14. Toroidal Equilibria

- Ideal MHD equilibria

① $\underline{\vec{F} = \vec{j} \times \vec{B} - \vec{\nabla} p = 0}$ ② $\underline{\vec{\nabla} \times \vec{B} = \mu_0 \vec{j}}$, ③ $\underline{\vec{\nabla} \cdot \vec{B} = 0}$ … (1)

- Flux coordinate representation for equilibrium

use arbitrary curvilinear coordinates (ρ, θ, φ) with $\rho = \rho(\psi)$,

$$\underline{\vec{F} = \sqrt{g}\,(j^\theta B^\varphi - j^\varphi B^\theta)\vec{\nabla}\rho + \sqrt{g}\, j^\rho B^\theta\, \vec{\nabla}\varphi - \sqrt{g}\, j^\rho B^\varphi \vec{\nabla}\theta - p'\vec{\nabla}\rho = \vec{0}} \quad \cdots (2)$$

$$\vec{F} = F_\rho \vec{\nabla}\rho + F_\beta \vec{\beta} \quad \text{where} \left(\begin{array}{l} F_\rho = \sqrt{g}\,(j^\theta B^\varphi - j^\varphi B^\theta) - p' \\ F_\beta = \sqrt{g}\, j^\rho, \quad \vec{\beta} = B^\theta \vec{\nabla}\varphi - B^\varphi \vec{\nabla}\theta \end{array} \right)$$

solving equilibrium means $F_\rho = 0$, $F_\beta = 0$

- Contravariant representation of fields and currents

let (ϑ, φ) SFL angle, and $\vartheta = \theta + \lambda(\rho, \vartheta, \varphi)$

$$\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\alpha = \vec{\nabla}\psi \times \vec{\nabla}(\vartheta - \iota\varphi) = \psi' \vec{\nabla}\rho \times \vec{\nabla}(\theta - \iota\varphi + \lambda) \quad \cdots (6)$$

$$= \psi'\left(1 + \frac{d\lambda}{d\theta}\right)\vec{\nabla}\rho \times \vec{\nabla}\theta + \psi'\left(\iota - \frac{d\lambda}{d\varphi}\right)\vec{\nabla}\varphi \times \vec{\nabla}\rho \quad \cdots (7)$$

$$\Rightarrow \quad \underline{B^\theta = \frac{\psi'}{\sqrt{g}}\left(\iota - \frac{d\lambda}{d\varphi}\right)}, \quad \underline{B^\varphi = \frac{\psi'}{\sqrt{g}}\left(1 + \frac{d\lambda}{d\theta}\right)} \quad \cdots (8)$$

similarly, $\underline{j^\theta = \frac{\psi'}{\mu_0\sqrt{g}}\left(\frac{dK}{d\varphi} - \frac{dG}{d\psi}\right)}$, $\underline{j^\varphi = \frac{\psi'}{\mu_0\sqrt{g}}\left(\frac{dI}{d\psi} - \frac{dK}{d\theta}\right)}$ … (9)

- Kruskal-Kulsrud average equilibrium

radial force balance becomes

$$\frac{(\psi')^2}{\mu_0\sqrt{g}}\left(\frac{dK}{d\varphi} - \frac{dG}{d\psi}\right)\left(1 + \frac{d\lambda}{d\theta}\right) - \frac{(\psi')^2}{\mu_0\sqrt{g}}\left(\frac{dI}{d\psi} - \frac{dK}{d\theta}\right)\left(\iota - \frac{d\lambda}{d\varphi}\right) = \frac{dp}{d\rho} \quad \cdots (10)$$

$$\left(\langle\,\rangle = \frac{1}{4\pi^2}\int d\theta \int d\varphi\right) \Rightarrow \quad \underline{\frac{d\psi}{d\rho}\frac{dG}{d\rho} + \frac{d\chi}{d\rho}\frac{dI}{d\rho} = -\mu_0 \sqrt{g}\,\frac{dp}{d\rho}} \quad \leftarrow \text{good benchmark.}$$

