Fusion Plasma Theory 2

Lecture 14 : Hot Plasma Wave Dispersion

① Perturbation during cyclotron motion in magnetized plasmas

- Kinetic effects in plasma waves : cyclotron harmonics and harmonic damping.
- Harmonics of cyclotron frequency = Finite-Larmor-Radius (FLR) effect.

  (particle이 ↺ 돌면서 다른 $\vec{E}, \vec{B}$ field를 느끼게 되고, average 했을 때 사라지지 않고 남으면 FLR.)
- $\vec{E} = \hat{x} E_x e^{i(kx-\omega t)} \Rightarrow \underline{\ddot{x} + \omega_c^2 x = \frac{q}{m} E_x e^{i(kx-\omega t)}}$
- Method of characteristics : $\underline{x = r_L \sin(\omega_c t)}$ ← unperturbed orbit.

  (무슨 일이 있든간에 particle의 orbit은 이미 정해져있다고 생각하기)

  $$\Rightarrow \underline{\ddot{x} + \omega_c^2 x = \frac{q}{m} E_x e^{i(k r_L \sin\omega_c t - \omega t)}}$$
- Incredible identity :

  $$\boxed{e^{iz\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(z) e^{in\phi}} \quad \left(\Leftarrow e^{z(t-\frac{1}{t})/2} = \sum_{n=-\infty}^{\infty} J_n(z) t^n \text{ with } t = e^{i\phi} \right)$$

② Cyclotron harmonic resonances

$$z = k r_L, \phi = \omega_c t \Rightarrow \underline{\ddot{x} + \omega_c^2 x = \frac{q}{m} E_x \sum_{n=-\infty}^{\infty} J_n(k r_L) e^{-i(\omega - n\omega_c)t}}$$

let $x_n(t) = A_n e^{-i(\omega-n\omega_c)t}$, then, $[\omega_c^2 - (\omega - n\omega_c)^2] A_n = \frac{q}{m} E_x J_n(k r_L)$

$$\rightarrow A_n = \frac{\frac{q}{m} E_x J_n(k r_L)}{\omega_c^2 - (\omega - n\omega_c)^2} \Longrightarrow \boxed{x(t) = \frac{q}{m} E_x \sum_{n=-\infty}^{\infty} \frac{J_n(k r_L) e^{-i(\omega - n\omega_c)t}}{\omega_c^2 - (\omega - n\omega_c)^2}}$$

- resonance occurs when $\omega = (n \pm 1)\omega_c$
- For small $k r_L \ll 1$, $J_n(k r_L) \sim \left(\frac{k r_L}{2}\right)^n / n!$ → higher harmonics has weaker amplitude by $n!$
- when $k r_L \to 0$, $J_n(k r_L) \to 0$ $(n \neq 0)$, $\to 1$ $(n = 0)$ → For $n=0$, resonance only occurs $\omega = \pm\omega_c$ (fluid limit)
- Particle gains energy so the wave will lose energy → cyclotron damping.

  It occurs always and does not require $df_0/dv < 0$ unlike Landau damping.

③ Vlasov equation for magnetized plasmas

Now consider kinetic waves through $\vec{B_0}$. (Assume $f = f_0 + f_1$ and $\vec{E} = \vec{E_1}$)

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\frac{\partial f}{\partial \vec{x}} + \frac{q}{m}(\vec{E}+\vec{v}\times\vec{B})\frac{\partial f}{\partial \vec{v}} = 0$$

$$\Downarrow$$

$$\frac{\partial f_1}{\partial t} + \vec{v}\cdot\frac{\partial f_1}{\partial \vec{x}} + \frac{q}{m}(\vec{v}\times\vec{B_0})\frac{\partial f_1}{\partial \vec{v}} + \frac{q}{m}(\vec{E_1}+\vec{v}\times\vec{B_1})\frac{\partial f_0}{\partial \vec{v}} = 0$$

$$\Rightarrow \frac{Df_1}{Dt} = \underbrace{\frac{\partial f_1}{\partial t} + \vec{v}\cdot\frac{\partial f_1}{\partial \vec{x}} + \frac{q}{m}(\vec{v}\times\vec{B_0})\frac{\partial f_1}{\partial \vec{v}}}_{f_1} = \underbrace{-\frac{q}{m}(\vec{E_1}+\vec{v}\times\vec{B_1})\frac{\partial f_0}{\partial \vec{v}}}_{f_0}$$

Direct Fourier (or Laplace) approach X work due to $\partial f_1/\partial \vec{v}$ which leads to non-linear equation.

