Fusion Plasma Theory 2.

Lecture 19 : Spitzer–Härm Theory of Conductivity.

① Collisional Transport in Bogoliubov Timescale Hierarchy.

$O(ns)$: $t \sim \omega_p^{-1}$: pair correlation w/ Debye shielding.

$O(ms)$: $t \sim \nu^{-1}$: local Maxwellian

$O(s)$: $t \sim L^2/D \sim L^2/\rho^2\nu = \nu^{-1}(L/\rho)^2$: collisional transport & diffusion

(or $\nu^{-1}(L/\lambda_{mfp})^2$ in parallel motion)

We will follow classical transport theories

Spitzer-Härm $\longrightarrow$ Chapmann - Enskog $\longrightarrow$ Braginskii.

② The Spitzer-Härm problem

This problem is the calculation of the response of a collisional, uniform, steady-state quasi-neutral plasma to an applied electric field in the absence of magnetic field.

$$\nu_0^{ei} \mathcal{L}[f_{e1}] = \underbrace{-\frac{e\vec{E}}{m_e}\cdot\frac{df_{Me}}{d\vec{v}}}_{(i)} = \underbrace{\frac{e\vec{E}\cdot\vec{v}}{T_e} f_{Me}}_{(ii)} \qquad \left(f_{Me} \propto \exp\left(-\frac{m_e v^2}{2T_e}\right),\ \frac{df_{Me}}{d\vec{v}} = -\frac{m_e\vec{v}}{T_e}f_{Me}\right)$$

$$\left(\begin{array}{l} (i)\ \vec{F} = m\vec{a} \rightarrow m_e\vec{a} = -e\vec{E} \rightarrow \vec{a} = -\frac{e\vec{E}}{m_e} \quad \rightarrow \quad \vec{a}\cdot\frac{df}{d\vec{v}} = -\frac{e\vec{E}}{m_e}\frac{df_{Me}}{d\vec{v}} \\ (ii)\ f_{Me} = n_e\left(\frac{m_e}{2\pi T_e}\right)^{3/2}\exp\left(-\frac{m_e v^2}{2T_e}\right),\quad \frac{df_{Me}}{d\vec{v}} = -\frac{m_e\vec{v}}{T_e}f_{Me} \end{array}\right)$$

giving $\eta_s = \alpha_e \eta$ with $\alpha_e = 3\pi/32$ where $\eta = m_e\nu_{ei}/n_e e^2$. (spitzer resistivity)

The Spitzer-Härm did better by including the full linear electron collisional operator.

$$\underbrace{C_{ee}^{l}[f_{e1}]}_{추가됨} + \underbrace{C_{ei}[f_{e1}]}_{기존 스피처.} = \frac{e\vec{E}\cdot\vec{v}}{T_e} f_{Me}$$

In the presence of ion flow $\vec{u_i}$, collisional operator becomes

$$C_{ee}^{l}[f_{e1}] + \nu_{ei}(v)\left[\mathcal{L}(f_{e1}) + \frac{m_e\vec{v}\cdot\vec{u}}{T_e} f_{Me}\right] = \frac{e\vec{E}\cdot\vec{v}}{T_e} f_{Me}$$

However, ion flow does not change the answer

First, notice $\mathcal{L}[\vec{v}] = -\vec{v}$, and $C_{ee}^{\ell}[(\vec{a}\cdot\vec{v}) f_{Me}] = 0$, one can write

$V_{\parallel}$ 성분을 숨아내는 operator. 상수 맥스웰

$$C_{ee}^{\lambda}\left[f_{e1} - \frac{m_e\vec{v}\cdot\vec{u_i}}{T_e} f_{Me}\right] + \nu_{ei}(v)\,\mathcal{L}\left[f_{e1} - \frac{m_e\vec{v}\cdot\vec{u_i}}{T_e} f_{Me}\right] = \frac{m_e\vec{E}\cdot\vec{v}}{T_e} f_{Me}$$

