고전역학 1.2 Mechanics of constrained systems.

① Holonomic constraints (제약조건이 위치, 시간에만 의존)

$$\boxed{\begin{array}{l} (1)\ f(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n, t) = 0 \\ (2)\ \displaystyle\sum_{\alpha=1}^{nd} g_\alpha(\mathbf{r})\,d\mathbf{r} = 0 \\ \left(\text{when } \dfrac{\partial h g_a}{\partial r_b} = \dfrac{\partial h g_b}{\partial r_a}\right) \end{array}}$$

증명1 $f=0 \Rightarrow \delta f = 0 \Rightarrow \displaystyle\sum_{\alpha=1}^{nd} \frac{\partial f}{\partial r_a} dr_a = 0 \Rightarrow \sum_{\alpha=1}^{nd} h g_a(\mathbf{r})\,d\mathbf{r} = 0$

증명2

증명2 : $\left(h g_a = \dfrac{\partial f}{\partial r_a} \right)$

$$\oint_c \sum_{\alpha=1}^{nd} h g_a\, dr_a = \iint_S \frac{\partial (h g_a)}{\partial r_b} - \frac{\partial (h g_b)}{\partial r_a}\, dr_a dr_b = 0$$

② D'Alembert's principle

holonomic constraints : $f_\alpha(\mathbf{r}, t) = 0 \quad \alpha = 1, \cdots, c$

- constraint force on a particle i : $\boxed{\mathbf{F}_{i,\alpha}^{(c)} = -\dfrac{\partial V_\alpha}{\partial \mathbf{r}_i} = -\lambda_\alpha \dfrac{\partial f}{\partial \mathbf{r}_i}}$

  (where constraint $\Leftrightarrow$ Equi-potential condition : $V_\alpha(\mathbf{r}, t) = \lambda_\alpha(t) f_\alpha(\mathbf{r}, t) = 0$

- virtual work done by constraint force : (while $f_\alpha = 0$) $= 0$

$$\sum_{i=1}^{n} \mathbf{F}_{i\alpha}^{(c)} \cdot \delta \mathbf{r}_i = -\lambda_\alpha \sum_{i=1}^{n} \frac{\partial f_\alpha}{\partial \mathbf{r}_i} \cdot \delta \mathbf{r}_i = -\lambda_\alpha \delta f_\alpha = 0$$

- D'Alembert's principle

알짜힘 · $\mathbf{F}_i = m\ddot{\mathbf{r}}_i$

합력 · $\mathbf{F}_i = \mathbf{F}_i^{(a)} + \displaystyle\sum_{\alpha=1}^{c} \mathbf{F}_{i,\alpha}^{(c)}$

가상일 · $\displaystyle\sum_{i=1}^{n} \mathbf{F}_{i,\alpha}^{(c)} \cdot \delta \mathbf{r}_i = 0$

$\Rightarrow \displaystyle\sum_{i=1}^{n} \sum_{\alpha=1}^{c} \mathbf{F}_{i,\alpha}^{(c)} \cdot \delta \mathbf{r}_i = \sum_{i=1}^{n} \left(\mathbf{F}_i - \mathbf{F}_i^{(a)} \right) \cdot \delta \mathbf{r}_i = 0$

$$\boxed{\therefore \sum_{i=1}^{n} \left(\mathbf{F}_i^{(a)} - m\ddot{\mathbf{r}}_i \right) = 0}$$

$\mathbf{F}_i^{(a)}$가 주어지면, 위식이 0이 되는 path를 물체는 따른다.

③ Lagrange's equation

n 입자, d 차원, C 제약 조건 → 일반화 좌표 : $q_1, \cdots, q_{nd-c}$ (nd−c 자유도)

$$\underline{r_i} = \underline{r_i}(\underline{q}, t) \rightarrow \dot{\underline{r_i}} = \sum_{j=1}^{nd-c} \frac{\partial \underline{r_i}}{\partial q_j} \dot{q}_j + \frac{\partial \underline{r_i}}{\partial t} \rightarrow \boxed{\frac{\partial \dot{r_i}}{\partial \dot{q}_j} = \frac{\partial r}{\partial q_j}}$$

- Derivation of Lagrange's equation. (from D'Alembert's principle : $\sum_{i=1}^{n} (\underbrace{\underline{F_i}^{(a)}}_{(1)} - \underbrace{m\ddot{\underline{r_i}}}_{(2)}) \cdot \delta \underline{r_i} = 0$)

(1) Force part

$$\sum_{i=1}^{n} \underline{F_i}^{(a)} \cdot \delta \underline{r_i} = \sum_{i=1}^{n} \sum_{j=1}^{nd-c} \underline{F_i}^{(a)} \frac{\partial \underline{r_i}}{\partial q_j} \delta q_j = \boxed{\sum_{j=1}^{nd-c} Q_j \, \delta q_j}$$

$$\left(Q_j \equiv \sum_{i=1}^{n} \underline{F_i}^{(a)} \frac{\partial \underline{r_i}}{\partial q_j} \right)$$

