Fusion Plasma Theory 2

Lecture 5 : Fokker - Planck - Landau

① Fokker - Planck collisional form.

$\phi(\vec{v}, \Delta\vec{v})$: probability to have $\vec{v} \to \vec{v} + \Delta\vec{v}$ by a collision. $\left(\int \phi(\vec{v}, \Delta\vec{v}) d\Delta\vec{v} = 1 \right)$

Distribution function change by a collision :

$$f(\vec{v}, t) = \int \phi(\vec{v} - \Delta\vec{v}, \Delta\vec{v}) \, f(\vec{v} - \Delta\vec{v}, t - \Delta t) \, d\Delta\vec{v}$$

(2) 충돌 확률 분포 (1) 이전 state (3) 모두 적분.

Expanding up to 2nd order

$$\phi(\vec{v} - \Delta\vec{v}, \Delta\vec{v}) = \phi(\vec{v}, \Delta\vec{v}) - \Delta\vec{v} \cdot \frac{\partial}{\partial \vec{v}} \phi(\vec{v}, \Delta\vec{v}) + \frac{1}{2} \Delta\vec{v}\Delta\vec{v} : \frac{\partial^2}{\partial\vec{v}\partial\vec{v}} \phi(\vec{v}, \Delta\vec{v})$$

$$f(\vec{v} - \Delta\vec{v}, t - \Delta t) = f(\vec{v}, t - \Delta t) - \Delta\vec{v} \cdot \frac{\partial}{\partial\vec{v}} f(\vec{v}, t - \Delta t) + \frac{1}{2} \Delta\vec{v}\Delta\vec{v} : \frac{\partial^2}{\partial\vec{v}\partial\vec{v}} f(\vec{v}, t - \Delta t)$$

$$\therefore f(\vec{v}, t) = f(\vec{v}, t - \Delta t) - \int \Delta\vec{v} \cdot \left(\frac{\partial f}{\partial\vec{v}} \phi + \frac{\partial\phi}{\partial\vec{v}} f \right) d\Delta\vec{v}$$
$$+ \frac{1}{2} \int \Delta\vec{v}\Delta\vec{v} : \left(\frac{\partial^2 f}{\partial\vec{v}\partial\vec{v}} \phi + 2 \frac{\partial f}{\partial\vec{v}} \frac{\partial\phi}{\partial\vec{v}} + \frac{\partial^2\phi}{\partial\vec{v}\partial\vec{v}} f \right) d\Delta\vec{v}$$

Distribution function change in time

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t} \right)_c = \frac{f(\vec{v}, t) - f(\vec{v}, t - \Delta t)}{\Delta t}$$

(Recall f is independent of $\Delta\vec{v}$)

$$= - \frac{\partial}{\partial\vec{v}} \cdot \left(\frac{\int (\phi \Delta\vec{v}) d\Delta\vec{v}}{\Delta t} f \right) + \frac{1}{2} \frac{\partial^2}{\partial\vec{v}\partial\vec{v}} : \left(\frac{\int (\phi \Delta\vec{v}\Delta\vec{v}) d\Delta\vec{v}}{\Delta t} f \right)$$

$$\equiv \frac{\partial}{\partial\vec{v}} \cdot \left(\vec{V} f \right) + \frac{\partial^2}{\partial\vec{v}\partial\vec{v}} : \left(\overleftrightarrow{D_v} f \right)$$

($\vec{V}$: dynamic friction , $\overleftrightarrow{D_v}$: velocity diffusion)

② Statistical collisional average

$$\vec{\nu} \equiv -\frac{\int(\phi\Delta\vec{v})\,d\Delta\vec{v}}{\Delta t} \equiv -\frac{\langle\Delta\vec{v}\rangle}{\Delta t}, \quad \overleftrightarrow{D}_v \equiv \frac{\int(\phi\Delta\vec{v}\Delta\vec{v})\,d\Delta\vec{v}}{2\Delta t} \equiv \frac{\langle\Delta\vec{v}\Delta\vec{v}\rangle}{2\Delta t}$$

* $\underline{\langle \mathbb{F}[\Delta\vec{v}]\rangle = \Delta t\int d\Omega\,\sigma(\theta,\phi)\int d\vec{v}'\,\mathbb{F}[\Delta\vec{v}]\,u f_b(\vec{v}')}$

($\Delta t\, d\Omega\, \sigma(\theta,\phi)\, u$: differential volume swept by incident particle
$d\vec{v}' f_b(\vec{v}')$: density of target species within $d\vec{v}'$.)