Let $f_{sh} \equiv f_{e1} - \frac{m_e\vec{v}\cdot\vec{u_i}}{T_e} f_{Me}$, then f_{sh} 를 새로운 distribution func.로 보는 문제로 바뀜.

$$\vec{j} = en_i\vec{u_i} - e\int f_{e1}\vec{v}\,d\vec{v} = en_i\vec{u_i} - en_e\vec{u_i} - e\int f_{sh}\vec{v}\,d\vec{v} = -e\int f_{sh}\vec{v}\,d\vec{v}$$

$$\left(\int f_{e1}\vec{v}\,d\vec{v} = \int f_{sh}\vec{v}\,d\vec{v} + \frac{m_e}{T_e}\int \vec{v}\vec{v} f_{Me}\,d\vec{v}\cdot\vec{u_i} = \int f_{sh}\vec{v}\,d\vec{v} + \frac{m_e}{T_e}\int \frac{1}{3}v^2 \overleftrightarrow{I} f_{Me}\,d\vec{v}\cdot\vec{u_i}\right.$$

$$\left. = \int f_{sh}\vec{v}\,d\vec{v} + \frac{P_e}{T_e}\overleftrightarrow{I}\cdot\vec{u_i} = \int f_{sh}\vec{v}\,d\vec{v} + n_e\vec{u_i}\right)$$

So the answer depends on f_{sh}, (or f_{e1}) ignoring the ion flow :

$$C_{ee}^{\ell}[f_{sh}] + \nu_{ei}(v)\,\mathcal{L}[f_{sh}] = \frac{e\vec{E}\cdot\vec{v}}{T_e} f_{Me}$$ ← 이거 풀어서 f_{sh} 구하면 $\vec{j}$ 구할수있음

Z	1	2	3	4	16	∞
α_e (Spitzer)	0.506	0.431		0.375	0.319	
α_e (Braginskii)	0.51	0.44	0.40	0.38		0.29

1) Lorentz operator 만 고려 → $\eta_s = \frac{\eta_c}{3.4}$

2) Lorentz + (e-e) collision 고려 → $\eta_B = \frac{\eta_c}{2.0}$

③ Laguerre polynomial expansion

The spitzer-Härm problem can be solved analytically, by expanding the perturbed distribution function (f_{sh} or f_{e1})

The appropriate basis of orthogonal functions is that of the associated Laguerre polynomials.
* note that this is not an eigen function of collision operator.

$$\underline{f_{e1} = \frac{2v_\parallel}{Nte^2} f_{Me} \sum_{k=0} a_k L_k^{(3/2)}(x)} \quad , \quad \underline{\text{where } x \equiv \frac{v^2}{v_{te}^2}}$$

- e.g.) $L_0^{(3/2)}(x) = 1$, $L_1^{(3/2)}(x) = \frac{5}{2} - x$, $L_2^{(3/2)}(x) = \frac{35}{8} - \frac{7}{2}x + \frac{1}{2}x^2$

- orthogonality relation) $\int_0^\infty x^{3/2} e^{-x} L_p^{(3/2)}(x) L_q^{(3/2)}(x)\, dx = \frac{\Gamma(p+5/2)}{\Gamma(p+1)} \delta_{pq}$

- first few integrals)
$$\int_0^\infty x^{3/2} e^{-x} \left(L_1^{(3/2)}(x)\right)^2 dx = \frac{3\sqrt{\pi}}{4}$$
$$\int_0^\infty x^{3/2} e^{-x} \left(L_2^{(3/2)}(x)\right)^2 dx = \frac{15\sqrt{\pi}}{8}$$
$$\int_0^\infty x^{3/2} e^{-x} \left(L_3^{(3/2)}(x)\right)^2 dx = \frac{105\sqrt{\pi}}{32}$$

④ Coefficients for Laguerre polynomial expansion

While the Laguerre polynomial are not eigenfunctions of Landau collision operator, they are well aligned with the truncated moment.

