Fusion Plasma Theory 2

Lecture 6 : Fokker-Planck for Maxwellian Plasmas

① Slowing down collisional frequency & Deflection frequency

$$C_{ab}[f_a, f_b] = -\frac{2L^{ab}}{v^3}\psi_b'\mathcal{L}(f_a) + L^{ab}\frac{d}{d\vec{v}}\cdot\Big(\underbrace{\frac{m_a}{m_b}\frac{\vec{v}}{v}\varphi_b' f_a}_{(1)} - \underbrace{\frac{\vec{v}\vec{v}}{v^2}\psi_b''\cdot\frac{df_a}{d\vec{v}}}_{(2)}\Big)$$

(1) $\vec{\mathcal{V}} = -\frac{\langle\Delta\vec{v}\rangle}{\Delta t} = \left(1+\frac{m_a}{m_b}\right)L^{ab}\frac{d\varphi_b}{d\vec{v}} = \left(1+\frac{m_a}{m_b}\right)L^{ab}\frac{\vec{v}}{v}\varphi_b'$

$$\boxed{\nu_s^{ab} \equiv -\frac{\langle\Delta v_\parallel/v\rangle}{\Delta t} = -\frac{\langle\Delta\vec{v}\rangle\cdot\hat{v}/v}{\Delta t} = \left(1+\frac{m_a}{m_b}\right)L^{ab}\frac{1}{v}\varphi_b'} \Rightarrow \boxed{\vec{\mathcal{V}} = \nu_s^{ab}\vec{v}}$$

∴ one can relate φ_b' with the slowing down collisional frequency ν_s^{ab}.

(2) $\overleftrightarrow{D}_v = \frac{\langle\Delta\vec{v}\Delta\vec{v}\rangle}{2\Delta t} = -L^{ab}\left(\overleftrightarrow{I}\psi_b' + \frac{\vec{v}\vec{v}}{v^2}\psi_b''\right) = -L^{ab}\left((\hat{x}\hat{x}+\hat{y}\hat{y})\frac{1}{v}\psi_b' + (\hat{z}\hat{z})\psi_b''\right)$

(where $\hat{z}$ is a local coordinate in velocity space aligned with $\vec{v}$, $\hat{x}, \hat{y}$ are arbitrary orthogonal direction.)

Parallel diffusion frequency : $\boxed{\nu_\parallel^{ab} \equiv \frac{\langle(\Delta v_\parallel/v)^2\rangle}{\Delta t} = -2L^{ab}\frac{\psi_b''}{v^2}}$

Deflection frequency : $\boxed{\nu_D^{ab} \equiv \frac{\langle(\Delta v_\perp/v)^2\rangle}{2\Delta t} = -2L^{ab}\frac{\psi_b'}{v^3}}$

$$\overleftrightarrow{D}_v = \begin{pmatrix} \frac{\langle\Delta v_x\Delta v_x\rangle}{2\Delta t} & 0 & 0 \\ 0 & \frac{\langle\Delta v_y\Delta v_y\rangle}{2\Delta t} & 0 \\ 0 & 0 & \frac{\langle\Delta v_z\Delta v_z\rangle}{2\Delta t} \end{pmatrix} = \begin{pmatrix} \frac{\langle\Delta v_\perp\Delta v_\perp\rangle}{4\Delta t} & 0 & 0 \\ 0 & \frac{\langle\Delta v_\perp\Delta v_\perp\rangle}{4\Delta t} & 0 \\ 0 & 0 & \frac{\langle\Delta v_\parallel\Delta v_\parallel\rangle}{2\Delta t} \end{pmatrix}$$

