8. Special theory of relativity

- Basic postulate
- Lorentz transformation
- Vectors and tensors
- 4-vectors in the Minkowski spacetime
- Manifestly covariant formalism for electromagnetism
- Lagrangian formulation of relativistic field theories

① Basic postulate

A. Galilean invariance

$$\underline{x}' = \underline{x} - \underline{v}t, \quad t' = t \qquad ((x,t) \text{ in } S, \ (x',t') \text{ in } S')$$

$$\underline{F} = m\frac{d^2x}{dt^2} = m\frac{d^2x'}{dt'^2} \qquad (\text{Galilean invariance})$$

$$\Delta t = t_2 - t_1 = t_2' - t_1' = \Delta t' \qquad (\text{universal elapsed time})$$

cf) $\underline{u}' = \frac{dx'}{dt} = \frac{dx}{dt} - \underline{v} = \underline{u} - \underline{v}$ ← (no universal velocity)

However, $c = c'$ (constancy of light velocity)

B. Invariant spacetime interval

- Einstein's basic postulates for special theory of relativity
  - (1) The laws of physics are the same to all inertial observers.
  - (2) Speed of light c is the same to all inertial observers.

- Minkowski spacetime

$$\boxed{\Delta S^2 = (c\Delta t)^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2)} \qquad \text{"Invariant spacetime interval"}$$

Different inertial observer use different spacetime coordinates,
but ΔS between two events in spacetime is the same to all of them

(Emission of light : $\Delta S = 0$)

- proper time interval (고유시간)

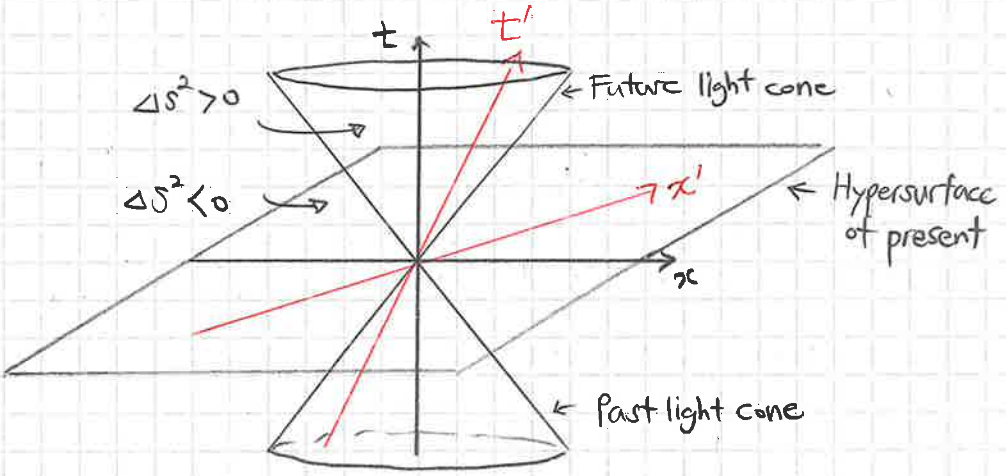
  두 사건을 한 장소에서 측정한 관찰자의 시간 ($\Delta s^2 > 0$)

  $$\boxed{\Delta s^2 = c^2 \Delta\tau^2}$$

- time dilation (시간 지연, 시간 팽창)

  $$\Delta s^2 = c^2\Delta\tau^2 = (c\Delta t)^2 - (v\Delta t)^2 \rightarrow \boxed{\Delta t = \frac{\Delta\tau}{(1-v^2/c^2)^{\frac{1}{2}}} = \frac{\Delta\tau}{(1-\beta^2)^{\frac{1}{2}}} > \Delta\tau}$$

  Define, $\beta = \frac{v}{c}$, $\gamma = \frac{1}{(1-\beta^2)^{1/2}}$ $\quad$ $\hookrightarrow \Delta t = \gamma \Delta\tau$

