2023년 9월 4일 (월) 14:00 ~ 15:15

Ch1. Continuum mechanics : force & motion

Ch2. Introductory to matrix algebra.

2.1 Matrices

$$\mathbb{A} = (A_{ij}) = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & & & \vdots \\ \vdots & & & \vdots \\ A_{m1} & \cdots & \cdots & A_{mn} \end{pmatrix},$$

$\mathbb{A}, \mathbb{B}, \mathbb{C} \rightarrow 3\times3$ matrix

$\mathbb{a}, \mathbb{b}, \mathbb{c} \rightarrow 3\times1$ column matrix

$\mathbb{a}^T, \mathbb{b}^T, \mathbb{c}^T \rightarrow 3\times1$ row matrix

Square matrix $\mathbb{A}$:

- Symmetric if $\mathbb{A} = \mathbb{A}^T$ or $A_{ij} = A_{ji}$
- anti-symmetric if $\mathbb{A}^T = -\mathbb{A}$ or $A_{ji} = -A_{ij}$
- $\mathbb{I} = \delta_{ij}$ $(i,j = 1,2,3)$
- $\mathrm{tr}\,\mathbb{A} = A_{ii} = A_{11} + A_{22} + A_{33}$
- $\det \mathbb{A} = \frac{1}{6} e_{ijk} e_{rst} A_{ir} A_{js} A_{kt}$
- $\det \mathbb{A} \neq 0 \rightleftharpoons A^{-1}$ exists.
- Orthogonal if $Q^T = Q^{-1}$. (we call orthogonal square matrix as Q)
  $\rightarrow Q^T Q = Q Q^T = \mathbb{I} \rightarrow \det Q = \pm 1$

2.2 The summation convention

If same index occurs in any expression, summation over the values 1, 2, and 3 of that index is automatically assumed.

(g) useful relation :

- $e_{ijp} e_{ijq} = 2\delta_{pq}$
- $e_{ijp} e_{rsp} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}$
- $e_{ijk} e_{rst} = \begin{vmatrix} e_{ir} & e_{is} & e_{it} \\ e_{jr} & e_{js} & e_{jt} \\ e_{kr} & e_{ks} & e_{kt} \end{vmatrix}$ $\Rightarrow$ ① True for $e_{123}e_{123}$ ② change $i \leftrightarrow j$
- $e_{mpq} \det \mathbb{A} = e_{ijk} A_{im} A_{jp} A_{kq}$
  $(mpq = 123 \rightarrow \det \mathbb{A} = e_{ijk} A_{i1} A_{j2} A_{k3})$

과제 : $e_{ijk} e_{rst} = \begin{vmatrix} e_{ir} & e_{is} & e_{it} \\ e_{jr} & e_{js} & e_{jt} \\ e_{kr} & e_{ks} & e_{kt} \end{vmatrix}$

$\det \mathbb{A} = \frac{1}{6} e_{ijk} e_{rst} A_{ir} A_{js} A_{kt}$ 를 이용하여 유도

$e_{mpq} \det \mathbb{A} = e_{ijk} A_{im} A_{jp} A_{kq}$

2.3 Eigenvalues and eigenvectors (3x3 matrix)

- $\mathbb{A}\mathbf{x} = \lambda\mathbf{x}$ or $(\mathbb{A}-\lambda\mathbb{I})\mathbf{x} = 0$
- it has non-trivial solution if $\det(\mathbb{A}-\lambda\mathbb{I}) = 0$

$$\det(\mathbb{A}-\lambda\mathbb{I}) = (\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda) \rightarrow \det\mathbb{A} = \lambda_1\lambda_2\lambda_3$$