⇒ Incident particle 이 날아갈 때, target species 의 분포로 ensemble average 하는 operator.

Recall : $\Delta\vec{v} = \left(\frac{m_r}{m_a}\right)\Delta\vec{u}$, $\Delta\vec{u} = u\sin\theta\,\hat{n} - (1-\cos\theta)\vec{u}$.

③ Dynamic friction

azimuthal average 하면 $\hat{n}$은 상쇄 → 0

$$\vec{\nu} = -\frac{\langle\Delta\vec{v}\rangle}{\Delta t} = -\frac{m_r}{m_a}\int d\phi\int\sin\theta\, d\theta\int d\vec{v}'\,\frac{b_{90}^2}{4\sin^4(\theta/2)}\left(u\sin\theta\,\hat{n} - (1-\cos\theta)\vec{u}\right)u f_b(\vec{v}')$$

(*note $b_{90}^2 = \left(\frac{e_a e_b}{4\pi\epsilon_0 m_r u^2}\right)^2$, $\ln\Lambda = \frac{1}{2}\int d\theta\cot\frac{\theta}{2}$.)

$\sin\theta(1-\cos\theta)/4\sin^4\frac{\theta}{2} = \cot\frac{\theta}{2}$

$$\vec{\nu} = 2\pi\left(\frac{m_r}{m_a}\right)\left(\frac{e_a e_b}{4\pi\epsilon_0 m_r}\right)^2\int d\theta\cot\left(\frac{\theta}{2}\right)\int d\vec{v}'\,\frac{\vec{u}}{u^3}f_b(\vec{v}')$$

$$= 4\pi\left(\frac{m_r}{m_a}\right)\left(\frac{e_a e_b}{4\pi\epsilon_0 m_r}\right)^2\ln\Lambda\int d\vec{v}'\,\frac{\vec{u}}{u^3}f_b(\vec{v}')$$

Define $L^{ab} \equiv \left(\frac{e_a e_b}{m_a \epsilon_0}\right)^2 \ln\Lambda$, then

$$\boxed{\vec{\nu} = \frac{L^{ab}}{4\pi}\left(1+\frac{m_a}{m_b}\right)\int d\vec{v}'\,\frac{\vec{u}}{u^3}f_b(\vec{v}')}$$

④ Velocity Diffusion

For velocity diffusion, consider first

$$\int d\phi \, \Delta\vec{u}\Delta\vec{u} = u^2 \sin^2\theta \int \hat{n}\hat{n}\, d\phi + 2\pi(1-\cos\theta)^2 \vec{u}\vec{u}$$

$$= u^2\sin^2\theta \int (\hat{x}\hat{x}\cos^2\phi + \hat{y}\hat{y}\sin^2\phi)\, d\phi + 2\pi(1-\cos\theta)^2 \vec{u}\vec{u}$$

$$= \pi u^2 \sin^2\theta\,(\hat{x}\hat{x}+\hat{y}\hat{y}) + 2\pi(1-\cos\theta)^2\vec{u}\vec{u}$$

Note that one can write

$$\hat{x}\hat{x} + \hat{y}\hat{y} = \overleftrightarrow{I} - \hat{z}\hat{z} = \frac{u^2\overleftrightarrow{I} - \vec{u}\vec{u}}{u^2}$$

giving

$$\int d\phi\,\Delta\vec{u}\Delta\vec{u} = \underbrace{\pi \sin^2\theta\,(u^2\overleftrightarrow{I} - \vec{u}\vec{u})}_{\text{dominant : } \overleftrightarrow{D_v}} + \underbrace{2\pi(1-\cos\theta)^2\vec{u}\vec{u}}_{\text{correction : } \overleftrightarrow{D_{vc}}}$$

(in small angle θ, $\overleftrightarrow{D_{vc}} \sim \theta^4$, but $\overleftrightarrow{D_v} \sim \theta^2$)

The conventional velocity diffusion can then be written:

$$\overleftrightarrow{D_v} = \frac{\langle \Delta\vec{v}\Delta\vec{v}\rangle}{2\Delta t} = \frac{\pi}{2}\left(\frac{m_r}{m_a}\right)^2 \int \sin\theta\, d\theta \int d\vec{v}' \frac{b_{90}^2}{4\sin^4(\theta/2)} \sin^2\theta\,(u^2\overleftrightarrow{I} - \vec{u}\vec{u})\, u f_b(\vec{v}')$$

