5. Kinematics of rigid-body rotation

① Body rotation as an orthogonal transformation

- Rigid body
- Orthogonal transformations
  - Direction cosine : $\cos\theta_{ij} = \hat{\underline{x}}_i' \cdot \hat{\underline{x}}_j$
  - Change of axes : $\hat{\underline{x}}_i' = (\hat{\underline{x}}_i' \cdot \hat{\underline{x}}_j)\hat{\underline{x}}_j = (\cos\theta_{ij})\hat{\underline{x}}_j$
  - Change of coordinates :

    if $\underline{v} = v_i \hat{\underline{x}}_i = v_i' \hat{\underline{x}}_i'$, then $v_i' = \underline{v} \cdot \hat{\underline{x}}_i' = (\cos\theta_{ij})\underline{v}\cdot\hat{\underline{x}}_j = (\cos\theta_{ij}) v_j$

    in matrix form, $\boxed{\underline{v}' = \underline{\underline{R}}\,\underline{v}}$ $(\underline{\underline{R}}_{ij} = \cos\theta_{ij})$

  - orthogonality : $\hat{\underline{x}}_i \cdot \hat{\underline{x}}_j = \hat{\underline{x}}_i' \cdot \hat{\underline{x}}_j' = \delta_{ij} \Rightarrow \delta_{ij} = \hat{\underline{x}}_i' \hat{\underline{x}}_j' = R_{ik}R_{jl}\hat{\underline{x}}_k \hat{\underline{x}}_l$

    $= R_{ik}R_{jl}\delta_{kl} = R_{ik}R_{jk}$

    $\Rightarrow \underline{\underline{I}} = \underline{\underline{R}}^T\underline{\underline{R}}$ (orthogonal transformation)

    $\boxed{\cdot}$ introduces 6 constraints , $\Rightarrow$ 3 independent angle

  - If $\underline{u} = \underline{\underline{A}}\,\underline{v}$, $\underline{u}' = \underline{\underline{R}}\,\underline{u} = \underline{\underline{R}}\,\underline{\underline{A}}\,\underline{v} = \underline{\underline{R}}\,\underline{\underline{A}}\,\underline{\underline{R}}^{-1}\underline{v}' = \underline{\underline{A}}'\underline{v}'$

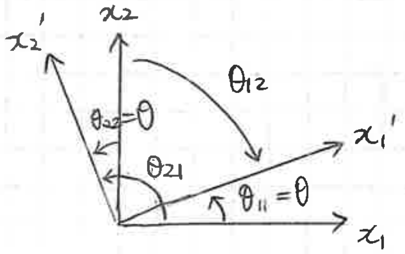
    $\therefore \boxed{\underline{\underline{A}}' = \underline{\underline{R}}\,\underline{\underline{A}}\,\underline{\underline{R}}^{-1}}$ ($\underline{\underline{A}}$ in new coordinates)

- Proper and Improper transformations
  - $|\underline{\underline{A}}| = \epsilon_{ijk}A_{1i}A_{2j}A_{3k}$, $|\underline{\underline{A}}^T| = |\underline{\underline{A}}|$, $|\underline{\underline{A}}\,\underline{\underline{B}}| = |\underline{\underline{A}}||\underline{\underline{B}}|$

    $\hookrightarrow |\underline{\underline{A}}'| = |\underline{\underline{R}}||\underline{\underline{A}}||\underline{\underline{R}}^{-1}| = |\underline{\underline{A}}|$ ∴ (similar matrix has same det.)

  - $|\underline{\underline{R}}||\underline{\underline{R}}^T| = 1 \rightarrow |\underline{\underline{R}}| = \pm 1 = \begin{cases} 1 & \text{(proper)} \\ -1 & \text{(improper)} \end{cases}$

∘ Two-dimensional transformation



$$\underline{R}_3(\theta)=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

좌표계가 반시계로 회전하면, 벡터 변환은 시계로 회전한다. Vice versa.

② Euler's theorem

∘ A real orthogonal matrix representing a proper transformation always has $\lambda=1$

(이 변환을 가해도 절대 크기와 방향이 변하지 않는 벡터가 하나 있다.