$(\Delta v_\perp^2 = (\Delta v_x)^2 + (\Delta v_y)^2)$

$$= \frac{v^2}{2}\begin{pmatrix} \frac{\langle(\Delta v_\perp/v)^2\rangle}{2\Delta t} & 0 & 0 \\ 0 & \frac{\langle(\Delta v_\perp/v)^2\rangle}{2\Delta t} & 0 \\ 0 & 0 & \frac{\langle(\Delta v_\parallel/v)^2\rangle}{\Delta t} \end{pmatrix} = \frac{v^2}{2}\begin{pmatrix} \nu_D^{ab} & 0 & 0 \\ 0 & \nu_D^{ab} & 0 \\ 0 & 0 & \nu_\parallel^{ab} \end{pmatrix}$$

$$\Rightarrow \boxed{\overleftrightarrow{D}_v = \frac{v^2}{2}\left(\nu_D^{ab}(\overleftrightarrow{I} - \hat{v}\hat{v}) + \nu_\parallel^{ab}\hat{v}\hat{v}\right)}$$

Let's find $\nu_D^{ab}, \nu_\parallel^{ab}, \nu_s^{ab}$ from now on!

② Collisional operator with characteristic frequencies.

$$C_{ab}[f_a,f_b] = \nu_D^{ab}\mathcal{L}[f_a] + \frac{\partial}{\partial\vec{v}}\cdot\left(\frac{m_a}{m_a+m_b}\nu_s^{ab}\vec{v}f_a + \frac{1}{2}\nu_{\parallel}^{ab}\vec{v}\vec{v}\cdot\frac{\partial f_a}{\partial\vec{v}}\right)$$

If test (incident) particle species (a) is also isotropically distributed, it becomes:

$$C_{ab}[f_a,f_b] = \nu_D^{ab}\mathcal{L}[f_a] + \frac{1}{v^2}\frac{\partial}{\partial v}\left[v^3\left(\frac{m_a}{m_a+m_b}\nu_s^{ab}f_a + \frac{1}{2}\nu_{\parallel}^{ab}v\frac{\partial f_a}{\partial v}\right)\right]$$

Note that it is also useful to represent it as: $C_{ab}[f_a,f_b] = \frac{\partial}{\partial\vec{v}}\cdot\vec{J_v}$

Using the characteristic frequencies, the velocity flux for the isotropic background (b) is given by:

$$\vec{J_v} = \frac{m_a}{m_a+m_b}\nu_s^{ab}\vec{v}f_a + \left(\frac{1}{2}\nu_D^{ab}(v^2\overleftrightarrow{I} - \vec{v}\vec{v}) + \frac{1}{2}\nu_{\parallel}^{ab}\vec{v}\vec{v}\right)\cdot\frac{\partial f_a}{\partial\vec{v}}$$

③ Rosenbluth potential due to Maxwellian - Chandrasekhar function

(1) Rosenbluth potentials can be often easily obtained by solving Laplace equation.

In the case of Maxwellian:

$$\nabla^2\varphi_b = \frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\frac{\partial\varphi_b}{\partial v}\right) = f_{Mb}(v) = \frac{n_b}{(\sqrt{\pi}v_{tb})^3}e^{-v^2/v_{tb}^2}$$

Integrating:

$$\frac{1}{v^2}\frac{\partial}{\partial v}\left(v^2\frac{\partial\varphi_b}{\partial v}\right) = \frac{n_b}{(\sqrt{\pi}v_{tb})^3}e^{-v^2/v_{tb}^2} \rightarrow v^2\frac{\partial\varphi_b}{\partial v} = \frac{n_b}{(\sqrt{\pi}v_{tb})^3}\int_0^v t^2e^{-t^2/v_{tb}^2}dt$$

$$\therefore \varphi_b' = \frac{\partial\varphi_b}{\partial v} = \frac{n_b}{(\sqrt{\pi}v_{tb})^3}\frac{1}{v^2}\int_0^v t^2e^{-t^2/v_{tb}^2}dt \underset{\uparrow}{=} \frac{n_b}{(\sqrt{\pi})^3}\frac{1}{v^2}\int_0^x t^2e^{-t^2}dt$$