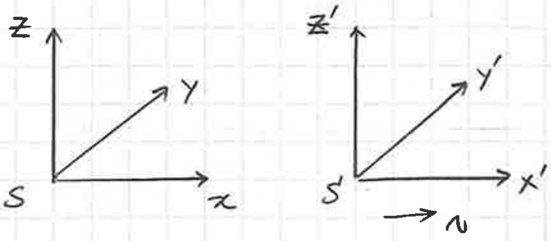


$\Delta s^2 > 0$: 모든 관찰자들은 동일한 사건 선후 관계를 관찰

$\Delta s^2 < 0$: 관찰자에 따라 사건 전후가 바뀔수 있음

② Lorentz transformations

A. Formula for the Lorentz boost in the x-direction



By symmetry : $\underline{dy = dy'}$, $\underline{dz = dz'}$

Invariance of spacetime interval : $\underline{c^2dt^2 - dx^2 = c^2dt'^2 - dx'^2}$

$$\begin{bmatrix} c\,dt' \\ dx' \end{bmatrix} = \underbrace{\begin{bmatrix} L_{00} & L_{01} \\ L_{10} & L_{11} \end{bmatrix}}_{\underline{\underline{L}}} \begin{bmatrix} c\,dt \\ dx \end{bmatrix}$$

which satisfies $\underset{a^T}{\underline{\underline{L}}^T} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \underset{a}{\underline{\underline{L}}} = \underset{a^T}{} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \underset{a}{}$ $\left(a = \begin{bmatrix} c\,dt \\ dx \end{bmatrix}\right)$

to ensure $c^2dt^2 - dx^2 = c^2dt'^2 - dx'^2$.

$\Rightarrow L_{00}^2 - L_{10}^2 = 1$, $L_{01}^2 - L_{11}^2 = -1$, $L_{00}L_{01} - L_{10}L_{11} = 0$

$\Rightarrow$ General solution is given by $\underline{L_{00} = L_{11} = \cosh\psi}$, $\underline{L_{10} = L_{01} = \sinh\psi}$

$(\because \cosh^2\psi - \sinh^2\psi = 1)$

$\Rightarrow$ For two events that occur at the origin of frame S',

$dx' = L_{10}c\,dt + L_{11}\,dx \xrightarrow[(\text{양변을 } \frac{t}{dt} \text{ 곱하기})]{} 0 = ct\sinh\psi + vt\cosh\psi$

$\therefore \underline{\tanh\psi = -\frac{v}{c} = -\beta}$, $\underline{\cosh\psi = (1-\tanh^2\psi)^{-1/2} = \gamma}$, $\underline{\sinh\psi = -\beta\gamma}$

∴ Lorentz boost in the x-direction

$$\boxed{\begin{bmatrix} c\,dt' \\ dx' \\ dy' \\ dz' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\,dt \\ dx \\ dy \\ dz \end{bmatrix}}$$

B. Formula for the general boosts

- Suppose that observer S' is moving at $\underline{v}$ w.r.t observer S.

  Then, defining $\underline{\beta} = \underline{v}/c$, gives the relation:

$$\begin{bmatrix} c\,dt' \\ \frac{1}{\beta^2}\underline{\beta}\,\underline{\beta}\cdot d\underline{x}'\,\underline{\beta} \\ d\underline{x}' - \frac{1}{\beta^2}\underline{\beta}\,\underline{\beta}\cdot d\underline{x}'\,\underline{\beta} \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma\underline{\beta}^T & 0 \\ -\gamma\underline{\beta} & \gamma\underline{\underline{1}} & 0 \\ 0 & 0 & \underline{\underline{1}} \end{bmatrix} \begin{bmatrix} c\,dt \\ \frac{1}{\beta^2}\underline{\beta}\cdot d\underline{x}\,\underline{\beta} \\ d\underline{x} - \frac{1}{\beta^2}\underline{\beta}\cdot d\underline{x}\,\underline{\beta} \end{bmatrix}$$

$$\Rightarrow \boxed{c\,dt' = \gamma\,(c\,dt - \underline{\beta}\cdot d\underline{x})}$$

$$d\underline{x}' = \frac{1}{\beta^2}\underline{\beta}\cdot d\underline{x}'\,\underline{\beta} + \left(d\underline{x}' - \frac{1}{\beta^2}\underline{\beta}\cdot d\underline{x}'\,\underline{\beta}\right)$$

