Fusion Plasma Theory 2

Lecture 7 : Fast Ion Collisions and Resistivity

① Qualitative overview of fast-ion collisions

- Fast or energetic ions :
  - Neutral beam ions : $O(100\,\text{keV}) \sim O(1\,\text{MeV})$
  - α particles : $O(3.5\,\text{MeV})$
  - Hydrogen isotope after fusion reactions : $O(1\,\text{MeV})$
- Fast ion heats e^- first, then after losing sufficient energy, heats thermal ion
- Only slowing down is important between fast ions and electrons.
  Deflection as well as slowing down becomes important between fast and thermal ion
- It's important to include fast ion distribution in fusion burning prediction.

② Collision frequencies for fast ions

Recall for Maxwellian :

$$C_{ab}[f_a, f_b] = \nu_D^{ab}\mathcal{L}[f_a] + \frac{1}{v^2}\frac{d}{dv}\left[v^3\left(\frac{m_a}{m_a+m_b}\nu_s^{ab} f_a + \frac{1}{2}\nu_{\parallel}^{ab} v \frac{df_a}{dv}\right)\right]$$

For a single incident particle with v,

$$\left\{\begin{aligned} \nu_D^{ab} &= \frac{n_b e_a^2 e_b^2 \ln\Lambda}{4\pi\epsilon_0^2 m_a^2}\frac{\phi(x_b)-G(x_b)}{v^3} \\ \frac{m_a}{m_a+m_b}\nu_s^{ab} &= \frac{n_b e_a^2 e_b^2 \ln\Lambda}{4\pi\epsilon_0^2 m_a^2}\left(\frac{m_a}{m_b}\right)\frac{2G(x_b)}{v v_{tb}^2} \\ \nu_{\parallel}^{ab} &= \frac{n_b e_a^2 e_b^2 \ln\Lambda}{4\pi\epsilon_0^2 m_a}\frac{2G(x_b)}{v^3} \end{aligned}\right.$$

$\leftarrow$ when $x \to 0$, $\phi \simeq 3G$

$\leftarrow$ we can guess, when $a=\alpha, b=e^-$, slowing down becomes the largest.

Label fast ion species as $a=\alpha$. With e^- and i, one can assume :

$$v_{ti} \ll v \ll v_{te} \quad \longrightarrow \quad x_i \gg 1 \gg x_e$$

so that one can take each asymptotic limit of ϕ and G.

③ Comparison for collisional frequencies for fast ions

For α-to-e, take the limit of $x_e \to 0$:

$$\frac{m_\alpha}{m_\alpha+m_e}\nu_s^{\alpha e} : \nu_D^{\alpha e} : \nu_{\parallel}^{\alpha e} = \frac{m_\alpha}{m_e}\frac{1}{v v_{te}^2} : \frac{1}{v^3} : \frac{1}{v^3} = \frac{m_\alpha v^2}{m_e v_{te}^2} : 1 : 1 = \frac{\varepsilon}{T_e} : 1 : 1$$

$$\therefore \frac{m_\alpha}{m_\alpha+m_e}\nu_s^{\alpha e} : \nu_D^{\alpha e} : \nu_{\parallel}^{\alpha e} = \frac{\varepsilon}{T_e} : 1 : 1 \quad (\varepsilon \sim O(1\text{MeV}),\ T_e \sim O(10\text{keV}))$$

$$\Rightarrow \underline{\frac{m_\alpha}{m_\alpha+m_e}\nu_s^{\alpha e} \gg \nu_D^{\alpha e} \sim \nu_{\parallel}^{\alpha e}}$$ (only slowing down matters when α-to-e^-)