$(x = t/v_{tb},\ dx = dt/v_{tb})$

Integration by parts $\left(\begin{matrix}u=t,\\ v=te^{-t^2}\end{matrix}\right)$

$$= \frac{n_b}{(\sqrt{\pi})^3}\frac{1}{v^2}\left(-\frac{1}{2}xe^{-x^2} + \frac{1}{2}\int_0^x e^{-t^2}dt\right) = \frac{n_b}{2\pi v_{tb}^2}G(x_b), \Rightarrow \boxed{\varphi_b' = \frac{n_b}{2\pi v_{tb}^2}G(x_b)}$$

The Chandrasekhar function $G(x)$ is defined: $\boxed{G(x) \equiv \frac{\phi(x) - x\phi'(x)}{2x^2}}$

with the error function $\boxed{\phi(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt}$

(Note that $' = d/dx$, $x_b = v/v_{tb}$, $x_a = v/v_{ta}$)

(2) 2nd Rosenbluth potential,

$$\nabla^2 \psi_b = \frac{1}{v^2}\frac{d}{dv}\left(v^2 \frac{d\psi_b}{dv}\right) = \varphi_b = \frac{n_b}{2\pi v_{tb}} \int G(x_b)\, dx_b$$

Some interesting properties are :

$$\text{(i)}\ -\left(\frac{\phi}{2x}\right)' = G \quad \left(G \equiv \frac{\phi - x\phi'}{2x^2}\right), \quad \text{(ii)} \int x\phi'\, dx = -\frac{\phi'}{2} \quad (\phi'' + 2x\phi' = 0), \quad \text{(iii)}\ \phi' - G' = 2\frac{G}{x}$$

Using (i), we get $\underline{\frac{1}{v^2}\frac{d}{dv}\left(v^2\frac{d\psi_b}{dv}\right)} = \frac{n_b}{2\pi v_{tb}}\left(-\frac{\phi(x_b)}{2x_b}\right) = \underline{-\frac{n_b}{4\pi v_{tb}}\frac{\phi(x_b)}{x_b}}$

Changing the variable between v and x_b, and integrating by parts :

$$x = x_b = v/v_{tb}$$

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\psi_b}{dx}\right) = -\frac{n_b}{4\pi}\frac{\phi}{x}$$

$$\left(* \int x^2\phi'\,dx = \int x \cdot x\phi'\,dx = x\int x\phi'\,dx - \int\left[\int x\phi'\,dx\right]dx = x\left(-\frac{\phi'}{2}\right) + \int\frac{\phi'}{2}\,dx\right)$$

$$x^2\frac{d\psi_b}{dx} = -\frac{n_b}{4\pi}\int x\phi\,dx = -\frac{n_b}{4\pi}\left[\frac{1}{2}x^2\phi - \frac{1}{2}\int x^2\phi'\,dx\right]$$

$$= -\frac{n_b}{8\pi}\left[x^2\phi - \left(-x\frac{\phi'}{2} + \int\frac{\phi'}{2}dx\right)\right] = -\frac{n_b}{8\pi}x^2\left[\phi + \frac{x\phi' - \phi}{2x^2}\right] = -\frac{n_b}{8\pi}x^2\left[\phi - G\right]$$

$$\therefore \boxed{\begin{aligned} \psi_b' &= \frac{d\psi_b}{dv} = -\frac{n_b}{8\pi}\left(\phi(x_b) - G(x_b)\right) \\ \psi_b'' &= \frac{d^2\psi_b}{dv^2} = -\frac{n_b}{4\pi v_{tb}}\frac{\phi(x_b)}{x_b} \end{aligned}}$$

④ Velocity dependence of collisional frequencies

$$\boxed{\begin{aligned} \nu_s^{ab} &= \hat{\nu}^{ab}\frac{T_a}{T_b}\left(1 + \frac{m_b}{m_a}\right)\frac{2G(x_b)}{x_a} \\ \nu_D^{ab} &= \hat{\nu}^{ab}\frac{\phi(x_b) - G(x_b)}{x_a^3} \\ \nu_{\parallel}^{ab} &= \hat{\nu}^{ab}\frac{2G(x_b)}{x_a^3} \end{aligned}}$$

where $\boxed{\hat{\nu}^{ab} = \frac{n_b e_a^2 e_b^2 \ln\Lambda}{4\pi\epsilon_0^2 m_a^2 v_{ta}^3}}$

$\underline{\text{basic collisional frequency}}$

: 90° deflection frequency for a fixed target with thermal velocity.

