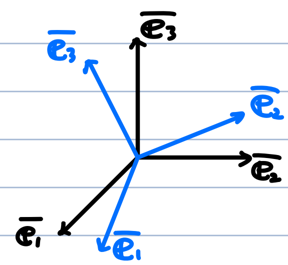
Ch3. Vectors & Cartesian tensor

3.1 Vectors

- $\mathbb{a}, \mathbb{b}$ ← Italic , ($\mathbb{a}, \mathbb{b}$: column matrix)
  (좌표변환과 같이 coordinate 같이 변하는 물리량 ; 본질적 값은 같음)
- $\mathbb{a} = a_i \mathbb{e}_i$
- $\mathbb{e}_i \cdot \mathbb{e}_j = \delta_{ij}$
- scalar product : $\mathbb{a} \cdot \mathbb{b} = ab\cos\theta = a_i b_i$
- vector product : $\mathbb{a} \times \mathbb{b} = \begin{vmatrix} \mathbb{e}_1 & \mathbb{e}_2 & \mathbb{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = e_{ijk} a_j b_k$
- $(\mathbb{a} \times \mathbb{b}) \cdot \mathbb{c} = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = e_{ijk} a_i b_j c_k$

3.2 Coordinate transformation (rotation)



- $\mathbb{a} = a_i \mathbb{e}_i = \bar{a}_i \bar{\mathbb{e}}_i$, $M_{ij} = \bar{\mathbb{e}}_i \cdot \mathbb{e}_j$
- $\bar{\mathbb{e}}_i = M_{ij} \mathbb{e}_j$ ($M_{ij}\mathbb{e}_j = \mathbb{e}_j(\mathbb{e}_j \cdot \bar{\mathbb{e}}_i) = \bar{\mathbb{e}}_i$)
- $M_{ir} M_{jr} = \delta_{ij}$ ($\delta_{ij} = \bar{\mathbb{e}}_i \cdot \bar{\mathbb{e}}_j = M_{ir}\mathbb{e}_r \cdot M_{js}\mathbb{e}_s = M_{ir}M_{js}\,\mathbb{e}_r \cdot \mathbb{e}_s = M_{ir}M_{js}\delta_{rs} = M_{ir}M_{jr}$)
  ↳ $\mathbb{M} \cdot \mathbb{M}^T = \mathbb{I}$ (∴ rotation transform $\mathbb{M}$ is an orthogonal matrix)

- $\bar{\mathbb{e}}_i = M_{ij} \mathbb{e}_j \longrightarrow \mathbb{e}_j = M_{ij} \bar{\mathbb{e}}_i$
  ($M_{ik} \bar{\mathbb{e}}_i = M_{ik} M_{ij} \mathbb{e}_j = \delta_{kj} \mathbb{e}_j = \mathbb{e}_k$)
- $\bar{a}_i \bar{\mathbb{e}}_i = a_j \mathbb{e}_j = a_j M_{ij} \bar{\mathbb{e}}_i$ ∴ $\bar{a}_i = M_{ij} a_j$

basis와 vector는 동일하게 transform 된다.

* proper / improper rotation
(회전) (반사,회전)

$\det \mathbb{M} = \pm 1$

$$\begin{cases} \text{vector} : \bar{a}_i = M_{ij} a_j \\ \text{pseudo-vector} : \bar{a}_i = (\det \mathbb{M}) M_{ij} a_j \end{cases}$$

different if $\det \mathbb{M} = -1$.

examples of pseudo vector : $\mathbb{a} \times \mathbb{b}$
regarding rotation
(각속도, 돌림힘 …)

3.3 The dyadic product

- $\mathbb{a}, \mathbb{b} \longrightarrow \mathbb{a} \cdot \mathbb{b}$, $\mathbb{a} \times \mathbb{b}$, $\mathbb{a} \otimes \mathbb{b}$
  (스칼라) (벡터) (second-order tensor)

- $\begin{cases} (\alpha \mathbb{a}) \otimes \mathbb{b} = \mathbb{a} \otimes (\alpha \mathbb{b}) = \alpha (\mathbb{a} \otimes \mathbb{b}) \\ \mathbb{a} \otimes (\mathbb{b} + \mathbb{c}) = \mathbb{a} \otimes \mathbb{b} + \mathbb{a} \otimes \mathbb{c} \\ (\mathbb{b} + \mathbb{c}) \otimes \mathbb{a} = \mathbb{b} \otimes \mathbb{a} + \mathbb{c} \otimes \mathbb{a} \end{cases}$

  ex) $\mathbb{e}_1 \otimes \mathbb{e}_3$ $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} (0\ 0\ 1) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

