§ 1.2 Mechanics of constrained systems

Keyword :
- Holonomic constraints
- D'Alembert's principle
- Lagrange's equation
- Special case of Lagrange's equation
- Examples

1. Holonomic constraints

def) $f(\underline{r_1}, \underline{r_2}, \cdots, \underline{r_n}, t) = 0$, 즉 제약조건이 위치/시간에 의해서만 정해짐

Lemma) $\sum_{a=1}^{nd} g_a(\underline{r}) d\underline{r_a} = 0$ and $\frac{d}{dr_a}(hg_b) = \frac{d}{dr_b}(hg_a)$ 인 h가 존재하면 $\Rightarrow$ $hg_a = \frac{df}{dr_a}$ 이므로 holonomic

느낌) $f(\underline{r_1}, \cdots, \underline{r_n}, t) = 0$ $\Rightarrow \sum_{a=1}^{nd} \frac{df}{dr_a} dr_a = 0 \Rightarrow \sum_{\alpha=1}^{nd} hg_a(\underline{r}) dr_a \Rightarrow \sum_{\alpha=1}^{nd} g_a(\underline{r}) dr_a$

$df = 0$

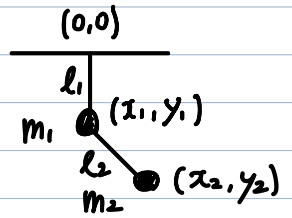
if $\frac{df}{dr_a} = hg_a$

증명) $\frac{d}{dr_b}(hg_a) = \frac{d}{dr_a}(hg_b)$ 인 h가 있다고 가정

→ any closed loop C, $\oint_C \sum_{\alpha=1}^{nd} hg_\alpha dr_\alpha = \iint_S \frac{d(hg_a)}{dr_b} - \frac{d(hg_b)}{dr_a} dr_a dr_b = 0$

∴ hg must be a gradient of some function f. (느낌: $\vec{\nabla} \times \vec{E} = 0 \to \vec{E} = -\vec{\nabla}\phi$)

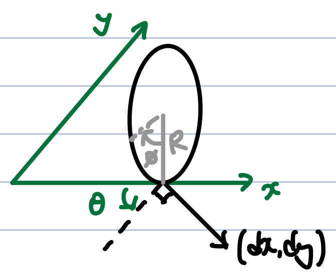
예시) Holonomic case



$x_1^2 + y_1^2 - l_1^2 = 0$

$(x_2 - x_1)^2 + (y_2 - y_1)^2 - l_2^2 = 0$

Nonholonomic case



$dx - R\sin\theta d\phi + 0 d\theta = 0 \to g_x = 1,\ g_\phi = -R\sin\theta,\ g_\theta = 0$

$dy + R\cos\theta d\phi + 0 d\theta = 0 \to g_y = 1,\ g_\phi = R\cos\theta,\ g_\theta = 0$

no h can satisfy $\frac{d}{d\theta}(-hR\sin\theta) = \frac{d}{d\phi}(h\cdot 0)$, $\frac{d}{d\theta}(h\cdot 1) = \frac{d}{dx}(h\cdot 0)$

In 1-D, this case is holonomic.

2. D'Alembert's principle

- 고전역학에서 제약력 ($f_\alpha=0$을 벗어나려는 힘)은 ∇f와 비례하는 힘이 되어야한다.
  제약력에 평행한 힘은 안 막아도 되지만, 제약력을 벗어나는 움직임을 막는 힘이 필요하다.

- $\lambda=\lambda(t)$인 이유 : $\lambda(\underline{r},t)$이면 $\underline{F}$와 수직이 아니므로.

