Fusion Plasma Theory 2

Lecture 12 : Landau damping

① Kinetic theory of waves

Fluid theory : Assumption of Maxwellian distribution in velocity-space

↳ becomes invalid in high-temperature due to infrequent collisions

↳ Non-Maxwellian can be developed and maintained under perturbation

Kinetic theory : To solve deviation from Maxwellian of the particle distribution

↳ Also essential when equilibrium is non-Maxwellian.

↳ Hot plasma theory : collisionless damping, finite gyroradius effects, wave echoes, trapped particle mode, etc.

② Method for hot wave dispersion in homogeneous media

- Electrostatic wave dispersion :

$$\vec{\nabla}\cdot\vec{E_1} = \frac{\rho_1}{\epsilon_0} \rightarrow i\vec{k}\cdot\vec{E_1} = \frac{\rho_1}{\epsilon_0} \rightarrow \underline{\vec{k}\cdot\vec{E_1} = -i\frac{\rho_1}{\epsilon_0}}$$

Also, in another form :

$$\frac{d\rho_1}{dt} + \vec{\nabla}\cdot\vec{j_1} = 0 \rightarrow -i\omega\rho_1 + i\vec{k}\cdot\vec{j_1} = 0 \;/\; \vec{j_1} = \overleftrightarrow{\sigma_1}\vec{E_1} \;/\; \vec{E_1} = -i\vec{k}\phi$$

$$i\vec{k}\epsilon_0\vec{E_1} = \rho_1 = \frac{\vec{k}\cdot\vec{j_1}}{\omega} = \frac{\vec{k}\cdot\overleftrightarrow{\sigma_1}\cdot\vec{E_1}}{\omega} \rightarrow i\vec{k}\cdot\epsilon_0\omega\overleftrightarrow{I}\cdot\vec{E_1} = \vec{k}\cdot\overleftrightarrow{\sigma_1}\cdot\vec{E_1}$$

$$\hookrightarrow \vec{k}\cdot\left(\overleftrightarrow{I} + \frac{i\overleftrightarrow{\sigma}}{\epsilon_0\omega}\right)\cdot\vec{k}\phi = 0 \quad \underline{\therefore \vec{k}\cdot\overleftrightarrow{\epsilon}\,\vec{k} = 0} \quad \underline{\text{where } \overleftrightarrow{\epsilon} = \overleftrightarrow{I} + \frac{i\overleftrightarrow{\sigma}}{\epsilon_0\omega}}$$

- Electromagnetic wave dispersion

$$\underline{\vec{k}(\vec{k}\cdot\vec{E_1}) - k^2\vec{E_1} + \frac{\omega^2}{c^2}\overleftrightarrow{\epsilon}\cdot\vec{E_1} = 0} \quad \text{where } \overleftrightarrow{\epsilon} = \overleftrightarrow{I} + \frac{i\overleftrightarrow{\sigma}}{\epsilon_0\omega}, \quad \vec{j_1} = \overleftrightarrow{\sigma_1}\cdot\vec{E_1}$$

For cold and warm plasma waves, we used fluid equations to solve ρ_1 and $\vec{j_1}$.

For hot plasma waves, we will obtain them directly from perturbed distribution.

$$\begin{cases} \rho_1 = \sum_s q_s \int f_1(\vec{x},\vec{v},t)\,d\vec{v} \\ \vec{j_1} = \sum_s q_s \int \vec{v} f_1(\vec{x},\vec{v},t)\,d\vec{v} \end{cases}$$

③ Vlasov equation

$$\frac{d\vec{x}}{dt} = \vec{v}, \quad \frac{d\vec{v}}{dt} = \frac{\vec{F}}{m}, \quad N = \int f(\vec{x},\vec{v},t)\, d\vec{v}\, d\vec{x}$$

$$\frac{dN}{dt} = \int \frac{\partial f}{\partial t} d\vec{v}\, d\vec{x} + \int f\, \vec{U}\cdot d\vec{S}$$

← additional volume captured by moving surface

$\downarrow$ divergence thm.

