\times\vec B \qquad \therefore \boxed{\vec v_E = \frac{\vec E\times\vec B}{B^2}}$

particular solution

p.8 Drift by arbitrary force

$\vec{E} = \vec{F}/q \;.\; \rightarrow \; \vec{V}_F = \dfrac{\vec{F}\times\vec{B}}{qB^2}$ (when $\vec{F}, \vec{B}$ is constant)

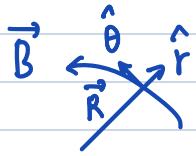
$\vec{F}$의 variation scale이 $\left(\begin{array}{l}\text{gyro-radius} \\ \text{gyro-frequency}\end{array}\right)$ 보다 $\left(\begin{array}{l}\text{크고} \\ \text{작으면}\end{array}\right)$ 항상 성립

$\vec{F} = m\vec{g} \;\rightarrow\; \vec{V}_g = \dfrac{m\vec{g}\times\vec{B}}{qB^2}$

p.9 Curvature drift

$\vec{F}_c = \dfrac{m v_{\parallel}^2}{R^2}\vec{R}\;,\quad \vec{V}_{cf} = \dfrac{m v_{\parallel}^2}{qB^2}\dfrac{\vec{R}\times\vec{B}}{R^2} = \dfrac{m v_{\parallel}^2}{qB^2}\dfrac{\vec{R}\times\vec{B}}{R^2}$

for general curvature, $\vec{F} = -m v_{\parallel}^2 \vec{\kappa}$ $\quad(\vec{\kappa} = (\hat{b}\cdot\vec{\nabla})\hat{b})$



$\vec{\kappa} = (\hat{b}\cdot\vec{\nabla})\hat{b} = \dfrac{d\hat{\theta}}{R\,d\theta} = \dfrac{-\hat{r}}{R} = -\dfrac{\vec{R}}{R^2}$

$\therefore$ Drift by curvature : $\vec{V}_R = \dfrac{2W_{\parallel}}{qB^2}(\vec{B}\times\vec{\kappa}) = \dfrac{2W_{\parallel}}{qB^2}(\vec{B}\times(\hat{b}\cdot\vec{\nabla})\hat{b})$

p.10 Mirror force

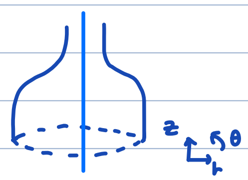
If $B = |\vec{B}|$ is varied along field line, $\vec{\nabla}B = \dfrac{dB_z}{dz}\hat{z}$

then, $\vec{\nabla}\cdot\vec{B} = 0 \rightarrow \dfrac{1}{r}\dfrac{d}{dr}(rB_r) + \dfrac{dB_z}{dz} = 0 \rightarrow B_r = -\dfrac{1}{2}r\dfrac{dB_z}{dz}$

$\therefore \rightarrow B_r \simeq -\dfrac{1}{2}\rho\dfrac{dB_z}{dz}$ (for a gyrating particle along $\vec{B}$)

* Inhomogeneous $\vec{B}$

$\vec{B} = B(z)\hat{z}$



assume $B_\theta = 0$ due to symmetry.

$\nabla\cdot\vec{B} = 0$

$\dfrac{1}{r}\dfrac{d}{dr}(rB_r) + \dfrac{dB_z}{dz} = 0$

$B_r = -\dfrac{1}{2}r\left(\dfrac{dB_z}{dz}\right)$

$\vec{F}_{\parallel} = -q v_\theta B_r \hat{z}$: mirror force*

$= \pm q v_\perp\left(-\dfrac{1}{2}r\dfrac{dB}{dz}\right)\hat{z}$

$v_\perp = -v_\theta$ (ion, e^- 생각!)

$= \mp\dfrac{1}{2}q r_L v_\perp\left(\dfrac{dB}{dz}\right) = \mp\dfrac{1}{2}q\dfrac{m v_\perp^2}{|q|B}\left(\dfrac{dB}{dz}\right)$

$= -\dfrac{\frac{1}{2}m v_\perp^2}{B}\left(\dfrac{dB}{dz}\right) \approx -\mu\vec{\nabla}_{\parallel}B$

$\boxed{\therefore \vec{F} = -\mu\vec{\nabla}B}$

Define $\dfrac{\frac{1}{2}m v_\perp^2}{B} \equiv \mu$: magnetic moment

$$\vec{V}_{\nabla B} = \frac{\mu \vec{B} \times \vec{\nabla} B}{qB^2} = \frac{W_\perp}{qB^2} (\hat{b} \times \vec{\nabla} B)$$

H.W : $(\hat{b} \times \vec{\nabla} B = (\hat{b} \cdot \vec{\nabla}) \vec{B}) \rightarrow \vec{V}_c + \vec{V}_{\nabla B} = \frac{2W_\parallel + W_\perp}{qB^2} (\hat{b} \times \vec{\nabla} B)$

$\vec{V}_c \parallel \vec{V}_{\nabla B}$ 증명

p.12 Rotational Transform for toroidal plasma

Curvature + gradB drift에 의해 이온-전자의 charge separation이 발생한다. 이것이 전기장을 유도하고, 유도된 전기장에 의한 ExB drift가 발생하여, 결국 하전입자들은 모두 R방향으로 빠져나간다.