- if $\mathbb{A}$ is real, symmetric,
  - λ's are real.
  - if $\lambda_1 \neq \lambda_2 \Rightarrow \mathbf{x}^{(1)T}\mathbf{x}^{(2)} = 0$
  - $\mathbb{P} = \begin{pmatrix} \mathbf{x}^{(1)T} \\ \mathbf{x}^{(2)T} \\ \mathbf{x}^{(3)T} \end{pmatrix}$, $\mathbb{P}^T = (\mathbf{x}^{(1)}\ \mathbf{x}^{(2)}\ \mathbf{x}^{(3)})$, $\mathbb{P}\mathbb{P}^T = \mathbb{I}$

$$\mathbb{P}\mathbb{A}\mathbb{P}^T = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix} \quad \text{(e.g)}\ \mathbb{P}\mathbb{A}\mathbb{P}^T\begin{pmatrix}1\\0\\0\end{pmatrix} = \mathbb{P}\mathbb{A}\mathbf{x}^{(1)} = \lambda_1\mathbb{P}\mathbf{x}^{(1)} = \lambda_1$$

- If $\mathbb{A}\mathbf{x} = \lambda\mathbf{x}$, $\mathbb{A}^2\mathbf{x} = \lambda^2\mathbf{x}$, $\cdots$ $\rightarrow$ λ^n is an eigenvalue of $\mathbb{A}^n$.

2.4 The Cayley-Hamilton theorem.

- $\underset{\substack{\| \\ \mathrm{tr}\,\mathbb{A}}}{\mathrm{tr}(\mathbb{P}\mathbb{A}\mathbb{P}^T)} = \lambda_1+\lambda_2+\lambda_3$, $\underset{\substack{\| \\ \mathrm{tr}\,\mathbb{A}^2}}{\mathrm{tr}((\mathbb{P}\mathbb{A}\mathbb{P}^T)^2)} = \lambda_1^2+\lambda_2^2+\lambda_3^2$ proof) 교재 11쪽 참고.

$$\det(\mathbb{A}-\lambda\mathbb{I}) = (\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda) = 0$$

$$\rightarrow \lambda^3 - \mathrm{tr}\,\mathbb{A}\,\lambda^2 + \tfrac{1}{2}((\mathrm{tr}\,\mathbb{A})^2 - \mathrm{tr}\,\mathbb{A}^2)\lambda - \det\mathbb{A} = 0$$

Theorem : λ를 $\mathbb{A}$로 바꾸어도 위 식은 성립한다.

$$\Rightarrow \underline{\mathbb{A}^3 - \mathbb{A}^2\,\mathrm{tr}\,\mathbb{A} + \tfrac{1}{2}\mathbb{A}((\mathrm{tr}\,\mathbb{A})^2 - \mathrm{tr}\,\mathbb{A}^2) - \det\mathbb{A} = 0}$$

proof) If eigenvectors form a basis,

$$f(\mathbb{A}) = (\lambda_1\mathbb{I}-\mathbb{A})(\lambda_2\mathbb{I}-\mathbb{A})(\lambda_3\mathbb{I}-\mathbb{A})$$

$\rightarrow f(\mathbb{A})\mathbf{x}^{(i)} = 0$ for $i = 1,2,3$

$\rightarrow$ for any $\mathbf{x}$, $f(\mathbb{A})\mathbf{x} = 0$ $\therefore f(\mathbb{A}) = 0$

2.5 The polar decomposition theorem

- $\mathbb{A}$ is positive-definite if $\mathbf{x}^T\mathbb{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq 0$ $\leftrightarrow$ λ's are all positive

proof) $\mathbb{A} = \mathbb{P}\mathbb{D}\mathbb{P}^T$, $\mathbf{y} = \mathbb{P}^T\mathbf{x}$

$$\mathbf{x}^T\mathbb{A}\mathbf{x} = \mathbf{x}^T\mathbb{P}\mathbb{D}\mathbb{P}^T\mathbf{x} = \mathbf{y}^T\mathbb{D}\mathbf{y} > 0 \quad (\text{for all } \lambda\text{'s} > 0)$$

- Non-singular, square matrix $\mathbb{F}$ can be decomposed, uniquely, into either of the products

