Fusion Plasma Theory 2

Lecture 18 - Linearized collisional operators

① Collisional operators for electrons and ions

Landau form : $C_{ab}[f_a, f_b] = \frac{m_a L^{ab}}{8\pi} \frac{\partial}{\partial \vec{v}} \cdot \int \overleftrightarrow{U} \cdot \left[\frac{f_b}{m_a} \frac{df_a}{d\vec{v}} - \frac{f_a}{m_b} \frac{df_b}{d\vec{v}'} \right] d\vec{v}'$

where $L^{ab} = \left(\frac{e_a e_b}{\varepsilon_0 m_a} \right)^2 \ln \Lambda$

Rosenbluth potentials : $C_{ab}[f_a, f_b] = L^{ab} \frac{d}{d\vec{v}} \cdot \left(\frac{m_a}{m_b} \frac{d\varphi_b}{d\vec{v}} f_a - \frac{d^2\psi_b}{d\vec{v} d\vec{v}} \cdot \frac{df_a}{d\vec{v}} \right)$

One can simplify them further for isotropic background or Maxwellian background, however transport processes are largely determined by deviations from Maxwellian.

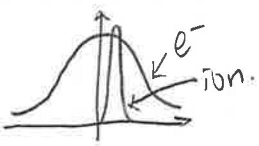
An important progress can be made by linearizing the collision operator around perturbed distributions. So we'll develop linearized (e-i), (i-e), (e-e), (i-i) collision operators which will be used to develop transport equations.

② Electron-ion collision operator

$\frac{m_e}{m_i} \ll 1 \Rightarrow C_{ei}[f_e, f_i] = -L^{ei} \frac{d}{d\vec{v}} \cdot \left(\frac{d^2\psi_i}{d\vec{v} d\vec{v}} \cdot \frac{df_e}{d\vec{v}} \right)$

For electrons, ion distribution is like delta function near the mean flow.

$v_{ti} \ll v_{te} \Rightarrow f_i(\vec{v}) \simeq n_i \delta(\vec{v} - \vec{u})$



$\varphi_i = -\frac{1}{4\pi} \int \frac{f(\vec{v}')}{|\vec{v} - \vec{v}'|} d\vec{v}' = -\frac{1}{4\pi} \int \frac{n_i \delta(\vec{v}' - \vec{u})}{|\vec{v} - \vec{v}'|} d\vec{v}' = -\frac{n_i}{4\pi} \frac{1}{|\vec{v} - \vec{u}|}$

$= -\frac{n_i}{4\pi} \left(v^2 - 2\vec{v} \cdot \vec{u} + \cancel{u^2} \right)^{-\frac{1}{2}} = -\frac{n_i}{4\pi v} \left(1 + \frac{\vec{v} \cdot \vec{u}}{v^2} \right)$

$\psi_i = -\frac{1}{8\pi} \int f(\vec{v}') |\vec{v} - \vec{v}'| d\vec{v}' = -\frac{1}{8\pi} \int n_i \delta(\vec{v}' - \vec{u}) |\vec{v} - \vec{v}'| d\vec{v}' = -\frac{n_i}{8\pi} |\vec{v} - \vec{u}|$

$= -\frac{n_i}{8\pi} \left(v^2 - 2\vec{v} \cdot \vec{u} + \cancel{u^2} \right)^{\frac{1}{2}} = -\frac{n_i v}{8\pi} \left(1 - \frac{\vec{v} \cdot \vec{u}}{v^2} \right)$

The second term in Ψ_i is the next order.

$$\therefore\ C_{ei}[f_e, f_i] = \frac{n_i L_{ei}}{8\pi}\frac{\partial}{\partial\vec{v}}\cdot\Big[\underbrace{\underset{\uparrow\,\text{0th}}{\frac{\partial^2 v}{\partial\vec{v}\partial\vec{v}}}\ \underset{\uparrow\,\text{1st}}{\frac{\partial f_{e1}}{\partial\vec{v}}}}_{\text{1st term}} - \underbrace{\frac{\partial^2}{\partial\vec{v}\partial\vec{v}}\underset{\uparrow\,\text{1st}}{\left(\frac{\vec{v}\cdot\vec{u}}{v}\right)}\cdot\underset{\uparrow\,\text{0th}}{\frac{\partial f_{Me}}{\partial\vec{v}}}}_{\text{2nd term}}\Big].$$