이를 방지하기 위해 자기장을 꼬아주는 Rotational Transform의 개념이 등장한다. 이는 inboard와 outboard를 번갈아가며 왔다갔다 함으로써, drift의 net effect를 상쇄시켜주는 역할을 한다.

실제 핵융합로는 ion-drift가 아래쪽 방향을 향하도록 하여, divertor pumping이 잘되도록 하는 방향으로 자기장 방향을 설정한다. Upper divertor에서 gas-pumping을 한다면, 자기장 방향을 반대방향으로 조절하기도 한다 (2025 KSTAR 운전 시, 텅스텐 불순물 때문에 의도적으로 B 방향을 반대방향으로 하여 운전하기도 하였다.

p.13 Time-varying electric field

$$\vec{E}_\perp = \vec{E}_\perp(\vec{r}, t)$$

$$m\dot{\vec{V}} = q\vec{E} + q\vec{V} \times \vec{B}$$

$$m\ddot{\vec{V}}_\perp = q\dot{\vec{E}}_\perp + \frac{q^2}{m}(\vec{E} + \vec{V}_\perp \times \vec{B}) \times \vec{B}$$

$$\ddot{\vec{V}}_\perp = -W_c^2 \vec{V}_\perp + \frac{q^2}{m^2} \vec{E} \times \vec{B} + \left(\frac{q}{m}\right) \dot{\vec{E}}_\perp$$

i) homogeneous part : gyro-motion

ii) particular part : $\vec{V}_\perp = \frac{\vec{E} \times \vec{B}}{B^2} + \underline{\frac{m}{qB^2} \frac{d\vec{E}_\perp}{dt}}$

$$\boxed{\vec{V}_{pol} = \frac{m}{qB^2} \frac{d\vec{E}_\perp}{dt}}$$

Polarization drift

p.14 Polarization currents

· Polarization current in plasma

$$\vec{j}_{pol} = \dot{\vec{P}}_\perp = \sum_s n_s q_s \vec{V}_{pol,s} = \frac{\sum_s m_s n_s}{B^2}\dot{\vec{E}}_\perp = \frac{\rho_m}{B^2}\dot{\vec{E}}_\perp$$

· Displacement current due to this polarization, in $\perp$ direction,

$$\vec{j}_d = \epsilon_0 \dot{\vec{E}}_\perp + \dot{\vec{P}}_\perp = \left(\epsilon_0 + \frac{\rho_m}{B^2}\right)\dot{\vec{E}}_\perp = \epsilon_\perp \dot{\vec{E}}_\perp$$

· Perpendicular dielectric constant in plasma

$$\boxed{\epsilon_\perp = \epsilon_0 + \rho_m / B^2}$$

($\epsilon_\perp \gg \epsilon_0$ by a factor of $10^3 \sim 10^4$)

$\Rightarrow$ plasma is highly anisotropic.

p.15 Charge-separating drift and consequential force

Suppose an arbitrary force $\vec{F}_\perp = m\vec{a}_\perp$, $\vec{V}_F = \frac{\vec{F} \times \vec{B}}{qB^2}$ ← charge separation

$\vec{V}_F$ will separate ion, and electron, generating $\vec{j}$ and charge accumulation ρ by

$$\frac{d\rho}{dt} + \vec{\nabla} \cdot \vec{j} = 0 \quad \cdots (1)$$

charge will increase the electric field by

$$\epsilon_\perp \vec{\nabla} \cdot \vec{E}_\perp = \rho \quad \cdots (2)$$

If $\epsilon_\perp = \text{const}$, $(1) + (2)$

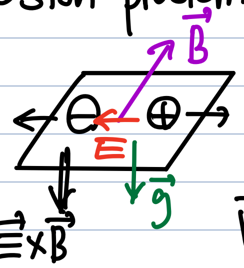
$$\frac{d}{dt}(\epsilon_\perp \vec{\nabla} \cdot \vec{E}_\perp) + \vec{\nabla} \cdot \vec{j} = 0 \Rightarrow \dot{\vec{E}}_\perp = -\vec{j}/\epsilon_\perp$$

$$\dot{\vec{E}}_\perp = -\frac{\vec{j}}{\epsilon_\perp} = -\frac{\sum_s n_s q_s \vec{V}_{F,s}}{\epsilon_\perp} = -\sum_s \frac{n_s q_s m_s \vec{a}_\perp \times \vec{B}}{qB^2 \epsilon_\perp} = -\left(\frac{\rho_m / B^2}{\epsilon_0 + \rho_m / B^2}\right)\vec{a}_\perp \times \vec{B}$$

$$\left(\epsilon_\perp = \epsilon_0 + \rho_m / B^2 = \epsilon_0 + \sum_s m_s n_s / B^2\right)$$

$$\vec{a}_E = \frac{\dot{\vec{E}}_\perp \times \vec{B}}{B^2} = -\left(\frac{\rho_m / B^2}{\epsilon_0 + \rho_m / B^2}\right)\frac{(\vec{a}_\perp \times \vec{B}) \times \vec{B}}{B^2} = \left(\frac{\rho_m / B^2}{\epsilon_0 + \rho_m / B^2}\right)\vec{a}_\perp \simeq \vec{a}_\perp$$

*Goldston problem



$\dot{\vec{V}}_E \sim \dot{\vec{g}}$

: 결국 applied force 방향, 가속도로 하전입자는 운동한다.

quasi-neutral + charge separation에 따라