(Define the velocity tensor $\overleftrightarrow{U} = U$ for simplification,

$$U \equiv \frac{u^2\overleftrightarrow{I} - \vec{u}\vec{u}}{u^3}, \quad U_{ij} = \frac{u^2\delta_{ij} - u_i u_j}{u^3} \;)$$

$$\therefore \overleftrightarrow{D_v} = \frac{\pi}{2}\left(\frac{m_r}{m_a}\right)^2 \left(\frac{e_a e_b}{4\pi\epsilon_0 m_r}\right)^2 \int d\theta \frac{\sin^3\theta}{4\sin^4(\theta/2)} \int d\vec{v}'\, U f_b(\vec{v}')$$

$$= \pi \left(\frac{e_a e_b}{4\pi\epsilon_0 m_a}\right)^2 \int d\theta \left(\cot\frac{\theta}{2} - \frac{1}{2}\sin\theta\right) \int d\vec{v}'\, U f_b(\vec{v}')$$

$$= \frac{1}{8\pi}\left(\frac{e_a e_b}{m_a \epsilon_0}\right)^2 \left(\ln\Lambda - \overset{\text{why?}}{\frac{1}{2}}\right) \int d\vec{v}'\, U f_b(\vec{v}')$$

$$\frac{\sin^3\theta}{4\sin^4(\theta/2)} = \frac{\cancel{8}^{\,2}\sin^{\cancel{3}}\theta/2\,\cos^3\theta/2}{\cancel{4}\sin^4\theta/2}$$

$$= 2\,\frac{\cos\theta/2}{\sin\theta/2}\cdot(1-\cos^2\theta/2)$$

$$= 2\left[\cot\theta/2 - \frac{\cos\theta/2 \cdot \sin^{\cancel{2}}\theta/2}{\cancel{\sin\theta/2}}\right]$$

$$= 2\cot\theta/2 - \sin\theta .$$

The correction part of the velocity diffusion can be written:

$$\overleftrightarrow{D}_{vc} = \pi\left(\frac{m_r}{m_a}\right)^2\int \sin\theta\, d\theta \int d\vec{v}'\,\frac{b_{90}^2}{4\sin^4(\theta/2)}(1-\cos\theta)^2(\vec{u}\vec{u})\,u f_b(\vec{v}')$$

$$= \pi\left(\frac{e_a e_b}{4\pi\varepsilon_0 m_a}\right)^2\int d\theta\,\sin\theta\,\frac{\cancel{(2\sin^2(\theta/2))^2}}{\cancel{4\sin^4(\theta/2)}}\cdot\int d\vec{v}'\,\frac{\vec{u}\vec{u}}{u^3}f_b(\vec{v}')$$

$$= \frac{1}{8\pi}\left(\frac{e_a e_b}{m_a\varepsilon_0}\right)^2\int d\vec{v}'\,\frac{\vec{u}\vec{u}}{u^3}f_b(\vec{v}')$$

So the velocity diffusion in total

주로 생략

$$\overleftrightarrow{D}_v + \overleftrightarrow{D}_{vc} = \frac{1}{8\pi}\left(\frac{e_a e_b}{m_a\varepsilon_0}\right)^2\left[\left(\ln\Lambda - \cancel{\tfrac{1}{2}}\right)\int d\vec{v}'\,\overleftrightarrow{U} f_b(\vec{v}') + \cancel{\int d\vec{v}'\,\frac{\vec{u}\vec{u}}{u^3}f_b(\vec{v}')}\right]$$

$$\Rightarrow \boxed{\overleftrightarrow{D}_v = \frac{L^{ab}}{8\pi}\int d\vec{v}'\,\overleftrightarrow{U} f_b(\vec{v}')} \quad \left(L^{ab} \equiv \left(\frac{e_a e_b}{m_a\varepsilon_0}\right)^2\ln\Lambda,\quad \overleftrightarrow{U} \equiv \frac{u^2\overleftrightarrow{I} - \vec{u}\vec{u}}{u^3}\right)$$

⑤ Properties of the velocity gradient operations.

