11. Poisson brackets

- Poisson brackets
- Poisson-bracket representation of equations of motion
- Symmetries and conservation laws
- Mapping to quantum mechanics

1 Poisson brackets

- Definition : $\{f,g\}_{q,p} = \frac{\partial f}{\partial q}\frac{\partial g}{\partial p} - \frac{\partial f}{\partial p}\cdot\frac{\partial g}{\partial q}$, $\{f,g\}_{\underline{\gamma}} = \left(\frac{\partial f}{\partial \underline{\gamma}}\right)^T \underline{\underline{J}} \left(\frac{\partial f}{\partial \underline{\gamma}}\right)$

- Canonical invariance : $\underline{\gamma} \to \underline{\Gamma}$ with $M_{ij} = \partial\Gamma_i/\partial\gamma_j$ $\quad (\underline{\Gamma} = \underline{\underline{M}}\,\underline{\gamma})$

$$\{f,g\}_{\underline{\gamma}} = \left(\frac{\partial f}{\partial \underline{\gamma}}\right)^T \underline{\underline{J}} \left(\frac{\partial f}{\partial \underline{\gamma}}\right) = \left(\underline{\underline{M}}^T\frac{\partial f}{\partial \underline{\Gamma}}\right)^T \underline{\underline{J}} \left(\underline{\underline{M}}^T\frac{\partial f}{\partial \underline{\Gamma}}\right) = \left(\frac{\partial f}{\partial \underline{\Gamma}}\right)^T \underline{\underline{M}}\,\underline{\underline{J}}\,\underline{\underline{M}}^T \frac{\partial f}{\partial \underline{\Gamma}} = \left(\frac{\partial f}{\partial \underline{\Gamma}}\right)^T \underline{\underline{J}} \left(\frac{\partial f}{\partial \underline{\Gamma}}\right) = \{f,g\}_{\underline{\Gamma}}$$

$\Rightarrow$ Poisson bracket is invariant under canonical transformation : $\{f,g\}$

- Algebraic properties :

① Anticommutativity : $\{f,g\} = -\{g,f\}$

② Bilinearity : $\{af+bg,h\} = a\{f,h\} + b\{g,h\}$

③ Product rule : $\{fg,h\} = \{f,h\}g + f\{g,h\}$

④ Jacobi's identity : $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$

2 Poisson-bracket representation of equations of motion

- $f(\underline{\gamma},t) \Rightarrow \frac{df}{dt} = \frac{\partial f}{\partial \underline{\gamma}}\cdot\dot{\underline{\gamma}} + \frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial \underline{\gamma}}\right)\underline{\underline{J}}\left(\frac{\partial H}{\partial \underline{\gamma}}\right) + \frac{\partial f}{\partial t} = \{f,H\} + \frac{\partial f}{\partial t}$

Hamilton's eqn : $\left(\dot{\underline{\gamma}} = \underline{\underline{J}}\frac{\partial H}{\partial \underline{\gamma}}\right)$

$\rightarrow$ f is constant of motion iff $\{f,H\} = 0$

$\rightarrow$ H is conserved iff $\frac{\partial H}{\partial t} = 0$

$\rightarrow$ If f and g are conserved and $\partial f/\partial t = \partial g/\partial t = 0$, $\{f,g\}$ is also conserved

Proof) $\{H,\{f,g\}\} + \{g,\{\cancel{H},f\}\} + \{f,\{g,\cancel{H}\}\} = 0$ $\quad \therefore \frac{d}{dt}\{f,g\} = \{\{f,g\},H\} + \frac{\partial}{\partial t}\{f,g\} = 0$

- Hamilton's equations : $\dot{q} = \{q, H\}$, $\dot{p} = \{p, H\}$ $\iff$ $\dot{\mathbf{r}} = \{\mathbf{r}, H\}$

  $(\mathbf{r}, \mathbf{p})$는 시간에 명시적으로 의존하지 않음.