- e.g.) $n_e u_\parallel \simeq \int d\vec{v}\, v_\parallel f_{e1} = \int d\vec{v} \frac{2v_\parallel^2}{v_{te}^2} f_{Me} \left[a_0 L_0^{(3/2)}(x) + a_1 L_1^{(3/2)}(x) + \cdots \right]$

in (v, θ, ϕ) coordinates, and ϕ-symmetry, $\xi \equiv \cos\theta$:

$$\int d\vec{v} = 2\pi \int_{-1}^1 d\xi \int_0^\infty v^2 dv = \pi v_{te}^3 \int_{-1}^1 d\xi \int_0^\infty x^{\frac{1}{2}} dx$$

$$\left(v^2 = x v_{te}^2 \ ,\ 2v\,dv = v_{te}^2 dx \Rightarrow v^2 dv = (x v_{te}^2) \cdot \left(\frac{v_{te}^2 dx}{2x^{1/2} v_{te}} \right) = \frac{v_{te}^3}{2} x^{1/2} dx \right)$$

*note $v_\parallel = v\cos\theta = v\xi = x^{\frac{1}{2}} \xi v_{te}$

$$\Rightarrow \underline{n_e u_\parallel} = \int d\vec{v}\, v_\parallel f_{e1} = \pi v_{te}^3 \int_{-1}^1 d\xi \int_0^\infty x^{\frac{1}{2}} dx \frac{2x\xi^2 v_{te}^2}{v_{te}^2} \left(\frac{n_e}{\pi^{3/2} v_{te}^3} e^{-x} \right) \left[\cdots \right]$$

$$= \frac{2n_e}{\sqrt{\pi}} \int_{-1}^1 \xi^2 d\xi \int_0^\infty dx\, x^{\frac{3}{2}} e^{-x} \left[\cdots \right] = \frac{4n_e}{3\sqrt{\pi}} \int_0^\infty dx\, x^{\frac{3}{2}} e^{-x} L_0^{(3/2)}(x) \left[a_0 L_0^{(3/2)}(x) + \cdots \right] = \frac{4}{3\sqrt{\pi}} \frac{3\sqrt{\pi}}{4} \underline{n_e a_0}$$

Evidently, the zeroth Laguerre coefficient is the parallel electron flow.

Similarly, the parallel heat flux of electrons :

$$\underline{q_{\parallel}} \simeq \int d\vec{v}\, v_{\parallel} \left(\frac{m_e v^2}{2} - \frac{5T_e}{2} \right) f_{e1}$$

$$= T_e \int d\vec{v} \frac{2v^2 \xi^2}{V_{te}^2} \left(\frac{v^2}{V_{te}^2} - \frac{5}{2} \right) \frac{n_e}{\pi^{3/2} V_{te}^3} e^{-x} \left[a_0 L_0^{(3/2)}(x) + a_1 L_1^{(3/2)}(x) + \cdots \right]$$

$$= T_e \pi V_{te}^3 \int_{-1}^{1} d\xi \int_0^\infty x^{1/2} dx \left\{ 2x \xi^2 \left(x - \frac{5}{2} \right) \frac{n_e}{\pi^{3/2} V_{te}^3} e^{-x} \left[\cdots \right] \right.$$

$$= \frac{4\pi}{3} \frac{n_e T_e}{\pi^{3/2}} \int_0^\infty x^{\frac{3}{2}} e^{-x} \left(-L_1^{(3/2)}(x) \right) \left[a_0 L_0^{(3/2)}(x) + a_1 L_1^{(3/2)}(x) + \cdots \right] \qquad L_1^{(3/2)}(x) = \frac{5}{2} - x$$

$$= -\frac{4}{3\sqrt{\pi}} n_e T_e a_1 \times \frac{15\sqrt{\pi}}{8} = \underline{-\frac{5}{2} p_e a_1}$$

⇒ First Laguarre polynomial is related to the electron heat flux.