$\vec{u} = \vec{v} - \vec{v}'$

1) $$\vec{\nabla}_v v = \vec{\nabla}_v\left(v_x^2+v_y^2+v_z^2\right)^{1/2} = \frac{(v_x, v_y, v_z)}{(v_x^2+v_y^2+v_z^2)^{1/2}} = \frac{\vec{v}}{v}$$

$$\Rightarrow \vec{\nabla}_v v = \frac{\vec{v}}{v},\quad \vec{\nabla}_v u = -\vec{\nabla}_{v'} u = \frac{\vec{u}}{u}$$

2) $$\vec{\nabla}_v\left(\frac{\vec{u}}{u}\right) = \frac{d}{d\vec{v}}\left(\frac{\vec{u}}{u}\right) = \frac{1}{u}\frac{d\vec{u}}{d\vec{v}} - \frac{1}{u^2}\left(\frac{du}{d\vec{v}}\right)\vec{u} = \frac{1}{u}\overleftrightarrow{I} - \frac{\vec{u}\vec{u}}{u^3} = \overleftrightarrow{U}$$

$$\Rightarrow \vec{\nabla}_v\left(\frac{\vec{u}}{u}\right) = -\vec{\nabla}_{v'}\left(\frac{\vec{u}}{u}\right) = \overleftrightarrow{U}$$

3) $$\vec{\nabla}_v\vec{\nabla}_v u = \vec{\nabla}_v\left(\frac{\vec{u}}{u}\right) = \overleftrightarrow{U} \quad \Rightarrow \vec{\nabla}_v\vec{\nabla}_v u = \overleftrightarrow{U}$$

4) $$\frac{\partial}{\partial v_i}\left(\frac{1}{u}\delta_{ij} - \frac{u_i u_j}{u^3}\right) = -\frac{1}{u^2}\frac{\partial u}{\partial v_i}\delta_{ij} - \frac{\partial}{\partial v_i}\left(\frac{1}{u^3}\right)u_i u_j - \frac{\partial u_i}{\partial v_i}\frac{u_j}{u^3} - \frac{\partial u_j}{\partial v_i}\frac{u_i}{u^3}$$

From (1) $$= -\frac{u_i}{u^3}\delta_{ij} + \frac{3}{u^4}\frac{u_i}{u}u_i u_j - \frac{3u_j}{u^3} - \frac{u_i}{u^3}\delta_{ij} = -\frac{u_j}{u^3} + \frac{3}{u^3}u_j - \frac{3u_j}{u^3} - \frac{u_j}{u^3} = -\frac{2u_j}{u^3}$$

$$\Rightarrow \vec{\nabla}_v\cdot\overleftrightarrow{U} = -\vec{\nabla}_{v'}\cdot\overleftrightarrow{U} = -2\frac{\vec{u}}{u^3}$$

5) $\frac{\partial}{\partial v_i}\left(\frac{u_i u_j}{u^3}\right) = -\frac{u_j}{u^3} \Rightarrow \vec{\nabla}_v\cdot\left(\frac{\vec{u}\vec{u}}{u^3}\right) = -\vec{\nabla}_{v'}\cdot\left(\frac{\vec{u}\vec{u}}{u^3}\right) = -\frac{\vec{u}}{u^3}$

6) $\nabla_v^2 u = \vec{\nabla}_v\cdot\vec{\nabla}_v u = \vec{\nabla}_v\cdot\left(\frac{\vec{u}}{u}\right) = \frac{1}{u}\vec{\nabla}_v\cdot\vec{u} + \vec{\nabla}_v\left(\frac{1}{u}\right)\cdot\vec{u} = \frac{3}{u} - \frac{1}{u^2}\left(\frac{du}{d\vec{v}}\right)\cdot\vec{u}$

$= \frac{3}{u} - \frac{1}{u^2}\cdot\frac{\vec{u}}{u}\cdot\vec{u} = \frac{3}{u} - \frac{1}{u} = \frac{1}{u} \Rightarrow \nabla_v^2 u = \nabla_{v'}^2 u = \overleftrightarrow{U} : \overleftrightarrow{I} = \frac{2}{u}$

7) $\vec{u}\cdot\overleftrightarrow{U} = \overleftrightarrow{U}\cdot\vec{u} = \vec{0}$ ← $\overleftrightarrow{U}$ implies perpendicular diffusion w.r.t $\vec{u}$.