- Liouville's theorem

  $\rho(\mathbf{r}, t)$: phase space distribution / $\rho(\mathbf{r}, t)\, d^{2n}\mathbf{r}$: probability

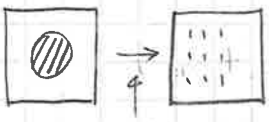
$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot (\rho \dot{\mathbf{r}}) \quad \text{(conservation of probability)}$$

$$= \frac{1}{2}(J_{ij} + J_{ji}) \frac{\partial^2 H}{\partial r_i \partial r_j} = 0$$

$$\hookrightarrow \frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial \mathbf{r}} \cdot \left(\rho \mathbb{J} \cdot \frac{\partial H}{\partial \mathbf{r}}\right) = -\left(\frac{\partial \rho}{\partial \mathbf{r}}\right)^T \mathbb{J} \frac{\partial H}{\partial \mathbf{r}} - \rho J_{ij} \frac{\partial^2 H}{\partial r_i \partial r_j} = -\{\rho, H\}$$

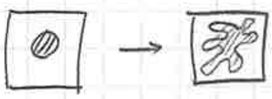
$$\Rightarrow \boxed{\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \{\rho, H\} = 0}$$

phase space distribution function is constant along the trajectories of Hamiltonian system.



Liouville 정리 위배

(① Liouville 정리
② 확률분포의 정규화조건)

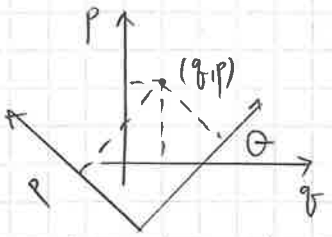


비율일정
큰 해상도에서
equal-a-priori 만족

3 Symmetries and conservation laws

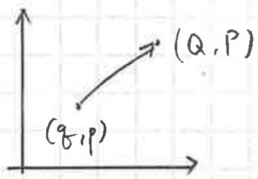
A. Two different interpresentation of canonical transformations.

Passive view



$f(\mathbf{r}) = F(\mathbf{r}'(\mathbf{r}))$

Active view



$f(\mathbf{r}) \rightarrow f(\mathbf{r}'(\mathbf{r}))$

B. Active view of infinitesimal canonical transformations

$\mathbf{r} = \mathbf{r} + \delta\mathbf{r}$, $\delta \mathbf{r} = \epsilon \mathbb{J} \frac{\partial G}{\partial \mathbf{r}}$ ($G(\mathbf{r}, t)$ is called the generator of the infinitesimal canonical transformation)

$$\delta f = f(\mathbf{r} + \delta\mathbf{r}, t) - f(\mathbf{r}, t) = \frac{\partial f}{\partial \mathbf{r}} \delta \mathbf{r} = \epsilon \left(\frac{\partial f}{\partial \mathbf{r}}\right) \mathbb{J} \left(\frac{\partial G}{\partial \mathbf{r}}\right) = \epsilon \{f, G\}$$

$\Rightarrow$ $\{f, G\}$ indicates the change of f generated by G.

C. Examples of infinitesimal transformations

- Positions and linear momenta

given function $f(q_I)$, $\underline{\epsilon\{f(q),p_j\} = \epsilon\frac{\partial f}{\partial q_j} = f(q_I+\epsilon\hat{e}_j) - f(q_I)}$

$\Rightarrow$ p_j generates infinitesimal translation of its conjugate position q in the direction of $\hat{e}_j$.

↳ Fundamental Poisson brackets $\{q_i, q_j\} = \{p_i, p_j\} = 0$, $\{q_i, p_j\} = \delta_{ij} \Leftrightarrow \{r_i, r_j\} = J_{ij}$

- Angular momenta

– system vector $\underline{F}$:= $\underline{F}(\underline{R}_{\hat{n}}(\theta)\underline{x}, \underline{R}_{\hat{n}}(\theta)\underline{p}) = \underline{R}_{\hat{n}}(\theta)\underline{F}(\underline{x},\underline{p})$

($\underline{R}_{\hat{n}}(\theta)$ form a group of $\underset{\uparrow \text{proper}}{S}\overset{\leftarrow\text{orthogonal}}{O}(3)_{\leftarrow 3\times3}$) : Lie groups.

– For any function $f(\underline{x},\underline{p})$ and angular momentum $\underline{L} \equiv \underline{r}\times\underline{p}$,

$$d\theta\{f, L_i\} = d\theta\{f, \epsilon_{ijk}x_j p_k\} = d\theta\, \epsilon_{ijk}\left(\{f,x_j\}p_k + x_j\{f,p_k\}\right)$$

$$= d\theta\,\epsilon_{ijk}\left(-\frac{\partial f}{\partial p_j}p_k + x_j\frac{\partial f}{\partial x_k}\right) = d\theta\,\epsilon_{ijk}\left(p_j\frac{\partial f}{\partial p_k} + x_j\frac{\partial f}{\partial x_k}\right)$$

$$= f\left((x_k + d\theta\,\epsilon_{ijk}x_j)\hat{e}_k, (p_k + d\theta\,\epsilon_{ijk}p_j)\hat{e}_k\right) - f(\underline{x},\underline{p})$$

$$= f(\underline{R}_{\hat{e}_i}(d\theta)\underline{x}, \underline{R}_{\hat{e}_i}(d\theta)\underline{p}) - f(\underline{x},\underline{p})$$