- $\mathbb{a} \otimes \mathbb{b} = (a_i \mathbb{e}_i) \otimes (b_j \mathbb{e}_j) = a_i b_j (\mathbb{e}_i \otimes \mathbb{e}_j) \neq b_i a_j \mathbb{e}_i \otimes \mathbb{e}_j = \mathbb{b} \otimes \mathbb{a}$ ∴ $\mathbb{a} \otimes \mathbb{b} \neq \mathbb{b} \otimes \mathbb{a}$

- $\mathbb{a} \otimes \mathbb{b} = a_i b_j \mathbb{e}_i \otimes \mathbb{e}_j = \bar{a}_i \bar{b}_j \bar{\mathbb{e}}_i \otimes \bar{\mathbb{e}}_j$

- Inner product with a vector
  $(\mathbb{a} \otimes \mathbb{b}) \cdot \mathbb{c} = \mathbb{a} (\mathbb{b} \cdot \mathbb{c})$
  $\mathbb{a} \cdot (\mathbb{b} \otimes \mathbb{c}) = \mathbb{c} (\mathbb{a} \cdot \mathbb{b})$

(+) Triadic product
$\mathbb{a} \otimes \mathbb{b} \otimes \mathbb{c} \equiv a_i b_j c_k \mathbb{e}_i \otimes \mathbb{e}_j \otimes \mathbb{e}_k$

3.4 Cartesian tensors

- 2^{nd} order cartesian tensor → linear combination of dyads.
- $\mathbb{A} = A_{ij}\, e_i \otimes e_j = \overline{A_{ij}}\, \overline{e_i} \otimes \overline{e_j}$

$\Rightarrow A_{ij}(M_{pi}\overline{e_p}) \otimes (M_{qj}\overline{e_q}) = A_{ij} M_{pi} M_{qj}\, \overline{e_p} \otimes \overline{e_q}$ $\quad \therefore \overline{A_{pq}} = M_{pi} M_{qj} A_{ij}$

- order-n tensor : $\mathbb{A} = \underbrace{A_{ij\cdots m}}_{\text{n indices}}\, e_i \otimes e_j \otimes \cdots \otimes e_m$
- $\overline{A_{pq\cdots t}} = M_{pi} M_{qj} \cdots M_{tm} A_{ij\cdots m} \leftrightarrow A_{ij\cdots m} = M_{pi} M_{qj} \cdots M_{tm} \overline{A_{pq\cdots t}}$
- Suppose 2nd-order tensor,

if $A_{ij} = A_{ji}$, $\quad \overline{A_{qp}} = M_{qi} M_{pj} A_{ij} = M_{pj} M_{qi} A_{ji} = \overline{A_{pq}}$ (symmetry conserved while transformation)

if $A_{ij} = -A_{ji}$, $\quad \overline{A_{qp}} = M_{qi} M_{pj} A_{ij} = M_{pj} M_{qi} (-A_{ji}) = -\overline{A_{pq}}$ (as same for anti-symmetry)

- Transpose definition,

$A_{ij}^T \equiv A_{ji}$, $\overline{A_{pq}}^T = \overline{A_{qp}}$

↳ $\overline{A_{pq}^T} = \overline{A_{qp}} = M_{qi} M_{pi} A_{ij} = M_{pi} M_{qj} A_{ij}^T$ (transpose & transform conserved)

- $\mathbb{A} = A_{ij}\, e_i \otimes e_j \longrightarrow \mathbb{A}^T \equiv A_{ji}\, e_i \otimes e_j$

3.5 Isotropic tensors

- Unit tensor

$\mathbb{I} = \delta_{ij}\, e_i \otimes e_j = \delta_{ij} M_{ri} M_{sj}\, \overline{e_r} \otimes \overline{e_s}$

$= M_{rj} M_{sj}\, \overline{e_r} \otimes \overline{e_s} = \delta_{rs}\, \overline{e_r} \otimes \overline{e_s} = \delta_{ij}\, \overline{e_i} \otimes \overline{e_j}$

(어떤 coordinate system 에서도 component가 같으면 ⇒ Isotropic tensor)

- Isotropic tensor

$p\mathbb{I}$ (p is scalar, and 2nd order tensor)

- Suppose M_{ij} : small angle $\delta\phi$ rotation.

