10. Canonical transformations

(Motivations and concepts
Generating function approach
Symplectic approach)

① Motivations and concepts

A. Invariance of Lagrangian mechanics under point transformations.

point transformation : $\underline{Q = Q(\underline{q}, t)}$

$$\dot{\underline{Q}} = \frac{\partial Q}{\partial \underline{q}} \dot{\underline{q}} + \frac{\partial Q}{\partial t}, \quad \frac{\partial}{\partial \dot{\underline{Q}}} = \frac{\partial}{\partial \dot{\underline{Q}} / \partial \dot{\underline{q}}} \frac{\partial}{\partial \dot{\underline{q}}} = \frac{\partial \underline{q}}{\partial \underline{Q}} \frac{\partial}{\partial \dot{\underline{q}}}$$

Thus, $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i}\right) = \frac{d}{dt}\left(\frac{\partial q_j}{\partial Q_i} \frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial q_j}{\partial Q_i} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \frac{d}{dt}\left(\frac{\partial q_j}{\partial Q_i}\right) \frac{\partial L}{\partial \dot{q}_j}$

서로 상쇄

$$\frac{\partial L}{\partial Q_i} = \frac{\partial q_j}{\partial Q_i} \frac{\partial L}{\partial q_j} + \frac{\partial \dot{q}_j}{\partial Q_i} \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial q_j}{\partial Q_i} \frac{\partial L}{\partial q_j} + \frac{d}{dt}\left(\frac{\partial q_j}{\partial Q_i}\right) \frac{\partial L}{\partial \dot{q}_j}$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i}\right) - \frac{\partial L}{\partial Q_i} = \frac{\partial q_j}{\partial Q_i}\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right) \qquad \therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_i}\right) - \frac{\partial L}{\partial Q_i} = 0 \Leftrightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

E-L eqn's are in same form!

B. Canonical transformations

- Definition : $\underline{(\underline{q}, \underline{p}) \to (\underline{Q}, \underline{P})}$ with $\underline{Q} = \underline{Q}(\underline{q}, \underline{p}, t)$, $\underline{P} = \underline{P}(\underline{q}, \underline{p}, t)$
  if it <u>preserves the form of Hamilton's equations</u>
- Condition : $\boxed{\lambda(\underline{p} \cdot \dot{\underline{q}} - L) = \underline{P} \cdot \dot{\underline{Q}} - K + \frac{dF}{dt}}$

  then $\underline{\dot{\underline{q}} = \frac{\partial H}{\partial \underline{p}}, \; \dot{\underline{p}} = -\frac{\partial H}{\partial \underline{q}}} \Leftrightarrow \underline{\dot{\underline{Q}} = \frac{\partial K}{\partial \underline{P}}, \; \dot{\underline{P}} = -\frac{\partial K}{\partial \underline{Q}}}$

* F를 old-new variable mixture의 함수로 정의하면,
Canonical transformation의 형태를 구체화할 수 있다.
이런 F를 <u>generating function</u> 이라고 한다.

② Generating function approach

A. Four basic types

i) $\boxed{F = F_1(q, Q, t)}$

$$p\cdot\dot{q} - H = P\cdot\dot{Q} - K + \frac{dF_1}{dt} = P\cdot\dot{Q} - K + \frac{\partial F_1}{\partial q}\dot{q} + \frac{\partial F_1}{\partial Q}\dot{Q} + \frac{\partial F_1}{\partial t}$$

$$= \frac{\partial F_1}{\partial q}\dot{q} + \left(P + \frac{\partial F_1}{\partial Q}\right)\dot{Q} - K + \frac{\partial F_1}{\partial t}$$

$$\Rightarrow p = \frac{\partial F_1}{\partial q}, \quad P = -\frac{\partial F_1}{\partial Q}, \quad K = H + \frac{\partial F_1}{\partial t}$$

ii) $\boxed{F = F_2(q, P, t) - Q\cdot P}$

$$p\cdot\dot{q} - H = P\cdot\dot{Q} - K + \frac{d}{dt}(F_2 - Q\cdot P) = \cancel{P\dot{Q}} - K + \frac{\partial F_2}{\partial q}\dot{q} + \frac{\partial F_2}{\partial P}\dot{P} + \frac{\partial F_2}{\partial t} - Q\dot{P} - \cancel{\dot{Q}P}$$

$$= \frac{\partial F_2}{\partial q}\dot{q} + \left(\frac{\partial F_2}{\partial P} - Q\right)\dot{P} - K + \frac{\partial F_2}{\partial t}$$

$$\Rightarrow p = \frac{\partial F_2}{\partial q}, \quad Q = \frac{\partial F_2}{\partial P}, \quad K = H + \frac{\partial F_2}{\partial t}$$