⑥ Einstein relation

$$\vec{\nu} = \frac{L^{ab}}{4\pi}\left(1+\frac{m_a}{m_b}\right)\int d\vec{v}'\,\frac{\vec{u}}{u^3}f_b(\vec{v}') \quad : \text{Dynamic friction}$$

$$\overleftrightarrow{D_v} = \frac{L^{ab}}{8\pi}\int d\vec{v}'\,\overleftrightarrow{U}f_b(\vec{v}') \quad : \text{Velocity diffusion}$$

$$\left(\vec{\nabla}_v\cdot\overleftrightarrow{U} = -\vec{\nabla}_{v'}\cdot\overleftrightarrow{U} = -2\frac{\vec{u}}{u^3}\right)$$

$$\Rightarrow \vec{\nu} = \frac{L^{ab}}{4\pi}\left(1+\frac{m_a}{m_b}\right)\int d\vec{v}'\left(-\frac{1}{2}\frac{\partial}{\partial\vec{v}}\cdot\overleftrightarrow{U}\right)f_b(\vec{v}')$$

$$= -\left(1+\frac{m_a}{m_b}\right)\frac{\partial}{\partial\vec{v}}\cdot\left(\frac{L^{ab}}{8\pi}\int d\vec{v}'\,\overleftrightarrow{U}f_b(\vec{v}')\right)$$

Giving the extended version of Einstein relation :

$$\boxed{\vec{\nu} = -\left(1+\frac{m_a}{m_b}\right)\vec{\nabla}_v\cdot\overleftrightarrow{D_v}}$$

← Divergence of $\overleftrightarrow{D_v}$ ⇒ Dynamic friction $\vec{\nu}$

⑰ Fokker - Planck - Landau form

$$\left(\frac{\delta f}{\delta t}\right)_c = \frac{\partial}{\partial \vec{v}} \cdot (\vec{\nu} f) + \frac{\partial^2}{\partial \vec{v} \partial \vec{v}} : (\overleftrightarrow{D}_v f) = \frac{\partial}{\partial \vec{v}} \cdot \left(\vec{\nu} f_a(\vec{v}) + \frac{\partial}{\partial \vec{v}} \cdot (\overleftrightarrow{D}_v f_a(\vec{v})) \right)$$

Inserting Einstein relation :

*note $\left\langle \begin{array}{l} f_a = f_a(\vec{v}) \\ f_b = f_b(\vec{v}') \end{array} \right\rangle$

$$\left(\frac{\delta f}{\delta t}\right)_c = \frac{\partial}{\partial \vec{v}} \cdot \left[-\left(1 + \frac{m_a}{m_b}\right)\left(\frac{\partial}{\partial \vec{v}} \cdot \overleftrightarrow{D}_v\right) f_a + \frac{\partial}{\partial \vec{v}} \cdot (\overleftrightarrow{D}_v f_a) \right]$$

$$= \frac{\partial}{\partial \vec{v}} \cdot \left[\overleftrightarrow{D}_v \cdot \frac{\partial}{\partial \vec{v}} f_a - \frac{m_a}{m_b}\left(\frac{\partial}{\partial \vec{v}} \cdot \overleftrightarrow{D}_v\right) f_a \right]$$

$$= \frac{L^{ab}}{8\pi} \frac{\partial}{\partial \vec{v}} \cdot \left[\int d\vec{v}' f_b \left(\overleftrightarrow{U} \cdot \frac{\partial}{\partial \vec{v}} f_a \right) - \frac{m_a}{m_b} \int d\vec{v}' \left(\frac{\partial}{\partial \vec{v}} \cdot \overleftrightarrow{U} \right) f_b f_a \right]$$

Using $\frac{\partial}{\partial \vec{v}} \cdot \overleftrightarrow{U} = - \frac{\partial}{\partial \vec{v}'} \cdot \overleftrightarrow{U}$ in the 2nd term, gives.

$$\left(\frac{\delta f}{\delta t}\right)_c = \frac{L^{ab}}{8\pi} \frac{\partial}{\partial \vec{v}} \cdot \left[\int d\vec{v}' f_b \left(\overleftrightarrow{U} \cdot \frac{\partial}{\partial \vec{v}} f_a \right) - \frac{m_a}{m_b} \int d\vec{v}' \left(- \frac{\partial}{\partial \vec{v}'} \cdot \overleftrightarrow{U} \right) f_b f_a \right]$$

$$= \frac{L^{ab}}{8\pi} \frac{\partial}{\partial \vec{v}} \cdot \int \overleftrightarrow{U} \cdot \left[f_b(\vec{v}') \frac{\partial f_a(\vec{v})}{\partial \vec{v}} - \frac{m_a}{m_b} f_a(\vec{v}) \frac{\partial f_b(\vec{v}')}{\partial \vec{v}'} \right] d\vec{v}'$$

Integration by parts.

