Fusion Plasma Theory 2

Lecture 8 : Basic Properties of Plasma Waves

① Waves and Instabilities

1) Waves : "Homogeneous" media

$$\text{Linearized equation} \xrightarrow[\text{Fourier} \ \sim e^{i(\vec{k}\cdot\vec{x}-\omega t)}]{} \text{"Algebraic" equations}$$

$$\vec{\nabla} \to i\vec{k}\ , \quad \frac{d}{dt} \to -i\omega$$

2) Instabilities : "Inhomogeneous media"

② Phase and Group velocities

1) Phase velocity : velocity of a constant phase for a wave

$$\psi(\vec{x},t) = A e^{i(\vec{k}\cdot\vec{x}-\omega t)}, \quad \frac{d}{dt}(\vec{k}\cdot\vec{x}-\omega t) = 0 \rightarrow \boxed{\vec{v_p} \equiv \frac{d\vec{x}}{dt} = \frac{\vec{k}}{k^2}\omega}$$

2) Group velocity : velocity of a envelope of the wave

$$\psi(\vec{x},t) = \int \phi(\vec{k}) e^{i(\vec{k}\cdot\vec{x}-\omega t)} d\vec{k}$$

(energy and information are carried by a wave packet : collection of many waves under the dispersion $\omega = \omega(\vec{k})$)

Expanding $\omega(\vec{k}) \simeq \omega_0 + (\vec{k}-\vec{k_0})\frac{d\omega}{d\vec{k}}\Big|_0$ near $\vec{k_0}$ for the packet :

$$\psi(\vec{x},t) \simeq \int \phi(\vec{k}) e^{i\vec{k}\cdot\vec{x}} e^{-i(\omega_0 + (\vec{k}-\vec{k_0})\frac{d\omega}{d\vec{k}})t} d\vec{k}$$

$$= e^{i(\vec{k_0}\cdot\vec{x}-\omega_0 t)} \int \phi(\vec{k}) e^{i(\vec{k}-\vec{k_0})\cdot(\vec{x}-\frac{d\omega}{d\vec{k}}\big|_0 t)} d\vec{k}$$

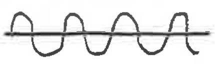
↑ 파동묶음을 현미경으로 봤을때 보이는 빠른물결

↑ 파동 묶음의 전체적 움직임 ⇒ $\frac{d\vec{x}}{dt} = \frac{d\omega}{d\vec{k}}\Big|_0 = \vec{v_g}$

$$\boxed{\vec{v_g} = \frac{d\omega}{d\vec{k}}}$$

③ Ray Tracing : Tracing wave pocket

wave



instability

ray tracing

; (waves propagate through inhomogeneous media)

$\Downarrow$

$\underline{\vec k = \vec k(\vec x, t)}$, $\underline{w = w(\vec k, \vec x, t)}$

But, assume still the variation is slow compared to $\vec k$ and w.

$\rightarrow \underline{\psi \sim e^{i\varphi(\vec x, t)}} \Rightarrow \boxed{\vec k \equiv \vec\nabla \varphi}, \boxed{w \equiv -\frac{\partial \varphi}{\partial t}}$ (∵ think of $e^{i(kx-wt)}$)

The variation of the wave vector :

$$\frac{\partial \vec k}{\partial t} = -\vec\nabla w = -\frac{\partial w}{\partial \vec k}\Big|_{\vec x} \cdot \frac{\partial \vec k}{\partial \vec x} - \frac{\partial w}{\partial \vec x}\Big|_{\vec k} = -\vec{V_g} \cdot \vec\nabla k - \frac{\partial w}{\partial \vec x}\Big|_{\vec k}$$

Note that we're tracing the wave packet, so all evaluation must be done near the packet center $\vec k \simeq \vec k_0$. It implies, along the group velocity $\frac{d\vec x}{dt} = \vec V_g$.

$$\frac{\partial \vec k}{\partial t} + \vec{V_g} \cdot \vec\nabla k = \frac{\partial \vec k}{\partial t} + \frac{d\vec x}{dt} \cdot \vec\nabla k = \boxed{\frac{d\vec k}{dt} = -\frac{\partial w}{\partial \vec x}\Big|_{\vec k}}$$

This also implies along the group velocity :

$$\frac{dw}{dt} = \frac{\partial w}{\partial t} + \cancel{\frac{\partial w}{\partial \vec k}\Big|_{\vec x}\frac{d\vec k}{dt}} + \cancel{\frac{\partial w}{\partial \vec x}\Big|_{\vec k}\frac{d\vec x}{dt}} = \frac{\partial w}{\partial t} \qquad \boxed{\frac{dw}{dt} = \frac{\partial w}{\partial t}}$$