$\vec{U}$ and $d\vec{S}$ is a six-vector in $(\vec{x},\vec{v})$ space

$$0 = \frac{dN}{dt} = \int \left(\frac{\partial f}{\partial t} + \vec{\nabla}\cdot(f\vec{U}) \right) d\vec{v}\, d\vec{x}$$

6-component divergence operator $\vec{\nabla} = (\vec{\nabla}_x, \vec{\nabla}_v)$

$\downarrow$

$$\frac{\partial f}{\partial t} + \vec{\nabla}\cdot(f\vec{U}) = 0$$

$\downarrow$ $\vec{\nabla}\cdot\vec{U} = \cancel{\vec{\nabla}_x\cdot\vec{v}} + \cancel{\vec{\nabla}_v\cdot\frac{\vec{F}}{m}} = 0$

- $\vec{F}$ is independent of $\vec{v}$.

$\vec{v}$ is independent of $\vec{x}$ (in 6-vector)

$\downarrow$

$$\frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f + \frac{\vec{F}}{m}\cdot\frac{\partial f}{\partial \vec{v}} = 0$$

$\downarrow$

$$\therefore \frac{Df}{Dt} = \frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla} f + \frac{q}{m}(\vec{E} + \vec{v}\times\vec{B})\cdot\frac{\partial f}{\partial \vec{v}} = 0 \quad \text{(w/o collision)}$$

<Vlasov equation in collisionless plasma> $\simeq C[f_a, f_b]$ (w/ collision)

(* note $\vec{F}_i$ is composed of macroscopic and slowly varying part, together with a microscopic and rapidly varying part (e.g. collisions)

Thus, we've treated the macroscopic $\vec{F}_i$ first in Vlasov equation, and added the collision term via Fokker-Planck collisions

~~④ Vlasov treatments for electrostatic wave~~

- consider a 1-D electrostatic wave through spatially uniform, quasi-neutral, unmagnetized plasma. $\vec{E_0} = \vec{B_0} = 0$, $\vec{B_1} = 0$, $f_0 = f_0(\vec{v}, t)$. $\vec{k} = k\hat{x}$, $\vec{v} = v\hat{x}$.

< Linearized Vlasov - Poisson equation >

$$\begin{cases} \dfrac{\partial f_1}{\partial t} + v\dfrac{\partial f_1}{\partial x} + \dfrac{qE_1}{m}\dfrac{\partial f_0}{\partial v} = 0 \\ \epsilon_0 \dfrac{\partial E_1}{\partial x} = q\displaystyle\int_{-\infty}^{\infty} f_1 dv \end{cases}$$

Assume $f_1 = \hat{f_1}(v)e^{i(kx-wt)}$, $E_1 = \hat{E_1}e^{i(kx-wt)}$, then two eqns above becomes

$$\begin{cases} -iwf_1 + vikf_1 + \dfrac{q}{m}\hat{E_1}\dfrac{\partial f_0}{\partial v} = 0 \\ -ik\epsilon_0 \hat{E_1} = q\displaystyle\int_{-\infty}^{\infty}\hat{f_1} dv \end{cases} \Rightarrow \begin{cases} \hat{f_1} = -i\dfrac{q\hat{E_1}}{m}\dfrac{df_0/dv}{w-kv} \\ \hat{E_1} = -i\dfrac{q}{k\epsilon_0}\displaystyle\int_{-\infty}^{\infty}\hat{f_1} dv \end{cases}$$

$$\Rightarrow \hat{E_1} = \left(-i\frac{q}{k\epsilon_0}\right)\cdot\left(-i\frac{q}{m}\hat{E_1}\right)\int_{-\infty}^{\infty}\frac{df_0/dv}{w-kv}dv$$

$$\Rightarrow D(k,w)E_1 = 0 \quad \text{where} \quad \boxed{D(k,w) = 1 + \frac{q^2}{mk\epsilon_0}\int_{-\infty}^{\infty}\frac{df_0/dv}{w-kv}dv = 0}$$

< plasma dispersion function by Vlasov (1945) >

$D(k,w) = 0$ has resonance near $w - kv \simeq 0$

↳ implies that true solution is on a complex w. ↗ grow ↘ damping.

