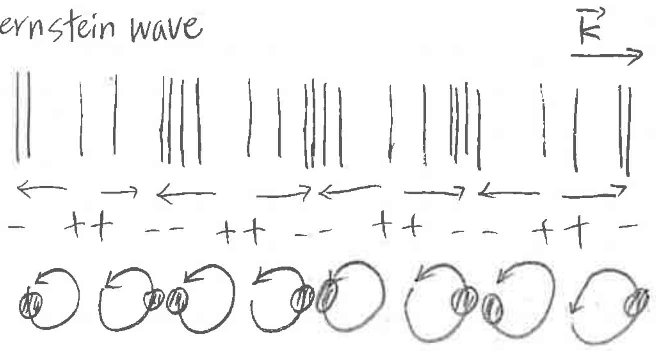
Fusion Plasma Theory 2

Lecture 15 : Bernstein Waves and Mode Conversion

① Bernstein wave



collective rotation in phase of the electrons around their guiding centers sustain the longitudinal electron Bernstein wave (EBW)

② Electrostatic kinetic wave dispersion

$\vec{k}\cdot\overleftrightarrow{\epsilon}\cdot\vec{k}=0$ w/ $\vec{k}=k_x\hat{x}+k_z\hat{z}=k_\perp\hat{x}+k_\parallel\hat{z}$

$$\Rightarrow k_x^2\epsilon_{xx}+2k_xk_z\epsilon_{xz}+k_z^2\epsilon_{zz}=0 \quad (\epsilon_{xz}=\epsilon_{zx})$$

$\Downarrow$ some tedious analysis using Eqs (21)–(24) in Lecture 14.

$$1+\sum_s\frac{\omega_{ps}^2}{k^2}\sum_{n=-\infty}^{\infty}\int\frac{J_n^2(z)}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\left(\frac{n\omega_{cs}}{v_\perp}\frac{df_{s0}}{dv_\perp}+k_\parallel\frac{df_{s0}}{dv_\parallel}\right)d\vec{v}=0$$

where $z=\frac{k_\perp v_\perp}{\omega_{cs}}$ and $k^2=k_\perp^2+k_\parallel^2$

< Harris dispersion relation >

③ Electrostatic kinetic wave dispersion under Maxwellian

$$\underbrace{k_x^2+k_z^2}_{I\text{ part in }\epsilon_{zz},\,\epsilon_{xx}}+\sum_s\frac{\omega_{ps}^2}{\omega^2}\zeta_{0s}e^{-\lambda_s}\sum_{n=-\infty}^{\infty}\left[k_x^2T_{xx}+2k_xk_zT_{xz}+k_z^2T_{zz}\right]$$

$$=k_x^2+k_z^2+\sum_s\frac{\omega_{ps}^2}{\omega^2}\zeta_{0s}e^{-\lambda_s}\sum_{n=-\infty}^{\infty}\left[k_x^2\frac{n^2}{\lambda_s}I_nZ-k_xk_z\left(\frac{2}{\lambda_s}\right)^{1/2}nZ'-k_z^2\zeta_{ns}Z'\right]=0$$

$\left(\lambda_s=\frac{k_x^2v_{ts}^2}{2\omega_{cs}^2},\ \zeta_{ns}=\frac{\omega-n\omega_c}{k_zv_{ts}},\ \text{and } Z'=-2(1+\zeta_{ns}Z)\right)$

$$-\sum_{n=-\infty}^{\infty}I_n\left(\frac{k_z(2\omega_{cs})}{v_{ts}}n+k_z^2\zeta_{ns}\right)Z'$$
$$=-\sum_{n=-\infty}^{\infty}I_nk_z^2\left(\frac{2n\omega_c}{k_zv_{ts}}+\frac{\omega-n\omega_c}{k_zv_{ts}}\right)Z'$$
$$=-\sum_{n=-\infty}^{\infty}I_nk_z^2\left(\frac{\omega+n\omega_c}{k_zv_{ts}}\right)Z'$$

$$0=k_x^2+k_z^2+\sum_s\frac{\omega_{ps}^2}{\omega^2}\zeta_{0s}e^{-\lambda_s}\sum_{n=-\infty}^{\infty}I_n\left[\frac{2\omega_c^2}{v_{ts}^2}n^2Z-k_z^2\zeta_{-ns}Z'\right]$$

$$=k_x^2+k_z^2+\sum_s\frac{\omega_{ps}^2}{\omega^2}\zeta_{0s}e^{-\lambda_s}\sum_{n=-\infty}^{\infty}I_n\left[\frac{2\omega_c^2}{v_{ts}^2}n^2Z+2k_z^2\zeta_{-ns}(1+\zeta_{ns})Z\right]$$

$$=k_x^2+k_z^2+\sum_s\frac{\omega_{ps}^2}{\omega^2}\zeta_{0s}e^{-\lambda_s}\sum_{n=-\infty}^{\infty}I_n\left[\cancel{\frac{2\omega_c^2}{v_{ts}^2}n^2Z}+2k_z^2\zeta_{0s}+2k_z^2\left(\zeta_{0s}^2Z-\cancel{\frac{n^2\omega_c^2}{k_z^2v_{ts}^2}Z}\right)\right]$$

$$=k_x^2+k_z^2+\sum_s\frac{2\omega_{ps}^2}{\omega^2}k_z^2\zeta_{0s}^2e^{-\lambda_s}\sum_{n=-\infty}^{\infty}I_n(1+\zeta_{0s}Z)$$

$$\sum_{n=-\infty}^{\infty}e^{-x}I_n=1$$

$$=k_x^2+k_z^2+\sum_s\frac{2\omega_{ps}^2}{\omega^2}k_z^2\zeta_{0s}^2\left(1+e^{-\lambda_s}\zeta_{0s}\sum_{n=-\infty}^{\infty}I_nZ\right)=0$$