∴ <u>Hamiltonian nature of wave propagation</u>

$$\left[\begin{array}{cccc} \frac{d\vec k}{dt} = -\frac{\partial w}{\partial \vec x} & \leftrightarrow & \frac{d\vec p}{dt} = -\frac{\partial H}{\partial q} & : \vec k = \vec p \\ \frac{d\vec x}{dt} = \frac{\partial w}{\partial \vec k} & \leftrightarrow & \frac{d\vec q}{dt} = \frac{\partial H}{\partial \vec p} & : \vec x = \vec q \\ \frac{dw}{dt} = \frac{\partial w}{\partial t} & \leftrightarrow & \frac{dH}{dt} = \frac{\partial H}{\partial t} & : w = H \end{array}\right]$$

when we think of $\vec p = \hbar \vec k$, $H = E = \hbar w$. it makes sense.

④ General categories of waves

Variation of E,M field $\nearrow \vec{E_1}$ $\searrow \vec{B_1}$

1) Electrostatic : $\vec{B_1}=0$ ⇔ Electromagnetic : $\vec{B_1}\neq 0$

2) Longitudinal : $\vec{k}\,/\!/\,\vec{E_1}$ ⇔ Transverse : $\vec{k}\perp\vec{E_1}$

c.f.) For $\psi \sim e^{i(\vec{k}\cdot\vec{x}-\omega t)}$, longitudinal wave is electrostatic.

$$\vec{\nabla}\times\vec{E_1} = i\vec{k}\times\vec{E_1} = 0 = -\frac{\partial \vec{B_1}}{\partial t} = i\omega B_1$$

3) Parallel : $\vec{k}\,/\!/\,\vec{B_0}$ ⇔ Perpendicular : $\vec{k}\perp\vec{B_0}$

⑤ Langmuir wave (High frequency wave)

$$\begin{cases} \frac{\partial n_e}{\partial t} + \vec{\nabla}\cdot(n_e\vec{u_e}) = 0 \\ m_e n_e\left(\frac{\partial \vec{u_e}}{\partial t} + \vec{u_e}\cdot\vec{\nabla}\vec{u_e}\right) = -en_e\vec{E} - \vec{\nabla}P_e \\ \vec{\nabla}P_e = \gamma T_e \vec{\nabla}n_e \\ \epsilon_0\vec{\nabla}\cdot\vec{E} = e(n_i - n_e) \end{cases}$$

$n_e \to n_{e0} + n_{e1}$

$u_e \to u_{e0} + u_{e1}$

$P_e \to P_{e0} + P_{e1}$

$E \to E_0 + E_1$

$u_{e0} = 0$,

$T_{e1} = 0$, $n_{i1} = 0$

$\gamma_e = 3$ (1D, adiabatic)

$$\Downarrow$$

$$\begin{Bmatrix} -i\omega n_1 + ikn_0u_1 = 0 \\ i\omega mn_0u_1 = en_0E_1 + i3kTn_1 \\ ik\epsilon_0E_1 = -en_1 \end{Bmatrix} \Rightarrow \begin{bmatrix} -i\omega & ikn_0 & 0 \\ i3kT & -i\omega mn_0 & en_0 \\ e & 0 & ik\epsilon_0 \end{bmatrix}\begin{bmatrix} n_1 \\ u_1 \\ E_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

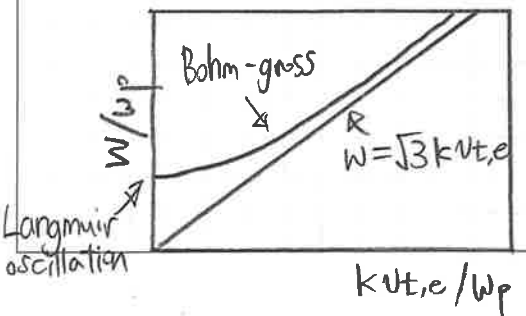
↑ let A

$$\det(A) = -i\omega^2 k mn_0\epsilon_0 + ikn_0^2e^2 + i3\overset{k^2}{\cancel{k^{3/2}}}\epsilon_0 n_0 T = 0$$

$$-\omega^2 mn_0\epsilon_0 + n_0^2e^2 + 3k^2\epsilon_0 n_0 T = 0$$

$$\omega^2 = \frac{ne^2}{m\epsilon_0} + 3k^2\frac{T_e}{m} \Rightarrow \boxed{\omega^2 = \omega_p^2 + 3k^2 v_{te}^2} \quad \left(\omega_p = \frac{ne^2}{m\epsilon_0},\ v_{t,e} = \sqrt{\frac{T_e}{m_e}}\right)$$

