Fusion Plasma Theory 2

Lecture 3 : Coulomb Scattering.

① Elastic Kinematics

Two particle species (a.b) gives $\vec{u} = \vec{v_a} - \vec{v_b}$ and $\Delta\vec{u} = \Delta\vec{v_a} - \Delta\vec{v_b}$

(1) Momentum conservation

$$m_a\Delta\vec{v_a} + m_b\Delta\vec{v_b} = 0 \rightarrow \begin{cases} \underline{\Delta\vec{p_a} = m_a\Delta\vec{v_a} = m_r\Delta\vec{u}} \\ \underline{\Delta\vec{p_b} = m_b\Delta\vec{v_b} = -m_r\Delta\vec{u}} \end{cases}$$

$$\underline{\left(m_r : \text{reduced mass} = \frac{m_a m_b}{m_a+m_b}\right)}$$

(2) Energy conservation

$$(\vec{u}+\Delta\vec{u})\cdot(\vec{u}+\Delta\vec{u}) = u^2 \rightarrow \underline{2\vec{u}\cdot\Delta\vec{u} + (\Delta\vec{u})^2 = 0}$$

(3) Energy change of two particle species

$$\underline{\Delta\epsilon_a = -\Delta\epsilon_b = m_r\vec{V}\cdot\Delta\vec{u}}$$

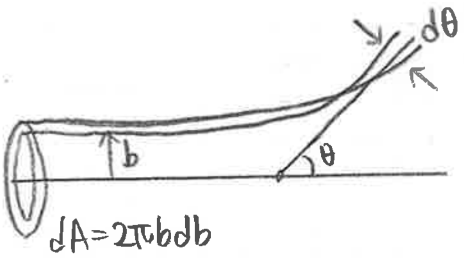
(proof) $\Delta\epsilon_a = \frac{1}{2}m_a(\vec{v_a}+\Delta\vec{v_a})^2 - \frac{1}{2}m_a v_a^2 = m_a\vec{v_a}\cdot\Delta\vec{v_a} + \frac{1}{2}m_a(\Delta\vec{v_a})^2$

(using $m_a\Delta v_a = m_r\Delta u$) $\rightarrow$ $= m_r\vec{v_a}\Delta\vec{u} + \frac{1}{2}m_a\left(\frac{m_r}{m_a}\Delta u\right)^2 = m_r\vec{v_a}\Delta\vec{u} + \frac{1}{2}\frac{m_r^2}{m_a}(\Delta u)^2$

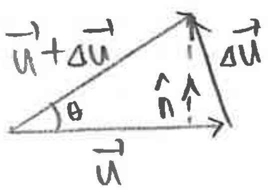
(using $2\vec{u}\cdot\Delta\vec{u} + (\Delta\vec{u})^2 = 0$) $\rightarrow$ $= m_r\vec{v_a}\Delta\vec{u} - \frac{m_r^2}{m_a}\vec{u}\cdot\Delta\vec{u} = m_r\Delta\vec{u}\cdot\left(\vec{v_a} - \frac{m_r}{m_a}(\vec{v_a}-\vec{v_b})\right)$

$= \underline{m_r\Delta\vec{u}\cdot\vec{V}}$ (where $\vec{V} = \frac{m_a\vec{v_a}+m_b\vec{v_b}}{m_a+m_b}$: COM velocity)

② Momentum and energy change along with the deflection angle



$$\begin{cases} \Delta\vec{u} = \hat{n}(u\sin\theta) - \vec{u}(1-\cos\theta) \\ \Delta\vec{p_a} = m_r\Delta\vec{u} = \hat{n}(m_r u\sin\theta) - m_r\vec{u}(1-\cos\theta) \\ \Delta\epsilon_a = m_r\vec{V}\cdot\Delta\vec{u} \\ \quad = (\hat{n}\cdot\vec{V})(m_r u\sin\theta) - m_r(\vec{u}\cdot\vec{V})(1-\cos\theta) \end{cases}$$