But Vlasov elude the resonance problem by taking only Principal value:

$$1 + \frac{q^2}{mk\epsilon_0}P\int_{-\infty}^{\infty}\frac{df_0/dv}{w-kv}dv = 0$$

So we'll ignore the singularity at the moment to follow Vlasov's results.

Different species can add to the perturbed electric field

$$D(k,w) \equiv 1 + \sum_s \frac{q^2}{m_s k \epsilon_0} \int_{-\infty}^{\infty} \frac{df_{s0}/dv}{w - kv}\, dv = 0$$

Assume $f_{s0} = f_{Ms}$ (Maxwellian background) and some useful relations are
even function

$$\left(\begin{array}{lll} \int_{-\infty}^{\infty} \frac{df_0}{dv}\, dv = 0, & \int_{-\infty}^{\infty} \frac{df_0}{dv} v\, dv = -n & \\ \int_{-\infty}^{\infty} \frac{df_0}{dv} v^2\, dv = 0 & \int_{-\infty}^{\infty} \frac{df_0}{dv} v^3 dv = -3n v_t^2, & \int_{-\infty}^{\infty} \frac{df_0}{dv} \frac{1}{v} = -\frac{n}{v_t^2} \end{array} \right)$$

In high frequency range, $w \gg kv$,

$$\frac{1}{w-kv} = \frac{1}{w}\left[\frac{1}{1-kv/w}\right] = \frac{1}{w}\left[1 + \frac{kv}{w} + \left(\frac{kv}{w}\right)^2 + \left(\frac{kv}{w}\right)^3 + \cdots\right]$$

$$= \frac{1}{w} + \frac{kv}{w^2} + \frac{k^2v^2}{w^3} + \frac{k^3v^3}{w^4} + \cdots$$

$$\therefore D(k,w) \simeq 1 + \frac{e^2}{m_e k \epsilon_0}\left(-\frac{k}{w^2} n - \frac{k^3}{w^4} 3n v_t^2\right) = 1 - \frac{w_{pe}^2}{w^2}\left(1 + \frac{3k^2 v_{t,e}^2}{w^2}\right) = 0$$

$$\Rightarrow \boxed{D(k,w) = 1 - \frac{w_{pe}^2}{w^2}\left(1 + \frac{3k^2 v_{t,e}^2}{w^2}\right) = 0}$$

when we only consider the leading order $w = w_{pe}$,

$$w^2 = w_{pe}^2 + 3k^2 v_{t,e}^2 \quad (1) \quad \text{<Bohm-Gross dispersion relation>}$$

In $k v_{t,i} \ll w \ll k v_{t,e}$, $\frac{1}{w-kv} = \frac{1}{w} + \frac{kv}{w^2}$ (ion), $\frac{1}{w-kv} = -\frac{1}{kv}$ (electron)

$$\therefore D(k,w) = 1 + \frac{Z^2e^2}{m_i k \epsilon_0}\left(-\frac{k}{w^2} n\right) + \frac{e^2}{m_e k \epsilon_0}\left(-\frac{1}{k}\right)\cdot\left(-\frac{n}{v_{t,e}^2}\right) = 1 - \frac{w_{pi}^2}{w^2} + \frac{w_{pe}^2}{k^2 v_{t,e}^2} = 0$$

$$\Rightarrow \boxed{D(k,w) = 1 - \frac{w_{pi}^2}{w^2} + \frac{w_{pe}^2}{k^2 v_{t,e}^2} = 0}$$

using $v_{te}^2 / w_{pe}^2 = \lambda_D^2$, $\quad 1 - \frac{w_{pi}^2}{w^2} + \frac{1}{k^2\lambda_D^2} = 0 \quad \frac{w_{pi}^2}{w^2} k^2 \lambda_D^2 = 1 + k^2\lambda_D^2$

$$\frac{k^2}{w^2} \frac{\cancel{n e^2}}{M \cancel{\epsilon_0}} \cdot \frac{\cancel{\epsilon_0} T}{\cancel{n e^2}} = 1 + k^2 \lambda_D^2 \Rightarrow \frac{w^2}{k^2} = \frac{T_e/M}{1 + k^2\lambda_D^2} \quad (2) \quad \text{<ion-accoustic wave>}$$