$$= -\underline{\beta}\gamma c\,dt + \gamma\frac{1}{\beta^2}\underline{\beta}\cdot d\underline{x}\,\underline{\beta} + \left(d\underline{x} - \frac{1}{\beta^2}\underline{\beta}\cdot d\underline{x}\,\underline{\beta}\right)$$

$$\boxed{d\underline{x}' = d\underline{x} + \frac{\gamma-1}{\beta^2}\underline{\beta}\cdot d\underline{x}\,\underline{\beta} - \underline{\beta}\gamma c\,dt}$$

(note that for $\beta \ll 1$, $d\underline{x}' = d\underline{x} - \underline{v}\,dt$)
($\gamma \simeq 1$)

- Relativistic addition of velocities

$$\boxed{\underline{u}' = \frac{d\underline{x}'}{dt'} = \frac{\underline{u} + (\gamma-1)(\underline{\beta}\cdot\underline{u})\underline{\beta}/\beta^2 - \underline{\beta}\gamma c}{\gamma\,(1 - \underline{\beta}\cdot\underline{u}/c)}}$$

(note that for $\beta \ll 1$, $\underline{u}' = \underline{u} - \underline{v}$)

③ Vectors and tensors

A. Covariant and contravariant components of vectors

$$\left\{\begin{array}{l}\text{basis vector } \{\underline{e}_\lambda\},\\ \text{inverse basis vector } \{\underline{e}^\lambda\}\end{array}\right\} \Rightarrow \underline{e}_\lambda \cdot \underline{e}^\mu = \delta^\mu_\lambda$$

$$\left\{\begin{array}{ll}\text{covariant components} & : V_\lambda = \underline{V}\cdot\underline{e}_\lambda \Leftrightarrow \underline{V} = V^\lambda \underline{e}_\lambda\\ \text{contravariant components} & : V^\lambda = \underline{V}\cdot\underline{e}^\lambda \Leftrightarrow \underline{V} = V_\lambda \underline{e}^\lambda\end{array}\right.$$

- Change of basis :

$$\left.\begin{array}{l}\underline{e}'_\lambda = \Lambda^\mu_\lambda \underline{e}_\mu\\ \underline{e}'^\lambda = \tilde{\Lambda}^\lambda_\mu \underline{e}^\mu\end{array}\right\} \Rightarrow \delta^\lambda_\mu = \underline{e}'^\lambda \cdot \underline{e}'_\mu = \tilde{\Lambda}^\lambda_\nu \Lambda^\rho_\mu \underline{e}^\nu \underline{e}_\rho = \tilde{\Lambda}^\lambda_\nu \Lambda^\nu_\mu \Rightarrow \boxed{\begin{array}{l}\underline{e}'_\lambda = \Lambda^\mu_\lambda \underline{e}_\mu\\ \underline{e}'^\lambda = (\Lambda^{-1})^\lambda_\mu \underline{e}^\mu\end{array}}$$

$$\therefore \tilde{\Lambda}^\lambda_\mu = (\Lambda^{-1})^\lambda_\mu \rightarrow \underline{\underline{e}'^\lambda = (\Lambda^{-1})^\lambda_\mu \underline{e}^\mu}$$

Thus $\boxed{\begin{array}{l}V'_\lambda = \underline{V}\cdot\underline{e}'_\lambda = \Lambda^\mu_\lambda \underline{V}\cdot\underline{e}_\mu = \Lambda^\mu_\lambda V_\mu,\\ V'^\lambda = \underline{V}\cdot\underline{e}'^\lambda = (\Lambda^{-1})^\lambda_\mu \underline{V}\cdot\underline{e}^\mu = (\Lambda^{-1})^\lambda_\mu V^\mu\end{array}}$ ← covariant / ← contravariant.