  $f_\alpha(\underline{r})$에서 $df=0 \rightarrow \frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz=0 \rightarrow \nabla f(\underline{r})\cdot d\underline{r}=0$

  구속력은 $f=0$ 면에 수직이므로 $\underline{F}=\lambda\nabla f \rightarrow F_{i,\alpha}^{(c)}=-\lambda\frac{\partial f}{\partial \underline{r}_i}$

- def) 가상변위 "$dt=0$" 동안 만들어낸 변위 $\delta\underline{r}$. $(d\underline{r}\neq\delta\underline{r})$

  만약 $f_\alpha(\underline{r},t)=0$ 이라면, $\nabla f(\underline{r})\cdot d\underline{r}+\frac{\partial f}{\partial t}=0$, 구속력이 $f=0$ 면에 수직이 아님

  하지만 '가상변위'에서는 $\frac{\partial f}{\partial t}=0$ 이므로, $\delta f=\nabla f(\underline{r})\cdot\delta\underline{r}=0 \rightarrow$ 구속력이 $f=0$면과 수직

- Thus, if $f=f(\underline{r},t)$, 구속력이 한 일 W는 $\begin{cases} \text{실제변위} & W=\underline{F}\cdot d\underline{r}\neq 0 \\ \text{가상변위} & W=\underline{F}\cdot\delta\underline{r}=0 \end{cases}$

  $$\sum_{i=1}^{n}\underline{F}_{i,\alpha}^{(c)}\cdot\delta r_i=-\lambda_i\sum_{i=1}^{n}\frac{\partial f_\alpha}{\partial r_i}\cdot\delta\underline{r}_i=-\lambda_\alpha\delta f_\alpha=0 \quad \text{for each } \alpha$$

- 또 다른 해석) $f_\alpha(\underline{r},t)=0$, $(\alpha=1,\cdots,c) \Longleftrightarrow V_\alpha(\underline{r},t)=\lambda(t)f(\underline{r},t)=0$

  holonomic constraints　　　　equi-potential condition

  $\hookrightarrow F_{i,\alpha}^{(c)}=-\frac{\partial V_\alpha}{\partial \underline{r}_i}=-\lambda_i\frac{\partial f_\alpha}{\partial \underline{r}_i}$

- D'Alembert's principle

  i) $\underline{F}_i=m\ddot{\underline{r}}_i$ (실제 힘)

  ii) $\underline{F}_i=\underline{F}_i^{(\alpha)}+\sum_{\alpha=1}^{c}F_{i,\alpha}^{(c)}$ (실제힘 = applied force + 구속력)

  iii) $\sum_{i=1}^{n}F_{i,\alpha}^{(c)}\cdot\delta\underline{r}_i=0$ (α 구속력에 대한 가상힘=0)

  $$\Rightarrow \sum_{i=1}^{n}\sum_{\alpha=1}^{c}\underline{F}_{i,\alpha}^{(c)}\cdot\delta\underline{r}_i=\sum_{i=1}^{n}(\underline{F}_i^{(\alpha)}-\underline{F}_i)\cdot\delta\underline{r}_i=\boxed{\sum_{i=1}^{n}(\underline{F}_i^{(\alpha)}-m\ddot{\underline{r}}_i)\cdot\delta\underline{r}_i=0}$$

  Applied force　　　가상변위

  <u>가상변위를 사용하면, 구속력을 제외하고 동역학을 기술할 수 있음</u>

3. Lagrange's equation

- 입자 n, 자유도 d, 제약조건 c → nd−c개의 generalized coordinate q를 만들자.
- $r_i = r_i(q,t) \Rightarrow \dot{r}_i = \sum_{j=1}^{nd-c} \frac{\partial r_i}{\partial q_j}\dot{q}_j + \frac{\partial r_i}{\partial t} \xrightarrow[\text{양변을 } \dot{q}_j\text{로 미분}]{} \boxed{\frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \frac{\partial r_i}{\partial q_j}}$

- From D'Alembert, $\sum_{i=1}^{n}(\underline{F}_i^{(\alpha)} - m\ddot{\underline{r}}_i)\cdot\delta\underline{r}_i = 0$ ← δr_i가 dependent 하므로 항상 $\underline{F}_i^{(\alpha)} = m\ddot{\underline{r}}_i$은 아니다

① $\sum_{i=1}^{n}\underline{F}_i^{(\alpha)}\cdot\delta\underline{r}_i = \sum_{i=1}^{n}\sum_{j=1}^{nd-c}\underline{F}_i^{(\alpha)}\frac{\partial r_i}{\partial q_j}\delta q_j = \underline{\sum_{j=1}^{nd-c}\sum_{i=1}^{n}F_i^{(\alpha)}\frac{\partial r_i}{\partial q_j}\delta q_j = \sum_{j=1}^{nd-c}Q_j\delta q_j}$

def) Q_j(generalized force) : generalized coordinate 에서의 force $= \sum_{i=1}^{n}F_i^{(\alpha)}\frac{\partial r_i}{\partial q_j}$