○ Inverse 3D equilibrium

$$\vec{j} = \frac{1}{\mu_0}\vec{\nabla}\times\vec{B} = \underbrace{\frac{1}{\mu_0\sqrt{g}}\left(\frac{dB_\varphi}{d\theta} - \frac{dB_\theta}{d\varphi}\right)}_{j^\rho}\vec{e_\rho} + \underbrace{\frac{1}{\mu_0\sqrt{g}}\left(\frac{dB_\rho}{d\varphi} - \frac{dB_\varphi}{d\rho}\right)}_{j^\theta}\vec{e_\theta} + \underbrace{\frac{1}{\mu_0\sqrt{g}}\left(\frac{dB_\theta}{d\rho} - \frac{dB_\rho}{d\theta}\right)}_{j^\varphi}\vec{e_\varphi} \quad \cdots (13)$$

∴ Two force balance becomes,

$$F_\rho = \frac{1}{\mu_0}\left(\frac{dB_\rho}{d\varphi} - \frac{dB_\varphi}{d\rho}\right)B^\varphi - \frac{1}{\mu_0}\left(\frac{dB_\theta}{d\rho} - \frac{dB_\rho}{d\theta}\right)B^\theta - p' = 0 \quad \cdots (14)$$

$$F_\beta = \frac{1}{\mu_0}\left(\frac{dB_\varphi}{d\theta} - \frac{dB_\theta}{d\varphi}\right) = 0 \quad \cdots (15)$$

However, we only know contravariant $\vec{B}$. Thus, we should know metric tensor! ((ex) $B_\rho = g_{\rho\theta}B^\theta + g_{\rho\varphi}B^\varphi$)

with Eq(8) : $B^\theta = \frac{\psi'}{\sqrt{g}}\left(2 - \frac{d\lambda}{d\varphi}\right)$, $B^\varphi = \frac{\psi'}{\sqrt{g}}\left(1 + \frac{d\lambda}{d\theta}\right)$,

we obtain the equations of 3D inverse equilibrium ,

① $$\left(\frac{d}{d\varphi}\left(g_{\rho\theta}B^\theta + g_{\rho\varphi}B^\varphi\right) - \frac{d}{d\rho}\left(g_{\varphi\theta}B^\theta + g_{\varphi\varphi}B^\varphi\right)\right)B^\varphi$$
$$- \left(\frac{d}{d\rho}\left(g_{\theta\theta}B^\theta + g_{\theta\varphi}B^\varphi\right) - \frac{d}{d\theta}\left(g_{\rho\theta}B^\theta + g_{\rho\varphi}B^\varphi\right)\right)B^\theta - \mu_0 p' = 0$$

$$\Rightarrow \left\{\frac{d}{d\varphi}\left[\frac{g_{\rho\theta}}{\sqrt{g}}\psi'\left(2 - \frac{d\lambda}{d\varphi}\right) + \frac{g_{\rho\varphi}}{\sqrt{g}}\psi'\left(1 + \frac{d\lambda}{d\theta}\right)\right] - \frac{d}{d\rho}\left[\frac{g_{\varphi\theta}}{\sqrt{g}}\psi'\left(2 - \frac{d\lambda}{d\varphi}\right) + \frac{g_{\varphi\varphi}}{\sqrt{g}}\psi'\left(1 + \frac{d\lambda}{d\theta}\right)\right]\right\} \cdot \frac{\psi'}{\sqrt{g}}\left(1 + \frac{d\lambda}{d\theta}\right)$$
$$- \left\{\frac{d}{d\rho}\left[\frac{g_{\theta\theta}}{\sqrt{g}}\psi'\left(2 - \frac{d\lambda}{d\rho}\right) + \frac{g_{\theta\varphi}}{\sqrt{g}}\psi'\left(1 + \frac{d\lambda}{d\theta}\right)\right] - \frac{d}{d\theta}\left[\frac{g_{\rho\theta}}{\sqrt{g}}\psi'\left(2 - \frac{d\lambda}{d\varphi}\right) + \frac{g_{\rho\varphi}}{\sqrt{g}}\psi'\left(1 + \frac{d\lambda}{d\theta}\right)\right]\right\} \cdot \frac{\psi'}{\sqrt{g}}\left(2 - \frac{d\lambda}{d\varphi}\right)$$
$$- \mu_0 p' = 0$$