(3) Two stream instability (interaction between two beams)

$$f_0 = \frac{n}{2}\left[\delta(v-v_0) + \delta(v+v_0)\right]$$

$$\int_{-\infty}^{\infty} \frac{df_0/dv}{w-kv}\,dv = \cancel{\frac{f_0}{w-kv}\Big|_{-\infty}^{\infty}} - k\int_{-\infty}^{\infty} \frac{f_0}{(w-kv)^2}\,dv = -\frac{nk}{2}\left[\frac{1}{(w-kv_0)^2} + \frac{1}{(w+kv_0)^2}\right]$$

$$D(k,w) = 1 - \frac{w_{ps}^2}{2}\left[\frac{1}{(w-kv_0)^2} + \frac{1}{(w+kv_0)^2}\right] = 0$$

The solution becomes :

$$2w^2 = (w_{ps}^2 + 2k^2v_0^2) \pm \sqrt{(w_{ps}^2 + 2k^2v_0^2)^2 + 4k^2v_0^2(w_{ps}^2 - k^2v_0^2)}$$

If $w_{ps} > kv_0$, w must have imaginary number, indicating two-stream instability. (instability for all sufficiently long wave-length).

(1) Bohm-Gross dispersion
(2) Ion-accoustic wave
(3) Two stream instability

From Vlasov's approach.

However. Vlasov elude the problem of singularity in the integral, indicating a necessity of correction.

⑤ Landau's initial value approach.

* $t = -\infty, \infty$ integral will ignore the damping term at $t=0$ through $t = -\infty, 0$

Thus, $\left\{ \begin{array}{l} \text{space : Fourier transform} \\ \text{time : Laplace transform} \end{array} \right\}$ was performed by Landau.

* some mathematics.

[1] Laplace transformation

$$\tilde{f}(s) = \int_0^\infty f(t) e^{-st} dt$$

: defined only for complex s with $Re(s)$ so that the integral converges at $t \to \infty$.

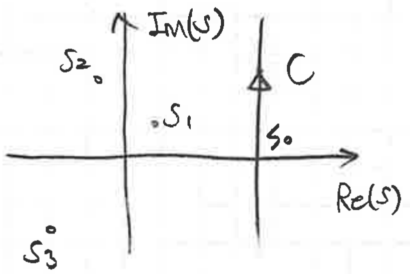
$$\tilde{\dot{f}}(s) = \int_0^\infty \frac{df(t)}{dt} \cdot e^{-st} dt = f(t) e^{-st} \Big|_0^\infty - \int_0^\infty f(t) (-s e^{-st}) dt$$

$$= -f(0) + s \int_0^\infty f(t) e^{-st} dt = s\tilde{f}(s) - f(0) \quad \therefore \tilde{\dot{f}}(s) = s\tilde{f}(s) - f(0)$$

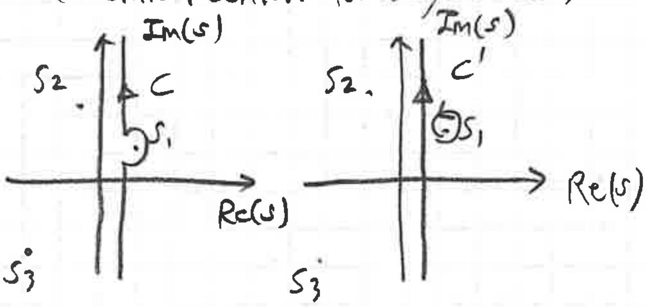
[2] Inverse Laplace transform

$$f(t) = \frac{1}{2\pi i} \int_c e^{st} \tilde{f}(s) ds$$

<contour of Bromwich Integral>



<Identical contour for asymtotics>

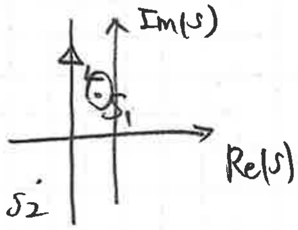


If not, consider $\tilde{E}_1(s) = \int_0^\infty E_1(t) e^{-st} dt$. $E_1(t) \sim e^{s_1 t}$ when $E_1(s) \sim (s - s_1)^{-1}$