⑤ Asymptotics of Chandrasekhar functions $\left(G(x) \equiv \frac{\phi - x\phi'}{2x^2} \right)$

$$\lim_{x\to 0} \phi(x) \to \frac{2}{\sqrt{\pi}} x, \quad \lim_{x\to\infty} \phi(x) \to 1$$

$$\lim_{x\to 0} G(x) \to \frac{2x}{3\sqrt{\pi}}, \quad \lim_{x\to\infty} G(x) \to \frac{1}{2x^2}$$

In ν_s^{ab}, ν_D^{ab}, $\nu_\parallel^{ab}$, all nominator has $G(x)$.

$\Rightarrow$ Thus, high E particles are getting harder and harder to slow down.
(e.g. runaway electron)

⑥ Collisional friction force (change of momentum due to collisions)

$$\vec{R}_{ab} \equiv \int m_a \vec{v}\, C_{ab}[f_a, f_b]\, d\vec{v}$$

Assume $f_a = f_{a0} + f_{a1}$, $f_b = f_{b0} + f_{b1}$ where f_{i0} is maxwellian.

$$\vec{R}_{ab} = \int m_a \vec{v} \left(C_{ab}[f_{a1}, f_{b0}] + C_{ab}[f_{a0}, f_{b1}] \right) d\vec{v}$$

(note $\int m_a \vec{v}\, C_{ab}[f_{a0}, f_{b0}]\, d\vec{v} = 0$ for Maxwellian)

The first part is relatively easy since their Rosenbluth potentials are known:

$$\int m_a \vec{v}\, C_{ab}[f_a, f_{b0}]\, d\vec{v} = \int m_a \vec{v} \frac{\partial}{\partial \vec{v}} \cdot \vec{J}_v\, d\vec{v} = -\int m_a \vec{J}_v\, d\vec{v}$$

$$\simeq -\int m_a \vec{U} f_a\, d\vec{v} = -\int m_a \vec{v}\, \nu_s^{ab} f_a\, d\vec{v}$$

The second part would require the evaluation of Rosenbluth potentials for non-Maxwellian, but this difficulty can be avoided by the momentum conservation.

$$\int m_a \vec{v}\, C_{ab}[f_{a0}, f_{b1}]\, d\vec{v} = -\int m_b \vec{v}\, C_{ba}[f_{b1}, f_{a0}]\, d\vec{v}$$

$$\Rightarrow \boxed{\vec{R}_{ab} = -\int \vec{v} \left(m_a \nu_s^{ab} f_{a1} + m_b \nu_s^{ba} f_{b1} \right) d\vec{v}}$$

⑦ Collisional heat exchange

integration by parts

$$Q_{ab} \equiv \frac{1}{2}\int m_a v^2 C_{ab}[f_a, f_{b0}]\,d\vec{v} = \frac{1}{2}\int m_a v^2 \frac{d}{d\vec{v}}\cdot\vec{J}_v\,d\vec{v} = -\int m_a\vec{v}\cdot\vec{J}_v\,d\vec{v}$$

$$= -\int m_a\vec{v}\cdot\left(\vec{V}f_a + \frac{d}{d\vec{v}}\cdot(\overleftrightarrow{D_v}f_a)\right) = -\int m_a\left(\vec{v}\cdot\vec{V} - Tr(\overleftrightarrow{D_v})\right)f_a$$

$$= -\int\frac{1}{2}m_a v^2\left(2\nu_s^{ab} - 2\nu_D^{ab} - \nu_{\parallel}^{ab}\right)f_a \equiv \boxed{-\int\frac{1}{2}m_a v^2\nu_E^{ab}f_a}$$

where the energy exchange frequency is defined by $\boxed{\nu_E \equiv 2\nu_s^{ab} - 2\nu_D^{ab} - \nu_{\parallel}^{ab}}$