③ $p_{\parallel}$, $p_{\perp}$, ε exchange with a fixed target

(Assume) $\vec{v_b} = 0$, (Then) $\vec{u} = \vec{v_a}$, $\vec{V} = \frac{m_r}{m_b}\vec{u}$, $\Delta\vec{p} = m_r \Delta\vec{u}$

(1) Parallel momentum change (Slowing down)

$$\frac{\Delta p_{\parallel}}{p_{\parallel}} = \frac{m_r \Delta\vec{u}\cdot\vec{u}}{m_a u^2} = -\frac{m_r}{m_a}(1-\cos\theta) = -\frac{m_b}{m_a+m_b}(1-\cos\theta)$$

$$\left(* \text{ Also } \Delta p_{\parallel} = \frac{\Delta\vec{p}\cdot\vec{u}}{u} = \frac{m_r \Delta\vec{u}\cdot\vec{u}}{u} = -\frac{m_r}{2u}(\Delta\vec{u})^2 = -\frac{(\Delta\vec{p})^2}{2m_r u}\right.$$

$\Rightarrow$ $\Delta p_{\parallel}$ is in higher order than the $\Delta p_{\perp}$ and Δp)

(2) Perpendicular momentum change (diffusion)

$$\frac{(\Delta p_{\perp})^2}{p^2} = \frac{(m_r \Delta u)^2}{(m_a\vec{u})^2} = \frac{m_r^2(\Delta u)^2}{m_a^2 u^2} = -\frac{2m_r^2 \Delta\vec{u}\cdot\vec{u}}{m_a^2 u^2} = \frac{2m_r}{m_a}\left(\frac{\Delta p_{\parallel}}{p_{\parallel}}\right) = \frac{2m_b}{m_a+m_b}\left(\frac{\Delta p_{\parallel}}{p_{\parallel}}\right)$$

(3) Energy change (transfer)

$$\frac{\Delta\varepsilon}{\varepsilon} = \frac{-m_r(\vec{u}\cdot\vec{V})(1-\cos\theta)}{\frac{1}{2}m_a u^2} = \frac{-m_r \frac{m_r}{m_b} u^2(1-\cos\theta)}{\frac{1}{2}m_a u^2} = -\frac{2m_r^2}{m_a m_b}(1-\cos\theta) = \frac{2m_a}{m_a+m_b}\left(\frac{\Delta p_{\parallel}}{p_{\parallel}}\right)$$

{ 3 characteristic collisional frequencies become identical for like-particle collision.

$\nu_{ee}^{\perp} = \nu_{ee}^{\varepsilon} = \nu_{ee}$, $\nu_{ii}^{\perp} = \nu_{ii}^{\varepsilon} = \nu_{ii}$

(a) electron, (b) ion $\rightarrow$ $\nu_{ei}^{\perp} = 2\nu_{ei}$, $\nu_{ei}^{\varepsilon} = \left(\frac{2m_e}{m_i}\right)\nu_{ei}$

(a) ion, (b) electron $\rightarrow$ $\nu_{ie}^{\perp} = \left(\frac{2m_e}{m_i}\right)\nu_{ie}$, $\nu_{ie}^{\varepsilon} = 2\nu_{ie}$ }

Summary: $\boxed{\nu_{ab}^{\perp} = \frac{2m_b}{m_a+m_b}\nu_{ab} , \quad \nu_{ab}^{\epsilon} = \frac{2m_a}{m_a+m_b}\nu_{ab}}$ $\boxed{\frac{\Delta p_{\parallel}}{p_{\parallel}} = -\frac{m_b}{m_a+m_b}(1-\cos\theta)}$

④ Differential cross-section by central force.

