Fusion Plasma Theory 2

Lecture 9 Cold Plasma Waves in Magnetized Plasmas

$\vec{B}_0$ introduces strong anisotropy of the propagation of wave $\vec{k}$.

Recall : $\underline{k^2\vec{E}_1 - \vec{k}(\vec{k}\cdot\vec{E}_1) = \frac{w^2}{c^2}\left(\vec{E}_1 + i\frac{\vec{j}_1}{\epsilon_0 w}\right)}$

All the information about plasma responses is essentially contained in $\underline{\vec{j}_1 = \overleftrightarrow{\sigma}\cdot\vec{E}_1}$

$\underline{\overleftrightarrow{\sigma} : \text{conductivity tensor}}$ $\longrightarrow$ $\underline{\vec{k}(\vec{k}\cdot\vec{E}_1) - k^2\vec{E}_1 + \frac{w^2}{c^2}\vec{E}_1 + iw\mu_0\overleftrightarrow{\sigma}\cdot\vec{E}_1 = 0}$

① Dielectric and susceptibility tensors

Dispersion relation can be rewritten :

$$\vec{k}(\vec{k}\cdot\vec{E}_1) - k^2\vec{E}_1 + \frac{w^2}{\epsilon_0 c^2}\vec{D}_1 = 0 \quad \text{where } \vec{D}_1 = \epsilon_0\left(\overleftrightarrow{I} + \frac{i}{w\epsilon_0}\overleftrightarrow{\sigma}\right)\cdot\vec{E}_1 = \epsilon_0\overleftrightarrow{\epsilon}\cdot\vec{E}_1$$

⇒ Dielectric tensor : $\overleftrightarrow{\epsilon} \equiv \overleftrightarrow{I} + \frac{i}{w\epsilon_0}\overleftrightarrow{\sigma} = \overleftrightarrow{I} + \overleftrightarrow{\chi}$

⇒ susceptibility : $\overleftrightarrow{\chi} \equiv \frac{i}{w\epsilon_0}\overleftrightarrow{\sigma}$.

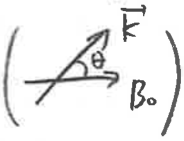
Note that $\overleftrightarrow{\epsilon} = \overleftrightarrow{\epsilon}(\vec{k},w)$, $\overleftrightarrow{\sigma} = \overleftrightarrow{\sigma}(\vec{k},w)$, $\overleftrightarrow{\chi} = \overleftrightarrow{\chi}(\vec{k},w)$

② Conductivity of a warm collisionless plasma / Conductivity dependence of a wave $\vec{k}$.

$$\begin{cases} \text{EoM} : -iwm_s n_{s0}\vec{u}_{s1} = q_s n_s(\vec{E}_1 + \vec{u}_{s1}\times\vec{B}_0) - i\vec{k}\gamma_s T_{s0} n_{s1} \\ \text{continuity} : -iwn_{s1} = -in_{s0}\vec{k}\cdot\vec{u}_{s1} \end{cases}$$

↳ $\underline{-iwm_s\vec{u}_{s1} = q_s(\vec{E}_1 + \vec{u}_{s1}\times\vec{B}_0) - i\frac{\vec{k}(\vec{k}\cdot\vec{u}_{s1})}{w}\gamma_s T_{s0}}$

$\left\langle \begin{array}{l} T_{s0} \simeq 0 : \text{Cold plasma dispersion} \\ T_{s0} \neq 0 : \text{Warm plasma dispersion} \end{array} \right\rangle$ // Hot plasma ⇐ Kinetic treatment

Let $\vec{B} = B_0\hat{z}$, and w/o loss of generality $k_\perp = k_x = k\sin\theta$, $k_\parallel = k_z = k\cos\theta$ ()

: $\vec{k} = k_x\hat{x} + k_z\hat{z} = k\sin\theta\hat{x} + k\cos\theta\hat{z}$

Then,

$$\begin{cases} -iwmu_{x1} = q(E_{x1} + u_{y1}B_0) - i\frac{k^2}{w}\gamma T(u_{x1}\sin^2\theta + u_{z1}\cos\theta\sin\theta) \\ -iwmu_{y1} = q(E_{y1} - u_{x1}B_0) \\ -iwmu_{z1} = qE_{z1} - i\frac{k^2}{w}\gamma T(u_{x1}\cos\theta\sin\theta + u_{z1}\sin^2\theta) \end{cases}$$

Notice $\vec{k}$ enters the conductivity (dielectric tensor) only through finite T.