To sum-up, $\underline{f_{e1} = \frac{2v_{\parallel}}{V_{te}^2} f_{Me} \left[u_{e\parallel} L_0^{(3/2)}(x) - \frac{2}{5} \frac{q_{\parallel e}}{p_e} L_1^{(3/2)}(x) + \cdots \right]}$

⑤ Expansion coefficients for distribution function

Recall : $C_{ee}^{l} [f_{eh}] + \nu_{ei}(v) \mathcal{L} [f_{eh}] = \frac{e\vec{E}\cdot\vec{v}}{T_e} f_{Me}$

Take the k-th moment to determine the coefficients :

$$\int d\vec{v}\, v_{\parallel} L_k^{(3/2)}(x) \times \left\{ C_{ee}^{l} [f_{eh}] + \nu_{ei}(v) \mathcal{L} [f_{eh}] = \frac{e\vec{E}\cdot\vec{v}}{T_e} f_{Me} \right\}$$

RHS : $\underline{\int d\vec{v} \frac{eE}{m_e} \frac{2v_{\parallel}^2}{V_{te}^2} f_{Me} L_k^{(3/2)}(x)} = \pi V_{te}^3 \int_{-1}^{1} d\xi \int_0^\infty x^{\frac{1}{2}} dx \frac{eE}{m_e} \frac{2x^2 \xi^2}{V_{te}^2} \frac{n_e}{\pi^{3/2} V_{te}^3} e^{-x} L_k^{(3/2)}(x)$

$$= \frac{4\pi}{3} \frac{eE}{m_e} \frac{n_e}{\pi^{3/2}} \int_0^\infty x^{3/2} e^{-x} L_k^{(3/2)}(x)\, dx = \frac{\cancel{4}}{3\sqrt{\pi}} \frac{n_e eE}{m_e} \cdot \frac{3\sqrt{\pi}}{\cancel{4}} \delta_{k0} = \underline{\frac{n_e eE}{m_e} \delta_{k0}}$$

LHS : is highly complicated. Let's take the more systematic scheme.

⑥ Laguerre moments by Sonine generating function.

$$\int d\vec{v}\, v_{\parallel} \sum_{k=0}^{\infty} a_k L_k^{(3/2)}(x) \left(C_{ee}^{l} + C_{ei}\right)\left[\frac{2v_{\parallel}}{v_{te}^2} f_{Me} \sum_{s=0}^{\infty} a_s L_s^{(3/2)}(x)\right]$$

$$= 2\sum_{k=0}^{\infty}\sum_{s=0}^{\infty} a_k a_s \left\langle \xi x^{1/2} L_k^{(3/2)}(x) f_{Me},\ \left(C_{ee}^{l} + C_{ei}\right)\left[\xi x^{1/2} L_s^{(3/2)}(x) f_{Me}\right]\right\rangle$$

$$= -n_e \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} a_k a_s \left(2\hat{\nu}_{ee} H_{ks}^{ee} + \hat{\nu}_{ei} H_{ks}^{ei}\right)$$

where
$$H_{ks}^{ee} = -\frac{1}{n_e \hat{\nu}_{ee}} \left\langle \xi x^{1/2} L_k^{(3/2)}(x) f_{Me},\ C_{ee}^{l}\left[\xi x^{1/2} L_s^{(3/2)}(x) f_{Me}\right]\right\rangle$$

$$H_{ks}^{ei} = -\frac{2}{n_e \hat{\nu}_{ei}} \left\langle \xi x^{1/2} L_k^{(3/2)}(x) f_{Me},\ C_{ei}\left[\xi x^{1/2} L_s^{(3/2)}(x) f_{Me}\right]\right\rangle$$

It is much easier to use generating function at this point.