Integrate the RHS along w/ unperturbed trajectory $(\vec{x}', \vec{v}', t)$ when calculating $\vec{E_1}$ and $\vec{B_1}$

$$f_1(\vec{x},\vec{v},t) = -\frac{q}{m}\int_{-\infty}^{t}\left(\vec{E_1}(\vec{x}',t') + \vec{v}'\times\vec{B_1}(\vec{x}',t')\right)\frac{\partial f_0(\vec{v}')}{\partial \vec{v}'}dt'$$

(★ f_1을 구하려고 하지만, f_1이 움직이는 경로는 f_0을 지배하는 background에 의해 결정된다)

Unperturbed trajectory $(\vec{x}', \vec{v}')$ becomes $(\vec{x}, \vec{v})$ as $t' \to t$

The unperturbed trajectory in uniform plasma:

$$\frac{d\vec{x}}{dt} = \vec{v}, \quad \frac{d\vec{v}}{dt} = \frac{q}{m}\vec{v}\times\vec{B_0} \quad \leftarrow$$ * 입자의 궤적 자체는 B_0에 의한 Helix with gyration 만 한다.

④ Unperturbed particle trajectory

$$\vec{B_0} = \hat{z}B_0, \quad \vec{v} = (v_\perp\cos\phi, v_\perp\sin\phi, v_\parallel) \longrightarrow \vec{v}' = (v_\perp\cos(\omega_c\tau+\phi), v_\perp\sin(\omega_c\tau+\phi), v_\parallel)$$

where $\tau \equiv t - t'$

Integrating in time:

$$\begin{pmatrix} x' - x = -\frac{v_\perp}{\omega_c}(\sin(\omega_c\tau+\phi) - \sin\phi) \\ y' - y = \frac{v_\perp}{\omega_c}(\cos(\omega_c\tau+\phi) - \cos\phi) \\ z' - z = -v_\parallel\tau \end{pmatrix}$$

* $\tau = 0$ 이면 $x = x'$, $y = y'$, $z = z'$

In our approach $f_0 = f_0(v_\perp, v_\parallel)$ anisotropy with $v_\perp = (v_x'^2 + v_y'^2)^{1/2}$

$$\begin{pmatrix} \frac{\partial f_0}{\partial v_x'} = \frac{\partial v_\perp}{\partial v_x'}\frac{\partial f_0}{\partial v_\perp} = \frac{v_x'}{v_\perp}\frac{\partial f_0}{\partial v_\perp} = \cos(\omega_c\tau+\phi)\frac{\partial f_0}{\partial v_\perp} \\ \frac{\partial f_0}{\partial v_y'} = \frac{\partial v_\perp}{\partial v_y'}\frac{\partial f_0}{\partial v_\perp} = \frac{v_y'}{v_\perp}\frac{\partial f_0}{\partial v_\perp} = \sin(\omega_c\tau+\phi)\frac{\partial f_0}{\partial v_\perp} \\ \frac{\partial f_0}{\partial v_z'} = \frac{\partial f_0}{\partial v_\parallel} \end{pmatrix}$$

⑤ Perturbed distribution function by waves

Now consider perturbations $\sim e^{i(\vec{k}\cdot\vec{x}-wt)}$. Let $\vec{k}=k_\perp\hat{x}+k_\parallel\hat{z}$.