$\mathbb{F} = \mathbb{R}\mathbb{U} = \mathbb{V}\mathbb{R}$ ($\mathbb{R}$: orthogonal, $\mathbb{U}, \mathbb{V}$: positive-definite symmetric)

proof) Let $\mathbb{C} \equiv \mathbb{F}^T\mathbb{F}$ and let $\bar{\mathbf{x}} = \mathbb{F}\mathbf{x}$

then $\mathbb{C}$ is symmetric (trivial),

and positive definite. $(\mathbf{x}^T\mathbb{C}\mathbf{x} = \mathbf{x}^T\mathbb{F}^T\mathbb{F}\mathbf{x} = (\mathbb{F}\mathbf{x})^2 = \bar{x}^2 > 0)$

Put $\mathbb{C}$'s eigenvalues are $\lambda_1^2, \lambda_2^2, \lambda_3^2$ assuming that $\lambda_1, \lambda_2, \lambda_3 > 0$.

So, $\mathbb{P}\mathbb{C}\mathbb{P}^T = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}$.

We define $\mathbb{U} \equiv \mathbb{P}^T \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \mathbb{P}$, which is symmetric, and positive-definite.

$\left(\mathbb{U} = \mathbb{U}^T, \ \mathbf{x}^T\mathbb{U}\mathbf{x} = \mathbf{x}^T\mathbb{P}^T \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \mathbb{P}\mathbf{x} > 0 \right)$

then $\mathbb{U}^T\mathbb{U} = \mathbb{P}^T \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \mathbb{P}\mathbb{P}^T \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \mathbb{P} = \mathbb{P}^T \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix} \mathbb{P} = \mathbb{C}$.

We define one more, $\mathbb{R} = \mathbb{F}\mathbb{U}^{-1}$

then $\mathbb{R}^T\mathbb{R} = \mathbb{U}^{-1}\mathbb{F}^T\mathbb{F}\mathbb{U}^{-1} = \mathbb{U}^{-1}\mathbb{C}\mathbb{U}^{-1} = \mathbb{U}^{-1}\mathbb{U}^2\mathbb{U}^{-1} = \mathbb{I}$.

$\therefore$ $\mathbb{R}$ is orthogonal. $\rightarrow$ $\mathbb{F} = \mathbb{R}\mathbb{U}$ where $\mathbb{R}$ is orthogonal, and $\mathbb{U}$ is positive-definite symmetric

And for $\mathbb{V} \equiv \mathbb{R}\mathbb{U}\mathbb{R}^T$, which can have $\mathbb{F} = \mathbb{R}\mathbb{U} = \mathbb{V}\mathbb{R}$

(i) $\mathbf{x}^T\mathbb{V}\mathbf{x} = \mathbf{x}^T\mathbb{R}\mathbb{U}\mathbb{R}^T\mathbf{x} = (\mathbb{R}^T\mathbf{x})^T\mathbb{U}(\mathbb{R}^T\mathbf{x}) > 0$ ($\because$ $\mathbb{U}$ is positive-definite)

(ii) $\mathbb{V}^T = (\mathbb{R}\mathbb{U}\mathbb{R}^T)^T = \mathbb{R}\mathbb{U}\mathbb{R}^T = \mathbb{V}$

$\therefore$ $\mathbb{V}$ is also positive-definite-symmetric matrix.

Let's prove its uniqueness, next.

If $\mathbb{F} = \mathbb{R}_1\mathbb{U}_1$ ($\mathbb{R}_1$ = orthogonal, $\mathbb{U}_1$ = symmetric, positive-definite),

then $\mathbb{F}^T\mathbb{F} = \mathbb{C} = \mathbb{U}_1^2$ (same for the above except $\mathbb{U} \rightarrow \mathbb{U}_1$)

and $\mathbb{P}\mathbb{U}_1^2\mathbb{P}^T = (\mathbb{P}\mathbb{U}_1\mathbb{P}^T)(\mathbb{P}\mathbb{U}_1\mathbb{P}^T) = \begin{pmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{pmatrix}$