For α-to-i, take the limit of $x_i \to \infty$:

$$\frac{m_\alpha}{m_\alpha+m_i}\nu_s^{\alpha i} : \nu_D^{\alpha i} : \nu_{\parallel}^{\alpha i} = \frac{m_\alpha}{m_i}\frac{1}{v v_{ti}^2}\frac{2}{v^2/v_{ti}^2} : \frac{1}{v^3} : \frac{1}{v^3}\frac{2}{v^2/v_{ti}^2}$$

$$\simeq \frac{m_\alpha}{m_i}\frac{1}{v^3} : \frac{1}{v^3} : \frac{1}{v^3}\frac{v_{ti}^2}{v^2} = \frac{m_\alpha}{m_i}\frac{1}{v^3} : \frac{1}{v^3} : \frac{1}{v^3}\frac{1}{x_i^2} \to 0$$

$$\Rightarrow \underline{\frac{m_\alpha}{m_\alpha+m_i}\nu_s^{\alpha i} \sim \nu_D^{\alpha i} \gg \nu_{\parallel}^{\alpha i}}$$ (both slowing down and deflection matters when α-to-i)

④ Slowing down collisional frequencies for fast ions

For α-to-e slowing down :

$$\underline{\frac{m_\alpha}{m_\alpha+m_e}\nu_s^{\alpha e} \sim \nu_s^{\alpha e} \sim \frac{n_e Z_\alpha^2 e^4 \ln\Lambda}{4\pi\epsilon_0^2 m_\alpha^2}\frac{m_\alpha}{m_e}\frac{4v}{v\, v_{te}^3 \cdot 3\sqrt{\pi}} = \frac{n_e Z_\alpha^2 e^4 m_e^{1/2} \ln\Lambda}{3(2\pi)^{3/2}\epsilon_0^2 m_\alpha T_e^{3/2}}}$$

* Note that slowing down by electrons depend only on T_e, not on ε.
It takes more time for slowing down, or electron heating, when $T_e \gg 1$.

For α-to-i slowing down :

$$\underline{\frac{m_\alpha}{m_\alpha+m_i}\nu_s^{\alpha i} \sim \frac{n_i Z_i^2 Z_\alpha^2 e^4 \ln\Lambda}{4\pi\epsilon_0^2 m_\alpha^2}\frac{m_\alpha}{m_i}\frac{1}{v^3} = \nu_s^{\alpha e}\left(\frac{v_c}{v}\right)^3}$$

$$\underline{v_c \equiv \left(\frac{3\pi^{1/2} m_e}{4 m_i}\frac{n_i Z_i^2}{n_e}\right)^{1/3} v_{te}}$$ ← critical fast ion energy
where e^- heating and ion heating becomes equal

⑤ Fast ion collisional operator and distribution

$$\nu_D^{\alpha i} \sim \frac{n_i Z_\alpha^2 Z_i^2 e^4 \ln\Lambda}{4\pi\epsilon_0^2 m_\alpha^2}\frac{1}{v^3} = \nu_s^{\alpha e}\left(\frac{v_d}{v}\right)^3 \quad ; \quad v_d \equiv \left(\frac{m_i}{m_\alpha}\right)^{1/3} v_c$$

From eq (1), combining all gives fast ion collision operator

$$\boxed{C_\alpha[f_\alpha] = \nu_s^{\alpha e}\frac{1}{v^2}\frac{\partial}{\partial v}\left[(\underbrace{v^3}_{\substack{\text{election}\\\text{(slowing down)}}}+\underbrace{v_c^3}_{\substack{\text{ion}\\\text{(slowing down)}}})f_\alpha\right] + \nu_s^{\alpha e}\underbrace{\frac{v_d^3}{v^3}}_{\text{deflection}}\mathcal{L}[f_\alpha]}$$

In the case of isotropic source S of fast ions, as expected from fusion burning

$$\frac{\partial f_\alpha}{\partial t} = \nu_s^{\alpha e}\frac{1}{v^2}\frac{\partial}{\partial v}\left[(v^3+v_c^3)f_\alpha\right] + \frac{S\delta(v-v_0)}{4\pi v_0^2} = 0$$

(v_0 = born velocity)

($\mathcal{L}$ vanishes for isotropic)

Giving the popular form of fast ion distribution:

$$\frac{1}{v^2}\frac{\partial}{\partial v}\left[(v^3+v_c^3)f_\alpha\right] = -\frac{S\tau_{\alpha e}}{4\pi v_0^2}\delta(v-v_0)$$

$$(v^3+v_c^3)f_\alpha = C - \frac{S\tau_{\alpha e}}{4\pi v_0^2}\int_0^v v'^2\delta(v'-v_0)dv' = \begin{cases} C - \dfrac{S\tau_{\alpha e}}{4\pi} = 0 & (v \geq v_0) \\ C - 0 = 0 & (v < v_0) \end{cases}$$

