Lecture 20. Internal and External modes

- Newcomb's analysis for screw pinch instabilities

① Suydam criterion

- localized interchange instability occurs near rational surfaces

② No zero crossing in $\xi(r)$ propile.

- internal (interchange) instability

③ $\xi(r)$ must be tested against external mode

taking into account vacuum energy.

This lecture is about internal and external mode.

We'll ignore δW_s by assuming no boundary surface current $r=a$.

- Trial function for δW

- Energy principle : δW must be positive for any $\vec{\xi}(\vec{x})$ profile for plasma to be stable. If one can find any one trial function $\vec{\xi}(\vec{x})$ that makes $\delta W<0$, plasma is ideally unstable.

- Screw pinch : since $\xi_{\parallel}, \xi_{\alpha}$ are removed in the minimization process, one can consider a trial function only in the form $\xi(r)$ for each m, n.

Also, one can assume $\xi(r)$ is real, since f, g are real function.

For most unstable $\xi(r)$:

$$\delta \hat{W}_F = \frac{\delta W_F}{2\pi^2 R_0/\mu_0} = \int_0^a f\xi'\xi' dr + \int_0^a g\xi\xi dr$$

$$= \left[f\xi'\xi\right]_0^a - \int_0^a \xi\left[\cancel{(f\xi')' + g\xi}\right] dr = f\xi'\xi\Big|_0^a$$

$(\because$ E-L eqn$)$

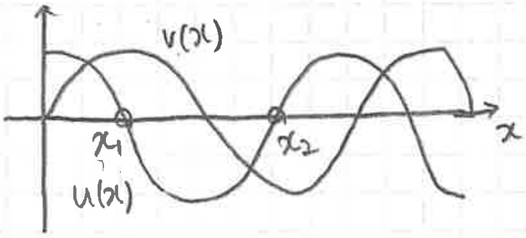
- Sturm separation theorem

zeros of two linearly independent solutions of the 2nd ODE must alternate.

proof) For two linearly independent solutions $u(x)$, $v(x)$ for I,

the Wronskian must be non-zero, i.e. $W = uv' - vu' \neq 0$.

Say $u(x)$ has two zeros at x_1 and x_2, Then $v(x)$ must have exactly one zero in the open interval.



(Wronskian $\neq 0$ 이면 두 해는 선형독립이다) $\begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = uv' - vu'$

At $x = x_1$, $W(x_1) = -u'(x_1)v(x_1)$

let's say $W(x_1) > 0$ and $u'(x_1) < 0$, $v(x) > 0$ w/o loss of generality

At $x = x_2$, $W(x_2) = -u'(x_2)v(x_2)$

For u and v to be linearly independent, $W(x_2) > 0$.

One can see $u'(x_2) > 0$, $v(x) < 0$, meaning that

$v(x)$ must cross zero between x_1 and x_2.

- Zero crossing condition for internal mode

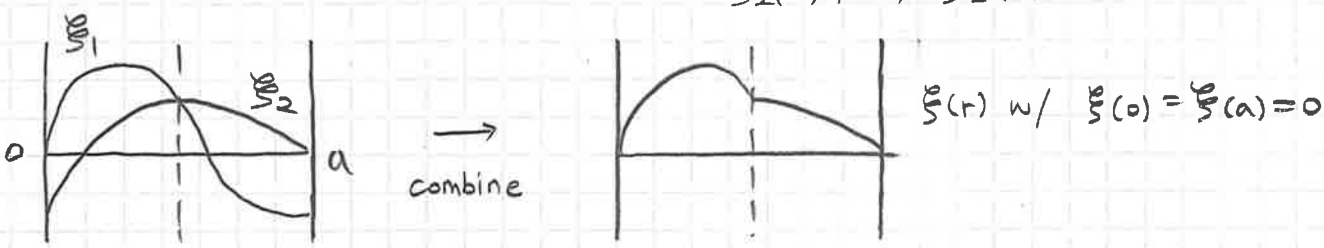
If a regular (at $r=0$) solution $\xi_1(r)$ has a zero crossing in $r<a$,

the system is unstable by an internal mode

⟨* no singular surface⟩

w/o loss of generality, a general solution of Newcomb equation :

$$\xi(r) = C_1\xi_1(r) + C_2\xi_2(r) \quad \text{with} \quad \begin{array}{l} \xi_1(0)=0, \ \xi_1(a)\neq 0 \\ \xi_2(0)\neq 0, \ \xi_2(a)=0 \end{array}$$



