2. Gyro-expansion

p.2 small m/q expansion

$$\frac{m}{q}\frac{d\vec{v}}{dt} = \vec{E} + \vec{v}\times\vec{B}$$

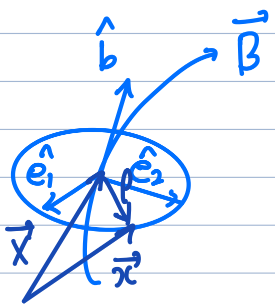
small "$\frac{m}{q}$" expansion

① $\frac{m}{q} = \epsilon \quad (\epsilon \ll 1)$

② $\frac{m}{q} \to \epsilon\frac{m}{q} \quad (\epsilon = 1) \quad \left(\text{eg. } \begin{matrix} \omega = \epsilon^{-1}\omega \\ \rho = \epsilon\rho \end{matrix}\right)$

$$\Rightarrow \boxed{\epsilon\frac{m}{q}\frac{d\vec{v}}{dt} = \vec{E} + \vec{v}\times\vec{B}}$$

p.3 Expansion from guiding center



$$\vec{x} = \vec{X} + \epsilon\vec{\rho}$$

$$\vec{B}(\vec{X}+\vec{\rho}) = \vec{B}(\vec{X}) + \epsilon(\vec{\rho}\cdot\vec{\nabla})\vec{B} + O(\epsilon^2)$$

$$\vec{E_\perp}(\vec{X}+\vec{\rho}) = \vec{E_\perp}(\vec{X}) + \epsilon(\vec{\rho}\cdot\vec{\nabla})\vec{E_\perp} + O(\epsilon^2)$$

$$\vec{E_\parallel}(\vec{X}+\vec{\rho}) = \epsilon\vec{E_\parallel}(\vec{X}) + \epsilon^2(\vec{\rho}\cdot\vec{\nabla})\vec{E_\parallel} + O(\epsilon^3)$$

p.4 Multiple time-scale in expansion

$$\vec{\rho_0} = \rho_0(\hat{e}_1\sin(\omega_0 t) + \hat{e}_2\cos(\omega_0 t))$$

$= \rho_0(\hat{e}_1\sin\gamma + \hat{e}_2\cos\gamma)$: gyrophase $\gamma = \gamma(t)$ $(\gamma_0 = \omega_0 t)$

$$\vec{\rho}(\vec{X},t) \longrightarrow \vec{\rho}(\vec{X},t,\gamma)$$

p.5 Time derivatives in gyroexpansion

$$\vec{v} = \frac{d\vec{X}}{dt} + \frac{d\vec{\rho}}{dt} \; :$$

$$\frac{d\vec{X}}{dt} = \underline{\vec{V_0} + \epsilon\vec{V_1}} + O(\epsilon^2)$$ ← we want to find $\vec{V_i}$ (various drift velocity)

$$\frac{d\vec{\rho}}{dt} = \left(\frac{d}{dt} + \dot{\gamma}\frac{d}{d\gamma}\right)\vec{\rho} = \left(\frac{d}{dt} + \omega\frac{d}{d\gamma}\right)\vec{\rho}$$ $(\omega(\vec{x},t) \equiv d\gamma/dt)$

$$= \left(\frac{d}{dt} + \epsilon^{-1}\omega\frac{d}{d\gamma}\right)\epsilon\vec{\rho} = \underline{\epsilon\frac{d\vec{\rho}}{dt} + \omega\frac{d\vec{\rho}}{d\gamma}}$$

Also. $\vec{\rho} = \vec{\rho_0} + \epsilon\vec{\rho_1} + O(\epsilon^2)$, $\omega = \omega_0 + \epsilon\omega_1 + O(\epsilon^2)$

$$\rightarrow \vec{v} = \vec{V_0} + \epsilon\vec{V_1} + \epsilon\frac{d\rho_0}{dt} + (\omega_0 + \epsilon\omega_1)\frac{d(\vec{\rho_0} + \epsilon\vec{\rho_1})}{d\gamma}$$

$$= \vec{V_0} + \epsilon\vec{V_1} + \omega_0\frac{d\vec{\rho_0}}{d\gamma} + \epsilon\left(\frac{d\vec{\rho_0}}{dt} + (\vec{V_0}\cdot\vec{\nabla})\vec{\rho_0} + \omega_0\frac{d\vec{\rho_1}}{d\gamma} + \omega_1\frac{d\vec{\rho_0}}{d\gamma}\right) + O(\epsilon^2)$$

$$\rightarrow \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{d\vec{X}}{dt} + \frac{d\vec{\rho}}{dt}\right) = \frac{d}{dt}\left(\vec{V_0} + \epsilon\vec{V_1}\right) + \frac{d}{dt}\left(\epsilon\frac{d\vec{\rho}}{dt} + \omega\frac{d\vec{\rho}}{d\gamma}\right)$$