1st term: $\dfrac{\partial^2 v}{\partial\vec{v}\partial\vec{v}} = \dfrac{\partial}{\partial\vec{v}}\left(\dfrac{\vec{v}}{v}\right) = \overleftrightarrow{I}\dfrac{1}{v} - \vec{v}\vec{v}\dfrac{1}{v^2} = \dfrac{1}{v^3}\left(v^2\overleftrightarrow{I} - \vec{v}\vec{v}\right) = \overleftrightarrow{V}$.

$$\frac{\partial}{\partial\vec{v}}\cdot\left[\overleftrightarrow{V}\cdot\frac{\partial f}{\partial\vec{v}}\right] = \frac{\partial}{\partial\vec{v}}\cdot\left[\frac{1}{v}\frac{\partial f}{\partial\vec{v}} - \frac{\vec{v}}{v^3}\left(\vec{v}\cdot\frac{\partial f}{\partial\vec{v}}\right)\right]$$ ← perpendicular to $\vec{v}$ since it subtracts the $\vec{v}$ component off

$$\left(\text{*note } \frac{\partial f}{\partial\vec{v}} = \frac{\partial f}{\partial v}\hat{v} + \frac{1}{v}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{v\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi}\right)$$

$$\therefore \frac{\partial}{\partial\vec{v}}\cdot\left[\overleftrightarrow{V}\cdot\frac{\partial f}{\partial\vec{v}}\right] = \frac{\partial}{\partial\vec{v}}\cdot\left[\frac{1}{v^2}\frac{\partial f}{\partial\theta}\hat{\theta} + \frac{1}{v^2\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi}\right] = \frac{1}{v^2\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{v}\frac{\partial f}{\partial\theta}\right) + \left(\frac{1}{v\sin\theta}\frac{\partial^2 f}{\partial\phi}\right)\right]$$

$$\equiv \frac{2}{v^3}\mathcal{L}(f_a) \quad \text{where } \mathcal{L}(f_a) \equiv \frac{1}{2}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 f}{\partial\phi}\right]$$

$\therefore$ To conclude, $\dfrac{n_i L_{ei}}{8\pi}\dfrac{\partial}{\partial\vec{v}}\cdot\left[\dfrac{\partial^2 v}{\partial\vec{v}\partial\vec{v}}\cdot\dfrac{\partial f_{e1}}{\partial\vec{v}}\right] = \dfrac{n_i L_{ei}}{8\pi}\dfrac{\partial}{\partial\vec{v}}\cdot\left[\overleftrightarrow{V}\cdot\dfrac{\partial f_{e1}}{\partial\vec{v}}\right] = \dfrac{n_i L_{ei}}{4\pi v^3}\mathcal{L}(f_{e1})$

Second term: $\dfrac{\partial^2}{\partial\vec{v}\partial\vec{v}}\left(\dfrac{\vec{v}\cdot\vec{u}}{v}\right) = \dfrac{\partial}{\partial\vec{v}}\left(\dfrac{\partial}{\partial\vec{v}}\cdot\hat{v}\right)\vec{u} = \dfrac{\partial}{\partial\vec{v}}\left(\overleftrightarrow{V}\cdot\vec{u}\right)$

$$\frac{\partial}{\partial\vec{v}}\left(\overleftrightarrow{V}\cdot\vec{u}\right)\cdot\frac{\partial f_{Me}}{\partial\vec{v}} = \frac{\partial}{\partial\vec{v}}\left(\overleftrightarrow{V}\cdot\vec{u}\right)\left(-\frac{m_e}{T_e}\vec{v}\right)f_{Me}$$

