Fusion Plasma Theory 2

Lecture 10 : Parallel / Ion Waves, CMA.

① High frequency parallel waves

Two parallel, R and L, waves : $n_{\parallel}^2 = n_{R,L}^2 = 1 - \dfrac{\omega_{pe}^2}{\omega(\omega \pm \omega_{ce})}$ (R : +, L : −)

- Resonance for L-wave ✗
- Resonance for R-wave at $\omega = -\omega_{ce}$

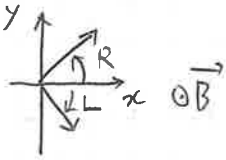
  electrons rotate around $\vec{B}$ along with right hand direction.
  So only R-wave can resonate with electron motion.

- cutoffs $\left(n_{\parallel}^2 = \dfrac{\omega^2 \pm \omega_{ce}\omega - \omega_{pe}^2}{\omega(\omega \pm \omega_{ce})} \right)$

$$\omega_{R,L} = \frac{1}{2}\left[\mp \omega_{ce} + \left(\omega_{ce}^2 + 4\omega_{pe}^2 \right)^{1/2} \right]$$

② Polarization of plasma waves

∵ is defined by relative amplitude and phase between the electric field components perpendicular to the background $\vec{B_0}$.



$\vec{E} = \hat{x}E_x + \hat{y}E_y$

R : $\vec{E} = (\hat{x} + i\hat{y})E_{x_0} e^{i(kz - \omega t)}$
$= (\hat{x} + i\hat{y}) e^{-i\omega t} E_{x_0} e^{ikz}$

$Re(\vec{E}) = (\hat{x}\cos\omega t + \hat{y}\sin\omega t)E_{x_0}$

From cold plasma dispersion

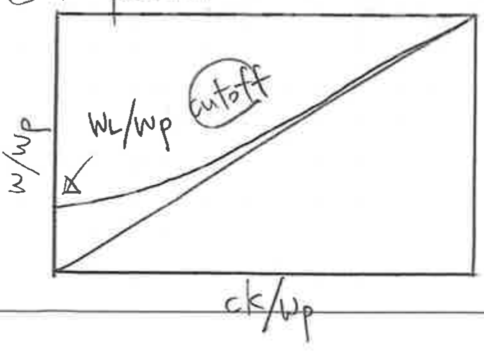
$$\begin{pmatrix} -n^2\cos^2\theta + S & -iD & n^2\sin\theta\cos\theta \\ iD & -n^2 + S & 0 \\ n^2\sin\theta\cos\theta & 0 & -n^2\sin^2\theta + P \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0 \Rightarrow E_y = i\frac{D}{n^2 - S}E_x$$

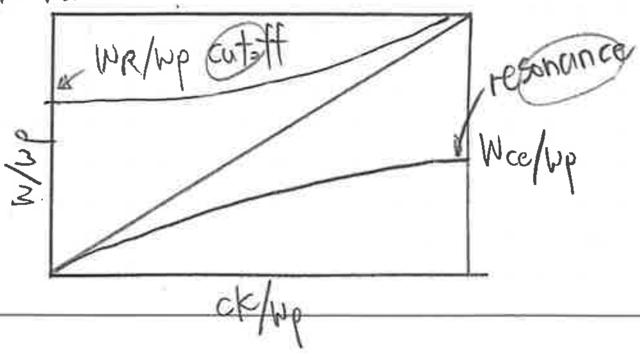
For R, L wave ($n_{\parallel}^2 = R$, $n_{\parallel}^2 = L$),

$E_y = \pm i E_x = e^{\pm i\frac{\pi}{2}} E_x$ → ∴ R and L waves are circularly polarized waves.

Also, R-wave rotates along with RH direction,
and L-wave rotates along with LH direction

③ Dispersion curve of L and R waves





When $\omega > \omega_R$
R wave is always faster than L-wave
(w.r.t. phase velocity)
⇒ Faraday rotation

④ Faraday rotation

A linearly polarized wave $\vec{E} = E_0 \hat{x} e^{-i\omega t}$ at $z=0$

can be decomposed by R and L wave electric fields : $\vec{E} = \vec{E_R} + \vec{E_L}$

$$\vec{E_R}(z=0) = \frac{E_0}{2}(\hat{x}+i\hat{y})e^{-i\omega t}, \quad \vec{E_L} = \frac{E_0}{2}(\hat{x}-i\hat{y})e^{-i\omega t}.$$

$$\Downarrow$$

$$\vec{E_R}(z) = \frac{E_0}{2}(\hat{x}+i\hat{y})e^{ik_R z - i\omega t}, \quad \vec{E_L}(z) = \frac{E_0}{2}(\hat{x}-i\hat{y})e^{ik_L z - i\omega t} \qquad \left(k_R = \frac{\omega}{c}n_R,\ k_L = \frac{\omega}{c}n_L\right)$$