$$= \frac{d\vec{V_0}}{dt} + \epsilon\frac{d\vec{V_1}}{dt} + \left(\frac{d}{dt} + (\vec{V_0} + \epsilon\vec{V_1} + \cdots)\cdot\vec{\nabla} + \epsilon^{-1}\omega\frac{d}{d\gamma}\right)\left(\epsilon\frac{d\vec{\rho}}{dt} + \omega\frac{d\vec{\rho}}{d\gamma}\right)$$

$$= \frac{d\vec{V_0}}{dt} + \epsilon\frac{d\vec{V_1}}{dt} + \left[\underline{\epsilon^{-1}\omega_0\frac{d}{d\gamma}} + \underline{\omega_1\frac{d}{d\gamma} + \frac{d}{dt} + \vec{V_0}\cdot\vec{\nabla}}\right] \times$$

$$\left[\underline{\omega_0\frac{d\vec{\rho_0}}{d\gamma}} + \epsilon\underline{\left(\frac{d\vec{\rho_0}}{dt} + (\vec{V_0}\cdot\vec{\nabla})\vec{\rho_0} + \omega_0\frac{d\vec{\rho_1}}{d\gamma} + \omega_1\frac{d\vec{\rho_0}}{d\gamma}\right)}\right] + \cdots$$

$$= \epsilon^{-1}\omega_0^2\frac{d^2\vec{\rho_0}}{d\gamma^2} + \frac{d\vec{V_0}}{dt} + \underline{\omega_0\left(\frac{d}{dt} + (\vec{V_0}\cdot\vec{\nabla})\right)\frac{d\vec{\rho_0}}{d\gamma} + \omega_0^2\frac{d^2\vec{\rho_1}}{d\gamma^2} + \omega_0\omega_1\frac{d^2\vec{\rho_0}}{d\gamma^2}}$$

$$+ \underline{\omega_0\omega_1\frac{d^2\vec{\rho_0}}{d\gamma^2} + \omega_0\left(\frac{d}{dt} + \vec{V_0}\cdot\vec{\nabla}\right)\frac{d\vec{\rho_0}}{d\gamma}} + \underline{\frac{d\vec{\rho_0}}{d\gamma}\left(\frac{d}{dt} + \vec{V_0}\cdot\vec{\nabla}\right)\omega_0}$$ ← $\omega = \omega(\vec{x},t)$

$$= \underbrace{\epsilon^{-1}\omega_0^2\frac{d^2\vec{\rho_0}}{d\gamma^2}}_{O(\epsilon^{-1})} + \underline{\frac{d\vec{V_0}}{dt} + 2\omega_0\left(\frac{d}{dt} + \vec{V_0}\cdot\vec{\nabla}\right)\frac{d\vec{\rho_0}}{d\gamma} + \omega_0^2\frac{d^2\vec{\rho_1}}{d\gamma^2} + 2\omega_0\omega_1\frac{d^2\vec{\rho_0}}{d\gamma^2}}$$

$$\underline{+ \frac{d\vec{\rho_0}}{d\gamma}\left(\frac{d}{dt} + \vec{V_0}\cdot\vec{\nabla}\right)\omega_0} + O(\epsilon)$$

$$\therefore \epsilon\frac{m}{q}\frac{d\vec{v}}{dt} = \frac{m}{q}\left[\omega_0^2\frac{d^2\vec{\rho_0}}{d\gamma^2} + \epsilon\left(\frac{d\vec{V_0}}{dt} + 2\omega_0\left(\frac{d}{dt} + \vec{V_0}\cdot\vec{\nabla}\right)\frac{d\vec{\rho_0}}{d\gamma} + \omega_0^2\frac{d^2\vec{\rho_1}}{d\gamma^2} + 2\omega_0\omega_1\frac{d^2\vec{\rho_0}}{d\gamma^2}\right)\right] \quad \cdots ①$$

p.8 Expansion for electromagnetic fields

$$\vec{E}+\vec{v}\times\vec{B}=\vec{E_\perp}+\varepsilon(\vec{\rho}\cdot\vec{\nabla})\vec{E_\perp}+\varepsilon\vec{E_\parallel}$$

