Ch10. Non linear constituitive equation

10.2 The theory of finite elastic deformation

Linear theory of elasticity is limited to cases the deformation gradient $\mathbb{F}$ are small.

Rubbers : behave elastically while undergoing large deformation.

$W = \rho_0 e$: strain-energy function

$W = W(F_{iR}) = W(\mathbb{F})$ ← arbitrary dependence on the deformation.

<u>Recall</u> $\dfrac{D}{Dt}F_{iR} = \dfrac{D}{Dt}\left(\dfrac{dx_i}{dX_R}\right) = \dfrac{dv_i}{dX_R} = \dfrac{dv_i}{dx_j}\dfrac{dx_j}{dX_R} = L_{ij}F_{jR}$

$$\rho\frac{De}{Dt} = T_{ij}\frac{dv_i}{dx_j} - \cancel{\frac{dq_j}{dx_j}}$$ 무시

$$\Rightarrow T_{ij}\frac{dv_i}{dx_j} = \frac{\rho}{\rho_0}\frac{DW}{Dt} = \frac{\rho}{\rho_0}\frac{dW}{dF_{iR}}\frac{DF_{iR}}{Dt} = \frac{\rho}{\rho_0}\frac{dW}{dF_{iR}}\frac{dx_j}{dX_R}\frac{dv_i}{dx_j} \quad \left(\text{for all } \frac{dv_i}{dx_j}\right)$$

$$\hookrightarrow \therefore \boxed{T_{ij} = \frac{\rho}{\rho_0}\frac{dW}{dF_{iR}}\frac{dx_j}{dX_R}}$$ Constituitive equation for finite elasticity.

<u>$W(F_{iR})$ shouldn't change</u> if a <u>rigid-body rotation</u> is superposed on the deformation

$\bar{x} = \mathbb{M}\cdot x$ ← rotation

$$\bar{F}_{iR} = \frac{d\bar{x}_i}{dX_R} = M_{ij}\frac{dx_j}{dX_R} = M_{ij}F_{jR} \quad \text{or} \quad \bar{\mathbb{F}} = \mathbb{M}\cdot\mathbb{F}$$ ← (좌표계가 아닌, 실제 물체를 돌린 것)

$\hookrightarrow$ <u>$W(\mathbb{F}) = W(\mathbb{M}\cdot\mathbb{F})$</u> for any $\mathbb{M}$.

$W(\mathbb{F}) = W(\mathbb{M}\cdot\mathbb{R}\cdot\mathbb{U})$ for any $\mathbb{M}$, especially $\mathbb{M} = \mathbb{R}^T$

$= W(\mathbb{U}) \longrightarrow \boxed{W = W(\mathbb{C})}$ 결론: rigid-body motion $\longrightarrow W = W(\mathbb{C})$

↳ 6 components

$$\frac{DW}{Dt} = \frac{dW}{dC_{RS}}\frac{DC_{RS}}{Dt} = \frac{dW}{dC_{RS}}\frac{D}{Dt}\left(\frac{dx_i}{dX_R}\frac{dx_i}{dX_S}\right) = \frac{dW}{dC_{RS}}\left(\frac{dv_i}{dX_R}\frac{dx_i}{dX_S} + \frac{dx_i}{dX_R}\frac{dv_i}{dX_S}\right)$$

$$= \left(\frac{dW}{dC_{RS}} + \frac{dW}{dC_{SR}}\right)\frac{dx_i}{dX_R}\frac{dv_i}{dX_S} = \left(\frac{dW}{dX_{RS}} + \frac{dW}{dX_{SR}}\right)\frac{dx_i}{dX_R}\frac{dx_j}{dX_S}\frac{dv_i}{dx_j}$$

Since $T_{ij}\dfrac{dv_i}{dx_j} = \dfrac{\rho}{\rho_0}\dfrac{D}{Dt}W \longrightarrow \boxed{T_{ij} = \dfrac{\rho}{\rho_0}\dfrac{dx_i}{dX_R}\dfrac{dx_j}{dX_S}\left(\dfrac{dW}{dX_{RS}} + \dfrac{dW}{dX_{SR}}\right)}$ Required general form of Constituitive equation for finite elasticic solid.