Adding two, one has

$$\vec{E} = \frac{E_0}{2}\left[\hat{x}(e^{ik_R z}+e^{ik_L z}) + i\hat{y}(e^{ik_R z}-e^{ik_L z})\right]e^{-i\omega t} \qquad \left(\begin{aligned} & e^{ik_R z} \pm e^{ik_L z} \\ &= e^{i\left(\frac{k_L+k_R}{2}\right)z}\left[e^{i\left(\frac{k_R-k_L}{2}\right)z} \pm e^{-i\left(\frac{k_R-k_L}{2}\right)z}\right]\end{aligned}\right)$$

$$= \frac{E_0}{2} e^{i\left(\frac{k_R+k_L}{2}\right)z}\left[\hat{x}\, 2\cos\left(\frac{k_R-k_L}{2}z\right) + i\hat{y}\, 2i\sin\left(\frac{k_R-k_L}{2}z\right)\right]e^{-i\omega t}$$

$$= E_0\left[\hat{x}\cos\left(\frac{k_L-k_R}{2}\right)z + \hat{y}\sin\left(\frac{k_L-k_R}{2}\right)z\right]\exp\left[i\left(\frac{k_L+k_R}{2}\right)z - i\omega t\right]$$

So the parallel wave remains linearly polarized, but rotated by an angle

$$\Delta\phi \equiv \left(\frac{k_L-k_R}{2}\right)z = (n_L - n_R)\frac{\omega}{2c}z$$

For $\omega \gg \omega_{pe}, \omega_{ce}$

$$n_{R,L} = \left(1 - \frac{\omega_{pe}^2}{\omega^2 \pm \omega\omega_{ce}}\right)^{1/2} \simeq 1 - \frac{\omega_{pe}^2}{2\omega(\omega\pm\omega_{ce})}, \quad \left(\frac{1}{\omega\pm\omega_{ce}} = \frac{1}{\omega}\left(1\pm\frac{\omega_{ce}}{\omega}\right)^{-1} = \frac{1}{\omega}\left(1\mp\frac{\omega_{ce}}{\omega}\right) = \frac{1}{\omega}\mp\frac{\omega_{ce}}{\omega^2}\right)$$

$$= 1 - \frac{\omega_{pe}^2}{2\omega}\left(\frac{1}{\omega}\mp\frac{\omega_{ce}}{\omega^2}\right) = 1 - \frac{\omega_{pe}^2}{2\omega^2} \pm \frac{\omega_{pe}^2\omega_{ce}}{2\omega^3}.$$

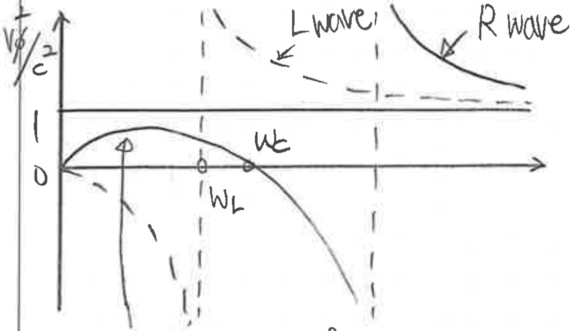
∴ Faraday rotation is approximately given by $\boxed{\Delta\phi = -\frac{\omega_{pe}^2\omega_{ce}}{2\omega^3 c}z}$

$$\boxed{\Delta\phi = \frac{e}{2m_e c n_c} n_e B z} \quad \text{where } n_c = \frac{\omega^2 m \epsilon_0}{e^2}$$

$$\Delta\phi \simeq \frac{e}{2m_e c}\int \frac{n_e}{n_c}\vec{B}\cdot d\vec{l}$$

(It can be used to measure the density and also magnetic field profiles.)

⑤ Whistler waves



low ω arrive later.
giving rise to the descending tone like a whistle.

whistler wave : One can hear even at very low frequency.

⑥ Low-frequency two-component plasma dispersion relation.