(1) Constants of motion

$$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - U(r) \;\rightarrow\; \frac{\partial L}{\partial \theta} = 0 \;\rightarrow\; \ell = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta}$$

$$h = E = \sum \vec{p}\cdot\dot{\vec{q}} - L = \frac{\partial L}{\partial \dot{\theta}}\dot{\theta} + \frac{\partial L}{\partial \dot{r}}\dot{r} - L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r)$$

(2) Formal solution of orbit geometry.

$$\ell = \mu r^2 \dot{\theta} \;\rightarrow\; dt = \frac{\mu r^2}{\ell} d\theta = \frac{\mu}{\ell x^2} d\theta \qquad (x = 1/r)$$

$$E = \frac{1}{2}\mu\dot{r}^2 + \frac{\ell^2}{2\mu r^2} + U(r) = \frac{\ell^2}{2\mu}\left(\frac{dx}{d\theta}\right)^2 + \frac{\ell^2}{2\mu}x^2 + U(x^{-1})$$

$$\left(\frac{dr}{dt} = \left(-\frac{1}{x^2}\right)\frac{dx}{dt} = \left(-\frac{1}{x^2}\right)\frac{dx}{1}\cdot\frac{\ell x^2}{\mu d\theta} = -\frac{\ell}{\mu}\frac{dx}{d\theta}\right)$$

$$\therefore \left(\frac{dx}{d\theta}\right)^2 = \frac{2\mu}{\ell^2}\left[E - U(x^{-1})\right] - x^2 \;\rightarrow\; \frac{dx}{d\theta} = -\left\{\frac{2\mu}{\ell^2}\left[E - V(x^{-1})\right] - x^2\right\}^{1/2}$$

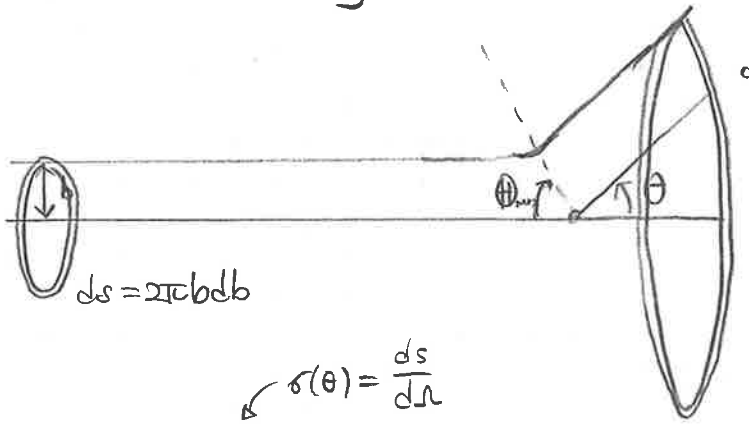
$$\therefore \theta = -\int_{r_{min}}^{r^{-1}} \frac{dx}{\left\{\frac{2\mu}{\ell^2}\left[E - V(x^{-1})\right] - x^2\right\}^{1/2}} = -\int_{r_{min}}^{r^{-1}} \frac{dx}{\sqrt{\frac{2\mu E}{\ell^2} - \frac{2\mu k}{\ell^2}x - x^2}}$$

$$\left(V(x^{-1}) = -k/r = -kx\right)$$

$$= \arccos \left.\frac{2x + \frac{2\mu k}{\ell^2}}{\sqrt{\frac{4\mu^2 k^2}{\ell^4} + \frac{8\mu E}{\ell^2}}}\right|_{x = x_{min}}^{x = x} = \arccos \frac{\frac{\ell^2}{\mu k}x + 1}{\sqrt{1 + \frac{2E\ell^2}{\mu k^2}}} = \arccos \frac{\frac{\ell^2}{\mu k r} + 1}{e}$$

$$\therefore r = \frac{\ell^2}{\mu k}\,\frac{1}{e\cos\theta - 1}$$

(3) Differential scattering cross-section in central force.