iii) $\boxed{F = F_3(p, Q, t) + q\cdot p}$

$$p\cdot\dot{q} - H = P\dot{Q} - K + \frac{d}{dt}(F_3 + q p) = P\cdot\dot{Q} - K + \frac{\partial F_3}{\partial p}\dot{p} + \frac{\partial F_3}{\partial Q}\dot{Q} + \frac{\partial F_3}{\partial t} + \dot{q}p + q\dot{p}$$

$$= p\dot{q} + \left(q + \frac{\partial F_3}{\partial p}\right)\dot{p} + \left(P + \frac{\partial F_3}{\partial Q}\right)\dot{Q} - K + \frac{\partial F_3}{\partial t}$$

$$\Rightarrow q = -\frac{\partial F_3}{\partial p}, \quad P = -\frac{\partial F_3}{\partial Q}, \quad K = H + \frac{\partial F_3}{\partial t}$$

iv) $\boxed{F = F_4(p, P, t) + q\cdot p - Q\cdot P}$

$$p\cdot\dot{q} - H = P\cdot\dot{Q} - K + \frac{d}{dt}(F_4 + q p - Q\cdot P) =$$

$$= \cancel{P\dot{Q}} - K + \frac{\partial F_4}{\partial p}\dot{p} + \frac{\partial F_4}{\partial P}\dot{P} + \frac{\partial F_4}{\partial t} + \dot{q}p + q\dot{p} - \cancel{\dot{Q}P} - Q\dot{P}$$

$$= p\cdot\dot{q} + \left(\frac{\partial F_4}{\partial p} + q\right)\dot{p} + \left(\frac{\partial F_4}{\partial P} - Q\right)\dot{P} - K + \frac{\partial F_4}{\partial t}$$

$$\Rightarrow q = -\frac{\partial F_4}{\partial p}, \quad Q = \frac{\partial F_4}{\partial P}, \quad K = H + \frac{\partial F_4}{\partial t}$$

* Remarks

- generating function으로부터 얻은 3개 식으로 $\Rightarrow$ $\underline{Q} = \underline{Q}(\underline{q}, \underline{p}, t)$, $\underline{P} = \underline{P}(\underline{q}, \underline{p}, t)$, $K = K(\underline{q}, \underline{p}, t)$
- Restricted canonical transformation : do not explicitly involve time $\Rightarrow$ $K = H$
- 4가지 외에 다른 generating function도 존재할 수 있음

B. Simple examples

- Point transformations

$$F_2(\underline{q}, \underline{P}, t) = f(\underline{q}, t) \cdot \underline{P} + g(\underline{q}, t)$$

$$\Rightarrow p_i = \frac{\partial F_2}{\partial q_i} = \frac{\partial f_j}{\partial q_i} P_j + \frac{\partial g}{\partial q_i}, \quad Q_i = \frac{\partial F_2}{\partial P_i} = f_i(\underline{q}, t)$$

$$\therefore \underline{Q} = \underline{f}(\underline{q}, t), \quad \underline{P} = \left[\frac{\partial f}{\partial q_i}\right]^{-1}\left(\underline{p} - \frac{\partial g}{\partial \underline{q}}\right)$$

$$F_2(\underline{q}, \underline{P}, t) = \underline{q} \cdot \underline{P}$$

$$\Rightarrow p_i = \frac{\partial F_2}{\partial q_i} = P_i, \quad Q_i = \frac{\partial F_2}{\partial P_i} = q_i \qquad \therefore \underline{Q} = \underline{q}, \ \underline{P} = \underline{p} \ \text{(identity)}$$

- Point-momentum exchange

$$F_1(\underline{q}, \underline{Q}, t) = \underline{q} \cdot \underline{Q}$$

$$\Rightarrow p_i = \frac{\partial F_1}{\partial q_i} = Q_i, \quad P_i = -\frac{\partial F_1}{\partial Q_i} = -q_i \qquad \therefore \underline{p} = \underline{Q}, \ \underline{P} = -\underline{q}$$

C. Example : 1-d harmonic oscillator

$$\underline{H(q,p) = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2} \qquad \left(\omega = \sqrt{\frac{k}{m}}\right)$$

position이 cyclic 하고, momentum은 보존되는 좌표를 찾아보자!

- $(q,p) \to (Q,P)$ satisfying $\underline{p = f(P)\cos Q, \quad q = \frac{1}{m\omega} f(P)\sin Q}$

  so that $\underline{K = H = \frac{1}{2m} f(P)^2}$ making Q cyclic.