In low frequency range, $\omega < (\omega_{pe}, \omega_{ce})$, ion physics can play an important role.

$$n_{\parallel}^2 = n_{R,L}^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega \pm \omega_{ce})} - \frac{\omega_{pi}^2}{\omega(\omega \pm \omega_{ci})} \approx \frac{\omega^2 \pm (\omega_{ce} + \cancel{\omega_{ci}})\omega + \omega_{ce}\omega_{ci} - \omega_{pe}^2 \frac{1}{\omega}(\omega + \omega_{ci}) - \omega_{pi}^2 \frac{1}{\omega}(\omega + \omega_{ce})}{(\omega \pm \omega_{ce})(\omega \pm \omega_{ci})}$$

$$\simeq \frac{\omega^2 \pm \omega\omega_{ce} + \omega_{ce}\omega_{ci} - \omega_{pe}^2}{(\omega \pm \omega_{ce})(\omega \pm \omega_{ci})}$$

O-wave remains same : $n_\perp^2 = P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega_s^2}$

X-wave becomes quite complicated

$$n_\perp^2 = \frac{RL}{S} = \frac{(\omega^2 + \omega\omega_{ce} + \omega_{ce}\omega_{ci} - \omega_{pe}^2)(\omega^2 - \omega\omega_{ce} + \omega_{ce}\omega_{ci} - \omega_{pe}^2)}{(\omega^2 - \omega_{UH}^2)(\omega^2 - \omega_{LH}^2)}$$

$$\boxed{\omega_{UH}^2 = \omega_{pe}^2 + \omega_{ce}^2}, \quad \boxed{\frac{1}{\omega_{LH}^2} \equiv \frac{1}{\omega_{pi}^2 + \omega_{ci}^2} + \frac{1}{|\omega_{ci}\omega_{ce}|}}$$

⑦ Low-frequency parallel (shear Alfvén) R-wave

$$\left(-\frac{\omega_{pe}^2}{\omega_{ce}} = \frac{\omega_{pi}^2}{\omega_{ci}}\right) \qquad \text{for R-wave}$$

$\omega \ll \omega_{ce}$, $$n_{R,L}^2 \simeq \frac{\cancel{\omega^2} \pm \cancel{\omega}\omega_{ce} + \omega_{ce}\omega_{ci} - \omega_{pe}^2}{(\cancel{\omega} \pm \omega_{ce})(\omega \pm \omega_{ci})} = 1 \mp \frac{\omega_{pe}^2}{\omega_{ce}(\omega \pm \omega_{ci})} = 1 \pm \frac{\omega_{pi}^2}{\omega_{ci}(\omega \pm \omega_{ci})} = 1 + \frac{\omega_{pi}^2}{\omega_{ci}(\omega + \omega_{ci})}$$

one can see there's no resonance or cutoff in R-wave. in this frequency range.

$$n_R^2 \simeq 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2} = 1 + \frac{c^2}{v_A^2} \simeq \frac{c^2}{v_A^2} \longrightarrow \omega = k v_A$$

where $$v_A = c\frac{\omega_{ci}}{\omega_{pi}} = c\frac{eB}{M}\sqrt{\frac{M\epsilon_0}{ne^2}} = \frac{B}{\sqrt{\mu_0 M n}} = \frac{B}{\sqrt{\mu_0 \rho}}$$

⑧ Low-frequency parallel (shear Alfvén) L-wave

$$n_L^2 \simeq 1 - \frac{\omega_{pi}^2}{\omega_{ci}(\omega - \omega_{ci})}$$

same dispersion in the very low frequency range : $n_{R,L}^2 = 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2} = 1 + \frac{c^2}{v_A^2}$ (same)

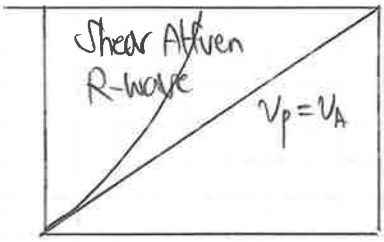
However, compared to R-wave, it has cutoff and resonance.