The associated Laguerre polynomials $L_k^{(\gamma)}(x)$, as known as Sonine polynomials, have generating function.

$$S_\gamma(\xi, x) = \frac{1}{(1-\xi)^{\gamma+1}} \exp\left(-\frac{x\xi}{1-\xi}\right) = \sum_{k=0}^{\infty} \xi^k L_k^{(\gamma)}(x)$$

↳ L_k를 포함한 복잡한 적분을 k, s의 모든 조합에 대해 수행하는 대신

$S_\gamma(\xi, x) = \sum_{k=0}^{\infty} \xi^k L_k^{(\gamma)}(x)$ 로 묶어서, 지수함수 적분으로 문제를 바꾸었다

If one evaluates:

$$G^{ee}(\xi, \eta) = -\frac{\sqrt{2}}{n_e \hat{\nu}_{ee}} \left\langle \xi x^{1/2} S_{3/2}(\xi, x) f_{Me},\ C_{ee}^{l}\left[\xi x^{1/2} S_{3/2}(\eta, x) f_{Me}\right]\right\rangle$$

$$G^{ei}(\xi, \eta) = -\frac{2}{n_e \hat{\nu}_{ei}} \left\langle \xi x^{1/2} S_{3/2}(\xi, x) f_{Me},\ C_{ei}^{l}\left[\xi x^{1/2} S_{3/2}(\eta, x) f_{Me}\right]\right\rangle$$

one can obtain H coefficient values by (ξ, η) series expansion:

$$G^{ee}(\xi, \eta) = \sqrt{2}\sum_{k=0}^{\infty}\sum_{s=0}^{\infty} H_{ks}^{ee} \xi^k \eta^s, \quad G^{ei}(\xi, \eta) = \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} H_{ks}^{ei} \xi^k \eta^s$$

☆ ⇒ 복잡한 라게르 다항식 적분을 일일이 하지 말고, 지수함수 꼴인 생성함수 (G)를 한번만 적분한 뒤, 테일러 전개하여 계수 (H)를 뽑아내자.

* Recall

$$\langle g_1, C_{aa}^{\ell}[f_1]\rangle = -\frac{L_{aa}}{8\pi}\int d\vec{v}\int d\vec{v}' f_{Ma}(\vec{v}) f_{Ma}(\vec{v}')\frac{d\hat{g}_a(\vec{v})}{d\vec{v}}\cdot\overleftrightarrow{U}\cdot\left(\frac{d\hat{f}_a(\vec{v})}{d\vec{v}} - \frac{d\hat{f}_a(\vec{v}')}{d\vec{v}'}\right)$$

to evaluate $\langle g_1, C_{aa}^{\ell}[f_1]\rangle$, one will need to evaluate

$$\frac{d}{d\vec{v}}\left(\xi x^{1/2} S_{3/2}(\xi,x)\right) = V_{te}^{-1}\frac{d}{d\vec{v}}\left[S_{3/2}(\xi,x)V_z\right] = V_{te}^{-1}\left[S_{3/2}(\xi,x)\frac{dV_z}{d\vec{v}} + V_z\frac{d}{d\vec{v}}\left(S_{3/2}(\xi,x)\right)\right]$$

$$= V_{te}^{-1}\left[S_{3/2}(\xi,x)\hat{e}_3 + v_z\frac{dx}{d\vec{v}}\frac{dS}{dx}\right] = \frac{S_{3/2}}{V_{te}}\left[\hat{e}_3 - V_z\frac{2\vec{v}}{V_{te}^2}\frac{\xi}{1-\xi}\right]$$

$$= V_{te}^{-1}S_{3/2}(\xi,x)\,\vec{S}(\xi,\vec{v}) \quad \text{where} \quad \vec{S}(\xi,\vec{v}) \equiv \hat{e}_3 - \frac{\xi}{1-\xi}\frac{m_e V_z\vec{v}}{T_e}$$