(all 1st order)

$$+\left(\vec{v_0}+\epsilon\vec{v_1}+\epsilon\frac{d\vec{\rho}}{dt}+\omega\underline{\frac{d\vec{\rho}}{d\gamma}}\right)\times\left(\vec{B}+\epsilon(\vec{\rho}\cdot\vec{\nabla})\vec{B}\right)$$

$$=\vec{E_\perp}+\vec{v_0}\times\vec{B}+\omega_0\frac{d\vec{\rho_0}}{d\gamma}\times\vec{B}$$

$$+\epsilon\Big[(\vec{\rho_0}\cdot\vec{\nabla})\vec{E_\perp}+\vec{E_\parallel}+\vec{v_0}\times(\vec{\rho_0}\cdot\vec{\nabla})\vec{B}+\vec{v_1}\times\vec{B}$$

$$+\left(\frac{d}{dt}+\vec{v_0}\cdot\vec{\nabla}\right)\vec{\rho_0}\times\vec{B}+\omega_0\frac{d\vec{\rho_0}}{d\gamma}\times(\vec{\rho_0}\cdot\vec{\nabla})\vec{B}+\underline{\omega_0\frac{d\vec{\rho_1}}{dt}\times\vec{B}+\omega_1\frac{d\vec{\rho_0}}{dt}\times\vec{B}}\Big]$$

$\cdots$ ②

p.9 Zeroth-order drift motion

① + ② for $\sim O(1)$

$$\omega_0^2\frac{d^2\vec{\rho_0}}{d\gamma^2}+\frac{q}{m}\vec{B}\times\frac{d\vec{\rho_0}}{d\gamma}=\frac{q}{m}\left(\vec{E_\perp}+\vec{v_0}\times\vec{B}\right)$$

- homogeneous solution : $\vec{\rho_0}=\rho_0(\hat{e}_1\sin\gamma+\hat{e}_2\cos\gamma)$
- particular solution : $\vec{v_{\perp 0}}=\dfrac{\vec{E}\times\vec{B}}{B^2}\equiv\vec{v_E}$: leading-order drift

(taking gyro-average → LHS = 0)

p.10 First-order (correction) equation

$$\frac{d\vec{v_0}}{dt}+2\omega_0\left(\frac{d}{dt}+\vec{v_0}\cdot\vec{\nabla}\right)\frac{d\vec{\rho_0}}{d\gamma}+\omega_0^2\frac{d^2\vec{\rho_1}}{d\gamma^2}+2\omega_0\omega_1\frac{d^2\vec{\rho_0}}{d\gamma^2}$$

$$=\frac{q}{m}\Big[(\vec{\rho_0}\cdot\vec{\nabla})\vec{E_\perp}+\vec{E_\parallel}+\vec{v_0}\times(\vec{\rho_0}\cdot\vec{\nabla})\vec{B}+\vec{v_1}\times\vec{B}$$

$$+\left(\frac{d}{dt}+\vec{v_0}\cdot\vec{\nabla}\right)\vec{\rho_0}\times\vec{B}+\omega_0\frac{d\vec{\rho_0}}{d\gamma}\times(\vec{\rho_0}\cdot\vec{\nabla})\vec{B}+\omega_0\frac{d\vec{\rho_1}}{dt}\times\vec{B}+\omega_1\frac{d\vec{\rho_0}}{dt}\times\vec{B}\Big]$$

Take gyro-average : $$\frac{d\vec{v_0}}{dt}=\frac{q}{m}\Big[\vec{E_\parallel}+\vec{v_1}\times\vec{B}+\omega_0\underbrace{\left\langle\frac{d\vec{\rho_0}}{d\gamma}\times(\vec{\rho_0}\cdot\vec{\nabla})\vec{B}\right\rangle}_{(*)}\Big]$$

$$\left\langle \frac{d\vec{\rho}_0}{d\nu}\times(\vec{\rho}_0\cdot\vec{\nabla})\vec{B}\right\rangle \qquad (\vec{\rho}_0=\hat{x}\sin\nu+\hat{y}\cos\nu)$$

$$=\rho_0^2\left\langle(\hat{x}\cos\nu-\hat{y}\sin\nu)\times\left(\sin\nu\frac{d}{dx}+\cos\nu\frac{d}{dy}\right)(B_x\hat{x}+B_y\hat{y}+B_z\hat{z})\right\rangle$$

$$=\frac{1}{2}\rho_0^2\left(\hat{x}\frac{d}{dy}-\hat{y}\frac{d}{dx}\right)\times(B_x\hat{x}+B_y\hat{y}+B_z\hat{z})$$

$$=\frac{1}{2}\rho_0^2\left(\frac{dB_y}{dy}\hat{z}-\frac{dB_z}{dy}\hat{y}+\frac{dB_x}{dx}\hat{z}-\frac{dB_z}{dx}\hat{x}\right)$$

$$=\frac{1}{2}\rho_0^2\left(\left(\frac{dB_x}{dx}+\frac{dB_y}{dy}+\underline{\frac{dB_z}{dz}}\right)\hat{z}-\left(\frac{dB_z}{dx}+\frac{dB_z}{dy}+\underline{\frac{dB_z}{dz}}\right)\hat{z}\right)=-\frac{1}{2}\rho_0^2\vec{\nabla}B$$