$d\Omega = 2\pi \sin\theta\, d\theta$

$(\theta = \pi - 2\Theta_{max})$

$\sigma(\theta) = \dfrac{ds}{d\Omega}$

$$2\pi b\,db = \sigma(\theta)\,d\Omega = 2\pi\sigma(\theta)\sin\theta\,d\theta \longrightarrow \sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|$$

i) $\sin\dfrac{\theta}{2} = \sin\left(\dfrac{\pi}{2} - \Theta_{max}\right) = \cos\Theta_{max} = \dfrac{1}{e}$

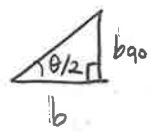
ii) $\cot^2\dfrac{\theta}{2} = \csc^2\dfrac{\theta}{2} - 1 = e^2 - 1 = \dfrac{2El^2}{mk^2} = \dfrac{2Eb^2 2mE}{mk^2} = \dfrac{4b^2E^2}{k^2} \rightarrow \cot\dfrac{\theta}{2} = \dfrac{2Eb}{k}$

$l = bmv_0 = b\sqrt{2mE}$

$b = \dfrac{k}{2E}\cot\dfrac{\theta}{2} = b_{90}\cot\dfrac{\theta}{2}$

differentiate

iii) $-\csc^2\dfrac{\theta}{2}\dfrac{d\theta}{2} = \dfrac{2E}{k}db \rightarrow \dfrac{db}{d\theta} = -\dfrac{k}{4E}\csc^2\dfrac{\theta}{2}$



iv) $\sin\theta = 2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$

$$\Rightarrow \sigma(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| = \left(\frac{k}{2E}\cot\frac{\theta}{2}\right)\left(\frac{1}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}\right)\left(\frac{k}{4E}\csc^2\frac{\theta}{2}\right)$$

$$\boxed{\sigma(\theta) = \frac{k^2}{16E^2}\frac{1}{\sin^4\frac{\theta}{2}} = \frac{b_{90}^2}{4\sin^4\frac{\theta}{2}}}$$

⑤ Collision frequency and coulomb logarithm

$$\nu = n_b v_a \int 2\pi b\, db = n_b v_a \int 2\pi \sigma(\theta) \sin\theta\, d\theta \qquad (\nu = n_b v_a \sigma)$$

↑ conventional collision frequency.

In Coulomb scattering, let's define

$$\nu_{ab} = n_b v_a \int 2\pi b\, db \left(-\frac{\Delta p_{\parallel}}{p}\right) = n_b v_a \int 2\pi \sigma(\theta) \sin\theta\, d\theta \left(-\frac{\Delta p_{\parallel}}{p}\right)$$

(by considering the fact that as b increases, slowing down decreases)

Using $\frac{\Delta p_{\parallel}}{p} = -\frac{m_b}{m_a+m_b}(1-\cos\theta)$ and fixed target b ($\vec{u} = \vec{v_a}$),

$$\Rightarrow \nu_{ab} = n_b u \left(\frac{m_r}{m_a}\right) \int 2\pi \sigma(\theta) \sin\theta (1-\cos\theta)\, d\theta$$

Using $\sigma(\theta) = b_{90}^2 / 4\sin^4\frac{\theta}{2}$,

$$\Rightarrow \nu_{ab} = n_b u \left(\frac{m_r}{m_a}\right) 2\pi b_{90}^2 \int \frac{\sin\theta (1-\cos\theta)}{4\sin^4\frac{\theta}{2}}\, d\theta \qquad \frac{\sin\theta(1-\cos\theta)}{4\sin^4\frac{\theta}{2}} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot 2\sin^2\frac{\theta}{2}}{4\sin^4\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

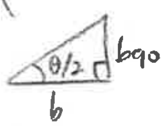
$$\Rightarrow \nu_{ab} = n_b u \left(\frac{m_r}{m_a}\right) 2\pi b_{90}^2 \int \cot\frac{\theta}{2}\, d\theta$$

✱ one can write this as

$$\nu_{ab} = n_b u \sigma_{ab}, \quad \sigma_{ab} = \sigma_{90}\left(2\int \cot\frac{\theta}{2} d\theta\right), \quad \sigma_{90} = \left(\frac{m_r}{m_a}\right)\pi b_{90}^2$$

✱ coulomb logarithm is defined as

$$\frac{1}{2}\int \cot\left(\frac{\theta}{2}\right) d\theta = \ln\left(\sin\frac{\theta}{2}\right)\Big|_{\theta=\theta_{min}}^{\theta=\theta_{max}} \equiv \ln\Lambda = \frac{1}{2}\ln\left(\frac{1+(b_{max}/b_{90})^2}{1+(b_{min}/b_{90})^2}\right)$$