$$\overleftrightarrow{V}\cdot\vec{u} = \frac{v^2 u_\alpha - v_\alpha v_\beta u_\beta}{v^3} = \frac{u_\alpha}{v} - \frac{v_\alpha(v_\beta u_\beta)}{v^3}$$

$$\frac{\partial}{\partial\vec{v}}\left(\overleftrightarrow{V}\cdot\vec{u}\right)\cdot\vec{v} = \frac{\partial}{\partial v_\gamma}\left(\frac{u_\alpha}{v} - \frac{v_\alpha(v_\beta u_\beta)}{v^3}\right)v_\gamma \qquad \text{*note } \frac{\partial}{\partial v_\gamma}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\frac{v_\gamma}{v}$$

$$= \left(-\frac{1}{v^2}\frac{v_\gamma}{v}u_\alpha - \frac{\delta_{\alpha\gamma}(u_\beta v_\beta)}{v^3} - \frac{v_\alpha\delta_{\beta\gamma}u_\beta}{v^3} + \frac{3}{v^4}\frac{v_\gamma}{v}(v_\alpha(v_\beta u_\beta))\right)v_\gamma$$

$$= -\frac{u_\alpha}{v} - \frac{(u_\beta v_\beta)v_\alpha}{v^3} - \frac{(u_\beta v_\beta)v_\alpha}{v^3} + \frac{3}{v^3}(u_\beta v_\beta)v_\alpha$$

$$= -\frac{u_\alpha}{v} + \frac{(u_\beta v_\beta)v_\alpha}{v^3} = -\left(\overleftrightarrow{V}\cdot\vec{u}\right)$$

$$\frac{\partial^2}{\partial\vec{v}\partial\vec{v}}\left(\frac{\vec{v}\cdot\vec{u}}{v}\right)\cdot\frac{\partial f_{Me}}{\partial\vec{v}} = \frac{\partial}{\partial\vec{v}}\left(\overleftrightarrow{V}\cdot\vec{u}\right)\cdot\frac{\partial f_{Me}}{\partial\vec{v}} = -\frac{m_e}{T_e}f_{Me}\cdot\left(\frac{\partial}{\partial\vec{v}}\left(\overleftrightarrow{V}\cdot\vec{u}\right)\cdot\vec{v}\right) = \left(\overleftrightarrow{V}\cdot\vec{u}\right)\frac{m_e}{T_e}f_{Me}$$

$$\frac{d}{d\vec{v}}\left(\overleftrightarrow{V}\right) = -\frac{2\vec{v}}{v^3}$$

To conclude, $-\frac{n_i L_{ei}}{8\pi}\frac{d}{d\vec{v}}\cdot\left[\left(\overleftrightarrow{V}\cdot\vec{u}\right)\frac{m_e}{T_e}f_{Me}\right] = \frac{n_i L_{ei}}{4\pi v^3}\frac{m_e \vec{v}\cdot\vec{u}}{T_e}f_{Me}$

Adding together, one can obtain the electron-ion collisional operator:

$$C_{ei}[f_e, f_i] = \frac{n_i L_{ei}}{4\pi v^3}\left[\mathcal{L}(f_{e1}) + \frac{m_e \vec{v}\cdot\vec{u}}{T_e}f_{Me}\right] = \nu_{ei}(v)\left[\underset{\substack{\uparrow \\ \text{electron perturbed} \\ \text{distribution}}}{\mathcal{L}(f_{e1})} + \underset{\substack{\uparrow \\ \text{frictional term} \\ \text{due to shiftedness.}}}{\frac{m_e \vec{v}\cdot\vec{u}}{T_e}f_{Me}}\right]$$

with $\nu_{ei}(v) = \frac{n_i L^{ei}}{4\pi v^3} = \frac{3\sqrt{\pi}}{4}\hat{\nu}_{ei}\left(\frac{v_{te}}{v}\right)^3$

Note we use the standard collisional frequency.

$$\hat{\nu}_{ei} = \frac{2^{1/2} n_i Z^2 e^4 \ln\Lambda}{12\pi^{3/2}\epsilon_0^2 m_e^{1/2} T_e^{3/2}} = \frac{4}{3\sqrt{\pi}}\frac{n_i Z^2 e^4 \ln\Lambda}{4\pi\epsilon_0^2 m_e^2 v_{te}^3}$$

↳ C_{ei} does not depend on the ion mass but only the charge of ions.