→ 항상 회전축이 존재한다 → orthogonal transform = rotation.)

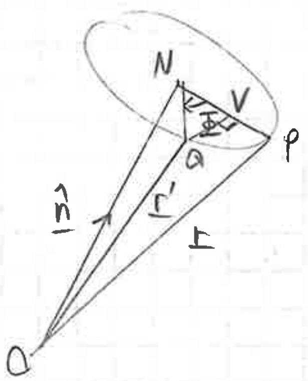
proof) $\underline{R}\,\underline{R}^T=1$, $(\underline{R}-\underline{1})\underline{R}^T=1-\underline{R}^T \Rightarrow |\underline{R}-\underline{1}||\underline{R}^T|=|1-\underline{R}^T|$

$\Rightarrow |\underline{R}-\underline{1}|=|\underline{1}-\underline{R}|=(-1)^3|\underline{R}-\underline{1}| \quad \therefore |\underline{R}-\underline{1}|=0 \rightarrow \lambda=1$

corollary) Chasle's theorem : Rigid body motion = Translation + Rotation

(6 coordinates : $\underline{3}$ translation, $\underline{2}$ rotation axis, $\underline{1}$ angle)

∘ rotation formula : $\boxed{\underline{r}'=\underline{\hat{n}}(\underline{\hat{n}}\cdot\underline{r})+[\underline{r}-\underline{\hat{n}}(\underline{\hat{n}}\cdot\underline{r})]\cos\Phi+(\underline{r}\times\underline{\hat{n}})\sin\Phi}$



$\underline{r}'=\underline{ON}+\underline{NV}+\underline{VQ}$

$=\underline{\hat{n}}(\underline{\hat{n}}\cdot\underline{r})+[\underline{r}-\underline{\hat{n}}(\underline{\hat{n}}\cdot\underline{r})]\cos\Phi+(\underline{r}\times\underline{\hat{n}})\sin\Phi$

③ Euler angles

Three independent variables to describe the rotation of a rigid body.

i) x, y만 반시계 방향으로 회전 $(x, y, z) \rightarrow (x', y', z')$

$$\underline{\underline{R_z}}(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 1 \\ -\sin\phi & \cos\phi & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

ii) y', z'만 반시계 방향으로 회전 $(x', y', z') \rightarrow (x'', y'', z'')$

$$\underline{\underline{R_x}}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

iii) x'', y''만 반시계 방향으로 회전

$$\underline{\underline{R_z}}(\psi) = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \boxed{\underline{\underline{R}}(\phi, \theta, \psi) = \underline{\underline{R_z}}(\psi)\, \underline{\underline{R_x}}(\theta)\, \underline{\underline{R_z}}(\phi)}$$

$0 \le \phi, \psi \le 2\pi$ (x-y plane 회전)

$0 \le \theta \le \pi$ (y, z plane 회전)

④ Vector representation of infinitesimal rotations

Note that $\underline{r}' = \hat{n}(\hat{n}\cdot\underline{r}) + [\underline{r} - \underline{n}(\underline{n}\cdot\underline{r})]\cos\Phi + (\underline{r}\times\hat{n})\sin\Phi \quad \cdots (*)$

(A) Properties of infinitesimal rotations

- Commutavity : $\underline{\underline{R_a}} = \underline{\underline{1}} + \underline{\underline{E_a}}$, $\underline{\underline{R_b}} = \underline{\underline{1}} + \underline{\underline{E_b}}$

$$\underline{\underline{R_a}}\,\underline{\underline{R_b}} = (\underline{\underline{1}}+\underline{\underline{E_a}})(\underline{\underline{1}}+\underline{\underline{E_b}}) = \underline{\underline{1}} + \underline{\underline{E_a}}\,\underline{\underline{E_b}} = (\underline{\underline{1}}+\underline{\underline{E_b}})(\underline{\underline{1}}+\underline{\underline{E_a}}) = \underline{\underline{R_b}}\,\underline{\underline{R_a}}$$