- Determining $f(P)$

$$p\dot{q} - H \underset{\uparrow}{=} (qm\omega\cot Q)\dot{q} - \frac{f(P)^2}{2m} = P\dot{Q} - \frac{f(P)^2}{2m} + \underbrace{(qm\omega\cot Q)\dot{q} - P\dot{Q}}_{= \frac{d}{dt}F_1(q,Q)}$$

$(p/q = m\omega\cot Q)$

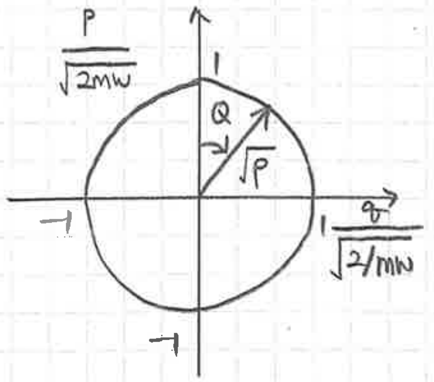
$$\underline{p = \frac{\partial F_1}{\partial q} = qm\omega\cot Q}, \quad P = -\frac{\partial F_1}{\partial Q} \Rightarrow \underline{F_1(q,Q) = \frac{1}{2}m\omega q^2\cot Q, \quad P = \frac{m\omega q^2}{2\sin^2 Q}}$$

$$\left\{\begin{aligned} q &= \sqrt{\frac{2P}{m\omega}}\sin Q, \\ p &= \sqrt{2m\omega P}\cos Q \end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} Q &= \arctan\frac{q/\sqrt{2/m\omega}}{p/\sqrt{2m\omega}} \\ P &= \left(\frac{q}{\sqrt{2/m\omega}}\right)^2 + \left(\frac{p}{\sqrt{2m\omega}}\right)^2 \end{aligned}\right\}$$

which corresponds to $\underline{f(P) = \sqrt{2m\omega P}}$, $\underline{K = H = \omega P}$

- New Hamilton's equations :

$$\underline{\underline{\dot{Q} = \frac{\partial K}{\partial P} = \omega, \quad \dot{P} = -\frac{\partial K}{\partial Q} = 0}}$$



$\left(\begin{aligned} &P : \text{action} \\ &Q : \text{angle} \end{aligned}\right)$

③ Symplectic approach

· $\underline{\gamma} = [q_1 \cdots q_n \; p_1 \cdots p_n]^T$, $\underline{\underline{J}} = \begin{bmatrix} \underline{\underline{0}}_n & \underline{\underline{1}}_n \\ -\underline{\underline{1}}_n & \underline{\underline{0}}_n \end{bmatrix}$ (2n×2n matrix)

($\underbrace{\underline{\underline{J}}^T = -\underline{\underline{J}}}_{\text{(anti-sym.)}}$, $\underbrace{\underline{\underline{J}}^T = \underline{\underline{J}}^{-1}}_{\text{(orthogonal)}}$, $\underline{|\underline{\underline{J}}| = 1}$)

⇒ Hamilton's equation : $\boxed{\dot{\underline{\gamma}} = \underline{\underline{J}} \frac{\partial H}{\partial \underline{\gamma}}}$ $\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \partial H/\partial q \\ \partial H/\partial p \end{pmatrix}$

(i) Symplectic matrices

① - restricted canonical transformation : $\underline{\Gamma} = [Q_1 \cdots Q_n \; P_1 \cdots P_n]^T = \underline{\Gamma}(\underline{\gamma})$

- Matrix : $M_{ij} = \partial \Gamma_i / \partial \gamma_j$

$$\Rightarrow \dot{\underline{\Gamma}} = \underline{\underline{M}} \dot{\underline{\gamma}} = \underline{\underline{M}} \underline{\underline{J}} \frac{\partial H}{\partial \underline{\gamma}} = \underline{\underline{M}} \underline{\underline{J}} \underline{\underline{M}}^T \frac{\partial H}{\partial \underline{\Gamma}} \left(= \underline{\underline{J}} \frac{\partial K}{\partial \underline{\Gamma}} = \underline{\underline{J}} \frac{\partial H}{\partial \underline{\Gamma}} \right) \Rightarrow \boxed{\underline{\underline{M}} \underline{\underline{J}} \underline{\underline{M}}^T = \underline{\underline{J}}}$$

"Symplectic condition"

($\underline{\Gamma} = \underline{\Gamma}(\underline{\gamma})$ is a restricted canonical transformation

$\Leftrightarrow$ $M_{ij} = \partial \Gamma_i / \partial \gamma_j$ is a symplectic matrix.)

