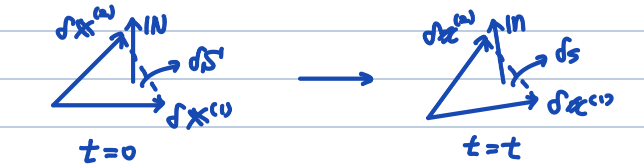
Ch9. Further Analysis of finite deformation

9.1 Deformation of a surface element

* area expansion & surface normal direction



$$\mathbb{N}\,dS = \frac{1}{2}\, d\mathbb{X}^{(1)} \times d\mathbb{X}^{(2)} \longleftrightarrow N_R\, dS = \frac{1}{2} e_{RST}\, dX_S^{(1)} dX_T^{(2)}$$

$$\mathbb{n}\, ds = \frac{1}{2}\, dx^{(1)} \times dx^{(2)} \longleftrightarrow n_i\, ds = \frac{1}{2} e_{ijk}\, dx_j^{(1)} dx_k^{(2)}$$

$$= \frac{1}{2} e_{ijk} \frac{dx_j}{dX_S}\frac{dx_k}{dX_T}\, dX_S^{(1)} dX_T^{(2)} + O(dX^3)$$

$$n_i \frac{dx_i}{dX_R}\, ds = \frac{1}{2} e_{ijk} \frac{dx_i}{dX_R}\frac{dx_j}{dX_S}\frac{dx_k}{dX_T}\, dX_S^{(1)} dX_T^{(2)} + O(dX^3)$$

$$= \frac{1}{2} e_{RST} \det \mathbb{F}\, dX_S^{(1)} dX_T^{(2)} + O(dX^3)$$

$$= \det \mathbb{F}\, N_R\, dS + O(dX^3)$$

$dX^{(1)}, dX^{(2)} \to 0$, $\quad n_i \frac{dx_i}{dX_R}\frac{ds}{dS} = \det \mathbb{F}\, N_R \longrightarrow N_R N_R = 1 = (\det \mathbb{F})^{-2}\, n_i \frac{dx_i}{dX_R}\, n_j \frac{dx_j}{dX_R} \left(\frac{ds}{dS}\right)^2$

$$\to \boxed{\left(\frac{ds}{dS}\right)^2 = \frac{(\det \mathbb{F})^2}{n_i B_{ij} n_j}} \quad \text{or} \quad \boxed{\mathbb{N} \det \mathbb{F} = \mathbb{n}\cdot\mathbb{F}\left(\frac{ds}{dS}\right) \ \& \ \left(\frac{ds}{dS}\right)^2 = \frac{(\det \mathbb{F})^2}{\mathbb{n}\cdot\mathbb{B}\cdot\mathbb{n}}}$$

Similarly, $\boxed{\mathbb{n} \det \mathbb{F}^{-1} = \mathbb{N}\cdot\mathbb{F}^{-1}\frac{dS}{ds} \quad \underline{\text{and}} \quad \left(\frac{dS}{ds}\right)^2 = \frac{(\det \mathbb{F}^{-1})^2}{\mathbb{N}\cdot\mathbb{C}^{-1}\cdot\mathbb{N}}}$ 10/30 과제

9.2 Decomposition of a deformation

- Recall polar decomposition theorem

$\mathbb{F} = \mathbb{R}\cdot\mathbb{U} = \mathbb{V}\cdot\mathbb{R}$ ($\mathbb{R}$: orthogonal, $\mathbb{U}$ & $\mathbb{V}$: positive-definite, symmetric)

Also, $\det \mathbb{F} > 0$ (since $\det \mathbb{F} = \frac{\rho}{\rho_0}$) $\to$ $\mathbb{R}$: proper rotation (no inversion)

$\mathbb{U} = \mathbb{R}^T \mathbb{V} \mathbb{R} \longleftrightarrow \mathbb{V} = \mathbb{R}\mathbb{U}\mathbb{R}^T$.

Components of $\mathbb{F}$: all constant.