〈Bohm - Gross dispersion〉



(low k): $\omega^2 = \omega_{pe}^2$: pure plasma oscillation
(long wavelength)

$v_p = \omega/k \gg v_{t,e}$ even $v_p > c$

↳ Thermal conduction speed ≪ wave speed
: adiabatic

(short wavelength)
(high k) $\omega = \sqrt{3}kv_{t,e}$ → electron sound wave due to T_e

⑥ Low-frequency wave in unmagnetized plasmas

Now, let's consider ions. (previous, they were assumed to be fixed)

$E_1 = -ik\phi_1$, $n_e = n_0 \exp(e\phi_1/T_e) \approx n_0(1+e\phi/T_e) = n_0 + n_1.$

$$\left\{\begin{array}{l} -i\omega n_1 + ikn_0u_1 = 0 \\ i\omega Mn_0u_1 = en_0ik\phi_1 + i\gamma_i kTn_1 \\ \epsilon_0 k^2\phi_1 = e(n_1 - n_0(e\phi_1/T_e)) \end{array}\right\} \Rightarrow \begin{bmatrix} -\omega & kn_0 & 0 \\ \gamma_i kT_i & -\omega Mn_0 & en_0k \\ -e & 0 & \epsilon_0k^2 + \frac{n_0e^2}{T_e} \end{bmatrix}\begin{bmatrix} n_1 \\ u_1 \\ \phi_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\det(A) = \omega^2 Mn_0\epsilon_0\left(k^2 + \frac{n_0e^2}{\epsilon_0T_e}\right) - n_0^2e^2k^2 - \gamma_i k^2 n_0T_i\epsilon_0\left(k^2 + \frac{n_0e^2}{\epsilon_0T_e}\right) = 0$$

$$\omega^2 M\cancel{n_0}\epsilon_0\left(1 + \frac{1}{\lambda_D^2k^2}\right) - n_0^{\cancel{2}}e^2 - \gamma_i\cancel{n_0}T_i\epsilon_0k^2\left(1 + \frac{1}{\lambda_D^2k^2}\right) = 0$$

$$\omega^2\left(1 + \frac{1}{\lambda_D^2k^2}\right) - \frac{n_0e^2}{M\epsilon_0} - \frac{\gamma_iT_i}{M}k^2\left(1 + \frac{1}{\lambda_D^2k^2}\right) = 0$$

$$\frac{\omega^2}{k^2} = \frac{\gamma_iT_i}{M} + \frac{n_0e^2}{M\epsilon_0}\frac{1}{\cancel{k^2}}\cdot\frac{\lambda_D^2\cancel{k^2}}{1+\lambda_D^2k^2} = \frac{\gamma_iT_i}{M} + \frac{\cancel{n_0e^2}}{M\cancel{\epsilon_0}}\,\frac{\cancel{\epsilon_0}T_e}{\cancel{n_0e^2}}\,\frac{1}{1+\lambda_D^2k^2}$$

$$\therefore \boxed{\frac{\omega^2}{k^2} = \frac{T_e/M}{1+k^2\lambda_D^2} + \frac{\gamma_iT_i}{M}}$$

- In the limit of long wave length (low k), $\boxed{\frac{\omega^2}{k^2} = \frac{T_e}{M} + \frac{\gamma_iT_i}{M}}$

  < ion sound wave >.

In many laboratory plasma ($T_i \ll T_e$)

$\boxed{\frac{\omega^2}{k^2} \approx C_s = \frac{T_e}{M}}$ ion sound speed = (electron temperature / ion mass)$^{1/2}$

- In the limit of short wave length (high k) $\boxed{\omega^2 = \frac{T_e/M}{\lambda_D^2} = \omega_{pi}^2 \equiv \frac{ne^2}{\epsilon_0 M}}$

Note when $\omega > \omega_{pe}$ $(k\uparrow) \rightarrow$ (oscillation $\rightarrow$ e^- sound wave)

$\omega < \omega_{pi}$ $(k\uparrow) \rightarrow$ (ion sound wave $\rightarrow$ oscillation)

<summary>

⇨ electron $\omega^2 = \omega_p^2 + 3k^2v_{t,e}^2$, ⇨ ion $\omega^2/k^2 = \frac{T_e/M}{1+k^2\lambda_D^2} + \frac{\gamma_iT_i}{M}$