B. Scalars and tensors

- Scalar remains invariant under a change of basis.
- tensor of rank (n,p) $T^{\lambda_1\cdots\lambda_n}_{\mu_1\cdots\mu_p} \rightarrow \boxed{T'^{\lambda_1\cdots\lambda_n}_{\mu_1\cdots\mu_p} = (\Lambda^{-1})^{\lambda_1}_{\nu_1}\cdots(\Lambda^{-1})^{\lambda_n}_{\nu_n}\Lambda^{\rho_1}_{\mu_1}\cdots\Lambda^{\rho_p}_{\mu_p} T^{\nu_1\cdots\nu_n}_{\rho_1\cdots\rho_p}}$

↳ 이렇게 transform 하는 대상을 ≡ "텐서"

- $$\underline{u}\cdot\underline{v} = \left\{\begin{array}{l}(u^\lambda \underline{e}_\lambda)\cdot(v^\mu \underline{e}_\mu) = g_{\lambda\mu} u^\lambda v^\mu\\ (u_\lambda \underline{e}^\lambda)\cdot(v_\mu \underline{e}^\mu) = g^{\lambda\mu} u_\lambda v_\mu\\ (u^\lambda e_\lambda)\cdot(v_\mu \underline{e}^\mu) = u^\lambda v_\lambda\\ (u_\lambda e^\lambda)\cdot(v^\mu \underline{e}_\mu) = u_\lambda v^\lambda\end{array}\right. \quad v_\lambda = g_{\lambda\mu} e^\mu, \quad v^\lambda = g^{\lambda\mu} e_\mu$$

where $\boxed{g_{\lambda\mu} \equiv \underline{e}_\lambda\cdot\underline{e}_\mu}$, $\boxed{g^{\lambda\mu} \equiv \underline{e}^\lambda\cdot\underline{e}^\mu}$ ((inverse) metric tensor)

④ 4-vectors in the Minkowski spacetime

- $x^\lambda = (x^0, x^1, x^2, x^3) = (ct, x, y, z)$
- $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 = dx^\lambda dx_\lambda \to dx_\lambda = (cdt, -dx, -dy, -dz)$

$\Rightarrow \underline{x^\lambda = (ct, x, y, z),\ x_\lambda = (ct, -x, -y, -z)} \Rightarrow \boxed{\underline{\underline{g}} = \begin{bmatrix} 1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{bmatrix}}$

- <u>Del operator</u>

$\dfrac{dx^\lambda}{dx^\mu} = \delta^\lambda_\mu$, if $x'^\lambda = (\Lambda^{-1})^\lambda_\mu x^\mu$, then $\dfrac{d}{dx'^\lambda} = \Lambda^\mu_\lambda \dfrac{d}{dx^\mu}$ ⇐ covariant

$$\Rightarrow \boxed{\partial_\lambda \equiv \frac{\partial}{\partial x^\lambda} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right),\quad \partial^\lambda \equiv \frac{\partial}{\partial x_\lambda} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right)}$$

(covariant del operator) (contravariant del operator)

<u>4-Laplacian (d'Alembertian) operator : $\Box \equiv \partial^\lambda \partial_\lambda = \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2} - \nabla^2$</u>

- <u>4-velocity</u>

$u^\lambda \equiv \dfrac{dx^\lambda}{d\tau}$ (where τ is proper time)

$dt = \gamma d\tau \to u^\lambda = \dfrac{dx^\lambda}{dt}\dfrac{dt}{d\tau} = \gamma \dfrac{dx^\lambda}{dt}$

$\therefore \boxed{u^\lambda = \gamma(c, \dot{x}, \dot{y}, \dot{z}),\ u_\lambda = \gamma(c, -\dot{x}, -\dot{y}, -\dot{z})}$ $(u^\lambda u_\lambda = \gamma^2(c^2 - v^2) = c^2)$

- <u>4-momentum</u> and <u>covariant force</u>

i) $p^\lambda \equiv m u^\lambda \to \boxed{p^\lambda = \gamma m(c, \dot{x}, \dot{y}, \dot{z}),\ p_\lambda = \gamma m(c, -\dot{x}, -\dot{y}, -\dot{z})}$, $(p^\lambda p_\lambda = m^2c^2)$