$\vec{\nabla}\cdot\vec{B}=0$

<magnetic moment>

$$\mu\equiv\frac{1}{2}q\omega_0\rho_0^2\to\frac{mv_\perp^2}{2B}+O(\epsilon)$$

$$\therefore\ \frac{d\vec{v}_0}{dt}=\frac{q}{m}\left[\vec{E}_\parallel+\vec{v}_1\times\vec{B}-\frac{1}{2}\omega_0\rho_0^2\vec{\nabla}B\right]$$

$$m\frac{d\vec{v}_0}{dt}=q\left[\vec{E}_\parallel+\vec{v}_1\times\vec{B}-\frac{1}{2}\omega_0\rho_0^2\vec{\nabla}B\right]=q\left[\vec{E}_\parallel+\vec{v}_1\times\vec{B}\right]-\mu\vec{\nabla}B\ \cdots ③$$

$\vec{B}\cdot③$

i) parallel : $m\dfrac{dv_{\parallel 0}}{dt}=q\vec{E}_\parallel-\mu(\hat{b}\cdot\vec{\nabla})B$

$\vec{B}\times③$

ii) perpendicular : $\vec{B}\times m\dfrac{d\vec{v}_0}{dt}=q\vec{B}\times(\vec{v}_1\times\vec{B})-\mu\vec{B}\times\vec{\nabla}B$

$$=q\vec{v}_{1\perp}B^2-\mu\vec{B}\times\vec{\nabla}B$$

$$\therefore\ \vec{v}_{1\perp}=\underline{\frac{m}{qB^2}\left(\vec{B}\times\frac{d\vec{v}_0}{dt}\right)}+\mu\frac{\vec{B}\times\vec{\nabla}B}{qB^2}\quad\cdots ④$$

← inertial drift.

← curvature drift가 안보이네? ⇒ First-order correction for drift motion

p.12 First-order correction for drift motion

$$\vec{v}_0=\vec{v}_{\parallel 0}\hat{b}+\frac{\vec{E}\times\vec{B}}{B^2}=\vec{v}_{\parallel 0}\hat{b}+\vec{v}_E$$

$$\frac{d\vec{v}_0}{dt}=\left(\frac{d}{dt}+v_{\parallel 0}\hat{b}\cdot\vec{\nabla}+\vec{v}_E\cdot\vec{\nabla}\right)(v_{\parallel 0}\hat{b}+\vec{v}_E)=\frac{d}{dt}(v_{\parallel 0}\hat{b})+\underbrace{(v_{\parallel 0}\hat{b}+\vec{v}_E)\cdot\vec{\nabla}(v_{\parallel 0}\hat{b})}_{\text{curvature drift}}+\underbrace{\frac{d\vec{v}_E}{dt}}_{\text{polarization drift}}\quad\cdots ⑤$$

($\vec{v}_E\cdot\vec{\nabla}\vec{v}_E$ are new and can be potentially important as Northrop discussed.)

combine ⑤ to ④, then

$$\vec{V}_{d1} = \frac{m}{qB^2}\left(\vec{B}\times\left(\frac{d}{dt}(V_{\parallel 0}\hat{b}) + (V_{\parallel 0}\hat{b}+\vec{V_E})\cdot\vec{\nabla}(V_{\parallel 0}\hat{b}) + \frac{d\vec{V_E}}{dt}\right)\right) + \mu\frac{\vec{B}\times\vec{\nabla}B}{qB^2}$$

$$= \frac{mV_{\parallel 0}^2}{qB^2}(\vec{B}\times\vec{K}) + \mu\frac{\vec{B}\times\vec{\nabla}B}{qB^2} + \underline{\frac{m}{qB^2}\frac{d\vec{E_\perp}}{dt}}$$

$$+ \frac{mV_{\parallel 0}}{qB^2}\vec{B}\times\left(\frac{d}{dt}+\vec{V_E}\cdot\vec{\nabla}\right)\hat{b} - \underline{\frac{m\vec{E_\perp}}{qB^3}\frac{dB}{dt}}$$

(x)

$$\frac{m}{qB^2}\vec{B}\times\frac{d}{dt}\left(\frac{\vec{E}\times\vec{B}}{B^2}\right) = \frac{m}{qB^2}\vec{B}\times\left(\frac{(\dot{\vec{E_\perp}}\times\vec{B})+(\vec{E_\perp}\times\dot{\vec{B}})}{B^2} - \frac{2}{B^3}(\vec{E}\times\vec{B})\frac{dB}{dt}\right)$$

$$= \frac{m}{qB^2}\dot{\vec{E_\perp}} + \frac{m\vec{E_\perp}}{qB^2}\frac{(\vec{B}\cdot\dot{\vec{B}})}{B^2} - \frac{m\vec{E}}{qB^2}\frac{2}{\cancel{B^3}}\cancel{B}\frac{dB}{dt} = \frac{m}{qB^2}\dot{\vec{E_\perp}} - \frac{m}{qB^3}\dot{B}$$