Recall that $\mathbb{T\!I} = (\det\mathbb{F})\,\mathbb{F}^{-1}\,\mathbb{T}$

From $T_{ij} = \dfrac{\rho}{\rho_0}\dfrac{dW}{dF_{iR}}F_{jR}$, $T_{ji} = \dfrac{\rho}{\rho_0}F_{iR}\dfrac{dW}{dF_{jR}} \longrightarrow \left[(\det\mathbb{F})\,\mathbb{F}^{-1}\cdot\mathbb{T}\right]_{Ri} = \dfrac{dW}{dF_{iR}} = \pi_{Ri}$

Also, $\mathbb{P} = \mathbb{T\!I}\cdot(\mathbb{F}^{-1})^T = (\det\mathbb{F})\,\mathbb{F}^{-1}\cdot\mathbb{T}\cdot(\mathbb{F}^{-1})^T$

From $T_{ij} = \dfrac{\rho}{\rho_0}\dfrac{dx_i}{dX_R}\left(\dfrac{dW}{dC_{RS}} + \dfrac{dW}{dC_{SR}}\right)\dfrac{dx_i}{dX_S}$, $P_{RS} = \left(\dfrac{dW}{dC_{RS}} + \dfrac{dW}{dC_{SR}}\right)$

Material symm: If rotational symmetry by $\mathbb{Q}$

$$\mathbb{F} \to \mathbb{Q}^T \mathbb{F} \mathbb{Q} \quad \Rightarrow \quad \mathbb{C} = \mathbb{F}^T \mathbb{F} \to \mathbb{Q}^T \mathbb{C} \mathbb{Q}$$

$W(\mathbb{C}) = W(\mathbb{Q}^T \mathbb{C} \mathbb{Q})$ (if isotropic, this holds for all $\mathbb{Q}$)

↳ W is invariant of $\mathbb{C}$ → $W = W(I_1, I_2, I_3)$ Similarly, $W(\mathbb{R}) = W(\mathbb{Q}^T \mathbb{R} \mathbb{Q})$

$$\frac{dW}{dC_{RS}} = \frac{dW}{dI_1}\frac{dI_1}{dC_{RS}} + \frac{dW}{dI_2}\frac{dI_2}{dC_{RS}} + \frac{dW}{dI_3}\frac{dI_3}{dC_{RS}}$$

$$\left(\begin{array}{l} I_1 = \operatorname{tr}\mathbb{C} = C_{RR} \\ I_2 = \frac{1}{2}\{(\operatorname{tr}\mathbb{C})^2 - \operatorname{tr}\mathbb{C}^2\} = \frac{1}{2}\{C_{RR}C_{SS} - C_{RS}C_{RS}\} \\ I_3 = \det \mathbb{C} \end{array}\right)$$

$$\frac{dI_1}{dC_{RS}} = \frac{dC_{PP}}{dC_{RS}} = \delta_{PR}\delta_{PS} = \delta_{RS}$$

$$\frac{dI_2}{dC_{RS}} = \frac{1}{2}\frac{d}{dC_{RS}}\{C_{PP}C_{QQ} - C_{PQ}C_{PQ}\} = \frac{1}{2}\{\delta_{PR}\delta_{PS}C_{QQ} + \delta_{QR}\delta_{QS}C_{PP} - 2\delta_{PR}\delta_{QS}C_{PQ}\}$$

$$= \delta_{RS}C_{PP} - C_{RS} = I_1\delta_{RS} - C_{RS}$$

$$I_3 = \frac{1}{3}\{\operatorname{tr}\mathbb{C}^3 - I_1\operatorname{tr}\mathbb{C}^2 + I_2\operatorname{tr}\mathbb{C}\} = \frac{1}{3}\{C_{AB}C_{BC}C_{CA} - I_1 C_{AB}C_{BA} + I_2 C_{AA}\}$$

$$\frac{dI_3}{dC_{RS}} = \frac{1}{3}\frac{d}{dC_{RS}}\{C_{AB}C_{BC}C_{CA} - I_1 C_{AB}C_{BA} + I_2 C_{AA}\}$$

$$= C_{RP}C_{SP} - I_1 C_{RS} + I_2\delta_{RS}$$

$$\boxed{T_{ij} = 2\frac{\rho}{\rho_0}\frac{dx_i}{dX_R}\frac{dx_j}{dX_S}\left\{\left(\frac{dW}{dI_1} + I_1\frac{dW}{dI_2} + I_2\frac{dW}{dI_3}\right)\delta_{RS} - \left(\frac{dW}{dI_2} + I_1\frac{dW}{dI_3}\right)C_{RS} + \frac{dW}{dI_3}C_{RP}C_{PS}\right\}}$$