(dependence of wave propagation : $\vec{k}$)

③ Cold plasma approximation

$$T=0 \quad \left\{ \begin{aligned} -i\omega m u_{x1} &= q(E_{x1} + u_{y1}B_0) \\ -i\omega m u_{y1} &= q(E_{y1} - u_{x1}B_0) \\ -i\omega m u_{z1} &= q E_{z1} \end{aligned} \right\}$$

consider $\vec{u} = \vec{u}(\vec{E_1})$ and $\vec{j} = \sum_s q_s n_s \vec{u_{s1}}$ leads to the conductivity.

$$\overleftrightarrow{\sigma} = \sum_s \begin{pmatrix} i\dfrac{q_s^2 n_s}{m_s}\dfrac{\omega}{\omega^2-\omega_{cs}^2} & -\dfrac{q_s^2 n_s}{m_s}\dfrac{\omega_{cs}}{\omega^2-\omega_{cs}^2} & 0 \\ \dfrac{q_s^2 n_s}{m_s}\dfrac{\omega_{cs}}{\omega^2-\omega_{cs}^2} & i\dfrac{q_s^2 n_s}{m_s}\dfrac{\omega}{\omega^2-\omega_{cs}^2} & 0 \\ 0 & 0 & i\dfrac{q_s^2 n_s}{m_s\omega} \end{pmatrix}$$

where $\omega_{cs} = \dfrac{q_s B_0}{m_s}$, <u>Note that it includes the sign of q_s</u>

④ Cold plasma wave dielectric tensor

$$\overleftrightarrow{\epsilon} = \overleftrightarrow{I} + \frac{i}{\epsilon_0 \omega}\overleftrightarrow{\sigma}, \qquad \overleftrightarrow{\epsilon} = \begin{pmatrix} S & -iD & 0 \\ iD & S & 0 \\ 0 & 0 & P \end{pmatrix}$$

$S = \epsilon_\perp$, $P = \epsilon_\parallel$

D : off diagonal part

$$S \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2 - \omega_{cs}^2} = \frac{1}{2}(R+L)$$

$$D \equiv \sum_s \left(\frac{\omega_{cs}}{\omega}\right)\frac{\omega_{ps}^2}{\omega^2-\omega_{cs}^2} = \frac{1}{2}(R-L)$$

$$P \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}$$

where

$$R \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega+\omega_{cs})}$$

$$L \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega-\omega_{cs})}$$

⑤ Cold plasma dispersion

Use the refractive index $\vec{n} = c\vec{k}/w$

$$\vec{k}(\vec{k}\cdot\vec{E_1}) - k^2\vec{E_1} + \frac{w^2}{c^2}\overleftrightarrow{\epsilon}\cdot\vec{E_1} = 0 \;\rightarrow\; \vec{n}(\vec{n}\cdot\vec{E_1}) - n^2\vec{E_1} + \overleftrightarrow{\epsilon}\cdot\vec{E_1} = 0$$

Final matrix becomes

$$\begin{array}{c} \hat{x}: \\ \hat{y}: \\ \hat{z}: \end{array} \begin{pmatrix} n^2\sin^2\theta - n^2 + S & iD & n^2\sin\theta\cos\theta \\ 0 & S & 0 \\ n^2\sin\theta\cos\theta & 0 & n^2\cos^2\theta - n^2 + P \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

$$\Downarrow$$

$$\begin{pmatrix} -n^2\cos^2\theta + S & iD & n^2\sin\theta\cos\theta \\ 0 & S & 0 \\ n^2\sin\theta\cos\theta & 0 & -n^2\sin^2\theta + P \end{pmatrix}\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = 0$$

Zero determinant condition gives

$$\boxed{An^4 - Bn^2 + C = 0} \qquad \left(\begin{aligned} A &\equiv S\sin^2\theta + P\cos^2\theta \\ B &\equiv RL\sin^2\theta + PS(1+\cos^2\theta) \\ C &\equiv PRL \end{aligned} \right)$$