Expression by Landau (1936) :

$$\boxed{\left(\frac{\delta f}{\delta t}\right)_c \equiv C_{ab}[f_a, f_b] = \frac{e_a^2 e_b^2 \ln \Lambda}{8\pi \epsilon_0^2 m_a} \frac{\partial}{\partial \vec{v}} \cdot \int \overleftrightarrow{U} \cdot \left[\frac{f_b(\vec{v}')}{m_a} \frac{\partial f_a(\vec{v})}{\partial \vec{v}} - \frac{f_a(\vec{v})}{m_b} \frac{\partial f_b(\vec{v}')}{\partial \vec{v}'} \right] d\vec{v}'}$$

☆

1) Particle conservation : $\int C_{ab}[f_a, f_b] \, d\vec{v} = 0$

2) Momentum conservation : $\int m_a \vec{v} C_{ab}[f_a, f_b] d\vec{v} + \int m_b \vec{v} C_{ba}[f_b, f_a] \, d\vec{v} = 0$

3) Energy conservation : $\frac{1}{2}\int m_a v^2 C_{ab}[f_a, f_b] \, d\vec{v} + \frac{1}{2}\int m_b v^2 C_{ba}[f_b, f_a] \, d\vec{v} = 0$

4) Invariance to Galilean transformation : $\vec{v} \to \vec{v} - \vec{V}$

5) In equal T, Landau integral vanishes by Maxwellian :

$$C_{ab}[f_{a0}, f_{b0}] = 0 \quad \text{if} \quad T_a = T_b$$

⑧ Landau integral with Rosenbluth potentials

One of difficulties in the full Coulomb collisional operator is the integral dependency on $\vec{u}$. In general, this can be avoided by solving differential equation rather than the integral, as suggested by M.N. Rosenbluth.

1) First Rosenbluth potential : f_b source $\rightarrow$ φ_b potential

$$\varphi_b(\vec{v}) \equiv -\frac{1}{4\pi}\int \frac{f_b(\vec{v}')}{u} d\vec{v}' \quad \leftrightarrow \quad \nabla_v^2 \varphi_b = f_b(\vec{v})$$

2) Second Rosenbluth potential : φ_b : source $\rightarrow$ ψ_b : potential

$$\psi_b(\vec{v}) \equiv -\frac{1}{8\pi}\int u f_b(\vec{v}') d\vec{v}' \quad \leftrightarrow \quad \left(\nabla_v^2 u = \frac{2}{u}\right) : \nabla_v^2 \psi_b = \varphi_b(\vec{v})$$

so that all techniques solving Poisson eqn can be adopted in principle.

Using $\vec{\nabla}_v\left(\frac{1}{u}\right) = -\frac{\vec{u}}{u^3}$,

$$\vec{\nu} = \frac{L^{ab}}{4\pi}\left(1+\frac{m_a}{m_b}\right)\int d\vec{v}' \frac{\vec{u}}{u^3} f_b(\vec{v}') = \left(1+\frac{m_a}{m_b}\right) L^{ab} \frac{d\varphi_b}{d\vec{v}}$$

Using $\vec{\nabla}_v \vec{\nabla}_v u = \overleftrightarrow{U}$,

$$\vec{\nabla}_v \cdot \overleftrightarrow{D}_v = -\left(\frac{m_b}{m_a+m_b}\right)\vec{\nu}$$

$$\overleftrightarrow{D} = \frac{L^{ab}}{8\pi}\int d\vec{v}' \overleftrightarrow{U} f_b(\vec{v}') = -L^{ab}\frac{d^2\psi_b}{d\vec{v}\,d\vec{v}}$$

$$-\left(\frac{m_a+m_b}{m_b}\right)\vec{\nabla}_v \cdot \overleftrightarrow{D}_v$$

combining the two and noting the Einstein relation. $\left(\vec{\nu} = -\left(1+\frac{m_a}{m_b}\right)\vec{\nabla}_v \cdot \overleftrightarrow{D}_v\right)$