So one can write the G^{ee} integral: $\left(\text{* note } L^{ab} = \left(\frac{e_a e_b}{m_a\epsilon_0}\right)^2\ln\Lambda,\ \hat{\nu}_{ee} \equiv \frac{n_e e^4\ln\Lambda}{12\pi^{3/2}\epsilon_0^2 m_e^{1/2} T_e^{3/2}}\right)$

$$G^{ee} = \frac{\sqrt{2}}{n_e\hat{\nu}_{ee}}\frac{L_{ee}}{8\pi}\int d\vec{v}\int d\vec{v}' f_{Me}(\vec{v})f_{Me}(\vec{v}')\frac{S_{3/2}(\xi,x)\vec{S}(\xi,\vec{v})}{V_{te}}\cdot\overleftrightarrow{U}\cdot\left(\frac{S_{3/2}(\eta,x)\vec{S}(\eta,\vec{v})}{V_{te}} - \frac{S_{3/2}(\eta,x')\vec{S}(\eta,\vec{v}')}{V_{te}}\right)$$

$$= \frac{3\sqrt{\pi}}{4}V_{te}\frac{1}{n_e^2}\int d\vec{v}\int d\vec{v}' f_{Me}(\vec{v})f_{Me}(\vec{v}')$$

$$\times\left[S_{3/2}(\xi,x)S_{3/2}(\eta,x)\vec{S}(\xi,\vec{v})\cdot\overleftrightarrow{U}\cdot\vec{S}(\eta,\vec{v}) - S_{3/2}(\xi,x)S_{3/2}(\eta,x')\vec{S}(\xi,\vec{v})\cdot\overleftrightarrow{U}\cdot\vec{S}(\eta,\vec{v}')\right]$$

* note that

$$S_{3/2}(\xi,x) = \frac{1}{(1-\xi)^{5/2}}\exp\left(-\frac{x\xi}{1-\xi}\right),\quad S_{3/2}(\eta,x) = \frac{1}{(1-\eta)^{5/2}}\exp\left(-\frac{x\eta}{1-\eta}\right)$$

$$f_{Me} = n_e\left(\frac{1}{\pi V_{te}^2}\right)^{3/2}\exp\left(-\frac{v^2}{V_{te}^2}\right) = \frac{n_e}{\pi^{3/2}V_{te}^3}\exp(-x)$$

$$\Rightarrow G^{ee} = \frac{1}{(1-\xi)^{5/2}(1-\eta)^{5/2}}\left(I_1^{ee} + I_2^{ee}\right)$$

이거 단순화 할 예정!

where

$$\begin{cases} I_1^{ee} = \dfrac{3}{4\pi^{5/2}}\left(\dfrac{m_e}{2T_e}\right)^{5/2}\displaystyle\int d\vec{v}\int d\vec{v}'\exp\Big(\underbrace{-\frac{1-\xi\eta}{(1-\xi)(1-\eta)}x - x'}\Big)\vec{S}(\xi,\vec{v})\cdot\overleftrightarrow{U}\cdot\vec{S}(\eta,\vec{v}) \\ I_2^{ee} = -\dfrac{3}{4\pi^{5/2}}\left(\dfrac{m_e}{2T_e}\right)^{5/2}\displaystyle\int d\vec{v}\int d\vec{v}'\exp\left(-\frac{1}{1-\xi}x - \frac{1}{1-\eta}x'\right)\vec{S}(\xi,\vec{v})\cdot\overleftrightarrow{U}\cdot\vec{S}(\eta,\vec{v}') \end{cases}$$

Let $\vec{u} = \vec{v} - \vec{v}'$, $x = \frac{m_e v^2}{2T_e}$, $x' = \frac{m_e v'^2}{2T_e}$, $\vec{Y} = \frac{m_1\vec{v} + m_e\vec{v}'}{m_1+m_e}$

and $m_1 \equiv \frac{1-\xi\eta}{(1-\xi)(1-\eta)} m_e$, $\vec{v} = \vec{Y} + \frac{m_e}{m_1+m_e}\vec{u}$, $\vec{v}' = \vec{Y} - \frac{m_1}{m_1+m_e}\vec{u}$

$$\exp\left(-\frac{m_1 v^2}{2T_e} - \frac{m_e v'^2}{2T_e}\right) = \exp\Big[-\Big(m_1 Y^2 + \frac{m_1 m_e^2}{(m_1+m_e)^2}u^2 + \cancel{2\vec{Y}\cdot\vec{u}\frac{m_1 m_e}{m_1+m_e}}\Big)/2T_e$$