If all these ions are stationary and multi-species,

$$C_{ei}[f_e, f_i] = \sum_j \nu_{ej}\mathcal{L}(f_{e1}) = Z_{eff}\frac{3\sqrt{2\pi}}{4}\hat{\nu}_{ee}\left(\frac{v_{te}}{v}\right)^3\mathcal{L}(f_{e1})$$

where $Z_{eff} = \frac{\sum_j n_j Z_j^2}{n_e} = \frac{\sum_j n_j Z_j^2}{\sum_j n_j Z_j}$.

To arrive this results, we only used $m_e \ll m_i$ assumption, so same treatment can be applied to $m_Z \gg m_i$ (such as W impurity)

③ Ion-electron collision operator

We shift the background electron Maxwellian but it turns out the effect by f_{e1} is also comparable.

So let : $\quad f_e(\vec{v}) = f_{Me}(\vec{v}-\vec{u}) + f_{e1}(\vec{v})$

Then, $\quad C_{ie}[f_i, f_e] = C_{ie}[f_i, f_{Me}(\vec{v}-\vec{u})] + C_{ie}[f_i, f_{e1}]$

(1) 2nd part

$$(m_i \gg m_e) \rightarrow \quad C_{ie}[f_i, f_{e1}] \simeq L^{ie}\frac{d}{d\vec{v}}\cdot\left(\frac{m_i}{m_e}\frac{d\varphi_e}{d\vec{v}} f_i\right) \simeq \frac{d}{d\vec{v}}\cdot(\vec{\nu} f_i)$$

The dynamic friction $\vec{\nu}$ for an ion is created by f_{e1} electrons.

only representation is changed from $\vec{\nu}$ to $\vec{R_{ei}}$

$$(v_{te} \gg v_{ti}) \rightarrow \quad \vec{R_{ie}} = -m_i\int \vec{\nu} f_i \, d\vec{v} \simeq -m_i n_i \vec{\nu}$$

$$(\vec{R_{ie}} = -\vec{R_{ei}}) \rightarrow \quad C_{ie}[f_i, f_{e1}] = \frac{\vec{R_{ei}}}{m_i n_i}\cdot\frac{df_i}{d\vec{v}} \qquad (\vec{R_{ei}} \text{ can be obtained after solving } f_{e1})$$

(2) 1st part

$$C_{ie}[f_i, f_{Me}(\vec{v}-\vec{u})] = L^{ie}\frac{d}{d\vec{v}}\cdot\left(\frac{m_i}{m_e}\frac{d\varphi_e}{d\vec{v}} f_i - \frac{d^2\psi_e}{d\vec{v}d\vec{v}}\frac{df_i}{d\vec{v}}\right) \leftarrow \text{full representation}$$

i) Maxwellian background and $G(x\to 0) \rightarrow \frac{2x}{3\sqrt{\pi}}$ $\quad (x = \frac{v}{v_t})$ gives :

$$\frac{d\varphi_e}{d\vec{v}} = \frac{n_e}{2\pi v_{te}^2}G(x_e) = \frac{n_e}{2\pi v_{te}^2}\frac{2}{3\sqrt{\pi}}\left(\frac{\vec{v}-\vec{u}}{v_{te}}\right) = \frac{n_e}{3}\left(\frac{1}{\pi v_{te}^2}\right)^{3/2}(\vec{v}-\vec{u}) = \frac{n_e}{3}\left(\frac{m_e}{2\pi T_e}\right)^{3/2}(\vec{v}-\vec{u})$$

ii)
$$\nabla_v^2\psi_e = \varphi_e(\vec{v}) \simeq \varphi_e(\vec{u}) = -\frac{1}{4\pi}\int\frac{f_{Me}(v')}{v'}dv' = -\frac{n_e}{(2\pi)^{3/2}}\left(\frac{m_e}{T_e}\right)^{1/2}$$