∘ Symplectic group 의 특징

i) Closure : If $\underline{\underline{A}} \underline{\underline{J}} \underline{\underline{A}}^T = \underline{\underline{J}}$, $\underline{\underline{B}} \underline{\underline{J}} \underline{\underline{B}}^T = \underline{\underline{J}}$,

then $(\underline{\underline{A}} \underline{\underline{B}}) \underline{\underline{J}} (\underline{\underline{A}} \underline{\underline{B}})^T = \underline{\underline{A}} \underline{\underline{B}} \underline{\underline{J}} \underline{\underline{B}}^T \underline{\underline{A}}^T = \underline{\underline{A}} \underline{\underline{J}} \underline{\underline{A}}^T = \underline{\underline{J}}$

ii) Associativity : $(\underline{\underline{A}} \underline{\underline{B}}) \underline{\underline{C}} = \underline{\underline{A}} (\underline{\underline{B}} \underline{\underline{C}})$ ← 자명

iii) Identity : $\underline{\underline{1}}_{2n} \underline{\underline{J}} \underline{\underline{1}}_{2n}^T = \underline{\underline{J}}$

iv) Inverse : If $\underline{\underline{A}} \underline{\underline{J}} \underline{\underline{A}}^T = \underline{\underline{J}}$,

then $\underline{\underline{A}}^{-1} \underline{\underline{J}} (\underline{\underline{A}}^{-1})^T = \underline{\underline{J}} \underline{\underline{A}}^T (\underline{\underline{A}}^{-1})^T = \underline{\underline{J}}$

②

- symplecticity of infinitesimal canonical tranformations

$$F_2(\underline{q},\underline{P},t)=\underline{q}\cdot\underline{P}+\epsilon G(\underline{q},\underline{P},t) \Rightarrow \underline{\gamma}\to\underline{\Gamma}=\underline{\gamma}+\delta\underline{\gamma}$$

$$\underline{p}=\frac{\partial F_2}{\partial \underline{q}}=\underline{P}+\underbrace{\epsilon\frac{\partial G}{\partial \underline{q}}}_{=-\delta\underline{p}}, \quad \underline{Q}=\frac{\partial F_2}{\partial \underline{P}}=\underline{q}+\epsilon\frac{\partial G}{\partial \underline{P}}\simeq\underline{q}+\underbrace{\epsilon\frac{\partial G}{\partial \underline{p}}}_{=\delta\underline{q}} \Rightarrow \delta\underline{\gamma}=\epsilon\underline{\underline{J}}\frac{\partial G}{\partial \underline{\gamma}}$$

$$M_{ij}=\frac{\partial \Gamma_i}{\partial \gamma_j}=\delta_{ij}+\frac{\partial}{\partial \gamma_j}\delta\gamma_i=\delta_{ij}+\epsilon J_{ik}\frac{\partial^2 G}{\partial\gamma_j\partial\gamma_k} \Rightarrow \underline{\underline{M}}=\underline{\underline{I}}+\epsilon\underline{\underline{J}}\frac{\partial^2 G}{\partial\underline{\gamma}\partial\underline{\gamma}}$$

$$(\text{using } \underline{\underline{J}}^T=-\underline{\underline{J}}) \quad \underline{\underline{M}}^T=\underline{\underline{I}}-\epsilon\underline{\underline{J}}\frac{\partial^2 G}{\partial\underline{\gamma}\partial\underline{\gamma}}$$

$$\therefore \underline{\underline{M}}\,\underline{\underline{J}}\,\underline{\underline{M}}^T=\left(\underline{\underline{I}}+\epsilon\underline{\underline{J}}\frac{\partial^2 G}{\partial\underline{\gamma}\partial\underline{\gamma}}\right)\underline{\underline{J}}\left(\underline{\underline{I}}-\epsilon\underline{\underline{J}}\frac{\partial^2 G}{\partial\underline{\gamma}\partial\underline{\gamma}}\right)$$

$$\simeq \underline{\underline{J}}+\cancel{\epsilon\underline{\underline{J}}\frac{\partial^2 G}{\partial\underline{\gamma}\partial\underline{\gamma}}\underline{\underline{J}}}-\cancel{\epsilon\underline{\underline{J}}\frac{\partial^2 G}{\partial\underline{\gamma}\partial\underline{\gamma}}\underline{\underline{I}}}=\underline{\underline{J}} \qquad \boxed{\therefore \underline{\underline{M}}\,\underline{\underline{J}}\,\underline{\underline{M}}^T=\underline{\underline{J}}}$$

$\Rightarrow$ Any infinitesimal canonical transformation satisfies symplectic condition

- $\gamma\to\Gamma_{t_0}=\Gamma(\gamma,t_0)\to\Gamma_t$ ($\Gamma_s\to\Gamma_{s+ds}$ 의 연속적인 합성)

by closure of symplectic group, $\Gamma_{t_0}\to\Gamma_t$ satisfies symplectic condition

$\Rightarrow$ Any canonical transformation $\underline{\gamma}\to\underline{\Gamma}_t=\underline{\Gamma}(\underline{\gamma},t)$ satisfies symplectic condition