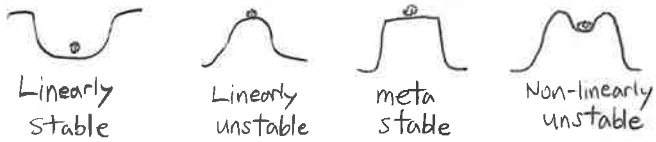
Fusion Plasma Theory I : Lecture 17 : MHD Wave and Rayleigh–Taylor

· Stability of equilibrium



∘ Approaches for stability assessment

$$\vec A(\vec x,t)=\vec A_0(\vec x)+\vec A_1(\vec x,t)$$

wave vs. instability is similar but differs by medium inhomogenity

$$\vec A_1(\vec x,t)=\vec A_1(\vec x)\exp(-i\omega t) : \text{instability in inhomogeneous media (diff. eq)}$$

$$=\vec A_1\exp[i(\vec k\cdot\vec x-\omega t)] : \text{wave in homogeneous media (algebraic eq)}$$

∘ Linearized MHD equations in homogeneous plasma

$$\left(\begin{array}{l}\vec B_0=B_0\hat z,\ \vec j_0=0,\ \vec u_0=0,\ \vec\nabla\rho_0=0,\ \vec\nabla p_0=0\\ \rho_1, p_1, \vec B_1, \vec j_1, \vec u_1 \propto e^{i(\vec k\cdot\vec x-\omega t)},\ \frac{\partial}{\partial t}=-i\omega,\ \vec\nabla=i\vec k\end{array}\right)$$

$$\frac{\partial\rho}{\partial t}+\vec u\cdot\vec\nabla\rho+\rho(\vec\nabla\cdot\vec u)=0 \longrightarrow \omega\rho_1=\rho_0(\vec k\cdot\vec u_1) \quad (4)$$

$$\frac{\partial p}{\partial t}+\vec u\cdot\vec\nabla p+\gamma p(\vec\nabla\cdot\vec u)=0 \longrightarrow \omega p_1=\gamma p_0(\vec k\cdot\vec u_1) \quad (5)$$

$$\frac{\partial\vec B}{\partial t}=-\vec\nabla\times\vec E=\vec\nabla\times(\vec u\times\vec B)=0 \longrightarrow \omega\vec B_1=-\vec k\times(\vec u_1\times\vec B_0) \quad (6)$$

$$\mu_0\vec j=\vec\nabla\times\vec B \longrightarrow \omega\mu_0\vec j_1=-i\vec k\times(\vec k\times(\vec u_1\times\vec B_0)) \quad (7)$$

$$\rho\left(\frac{\partial\vec u}{\partial t}+\cancel{\vec u\cdot\vec\nabla u}\right)=\vec j\times\vec B-\vec\nabla p \longrightarrow \omega\rho_0\vec u_1=i(\vec j_1\times\vec B_0)+\vec k p_1 \quad (8)$$

Ideal MHD wave dispersion

$$(6) \rightarrow \omega \vec{B_1} = -\vec{u_1}(\vec{k}\cdot\vec{B_0}) + \vec{B_0}(\vec{k}\cdot\vec{u_1}) \qquad (9)$$

$$(7) \rightarrow \omega\mu_0 \vec{j_1} = -i(\vec{k}\times\vec{u_1})(\vec{k}\cdot\vec{B_0}) + i(\vec{k}\times\vec{B_0})(\vec{k}\cdot\vec{u_1}) \qquad (10)$$

Plugging Eqs (5), (10) into ($\omega\mu_0 \cdot$ Eq (8)) to remove p_1 and $\vec{j_1}$:

$$\omega^2 \rho_0 \mu_0 \vec{u_1} = i\omega\mu_0(\vec{j_1}\times\vec{B_0}) + \omega\mu_0 \vec{k} p_1$$

$$\text{(L1)} \quad = (\vec{k}\times\vec{u_1})(\vec{k}\cdot\vec{B_0})\times\vec{B_0} - (\vec{k}\times\vec{B_0})(\vec{k}\cdot\vec{u_1})\times\vec{B_0} + \gamma\mu_0 p_0 \vec{k}(\vec{k}\cdot\vec{u_1})$$

$$= \Big[\underset{\text{R1}}{\vec{u_1}(\vec{k}\cdot\vec{B_0})} - \underset{\text{R2}}{\vec{k}(\vec{B_0}\cdot\vec{u_1})}\Big](\vec{k}\cdot\vec{B_0}) - \Big[\underbrace{\vec{B_0}(\vec{k}\cdot\vec{B_0}) - \vec{k}B_0^2}_{\text{R3,4} \; = -\vec{k_\perp}B_0^2}\Big](\vec{k}\cdot\vec{u_1}) + \underset{\text{R5}}{\gamma\mu_0 p_0 \vec{k}(\vec{k}\cdot\vec{u_1})} \qquad (11)$$

Define Alfvén velocity $\boxed{v_A \equiv \dfrac{B_0}{\sqrt{\mu_0\rho_0}}}$, and sound speed $\boxed{C_s \equiv \sqrt{\dfrac{\gamma p_0}{\rho_0}}}$

Then, the full dispersion (Eq. (11) $/\mu_0\rho_0$) becomes:

$$\boxed{(\underset{\text{L1}}{\omega^2} - \underset{\text{R1}}{k_\parallel^2 v_A^2})\vec{u_1} = \underset{\text{R3,4}}{\vec{k_\perp}(\vec{k}\cdot\vec{u_1})v_A^2} - \underset{\text{R2}}{\vec{k}k_\parallel u_\parallel v_A^2} + \underset{\text{R5}}{\vec{k}(\vec{k}\cdot\vec{u_1})C_s^2}}$$