$\mathbb{M}\mathbb{C} = \mathbb{C} + \delta\phi \times \mathbb{C} = \begin{bmatrix} 1 & -\delta\phi_z & \delta\phi_y \\ \delta\phi_z & 1 & -\delta\phi_x \\ -\delta\phi_y & \delta\phi_x & 1 \end{bmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} \leftrightarrow \quad \overline{\mathbb{C}} = \overline{\mathbb{C}} + \delta\phi \times \overline{\mathbb{C}}$

$\overline{c_i} = c_i + e_{ijk}\delta\theta_j c_k$

$\overline{A_{ij}} = M_{ia} M_{jb} A_{ab} = A_{ij}$ ← isotropic tensor의 조건

↳ 증명 : (과제)

- 3rd-order isotropic tensor

$e_{ijk}\, e_i \otimes e_j \otimes e_k$ ← all isotropic tensors are of this form.

$\overline{e_{ijk}} = M_{ia} M_{jb} M_{kc} e_{abc} = e_{abc} M_{ai}^T M_{bj}^T M_{ck}^T = e_{ijk} \det M^T = e_{ijk} \det M = e_{ijk}$ (if $\det M = 1$)

- 4th-order isotropic tensor

$(\lambda \delta_{ij}\delta_{kl} + \mu \delta_{ik}\delta_{jl} + \nu \delta_{il}\delta_{jk})\, e_i \otimes e_j \otimes e_k \otimes e_l \longrightarrow$ (과제)

3.6 Multiplication of tensors

$\mathbb{a} = a_i \mathbb{e}_i \ , \ \mathbb{B} = B_{ij} \mathbb{e}_i \otimes \mathbb{e}_j \quad \Rightarrow \mathbb{C} = a_i B_{jk} \mathbb{e}_i \otimes \mathbb{e}_j \otimes \mathbb{e}_k$

$\mathbb{C} = \mathbb{a} \otimes \mathbb{B} \longrightarrow C_{ijk} = a_i B_{jk}$

$\bar{C}_{pqr} = M_{pi} M_{qj} M_{rk} a_i B_{jk} = M_{pi} M_{qj} M_{rk} C_{ijk}$

Likewise, $\mathbb{A} \otimes \mathbb{B} = \mathbb{D} \ , \ D_{ijkl} = A_{ij} B_{kl}$ (4th order tensor)

Contraction : $A_{ii} \longrightarrow \bar{A}_{ii} = M_{ij} M_{ik} A_{jk} = \delta_{jk} A_{jk} = A_{jj}$

sum 하면서 order가 2개 준다.

$(\tilde{C}_i = C_{ijj})$ $C_{ijj} \longrightarrow \bar{C}_{ijj} = M_{il} M_{jm} M_{jn} C_{lmn} = M_{il} \delta_{mn} C_{lmn} = \underline{M_{il} C_{lmm}}$

Likewise, $\mathbb{a} \cdot \mathbb{B} = (a_i \mathbb{e}_i) \cdot (B_{jk} \mathbb{e}_j \otimes \mathbb{e}_k) = \underline{a_i B_{ij} \mathbb{e}_j}$

$\mathbb{B} \cdot \mathbb{a} = (B_{ij} \mathbb{e}_i \otimes \mathbb{e}_j) \cdot (a_k \mathbb{e}_k) = \underline{a_i B_{ji} \mathbb{e}_j}$

vector

$\mathbb{A} \cdot \mathbb{B} = A_{ij} B_{jk} \mathbb{e}_i \otimes \mathbb{e}_k$

$\mathbb{A}^T \cdot \mathbb{B} = A_{ji} B_{jk} \mathbb{e}_i \otimes \mathbb{e}_k$

2nd-order tensor

$(\mathbb{A} \cdot \mathbb{B})^T = B_{ji} A_{kj} \mathbb{e}_i \otimes \mathbb{e}_k = \mathbb{B}^T \cdot \mathbb{A}^T$

Special case, If $\exists \mathbb{A}^{-1}$ s.t $\mathbb{A}^{-1} \cdot \mathbb{A} = \mathbb{I}, \mathbb{A} \cdot \mathbb{A}^{-1} = \mathbb{I}$,

$\mathbb{A}^{-1}$ is called the inverse tensor to $\mathbb{A}$.

If $\mathbb{A}^{-1} = \mathbb{A}^T$, $\mathbb{A}$ is an orthogonal tensor.