② $\sum_{i=1}^{n}m\ddot{\underline{r}}_i\cdot\delta\underline{r}_i = \sum_{i=1}^{n}\sum_{j=1}^{nd-c}m\ddot{\underline{r}}_i\cdot\frac{\partial r_i}{\partial q_j}\delta q_j = \sum_{i=1}^{n}\sum_{j=1}^{nd-c}\left[\frac{d}{dt}\left(m\dot{\underline{r}}_i\cdot\frac{\partial r_i}{\partial q_j}\right) - m\dot{\underline{r}}_i\cdot\frac{d}{dt}\frac{\partial r_i}{\partial q_j}\right]\delta q_j$

$= \sum_{i=1}^{n}\sum_{j=1}^{nd-c}\left[\frac{d}{dt}\left(m\dot{r}_i\cdot\frac{\partial \dot{r}_i}{\partial \dot{q}_j}\right) - m\dot{r}_i\cdot\frac{\partial \dot{r}_i}{\partial q_j}\right]\delta q_j = \underline{\sum_{j=1}^{nd-c}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j}\right]\delta q_j}$

$\Rightarrow \sum_{j=1}^{nd-c}\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j\right)\delta q_j = 0 \rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j \quad \text{for } j=1,\cdots,nd-c}$

Lagrange's equation

- Special case of Lagrange's equation

① Applied force are all conservative, $V = V(\underline{r})$

$F_i^{(\alpha)} = -\frac{\partial V}{\partial r_i} \rightarrow Q_j = \sum_{i=1}^{n}F_i^{(\alpha)}\frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{n}\left(-\frac{\partial V}{\partial r_i}\right)\frac{\partial r_i}{\partial q_j} = -\frac{\partial V}{\partial q_j}$

$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0 \rightarrow \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0 \rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$

Define the Lagrangian $\underline{L(q,\dot{q},t) = T(q,\dot{q},t) - V(q,t)}$

② Monogenic system : 모든 힘이 V로 부터 나오는 시스템 $V = V(q,\dot{q},t)$

$Q_j = \sum_{i=1}^{n}F_i^{(\alpha)}\frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{n}\left(-\frac{\partial V}{\partial r_i} + \underline{\quad\quad}\right)\frac{\partial r_i}{\partial q_j} = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j}\right)$

$\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial V}{\partial \dot{q}_j}\right) \rightarrow \frac{d}{dt}\left(\frac{\partial (T-V)}{\partial \dot{q}_j}\right) - \frac{\partial (T-V)}{\partial q_j} = 0$

$\rightarrow \boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$

③ Polygenic system : 마찰력처럼 applied force가 V로만 정의되지는 않는 시스템

ex) special case : drag force (friction) $\underline{F}_i^{(f)} = -\gamma_i \dot{\underline{r}}_i$

extra force : $Q_j^{(ex)} = \sum_{i=1}^{n} F_i^{(f)} \cdot \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{n} \left(-\gamma_i \dot{\underline{r}}_i \cdot \frac{\partial \dot{r}_i}{\partial \dot{q}_j}\right) = -\frac{\partial}{\partial \dot{q}_j}\left(\frac{1}{2}\gamma_i \dot{\underline{r}}_i^2\right) = -\frac{\partial G}{\partial \dot{q}_j}$

(Define Rayleigh dissipation function : $G(\dot{\underline{r}}_1, \cdots \dot{\underline{r}}_n) \equiv \frac{1}{2}\sum_{i=1}^{n} \gamma_i \dot{\underline{r}}_i^2$)

In this case, $\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = -\frac{\partial G}{\partial \dot{q}_j}}$

④ Extra constraints in $\underline{q} = (q_1, \cdots, q_{\tilde{n}})$ coordinates

$$\sum_{j=1}^{\tilde{n}} \left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j\right) \delta q_j = 0 \quad , \text{where } f_1(\underline{q}) = \cdots = f_{\tilde{c}}(\underline{q}) = 0$$

- extra constraints : $f_1(\underline{q}) = \cdots = f_{\tilde{c}}(\underline{q}) = 0$ 과 $q_1, \cdots, q_{\tilde{n}}$ 에 대해