Constitutive equation for an isotropic finite elastic solid.

$$\mathbb{T} = 2(I_3)^{-\frac{1}{2}}\mathbb{F}\cdot\{(W_1 + I_1W_2 + I_2W_3)\mathbb{I} - (W_2 + I_1W_3)\mathbb{C} + W_3\mathbb{C}^2\}\cdot\mathbb{F}^T$$

(using $\mathbb{B} = \mathbb{F}\cdot\mathbb{F}^T$ & $\mathbb{C} = \mathbb{F}^T\mathbb{F}$)

$$\mathbb{T} = 2(I_3)^{-\frac{1}{2}}\{(W_1 + I_1W_2 + I_2W_3)\mathbb{B} - (W_2 + I_1W_3)\mathbb{B}^2 + W_3\mathbb{B}^3\}$$

(using $\mathbb{B}^3 - I_1\mathbb{B}^2 + I_2\mathbb{B} - I_3\mathbb{I} = 0$, $\operatorname{tr}\mathbb{B}^n = \operatorname{tr}\mathbb{C}^n$)

$$\mathbb{T} = 2(I_3)^{-\frac{1}{2}}\{I_3W_3\mathbb{I} + (W_1 + I_1W_2)\mathbb{B} - W_2\mathbb{B}^2\}$$

10.3 A non linear viscous fluids

Recall : Newtonian viscous fluid

$$T_{ij} = -p(\rho,\theta)\delta_{ij} + B_{ijkl}(\rho,\theta)\, D_{kl}$$

If fluid is at rest, $T_{ij} = -p(\rho,\theta)\delta_{ij}$

If motion is rotated, $\mathbb{F} = \left(\frac{dx_i}{dX_k}\right) \Rightarrow \underline{\mathbb{F} \to Q^T \mathbb{F} Q}$

Likewise, $\underline{\frac{D}{Dt}\mathbb{F} \longrightarrow Q^T \frac{D}{Dt}\mathbb{F} Q}$

Also, $\underline{\mathbb{F}^{-1} \to Q^T \mathbb{F}^{-1} Q}$

$$D_{kl} = \frac{1}{2}\left(\frac{dv_k}{dx_l} + \frac{dv_l}{dx_k}\right) = \frac{1}{2}\left(\frac{dv_k}{dX_R}\frac{dX_R}{dx_l} + \frac{dv_l}{dX_R}\frac{dX_R}{dx_k}\right) = \frac{1}{2}\left(\frac{d\mathbb{F}}{Dt}\mathbb{F}^{-1} + (\mathbb{F}^{-1})^T\frac{D\mathbb{F}^T}{Dt}\right)_{kl}$$

$\therefore \underline{\mathbb{D} \to Q^T \mathbb{D} Q}$

If isotropic, $\mathbb{T} \to Q^T \mathbb{T} Q$

$$\therefore Q^T_{ia} T_{ab} Q_{bj} = -p\delta_{ij} + B_{ijkl} Q^T_{kc} D_{cd} Q_{dl}$$

$$Q_{ei}\left(Q^T_{ia} T_{ab} Q_{bj}\right) Q^T_{jf} \Rightarrow T_{ef} = -p\delta_{ef} + \underbrace{B_{ijkl} Q_{ei} Q_{fj} Q_{ck} Q_{dl}}_{B_{efcd}} D_{cd}$$

$\therefore B_{ijkl}$: isotropic ,

$$T_{ij} = \{-p + \lambda D_{kk}\}\delta_{ij} + 2\mu D_{ij} \quad \text{, or,} \quad \mathbb{T} = (-p + \lambda\, \mathrm{tr}\mathbb{D})\mathbb{I} + 2\mu\mathbb{D}.$$

* Non-newtonian fluid

$T_{ij} = T_{ij}\left(\frac{dv_p}{dx_q}, \rho, \theta\right)$, or , $\underline{\mathbb{T} = \mathbb{T}(\mathbb{L}, \rho, \theta)}$