Polar decomposition : $\mathbb{F} \rightarrow F_{ij} = R_{ik} U_{kj} = V_{ik} R_{kj}$

$(\underline{\mathbb{R}} = R_{ij} \mathbb{e}_i \otimes \mathbb{e}_j \ , \ \underline{\mathbb{U}} = U_{ij} \mathbb{e}_i \otimes \mathbb{e}_j \ , \ \underline{\mathbb{V}} = V_{ij} \mathbb{e}_i \otimes \mathbb{e}_j)$

(orthogonal tensor) (positive definite symmetric tensor)

3.7 Tensor and matrix notation

$\mathbb{a}$: vector $\longrightarrow a_i$: component $\longrightarrow \mathrm{a}$: column matrix

$\mathbb{A}$: tensor $\longrightarrow A_{ij}$: component $\longrightarrow \mathrm{A}$: matrix

coordinate independent

coordinate dependent

$\bar{a}_i = M_{ij} a_j \longrightarrow \bar{\mathrm{a}} = \mathbb{M} \mathrm{a} \longleftrightarrow \mathrm{a} = \mathbb{M}^T \mathrm{a}$

$\bar{A}_{ij} = M_{ik} M_{jl} A_{kl} = M_{ik} A_{kl} (M^T)_{lj} \longrightarrow \bar{\mathbb{A}} = \mathbb{M} \mathbb{A} \mathbb{M}^T \longleftrightarrow \mathbb{A} = \mathbb{M}^T \bar{\mathbb{A}} \mathbb{M}$

Table 3.1 Examples of tensor and matrix notation

Direct tensor notation	Tensor component notation	Matrix notation
$\alpha = \boldsymbol{a} \cdot \boldsymbol{b}$	$\alpha = a_i b_i$	$(\alpha) = \mathbf{a}^T \mathbf{b}$
$\mathbf{A} = \boldsymbol{a} \otimes \mathbf{b}$	$A_{ij} = a_i b_j$	$\mathbf{A} = \mathbf{a} \mathbf{b}^T$
$\boldsymbol{b} = \mathbf{A} \cdot \boldsymbol{a}$	$b_i = A_{ij} a_j$	$\mathbf{b} = \mathbf{A} \mathbf{a}$
$\boldsymbol{b} = \boldsymbol{a} \cdot \boldsymbol{A}$	$b_j = a_i A_{ij}$	$\mathbf{b}^T = \mathbf{a}^T A$
$\alpha = \boldsymbol{a} \cdot \mathbf{A} \cdot \boldsymbol{b}$	$\alpha = a_i A_{ij} b_j$	$(\alpha) = \mathbf{a}^T \mathbf{A} \mathbf{b}$
$\boldsymbol{C} = \mathbf{A} \cdot \boldsymbol{B}$	$C_{ij} = A_{ik} B_{kj}$	$\mathbf{C} = \mathbf{A} \mathbf{B}$
$\boldsymbol{C} = \mathbf{A} \cdot \boldsymbol{B}^T$	$C_{ij} = A_{ik} B_{jk}$	$\mathbf{C} = \mathbf{A} \mathbf{B}^T$
$\boldsymbol{D} = \mathbf{A} \cdot \boldsymbol{B} \cdot \boldsymbol{C}$	$D_{ij} = A_{ik} B_{km} C_{mj}$	$\mathbf{D} = \mathbf{A} \mathbf{B} \mathbf{C}$

3.8 Invariants of a 2nd order tensor

$\mathbb{A} \begin{matrix} \nearrow \bar{\mathbb{A}} \\ \searrow \mathbb{A} \end{matrix} \Big\}$ λ's are the same. (Eigenvalues are intrinsic to the tensor)

$$0 = \det(\mathbb{A} - \lambda \mathbb{I}) = \det(\mathbb{M}\mathbb{A}\mathbb{M}^T - \lambda \mathbb{M}\mathbb{M}^T \mathbb{I}) = \det(\mathbb{M}(\mathbb{A} - \lambda \mathbb{I})\mathbb{M}^T) = \det(\mathbb{A} - \lambda \mathbb{I}) \cdot (\det \mathbb{M})^2$$

If $\mathbb{A}$ is symmetric, λ's are real and they are called the principal components,

and denote it $\longrightarrow A_1, A_2, A_3$

If A_1, A_2, A_3 is distinct, then the normalized eigenvectors $x^{(1)}, x^{(2)}, x^{(3)}$ are unique and mutually orthogonal, and

$\mathbb{A} \cdot x_i = A_i x_i$ (no summation)