$$- \Big(m_e Y^2 + \frac{m_e m_1^2}{(m_1+m_e)^2}u^2 - \cancel{2\vec{Y}\cdot\vec{u}\frac{m_1 m_e}{m_1+m_e}}\Big)/2T_e\Big]$$

$$= \exp\left[\left(-(m_1+m_e)Y^2 - \frac{m_1 m_e}{m_1+m_e}u^2\right)/2T_e\right]$$

$$\vec{S}(\xi,\vec{v}) \equiv \hat{e}_3 - \frac{\xi}{1-\xi}\frac{m_e v_z \vec{v}}{T_e} = \hat{e}_3 - \frac{\xi}{1-\xi}\frac{m_e}{T_e}\left(Y_z + \frac{m_e}{m_1+m_e}u_z\right)\left(\vec{Y} + \frac{m_e}{m_1+m_e}\vec{u}\right)$$

$$= \hat{e}_3 - \frac{\xi}{1-\xi}\frac{m_e}{T_e}\left(Y_z + \frac{m_e}{m_1+m_e}u_z\right)\vec{Y} \quad (\because \text{ no contribution of } \vec{u} \text{ due to } \vec{u}\cdot\overleftrightarrow{U} = 0)$$

$$\therefore I_1^{ee} = \frac{3}{4\pi^{5/2}}\left(\frac{m_e}{2T_e}\right)^{5/2}\int d\vec{Y}\int d\vec{u}\,\exp\left[\left(-(m_1+m_e)Y^2 - \frac{m_1 m_e}{m_1+m_e}u^2\right)/2T_e\right]$$
$$\times \vec{S}_1(\xi,\vec{Y},\vec{u})\cdot\overleftrightarrow{U}\cdot\vec{S}(\xi,\vec{Y},\vec{u})$$

For example, one can obtain:

$$\int d\vec{u}\,\exp\left(-\frac{m_1 m_e}{m_1+m_e}\frac{u^2}{2T_e}\right)\left(\frac{u^2\overleftrightarrow{I} - \vec{u}\vec{u}}{u^3}\right) = \frac{8\pi(m_1+m_e)T_e}{3m_1 m_e}\overleftrightarrow{I}$$

$$\int d\vec{u}\,\exp\left(-\frac{m_1 m_e}{m_1+m_e}\frac{u^2}{2T_e}\right)\left(\frac{u^2\overleftrightarrow{I} - \vec{u}\vec{u}}{u^3}\right)u_z = 0$$

$$\int d\vec{u}\,\exp\left(-\frac{m_1 m_e}{m_1+m_e}\frac{u^2}{2T_e}\right)\left(\frac{u^2\overleftrightarrow{I} - \vec{u}\vec{u}}{u^3}\right)u_z^2 = \frac{32\pi(m_1+m_e)^2 T_e^2}{15 m_1^2 m_e^2}\left(\overleftrightarrow{I} - \frac{1}{2}\hat{z}\hat{z}\right).$$

Then, one can complete the integral for $\vec{Y}$. Then we obtain

$$\left\{\begin{aligned} I_1^{ee} &= \frac{(1-\xi)^{5/2}(1-\eta)^{5/2}}{(1-\xi\eta)^2(2-\xi-\eta)^{5/2}}\left(4-2\xi-2\eta+3\xi\eta-3\xi^2\eta^2\right) \\ I_2^{ee} &= -\frac{2(1-\xi)^{5/2}(1-\eta)^{5/2}}{(2-\xi-\eta)^{5/2}}\left(4-2\xi-2\eta+3\xi\eta\right) \end{aligned}\right\}$$

I_2^{ee} similarly as done in I_1^{ee}.