*note
$$f_{Me}(\vec{v}) = n_e\left(\frac{m_e}{2\pi T_e}\right)^{3/2}\exp\left(-\frac{m_e v^2}{2T_e}\right)$$

$$\int\frac{1}{v}f_{Me}(\vec{v})\,d\vec{v} = 4\pi\int f_{Me}(\vec{v})\, v\, dv = 4\pi n_e\left(\frac{m_e}{2\pi T_e}\right)^{3/2}\int_0^\infty v\exp\left(-\frac{m_e v^2}{2T_e}\right)dv$$

$$(av^2 = t,\ 2av\,dv = dt) \Rightarrow \quad 4\pi n_e\left(\frac{m_e}{2\pi T_e}\right)^{3/2}\int_0^\infty\frac{1}{2a}e^{-t}dt$$

$$= 4\pi n_e\left(\frac{m_e}{2\pi T_e}\right)^{3/2}\left(\frac{T_e}{m_e}\right) = 2n_e\left(\frac{m_e}{2\pi T_e}\right)^{1/2}$$

Substituting all, one can obtain the ion-electron collision operator:

$$C_{ie}[f_i, f_e] = \underbrace{\frac{\vec{R_{ei}}}{m_i n_i}\cdot\frac{df_i}{d\vec{v}}}_{\text{friction}} + \frac{m_e n_e}{m_i n_i \tau_e}\frac{d}{d\vec{v}}\cdot\underbrace{\left[(\vec{v}-\vec{u})f_i + \frac{T_e}{m_i}\frac{df_i}{d\vec{v}}\right]}_{\text{ion-electron energy exchange}}$$

④ Linearized like-particle collision operator

No simplification can be made.

$$C_{aa}^{l}[f_a] = C_{aa}[f_a, f_a] = C_{aa}[f_{a1}, f_{Ma}] + C_{aa}[f_{Ma}, f_{a1}] \quad (\because C_{aa}[f_{Ma}, f_{Ma}] = 0)$$

(1) 1st-term

Using the Landau form.

$$C_{aa}[f_{a1}, f_{Ma}] = \frac{L^{aa}}{8\pi}\frac{d}{d\vec{v}}\cdot\int \overleftrightarrow{U}\cdot\Big[\underbrace{f_{Ma}(\vec{v}')\frac{df_{a1}(\vec{v})}{d\vec{v}}}_{\text{1st-part}} - \underbrace{f_{a1}(\vec{v})\frac{df_{Ma}(\vec{v}')}{d\vec{v}'}}_{\text{2nd part}}\Big]d\vec{v}'$$

$(\overleftrightarrow{U}\cdot\vec{u} = 0 \Leftrightarrow \overleftrightarrow{U}\cdot\vec{v} = \overleftrightarrow{U}\cdot\vec{v}')$ $\quad \overleftrightarrow{U}\cdot\vec{u} = 0$

(i) 2nd-part becomes $-\overleftrightarrow{U}\cdot\left(-\frac{m_a}{T_a}\right)(\vec{v}'-\vec{u})f_{Ma}(\vec{v}')f_{a1}(\vec{v})$

$$= -\overleftrightarrow{U}\cdot\left(-\frac{m_a}{T_a}\right)(\vec{v}-\vec{u})f_{Ma}(\vec{v}')f_{a1}(\vec{v}) = \overleftrightarrow{U}\cdot f_{Ma}(\vec{v}')f_{Ma}(\vec{v})f_{a1}(\vec{v})\frac{d}{d\vec{v}}\left(\frac{1}{f_{Ma}(\vec{v})}\right)$$

$$\left(-\frac{m_a}{T_a}(\vec{v}-\vec{u}) = \frac{1}{f_{Ma}}\frac{df_{Ma}}{d\vec{v}} = -f_{Ma}\frac{d}{d\vec{v}}\left(\frac{1}{f_{Ma}}\right)\right)$$

$\therefore$ Landau form can be put into :

$$C_{aa}[f_{a1}, f_{Ma}] = \frac{L^{aa}}{8\pi}\frac{d}{d\vec{v}}\cdot\int\overleftrightarrow{U}\cdot f_{Ma}(\vec{v})f_{Ma}(\vec{v}')\frac{d}{d\vec{v}}\left(\frac{f_{a1}(\vec{v})}{f_{Ma}(\vec{v})}\right)d\vec{v}'$$

(2) 2nd-term.