Cancel

$$f_1=-\frac{q}{m}\int_0^\infty\left[\left(E_x+\cancel{V_y'B_z}-V_z'B_y\right)\frac{df_0}{dV_x'}+\left(E_y+V_z'B_x-\cancel{V_x'B_z}\right)\frac{df_0}{dV_y'}+\left(E_z+V_x'B_y-V_y'B_x\right)\frac{df_0}{dV_z'}\right]\cdot e^{i(\vec{k}\cdot(\vec{x}'-\vec{x})-w(t'-t))}d\tau$$

$$=-\frac{q}{m}\int_0^\infty\Bigg[\left(E_x-V_\parallel B_y\right)\cos(w_c\tau+\phi)\frac{df_0}{dV_\perp}+\left(E_y+V_\parallel B_x\right)\sin(w_c\tau+\phi)\frac{df_0}{dV_\perp}$$

$$+\left(E_z+V_\perp\cos(w_c\tau+\phi)B_y-V_\perp\sin(w_c\tau+\phi)B_x\right)\frac{df_0}{dV_\parallel}\Bigg]\times\exp\left[-i\frac{V_\perp}{w_c}\left(\sin(w_c\tau+\phi)-\sin\phi\right)+i(w-k_\parallel V_\parallel)\tau\right]d\tau$$

*note that $\frac{d\vec{B}_1}{dt}=-\vec{\nabla}\times\vec{E}\to w\vec{B}_1=\vec{k}\times\vec{E}_1 \Rightarrow B_x=-k_zE_y/w,\ \ B_y=(k_zE_x-k_xE_z)/w,\ \ B_z=k_xB_y/w$

Then, one can estimate j_1 for dielectric tensor $(\vec{j}_1=\overleftrightarrow{\sigma}\cdot\vec{E}_1)$

$$\vec{j}_1=q\int d\vec{v}\left(\hat{x}v_\perp\cos\phi+\hat{y}v_\perp\sin\phi+\hat{z}v_\parallel\right)f_1$$

⑥ Appendix - Bessel identities for kinetic integral. (결국 해야하는 integral)

$$e^{iz\sin\phi}=\sum_{n=-\infty}^{\infty}J_n(z)e^{in\phi} \qquad \int d\vec{v}=\int_0^{2\pi}d\phi\int dv_\perp\int dv_\parallel$$

← 일단은 이거먼저하자.

$$\boxed{\int_0^{2\pi}d\phi\, e^{iz\sin\phi-iz\sin(\phi+w_c\tau)}\begin{pmatrix}\sin\phi\sin(\phi+w_c\tau)\\ \sin\phi\cos(\phi+w_c\tau)\\ \cos\phi\sin(\phi+w_c\tau)\\ \cos\phi\cos(\phi+w_c\tau)\\ 1\\ \sin\phi\\ \cos\phi\\ \sin(\phi+w_c\tau)\\ \cos(\phi+w_c\tau)\end{pmatrix}=2\pi\sum_{n=-\infty}^{\infty}e^{-inw_c\tau}\begin{pmatrix}J_nJ_n'\\ -i(n/z)J_nJ_n'\\ i(n/z)J_nJ_n'\\ (n^2/z^2)J_nJ_n\\ J_nJ_n\\ -iJ_nJ_n'\\ (n/z)J_nJ_n\\ iJ_nJ_n'\\ (n/z)J_nJ_n\end{pmatrix}}$$

example) (i) $\int d\phi\, e^{iz\sin\phi-iz\sin(\phi+w_c\tau)}=\int d\phi\sum_n J_ne^{in\phi}\sum_m J_m e^{-im(\phi+w_c\tau)}$

$$=\begin{cases}0 & (m\neq n)\\ 2\pi\sum_n J_nJ_n e^{-inw_c\tau} & (m=n)\end{cases}$$