$\delta\hat{W}_F$ by these trial functions becomes :

$$\delta\hat{W}_F = f\xi'\xi\Big|_0^{r_0} + f\xi'\xi\Big|_{r_0}^{a} = f(r_0)\xi(r_0)\left[\xi_1'(r_0) - \xi_2'(r_0)\right]$$

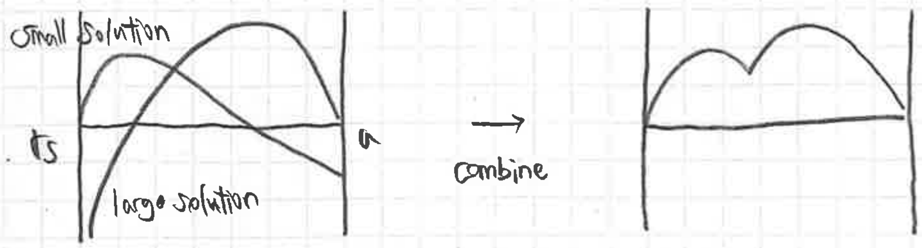
If ξ_1 has zero crossing, one can see $\xi_1'(r_0) < \xi_2'(r_0) \Rightarrow \delta W_F < 0$

On the other hand, $\delta\hat{W}_F > 0$ in the ξ_2 no zero crossing cannot satisfy Sturm separation theorem.

This is how Newcomb concluded that " an internal mode is unstable iff there is zero crossing of a regular solution $\xi(r)$.

⟨* with singular surface, ($r=r_s$)⟩

Solution is separated by the singular surface. Let's discuss solution for $r\in[r_s, a]$



Same logic : $\delta\hat{W}_F = f(r_0)\xi(r_0)\left[\xi_1'(r_0) - \xi_2'(r_0)\right] < 0 \rightarrow$ unstable.

Jump in ξ' is understood physically as a development of a long parallel current when internal mode grows.

- <u>Perturbed energy for external mode</u>
  - If plasma profiles meet Suydam criterion and do not have a zero-crossing, what remains is the stability against external modes with $\xi(a) \neq 0$.
  - $\xi(a)$ comes with $B_{1r} = \hat{r} \cdot \vec{\nabla} \times (\vec{\xi}_\perp \times \vec{B}) = i \frac{B_\theta}{r}(m - nq)\xi = iF\xi$

    so, $B_{1r}(a) = iF_a \xi_a$
  - This creates perturbed magnetic field through vacuum up to the conducting vessel wall. If an ideal wall is assumed at $r=b$, $B_{1r}(b) = 0$.
  - $B_{1r}(a)$, $B_{1r}(b)$ can be used to determine B_{1r} in vacuum.

    Also, note that $\vec{B}_1 = \vec{\nabla}\chi$ since $\vec{\nabla} \times \vec{B}_1 = 0$
  - Then, it becomes Neumann problem of Laplace equation:

    <u>eqn</u>: $\vec{\nabla} \cdot \vec{B}_1 = \nabla^2 \chi = 0$ with $\left.\frac{d\chi}{dr}\right|_a = iF_a\xi_a$, $\left.\frac{d\chi}{dr}\right|_b = 0$

    <u>sol</u>: $\chi(r,\theta,z) = \left[C_1 K_m(|k|r) + C_2 I_m(|k|r)\right] e^{i(m\theta + kz)}$

    (K_m, I_m are the modified Bessel function)

    Boundary condition $\Rightarrow$ $\chi = \frac{iF_a\xi_a}{|k|} \frac{K_r - (K_b'/I_b')I_r}{K_a' - (K_b'/I_b')I_a'}$
  - Now vacuum energy due to $\vec{B}_1$ is given by:

$$\delta W_v = \frac{1}{2\mu_0}\int \vec{B}_1^* \cdot \vec{B}_1 \, dV = \frac{1}{2\mu_0}\int \vec{\nabla}\cdot(\chi^* \vec{\nabla}\chi)\, dV = -\frac{1}{2\mu_0}\int_{\underset{\text{plasma surf.}}{S}} \chi^*(\hat{n}\cdot\vec{\nabla}\chi)\, dS$$

$$= -\frac{2\pi R_0^2 a}{\mu_0}\left(\chi^* \frac{d\chi}{dr}\right)_a$$

$$\therefore \boxed{\delta \hat{W}_v \equiv \frac{\delta W_v}{2\pi R_0^2 a/\mu_0} = \frac{a^2 F_a^2 \Lambda}{m}\xi_a^2}, \quad \text{where } \Lambda \equiv -\frac{m K_a}{|ka| K_a'}\left[\frac{1-(K_b' I_a / I_b' K_a)}{1-(K_b' I_a' / I_b' K_a')}\right]$$