The integral does not converge unless $s > Re(s_1)$

Dominant dynamics can be extracted by the deformed contour, such as

$$E_1(t) = Res(s_1) e^{s_1 t} + \frac{1}{2\pi i} \int_{c'} \tilde{E}_1(s) e^{st} ds$$



↑ singularity $s_1(k)$ gives dominant dynamics in dispersion relation

Now, consider $E_1(x,t) = \hat{E}_1(t)e^{ikx}$ and $f_1(x,v,t) = \hat{f}_1(v,t)e^{ikx}$

Vlasov equation becomes : $\dfrac{d\hat{f}_1}{dt} + ikv\hat{f}_1 + \dfrac{q\hat{E}_1}{m}\dfrac{df_0}{dv} = 0$.

Do the laplace transform $\tilde{f}_1(v,s) = \int_0^\infty \hat{f}_1(v,t)e^{-st}dt$:

$$\begin{cases} s\tilde{f}_1(v,s) - \hat{f}_1(v,0) + ikv\tilde{f}_1(v,s) + \dfrac{q\tilde{E}_1(s)}{m}\dfrac{df_0}{dv} = 0 \\ \tilde{E}_1(s) = -i\dfrac{q}{k\epsilon_0}\displaystyle\int_{-\infty}^{\infty}\tilde{f}_1(v,s)\,dv \end{cases}$$

$$\Rightarrow \quad (s+ikv)\tilde{f}_1(v,s) = \hat{f}_1(v,0) - \frac{q\tilde{E}_1(s)}{m}\frac{df_0}{dv}$$

$$\tilde{E}_1(s) = -i\frac{q}{k\epsilon_0}\int_{-\infty}^{\infty}\frac{1}{s+ikv}\left(\hat{f}_1(v,0) - \frac{q\tilde{E}_1(s)}{m}\frac{df_0}{dv}\right)dv$$

$$\boxed{\tilde{E}_1(s) = \frac{-i\dfrac{q}{k\epsilon_0}\displaystyle\int_{-\infty}^{\infty}\frac{\hat{f}_1(v,0)}{s+ikv}dv}{1 - i\dfrac{q^2}{mk\epsilon_0}\displaystyle\int_{-\infty}^{\infty}\frac{df_0/dv}{s+ikv}dv}}$$

$s = -i\omega$ 와 동일.

where the denominator is a version of dielectric function by Laplace transform.

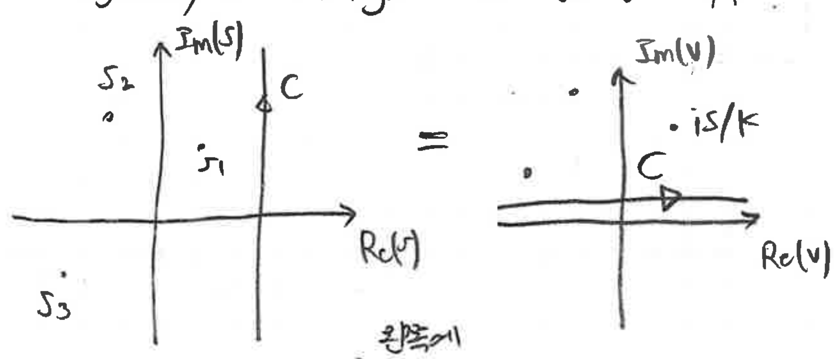
$$D(k,s) \equiv 1 - i\frac{q^2}{mk\epsilon_0}\int_{-\infty}^{\infty}\frac{df_0/dv}{s+ikv}dv$$

Inverse Laplace transform :

$$E_1(t) = \frac{1}{2\pi i}\int_{s_0-i\infty}^{s_0+i\infty}\tilde{E}_1(s)e^{st}ds = \frac{1}{2\pi i}\int_{s_0-i\infty}^{s_0+i\infty}\left[\frac{-i\dfrac{q^2}{k\epsilon_0}\displaystyle\int_{-\infty}^{\infty}\frac{\hat{f}_1(v,0)}{s+ikv}dv}{1 - i\dfrac{q^2}{mk\epsilon_0}\displaystyle\int_{-\infty}^{\infty}\frac{df_0/dv}{s+ikv}dv}\right]e^{st}ds$$