First, Consider the <u>homogeneous</u> deformation $\underline{x = \mathbb{F}\mathbb{X}}$

Consider the two successive homogeneous deformation ($\mathbb{X} \to \hat{x} \to x$)

$$\hat{x} = \underbrace{\mathbb{U}}_{\text{Deformation}}\cdot\mathbb{X}, \quad x = \underbrace{\mathbb{R}}_{\text{Rotation}}\cdot\hat{x} \longrightarrow x = \mathbb{R}\cdot\hat{x} = \mathbb{R}\mathbb{U}\mathbb{X} = \mathbb{F}\mathbb{X}$$

$\therefore$ Any homogeneous deformation $\mathbb{F}$ can be decomposed into

① a deformation which correspond to the symmetric tensor, $\mathbb{U}$

followed by the rotation $\mathbb{R}$. (늘리고, 돌리기)

② the same rotation $\mathbb{R}$ followed by a deformation, which correspond

to the symmetric tensor $\mathbb{V}$. (돌리고, 늘리기)

Next, if not homogeneous, $d\mathbb{x} = \mathbb{F}\, d\mathbb{X}$

$\mathbb{F}, \mathbb{R}, \mathbb{U}, \mathbb{V}$: function of position

local decomposition and rotation

$\mathbb{R}$: rotation tensor

$\left.\begin{array}{l}\mathbb{U}\text{: right stretch tensor} \\ \mathbb{V}\text{: left stretch tensor}\end{array}\right\}$ $\quad \begin{array}{l}\mathbb{C} = \mathbb{F}^T\mathbb{F} = \mathbb{U}^2 \\ \mathbb{B} = \mathbb{F}\mathbb{F}^T = \mathbb{V}^2\end{array}$

(square root of eigenvalue)

Because $\mathbb{U}, \mathbb{V}$ are symmetric & positive definite, we can recover $\mathbb{U}$ from ①, $\mathbb{V}$ from ②

$\therefore \mathbb{U} \leftrightarrow \mathbb{C} \quad (\mathbb{C} = \mathbb{U}^2)$

$\mathbb{V} \leftrightarrow \mathbb{B} \quad (\mathbb{B} = \mathbb{V}^2)$

direct physical meaning

easier to calculate

$\mathbb{F} = \mathbb{I} + \underline{\mathbb{E}} + \underline{\mathbb{\Omega}}$

infinitesimal strain tensor $\left(= \frac{\mathbb{F}+\mathbb{F}^T}{2} - \mathbb{I}\right)$

infinitesimal rotation tensor $\left(= \frac{\mathbb{F}-\mathbb{F}^T}{2}\right)$

$\mathbb{U}^2 = \mathbb{F}^T\mathbb{F} = (\mathbb{I}+\mathbb{E}-\mathbb{\Omega})(\mathbb{I}+\mathbb{E}+\mathbb{\Omega}) \simeq \mathbb{I} + 2\mathbb{E}$

$\rightarrow \boxed{\mathbb{U} \simeq \mathbb{I}+\mathbb{E}}$ Similarly, $\boxed{\mathbb{V} \simeq \mathbb{I}+\mathbb{E}}$

$\therefore \mathbb{U} - \mathbb{I} = \mathbb{V} - \mathbb{I} = \mathbb{E}$ (small deformation rotation)

Also, $\mathbb{U}^{-1} = \mathbb{I} - \mathbb{E}$

$\mathbb{R} = \mathbb{F}\mathbb{U}^{-1} \simeq (\mathbb{I}+\mathbb{E}+\mathbb{\Omega})\cdot(\mathbb{I}-\mathbb{E}) \simeq \mathbb{I} + \mathbb{\Omega}$ $\quad \boxed{\therefore \mathbb{R} - \mathbb{I} \simeq \mathbb{\Omega}}$

9.3 Principal stretch & principal axes of deformation

- Recall : line element, $\underbrace{\lambda}_{\text{stretch}} \tilde{a} = \mathbb{F}\cdot\tilde{\mathbb{A}}$ unit vector ,

$\mathbb{F} \rightarrow \mathbb{U}$, $\lambda a = \mathbb{U}\cdot\mathbb{A}$

If $a = \mathbb{A}$ (no rotation during $\mathbb{U}$), $\underline{(\mathbb{U} - \lambda\mathbb{I})\mathbb{A} = \mathbb{O}}$

λ: principal value of $\mathbb{U}$ (λ : real and positive)

$\mathbb{A}$: principal direction of $\mathbb{U}$.