One can write.

$$\sigma_{ab} = \sigma_{90}(4\ln\Lambda)$$

∴ σ_{ab} and ν_{ab} increase by a factor of $4\ln\Lambda$, compared to 90° (large-angle) scattering, comes dominantly from the accumulated "small-angle" scattering.

ibis

⑥ Evaluation of Coulomb Logarithm

(Assume) $b_{min} = \begin{cases} 0 & \text{(ion)} \\ \underbrace{\hbar / m_r u}_{\text{deBrogile wave. } (\lambda = \frac{h}{p})} & \text{(electron)} \end{cases}$, $b_{max} = \lambda_D = \sqrt{\frac{\varepsilon_0 T}{n e^2}}$ $\Rightarrow \Lambda = \frac{\lambda_D}{b_{90}}$

(Assume) $m_r u^2 = 3T$, $e_a = e_b = e$, $b_{90} = \frac{k}{2E} = \frac{e^2}{4\pi\varepsilon_0} \frac{1}{3T}$

$$\Rightarrow \underline{\Lambda = \left(\frac{\varepsilon_0 T}{n_e e^2}\right)^{1/2} \left(\frac{e^2}{12\pi\varepsilon_0 T}\right)^{-1} = 12\pi n_e \lambda_D^3 = 9 N_D} \quad (N_D = \tfrac{4}{3}\pi n_e \lambda_D^3)$$

(* Λ is very large ($\because N_D \gg 1$), but $\ln\Lambda$ changes little.)

- Ion-ion coulomb logarithm.

$$\ln\Lambda = 17.3 - \frac{1}{2}\ln(n_e/10^{20}) + \frac{3}{2}\ln T\,[\mathrm{keV}] \quad \sim (17.20)$$

- $e^- - e^-$ coulomb logarithm

$$\ln\Lambda = 14.9 - \frac{1}{2}\ln(n_e/10^{20}) + \ln T_e\,[\mathrm{keV}]$$

- e^--ion coulomb logarithm

for fusion plasma

$$\ln\Lambda = 15.2 - \frac{1}{2}\ln(n_e/10^{20}) + \ln T_e\,[\mathrm{keV}] \quad \sim (17.18)$$

⑦ Slowing-down collisional frequency

All together, $\left(\nu_{ab} = n_b u \sigma_{ab},\ \sigma_{ab} = \sigma_{90}(4\ln\Lambda),\ \sigma_{90} = \left(\frac{m_r}{m_a}\right)\pi b_{90}^2 \right.$
$\left. b_{90} = \frac{e^2}{4\pi\varepsilon_0}\frac{1}{3T},\ m_r u^2 = 3T \right)$

$$\Rightarrow \boxed{\nu_{ab}} = n_b u (4\ln\Lambda)\left(\frac{m_r}{m_a}\right)\pi \left(\frac{e^2}{4\pi\varepsilon_0}\frac{1}{m_r u^2}\right)^2 = \boxed{\frac{n_b e_a^2 e_b^2 \ln\Lambda}{4\pi\varepsilon_0^2 m_r m_a u^3} = \frac{n_b e_a^2 e_b^2 m_r^{1/2} \ln\Lambda}{12\sqrt{3}\pi\varepsilon_0^2 m_a T_a^{3/2}}}$$

⑧ Accounting Maxwell distribution of incident particles.

$$\vec{R}_{ab} = -n_a m_a \langle \nu_{ab} \vec{v_a} \rangle = -n_a m_a \hat{\nu}_{ab} \vec{V_a} \quad (\vec{V_a} : \text{fluid velocity})$$

∴ We want to know this!

Let $\vec{V_a} = V_z \hat{z}$ w/o loss of generality.