↳ $p^0 = \gamma mc = \left(1 - \dfrac{v^2}{c^2}\right)^{-\frac{1}{2}} mc \simeq \left(1 + \dfrac{v^2}{2c^2}\right) mc = \dfrac{1}{c}\left(mc^2 + \dfrac{1}{2}mv^2\right) = \dfrac{E}{c}$

$\therefore \boxed{p^\lambda = (E/c, \underline{p}),\ p_\lambda = (E/c, -\underline{p})}$. $\left(p^\lambda p_\lambda = \dfrac{1}{c^2}(E^2 - p^2c^2) = m^2c^2\right)$

$E^2 - p^2c^2 = m^2c^4$, $E_0 = mc^2$

ii) $K^\lambda \equiv \dfrac{dp^\lambda}{d\tau} \Rightarrow \boxed{K^\lambda = \gamma\left(\dfrac{\dot{E}}{c}, \underline{F}\right),\ K_\lambda = \gamma\left(\dfrac{\dot{E}}{c}, -\underline{F}\right)}$

⑤ Manifestly covariant formalism for electromagnetism

- 4-potential : $A^\lambda = (\phi/c, \underline{A})$, $A_\lambda = (\phi/c, -\underline{A})$
- Electromagnetic field tensor (Faraday tensor) : $F_{\lambda\mu} = \partial_\lambda A_\mu - \partial_\mu A_\lambda$ $(= -F_{\mu\lambda})$
  - Gauge transformation : $A_\lambda \to A_\lambda' - \partial_\lambda X$ do not change $F_{\lambda\mu}$
  - Skew-symmetric : $\underline{F_{\lambda\mu} = -F_{\mu\lambda}}$ (diagonal entries $=0$)
  - For $i=1,2,3$; $\underline{F_{0i} = -F_{i0} = -\frac{1}{c}\frac{\partial A_i}{\partial t} - \frac{1}{c}\frac{\partial \phi}{\partial x_i} = \frac{E_i}{c}}$

    For $i,j \in \{1,2,3\}$: $\underline{F_{ij} = -F_{ji} = -\frac{\partial A_j}{\partial x^i} + \frac{\partial A_i}{\partial x^j} = -\epsilon_{ijk}(\partial_j A_k) = -\epsilon_{ijk}B_k}$
  - Contravariant form : $F^{\lambda\mu} = g^{\lambda\nu}F_{\nu\rho}g^{\rho\mu}$

    For $i,j \in \{1,2,3\}$, $\underbrace{F^{0i} = -F^{i0} = -E_i/c}_{\text{(부호 반대)}}$, $\underbrace{F^{ij} = -F^{ji} = -\epsilon_{ijk}B_k}_{\text{(그대로)}}$

  - $$F_{\lambda\mu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix}, \quad F^{\lambda\mu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}$$

- Covariant Lorentz force law

$$\boxed{K^\lambda = q F^{\lambda\mu} u_\mu}$$

$$\Rightarrow K^0 = q F^{0i} u_i = q\left(-\frac{E_i}{c}\right)(-\gamma v_i) = \frac{\gamma q}{c}\underline{E}\cdot\underline{v}$$

$$K^i = q(F^{i0}u_0 + F^{ij}u_j) = q\left(\frac{E_i}{\cancel{c}}\gamma \cancel{c} + \epsilon_{ijk}\gamma v_j B_k\right) = \gamma q(E_i + \epsilon_{ijk}v_j B_k)$$

$$\Rightarrow \dot{E} = q\vec{E}\cdot\vec{v}, \quad \vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$$

- Covariant Maxwell's equation

  - Homogeneous Maxwell's equation

$$\partial_\lambda F_{\mu\nu} + \partial_\nu F_{\lambda\mu} + \partial_\mu F_{\nu\lambda} = 0$$