Therefore, $\mathbb{P}\mathbb{U}_1\mathbb{P}^T = \begin{pmatrix} \pm\lambda_1 & 0 & 0 \\ 0 & \pm\lambda_2 & 0 \\ 0 & 0 & \pm\lambda_3 \end{pmatrix}$, which is $\mathbb{U}_1 = \mathbb{P}^T \begin{pmatrix} \pm\lambda_1 & 0 & 0 \\ 0 & \pm\lambda_2 & 0 \\ 0 & 0 & \pm\lambda_3 \end{pmatrix} \mathbb{P}$

but regarding that $\mathbb{U}_1$ is positive-definite, only (+) sign is allowed.

$\therefore$ $\mathbb{U}_1 = \mathbb{P}^T \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \mathbb{P} = \mathbb{U}$ $\therefore$ $\mathbb{U}$ is unique,

and $\mathbb{R}, \mathbb{V}$'s uniqueness is followed by definition.

유체역학 9월 4일 과제 (due : 9/5 23:59) 2019-15767 정태근

(1) 식 (2.14) : $\det \mathbb{A} = \frac{1}{6} e_{ijk} e_{rst} A_{ir} A_{js} A_{kt}$ 와

식 (2.21) : $e_{ijk} e_{rst} = \begin{vmatrix} \delta_{ir} & \delta_{is} & \delta_{it} \\ \delta_{jr} & \delta_{js} & \delta_{jt} \\ \delta_{kr} & \delta_{ks} & \delta_{kt} \end{vmatrix}$ 로부터

식 (2.22) : $e_{mpq} \det \mathbb{A} = e_{ijk} A_{im} A_{jp} A_{kq}$ 를 유도하시오.

sol) $e_{mpq} \det \mathbb{A} = e_{mpq} \frac{1}{6} e_{ijk} e_{rst} A_{ir} A_{js} A_{kt}$

$$= \frac{1}{6} e_{ijk} (e_{mpq} e_{rst}) A_{ir} A_{js} A_{kt}$$

$$= \frac{1}{6} e_{ijk} \begin{vmatrix} \delta_{mr} & \delta_{ms} & \delta_{mt} \\ \delta_{pr} & \delta_{ps} & \delta_{pt} \\ \delta_{qr} & \delta_{qs} & \delta_{qt} \end{vmatrix}_{\equiv \text{ put } \mathbb{B}} A_{ir} A_{js} A_{kt}$$

(if $r \neq m$ or $s \neq p$ or $t \neq q$, $\det \mathbb{B} = 0$ which does not contribute to the result.)

$$= \frac{1}{6} e_{ijk} \underbrace{\begin{vmatrix} \delta_{mm} & \delta_{mp} & \delta_{mq} \\ \delta_{pm} & \delta_{pp} & \delta_{pq} \\ \delta_{qm} & \delta_{qp} & \delta_{qq} \end{vmatrix}}_{\text{Check!}} A_{im} A_{jp} A_{kq}$$

$$\left(\begin{vmatrix} \delta_{mm} & \delta_{mp} & \delta_{mq} \\ \delta_{pm} & \delta_{pp} & \delta_{pq} \\ \delta_{qm} & \delta_{qp} & \delta_{qq} \end{vmatrix} = \delta_{mm}\delta_{pp}\delta_{qq} + \delta_{pm}\delta_{mq}\delta_{qp} + \delta_{qm}\delta_{mp}\delta_{pq} - \delta_{pp}\delta_{mq}\delta_{qm} - \delta_{mm}\delta_{pq}\delta_{qp} - \delta_{qq}\delta_{mp}\delta_{pm} = 27 + 3 + 3 - 9 - 9 - 9 = 6 \right)$$

$\therefore$ $e_{mpq} \det \mathbb{A} = \frac{1}{\cancel{6}} e_{ijk} \cancel{6} A_{im} A_{jp} A_{kq} = e_{ijk} A_{im} A_{jp} A_{kq}$ ∎

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and $\mathbb{R}, \mathbb{V}$'s uniqueness is followed by definition.