Adding I_1^{ee} and I_2^{ee}, one can finally obtain :

$$\begin{cases} G^{ee}(\xi,\eta) = \dfrac{\xi\eta}{(1-\xi\eta)^2(2-\xi-\eta)^{5/2}}\left(8-4\xi-4\eta-\xi\eta+2\xi^2\eta+2\xi\eta^2-3\xi^2\eta^2\right) \\ G^{ei}(\xi,\eta) = \dfrac{3m_e}{8\pi T_e}\dfrac{1}{(1-\xi)^{5/2}(1-\eta)^{5/2}}\displaystyle\int d\vec{v}\left(-\frac{1-\xi\eta}{(1-\xi)(1-\eta)}x\right)\frac{v^2-v_z^2}{v^3} \\ \qquad = \dfrac{1}{(1-\xi\eta)(1-\xi)^{3/2}(1-\eta)^{3/2}} \end{cases}$$

They are followed by Taylor expansion for (ξ,η) to obtain the H coefficients.

$$H_{ij}^{ee} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 3/4 & 15/32 \\ 0 & 3/4 & 45/16 & 309/128 \\ 0 & 15/32 & 309/128 & 5627/1024 \end{pmatrix}$$

$$H_{ij}^{ei} = \begin{pmatrix} 1 & 3/2 & 15/8 & 35/16 \\ 3/2 & 13/4 & 69/16 & 165/32 \\ 15/8 & 69/16 & 433/64 & 1077/128 \\ 35/16 & 165/32 & 1077/128 & 2957/256 \end{pmatrix}$$

Now, turn to $\displaystyle\int d\vec{v}\, v_\parallel L_k^{(3/2)}(x) \times \left\{ C_{ee}^{l}[f_{sh}] + \nu_{ei}(v)\mathcal{L}[f_{sh}] = \frac{e\vec{E}\cdot\vec{v}}{T_e} f_{Mc} \right\}$ ... (21)

$$= -n_e \sum_{k=0}^{\infty}\sum_{s=0}^{\infty} a_k a_s \left(2\hat{\nu}_{ee} H_{ks}^{ee} + \hat{\nu}_{ei} H_{ks}^{ei}\right) \quad \cdots \ (26)$$

$$\Rightarrow \quad -2n_e\hat{\nu}_{ee}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 3/4 \\ 0 & 3/4 & 45/16 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} - n_e\hat{\nu}_{ei}\begin{pmatrix} 1 & 3/2 & 15/8 \\ 3/2 & 13/4 & 69/16 \\ 15/8 & 69/16 & 433/64 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \frac{n_e e E}{m_e}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

⑰ Spitzer-Harm distribution and conductivity

Using $\hat{\nu}_{ee} = \hat{\nu}_{ei}/(\sqrt{2}Z)$ when $Zn_i = n_e$, one can invert:

$$\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = -\frac{eE}{\hat{\nu}_{ei} m_e} \times \frac{1}{1+\frac{61\sqrt{2}}{72}Z+\frac{2}{9}Z^2} \times \begin{pmatrix} 1+\frac{151\sqrt{2}}{72}Z+\frac{217}{288}Z^2 \\ -\frac{5\sqrt{2}}{8}Z-\frac{11}{24}Z^2 \\ -\frac{\sqrt{2}}{6}Z+\frac{1}{12}Z^2 \end{pmatrix}$$

Finally, conductivity is:

$$j_{\parallel} = -en_e u_{\parallel} = -en_e a_0 = \sigma E \cdot \frac{1+151\sqrt{2}/72\cdot Z+217/288\cdot Z^2}{1+61\sqrt{2}/72\cdot Z+2/9\cdot Z^2}$$

For $Z=1$, $j_{\parallel} \simeq 1.95\sigma E$ $\quad \left(\alpha_e = \frac{1}{1.95} \simeq 0.51\right)$

To finish, the Spitzer-Härm solution for distribution function:

$$f_{e1} = \frac{2v_{\parallel}}{v_{te}^2} f_{Me}\left[a_0 + a_1\left(\frac{5}{2}-\frac{v^2}{v_{te}}\right) + a_2\left(\frac{35}{8}-\frac{7}{2}\frac{v^2}{v_{te}^2}+\frac{1}{2}\frac{v^4}{v_{te}^4}\right)\right] \quad \blacksquare$$