(using levi-civita), $0 = \frac{1}{2}\varepsilon^{\lambda\mu\nu\rho}\partial_\mu F_{\nu\rho} = \partial_\mu \tilde{F}^{\lambda\mu} = -\partial_\mu \tilde{F}^{\mu\lambda}$

where $\tilde{F}^{\lambda\mu} = \frac{1}{2}\epsilon^{\lambda\mu\nu\rho}F_{\nu\rho} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{bmatrix}$

$$\boxed{\partial_\lambda \tilde{F}^{\lambda\mu} = 0} \Rightarrow \begin{cases} ① \ \mu=0, & \vec{\nabla}\cdot\vec{B} = 0 \\ ② \ \mu=1,2,3, & \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \end{cases}$$

These eqns are trivial mathematical identities stemming from $F_{\lambda\mu} = \partial_\lambda A_\mu - \partial_\mu A_\lambda$

  - Inhomogeneous Maxwell's equation

define: $j^\lambda \equiv (\rho c, \vec{j})$

$$\boxed{\partial_\lambda F^{\lambda\mu} = \mu_0 j^\mu} \Rightarrow \begin{cases} ① \ \mu=0, & \frac{1}{c}\vec{\nabla}\cdot\vec{E} = \mu_0\rho c \Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0} \\ ② \ \mu=1,2,3, & -\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t} + \vec{\nabla}\times\vec{B} = \mu_0\vec{j} \Rightarrow \vec{\nabla}\times\vec{B} = \mu_0\left(\vec{j} + \epsilon_0\frac{\partial\vec{E}}{\partial t}\right) \end{cases}$$

Maxwell's equations are covariant under Lorentz transformation

⑥ Lagrangian formulation of relativistic field theories

A. General formalism (follow the formalism of §7. continuum mechanics)

- covariant Hamilton's principle

$$\boxed{\delta \int_\Omega d^4x^\lambda \mathcal{L}(u_\rho, u_{\rho,\lambda} ; x^\lambda) = 0}$$

- covariant Euler-Lagrange equations

$$\boxed{\frac{\partial L}{\partial u_\rho} - \partial_\lambda \frac{\partial L}{\partial u_{\rho,\lambda}} = 0}$$

- Noether's theorem

$$\boxed{\partial_\lambda \left[\left(L\delta_\mu^\lambda - \frac{\partial L}{\partial u_{\rho,\lambda}} u_{\rho,\mu} \right) X^{\nu\mu} + \frac{\partial L}{\partial u_{\rho,\lambda}} U_\rho^\nu - G^{\nu\lambda} \right] = 0}$$

- covariant stress-energy tensor

$$\boxed{T_\mu^\lambda \equiv \frac{\partial L}{\partial u_{\rho,\lambda}} u_{\rho,\mu} - L\delta_\mu^\lambda , \quad T^{\mu\lambda} = g^{\mu\nu} T_\nu^\lambda = g^{\mu\nu} \frac{\partial L}{\partial u_{\rho,\lambda}} u_{\rho,\mu} - g^{\mu\lambda} L}$$

B. Application to the electromagnetic field

- Lagrangian density describing the electromagnetic field :

$$\boxed{L(A_\lambda, A_{\lambda,\mu} ; x^\mu) = -\frac{1}{4\mu_0} F_{\lambda\mu} F^{\lambda\mu} - j_\lambda(x^\mu) A^\lambda = \frac{\epsilon_0}{2}\left(\underline{E}^2 - c^2 \underline{B}^2\right) - \rho\phi + \underline{j}\cdot\underline{A}}$$

- Euler-Lagrange equations are given by

$$\frac{\partial L}{\partial A_\lambda} - \partial_\mu \frac{\partial L}{\partial A_{\lambda,\mu}} = -j^\lambda + \frac{1}{2\mu_0} \partial_\mu \left(F^{\nu\rho} \frac{\partial \overbrace{F_{\nu\rho}}^{A_{\rho,\nu} - A_{\nu,\rho}}}{\partial A_{\lambda,\mu}} \right) = -j^\lambda + \frac{1}{2\mu_0} \partial_\mu \left(F^{\mu\lambda} - F^{\lambda\mu} \right)$$

$$= -j^\lambda + \frac{1}{\mu_0} \partial_\mu F^{\mu\lambda} = 0 \qquad \therefore \boxed{\partial_\mu F^{\mu\lambda} = \mu_0 j^\lambda}$$