(2) Inertia part

$$\sum_{i=1}^{n} m_i \ddot{\underline{r_i}} \cdot \delta \underline{r_i} = \sum_{i=1}^{n} \sum_{j=1}^{nd-c} m_i \ddot{\underline{r_i}} \frac{\partial \underline{r_i}}{\partial q_j} \cdot \delta q_j = \sum_{i=1}^{n} \sum_{j=1}^{nd-c} \left[\frac{d}{dt}\left(m_i \dot{\underline{r_i}} \frac{\partial \underline{r_i}}{\partial q_j}\right) - m_i \dot{\underline{r_i}} \frac{d}{dt}\frac{\partial \underline{r_i}}{\partial q_j} \right] \delta q_j$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{nd-c} \left[\frac{d}{dt}\left(m_i \dot{\underline{r_i}} \frac{\partial \dot{\underline{r_i}}}{\partial \dot{q}_j}\right) - m_i \dot{\underline{r_i}} \frac{\partial \dot{\underline{r_i}}}{\partial q_j} \right] \cdot \delta q_j = \boxed{\sum_{j=1}^{nd-c} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right]}$$

$$\left(T = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{\underline{r_i}}^2 \right)$$

$$(1)+(2) \rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad \text{for } j = 1, \cdots, nd-c}$$

- special case

(1) conservative $\quad \underline{F_i}^{(a)} = -\frac{\partial V}{\partial \underline{r_i}}$

$$Q_j = \sum_{i=1}^{n} \left(-\frac{\partial V}{\partial \underline{r_i}}\right) \frac{\partial \underline{r_i}}{\partial q_j} = -\frac{\partial V}{\partial q_j} \rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j}$$

Define $L \equiv T - V$, then $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$

(2) monogenic $\quad V = V(\underline{q}, \dot{\underline{q}}, t)$

$$Q_j = \sum_{i=1}^{n} \left(-\frac{\partial V}{\partial \underline{r_i}} + \frac{d}{dt}\frac{\partial V}{\partial \dot{\underline{r_i}}} \right) \frac{\partial \underline{r_i}}{\partial q_j} = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_j} \Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$

(3) Polygenic (마찰력처럼 applied force가 V로만 표현되지 않는 경우, $\boxed{\underline{F}_i^{(f)} = -\gamma_i \dot{\underline{r}}_i}$)

$$Q_j^{(ex)} = \sum_{i=1}^{n} F_i^{(f)} \frac{\partial r_i}{\partial q_j} = -\sum_{i=1}^{n} \gamma_i \dot{\underline{r}}_i \frac{\partial r_i}{\partial q_j} = -\sum_{i=1}^{n} \gamma_i \dot{\underline{r}}_i \frac{\partial \dot{r}_i}{\partial \dot{q}_j} = -\frac{\partial G}{\partial \dot{q}_j}$$

$$\left(G(\dot{\underline{r}}_1, \cdots \dot{\underline{r}}_n) \equiv \frac{1}{2}\sum_{i=1}^{n} \gamma_i \dot{\underline{r}}_i^2 \right)$$

$$\therefore \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = -\frac{\partial G}{\partial \dot{q}_j}}$$

(4) Extra holonomic constraints (undetermined problem)

◦ $\sum_{j=1}^{\tilde{n}} \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} - Q_j\right) \cdot \delta q_j = 0$ where $f_1(q_I) = \cdots = f_{\tilde{c}}(q_I) = 0$

(δq_j ($\tilde{n}-\tilde{c}$ 차원) $\rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} - Q_j$ ($\tilde{c}$ 차원))

◦ $\tilde{c}$개의 holonomic constraints : $\sum_{j=1}^{\tilde{n}} \frac{\partial f_\alpha}{\partial q_j} \delta q_j = 0$ ← basis 역할.

$$\Rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = \sum_{\alpha=1}^{\tilde{c}} \lambda_\alpha(t) \frac{\partial f_\alpha}{\partial q_j}}$$

in case of monogenic case, $Q_j = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_j}$,

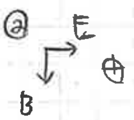
$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} - Q_j = \sum_{\alpha=1}^{\tilde{c}} \lambda_\alpha(t) \frac{\partial f_\alpha}{\partial q_j}}$$

(예시) ① 

$T = \frac{1}{2} m \dot{\underline{r}}^2$

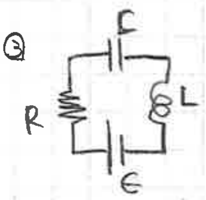
$V = -mgl\cos\theta$

$G = \frac{1}{2}\gamma \dot{\underline{r}}^2 = \frac{1}{2}\gamma l^2 \dot{\theta}^2$

② 

$T = \frac{1}{2} m \dot{\underline{r}}^2$

$V = q\phi - q\underline{A}\cdot\underline{r}$

③ 

$T = \frac{1}{2} L \dot{q}^2$

$V = \frac{1}{2C} q^2$

$G = \frac{1}{2} R \dot{q}^2$