(ii) $\int d\phi\, e^{iz\sin\phi-iz\sin(\phi+w_c\tau)}\sin\phi\cos(\phi+w_c\tau)$

$$\left(\begin{array}{l}\frac{d}{dz}\left(e^{iz\sin\phi}=\sum_n J_ne^{in\phi}\right)\to i\sin\phi\, e^{iz\sin\phi}=\sum_n J_n'e^{in\phi}\to\sin\phi\, e^{iz\sin\phi}=-i\sum_n J_n'e^{in\phi}\\ \frac{d}{d\phi}\left(e^{-iz\sin(\phi+w_c\tau)}=\sum_m J_me^{-im(\phi+w_c\tau)}\right)\to -iz\cos(\phi+w_c\tau)e^{-iz\sin(\phi+w_c\tau)}=-i\sum_m mJ_me^{-im(\phi+w_c\tau)}\end{array}\right)$$

$$=2\pi\sum_n e^{-inw_c\tau}\left[-i\left(\frac{n}{z}\right)J_nJ_n'\right]$$

Also, $\displaystyle\int_0^\infty \exp\left[i(\omega-k_\parallel v_\parallel-n\omega_c)\tau\right]d\tau = \left.\frac{\exp(i(\cdots)\tau)}{i(\omega-k_\parallel v_\parallel-n\omega_c)}\right|_0^\infty = \frac{i}{\omega-k_\parallel v_\parallel-n\omega_c}$ provided $\mathrm{Im}(\omega)>0$

이게 돼야 ∞에서 0이 됨

⑦ Kinetic dielectric tensor in magnetized plasmas

Then after performing $d\phi$ and $d\tau$ integral, we get:

$$\overleftrightarrow{\epsilon} = \overleftrightarrow{I} + \sum_s \frac{\omega_{ps}^2}{\omega^2}\sum_{n=-\infty}^{\infty}\int \frac{\overleftrightarrow{S}}{\omega-k_\parallel v_\parallel-n\omega_{cs}}\,2\pi\, dv_\perp dv_\parallel$$

where

$$\overleftrightarrow{S} = \begin{pmatrix} \frac{n^2J_n^2}{z^2}v_\perp U & i\frac{nJ_nJ_n'}{z}v_\perp U & \frac{nJ_n^2}{z}v_\perp W \\ -i\frac{nJ_nJ_n'}{z}v_\perp U & (J_n')^2 v_\perp U & -iJ_nJ_n' v_\perp W \\ \frac{nJ_n^2}{z}v_\parallel U & iJ_nJ_n' v_\parallel U & J_n^2 v_\parallel W \end{pmatrix}$$

with $U \equiv (\omega-k_\parallel v_\parallel)\frac{\partial f_{s0}}{\partial v_\perp} + k_\parallel v_\perp \frac{\partial f_{s0}}{\partial v_\parallel}$

$W \equiv \frac{n\omega_{cs}v_\parallel}{v_\perp}\frac{\partial f_{s0}}{\partial v_\perp} + (\omega-n\omega_{cs})\frac{\partial f_{s0}}{\partial v_\parallel}$

$\overleftrightarrow{S}$ and $\overleftrightarrow{\sigma}$ is in a Hermitian form when the integral is performed. (Onsager symmetry)

⑧ Hot plasma dielectric tensor in magnetized Maxwellian

Identity: $\displaystyle\int_0^\infty t J_\nu(at)J_\nu(bt)e^{-p^2t^2}dt = \frac{1}{2p^2}\exp\left(-\frac{a^2+b^2}{4p^2}\right)I_\nu\left(\frac{ab}{2p^2}\right)$

(I_ν = modified Bessel function of the first kind)

Kinetic dielectric tensor for Maxwellian plasmas:

$$\overleftrightarrow{\epsilon} = \overleftrightarrow{I} + \sum_s \frac{\omega_{ps}^2}{\omega^2}\zeta_{0s}e^{-\lambda_s}\sum_{n=-\infty}^{\infty}\overleftrightarrow{T}$$

where

$$\overleftrightarrow{T} = \begin{pmatrix} \frac{n^2 I_n}{\lambda_s}Z & in(I_n'-I_n)Z & -\frac{nI_n}{(2\lambda_s)^{1/2}}Z' \\ -in(I_n'-I_n)Z & \left(\frac{n^2I_n}{\lambda_s}+2I_n\lambda_s-2I_n'\lambda_s\right)Z & i\frac{(I_n'-I_n)\lambda_s^{1/2}}{2^{1/2}}Z' \\ -\frac{nI_n}{(2\lambda_s)^{1/2}}Z' & -i\frac{(I_n'-I_n)\lambda_s^{1/2}}{2^{1/2}}Z' & -I_nZ'\zeta_{ns} \end{pmatrix}$$