(since $\mathbb{U}$ is symmetric,)

$\lambda_1 \geq \lambda_2 \geq \lambda_3$: principal stretches

A_1, A_2, A_3 : principal axes (mutually orthogonal)

$\xrightarrow{\text{if chosen to be coord. axes.}} (U_{RS}) = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$

- If the principal axes of $\mathbb{U}$ are chosen to be the coord axes,

$$(U_{RS}) = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}$$

$\mathbb{F} = \mathbb{R}\mathbb{U}$: consists of these 3 extension followed by the rotation $\mathbb{R}$. ($\mathbb{U}\cdot\mathbb{A} = \lambda\mathbb{A}$)

$\mathbb{F} = \mathbb{V}\mathbb{R}$: rotation $\mathbb{R}$ followed by 3 extension represented by $\mathbb{V}$. ($\mathbb{F}\cdot\mathbb{A} = \lambda\mathbb{R}\cdot\mathbb{A}$)

$\mathbb{R}\mathbb{A}$ should be the principal axes of $\mathbb{V}$. $\mathbb{V}(\mathbb{R}\mathbb{A}) = \lambda(\mathbb{R}\mathbb{A})$

$\therefore \lambda_1, \lambda_2, \lambda_3$: principal stretches of $\mathbb{U}$, and also the principal values of $\mathbb{V}$,

and the corresponding principal directions are $\mathbb{R}\cdot\mathbb{A}_1$, $\mathbb{R}\cdot\mathbb{A}_2$, $\mathbb{R}\cdot\mathbb{A}_3$.

($\mathbb{U}$와 $\mathbb{V}$ 모두 principal stretch 동일
but principal axes는 $\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3$ 에서 $\mathbb{R}\mathbb{A}_1, \mathbb{R}\mathbb{A}_2, \mathbb{R}\mathbb{A}_3$ 로 변경)

$$\mathbb{C} = \mathbb{F}^T\mathbb{F} = \mathbb{U}^2, \quad \gamma = \frac{1}{2}(\mathbb{C} - \mathbb{I}) \leftarrow \mathbb{A}_i = \lambda_i^2, \frac{\lambda_i^2 - 1}{2}$$

$$\mathbb{B} = \mathbb{F}\mathbb{F}^T = \mathbb{V}^2, \quad \eta = \frac{1}{2}(\mathbb{I} - \mathbb{B}^{-1}) \leftarrow \mathbb{R}\cdot\mathbb{A}_1 = \lambda_i^2, \frac{1 - \lambda_i^{-2}}{2}$$

Given $\mathbb{F}$, easier to calculate $\mathbb{C}$ & $\mathbb{B}$ than $\mathbb{U}$ & $\mathbb{V}$.

- Alternative way of interpretation

From $\mathbb{U}\cdot\mathbb{A} = \lambda\mathbb{A}$, $\mathbb{A}\mathbb{U}^T\mathbb{U}\mathbb{A} = \lambda^2 \longrightarrow A_R A_S C_{RS} = \lambda^2$

Given direction $\mathbb{A}$, we can calculate λ, extension ratio.

Q. When is the extremal value of λ^2 or $A_R C_{RS} A_S$ under the constraint of $A_R A_R = 1$?

$$\frac{d}{dA_p}\left\{A_R A_S C_{RS} - \mu^2(A_R A_R - 1)\right\} = 0 \quad \text{(라그랑주 승수법)}$$

Using $C_{RS} = C_{SR}$, $(C_{PR} - \mu^2 \delta_{PR})A_R = 0$

$\therefore$ The directions of $\mathbb{A}$ for which λ^2 is extremal are the principal direction of $\mathbb{C}$.

The corresponding values of λ^2 are the principal values of $\mathbb{C}$.

Similar analysis for $\mathbb{B}$: $\mathbb{U}\mathbb{A} = \lambda\mathbb{A} \longrightarrow \mathbb{R}\mathbb{U}\mathbb{A} = \lambda\mathbb{R}\mathbb{A}$

$\mathbb{V}\cdot(\mathbb{R}\mathbb{A}) = \lambda(\mathbb{R}\mathbb{A})$, also principal values of $\mathbb{B}$.