- Anti-symmetry : since $\underline{\underline{R_a}}^T = \underline{\underline{R_a}}^{-1}$, $(\underline{\underline{1}}+\underline{\underline{E_a}})^T = (\underline{\underline{1}}+\underline{\underline{E_a}})^{-1}$

$$\Rightarrow \underline{\underline{E_a}}^T = -\underline{\underline{E_a}} \quad (\text{느낌: 테일러전개})$$

$\therefore$ $\underline{\underline{E_a}}^T = -\underline{\underline{E_a}}$ 이므로, $\underline{\underline{E_a}} = \begin{bmatrix} 0 & d\Omega_3 & -d\Omega_2 \\ -d\Omega_3 & 0 & d\Omega_1 \\ d\Omega_2 & -d\Omega_1 & 0 \end{bmatrix}$ (let's define like this)

then $\boxed{d\underline{r} = \underline{\underline{E_a}}\,\underline{r} = \underline{r}\times d\underline{\Omega}}$ where $\boxed{d\underline{\Omega} = \begin{bmatrix} d\Omega_1 \\ d\Omega_2 \\ d\Omega_3 \end{bmatrix}}$

$\Rightarrow$ Infinitesimal rotations can always be represented by vectors.

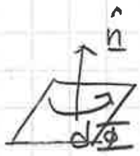
(B) Interpretations of vectors associated with infinitesimal vectors.

$\Phi \to d\Phi$ in $(*)$, $\to \underline{r}' = \underline{r} + (\underline{r}\times\hat{\underline{n}})\sin\Phi$, $\to d\underline{r} = (\underline{r}\times\hat{\underline{n}})\sin\Phi$

$\therefore d\underline{r} = \underline{r}\times d\underline{\Omega} = \underline{r}\times(\hat{\underline{n}}\sin\Phi) \to \boxed{d\underline{\Omega} = \hat{\underline{n}}\sin\Phi}$

($\Rightarrow$ Infinitesimal rotation vector의 방향은 axis 와 나란하며,
그 크기는 Infinitesimal rotation angle 에 의해 결정된다)

⑤ Fictitious forces in a rotating frame



(a) Rate of change of a vector $\underline{G}$

$$(d\underline{G})_{lab} = (d\underline{G})_{body} + (d\underline{G})_{rot}$$

lab frame에서는 회전방향 반대.

$$= (d\underline{G})_{body} + \underline{G}\times\hat{\underline{n}}(-d\Phi)$$

$$= (d\underline{G})_{body} + d\underline{\Omega}\times\underline{G}$$

$$\therefore \boxed{\left(\frac{d\underline{G}}{dt}\right)_{lab} = \left(\frac{d\underline{G}}{dt}\right)_{body} + \underline{\omega}\times\underline{G}} \qquad \left(\underline{\omega} = \frac{d\underline{\Omega}}{dt} \text{ : angular velocity vector}\right)$$

(b) Fictitious forces (in const $\underline{\omega}$)

$$\left(\text{note that : } \underline{v}_{lab} \equiv \left(\frac{d\underline{r}}{dt}\right)_{lab} = \left(\frac{d\underline{r}}{dt}\right)_{body} + \underline{\omega}\times\underline{r} = \underline{v}_{body} + \underline{\omega}\times\underline{r}\right)$$

$$\underline{F}_{lab} = m\left(\frac{d\underline{v}_{lab}}{dt}\right)_{lab} = m\left(\frac{d\underline{v}_{lab}}{dt}\right)_{body} + m\underline{\omega}\times\underline{v}_{lab}$$

$$= \left(m\left(\frac{d\underline{v}_{body}}{dt}\right)_{body} + m\underline{\omega}\times\underline{v}_{body}\right) + m\underline{\omega}\times(\underline{v}_{body} + \underline{\omega}\times\underline{r})$$

$$\therefore \boxed{\underline{F}_{lab} = \underbrace{m\left(\frac{d\underline{v}_{body}}{dt}\right)_{body}}_{\text{force in Body frame}} + \underbrace{2m\underline{\omega}\times\underline{v}_{body}}_{\text{코리올리 힘}} + \underbrace{m\underline{\omega}\times(\underline{\omega}\times\underline{r})}_{\text{원심력}}}$$