$$\Rightarrow \boxed{\begin{aligned} &\psi'\left(\frac{d}{d\varphi} + 2\frac{d}{d\theta} + \frac{d\lambda}{d\theta}\frac{d}{d\varphi} - \frac{d\lambda}{d\varphi}\frac{d}{d\theta}\right)\left[\frac{g_{\rho\theta}}{\sqrt{g}}\psi'\left(2 - \frac{d\lambda}{d\varphi}\right) + \frac{g_{\rho\varphi}}{\sqrt{g}}\psi'\left(1 + \frac{d\lambda}{d\theta}\right)\right] \\ &- \psi'\left(1 + \frac{d\lambda}{d\theta}\right)\frac{d}{d\rho}\left[\frac{g_{\varphi\theta}}{\sqrt{g}}\psi'\left(2 - \frac{d\lambda}{d\varphi}\right) + \frac{g_{\varphi\varphi}}{\sqrt{g}}\psi'\left(1 + \frac{d\lambda}{d\theta}\right)\right] \\ &- \psi'\left(2 - \frac{d\lambda}{d\varphi}\right)\frac{d}{d\rho}\left[\frac{g_{\theta\theta}}{\sqrt{g}}\psi'\left(2 - \frac{d\lambda}{d\rho}\right) + \frac{g_{\theta\varphi}}{\sqrt{g}}\psi'\left(1 + \frac{d\lambda}{d\theta}\right)\right] - \mu_0\sqrt{g}\,p' = 0 \end{aligned}} \quad \cdots (16)$$

②
$$\frac{\partial}{\partial\theta}\left(g_{\varphi\theta}B^{\theta}+g_{\varphi\varphi}B^{\varphi}\right)-\frac{\partial}{\partial\varphi}\left(g_{\theta\theta}B^{\theta}+g_{\theta\varphi}B^{\varphi}\right)$$
$$=\boxed{\frac{\partial}{\partial\theta}\left[\frac{g_{\varphi\theta}}{\sqrt{g}}\left(2-\frac{\partial\lambda}{\partial\varphi}\right)+\frac{g_{\varphi\varphi}}{\sqrt{g}}\left(1+\frac{\partial\lambda}{\partial\theta}\right)\right]-\frac{\partial}{\partial\varphi}\left[\frac{g_{\theta\theta}}{\sqrt{g}}\left(2-\frac{\partial\lambda}{\partial\varphi}\right)+\frac{g_{\theta\varphi}}{\sqrt{g}}\left(1+\frac{\partial\lambda}{\partial\theta}\right)\right]=0} \quad \cdots (17)$$

- Matrix representation for equilibrium

Now, we need to calculate the metric tensors.

Let's choose fixed coordinate system as $\vec{x}=\vec{x}(R,\phi,z)$, and $\varphi=-\phi$.

