9. Hamilton's equations

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Motivation

Legendre transformation

From Lagrange's equations to Hamilton's equations

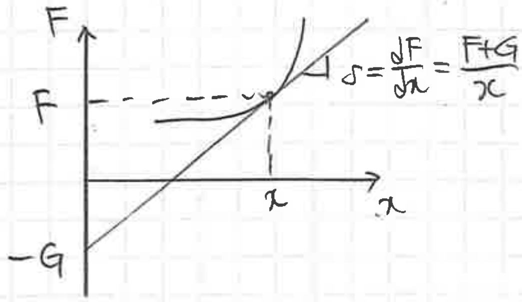
Modified Hamilton's principle
)

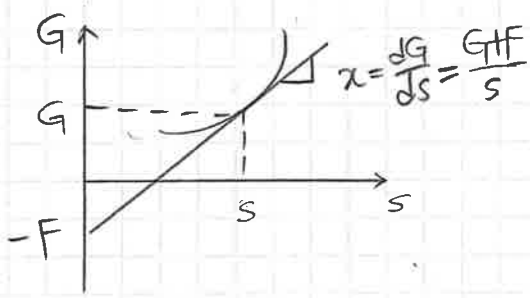
① Motivation

- Lagrangian mechanics : $L(q_r, \dot{q}_r, t)$: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_r} - \frac{\partial L}{\partial q_r} = 0$
  - momenta are treated as functions of positions and velocities : $p(q_r, \dot{q}_r, t) \equiv \frac{\partial L}{\partial \dot{q}_r}$
- Hamiltonian mechanics : $H(q_r, p, t)$: $\dot{q}_r = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q_r}$
  - positions and momenta are treated as independent variable.
- Advantage
  i) canonical transformation (정준변환)을 통해,
  dynamical variable을 자유롭게 고를수 있다
  ii) compatible with quantum mechanics
  iii) phase-space 의 개념 → 통계물리에서 유용

② Legendre transformation

Goal: Given a convex function $F(x)$, find a function $G(s)$; $s = F'(x)$
that preserve all features of $F(x)$.





$$G(s) = s\,x(s) - F(x(s)) \iff F(x) = x\,s(x) - G(s(x))$$

(1대1 대응)

③ From Lagrange's equations to Hamilton's equation

Typically, $\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}$ (inertia) is positive. $\Rightarrow \underline{p} \Leftrightarrow \dot{\underline{q}}$ 1대1 대응

$$\Rightarrow \boxed{H(\underline{q}, \underline{p}, t) = \underline{p} \cdot \dot{\underline{q}} - L(\underline{q}, \dot{\underline{q}}, t)}$$

- example

i) $T = \frac{1}{2} \dot{\underline{q}}^T \underline{\underline{M}}(\underline{q}) \dot{\underline{q}}$, $V = V(\underline{q})$. $L = T - V$

$$\Rightarrow H = \underline{p} \cdot \dot{\underline{q}} - L = T + V$$

ii) $L = \frac{1}{2} m \dot{\underline{x}}^2 - q\phi(\underline{x}) + q\underline{A}(\underline{x}) \cdot \dot{\underline{x}}$, $\frac{\partial L}{\partial \dot{\underline{x}}} = m\dot{\underline{x}} + q\underline{A} = \underline{p}$

$$\Rightarrow H = \underline{p} \cdot \dot{\underline{x}} - L = \frac{1}{2} m \dot{\underline{x}}^2 + q\phi = \frac{1}{2m}(\underline{p} - q\underline{A})^2 + q\phi$$

- Derivation of Hamilton's equations

From Legendre transformation: $H(\underline{q}, \underline{p}, t) = \underline{p} \cdot \dot{\underline{q}} - L(\underline{q}, \dot{\underline{q}}, t)$,

$$dH = \dot{\underline{q}} \cdot d\underline{p} + \cancel{\underline{p} \cdot d\dot{\underline{q}}} - \underbrace{\frac{\partial L}{\partial \underline{q}}}_{=\dot{\underline{p}}} \cdot d\underline{q} - \underbrace{\cancel{\frac{\partial L}{\partial \dot{\underline{q}}} \cdot d\dot{\underline{q}}}}_{=\underline{p}} - \frac{\partial L}{\partial t} dt = -\dot{\underline{p}}\, d\underline{q} + \dot{\underline{q}}\, d\underline{p} - \frac{\partial L}{\partial t} dt$$

From variation of Hamiltonian,

$$dH = \left(\frac{\partial H}{\partial \underline{q}}\right) d\underline{q} + \left(\frac{\partial H}{\partial \underline{p}}\right) d\underline{p} + \left(\frac{\partial H}{\partial t}\right) dt$$

$$\therefore \boxed{\dot{\underline{q}} = \frac{\partial H}{\partial \underline{p}}, \quad \dot{\underline{p}} = -\frac{\partial H}{\partial \underline{q}}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

- Cyclic coordinates and conservation law

  - $\frac{\partial L}{\partial q_i} = 0 \rightarrow \dot{p}_i = -\frac{\partial H}{\partial q_i} = \frac{\partial L}{\partial q_i} = 0$ ($q \xrightarrow{\text{cyclic}} L$, then also $q \xrightarrow{\text{cyclic}} H$, then conjugate p is conserved.)

  - $\frac{\partial L}{\partial t} = 0 \rightarrow \frac{dH}{dt} = \underbrace{\frac{\partial H}{\partial q}}_{=\dot{p}...}$

$$\frac{dH}{dt} = \frac{\partial H}{\partial q}\cdot\dot{q} + \frac{\partial H}{\partial p}\cdot\dot{p} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t} = 0$$

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

④ Modified Hamilton's principle

- Statement of the principle

$$I[q,p] \equiv \int_{t_1}^{t_2} dt \left[\underbrace{p\cdot\dot{q} - H(q,p,t)}_{\text{Lagrangian}} \right] \Leftarrow \text{stationary value for actual path.}$$

- Derivation of Hamilton's equations

integration by parts : $\cancel{[p\,\delta q]_{t_1}^{t_2}}^{\,0} - \int_{t_1}^{t_2} dt\, \dot{p}\, \delta q$

$$\delta I = \int_{t_1}^{t_2} dt \left(\dot{q}\,\delta p + p\,\delta\dot{q} - \frac{\partial H}{\partial q}\delta q - \frac{\partial H}{\partial p}\delta p \right)$$

$$= \int_{t_1}^{t_2} dt \left[\left(\dot{q} - \frac{\partial H}{\partial p}\right)\delta p - \left(\dot{p} + \frac{\partial H}{\partial q}\right)\delta q \right] = 0$$

$$\therefore \boxed{\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}}$$

- Remarks

  - Hamilton's principle / Modified Hamilton's principle explore different path spaces. nonetheless they predict the same actual path

  - $\delta p(t_2) = \delta p(t_1) = 0$ is not necessary, but imposing them allows us to treat them on a same footing.