Landau's analytic continuation of Vlasov integral

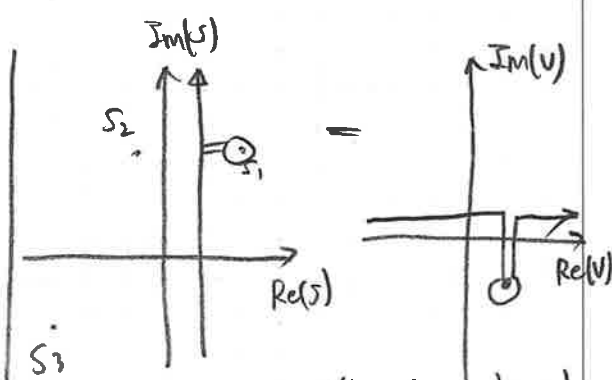
Singularity of the integral occurs at $v = is/k$.



C 왼쪽에 s_i가 s-plane 왼쪽에 있다면,

is/k는 v-plane에서 위에 있어야한다!

But



<Landau integral path>

ibis

The singularity cannot arise in the numerator of $\tilde{E}_1(s)$ which is an entire function when $\mathrm{Re}(s)>0$. as in the original Bromwich integral path, and remains so even for $\mathrm{Re}(s)<0$ by analytic continuation.

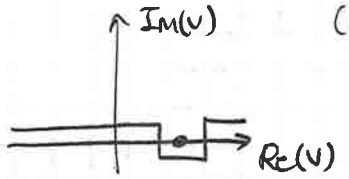
So singularity occurs only when $D(k,s)=0$. (zeros in $\tilde{E}_1(s)$ denominator)

In case $\mathrm{Re}(s_1)>0$, no change is needed in v-plane contour.

By $s=-i\omega$, it becomes just Vlasov dispersion. (Vlasov ~ instabilities)

In case $\mathrm{Re}(s_1)<0$, one needs to take Landau integral path.

The most interesting case : $\mathrm{Re}(s_1) \to -0$: weakly damped oscillation.



$(s=-i\omega)$

$$D(k,\omega) = 1+\frac{q^2}{mk\epsilon_0}\left[P\int_{-\infty}^{\infty}\frac{df_0/dv}{\omega-kv} - \frac{1}{k}\,\mathrm{Res}\int_{-\infty}^{\infty}\frac{df_0/dv}{v-\omega/k}\,dv\right]$$

$$\boxed{\therefore D(k,\omega) = 1+\frac{q^2}{mk\epsilon_0}\left[P\int_{-\infty}^{\infty}\frac{df_0/dv}{\omega-kv} - \frac{\pi i}{k}\frac{df_0}{dv}\bigg|_{v=\omega/k}\right]}$$

$\because$ residue theorem (half circle)

last term $-\frac{\pi i}{k}\frac{df_0}{dv}\Big|_{v=\omega/k}$ is a new correction to Vlasov, a new mechanism of damping.

$\Rightarrow$ Wave damping in an entirely collisionless system

Consider $\omega \gg kv$, over Maxwellian, then

$$D(k,\omega) = 1-\frac{\omega_{pe}^2}{\omega^2}+i\sqrt{\frac{\pi}{2}}\frac{\omega_{pe}^2\,\omega}{k^3 v_{t,e}^3}\exp\left(-\frac{\omega^2}{2k^2 v_{t,e}^2}\right)=0$$

Treating the correction term is small, one obtains

$$\omega = \omega_{pe} - \frac{i}{2}\sqrt{\frac{\pi}{2}}\frac{\omega_{pe}^4}{k^3 v_{t,e}^3}\exp\left(-\frac{\omega_{pe}^2}{2k^2 v_{t,e}^2}\right) = \omega_{pe}\left[1-\frac{i}{2}\sqrt{\frac{\pi}{2}}\frac{1}{k^3\lambda_D^3}\exp\left(-\frac{1}{2k^2\lambda_D^2}\right)\right]$$