Let $\vec{k} = k_\perp \hat{y} + k_\parallel \hat{z}$ without loss of generality. and $\vec{u_1} = u_x\hat{x} + u_y\hat{y} + u_z\hat{z}$.

$$\hat{x}: \quad (\omega^2 - k_\parallel^2 v_A^2)u_x = 0$$

$$\hat{y}: \quad (\omega^2 - k_\parallel^2 v_A^2)u_y = k_\perp(k_\perp u_y + \cancel{k_\parallel u_z})v_A^2 - \cancel{k_\perp k_\parallel u_z v_A^2} + k_\perp(k_\perp u_y + k_\parallel u_z)C_s^2$$

$$\Rightarrow (\omega^2 - k^2 v_A^2 - k_\perp^2 C_s^2)u_y - (k_\perp k_\parallel C_s^2)u_z = 0$$

$$\hat{z}: \quad (\omega^2 - \cancel{k_\parallel^2 v_A^2})u_z = 0 - \cancel{k_\parallel^2 u_\parallel v_A^2} + k_\parallel(k_\perp u_y + k_\parallel u_z)C_s^2$$

$$\Rightarrow -k_\perp k_\parallel C_s^2 u_y + (\omega^2 - k_\parallel^2 C_s^2)u_z = 0$$

$\hat{x}$ component is isolated, giving shear Alfvén wave dispersion.

$$\boxed{w^2 = k_\parallel^2 v_A^2}$$

$(\hat{y}, \hat{z})$ components are coupled, giving Magnetosonic wave dispersion

$$\begin{pmatrix} & & \\ & D & \\ & & \end{pmatrix}\begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow \det D = 0 \longrightarrow (w^2 - k_\parallel^2 v_A^2)(w^2 - \tfrac{1}{2}k^2 + \dots)$$

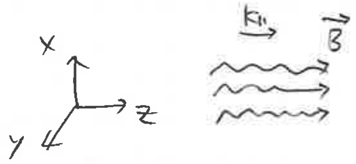
(to have non-trivial solution)

$$\boxed{w^2 = \frac{1}{2}k^2 (v_A^2 + c_s^2)\left(1 \pm \sqrt{1-\alpha^2}\right) \quad \text{where } 0 < \alpha \equiv \frac{k_\parallel}{k_\perp}\frac{2 v_A c_s}{v_A^2 + c_s^2} < 1}$$

with fast (+) and slow (−) branches.

Note that here all $w^2 > 0$, giving purely oscillatory propagating wave (no dissipation)

- Shear Alfvén wave $\boxed{w^2 = k_\parallel^2 v_A^2}$



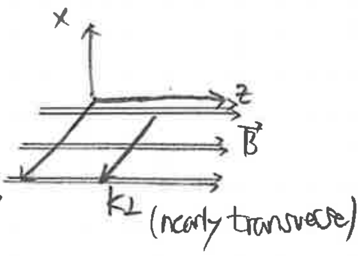
$\vec{u}_1 = \begin{pmatrix} u_x \\ 0 \\ 0 \end{pmatrix} \rightarrow \vec{\nabla}\cdot\vec{u}_1 = 0,\ \vec{k}\cdot\vec{u}_1 = 0 \rightarrow \rho_1 = p_1 = 0$ (from Eqs (4), (5))

Eq (9) $\rightarrow w\vec{B}_1 = -\vec{u}(\vec{k}\cdot\vec{B}_0) + \vec{B}_0 \cancel{(\vec{k}\cdot\vec{u}_1)}^{0} \rightarrow \vec{B}_1 = \begin{pmatrix} B_{1x} \\ 0 \\ 0 \end{pmatrix}$ (Transverse & parallel wave)

Perpendicular Kinetic $\leftrightarrow$ Magnetic bending energy

Most dangerous $\rightarrow B^2 = B_0^2 + \cancel{2\vec{B}_0\vec{B}_1}^{0} + B_1^2$: Easy to excite since $B_1^2 \ll 1$ (2nd-order)

- Compressional Alfvén wave



$\beta \ll 1$, $c_s \ll v_A$, $\alpha \ll 1$

$$\boxed{w^2 \approx k^2 v_A^2 = k_\perp^2 v_A^2}$$

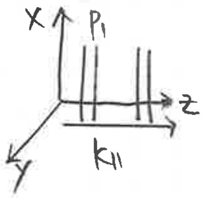
(assume $k_\perp \gg k_\parallel$)

$\vec{u}_1 = \begin{pmatrix} 0 \\ u_y \\ u_z \end{pmatrix}$ $(u_y \gg u_z)$, $p_1 \neq 0$, $\rho_1 \neq 0$

$\vec{B}_1 = \begin{pmatrix} 0 \\ B_{1y} \\ 0 \end{pmatrix}$

Kinetic $\leftrightarrow$ Magnetic energy

- Sound wave



$\beta \ll 1$, $\alpha \ll 1$

$$\boxed{w^2 \simeq k_\parallel^2 c_s^2}$$

$\vec{u}_1 = \begin{pmatrix} 0 \\ 0 \\ u_z \end{pmatrix}$, $\vec{B}_1 = 0$

(Thermal & longitudal parallel wave)

$$\boxed{k_\perp^2 v_A^2 \gg k_\parallel^2 v_A^2 \gg k_\parallel^2 c_s^2}$$

$\therefore$ Ideal MHD as fast as at least sound wave and mostly Alfven time scale

· Rayleigh - Taylor instability