Proper shifted Maxwellian distribution becomes

$$f_a = \frac{n_a}{(\sqrt{\pi} v_{ta})^3} \exp\left(- \frac{|\vec{v_a} - \vec{V_a}|^2}{v_{ta}^2} \right)$$

$$= \frac{n_a}{(\sqrt{\pi} v_{ta})^3} \exp\left(-\frac{v_a^2}{v_{ta}^2}\right) \exp\left(\frac{2\vec{v_a}\cdot\vec{V_a} - V_a^2}{v_{ta}^2}\right) = f_{a0}\left(1 + 2\frac{V_z v_z}{v_{ta}^2}\right)$$

$\frac{V_a}{v_{ta}} \ll 1$

(주어진 온도 v_{ta} 에서의 v_a 의 분포함수)

Averaging,

$$\langle \nu_{ab} \vec{v_a} \rangle = \frac{\int f_a \nu_{ab} \vec{v_a} d\vec{v_a}}{\int f_a d\vec{v_a}} = \hat{z} V_z \left[\frac{1}{n_a} \int d\vec{v}\, \nu_{ab} \left(\frac{2v_z^2}{v_{ta}^2}\right) f_{a0} \right]$$

$$\therefore \hat{\nu}_{ab} = \left[\frac{1}{n_a} \int d\vec{v}\, \nu_{ab} \left(\frac{2v_z^2}{v_{ta}^2}\right) f_{a0} \right] \quad : \text{Fluid slowing down frequency}$$

Using the isotropy in f_{a0}, $v_z^2 = v^2/3$, $u = v$ for the fixed target,

$$\hat{\nu}_{ab} = \frac{n_b e_a^2 e_b^2 \ln\Lambda}{4\pi\epsilon_0^2 m_r m_a} \int_0^\infty \frac{2}{3 v v_{ta}^2} \frac{1}{(\sqrt{\pi} v_{ta})^3} e^{-\frac{v^2}{v_{ta}^2}} (4\pi v^2 dv)$$

$$\boxed{\hat{\nu}_{ab} \equiv \frac{2^{1/2} n_b e_a^2 e_b^2 m_r^{1/2} \ln\Lambda}{12\pi^{3/2} \epsilon_0^2 m_a T_a^{3/2}}}$$

Integration by substitution.

This is what's to be used in Krook form : $\vec{R}_{ab} = -n_a m_a \hat{\nu}_{ab} \vec{V_a}$

Definition of collisional frequencies in community.

- Electron - ion $\hat{\nu}_{ei} \equiv \dfrac{2^{1/2} n_i Z^2 e^4 \ln\Lambda}{12\pi^{3/2} \epsilon_0^2 m_e^{1/2} T_e^{3/2}}$

- Electron - electron $\hat{\nu}_{ee} \equiv \dfrac{n_e e^4 \ln\Lambda}{12\pi^{3/2} \epsilon_0^2 m_e^{1/2} T_e^{3/2}}$ *note ($m_r^{1/2} = (\frac{1}{2})^{1/2}$)

- Ion - ion $\hat{\nu}_{ii} \equiv \dfrac{n_i Z^4 e^4 \ln\Lambda}{12\pi^{3/2} \epsilon_0^2 m_i^{1/2} T_i^{3/2}}$

- Ion - electron $\hat{\nu}_{ie} \equiv \dfrac{2^{1/2} n_e Z^2 e^4 m_e^{1/2} \ln\Lambda}{12\pi^{3/2} \epsilon_0^2 m_i T_i^{3/2}}$

Collision times.

- Electron collision time : $\tau_e \equiv \dfrac{1}{\hat{\nu}_{ei}} = 3(2\pi)^{3/2} \dfrac{\epsilon_0^2 m_e^{1/2} T_e^{3/2}}{n_i Z^2 e^4 \ln\Lambda}$

- Ion collision time : $\tau_i \equiv \dfrac{1}{\hat{\nu}_{ii}} = 12\pi^{3/2} \dfrac{\epsilon_0^2 m_i^{1/2} T_i^{3/2}}{n_i Z^4 e^4 \ln\Lambda}$

- Heat exchange time : $\tau_{ie} = \tau_{ei} = 3\sqrt{2}\,\pi^{3/2} \dfrac{\epsilon_0^2 m_i T_e^{3/2}}{n_i m_e^{1/2} Z^2 e^4 \ln\Lambda}$ ← How?