– If $\underline{F}$ is a system vector,

$$d\theta\{F_i, L_j\} = F_i(\underline{R}_{\hat{e}_j}(d\theta)\underline{x}, \underline{R}_{\hat{e}_j}(d\theta)\underline{p}) = [\underline{R}_{\hat{e}_j}(d\theta) - \underline{1}]_{ik}F_k(\underline{x},\underline{p})$$

$$= d\theta(\underline{J}_j)_{ik}F_k = -d\theta\,\epsilon_{ikj}F_k = d\theta\,\epsilon_{ijk}F_k$$

$$\therefore \underline{\{F_i, L_j\} = \epsilon_{ijk}F_k}$$

$$\underline{d\theta\{\underline{F}, \underline{L}\cdot\hat{n}\} = d\theta\,\hat{e}_i\{F_i, L_j\}\hat{n}_j = d\theta\,\epsilon_{ijk}\hat{e}_i\hat{n}_j F_k = d\theta\,\hat{n}\times\underline{F}}$$

↳ $\underline{\underline{L}\cdot\hat{n}\text{ generates infinitesimal rotation about }\hat{n}}$ / $\underline{\{L_i, L_j\} = \epsilon_{ijk}L_k}$

– If $\underline{F}, \underline{G}$ are system vectors, any function $f(\underline{F}\cdot\underline{G})$ satisfies

$$\underline{d\theta\{f, L_i\}} = f\left(\underline{F}(\underline{R}_{\hat{e}_i}(d\theta)\underline{x}, \underline{R}_{\hat{e}_i}(d\theta)\underline{p})\cdot\underline{G}(\underline{R}_{\hat{e}_i}(d\theta)\underline{x}, \underline{R}_{\hat{e}_i}(d\theta)\underline{p})\right) - f(\underline{F}\cdot\underline{G})$$

$$= f([\underline{R}_{\hat{e}_i}(d\theta)\underline{F}]\cdot[\underline{R}_{\hat{e}_i}(d\theta)\underline{G}]) - f(\underline{F}\cdot\underline{G})$$

$$= f(\underline{F}\cdot\underline{R}^T_{\hat{e}_i}(d\theta)\,\underline{R}_{\hat{e}_i}(d\theta)\,\underline{G}) - f(\underline{F}\cdot\underline{G}) = f(\underline{F}\cdot\underline{G}) - f(\underline{F}\cdot\underline{G}) = \underline{0}$$

* Proof of $\underline{L} = \underline{x} \times \underline{p}$: system vector

Let $\underline{C} = \underline{A} \times \underline{B}$, $\det \underline{R} = 1$

$$[\underline{R}^T(\underline{R}\underline{A} \times \underline{R}\underline{B})]_i = R_{ji}\,\varepsilon_{jkl}\,R_{km}A_m R_{ln}B_n = \varepsilon_{jkl}R_{ji}R_{km}R_{ln}A_mB_n \quad (\det \underline{R} = \varepsilon_{ijk}R_{i1}R_{j2}R_{k3})$$

$$= (\det \underline{R})\,\varepsilon_{imn}A_mB_n = (\underline{A}\times\underline{B})_i$$

Hence, $\underline{R}^T(\underline{R}\underline{A}\times\underline{R}\underline{B}) = \underline{A}\times\underline{B} \Rightarrow (\underline{R}\underline{A}\times\underline{R}\underline{B}) = \underline{R}(\underline{A}\times\underline{B})$

Thus, $\underline{L}(\underline{x},\underline{p}) \equiv \underline{x}\times\underline{p} \Rightarrow \underline{L}(\underline{R}\underline{x},\underline{R}\underline{p}) = \underline{R}\,\underline{L}(\underline{x},\underline{p})$: system vector

∘ Hamilton

For any function $f(\underline{\gamma})$, $\quad dt\{f,H\} \overset{\text{Eqn of motion}}{=} dt\,\dot{f} = df$

(note that $\frac{\partial f}{\partial t} = 0$) $\Rightarrow$ Hamiltonian generates infinitesimal time translation.

