Lecture 15. Quasisymmetry

◦ Symmetry and conservation

$$A = \oint \sqrt{2m(H-\mu B)}\, dl \rightarrow$$ Quasi-symmetry in $B \equiv |\vec{B}|$ allows magnetic confinement even without symmetry in actual $\vec{B}$

◦ Radial drift and Isodynamics

$$H = \frac{1}{2}mv_\parallel^2 + \mu B + e\phi = \text{conserved} \Rightarrow \vec{V_1} \equiv \dot{\vec{X}} = \frac{v_\parallel(\vec{B} + \vec{\nabla}\times(\rho_\parallel \vec{B}))}{B + \hat{b}\cdot\vec{\nabla}\times(\rho_\parallel \vec{B})} \simeq v_\parallel \hat{b} + \underbrace{\frac{v_\parallel}{B}\vec{\nabla}\times(\rho_\parallel \vec{B})}_{\text{" } \vec{v_d}}$$

(drift across field lines $\equiv \vec{v_d}$)

$$\vec{v_d} \equiv \vec{V_\perp} = \frac{v_\parallel}{B}\vec{\nabla}\times(\rho_\parallel \vec{B}) = \rho_\parallel \vec{\nabla}\times(v_\parallel \hat{b}) = \rho_\parallel\left(v_\parallel \vec{\nabla}\times\hat{b} + \vec{\nabla}v_\parallel \times \hat{b}\right)$$

$$= \rho_\parallel\left[v_\parallel \vec{\nabla}\times\hat{b} - \frac{1}{v_\parallel}\left(\frac{\mu}{m}\vec{\nabla}B - \frac{e}{m}\vec{\nabla}\phi\right)\times\hat{b}\right]$$

$$\left(\text{where } \vec{\nabla}v_\parallel = \pm\vec{\nabla}\sqrt{\frac{2}{m}(H-\mu B - e\phi)} = \pm\frac{1}{\cancel{2}v_\parallel}\vec{\nabla}\left(\frac{\cancel{2}}{M}(H-\mu B - e\phi)\right) = \frac{1}{v_\parallel}\left(\frac{\mu}{m}\vec{\nabla}B - \frac{e}{m}\vec{\nabla}\phi\right)\right)$$

(radial drift)

$$\vec{v_d}\cdot\vec{\nabla}\psi = \rho_\parallel v_\parallel (\vec{\nabla}\times\hat{b})\cdot\vec{\nabla}\psi + \frac{\rho_\parallel}{v_\parallel}\left[\frac{\mu}{m}(\hat{b}\times\vec{\nabla}B)\cdot\vec{\nabla}\psi\right]$$

$$\left(\begin{aligned} &\text{note, } \vec{\nabla}\times\hat{b} = \vec{\nabla}\times\left(\frac{\vec{B}}{B}\right) = \frac{\vec{\nabla}\times\vec{B}}{B} + \vec{\nabla}\left(\frac{1}{B}\right)\times\vec{B} = \frac{1}{B}\vec{\nabla}\times\vec{B} - \frac{\vec{\nabla}B\times\vec{B}}{B^2} \\ &\hat{b}\times\vec{\nabla}\ln B = \frac{\vec{B}}{B}\times\frac{\vec{\nabla}B}{B} = \frac{-\vec{\nabla}B\times\vec{B}}{B^2} \quad \rightarrow \quad \vec{\nabla}\times\hat{b} = \frac{1}{B}\vec{\nabla}\times\vec{B} + \hat{b}\times\vec{\nabla}\ln B \\ &= \frac{1}{B}(\hat{b}\times\vec{\nabla}B) \end{aligned}\right)$$

$$\therefore \vec{v_d}\cdot\vec{\nabla}\psi = \frac{mv_\parallel^2}{eB}\left[\frac{1}{B}\underbrace{\cancel{\vec{\nabla}\times\vec{B}}}_{=\mu_0 j} + \hat{b}\times\vec{\nabla}\ln B\right]\cdot\vec{\nabla}\psi + \frac{m}{eB}\frac{v_\perp^2}{2B}\cdot B(\hat{b}\times\vec{\nabla}\ln B)\cdot\vec{\nabla}\psi$$

$$= \frac{mv_\parallel^2}{eB}(\hat{b}\times\vec{\nabla}\ln B)\cdot\vec{\nabla}\psi + \frac{mv_\perp^2}{2eB}(\hat{b}\times\vec{\nabla}\ln B)\cdot\vec{\nabla}\psi = \frac{m(v_\parallel^2 + \frac{1}{2}v_\perp^2)}{eB}(\hat{b}\times\vec{\nabla}\ln B)\cdot\vec{\nabla}\psi$$

$\therefore$ isodynamic condition of equilibrium (no radial drift at all, $\vec{V_d}\cdot\vec{\nabla}\psi = 0$)

$$\boxed{\vec{\nabla}\psi\times(\hat{b}\cdot\vec{\nabla}\ln B) = 0}$$

in low β, isodynamic condition $\Leftrightarrow$ zero geodesic curvature $\vec{k_g} = (\hat{n}\times\hat{b})\cdot\vec{K} = 0$

$$\left(\hat{n} = \frac{\vec{\nabla}\psi}{|\vec{\nabla}\psi|},\ \vec{K} \simeq \vec{\nabla}\ln B\right)$$