(* note that only slows down : $f_\alpha(v \geq v_0) = 0$)

$$\therefore (v^3+v_c^3)f_\alpha = \begin{cases} \dfrac{S\tau_{\alpha e}}{4\pi} & (v \leq v_0) \\ 0 & (v > v_0) \end{cases} = \frac{S\tau_{\alpha e}}{4\pi}H(v_0 - v)$$

← Heaviside step function

$$\Longrightarrow \boxed{f_\alpha = \frac{S\tau_{\alpha e}}{4\pi(v^3+v_c^3)}H(v_0-v)}$$

⑥ Power transfer (heating) from fast ions

Power transfer to electrons by a single fast ion

$$\boxed{P_{\alpha e}} = \frac{1}{2}m_\alpha v^2 \nu_E^{\alpha e} \sim \frac{1}{2}m_\alpha v^2 (2\nu_s^{\alpha e}) = \boxed{2\nu_s^{\alpha e}\varepsilon} \qquad (\text{*note } \nu_E^{ab} = 2\nu_s^{ab} - 2\nu_D^{ab} - \nu_\parallel^{ab})$$

Power transfer to ions by a single fast ion

$$\boxed{P_{\alpha i}} = \frac{1}{2}m_\alpha v^2 \nu_E^{\alpha i} \sim \frac{1}{2}m_\alpha v^2 \underline{(2\nu_s^{\alpha i} - 2\nu_D^{\alpha i})} \sim 2\nu_s^{\alpha e}\varepsilon\left(\frac{v_c}{v}\right)^3 = \boxed{2\nu_s^{\alpha e}\varepsilon\left(\frac{\varepsilon_c}{\varepsilon}\right)^{3/2}}$$

$$\left(\nu_s^{\alpha i} - \nu_D^{\alpha i} = \frac{m_\alpha + m_i}{m_\alpha}\nu_s^{\alpha e}\left(\frac{v_c}{v}\right)^3 - \nu_s^{\alpha e}\left(\frac{v_d}{v}\right)^3 \quad v_d = \left(\frac{m_i}{m_\alpha}\right)^{1/3} v_c\right.$$

$$\left. = \frac{m_\alpha + m_i}{m_\alpha}\nu_s^{\alpha e}\left(\frac{v_c}{v}\right)^3 - \frac{m_i}{m_\alpha}\nu_s^{\alpha e}\left(\frac{v_c}{v}\right)^3 = \nu_s^{\alpha e}\left(\frac{v_c}{v}\right)^3\right)$$

∴ Total heating power :

$$\boxed{P_\alpha = P_{\alpha e} + P_{\alpha i} = \frac{2\varepsilon}{\tau_{\alpha e}}\left(1 + \frac{\varepsilon_c^{3/2}}{\varepsilon^{3/2}}\right)}$$

⑦ Time response of power transfer

Power transfer rate = loss rate of beam energy.

$$P = -\frac{d\varepsilon}{dt} = -\frac{2\varepsilon}{\tau_{\alpha e}}\left(1 + \frac{\varepsilon_c^{3/2}}{\varepsilon^{3/2}}\right) (= P_e + P_i)$$

Integrating the equation :

$$-\frac{d\varepsilon}{dt} = \frac{2}{\tau_{\alpha e}}\frac{\varepsilon^{3/2} + \varepsilon_c^{3/2}}{\varepsilon^{1/2}} \rightarrow \frac{\varepsilon^{1/2}}{\varepsilon^{3/2} + \varepsilon_c^{3/2}}d\varepsilon = -\frac{2}{\tau_{\alpha e}}dt$$

$$\left(\text{let } u = \varepsilon^{3/2} + \varepsilon_c^{3/2},\ du = \frac{3}{2}\varepsilon^{1/2}d\varepsilon\right)$$