D. Noether's theorem in the phase space

∘ Change of the Hamiltonian $\quad \underline{\gamma} \to \underline{\Gamma} = \underline{\gamma} + \delta\underline{\gamma}$ (generated by $G(\underline{\gamma},t)$)

$$\delta H = H(\underline{\gamma}+\delta\underline{\gamma},t) - H(\underline{\gamma},t) = \underbrace{H(\underline{\gamma}+\delta\underline{\gamma}) - K(\underline{\gamma}+\delta\underline{\gamma})}_{\equiv \bar{\delta}H} + \underbrace{K(\underline{\gamma}+\delta\underline{\gamma}) - H(\underline{\gamma},t)}_{\equiv \epsilon\, dG/dt}$$

$$\bar{\delta}H = \delta H - \epsilon\frac{dG}{dt} = \epsilon\{H,G\} - \epsilon\frac{\partial G}{\partial t} = -\epsilon\frac{dG}{dt}$$

∘ Noether's theorem in the phase space

$\bar{\delta}H = 0$ iff $\frac{dG}{dt} = 0$ $\quad$ ($\bar{\delta}H$: canonical 변환으로 같은 지점에서 측정한 Hamiltonian 값의 변화)

E. Symmetry group in the Kepler problem

- Hamiltonian : $H = \frac{p^2}{2m} - \frac{k}{r}$ $\left(r = \sqrt{\underline{r}\cdot\underline{r}},\ p = \sqrt{\underline{p}\cdot\underline{p}}\right)$

- Constants of motion

  - $\frac{\partial H}{\partial t} = 0$, H is conserved.
  - $\{H, L_i\} = \frac{1}{2m}\{\underline{p}\cdot\underline{p}, L_i\} - k\{(\underline{r}\cdot\underline{r})^{-1/2}, L_i\} = 0$, $\underline{L}$ is conserved.

    (각운동량 : 회전을 만드는 것 → 스칼라는 회전에 불변 : $\{H, L_i\} = 0$

  - $\underline{A} \equiv \underline{p}\times\underline{L} - mk\hat{\underline{r}}$ (Laplace-Runge-Lenz (LRL))

$$\{H, A_i\} = \frac{1}{2m}\underbrace{\{p^2, \epsilon_{ijk}p_jL_k\}}_{(i)} - \frac{k}{2}\underbrace{\{p^2, \frac{r_i}{r}\}}_{(ii)} - k\underbrace{\{\frac{1}{r}, (\underline{p}\times\underline{L})_i\}}_{(iii)} - k\cancel{\{\frac{1}{r}, \frac{r_i}{r}\}}^{0} = 0$$

(i) $\{p^2, p_iL_j\} = \cancel{\{p^2, p_i\}}^{0}L_j + p_i\cancel{\{p^2, L_j\}}^{0} = 0$ $\left(\{\text{scalar}, L_j\} = 0\right)$

(ii) $\{p^2, \frac{r_i}{r}\} = 2p_i \times \left(-\frac{r_j}{r^3}r_i + \frac{1}{r}\right) = \frac{2r_ir_j}{r^3}p_j - \frac{p_i}{2}$

(iii) $\{\frac{1}{r}, (\underline{p}\times\underline{L})_i\} = \{\frac{1}{r}, \epsilon_{ijk}p_jL_k\} = \epsilon_{ijk}L_k\{\frac{1}{r}, p_j\} = -\epsilon_{ijk}L_k\frac{r_j}{r^3}$

$$= -\epsilon_{ijk}\epsilon_{lmk}\frac{r_jr_lp_m}{r^3} = -\frac{r_ir_jp_j}{r^3} + \frac{p_i}{r}$$

Thus $\underline{A}$ is conserved.

- Symmetry group for bound orbits $(H<0)$

$\{A_i, A_j\} = -2mH\epsilon_{ijk}L_k$, define $D_i \equiv L_i/\sqrt{2m|H|}$, then

$\{L_i, L_j\} = \epsilon_{ijk}L_k$, $\{D_i, L_j\} = \epsilon_{ijk}D_k$, $\{D_i, D_j\} = \epsilon_{ijk}L_k$ ⇒ Lie-Algebra of what?