Since $\mathbb{L} = \mathbb{D} + \mathbb{W}$, $\underline{\mathbb{T} = \mathbb{T}(\mathbb{D}, \mathbb{W}, \rho, \theta)}$

Original motion : $x = x(X, t)$, $v = v(x, t)$

New motion : $\bar{x} = \underline{M(t)}\, x(X, t) \leftrightarrow x = M^T \bar{x}$

↳ time dependent rigid rotation

$\Rightarrow \underline{\bar{v} = \frac{D}{Dt}\bar{x} = \dot{M} x + M v}$

$$\bar{L}_{ij} = \frac{d\bar{v}_i}{d\bar{x}_j} = \frac{d\bar{v}_i}{dx_k}\frac{dx_k}{d\bar{x}_j} = \left(M_{ip}\delta_{pk} + M_{ip}\frac{dv_p}{dx_k}\right) M^T_{kj} \quad \text{, or,} \quad \underline{\bar{\mathbb{L}} = (\dot{M} + M\cdot\mathbb{L})M^T}$$

$$\bar{\mathbb{D}} = \frac{1}{2}(\bar{\mathbb{L}} + \bar{\mathbb{L}}^T) = \frac{1}{2}(\dot{M}\cdot M^T + M\cdot\dot{M}^T) + \frac{1}{2}M(\mathbb{L} + \mathbb{L}^T)M^T$$

$$\bar{\mathbb{W}} = \frac{1}{2}(\bar{\mathbb{L}} - \bar{\mathbb{L}}^T) = \frac{1}{2}(\dot{M}\cdot M^T - M\cdot\dot{M}^T) + \frac{1}{2}M(\mathbb{L} - \mathbb{L}^T)M^T$$

From $\mathbb{M}\cdot\mathbb{M}^T = \mathbb{I}$
$\rightarrow \dot{\mathbb{M}}\cdot\mathbb{M}^T + \mathbb{M}\cdot\dot{\mathbb{M}}^T = 0$ $\Rightarrow$ $\overline{\mathbb{D}} = \mathbb{M}\,\mathbb{D}\,\mathbb{M}^T$
$\overline{\mathbb{W}} = \mathbb{M}(\mathbb{M}^T\cdot\dot{\mathbb{M}} + \mathbb{W})\cdot\mathbb{M}^T$

We need $\overline{\mathbb{T}} = \mathbb{M}\cdot\mathbb{T}\cdot\mathbb{M}^T$, $\overline{\mathbb{T}} = \mathbb{T}(\overline{\mathbb{D}}, \overline{\mathbb{W}}, \rho, \theta)$

$\therefore \mathbb{T}\{\mathbb{M}\mathbb{D}\mathbb{M}^T, \mathbb{M}(\mathbb{M}^T\cdot\dot{\mathbb{M}} + \mathbb{W})\cdot\mathbb{M}^T, \rho, \theta\} = \mathbb{M}\cdot\mathbb{T}(\mathbb{D}, \mathbb{W}, \rho, \theta)\cdot\mathbb{M}^T$ for any $\mathbb{M}$.

Consider the case $\mathbb{M} = \mathbb{I}$, $\dot{\mathbb{M}} \neq 0$

$\mathbb{T}\{\mathbb{D}, \dot{\mathbb{M}} + \underline{\mathbb{W}}, \rho, \theta\} = \mathbb{T}(\mathbb{D}, \underline{\mathbb{W}}, \rho, \theta)$ $\rightarrow$ $\therefore$ $\mathbb{T}$ should be indep. of $\mathbb{W}$.

$\hookrightarrow$ $\boxed{\mathbb{T}(\mathbb{M}\cdot\mathbb{D}\cdot\mathbb{M}^T, \rho, \theta) = \mathbb{M}\,\mathbb{T}(\mathbb{D}, \rho, \theta)\,\mathbb{M}^T}$

In general, $\mathbb{T} = -p\mathbb{I} + \alpha\mathbb{D} + \beta\mathbb{D}^2$

where p, α, β are function of ρ, θ, and invariants of $\mathbb{D}$.