In the long wavelength approximation, ($ka \sim kb \ll 1$), $\boxed{\Lambda \simeq \frac{1+(a/b)^{2m}}{1-(a/b)^{2m}}}$

- No Wall and ideal wall limit of δW
  - Total $\delta\hat{W} = \delta\hat{W}_F + \delta\hat{W}_V$ is given by

$$\delta\hat{W} = \frac{\delta W}{2\pi^2 R_0/\mu_0} = \int_0^a [f\xi'^2 + g\xi]dr + \left(\frac{FF^+}{k_0^2} + \frac{r^2F^2\Lambda}{m}\right)_a \xi_a^2$$

  - For an external mode, $\boxed{\delta\hat{W} = \left(\frac{F^2 r\xi'}{k_0^2\xi} + \frac{FF^+}{k_0^2} + \frac{r^2F^2\Lambda}{m}\right)_a \xi_a^2 = \delta\hat{W}_b}$ <ideal wall limit>

기존 $\delta W_F = f\xi\xi'|_0^a$ 항　　새롭지 추가

  - the stability of an external mode can be determined by the boundary values ξ_a, ξ_a' but ξ_a' requires the whole solution of Newcomb equation.

This $\delta\hat{W}_b$ gives ideal wall limit of external stability with ideal conducting wall $(r=b)$.

  - In the limit of $b\to\infty$, $\boxed{\delta\hat{W}_\infty = \left(\frac{F^2 r\xi'}{k_0^2\xi} + \frac{FF^+}{k_0^2} + \frac{r^2F^2\Lambda_\infty}{m^2}\right)_a \xi_a^2}$ <no wall limit>

where $\Lambda_\infty = \lim_{b\to\infty}\Lambda = -\frac{mka}{|ka|K_a'} \simeq 1 \geq 1$

  - we know that as b grows, Λ decreases. $\therefore \delta\hat{W}_\infty < \delta\hat{W}_b$

showing wall stabilizing effect.

  - If the plasma is unstable with no-wall, $\delta W_\infty < 0$, a mode can grow through resistivity at the wall even if $\delta W_b > 0$.
  - Also no-wall limit becomes practically the ideal wall limit for high (m,n) mode since wall stabilizing effect becomes negligible.

(when $m\to\infty$, $\Lambda\to 1$ (no wall limit))

- Resistive wall mode (wall is not a perpect conductor)

$$\boxed{\gamma\tau_w = -\frac{\delta W_\infty}{\delta W_b}}$$

(stability boundary is not changed from no-wall stability boundary even with the wall, and just the growth rate becomes slow in the wall resistive time scale)

But, We assume that we can control through feedback control.

- **Straight tokamak approximation**

It ignores toroidicity or curvature, so becomes a poor approximation for pressure driven instabilities, but still explains the current driven instabilities.

$$\boxed{\delta\hat{W} = \frac{\delta W}{2\pi^2 B_z^2/\mu_0 R_0} = \int_0^a \left(\frac{n}{m}-\frac{1}{q}\right)^2\left[r^2\xi'^2 + (m^2-1)\xi^2\right] r\,dr + \left(\frac{n}{m}-\frac{1}{q_a}\right)\left[\left(\frac{n}{m}+\frac{1}{q_a}\right) + m\Lambda\left(\frac{n}{m}-\frac{1}{q_a}\right)\right] a^2\xi_a^2}$$

It can be obtained from $\delta\hat{W} = \int_0^a \left[f\xi'^2 + g\xi^2\right] dr + \left(\frac{FF^+}{k_0^2} + \frac{r^2F^2\Lambda}{m}\right)_a \xi_a^2$.

and f, g in lecture 19, by setting $k \to 0$ and $k_0 \simeq k^2 + \frac{m^2}{r^2}$ (HW or Exam)