Linearizing and using energy conservation, the collisional heating up to the 1st order is :

$$Q_{ab} = \frac{1}{2}\int m_a v^2\left(C_{ab}[f_{a0}, f_{b0}] + C_{ab}[f_{a0}, f_{b1}] + C_{ab}[f_{a1}, f_{b0}]\right)d\vec{v}$$

$$\Rightarrow \boxed{Q_{ab} = -\int\frac{1}{2}v^2\left(m_a\nu_E^{ab}(f_{a0}+f_{a1}) + m_b\nu_E^{ba}f_{b1}\right)d\vec{v}}$$

* note
leading order does not change, even for Maxwellian

⑧ collisional friction force by Shifted Maxwellian

consider the Maxwellian background $f_{b0} = f_{bM}$, but test particle species are under the shifted Maxwellian due to the flow $\vec{u} = \vec{u}_a$

$$f_a = \frac{n_a}{(\sqrt{\pi}v_{ta})^3}e^{-(\vec{v}-\vec{u})^2/v_{ta}^2} \simeq f_{aM}\left(1 + \frac{2\vec{v}\cdot\vec{u}}{v_{ta}^2}\right) = f_{a0} + f_{a1}$$

collisional friction force is then given and its first term

$$\vec{R}_{ab} = -m_a\int\nu_s^{ab}\vec{v}f_{a1}\,d\vec{v} = -\frac{1}{T_a}\int\nu_s^{ab}\vec{v}(\vec{v}\cdot\vec{u})f_{a0}\,d\vec{v} = -\frac{1}{3T_a}\vec{u}\int\nu_s^{ab}v^2 f_{a0}\,d\vec{v}$$

$v^2 \doteqdot$ 1방향만 고려

$$= -\frac{2\vec{u}}{3T_b}\left(1+\frac{m_b}{m_a}\right)\hat{\nu}^{ab}\int v^2\frac{G(x_b)}{x_a}f_{a0}\,d\vec{v}$$

$$= -\frac{2\vec{u}}{3T_b}\left(1+\frac{m_b}{m_a}\right)\hat{\nu}^{ab}\int v^2\frac{\phi(x_b) - x\phi'(x_b)}{2x_b^2 x_a}f_{a0}\,d\vec{v}$$

$$= -\frac{2\vec{u}}{3T_b}\left(1+\frac{m_b}{m_a}\right)\hat{\nu}^{ab}\frac{n_a}{(\sqrt{\pi}v_{ta})^3}\int 4\pi v^4\frac{\phi(x_b) - x_b\phi'(x_b)}{2x_b^2 x_a}e^{-x_a^2}\,dv$$

☆

Define $k \equiv v_{tb}/v_{ta}$ to treat both $x_a = v/v_{ta}$, $x_b = v/v_{tb}$, and rearrange it

$$\vec{R}_{ab} = -\frac{8}{3\sqrt{\pi}} n_a m_a \vec{u}\, \hat{\nu}^{ab} \left(\frac{m_a}{m_b}+1\right) k^2 \left[\int (x\phi - x^2\phi')e^{-k^2x^2}dx\right]$$

Integration by parts with the error function ϕ:

$$\vec{R}_{ab} = -\frac{8}{3\sqrt{\pi}} n_a m_a \vec{u}\, \hat{\nu}^{ab} \left(\frac{m_a}{m_b}+1\right) k^2 \left[\frac{1}{2k^2(k^2+1)^{3/2}}\right]$$

$$= -\frac{8}{3\sqrt{\pi}} n_a m_a \vec{u} \left(\frac{m_a}{m_b}+1\right) \frac{n_b e_a^2 e_b^2 \ln\Lambda}{4\pi\epsilon_0^2 m_a^2 v_{ta}^3} \frac{1}{2(k^2+1)^{3/2}}$$

$$\therefore \boxed{\vec{R}_{ab} = -m_a n_a \vec{u}_a \left[\frac{\sqrt{2}\, n_b e_a^2 e_b^2 \ln\Lambda}{12\pi^{3/2}\epsilon_0^2 m_a m_r}\left(\frac{T_a}{m_a}+\frac{T_b}{m_b}\right)^{-3/2}\right] \equiv -m_a n_a \vec{u}_a \nu_f^{ab}}$$