Landau integral can be expressed by

$$C_{ab}[f_a, f_b] = \frac{d}{d\vec{v}}\cdot\left(\vec{\nu} f_a(\vec{v}) + \frac{d}{d\vec{v}}\cdot\left(\overleftrightarrow{D}_v f_a\right)\right) = \frac{d}{d\vec{v}}\cdot\left(\vec{\nu} f_a(\vec{v}) + \overleftrightarrow{D}_v \cdot \frac{df_a}{d\vec{v}} + \left(\frac{d}{d\vec{v}}\cdot\overleftrightarrow{D}_v\right) f_a\right)$$

$$= \star$$

$$\boxed{C_{ab}[f_a, f_b] = L^{ab}\frac{d}{d\vec{v}}\cdot\left(\frac{m_a}{m_b}\frac{d\varphi_b}{d\vec{v}} f_a - \frac{d^2\psi_b}{d\vec{v}\,d\vec{v}}\cdot\frac{df_a}{d\vec{v}}\right)}$$

⑨ Rosenbluth potentials with Isotropic background distribution

Rosenbluth potentials become particularly simple when background distribution is isotropic, i.e. $f_b(\vec{v}) = f_b(v)$. Then, clearly $\varphi_b(\vec{v}) = \varphi_b(v)$, $\psi_b(\vec{v}) = \psi_b(v)$

1) $$\frac{d\varphi_b}{d\vec{v}} = \frac{d\varphi_b}{dv}\frac{dv}{d\vec{v}} = \frac{\vec{v}}{v}\varphi_b' \quad \left(' = \frac{d}{dv}\right)$$

2) $$\frac{d^2\psi_b}{d\vec{v}d\vec{v}} = \frac{d}{d\vec{v}}\left(\frac{\vec{v}}{v}\psi_b'\right) = \frac{d}{d\vec{v}}\left(\frac{\vec{v}}{v}\right)\psi_b' + \frac{\vec{v}\vec{v}}{v^2}\psi_b'' = \overleftrightarrow{V}\psi_b' + \frac{\vec{v}\vec{v}}{v^2}\psi_b'' \quad \left(\overleftrightarrow{V} = \frac{v^2\overleftrightarrow{I} - \vec{v}\vec{v}}{v^3}\right)$$

⑩ Lorentz operator

$$C_{ab}[f_a, f_b] = L^{ab}\frac{d}{d\vec{v}}\cdot\left(\frac{m_a}{m_b}\frac{d\varphi_b}{d\vec{v}}f_a - \frac{d^2\psi_b}{d\vec{v}d\vec{v}}\cdot\frac{df_a}{d\vec{v}}\right)$$

$$= \underline{L^{ab}\frac{d}{d\vec{v}}}\cdot\left[\frac{m_a}{m_b}\frac{\vec{v}}{v}\varphi_b' f_a - \left(\underline{\overleftrightarrow{V}\psi_b'} + \frac{\vec{v}\vec{v}}{v^2}\psi_b''\right)\cdot\underline{\frac{df_a}{d\vec{v}}}\right]$$

very important element

$$\rightarrow \frac{d}{d\vec{v}}\cdot\left[\overleftrightarrow{V}\cdot\frac{df_a}{d\vec{v}}\right] = \frac{d}{d\vec{v}}\cdot\left[\frac{1}{v}\frac{df_a}{d\vec{v}} - \frac{\vec{v}}{v^3}\left(\vec{v}\cdot\frac{df_a}{d\vec{v}}\right)\right]$$

This part only changes the direction : $\left(\vec{A} = \frac{df}{d\vec{v}},\ \ \frac{1}{v}\left(\vec{A} - \hat{v}(\hat{v}\cdot\vec{A})\right)\right)$

$\Rightarrow$ Lorentz scattering.

Using 'velocity spherical coordinates (v, θ, ϕ) (only component (θ, ϕ) matters)

$$\frac{df}{d\vec{v}} = \frac{df_a}{dv}\hat{v} + \frac{1}{v}\frac{df}{d\theta}\hat{\theta} + \frac{1}{v\sin\theta}\frac{df}{d\phi}\hat{\phi}$$

It can be re-written:

$$\frac{d}{d\vec{v}}\cdot\left[\overleftrightarrow{V}\cdot\frac{df_a}{d\vec{v}}\right] = \frac{d}{d\vec{v}}\cdot\left[\frac{1}{v^2}\frac{df_a}{d\theta}\hat{\theta} + \frac{1}{v^2\sin\theta}\frac{df_a}{d\phi}\hat{\phi}\right]$$

$$= \frac{1}{v^2\sin\theta}\left[\frac{d}{d\theta}\left(\frac{\sin\theta}{v}\frac{df_a}{d\theta}\right) + \left(\frac{1}{v\sin\theta}\frac{d^2f_a}{d\phi^2}\right)\right] \equiv \frac{2}{v^3}\mathcal{L}(f_a)$$

