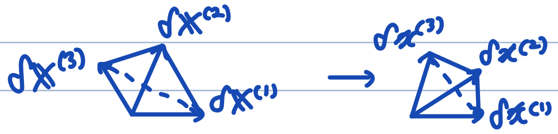
Ch7. Conservation laws

7.1 Conservation law of physics

- mass, charge, momentum, angular-momentum

7.2 Conservation law



$$\delta v = \frac{1}{6}\delta x^{(1)}\cdot(\delta x^{(2)}\times\delta x^{(3)}) = \frac{1}{6}e_{ijk}\delta x_i^{(1)}\delta x_j^{(2)}\delta x_k^{(3)}$$

$$\delta x_i^{(1)} = \frac{dx_i}{dX_R}\delta X_R^{(1)} + o((\delta X_R^{(1)})^2)$$

$$\delta v = \frac{1}{6}e_{ijk}\frac{dx_i}{dX_R}\frac{dx_j}{dX_S}\frac{dx_k}{dX_T}\delta X_R^{(1)}\delta X_S^{(2)}\delta X_T^{(3)} = \frac{1}{6}e_{RST}\det\mathbb{F}\,\delta X_R^{(1)}\delta X_S^{(2)}\delta X_T^{(3)}$$

$$\delta V = \frac{1}{6}e_{RST}\delta X_R^{(1)}\delta X_S^{(2)}\delta X_T^{(3)}$$

$$\Rightarrow \boxed{\frac{dv}{dV} = \det\mathbb{F}}$$

If incompressible, $\frac{dv}{dV} = 1 = \det\mathbb{F}$

For small deformation, $F_{iR} = \delta_{iR} + \frac{du_i}{dX_R}$,

$$\mathbb{F} = \begin{pmatrix} 1+\frac{du_1}{dX_1} & 0 & 0 \\ 0 & 1+\frac{du_2}{dX_2} & 0 \\ 0 & 0 & 1+\frac{du_3}{dX_3} \end{pmatrix}, \quad \det\mathbb{F} = 1 + \frac{du_i}{dX_i} = 1 + E_{ii}$$

* Conservation of mass

Lagrangian form : $\frac{\rho_0}{\rho} = \frac{dv}{dV} = \det\mathbb{F}$

Eulerian form : $\iiint_R \frac{\partial\rho}{\partial t}dV = -\iint_B \rho\mathbb{V}\cdot\mathbb{n}\,ds$

$$\rightarrow \iiint_R\left(\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbb{V})\right)dV = 0 \quad (\text{for any } R) \rightarrow \boxed{\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbb{V}) = 0}$$

$$\left(\begin{array}{l} \frac{\partial\rho}{\partial t} + (\mathbb{V}\cdot\nabla)\rho + \rho\nabla\cdot\mathbb{V} = 0 \\ \frac{D\rho}{Dt} + \rho\nabla\cdot\mathbb{V} = 0 \end{array}\right)$$

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$\nabla\cdot\mathbb{V} = 0$

7.3 The material time derivative of a volume integral.

$$\iiint_R \phi \rho \, dV \longrightarrow \underline{\frac{D}{Dt}\iiint_R \phi\rho \, dV} = \iiint_R \frac{D}{Dt}(\phi\rho)\, dV$$

( )/mass

$$(LHS) = \iiint_R \frac{\partial}{\partial t}(\phi\rho)\,dV + \iint_S \phi\rho \mathbf{v}\cdot\mathbf{n}\,dS = \underline{\iiint_R \left\{\frac{\partial}{\partial t}(\phi\rho) + \nabla\cdot(\phi\rho\mathbf{v})\right\}dV}$$

$$(RHS) = \iiint_R \left[\phi\underbrace{\left\{\frac{\partial}{\partial t}\rho + \nabla\cdot(\rho\mathbf{v})\right\}}_{0} + \rho\underbrace{\left\{\frac{\partial}{\partial t}\phi + \mathbf{v}\cdot\nabla\phi\right\}}_{D\phi/Dt}\right] \quad (\because \nabla\cdot(\phi\rho\mathbf{v}) = \phi\,\nabla\cdot(\rho\mathbf{v}) + \rho\mathbf{v}\cdot\nabla\phi)$$

$$\boxed{\therefore \frac{D}{Dt}\iiint_R \phi\rho\,dV = \iiint_R \rho\frac{D\phi}{Dt}\,dV}$$

if) $\phi = 1$. $\frac{D}{Dt}\iiint_R \rho\,dV = 0$

7.4 Conservation of linear momentum

$$\frac{D}{Dt}\iiint_R \rho\mathbf{v}\,dV = \iiint_R \rho\mathbf{b}\,dV + \iint_S \mathbf{t}^{(n)}\,dS \qquad t_j^{(n)} = n_i T_{ij}$$

$$\frac{D}{Dt}\iiint \rho v_j\,dV = \iiint_R \rho b_j\,dV + \iint_S n_i T_{ij}\,dS$$

$$\iiint_R \left\{\rho\frac{D}{Dt}v_j - \rho b_j - \frac{\partial}{\partial x_i}T_{ij}\right\}dV = 0$$

$$\boxed{\begin{array}{ll} \mathbf{f} : \text{acceleration} & \Rightarrow \rho f_j = \rho b_j + \frac{\partial}{\partial x_i}T_{ij} \\ (\text{in equilibrium}) & \Rightarrow \rho b_j + \frac{\partial}{\partial x_i}T_{ij} = 0 \end{array}}$$