$$\begin{cases} \vec{e_\rho}=\dfrac{\partial\vec{x}}{\partial\rho}=\dfrac{\partial R}{\partial\rho}\hat{e}_R+0\cdot\hat{e}_\phi+\dfrac{\partial Z}{\partial\rho}\hat{e}_Z \\ \vec{e_\theta}=\dfrac{\partial\vec{x}}{\partial\theta}=\dfrac{\partial R}{\partial\theta}\hat{e}_R+0\cdot\hat{e}_\phi+\dfrac{\partial Z}{\partial\theta}\hat{e}_Z \\ \vec{e_\varphi}=\dfrac{\partial\vec{x}}{\partial\varphi}=\dfrac{\partial R}{\partial\varphi}\hat{e}_R-R\hat{e}_\phi+\dfrac{\partial Z}{\partial\varphi}\hat{e}_Z \end{cases} \iff \vec{e_\rho}=\begin{bmatrix}\partial_\rho R\\0\\\partial_\rho Z\end{bmatrix},\ \vec{e_\theta}=\begin{bmatrix}\partial_\theta R\\0\\\partial_\theta Z\end{bmatrix},\ \vec{e_\varphi}=\begin{bmatrix}\partial_\varphi R\\R\\\partial_\varphi Z\end{bmatrix}$$

then, $g_{ij}=\vec{e_i}\cdot\vec{e_j}$, $\sqrt{g}=\det[g_{ij}]$ are obtained.

Typical strategy is to expand R, Z in Fourier Series as well as λ as

$$\left(\begin{aligned} R(\rho,\theta,\varphi)&=R_{mn}(\rho)\exp\left[i(m\theta-n\varphi)\right]\\ Z(\rho,\theta,\varphi)&=Z_{mn}(\rho)\exp\left[i(m\theta-n\varphi)\right]\\ \lambda(\rho,\theta,\varphi)&=\lambda_{mn}(\rho)\exp\left[i(m\theta-n\varphi)\right]\end{aligned}\right)$$

Then we can find $R(\rho,\theta,\varphi)$, $Z(\rho,\theta,\varphi)$, $\lambda(\rho,\theta,\varphi)$

by choosing $R_{mn}, Z_{mn}, \lambda_{mn}$ that minimizes the error.

Conclusion, in a given (ρ,θ,φ), we get $\underline{R(\rho,\theta,\varphi), Z(\rho,\theta,\varphi), \lambda(\rho,\theta,\varphi)}$ clearly.

Thus, at $\underline{R, Z \text{ point, we know } \vec{B} \text{ and } \vec{J} \text{ from Eqn (8), (13)}}$.

c.f.) In VMEC, $\rho=\dfrac{\psi}{\psi_a}$ / In DESC, $\rho=\sqrt{\dfrac{\psi}{\psi_a}}$ ← How do we specify ψ?
We don't know ψ before solving it.
Isn't it?

* Summary

$$B^\rho = 0, \quad B^\theta = \frac{\psi'}{2\pi\sqrt{g}}\left(\iota - \frac{\partial\lambda}{\partial\varphi}\right), \quad B^\varphi = \frac{\psi'}{2\pi\sqrt{g}}\left(1 + \frac{\partial\lambda}{\partial\theta}\right)$$

$$J^\rho = \frac{1}{\mu_0\sqrt{g}}\left(\frac{\partial B_\varphi}{\partial\theta} - \frac{\partial B_\theta}{\partial\varphi}\right), \quad J^\theta = \frac{1}{\mu_0\sqrt{g}}\left(\frac{\partial B_\rho}{\partial\varphi} - \frac{\partial B_\varphi}{\partial\theta}\right), \quad J^\varphi = \frac{1}{\mu_0\sqrt{g}}\left(\frac{\partial B_\theta}{\partial\rho} - \frac{\partial B_\rho}{\partial\theta}\right)$$

$$\Rightarrow \boxed{\begin{aligned} \vec{B}(\rho,\theta,\varphi) &= \vec{B}(R(\rho,\theta,\varphi), Z(\rho,\theta,\varphi), \lambda(\rho,\theta,\varphi)) \\ \vec{J}(\rho,\theta,\varphi) &= \vec{J}(R(\rho,\theta,\varphi), Z(\rho,\theta,\varphi), \lambda(\rho,\theta,\varphi)) \end{aligned}}$$

$\vec{B}$ and $\vec{J}$ are computed from the independent variables and inputs.