$$\left(\begin{array}{ll} I_n = I_n(\lambda_s), & \to \lambda_s \equiv \frac{k_\perp^2 v_{ts}^2}{2\omega_{cs}^2} = \frac{1}{2}k_\perp^2\rho^2 \ll 0 \\ Z = Z(\zeta_{ns}) & \to \zeta_{ns} = \frac{\omega-n\omega_{cs}}{k_\parallel v_{ts}} \gg 0 \end{array}\right)$$ ← λ dependency is FLR effect.

⑨ Hot plasma wave dispersion (Hot → cold)

$$\epsilon_{ij} = \delta_{ij} + \sum_s \frac{w_{ps}^2}{w^2} \zeta_{0s} e^{-\lambda_s} \sum_{n=-\infty}^{\infty} T_{ij}$$

$$\boxed{\begin{array}{l} \text{Assume } V_{ts} \to 0 \ : \lambda_s \to 0 \text{ and } \zeta_s \to \infty \\ \therefore I_n(\lambda_s) = \left(\frac{\lambda_s}{2}\right)^n, \quad Z(\zeta_s) = -\frac{1}{\zeta_s} \end{array}}$$

(1) ϵ_{xx}

$$T_{xx} = \frac{n^2 I_n}{\lambda_s} Z(\zeta_{ns}) \to n^2 \frac{1}{\lambda_s}\left(\frac{\lambda_s}{2}\right)^n Z(\zeta_{ns}) = \begin{cases} 0 & (n=0) \\ \frac{1}{2} Z(\zeta_{\pm 1s}) & (n=\pm 1) \\ 0 & (|n| \geq 2) \end{cases}$$

*note $I_n = I_{-n}$ so it's defined only for $n \geq 0$

$$\sum_{n=-\infty}^{\infty} T_{xx} = \frac{1}{2} Z(\zeta_{1s}) + \frac{1}{2} Z(\zeta_{-1s}) = -\frac{1}{2}\frac{1}{\zeta_{1s}} - \frac{1}{2}\frac{1}{\zeta_{-1s}}$$

$$\left(\zeta_{\pm 1s} = \frac{w \mp w_{cs}}{k_\parallel v_{ts}}, \quad \zeta_{0s} = \frac{w}{k_\parallel v_{ts}}\right)$$

$$\zeta_{0s} \sum_{n=-\infty}^{\infty} T_{xx} = \frac{w}{\cancel{k_\parallel v_{ts}}} \frac{1}{2}\left[-\frac{\cancel{k_\parallel v_{ts}}}{w - w_{cs}} - \frac{\cancel{k_\parallel v_{ts}}}{w + w_{cs}}\right] = -\frac{w}{2}\left(\frac{1}{w-w_{cs}} + \frac{1}{w+w_{cs}}\right) = -\frac{w}{\cancel{2}} \frac{\cancel{2}w}{w^2 - w_{cs}^2}$$

$$= -\frac{w^2}{w^2 - w_{cs}^2}$$

$$\therefore \epsilon_{xx} = 1 + \sum_s \frac{w_{ps}^2}{w^2}\left(-\frac{w^2}{w^2 - w_{cs}^2}\right) = 1 - \sum_s \frac{w_{ps}^2}{w^2 - w_{cs}^2} = S, \quad \text{where } S = 1 - \sum_s \frac{w_{ps}^2}{w^2 - w_{cs}^2}$$