- **Kruskal-Shafranov limit**

For, $m=1$ mode,

$$\delta\hat{W} = \int_0^a \left(n-\frac{1}{q}\right)^2\left[r^2\xi'^2\right] r\,dr + \left(n-\frac{1}{q_a}\right)\left[\left(n+\frac{1}{q_a}\right) + \Lambda\left(n-\frac{1}{q_a}\right)\right] a^2\xi_a^2$$

$$\delta\hat{W} = \int_0^a \left(n-\frac{1}{q}\right)^2\left[r^2\xi'^2\right] r\,dr + \left(n-\frac{1}{q_a}\right) 2na^2\xi_a^2 \quad \swarrow \Lambda \simeq 1$$

ξ profile minimizing $\delta\hat{W}$ is such that $\xi' = 0$.

$$\therefore \delta\hat{W} = \left(n-\frac{1}{q_a}\right)2na^2\xi_a^2 > 0 \ \rightarrow\ q_a > \frac{1}{n} \ \Rightarrow\ \boxed{q_a > 1}$$

considering $q_a \approx \frac{aB_\phi}{RB_{\theta a}} = \frac{aB_\phi}{R_0(\mu_0 I_p)/2\pi a} = \frac{2\pi a^2 B_\phi}{\mu_0 R_0 I_p} > 1 \Rightarrow \boxed{I_p < \frac{2\pi a^2 B_\phi}{\mu_0 R_0} = \frac{5a^2 B_\phi}{R_0}\ [MA]}$

<Kruskal-Shafranov Limit>

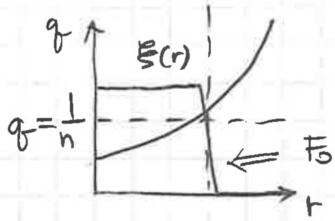
- Internal $m=1$ Kink

$$\delta \hat{W} = \int_0^a \left(\frac{n}{m} - \frac{1}{q}\right)^2 \left[r^2 \xi'^2 + (m^2-1)\xi^2\right] r\,dr$$

Internal mode is stable, except $m=1$, which can become

$$\delta \hat{W} = \int_0^a \left(n - \frac{1}{q}\right)^2 \left[r^2 \xi'^2\right] r\,dr \rightarrow 0$$

$\Rightarrow$ Internal $m=1$ kink is marginally stable. ($\delta W \rightarrow 0$)



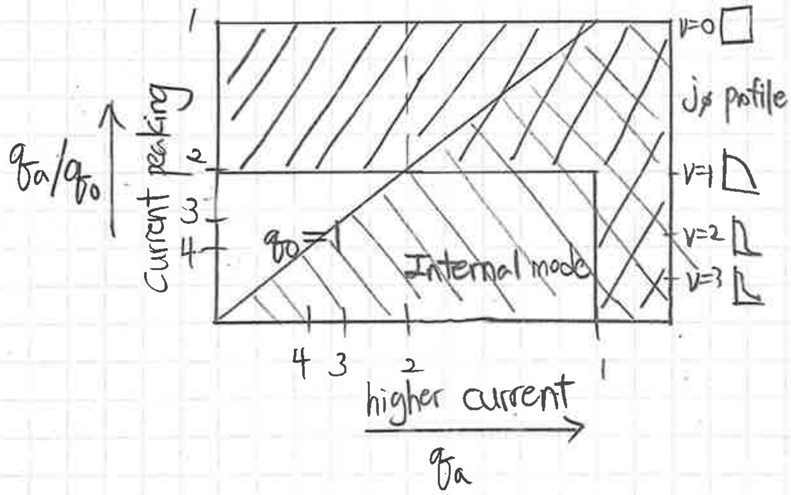
$\Leftarrow$ For this $\xi(r)$, $\delta \hat{W} \rightarrow 0$

But, Rosenbluth showed that higher order $\delta \hat{W} < 0$ for internal $m=1$ kink.

$\therefore$ Stable condition $\boxed{q_0 > 1}$

- External Kink modes

For $m>1$ external kink, sufficient condition for stability is $\boxed{q_a > \frac{m}{n}}$



$j = j_0 \left(1 - \left(\frac{r}{a}\right)^2\right)^\nu$

Internal mode ($q_0 < 1$)

Kink mode for $(m,n) = (2,1)$ stable ($q_a < 2$)