Divide A, B, C by $\cos^2\theta$ gives $\tan^2\theta$,

$$\boxed{\tan^2\theta = -\frac{P(n^2-R)(n^2-L)}{(Sn^2-RL)(n^2-P)}}$$

: < Aström and Allis dispersion >

represents the change of dispersion due to wave propagation angle

$\theta = 0°$ $(\vec{k} \parallel \vec{B_0})$

- Plasma oscillation : $P = 0$
- R-wave : $n_\parallel^2 = R$
- L-wave : $n_\parallel^2 = L$

$\theta = 90°$ $(\vec{k} \perp \vec{B_0})$

- O-wave : $n_\perp^2 = P$
- X-wave : $n_\perp^2 = RL/S$

⑥ Wave cutoff and Resonance

1) cutoff $k \to 0$ or $n \to 0$

$$An^4 - Bn^2 + C = 0 \quad \to \quad C = PRL = 0$$

i) $P = 0$: O-wave cutoff : $\omega^2 = \sum_s \omega_{ps}^2$

ii) $R = 0$: R-wave and X-wave cutoff : $\sum_s \frac{\omega_{ps}^2}{\omega(\omega + \omega_{cs})} = 1$

iii) $L = 0$: L-wave and X-wave cutoff : $\sum_s \frac{\omega_{ps}^2}{\omega(\omega - \omega_{cs})} = 1$

2) resonance $k \to \infty$ or $n \to \infty$

$$\tan^2\theta = -\frac{P(n^2-R)(n^2-L)}{(Sn^2-RL)(n^2-P)} \simeq -\frac{P}{S}$$ ($\because$ resonance depends on angle)

i) $\theta = 0$: $S = \frac{1}{2}(R+L) = \infty$

(i) $R \to \infty$: R-wave resonance : $\omega = -\omega_{cs}$ (ECR)

(ii) $L \to \infty$: L-wave resonance : $\omega = \omega_{cs}$ (ICR)

ii) $\theta = 90$: $S \to 0$: X-wave resonance : $\sum_s \frac{\omega_{ps}^2}{\omega^2 - \omega_{cs}^2} = 1$ (UHR, LHR)

⑦ High frequency perpendicular (O, X) wave in magnetized plasmas

In high frequency, one can consider only electrons by $\omega_{pe} \gg \omega_{pi}$, and $\omega_{ce} \gg \omega_{ci}$

In the case of $\vec{k} \perp \vec{B_0}$, we have O and X wave

1) O-wave $n_\perp^2 = P$: $\omega^2 = \omega_{pe}^2 + c^2 k^2$

(= ordinary EM plasma wave dispersion for unmagnetized plasma)

$\because$ only $\hat{z}\hat{z}$ components of dielectric tensor play a role for O-wave

That is, $\vec{u_1} \parallel \vec{B_0} \parallel \vec{E_1}$ and O-wave does not see the background $\vec{B}$.

2) X-wave : $n_\perp^2 = RL/S = \dfrac{2RL}{R+L}$

$$= \frac{2}{2-\dfrac{\omega_{ps}^2\cdot 2\cancel{\omega^2}}{\cancel{\omega^4}(\omega^2-\omega_{cs}^2)}}\cdot\left(1-\frac{\omega_{ps}^2}{\omega(\omega-\omega_{cs})}-\frac{\omega_{ps}^2}{\omega(\omega+\omega_{cs})}+\frac{\omega_{ps}^4}{\omega^2(\omega^2-\omega_{cs}^2)}\right)$$

$$= \frac{\cancel{\omega^2-\omega_{cs}^2}}{\omega^2-\omega_{cs}^2-\omega_{ps}^2}\cdot\left(\frac{\omega^2(\omega^2-\omega_{cs}^2)-\omega_{ps}^2\,\omega(\omega+\omega_{cs})-\omega_{ps}^2\,\omega(\omega-\omega_{cs})+\omega_{ps}^4}{\omega^2\cancel{(\omega^2-\omega_{cs}^2)}}\right)$$