$(\mathrm{tr}\,\mathbb{D}, \frac{1}{2}\{(\mathrm{tr}\,\mathbb{D})^2 - \mathrm{tr}\,\mathbb{D}^2\}, \det\mathbb{D})$
$(\mathrm{tr}\,\mathbb{D}, \mathrm{tr}\,\mathbb{D}^2, \mathrm{tr}\,\mathbb{D}^3)$

* Appendix : Representation theorem for an isotropic tensor function of a tensor

Theorem : $\mathbb{T}$ is an isotropic tensor function of $\mathbb{D}$ iff $\mathbb{T} = \alpha\mathbb{I} + \beta\mathbb{D} + \gamma\mathbb{D}^2$ where α, β, γ are scalar functions of $\mathrm{tr}\,\mathbb{D}, \mathrm{tr}\,\mathbb{D}^2, \mathrm{tr}\,\mathbb{D}^3$.

$\boxed{\mathbb{T}(\mathbb{M}\cdot\mathbb{D}\cdot\mathbb{M}^T) = \mathbb{M}\cdot\mathbb{T}(\mathbb{D})\cdot\mathbb{M}^T \text{ for all } \mathbb{M}.}$

(a) : Sufficiency (denote $\overline{\mathbb{T}} = \mathbb{M}\,\mathbb{T}\,\mathbb{M}^T$, $\overline{\mathbb{D}} = \mathbb{M}\,\mathbb{D}\,\mathbb{M}^T$)

$$\mathbb{M}\cdot\mathbb{T}(\mathbb{D})\,\mathbb{M}^T = \mathbb{M}(\alpha\mathbb{I} + \beta\mathbb{D} + \gamma\mathbb{D}^2)\,\mathbb{M}^T$$
$$= \alpha\mathbb{I} + \beta\,\mathbb{M}\mathbb{D}\mathbb{M}^T + \gamma\,\mathbb{M}\mathbb{D}\mathbb{M}^T\mathbb{M}\mathbb{D}\mathbb{M}^T$$
$$= \alpha\mathbb{I} + \beta\,\overline{\mathbb{D}} + \gamma\,\overline{\mathbb{D}}^2 = \mathbb{T}(\overline{\mathbb{D}})$$

(b) : necessity

Assume $\mathbb{T}(\mathbb{M}\cdot\mathbb{D}\cdot\mathbb{M}^T) = \mathbb{M}\,\mathbb{T}(\mathbb{D})\,\mathbb{M}^T$ for all $\mathbb{M}$.

Consider the case $\mathbb{D}$ is diagonal. $\mathbb{D} = \begin{pmatrix} D_1 & & \\ & D_2 & \\ & & D_3 \end{pmatrix}$ $\rightarrow T_{ij} = T_{ij}(D_1, D_2, D_3)$

(i) Choose $\mathbb{M} = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}$, then $\overline{\mathbb{D}} = \mathbb{M}\mathbb{D}\mathbb{M}^T = \mathbb{D}$, and $\overline{\mathbb{T}} = \begin{pmatrix} T_{11} & -T_{12} & -T_{13} \\ -T_{12} & T_{22} & T_{23} \\ -T_{13} & T_{23} & T_{33} \end{pmatrix} = \mathbb{T}(\overline{\mathbb{D}}) = \mathbb{T}(\mathbb{D})$

$\therefore T_{12} = T_{13} = 0$, Likewise $T_{23} = 0$ (by other choice of $\mathbb{M}$)

$\hookrightarrow$ If $\mathbb{D}$ is diagonal, so is $\mathbb{T}$. $\rightarrow$ $\mathbb{D}$ and $\mathbb{T}$ have the same principal axes.

$\rightarrow T_{11} = T_1 = F(D_1, D_2, D_3)$, $T_{22} = T_2 = F_2(D_1, D_2, D_3)$, $T_{33} = T_3 = F_3(D_1, D_2, D_3)$