④ Bernstein dispersion relation

Using $k_z^2 \zeta_{os}^2 = w^2/V_{ts}^2$, we obtain general dispersion for Bernstein wave:

$$\boxed{k^2 + \sum_s \frac{2w_{ps}^2}{V_{ts}^2}\left(1 + e^{-\lambda_s} \zeta_{os} \sum_{n=-\infty}^{\infty} I_n(\lambda_s) Z(\zeta_{ns})\right) = 0}$$

1) parallel propagating wave. $k_\perp \to 0$, $\lambda_s \to 0$

$$1 + \sum_s \frac{2w_{ps}^2}{k_{\parallel}^2 V_{ts}^2}\left(1 + \zeta_{os} Z(\zeta_{os})\right) = 1 - \sum_s \frac{w_{ps}^2}{k_\perp^2 V_{ts}^2} Z'(\zeta_{os}) = 0$$

2) Perpendicular propagating wave $k_\parallel \to 0$, $\zeta_{ns} \to \infty$

$$1 + \sum_s \frac{2w_{ps}^2}{k_\perp^2 V_{ts}^2} \underbrace{\left(1 - e^{-\lambda_s} \zeta_{os} \sum_{n=-\infty}^{\infty} \frac{I_n(\lambda_s)}{\zeta_{ns}}\right)}_{(*)} = 0$$

$$(*) = 1 - e^{-\lambda_s} \zeta_{os} \left(\frac{I_0}{\zeta_{os}} + \sum_{n=1}^{\infty} \left(\frac{1}{\zeta_{ns}} + \frac{1}{\zeta_{-ns}} \right) I_n \right)$$

$$= 1 - e^{-\lambda_s} I_0 - e^{-\lambda_s} \zeta_{so} \sum_{n=1}^{\infty} \frac{2w_{\parallel} k_\parallel V_{ts}}{w^2 - n^2 w_c^2} I_n \qquad \zeta_{so} = \frac{w}{k_\parallel V_{ts}}$$

$$= 1 - e^{-\lambda_s} I_0 - e^{-\lambda_s} \sum_{n=1}^{\infty} \frac{2w^2 - 2n^2 w_c^2 + 2n^2 w_c^2}{w^2 - n^2 w_c^2} I_n$$

$$= \underbrace{\cancel{1 - e^{-\lambda_s} \left(I_0 + 2\sum_{n=1}^{\infty} I_n\right)}}_{=1} - e^{-\lambda_s} \sum_{n=1}^{\infty} \frac{2n^2}{(w/w_c)^2 - n^2} I_n$$

(*note $\lambda_s = \frac{1}{2} k_\perp^2 \rho_s^2$)

$$\Rightarrow \boxed{1 = \sum_s \frac{w_{ps}^2}{w_{cs}^2} \frac{2}{\lambda_s} \sum_{n=1}^{\infty} e^{-\lambda_s} I_n(\lambda_s) \frac{n^2}{(w/w_{cs})^2 - n^2}} \quad \cdots (13)$$

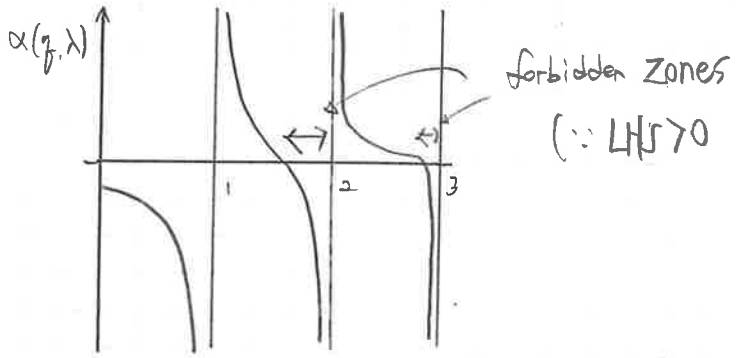
⑤ Frequency characteristics of Electron Bernstein wave

Consider the high frequency mode where the ion contribution can be ignored due to $\omega/\omega_{ci} \gg 1$. Define $q = \omega/\omega_{ce}$ and drop 'e' subscript.

$$\Rightarrow \boxed{\frac{k_\perp^2 V_{te}^2}{2\omega_{pe}^2} = 2\sum_{n=1}^{\infty} e^{-\lambda} I_n(\lambda) \frac{n^2}{q^2 - n^2} \equiv \alpha(q,\lambda)}$$

Given a $\lambda > 0$, α as a function of q is shown below.

↳ indicates that "its frequencies are just above each cyclotron harmonic frequences" and that "there are forbidden zones just below them".



($\because$ LHS > 0 so RHS cannot be below 0)

⑥ Dispersion characteristics of Electron Bernstein Wave

The fluid limit of EBW can be obtained by $\lambda \to 0$ and $I_n(\lambda) \sim \frac{1}{n!}\left(\frac{\lambda}{2}\right)^n$

From eq (13), only n=1 remains finite:

$$1 = \frac{\omega_{pe}^2}{\omega_{ce}^2} \cancel{\frac{2}{\lambda}} \cancel{\frac{\lambda}{2}} \frac{1}{(\omega/\omega_{ce})^2 - 1} = \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2}$$

→ upper hybrid resonance condition (X-wave resonance condition) ($S=0$)