$$= \frac{1}{\omega^2(\omega^2-\omega_{ce}^2-\omega_{pe}^2)}\left(\omega^4-\omega^2\omega_{ce}^2-2\omega_{pe}^2\omega^2+\omega_{pe}^4\right)$$

$$= \frac{(\omega^4-\omega^2\omega_{ce}^2-\omega_{pe}^2\omega^2)-\omega_{pe}^4-\omega_{pe}^2\omega^2}{\omega^2(\omega^2-\omega_{ce}^2-\omega_{pe}^2)} = 1-\frac{\omega_{pe}^2(\omega^2-\omega_{pe}^2)}{\omega^2(\omega^2-\omega_{UH}^2)}$$

$$\boxed{\therefore\ n_\perp^2 = 1-\frac{\omega_{pe}^2(\omega^2-\omega_{pe}^2)}{\omega^2(\omega^2-\omega_{UH}^2)}}$$

where $\omega_{UH}^2=\omega_{pe}^2+\omega_{ce}^2$

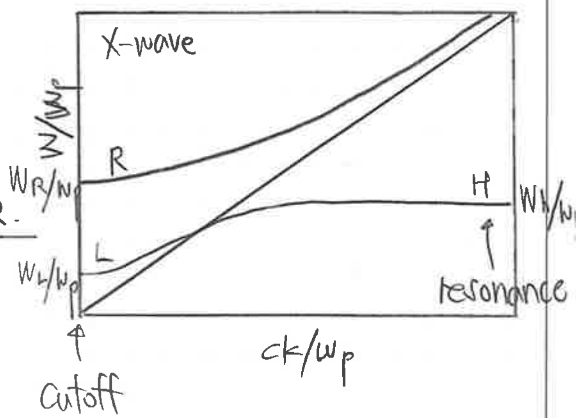
- The resonance ($k\to\infty$) at $\boxed{\omega=\omega_{UH}}$ (consistent with $n_\perp^2=RL/S$, $S=0$).
- Two cutoffs ($k\to 0$) at $RL=0$

$$RL=0 \ \to\ (\omega^4-\omega^2\omega_{ce}^2-2\omega_{pe}^2\omega^2+\omega_{pe}^4) = (\omega^2-\omega_{pe}^2+\omega\omega_{ce})(\omega^2-\omega_{pe}^2-\omega\omega_{ce})=0$$

$$\Rightarrow\ \boxed{\begin{aligned}\omega=\omega_R &\equiv \tfrac{1}{2}\left(-\omega_{ce}+(\omega_{ce}^2+4\omega_{pe}^2)^{1/2}\right)\\ =\omega_L &= \tfrac{1}{2}\left(\omega_{ce}+(\omega_{ce}^2+4\omega_{pe}^2)^{1/2}\right)\end{aligned}}$$

One can see $\omega_L<\omega_{UH}<\omega_H$.

X-wave only propagates when $\omega_L<\omega<\omega_{UH}$ and $\omega>\omega_R$.



Remember both ω_R and ω_{UH} increases with n, and B.

⑧ Physical picture of resonance and cutoff

(O, X) wave dispersion can be used to understand the details of cutoff and resonance

$$\left\langle \begin{array}{l} \text{cutoff : wave packet reflected} \\ \text{resonance : wave packet stop.} \end{array} \right\rangle$$

$$\frac{d}{dt}\vec{V_g} = \vec{V_g}\cdot\vec{\nabla}\vec{V_g} = \frac{1}{2}\vec{\nabla}V_g^2 \begin{cases} = 0 & \text{(resonance)} \\ \neq 0 & \text{(cutoff)} \end{cases}$$

EX) V_g of X-wave near resonance $\omega \simeq \omega_{UH}$:

$$V_g = \frac{d\omega}{dk} \simeq \frac{(\omega_{UH}^2 - \omega^2)^{3/2} c}{\omega_{pe}\omega_{ce}\omega_{UH}} \simeq 0 \quad (\text{when } \omega \simeq \omega_{UH})$$

EX) V_g of O-wave near the cutoff

$$V_g^2 = \frac{c^2}{1+\omega_{pe}^2/k^2c^2} = c\left(1-\omega_{pe}^2/\omega^2\right)$$