(ii) Choose $\mathbb{M} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$$\overline{\mathbb{D}} = \mathbb{M}\mathbb{D}\mathbb{M}^T = \begin{pmatrix} D_2 & & \\ & D_3 & \\ & & D_1 \end{pmatrix}, \quad \overline{\mathbb{T}} = \mathbb{M}\mathbb{T}\mathbb{M}^T = \begin{pmatrix} T_2 & & \\ & T_3 & \\ & & T_1 \end{pmatrix}$$

$$\therefore \mathbb{T}(\overline{\mathbb{D}}) = \begin{pmatrix} F(D_2, D_3, D_1) & & \\ & F_2(D_2, D_3, D_1) & \\ & & F_3(D_2, D_3, D_1) \end{pmatrix} = \overline{\mathbb{T}} = \begin{pmatrix} T_2 & & \\ & T_3 & \\ & & T_1 \end{pmatrix} = \begin{pmatrix} F_2(D_1, D_2, D_3) & & \\ & F_3(D_1, D_2, D_3) & \\ & & F(D_1, D_2, D_3) \end{pmatrix}$$

$$\hookrightarrow \begin{cases} F(D_2, D_3, D_1) = F_2(D_1, D_2, D_3) \\ F_2(D_2, D_3, D_1) = F_3(D_1, D_2, D_3) \\ F_3(D_2, D_3, D_1) = F(D_1, D_2, D_3) \end{cases} \Rightarrow \begin{aligned} &\underline{T_1 = F(D_1, D_2, D_3)} \\ &\underline{T_2 = F(D_2, D_3, D_1)} \\ &\underline{T_3 = F(D_3, D_1, D_2)} \end{aligned}$$

$$\alpha(D_1, D_2, D_3) + \beta(D_1, D_2, D_3)\, D_1 + \gamma(D_1, D_2, D_3)\, D_1^2 = F(D_1, D_2, D_3) = T_1$$
$$\alpha(D_1, D_2, D_3) + \beta(D_1, D_2, D_3)\, D_2 + \gamma(D_1, D_2, D_3)\, D_2^2 = F(D_2, D_3, D_1) = T_2$$
$$\alpha(D_1, D_2, D_3) + \beta(D_1, D_2, D_3)\, D_3 + \gamma(D_1, D_2, D_3)\, D_3^2 = F(D_3, D_1, D_2) = T_3$$

(iii) Choose $\mathbb{M} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$$\overline{\mathbb{D}} = \mathbb{M}\mathbb{D}\mathbb{M}^T = \begin{pmatrix} D_2 & & \\ & D_1 & \\ & & D_3 \end{pmatrix} \qquad \overline{\mathbb{T}} = \mathbb{M}\mathbb{T}\mathbb{M}^T = \begin{pmatrix} T_2 & & \\ & T_1 & \\ & & T_3 \end{pmatrix}$$

$$\mathbb{T}(\overline{\mathbb{D}}) = \begin{pmatrix} F(D_2, D_1, D_3) & & \\ & F_2(D_2, D_1, D_3) & \\ & & F_3(D_2, D_1, D_3) \end{pmatrix} = \overline{\mathbb{T}} = \begin{pmatrix} F_2(D_1, D_2, D_3) & & \\ & F(D_1, D_2, D_3) & \\ & & F_3(D_1, D_2, D_3) \end{pmatrix}$$

(Recall)

$$F_2(D_1, D_2, D_3) = F(D_2, D_3, D_1) \qquad \Rightarrow \underline{F(D_2, D_3, D_1) = F(D_2, D_1, D_3)}$$
$$F_3(D_1, D_2, D_3) = F(D_3, D_1, D_2)$$

결론: F는 맨 앞의 Argument D_i에 의존
(뒤의 D_j, D_k는 순서 바뀌어도 괜찮)

(iv) We can find $\alpha(D_1, D_2, D_3)$, $\beta(D_1, D_2, D_3)$, $\gamma(D_1, D_2, D_3)$ such that

$$\alpha + \beta D_1 + \gamma D_1^2 = F(D_1, D_2, D_3) = T_1$$
$$\alpha + \beta D_2 + \gamma D_2^2 = F(D_2, D_3, D_1) = T_2$$
$$\alpha + \beta D_3 + \gamma D_3^2 = F(D_3, D_1, D_2) = T_3$$

$$\begin{pmatrix} T_1 & & \\ & T_2 & \\ & & \end{pmatrix} = \alpha \begin{pmatrix} 1 & & \\ & 1 & \\ & & \end{pmatrix} + \beta \begin{pmatrix} D_1 & & \\ & D_2 & \\ & & \end{pmatrix} + \gamma \begin{pmatrix} D_1^2 & & \\ & D_2^2 & \\ & & \end{pmatrix}$$

T_3 1 D_3 D_3^2