(2) $\epsilon_{xy} = -\epsilon_{yx}$

$$T_{xy} = -in(I_n' - I_n) \to -in\Big(\overbrace{\frac{n}{2}\left(\frac{\lambda_s}{2}\right)^{n-1}}^{I_n'} - \overbrace{\left(\frac{\lambda_s}{2}\right)^n}^{I_n}\Big) Z(\zeta_{ns}) = \begin{cases} 0 & (n=0) \\ \mp i \frac{1}{2} Z(\zeta_{ns}) & (n=\pm 1) \\ 0 & (|n| \geq 2) \end{cases}$$

$$\sum_{n=-\infty}^{\infty} T_{xy} = -\frac{i}{2} Z(\zeta_{1s}) + \frac{i}{2} Z(\zeta_{-1s}) = +\frac{i}{2}\left(\frac{1}{\zeta_{1s}} - \frac{1}{\zeta_{-1s}}\right)$$

$$\zeta_{0s} \sum_{n=-\infty}^{\infty} T_{xy} = \frac{w}{\cancel{k_\parallel v_{ts}}} \cdot \frac{-i}{2}\left(\frac{\cancel{k_\parallel v_{ts}}}{w + w_c} - \frac{\cancel{k_\parallel v_{ts}}}{w - w_c}\right) = \frac{-wi}{\cancel{2}} \cdot \frac{-\cancel{2}w_c}{w^2 - w_c^2} = +i \frac{w w_c}{w^2 - w_c^2}$$

$$\therefore \epsilon_{xy} = \sum_s \frac{w_{ps}^2}{w^2}\left(+i \frac{w w_c}{w^2 - w_c^2}\right) = +i \sum_s \frac{w_c}{w} \frac{w_{ps}^2}{w^2 - w_c^2} = +iD \quad \text{where } D = \sum_s \frac{w_c}{w} \frac{w_{ps}^2}{w^2 - w_c^2}$$

(3) ϵ_{zz}

$$T_{zz} = -I_n Z' \zeta_{ns} \rightarrow -\left(\frac{\lambda_s}{2}\right)^n Z' \zeta_{ns} = \begin{cases} -Z' \zeta_{os} & (n=0) \\ 0 & (|n| \geq 1) \end{cases}$$

$$\sum_{n=-\infty}^{\infty} T_{zz} = -Z' \zeta_{os} = \frac{-1}{\zeta_{os}^2} \zeta_{os} = -\frac{1}{\zeta_{os}}$$

$$\zeta_{os} \sum_{n=-\infty}^{\infty} T_{zz} = \zeta_{os} \left(-\frac{1}{\zeta_{os}}\right) = -1$$

$$\therefore \epsilon_{zz} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2}(-1) = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} = P \quad \text{where } P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}$$

(4) $\epsilon_{xz} = \epsilon_{zx}$

$$T_{xz} = -\frac{n I_n}{(2\lambda_s)^{1/2}} Z' \rightarrow -\frac{n\left(\frac{\lambda_s}{2}\right)^n}{(2\lambda_s)^{1/2}} \sim \begin{cases} 0 & (n=0) \\ \sqrt{\lambda}\, Z' & (n=1) \sim \sqrt{\lambda}\left(\frac{1}{\zeta_{1s}^2}\right) \rightarrow 0 \\ \cancel{\lambda^{3/2} Z'} & (n=2) \sim \lambda^3 \sqrt{\lambda}\left(\frac{1}{\zeta_{1s}^2}\right) \rightarrow 0 \end{cases}$$

$$\therefore \epsilon_{xz} = \sum_s \frac{\omega_{ps}^2}{\omega^2} \cdot 0 = 0$$

(5) $\epsilon_{yz} = \epsilon_{zy}$

$$T_{yz} = -i \frac{(I_n' - I_n)\lambda_s^{1/2}}{2^{1/2}} Z' \rightarrow -i \frac{\left(\frac{n}{2}\left(\frac{\lambda_s}{2}\right)^{n-1} - \left(\frac{\lambda_s}{2}\right)^n\right)\lambda_s^{1/2}}{2^{1/2}} Z' \sim \begin{cases} -i\left(\frac{\lambda_s}{2}\right)^{1/2} & (n=0) \rightarrow 0 \\ -i\left(\frac{\lambda_s}{8}\right)^{1/2} & (n=1) \rightarrow 0 \\ \vdots & \end{cases}$$