- **Omnigenity**

Isodynamic : Ideal but impossible to meet everywhere

Omnigenity : $\langle \vec{v_d} \cdot \vec{\nabla}\psi \rangle_b = 0$

where bounce average : $\langle A \rangle_b = \dfrac{\oint A\,dt}{\oint dt} = \dfrac{\oint A \frac{dl}{v_\parallel}}{\oint \frac{dl}{v_\parallel}} = \omega_b \oint A \dfrac{dl}{v_\parallel}$ $\quad \left(\omega_b = \tau^{-1} = \dfrac{1}{\oint \frac{dl}{v_\parallel}}\right)$

- **Action variation**

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}(\vartheta - \iota\varphi) = \vec{\nabla}\psi \times \vec{\nabla}\alpha \qquad (\alpha = \vartheta - \iota\varphi)$$

one can make non-flux-coordinate (ψ, α, θ) where $\theta = \vartheta - \dfrac{N}{M}\varphi$.

Then, $\quad B^\theta = \vec{B} \cdot \vec{\nabla}\theta = (\vec{\nabla}\psi \times \vec{\nabla}\alpha) \cdot \vec{\nabla}\theta = \dfrac{1}{\sqrt{g}}$.

$$B_\theta = \vec{B} \cdot \vec{e_\theta} = (\vec{\nabla}\psi \times \vec{\nabla}\alpha) \cdot \sqrt{g}\,(\vec{\nabla}\psi \times \vec{\nabla}\alpha) = \sqrt{g}\,B^2$$

Expand radial drift to this (ψ, α, θ) coordinate.

$$\vec{v_d} = \frac{v_\parallel}{B} \vec{\nabla} \times (\rho_\parallel \vec{B}) = \frac{v_\parallel}{\sqrt{g}\,B}\left(\frac{\partial}{\partial\alpha}(\rho_\parallel B_\theta) - \frac{\partial}{\partial\theta}(\rho_\parallel B_\alpha)\right)\vec{e_\psi} + (\cdots)\vec{e_\alpha} + (\cdots)\vec{e_\theta}$$

$$\vec{v_d} \cdot \vec{\nabla}\psi = \frac{v_\parallel}{\sqrt{g}\,B}\left(\frac{\partial}{\partial\alpha}(\rho_\parallel \sqrt{g} B^2) - \frac{\partial}{\partial\theta}(\rho_\parallel B_\alpha)\right)$$

note, $\langle A \rangle_b = \omega_b \oint A \dfrac{dl}{v_\parallel} = \omega_b \oint A \dfrac{B}{v_\parallel}\dfrac{d\theta}{B^\theta} = \omega_b \oint d\theta \dfrac{\sqrt{g}\,B}{v_\parallel} A \quad (dl = \sqrt{g}\,B\,d\theta)$

$$\therefore \langle \vec{v_d} \cdot \vec{\nabla}\psi \rangle_b = \omega_b \oint d\theta \left(\frac{\partial}{\partial\alpha}(\rho_\parallel \sqrt{g} B^2) - \cancel{\frac{\partial}{\partial\theta}(\rho_\parallel B_\alpha)}\right)$$

$$= \omega_b \frac{\partial}{\partial\alpha} \oint d\theta \left(\frac{m v_\parallel}{eB}\sqrt{g} B^2\right) = \frac{\omega_b}{e}\frac{\partial}{\partial\alpha}\oint m v_\parallel\, dl = \frac{\omega_b}{e}\frac{\partial J}{\partial\alpha} \quad \left(J = \oint m v_\parallel\, dl\right)$$

$\Rightarrow$ average radial drift is proportional to the variation in the action $J = \oint m v_\parallel dl$

$$\therefore \underline{\langle \vec{v_d} \cdot \vec{\nabla}\psi \rangle_b = \frac{\omega_b}{e}\frac{\partial J}{\partial\alpha}}, \quad \text{similarly } \underline{\langle \vec{v_d} \cdot \vec{\nabla}\alpha \rangle_b = -\frac{\omega_b}{e}\frac{\partial J}{\partial\psi}}$$

$$\underline{\langle \vec{v_d} \cdot \vec{\nabla} \rangle_b J = \langle \vec{v_d} \cdot \vec{\nabla}\psi \rangle_b \frac{\partial J}{\partial\psi} + \langle \vec{v_d} \cdot \vec{\nabla}\alpha \rangle_b \frac{\partial J}{\partial\alpha} = 0} \quad \Leftarrow \text{2nd adiabatic invariance}$$

∘ Omnigenity and Quasisymmetry

omnigenity $\left\{ \begin{array}{ll} \text{radial orbit drift vanishes} & \langle v_d \cdot \vec{\nabla}\psi \rangle_b = 0 \\ \text{no action variation on flux surfaces} & dJ/d\alpha = 0 \end{array} \right\}$

$$\langle \vec{v_d} \cdot \vec{\nabla}\psi \rangle_b = W_b \oint d\theta \frac{d}{d\alpha} \left(\rho_{\parallel} \sqrt{g} B^2 \right)$$

If $\sqrt{g}$ depends only on B, the entire $\oint$ will depend only on B

⇒ Boozer coordinate !!