⑨ Collisional heating between Maxwellians

$$Q_{ab} = -\int \frac{1}{2} m_a v^2 \nu_E^{ab} f_{a0}\, d\vec{v}$$

$$\nu_E^{ab} = 2\nu_s^{ab} - 2\nu_D^{ab} - \nu_\parallel^{ab} = \hat{\nu}^{ab}\frac{m_a x_b^3}{m_b x_a^3}\left[\frac{\phi}{x_b^3} - \left(1+\frac{m_b}{m_a}\right)\frac{\phi'}{x_b^2}\right]$$

☆ Again define $k \equiv v_{tb}/v_{ta}$, integrate by parts and rearrange:

$$Q_{ab} = \frac{8}{\sqrt{\pi}}\frac{m_a^2}{m_b^2} n_b T_b \hat{\nu}^{ab} \int \left[\left(1+\frac{m_b}{m_a}\right)x^2\phi' - x\phi\right] e^{-k^2x^2}dx$$

$$= \frac{8}{\sqrt{\pi}}\frac{m_a^2}{m_b^2} n_b T_b \hat{\nu}^{ab} \frac{T_b - T_a}{2T_a k^2 (k^2+1)^{3/2}}$$

$$\therefore \boxed{Q_{ab} = \frac{n_a n_b e_a^2 e_b^2 \ln\Lambda}{2\sqrt{2}\pi^{3/2}\epsilon_0^2 m_a m_b}(T_b - T_a)\left(\frac{T_a}{m_a}+\frac{T_b}{m_b}\right)^{-3/2}}$$

⑩ Temperature equilibration between Maxwellians

$$\frac{3}{2} n_a \frac{dT_a}{dt} = Q_{ab} \quad , \quad \tau_{eq} = (\nu_\epsilon^{ab})^{-1} = \frac{T_b - T_a}{dT_a/dt} = \frac{3n_a(T_b - T_a)}{2\,Q_{ab}}$$

$$\Rightarrow \boxed{\tau_{eq} = \frac{6\pi^{3/2}\epsilon_0^2 m_a m_b}{\sqrt{2}\, n_b e_a^2 e_b^2 \ln\Lambda}\left(\frac{T_a}{m_a} + \frac{T_b}{m_b}\right)^{3/2}}$$

⑪ Collisional process between shifted Maxwellians in both species.

$$\therefore \vec{R_{ab}} = \int m_a \vec{v}\, C_{ab}\, d\vec{v} = -\int \vec{v}\left(m_a \nu_s^{ab} f_{a1} - m_b \nu_s^{ba} f_{b1}\right) d\vec{v} = -m_a n_a \nu_f^{ab}(\vec{u_a} - \vec{u_b})$$

$$\vec{Q_{ab}} = \int \frac{1}{2} m_a v^2 C_{ab}\, d\vec{v} = \int \frac{1}{2} m_a (\vec{v} - \vec{u_a})^2 C_{ab}\, d\vec{v} + \int m_a (\vec{v}\cdot\vec{u_a})\, C_{ab}\, d\vec{v}$$

$$\therefore \vec{Q_{ab}} = \vec{Q_{ab}} + \vec{u_a}\cdot\vec{R_{ab}} = \frac{3}{2} n_a \nu_\epsilon^{ab}(T_b - T_a) + m_a n_a \nu_f^{ab}\, \vec{u_a}\cdot(\vec{u_b} - \vec{u_a}).$$