$$\therefore \epsilon_{yz} = \epsilon_{zy} \simeq 0$$

(6) ϵ_{yy}

$$T_{yy} = \left(\frac{n^2 I_n}{\lambda_s} + 2 I_n \lambda_s - 2 I_n' \lambda_s\right) Z \rightarrow \left(\frac{n^2}{\lambda_s}\left(\frac{\lambda_s}{2}\right)^n + 2\left(\frac{\lambda_s}{2}\right)^n \lambda_s - n\left(\frac{\lambda_s}{2}\right)^{n-1}\lambda_s\right) Z$$

$$= \begin{cases} 0 & (n=0) \\ \frac{1}{2} Z(\zeta_{ns}) & (n=\pm 1) \\ 0 & (|n| \geq 2) \end{cases} \Rightarrow \sum_{n=-\infty}^{\infty} T_{yy} = \frac{1}{2} Z(\zeta_{1s}) + \frac{1}{2} Z(\zeta_{-1s}) = -\frac{1}{2}\frac{1}{\zeta_{1s}} - \frac{1}{2\zeta_{-1s}}$$

$$\zeta_{os} \sum_{n=-\infty}^{\infty} T_{yy} = \frac{\omega}{k_\parallel v_{ts}} \frac{1}{2}\left[-\frac{k_\parallel v_{ts}}{\omega - \omega_{cs}} - \frac{\cancel{k_\parallel v_{ts}}}{\omega + \omega_{cs}}\right] = -\frac{\omega}{2}\frac{2\omega}{\omega^2 - \omega_{cs}^2} = -\frac{\omega^2}{\omega^2 - \omega_{cs}^2}$$

$$\therefore \epsilon_{yy} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2}\left(-\frac{\omega^2}{\omega^2 - \omega_{cs}^2}\right) = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \omega_{cs}^2} = S$$

∴ In cold plasma limit, $\overleftrightarrow{\epsilon} \simeq \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}$

⑩ Hot parallel wave dispersion ✲$(k_\perp \to 0,\ \lambda_s \to 0)$ $O(\lambda_s)$ for $\overleftrightarrow{T}$ becomes

$$\overleftrightarrow{T} = \begin{pmatrix} (Z_{1s}+Z_{-1s})/2 & i(Z_{1s}-Z_{-1s})/2 & 0 \\ -i(Z_{1s}-Z_{-1s})/2 & (Z_{1s}+Z_{-1s})/2 & 0 \\ 0 & 0 & -Z'_{0s}\zeta_{0s} \end{pmatrix}$$

*recall

$$\overleftrightarrow{\epsilon} = \overleftrightarrow{I} + \sum_s \frac{\omega_{ps}^2}{\omega^2}\zeta_{0s} e^{-\lambda_s}\overleftrightarrow{T} \quad \text{and} \quad S = \frac{R+L}{2},\ D = \frac{R-L}{2}$$

$$\left\{\begin{aligned} S &= \epsilon_{xx} = \epsilon_{yy} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2}\zeta_{0s}\frac{Z_{1s}+Z_{-1s}}{2} \\ D &= -\epsilon_{xy} = \epsilon_{yx} = \sum_s \frac{\omega_{ps}^2}{\omega^2}\zeta_{0s}\left(\frac{-Z_{1s}+Z_{-1s}}{2}\right) \\ P &= \epsilon_{zz} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\zeta_{0s}^2 Z'(\zeta_{0s}) \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} R &= 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2}\zeta_{0s} Z_{-1s} \\ L &= 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2}\zeta_{0s} Z_{1s} \\ P &= 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\zeta_{0s}^2 Z'(\zeta_{0s}) \end{aligned}\right\}$$

⑪ Cyclotron damping of parallel wave

Hot plasma oscillation $P = 0 \rightarrow 1 = \sum_s \frac{\omega_{ps}^2}{\omega^2}\frac{\omega^2}{k_\parallel^2 v_{ts}^2} Z'(\zeta_{0s}) \rightarrow k_\parallel^2 = \sum_s \frac{\omega_{ps}^2}{v_{ts}^2} Z'(\zeta_{0s})$

R-wave $n_\parallel^2 = R \rightarrow n_\parallel^2 = 1 + \sum_s \frac{\omega_{ps}^2}{\omega k_\parallel v_{ts}} Z\left(\frac{\omega+\omega_{cs}}{k_\parallel v_{ts}}\right)$