Similarly in (1). $C_{aa}[f_{Ma}, f_{a1}] = -\frac{L^{aa}}{8\pi}\frac{d}{d\vec{v}}\cdot\int\overleftrightarrow{U}\cdot f_{Ma}(\vec{v})f_{Ma}(\vec{v}')\frac{d}{d\vec{v}'}\left(\frac{f_{a1}(\vec{v}')}{f_{Ma}(\vec{v}')}\right)d\vec{v}'$

Substituting all (using $\hat{f}_a \equiv f_{a1}/f_{Ma}$)

$$(1)+(2) \Rightarrow C_{aa}^{l}[\hat{f}_a] = \frac{L^{aa}}{8\pi}\frac{d}{d\vec{v}}\cdot\int\overleftrightarrow{U}\cdot f_{Ma}(\vec{v})f_{Ma}(\vec{v}')\left(\frac{d\hat{f}_a(\vec{v})}{d\vec{v}} - \frac{d\hat{f}_a(\vec{v}')}{d\vec{v}'}\right)d\vec{v}'$$

Note that $\hat{f}_a = (1, \vec{v}, v^2)$ is a solution for $C_{aa}^{l}[\hat{f}_a] = 0$.

① → $0-0=0$ / $\vec{v}$ → $1-1=0$ / v^2 → $\vec{v}-\vec{v}'=\vec{u}$ & $\overleftrightarrow{U}\cdot\vec{u}=0$

$f_{a1} = (A + \vec{B}\cdot\vec{v} + Cv^2)f_{Ma} \rightarrow C_{aa}^{l}[f_{a1}] = 0$

Maxwell form

Maxwellian → Maxwellian 으로 간
shifted Maxwellian 으로 간것뿐이다.

$$\delta f = \Big[\underset{\uparrow\,1}{\frac{\delta n}{n_0}} + \underset{\uparrow\,\vec{v}}{\frac{m_a}{T_0}\vec{v}\cdot\delta\vec{u}} + \underset{\uparrow\,v^2}{\left(\frac{m_a v^2}{2T_0} - \frac{3}{2}\right)\frac{\delta T}{T_0}}\Big]f_{Ma0} \rightarrow C_{aa}^{l}[\delta f] = 0$$

확실하게 이해해보기 (지금 90%)

어떤 변화에 대해서도 $C_{aa}^{l}[\delta f] = 0$, since $f_{Ma} = f_{Ma0} + \delta f$

$\delta n, \delta\vec{u}, \delta T$

⑤ Self-adjointness in collision operator.

Linearized collision operator has the property of self-adjointness w/ weighting f_M^{-1}.

$$\langle g_1, f_1 \rangle \equiv \int \frac{1}{f_M} g_1 f_1 d\vec{v}.$$

Let $g_1 = f_{Ma}\hat{g}_a$, $f_1 = f_{Ma}\hat{f}_a$. Then:

$$\langle g_1, C^l_{aa}[f_1] \rangle = \frac{L_{aa}}{8\pi}\int \hat{g}_a(\vec{v}) d\vec{v} \frac{\partial}{\partial \vec{v}} \cdot \int f_{Ma}(\vec{v}) f_{Ma}(\vec{v}') \overleftrightarrow{U} \cdot \left(\frac{d\hat{f}_a(\vec{v})}{d\vec{v}} - \frac{d\hat{f}_a(\vec{v}')}{d\vec{v}'} \right) d\vec{v}'$$