$$\int \frac{1}{u}\left(\frac{2}{3}du\right) = \frac{2}{3}\ln u \Big|_{\varepsilon_0}^{\varepsilon} = \frac{2}{3}\ln\left(\varepsilon^{3/2} + \varepsilon_c^{3/2}\right)\Big|_{\varepsilon_0}^{\varepsilon} = \frac{2}{3}\ln\left(\frac{\varepsilon^{3/2} + \varepsilon_c^{3/2}}{\varepsilon_0^{3/2} + \varepsilon_c^{3/2}}\right) = -\frac{2t}{\tau_{\alpha e}}$$

$$\therefore \frac{\varepsilon^{3/2} + \varepsilon_c^{3/2}}{\varepsilon_0^{3/2} + \varepsilon_c^{3/2}} = e^{-\frac{3t}{\tau_{\alpha e}}} \rightarrow \varepsilon^{3/2} + \varepsilon_c^{3/2} = e^{-\frac{3t}{\tau_{\alpha e}}}\left(\varepsilon_0^{3/2} + \varepsilon_c^{3/2}\right)$$

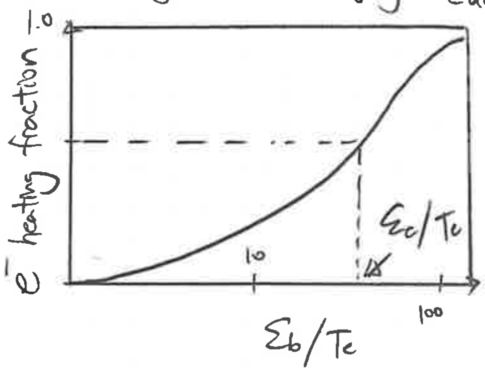
$$\therefore \boxed{\varepsilon(t) = \varepsilon_0\left[e^{-\frac{3t}{\tau_{\alpha e}}} - \frac{\varepsilon_c^{3/2}}{\varepsilon_0^{3/2}}\left(1 - e^{-\frac{3t}{\tau_{\alpha e}}}\right)\right]^{2/3}}$$

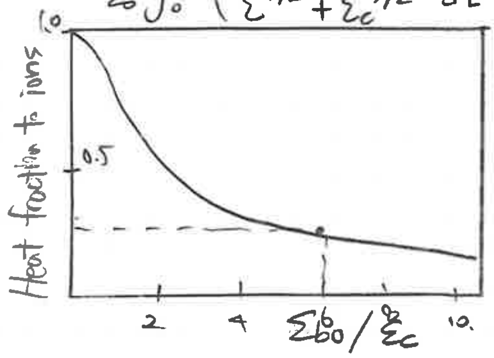
Time required for full power transfer ($\varepsilon = 0$) → $t = \frac{\tau_{\alpha e}}{3}\ln\left(1 + \frac{\varepsilon_0^{3/2}}{\varepsilon_c^{3/2}}\right)$

⑧ Electron and ion heating fraction

Define $F \equiv$ total fraction of ion heating by full slowing down from fast ion ε_0.

$$F \equiv \frac{1}{\varepsilon_0}\int P_i\,dt = \frac{1}{\varepsilon_0}\int \frac{2\varepsilon}{\tau_{\alpha e}}\frac{\varepsilon_c^{3/2}}{\varepsilon^{3/2}}\,dt = -\frac{1}{\varepsilon_0}\int_0^{\varepsilon_0}\left(\frac{\varepsilon_c^{3/2}}{\varepsilon^{3/2}+\varepsilon_c^{3/2}}\frac{d\varepsilon}{dt}\right)dt$$





⑨ Perturbed distribution functions

In general, to determine p.d.f. involves coupled PDE.

Vlasov operator: $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \vec{v}\cdot\vec{\nabla}f + \frac{q}{m}(\vec{E}+\vec{v}\times\vec{B})\cdot\frac{\partial f}{\partial \vec{v}}$

$$\frac{Df_{a0}}{Dt} = C_{ab}[f_{a1}, f_{b0}] + C_{ab}[f_{a0}, f_{b1}] + C_{aa}[f_{a1}, f_{a0}] + C_{aa}[f_{a0}, f_{a1}]$$