9.4 Strain invariants

$\lambda_1, \lambda_2, \lambda_3$: invariants (i.e, independent of the reference frame)

↳ principal values of $\mathbb{U}$ and $\mathbb{V}$.

$\lambda_1^2, \lambda_2^2, \lambda_3^2$: principal values of $\mathbb{C}$ and $\mathbb{B}$.

$\mathbb{R} \cdot$ (principal axes of $\mathbb{C}$) = (principal axes of $\mathbb{B}$)

or $\mathbb{U}^2 = \mathbb{C}$, $\mathbb{U} = \sqrt{\mathbb{C}}$, $\mathbb{R} = \mathbb{F}\mathbb{U}^{-1} = \mathbb{F}\mathbb{C}^{-\frac{1}{2}}$

$\mathbb{C} = \mathbb{M}^T \mathbb{D} \mathbb{M} \longrightarrow \mathbb{C}^{-\frac{1}{2}} = \mathbb{M}^T \mathbb{D}^{-\frac{1}{2}} \mathbb{M}$ (11/1 과제)

9.5 Alternative stress measures

T_{ij} : stress in the current config

In some cases, more convenient to describe in the reference config.

$$\begin{cases} \mathbb{N} \det \mathbb{F} = \mathbb{n}\,\mathbb{F} \dfrac{ds}{dS} \\ \mathbb{n} \det \mathbb{F}^{-1} = \mathbb{N} \cdot \mathbb{F}^{-1} \dfrac{dS}{ds} \end{cases}$$

reference config surface normal $\mathbb{e}_R \xrightarrow{\text{deform}} \mathbb{n}_R$ in current config.

$$\mathbb{n}_R = (\det \mathbb{F}) \frac{dS}{ds} \mathbb{e}_R \mathbb{F}^{-1}$$

The force on this deformed surface : $\Pi_R\, dS$ (정의)

$$\Pi_R = \Pi_{Ri}\mathbb{e}_i \longrightarrow \Pi_R\, dS = \mathbb{n}_R \cdot \mathbb{T}\, ds \longrightarrow \left[\Pi_{Rj} \cancel{\mathbb{e}_j}\, \cancel{dS} = (\det \mathbb{F}) \frac{\cancel{dS}}{\cancel{ds}} \cancel{\mathbb{e}_R} \cdot \mathbb{F}^{-1} \cdot \mathbb{T}\, \cancel{ds}\right] \mathbb{e}_i$$

$$\rightarrow \Pi_{Ri} = (\det \mathbb{F}) F^{-1}_{Rj} T_{ji} \text{ , or, } \boxed{\Pi = (\det \mathbb{F}) \mathbb{F}^{-1} \cdot \mathbb{T} \longleftrightarrow \mathbb{T} = (\det \mathbb{F})^{-1} \mathbb{F} \cdot \Pi}$$

Π : (nominal stress tensor
first Piola-Kirchhoff stress tensor)

$$\mathbb{t}^{(N)} = \mathbb{N} \cdot \Pi \quad (\text{cf. } \mathbb{t}^{(n)} = \mathbb{n} \cdot \mathbb{T})$$

Equation of motion : $\dfrac{\partial \Pi_{Ri}}{\partial X_R} + \rho_0 b_i = \rho_0 f_i$ $\quad$ (cf. $\dfrac{\partial}{\partial x_i} T_{ij} + \rho b_j = \rho f_j$)

Torque equation : should have references to x → NOT useful.

$$\Pi_{Ri} \neq \Pi_{iR}$$

The second Piola-Kirchhoff tensor $\mathbb{P} \rightarrow P_{RS}$

$\mathbb{P} = \Pi \cdot (\mathbb{F}^{-1})^T = (\det \mathbb{F}) \mathbb{F}^{-1} \cdot \mathbb{T} \cdot (\mathbb{F}^{-1})^T$ ⟶ Symmetric but no direct interpretation.

$$\rightarrow \Pi = \mathbb{P} \cdot \mathbb{F}^T , \quad \mathbb{T} = (\det \mathbb{F})^{-1} \mathbb{F}\,\mathbb{P}\,\mathbb{F}^T$$