  δq_j는 $\tilde{n} - \tilde{c}$ 차원에 존재하므로

  내적을 고려하면 ($\vec{a} \cdot \vec{b} = 0 \rightarrow \vec{a} \perp \vec{b}$)

  $\left(\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_1}\right) - \frac{\partial T}{\partial q_1} - Q_1, \cdots, \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_{\tilde{n}}}\right) - \frac{\partial T}{\partial q_{\tilde{n}}} - Q_1\right)$는 $\tilde{c}$차원의 hyperplane에 존재한다.

- 이 $\tilde{c}$개의 constraint force는 principle of virtual work에 의해

  $\sum_{j=1}^{\tilde{n}} \frac{\partial f_\alpha}{\partial q_j} \delta q_j$ ($\alpha = 1, \cdots, \tilde{c}$) 이고, hyperplane의 수직벡터는 $\frac{\partial \tilde{f}_\alpha}{\partial q_j}$ 이므로

  $\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} - Q_j = \sum_{\alpha=1}^{\tilde{n}} \lambda_\alpha(t) \frac{\partial \tilde{f}_\alpha}{\partial q_j}}$ (for $j = 1, \cdots, \tilde{n}$)

- If monogenic system, where $Q_j = -\frac{\partial V}{\partial q_j} + \frac{d}{dt}\frac{\partial V}{\partial \dot{q}_j}$

  $\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = \sum_{\alpha=1}^{\tilde{c}} \lambda_\alpha(t) \frac{\partial \tilde{f}_\alpha}{\partial q_j} \quad \text{for } j = 1, \cdots, \tilde{n}}$

  Lagrange's equation of the first kind

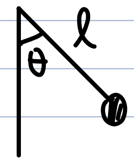
- λ_α: Lagrange multiplier → method for solving a constrained optimization problem

  Lagrange equation $\tilde{n}$개, constraints $\tilde{c}$개 : $\tilde{n} + \tilde{c}$개의 식

  $(q_1, \cdots, q_{\tilde{n}}) + (\lambda_1, \cdots, \lambda_{\tilde{c}})$: $\tilde{n} + \tilde{c}$개의 변수

(예시)

①



$$T = \frac{1}{2}m\dot{\underline{r}}^2 = \frac{1}{2}m\ell^2\dot{\theta}^2$$

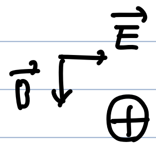
$$V = -mg\ell\cos\theta$$

$$G = \frac{1}{2}\gamma\dot{\underline{r}}^2 = \frac{1}{2}\gamma\ell^2\dot{\theta}^2$$

$$\frac{d}{dt}\frac{dL}{d\dot{\theta}} - \frac{dL}{d\theta} = -\frac{dG}{d\dot{\theta}}$$

$$\therefore\ \underline{\ddot{\theta} = -\frac{g}{\ell}\sin\theta - \frac{\gamma}{m}\dot{\theta}}$$

②



$$T = \frac{1}{2}m\dot{\underline{r}}^2$$

$$V = q\phi - q\underline{A}\cdot\underline{r}$$

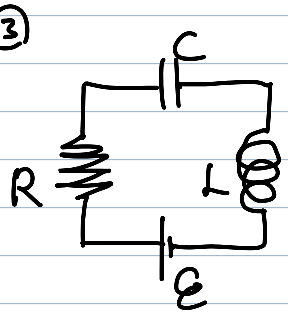
scalar potential : $q\phi$

vector potential : $\vec{r}\cdot\vec{A}$

$$\frac{d}{dt}\left(\frac{dL}{d\dot{\underline{r}}}\right) - \frac{dL}{d\underline{r}} = 0$$

$$\therefore\ \underline{m\ddot{\underline{r}} = q(\vec{E} + \vec{v}\times\vec{B})}$$

③



$$T = \frac{1}{2}L\dot{q}^2$$

$$V = -q\epsilon + \frac{1}{2C}q^2$$

$$G = \frac{1}{2}R\dot{q}^2$$

$$\therefore\ \underline{L\ddot{q} - \epsilon + \frac{1}{C}q = -R\dot{q}}$$