(i) (integration by parts)

$$= -\frac{L_{aa}}{8\pi}\int d\vec{v} \int d\vec{v}' f_{Ma}(\vec{v}) f_{Ma}(\vec{v}') \frac{d\hat{g}_a(\vec{v})}{d\vec{v}} \cdot \overleftrightarrow{U} \cdot \left(\frac{d\hat{f}_a(\vec{v})}{d\vec{v}} - \frac{d\hat{f}_a(\vec{v}')}{d\vec{v}'} \right)$$

(ii) ($\vec{v} \Leftrightarrow \vec{v}'$ switch by $\overleftrightarrow{U}\cdot\vec{u} = 0$)

$$= \frac{L_{aa}}{8\pi}\int d\vec{v} \int d\vec{v}' f_{Ma}(\vec{v}) f_{Ma}(\vec{v}') \frac{d\hat{g}_a(\vec{v}')}{d\vec{v}'} \cdot \overleftrightarrow{U} \cdot \left(\frac{d\hat{f}_a(\vec{v})}{d\vec{v}} - \frac{d\hat{f}_a(\vec{v}')}{d\vec{v}'} \right)$$

$\frac{1}{2}((i)+(ii))$

$$= -\frac{L_{aa}}{16\pi}\int d\vec{v} \int d\vec{v}' f_{Ma}(\vec{v}) f_{Ma}(\vec{v}') \left(\frac{d\hat{g}_a(\vec{v})}{d\vec{v}} - \frac{d\hat{g}_a(\vec{v}')}{d\vec{v}'} \right) \cdot \overleftrightarrow{U} \cdot \left(\frac{d\hat{f}_a(\vec{v})}{d\vec{v}} - \frac{d\hat{f}_a(\vec{v}')}{d\vec{v}'} \right)$$

∴ From this symmetric form, $\langle g_1, C^l_{aa}[f_1] \rangle = \langle f_1, C^l_{aa}[g_1] \rangle$.

Rewriting the self-adjoint property with $\hat{g}_a$ and $\hat{f}_a$:

$$\int \hat{g}_a C^l_{aa}[f_{Ma}\hat{f}_a] d\vec{v} = \int \hat{f}_a C^l_{aa}[f_{Ma}\hat{g}_a] d\vec{v}$$

$\boxed{1}$ Self adjointness + $\boxed{2}$ $\hat{f}_a = (1, \vec{v}, v^2)$ are solutions of $C^l_{aa}[\hat{f}_a]$ gives conservation.

1) particle conservation $\boxed{2}$

$$0 \overset{\boxed{2}}{=} \int \hat{g}_a C^l_{aa}[f_{Ma}\cdot 1] d\vec{v} \overset{\boxed{1}}{=} \int 1\, C^{ll}_{a}[f_{Ma}\hat{g}_a] d\vec{v} = \underbrace{\int C^l_{aa}[g_1] d\vec{v}}_{\text{particle}}$$

2) momentum conservation

$$0 \overset{\boxed{2}}{=} m_a \int \hat{g}_a C^l_{aa}[f_{Ma}\vec{v}] d\vec{v} \overset{\boxed{1}}{=} \underbrace{\int m\vec{v}\, C^l_{aa}[g_1] d\vec{v}}_{\text{momentum}}$$

3) energy conservation

$$0 \overset{\boxed{2}}{=} \frac{1}{2}m_a \int \hat{g}_a C^l_{aa}[f_{Ma}v^2] d\vec{v} \overset{\boxed{1}}{=} \underbrace{\int \frac{1}{2}m_a v^2 C^l_{aa}[g_1] d\vec{v}}_{\text{energy}}$$

*note

electron collisional operator remains self-adjoint

(∵ like-particle & Lorentz scattering are both self-adjoint)

$$\langle g_1, (C_{ee}^{\ell} + \mathcal{L}_{ei})[f_1] \rangle = \langle f_1, (C_{ee}^{\ell} + \mathcal{L}_{ei})[g_1] \rangle \quad \blacksquare$$