$\therefore$ The pitch angle scattering is represented by Lorentz operator :

$$\mathcal{L}(f_a) \equiv \frac{1}{2}\left[\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{df_a}{d\theta}\right) + \frac{1}{\sin^2\theta}\frac{d^2f_a}{d\phi^2}\right]$$

Note that the Lorentz operator is same as the total angular momentum operator but in velocity space, with an additional factor 1/2.

$$\mathcal{L} = \frac{1}{2}\hat{L}^2 = \frac{1}{2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]$$

So the eigen functions of Lorentz operator are spherical harmonics $Y_\ell^m(\theta,\phi)$

$$\mathcal{L}\left[Y_\ell^m(\theta,\phi)\right] = \ell(\ell+1)Y_\ell^m(\theta,\phi)$$

← useful when Lorentz ~~scattering~~ becomes important.

Typically, the angular dependency on ϕ is ignorable in strongly magnetized plasmas and then it becomes:

$$\mathcal{L}\left[P_\ell(\theta)\right] = \ell(\ell+1)P_\ell(\theta) \quad \text{where } P_\ell = \text{legendre polynomial.}$$

When ϕ is ignorable, we also often defines the pitch angle $\xi = v_\parallel/v = \cos\theta$

$$\mathcal{L} = \frac{1}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial}{\partial\xi}$$

$$\left(\because \frac{\partial}{\partial\theta} = \frac{\partial\xi}{\partial\theta}\frac{\partial}{\partial\xi} = -\sin\theta\frac{\partial}{\partial\xi}\right.$$

$$\left.\hat{\mathcal{L}} = \frac{1}{2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\right] = \frac{1}{2}\frac{\partial}{\partial\xi}\left[(1-\xi^2)\frac{\partial}{\partial\xi}\right]\right.$$

Also, when further assuming the particle kinetic energy is conserved, one can use $\mu = \frac{mv_\perp^2}{2B}$ as a variable, as found useful in bounce-averaged drift-kinetic theories:

$$L = \mu\frac{v_\parallel}{B}\frac{\partial}{\partial\mu}\left(v_\parallel\mu\frac{\partial}{\partial\mu}\right)$$

(11) Collisional operator for isotropic background

$$C_{ab}[f_a,f_b] = L^{ab}\frac{\partial}{\partial\vec{v}}\cdot\left(\frac{m_a}{m_b}\frac{\vec{v}}{v}\varphi_b' f_a - \left(\overleftrightarrow{V}\psi_b' + \frac{\vec{v}\vec{v}}{v^2}\psi_b''\right)\cdot\frac{\partial f_a}{\partial\vec{v}}\right)$$

$$= -\frac{2L^{ab}}{v^3}\psi_b'\mathcal{L}[f_a] + L^{ab}\frac{\partial}{\partial\vec{v}}\cdot\left(\frac{m_a}{m_b}\frac{\vec{v}}{v}\varphi_b' f_a - \frac{\vec{v}\vec{v}}{v^2}\psi_b''\cdot\frac{\partial f_a}{\partial\vec{v}}\right)$$

In the case $f_a(\vec{v}) = f_a(v)$, use: $\frac{\partial}{\partial\vec{v}}\cdot\left(A(\vec{v})\vec{v}\right) = \frac{1}{v^2}\frac{\partial}{\partial v}(v^3 A)$ ← trivial in spherical coord.

$$\therefore \boxed{C_{ab}[f_a,f_b] = -\frac{2L^{ab}}{v^3}\psi_b'\mathcal{L}(f_a) + \frac{L^{ab}}{v^2}\frac{\partial}{\partial v}\left[v^3\left(\frac{m_a}{m_b}\frac{\varphi_b'}{v}f_a - \frac{\psi_b''}{v}\frac{\partial f_a}{\partial\vec{v}}\right)\right]}$$

In many cases: the other parts than Lorentz operator are ignorable. Nonetheless, one can also see that the second part can be important especially for the cases $m_a \gg m_b$ (i.e. ion-electron or α-to-thermal collisions).