7.5 Conservation of angular momentum

$$\frac{D}{Dt}\iiint_R \rho\mathbf{x}\times\mathbf{v}\,dV = \iiint_R \rho\mathbf{x}\times\mathbf{b}\,dV + \iint_S \mathbf{x}\times\mathbf{t}^{(n)}\,dS$$

$$\frac{D}{Dt}\iiint_R \rho e_{ijk}x_j v_k\,dV = \iiint_R e_{ijk}\rho x_j b_k\,dV + \iint_S e_{ijk}x_j n_p T_{pk}\,dS$$

$$e_{ijk}\left\{\rho\frac{D}{Dt}(x_j v_k) - \rho x_j b_k - \frac{\partial}{\partial x_p}(x_j T_{pk})\right\} = 0$$

$$\cancel{\rho v_j v_k}^{\,0} + \underbrace{\rho x_j f_k - \rho x_j b_k - x_j\frac{\partial}{\partial x_p}T_{pk}}_{=0} - T_{jk} = 0$$

$$\boxed{\therefore e_{ijk}T_{jk} = 0 \longrightarrow T_{ij} = T_{ji}}$$

7.6 Conservation of energy

$K \equiv \frac{1}{2}\iiint_R \rho v_i v_i \, dV$: kinetic energy

$E \equiv \iiint \rho e \, dV$: internal energy

internal energy density

$\frac{D}{Dt}(K+E)$ = sum of the rate (at which mechanical work done by the body and surface forces acting in R) and the rate (at which other energies enter R.)

heat flux : $\mathbf{q}$ ($\mathbf{q}\cdot\mathbf{n}$)

$t_i^{(n)}$

$$\frac{D}{Dt}\iiint \rho\left(\tfrac{1}{2}v_i v_i + e\right)dV = \iiint_R \rho b_i v_i \, dV + \iint_S (T_{ji}v_i - q_j)n_j \, dS$$

$$\rho\frac{D}{Dt}\left(\tfrac{1}{2}v_i v_i + e\right) = \rho b_i v_i + \frac{\partial}{\partial x_j}(T_{ji}v_i - q_j)$$

$$\rho v_i f_i + \rho\frac{De}{Dt} = \rho b_i v_i + v_i\frac{\partial}{\partial x_j}T_{ji} + T_{ji}\frac{\partial v_i}{\partial x_j} - \frac{\partial q_j}{\partial x_j}$$

$$-v_i\underbrace{\left(\frac{\partial}{\partial x_j}T_{ji} + \rho b_i - \rho f_i\right)}_{=0} + \rho\frac{De}{Dt} - T_{ji}\frac{\partial v_i}{\partial x_j} + \frac{\partial q_j}{\partial x_j} = 0$$

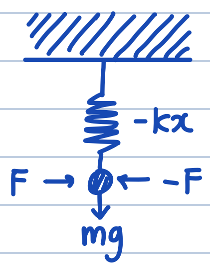
$$\rho\frac{De}{Dt} = T_{ji}\frac{\partial v_i}{\partial x_j} - \frac{\partial q_j}{\partial x_j} = T_{ji}D_{ji} - \frac{\partial q_i}{\partial x_i} \qquad \boxed{\therefore \rho\frac{De}{Dt} = T_{ij}D_{ij} - \frac{\partial q_i}{\partial x_i}}$$

$$T_{ji}\frac{\partial v_i}{\partial x_j} = \frac{1}{2}T_{ij}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

cf) $\mathbf{q} = -k\nabla T$

↳ thermal conductivity

7.7 The principle of virtual work



$d\mathbf{r}\cdot(\mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4) = 0$

suppose in equil : $\frac{\partial}{\partial x_i}T_{ij} + \rho b_j = 0$, $D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$

$$\iiint_R T_{ij}D_{ij}\,dV = \iiint_R T_{ij}\frac{\partial v_j}{\partial x_i}dV = \iiint_R\left\{\frac{\partial}{\partial x_i}(T_{ij}v_j) - v_j\frac{\partial}{\partial x_i}T_{ij}\right\}dV$$

$$= \iiint_R\left\{\frac{\partial}{\partial x_i}(T_{ij}v_j) + \rho v_j b_j\right\}dV = \iint_S T_{ij}v_j n_i \, dS + \iiint_R \rho v_j b_j \, dV$$

$$\therefore \boxed{\iiint_R T_{ij}D_{ij}\,dV = \iint \mathbf{t}^{(n)}\cdot\mathbf{v}\,dS + \iiint_R \rho\mathbf{v}\cdot\mathbf{b}\,dV}$$ work done on v_i field by T_{ij}

$$= \iiint_R\left(\rho\frac{De}{Dt} + \nabla\cdot\mathbf{q}\right)dV$$