⑰ Electromagnetic waves in unmagnetized plasmas

$\vec{B}_1 \neq 0$

$$\begin{aligned}\text{Ampere's}&:\\ \text{Faraday's}&:\end{aligned}\left(\begin{aligned}\vec{\nabla}\times\vec{E}&=-\frac{\partial\vec{B}}{\partial t}\\ \vec{\nabla}\times\vec{B}&=\mu_0\vec{j}+\frac{1}{c^2}\frac{\partial\vec{E}}{\partial t}\end{aligned}\right)\Rightarrow\left(\begin{aligned}i\vec{k}\times\vec{E}_1&=i\omega\vec{B}_1\\ i\vec{k}\times\vec{B}_1&=\mu_0\vec{j}_1-i\omega\vec{E}_1/c^2\end{aligned}\right)\Longrightarrow \text{take } \vec{k}\times$$

$$\vec{k}\times i(\vec{k}\times\vec{E}_1)=i\omega\,\vec{k}\times\vec{B}_1$$

$$-i\left(k^2\vec{E}_1-\vec{k}(\vec{k}\cdot\vec{E}_1)\right)=\omega\left(\mu_0\vec{j}_1-i\omega\vec{E}_1/c^2\right)$$

$$\underline{k^2\vec{E}_1-\vec{k}(\vec{k}\cdot\vec{E}_1)=\frac{\omega^2}{c^2}\left(\vec{E}_1+i\frac{\vec{j}_1}{\epsilon_0\omega}\right)}$$

To penerate the plasma,
$\omega > \omega_{pe} = \frac{ne^2}{m\epsilon_0}$
Thus, it's harder for wave to penetrate dense plasma. (core)
(cutoff!)

Let's only consider $\vec{k}\perp\vec{E}_1$. thus $\vec{k}\cdot\vec{E}_1=0$
Now we should express $\vec{j}_1$ using $\vec{E}_1$.

Consider $\vec{E}$ perturbation → Electron motion → current
(consider only high-frequency wave, ignoring ion motion)

$$\frac{d}{dt}(m\vec{u}_1)=-e\vec{E}_1\rightarrow\left(\begin{aligned}-i\omega m\vec{u}_1&=-e\vec{E}_1\\ \vec{j}_1&=-en_0\vec{u}_1\end{aligned}\right)\rightarrow\underline{\vec{j}_1=\frac{n_0e^2}{m\omega}i\vec{E}_1}$$

$$\Rightarrow k^2\vec{E}_1=\frac{\omega^2}{c^2}\left(\vec{E}_1-\frac{ne^2}{m\epsilon_0}\frac{1}{\omega^2}\vec{E}_1\right)\Rightarrow\omega^2=c^2k^2+\omega_{pe}^2\qquad\boxed{\therefore\omega^2=\omega_{pe}^2+c^2k^2}$$

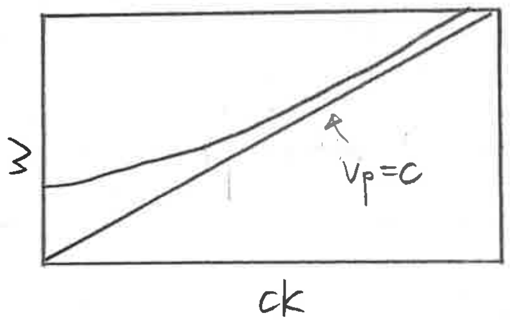
<EM plasma wave in unmagnetized plasma>

- Phase velocity of EM plasma wave

$$v_p=\omega/k=c\left(1+\omega_{pe}^2/c^2k^2\right)^{1/2}>c$$

- Group velocity of EM plasma wave

$$v_g=\frac{d\omega}{dk}=\frac{c}{\left(1+\omega_{pe}^2/c^2k^2\right)^{1/2}}<c$$



- <u>Cutoff density</u>

$$\omega^2\geq\omega_{pe}^2=\frac{ne^2}{m\epsilon_0}\rightarrow\boxed{n_c=\frac{m\epsilon_0\omega^2}{e^2}}$$

- Collisionless Skin-depth

$$k=(\omega^2-\omega_{pe}^2)^{1/2}/c=\pm i(\omega_{pe}^2-\omega^2)^{1/2}/c\ ,\quad\psi\sim e^{ikx}=e^{-x\frac{(\omega_{pe}^2-\omega^2)^{1/2}}{c}}\Rightarrow\boxed{\delta_e\equiv\frac{c}{\omega_{pe}}}$$