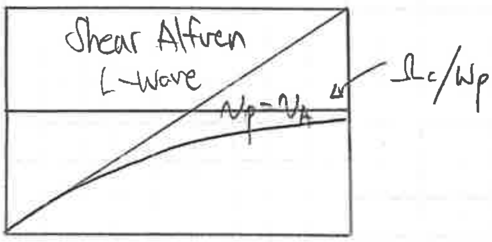
$$\left[\begin{array}{l} \text{Resonance} : \omega = \omega_{ci} \quad (\text{ion-lefthand direction} \Leftrightarrow \text{L-wave}) \\ \text{cutoff} : \omega_L = \omega_{ci} + \frac{\omega_{pi}^2}{\omega_{ci}} \end{array}\right.$$

Approximation (High frequency / Low frequency) in L-wave does not change the L-wave cutoff condition.

⑨ Shear Alfvén wave characteristics

Fluid frozen to the field perturbates.





⑩ Very low-frequency perpendicular (magnetosonic) X-wave.

In the very low frequency range, $R \approx L$, so $S \simeq R \simeq L$. (from ⑦ and ⑧)

$$\therefore \epsilon_\perp \simeq R \simeq L \simeq S \approx 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2} = 1 + \frac{c^2}{v_A^2} = 1 + \frac{\mu_0 \rho c^2}{B^2}$$

This means for perpendicular wave:

$$n_\perp^2 = RL/S \simeq R \simeq L \simeq S \simeq 1 + \frac{\omega_{pi}^2}{\omega_{ci}^2} = 1 + \frac{c^2}{v_A^2} \simeq n_\parallel^2$$

It indicates that the <u>low-frequency cold-plasma wave is isotropic</u> in nature

⑪ Low frequency perpendicular X-wave.

$$\omega_{ci} \ll \omega \ll (\omega_{ce}, \omega_{pe}), \qquad (R, L) = 1 - \frac{\omega_{pe}^2}{(\omega \pm \omega_{ce})(\omega \pm \omega_{ci})}$$

$$S = \frac{1}{2}(R+L) = 1 - \frac{\omega_{pe}^2}{2(\omega+\omega_{ce})(\omega+\omega_{ci})} - \frac{\omega_{pe}^2}{2(\omega-\omega_{ce})(\omega-\omega_{ci})} = 1 - \frac{\omega_{pe}^2(\omega^2+\omega_{ce}\omega_{ci})}{(\omega^2-\omega_{ce}^2)(\omega^2-\omega_{ci}^2)}$$

$$\simeq \frac{\omega^4 - (\omega_{pe}^2+\omega_{ce}^2)\omega^2 + (\omega_{ce}\omega_{ci} - \omega_{pe}^2)\omega_{ce}\omega_{ci}}{(\omega^2-\omega_{ce}^2)(\omega^2-\omega_{ci}^2)}$$

With approximation of $\omega_{pe}^2+\omega_{ce}^2 \gg (\omega_{pe}(\omega_{ce}\omega_{ci})^{1/2}, \omega_{ce}^2\omega_{ci}^2)$

quadratic eq. solution.

$$\omega^2 \approx \begin{cases} \omega_{pe}^2+\omega_{ce}^2 \equiv \omega_{UH}^2 \\ \dfrac{(\omega_{ce}\omega_{ci} - \omega_{pe}^2)\omega_{ce}\omega_{ci}}{\omega_{pe}^2+\omega_{ce}^2} \equiv \omega_{LH}^2 \end{cases}$$

⑫ On hybrid-resonances

$$S \approx \frac{(\omega^2-\omega_{UH}^2)(\omega^2-\omega_{LH}^2)}{(\omega^2-\omega_{ce}^2)(\omega^2-\omega_{ci}^2)}.$$

- In low n, one can assume $\omega_{ce}^2 \gg \omega_{pe}^2$,

thus $$\omega_{LH}^2 \simeq \frac{(\omega_{ce}\omega_{ci} - \omega_{pc}^2)\omega_{ce}\omega_{ci}}{\cancel{\omega_{pe}^2}+\omega_{ce}^2} \simeq \omega_{ci}^2 - \frac{\omega_{pe}^2\omega_{ci}}{\omega_{ce}} = \omega_{ci}^2 + \omega_{pi}^2$$

↳ = ion version of upper hybrid frequency

- In high n, one can assume $\omega_{pe}^2 \gg \omega_{ce}^2$.

thus $$\omega_{LH}^2 = \frac{(\omega_{ce}\omega_{ci} - \omega_{pe}^2)\omega_{ce}\omega_{ci}}{\omega_{pe}^2+\cancel{\omega_{ce}^2}} \simeq \omega_{ce}\omega_{ci} \quad \Rightarrow \omega_{LH} \simeq |\omega_{ce}\omega_{ci}|$$