That is; $\boxed{\begin{array}{l} \text{If } B \text{ is independent of } \alpha \text{ on Boozer coordinates,} \\ \quad B = B(\psi, \theta) = B(\psi, M\theta_B - N\varphi_B), \\ \text{the omnigenity will be automatically achieved} \quad \text{(Quasi-symmetry)} \end{array}}$

$(M, N) = (1, 0)$ (QA)

$= (1, \pm N \cdot nfp)$ (QH) ← Type of quasi-symmetry

$= (0, 1)$ (QP)

Basic principle : Particle will hang on a surface if there is a way to conserve its action

$$\langle \vec{v_d} \cdot \vec{\nabla} \rangle_b J = \langle \vec{v_d} \cdot \vec{\nabla}\psi \rangle_b \frac{dJ}{d\psi} + \langle \vec{v_d} \cdot \vec{\nabla}\alpha \rangle_b \frac{dJ}{d\alpha} = 0$$

If $\frac{dJ}{d\alpha} = 0$ → then, to conserve action, $\frac{dJ}{d\psi}$ must be zero.

- Isomorphism

Tokamaks and stellarators are indistinguishable to the leading order for particles.

$$L_{gc} = (mv_{\parallel}\hat{b} + e\vec{A})\cdot\dot{\vec{X}} + \frac{m}{e}\mu\dot{\gamma} - H_{gc} = e(\rho_{\parallel}\vec{B} + \vec{A})\cdot\dot{\vec{X}} + \frac{m}{e}\mu\dot{\gamma} - H_{gc}$$

Take Boozer coordinates, $\theta = \theta_B - \frac{N}{M}\varphi_B$, and assume quasi-symmetry.

Then, $B = B(\psi, \theta)$, $H_{gc} = H_{gc}(\psi, \theta)$.

$$\left(\begin{array}{ll} \vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\alpha = \vec{\nabla}\times(\psi\vec{\nabla}\alpha), & \Rightarrow \vec{A} = \psi\vec{\nabla}\alpha = \psi\vec{\nabla}\theta_B - \chi\vec{\nabla}\varphi_B \\ \vec{B} = I\vec{\nabla}\theta_B + G\vec{\nabla}\varphi_B + \cancel{K\vec{\nabla}\psi}, & \dot{\vec{X}} = \dot{\theta}_B\vec{e_{\theta_B}} + \dot{\varphi}_B\vec{e_{\varphi_B}} + \dot{\psi}\vec{e_\psi} \end{array}\right)$$

$$\Rightarrow L_{gc} = e(\rho_{\parallel}(I\vec{\nabla}\theta_B + G\vec{\nabla}\varphi_B) + \psi\vec{\nabla}\theta_B - \chi\vec{\nabla}\varphi_B)\cdot(\dot{\theta}_B\vec{e_{\theta_B}} + \dot{\varphi}_B\vec{e_{\varphi_B}} + \dot{\psi}\vec{e_\psi}) + \frac{m}{e}\mu\dot{\gamma} - H_{gc}$$

$$= e(\psi + \rho_{\parallel}I)\dot{\theta}_B + e(\rho_{\parallel}G - \chi)\dot{\varphi}_B + \frac{m}{e}\mu\dot{\gamma} - H_{gc}$$

note that this is a canonical form $L_{gc} = \sum_i p_i\dot{q}_i - H_{gc}$.

$\theta_B \to \theta$, with $\theta = \theta_B - \left(\frac{N}{M}\right)\varphi_B$, then.

$$L_{gc} = e(\psi + \rho_{\parallel}I)\dot{\theta} + e\left(\rho_{\parallel}\left(G + \frac{N}{M}I\right) - \left(\chi - \frac{N}{M}\psi\right)\right)\dot{\varphi}_B + \frac{m}{e}\mu\dot{\gamma} - H_{gc}(\psi,\theta)$$

$$p_{\varphi_B} = \frac{\partial L_{gc}}{\partial\dot{\varphi}_B} = e\left(\rho_{\parallel}\left[G + \frac{N}{M}I\right] - \left[\chi - \frac{N}{M}\psi\right]\right) = \text{const}$$

($\because$ G, I, χ, ψ 모두 flux function)

$$\left(\begin{array}{l} \theta_B \leftrightarrow \theta_B - \left(\frac{N}{M}\right)\varphi_B \\ G \leftrightarrow G + \left(\frac{N}{M}\right)I \\ \chi \leftrightarrow \chi - \left(\frac{N}{M}\right)\psi \end{array}\right)$$

Isomorphism

One can simply transform coordinates to make particle trajectory under quasisymmetry same as ones for tokamak. (only for guiding center particle)