$\underline{L}_1 = \begin{bmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&0&0&-1\\ 0&0&1&0 \end{bmatrix}$ $\underline{L}_2 = \begin{bmatrix} 0&0&0&0\\ 0&0&0&1\\ 0&0&0&0\\ 0&-1&0&0 \end{bmatrix}$ $\underline{L}_3 = \begin{bmatrix} 0&0&0&0\\ 0&0&-1&0\\ 0&1&0&0\\ 0&0&0&0 \end{bmatrix}$ ⇐ Infinitesimal rotation generator

$\left([\underline{L}_i, \underline{L}_j] = \epsilon_{ijk}\underline{L}_k\right)$

$\underline{D}_1 = \begin{bmatrix} 0&-1&0&0\\ 1&0&0&0\\ 0&0&0&0\\ 0&0&0&0 \end{bmatrix}$ $\underline{D}_2 = \begin{bmatrix} 0&0&-1&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{bmatrix}$ $\underline{D}_3 = \begin{bmatrix} 0&0&0&-1\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{bmatrix}$

$[\underline{L}_i, \underline{L}_j] = \epsilon_{ijk}\underline{L}_k$, $[\underline{D}_i, \underline{L}_j] = \epsilon_{ijk}\underline{D}_k$, $[\underline{D}_i, \underline{D}_j] = \epsilon_{ijk}\underline{L}_k$ ⇒ SO(4) group.

(Kepler problem is Symmetric under rotations in the SO(4) group
⇒ The symmetry group of bound Kepler problem is SO(4))

- Symmetry group for unbound orbits $(H \geq 0)$ (Hyperbolic)

$$\{A_i, A_j\} = -2mH\varepsilon_{ijk}L_k \ , \quad D_i \equiv L_i/\sqrt{2mH}$$

$(H>0)$ $\{L_i, L_j\} = \varepsilon_{ijk}L_k$, $\{D_i, L_j\} = \varepsilon_{ijk}D_k$, $\{D_i, D_j\} = -\varepsilon_{ijk}L_k$ $\Rightarrow$ SO(3,1) group

$(H=0)$ $\{L_i, L_j\} = \varepsilon_{ijk}L_k$, $\{A_i, L_j\} = \varepsilon_{ijk}A_k$, $\{A_i, A_j\} = 0$ $\Rightarrow$ SE(3) group

rotation + translation in 3-d

4 Mapping to quantum mechanics

A. Classical vs. quantum mechanics

- classical Mechanics → Quantum mechanics
  - state : phase-space coordinates → $|\psi\rangle$ in Hilbert space
  - observable : real valued function → Linear operator with real eigenvalues
  - $du = \epsilon\{u,v\}$ → ?

대응되는게 무엇일까?

B. Common theoretical template of classical and quantum mechanics

- $\{u,v\}_{QM}$ must satisfy the product rule.

$$\{u_1u_2, v_1v_2\}_{QM} = \{u_1, v_1v_2\}u_2 + u_1\{u_2, v_1v_2\}$$
$$= \underline{\{u_1,v_1\}v_2u_2} + v_1\{u_1,v_2\}u_2 + \underline{u_1v_1\{u_2,v_2\}} + u_1\{u_2,v_1\}u_2$$

$$\{u_1u_2, v_1v_2\}_{QM} = \{u_1u_2, v_1\}v_2 + v_1\{u_1u_2, v_2\}$$
$$= u_1\{u_2,v_1\}v_2 + \underline{\{u_1,v_1\}u_2v_2} + \underline{v_1u_1\{u_2,v_2\}} + v_1\{u_1,v_2\}v_2$$

$$\therefore \{u_1,v_1\}(u_2v_2 - v_2u_2) = (u_1v_1 - v_1u_1)\{u_2,v_2\}$$

$$\Rightarrow \underline{\{u,v\}_{QM} = c(uv - vu) = c[u,v]}$$

To specify c, (1) u,v : Hermitian → $\{u,v\}_{QM}$: Hermitian

$$\{u,v\}^\dagger = -c^*[u,v] = \{u,v\} = c[u,v] \qquad c^* = -c \text{ (pure imaginary)}$$

(2) dimension of $\{u,v\} = [u][v]/(q)(p)$ $\Rightarrow$ $[c] = 1/(q)(p)$

(1)+(2) → $c = i\hbar$ $\Rightarrow$ $\boxed{\{u,v\}_{QM} = i\hbar[u,v]}$

- Quantum analogs of classical poisson brackets

$$\begin{cases} [q_i, q_j] = 0, \quad [p_i, p_j] = 0, \quad [q_i, p_j] = i\hbar\delta_{ij} \\ [v_i, L_j] = i\hbar\,\varepsilon_{ijk} v_k \\ i\hbar\dot{u} = [u, H] \end{cases}$$

$[q_i, p_j] = i\hbar\delta_{ij}$ ← 동시에 측정 불가.

$i\hbar\dot{u} = [u, H]$ ← Heisenberg equation of motion