L-wave $n_\parallel^2 = L \rightarrow n_\parallel^2 = 1 + \sum_s \frac{\omega_{ps}^2}{\omega k_\parallel v_{ts}} Z\left(\frac{\omega-\omega_{cs}}{k_\parallel v_{ts}}\right)$

⇒ condition for significant R, L wave damping

R wave : $\omega - \omega_{ce} \leq k_\parallel v_{te}$

L wave : $\omega + \omega_{ci} \leq k v_{ti}$

⑪ Hot perpendicular wave dispersion $\left(k_{\parallel}\to 0,\ \xi_{ns}\to\infty,\ Z\to -\left(\frac{1}{\xi_{ns}}\right)\right)$

$$\begin{pmatrix} \epsilon_{xx} & \epsilon_{xy} & 0 \\ -\epsilon_{xy} & \epsilon_{yy}-n_{\perp}^2 & 0 \\ 0 & 0 & \epsilon_{zz}-n_{\perp}^2 \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

$$T_{zz} = -I_n Z' \xi_{ns} = -I_n \frac{1}{\xi_{ns}} = -I_n \frac{k_{\parallel} v_{ts}}{\omega - n\omega_{cs}}$$

$$\epsilon_{zz} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2}\xi_{os} e^{-\lambda_s} \sum_{n=-\infty}^{\infty} T_{zz} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega} e^{-\lambda_s} \sum_{n=-\infty}^{\infty} \frac{I_n(\lambda_s)}{\omega - n\omega_{cs}}$$

$\therefore$ O-wave $(n_{\perp}^2 = P)$ $\rightarrow$ $\left(n_{\perp}^2 = 1 - \sum_s \frac{\omega_{ps}^2}{\omega} e^{-\lambda_s} \sum_{n=-\infty}^{\infty} \frac{I_n(\lambda_s)}{\omega - n\omega_{cs}}\right)$

X-wave $\left(n_{\perp}^2 = \frac{RL}{S} = \epsilon_{yy} + \frac{\epsilon_{xy}^2}{\epsilon_{xx}}\right)$

$$\rightarrow \left(n_{\perp}^2 = 1 - \sum_s \frac{\omega_{ps}^2}{\omega}\frac{e^{-\lambda_s}}{\lambda_s}\sum_{n=-\infty}^{\infty}\frac{n^2 I_n(\lambda_s) + 2\lambda_s^2 I_n(\lambda_s) - 2\lambda_s^2 I_n'(\lambda_s)}{\omega - n\omega_{cs}} - \frac{\left[\sum_s \frac{\omega_{ps}^2}{\omega} e^{-\lambda_s}\sum_{n=-\infty}^{\infty}\frac{n(I_n' - I_n)}{\omega - n\omega_{cs}}\right]^2}{1 - \sum_s \frac{\omega_{ps}^2}{\omega}\frac{e^{-\lambda_s}}{\lambda_s}\sum_{n=-\infty}^{\infty}\frac{n^2 I_n}{\omega - n\omega_{cs}}} \right)$$

⑫ Cyclotron harmonic damping

In either O or X-wave, one can see new class of resonance due to cyclotron harmonic frequency $\omega \simeq n\omega_c$. These resonances are FLR effects.

$$\frac{1}{\omega - kv_{\parallel} - n\omega_{cs}} \rightarrow P\left(\frac{1}{\omega - kv_{\parallel} - n\omega_{cs}}\right) - i\pi\delta(\omega - kv_{\parallel} - n\omega_{cs})$$

There is another class of solution with small wavelength $n_{\perp} = k_{\perp}c/\omega \gg 1$

$$\epsilon_{xx} = 0 = 1 - \sum_s \frac{\omega_{ps}^2}{\omega}\frac{e^{-\lambda_s}}{\lambda_s}\sum_{n=-\infty}^{\infty}\frac{n^2 I_n(\lambda_s)}{\omega - n\omega_{cs}}$$: Bernstein wave