13. Flux-coordinates

1 Toroidal and poloidal flux

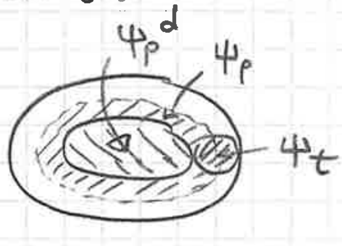
$$\underline{\psi_t(\rho)} = \int_{S_{tor}} \vec{B}\cdot d\vec{S} = \frac{1}{2\pi}\int dV\, \vec{\nabla}\cdot(B\varphi) = \underline{\frac{1}{2\pi}\int dV\, B^{\varphi}}$$

$$\left(\because \int dV\, \vec{\nabla}\cdot(B\varphi) = \int d\vec{S}\cdot\vec{B}(2\pi) - \int d\vec{S}\,\vec{B}\cdot(0) = 2\pi\psi_t\right)$$

$$\underline{\psi_p(\rho)} = \int_{S_{p\text{-}1}} \vec{B}\cdot d\vec{S} = \frac{1}{2\pi}\int dV \cdot \vec{B}\cdot(B\theta) = \underline{\frac{1}{2\pi}\int dV\, B^{\theta}}$$

$$\left(\because \int dV\, \vec{\nabla}\cdot(B\theta) = \int d\vec{S}\,\vec{B}(2\pi) - \int d\vec{S}\,\vec{B}\cdot(0) = 2\pi\psi_\theta\right)$$

$$\underline{\psi_p + \psi_p^{d}} = \text{const} \Rightarrow \vec{\nabla}\psi_p = -\vec{\nabla}\psi_p^{d}$$



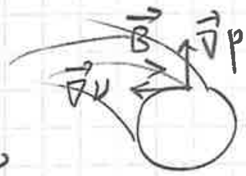
2 Clebsch representation

$$B^{\rho} = 0 \quad (\because \vec{B}\cdot\vec{\nabla}\rho = 0) \rightarrow \vec{B} = B^{\theta}\vec{e_\theta} + B^{\varphi}\vec{e_\varphi}$$

$$= JB^{\theta}(\vec{\nabla}\varphi\times\vec{\nabla}\rho) + JB^{\varphi}(\vec{\nabla}\rho\times\vec{\nabla}\theta)$$

$$\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(B^i\vec{e_i}) = \frac{1}{J}\frac{\partial}{\partial u_i}(JB^i) = \frac{1}{J}\left[\frac{\partial}{\partial\theta}(JB^{\theta}) + \frac{\partial}{\partial\varphi}(JB^{\varphi})\right] = 0$$

$$\rightarrow JB^{\theta} = -\frac{\partial\nu}{\partial\varphi}, \quad JB^{\varphi} = \frac{\partial\nu}{\partial\theta}$$



$$\therefore \vec{B} = \vec{\nabla}\rho\times\left(\frac{\partial\nu}{\partial\theta}\vec{\nabla}\theta + \frac{\partial\nu}{\partial\varphi}\vec{\nabla}\varphi + \left(\frac{\partial\nu}{\partial\rho}\vec{\nabla}\rho\right)\right) = \vec{\nabla}\rho\times\vec{\nabla}\nu$$

($\Rightarrow$ B line lies on a constant ν surfaces)

No.

3 Magnetic field representation

$\vec{B} = \vec{\nabla}\rho \times \vec{\nabla}\nu \quad \Rightarrow \nu(\rho,\theta,\varphi) = \alpha(\rho)\theta + \beta(\rho)\varphi + \tilde{\nu}(\rho,\theta,\varphi)$

* Toroidal flux

$$\underline{\psi_t} = \frac{1}{2\pi}\int d\rho\, d\theta\, d\varphi\, J B^{\varphi} = \frac{1}{2\pi}\int d\rho\, d\theta\, d\varphi\, \frac{\partial\nu}{\partial\theta}$$

$$= \frac{1}{2\pi}\int d\rho\, d\theta\, d\varphi \left(\alpha(\rho) + \frac{\partial\tilde{\nu}}{\partial\theta}\right) = \underline{2\pi\int d\rho\, \alpha(\rho)}$$

$$\underline{\psi_p} = \frac{1}{2\pi}\int d\rho\, d\theta\, d\varphi\, J B^{\theta} = \frac{1}{2\pi}\int d\rho\, d\theta\, d\varphi \left(-\frac{\partial\nu}{\partial\varphi}\right)$$

$$= -\frac{1}{2\pi}\int d\rho\, d\theta\, d\varphi \left(\beta(\rho) + \frac{\partial\tilde{\nu}}{\partial\varphi}\right) = \underline{-2\pi\int d\rho\, \beta(\varphi)}$$

$$\Rightarrow \underline{\alpha(\rho) = \frac{1}{2\pi}\frac{d\psi_t}{d\rho}}, \quad \underline{\beta(\rho) = -\frac{1}{2\pi}\frac{d\psi_p}{d\rho}}.$$

$$\therefore \vec{B} = \vec{\nabla}\rho \times \left(\frac{1}{2\pi}\frac{d\psi_t}{d\rho}\vec{\nabla}\theta - \frac{1}{2\pi}\frac{d\psi_p}{d\rho}\vec{\nabla}\varphi + \vec{\nabla}\tilde{\nu}\right)$$

$$= \frac{1}{2\pi}\vec{\nabla}\varphi \times \vec{\nabla}\psi_p + \frac{1}{2\pi}\vec{\nabla}\psi_t \times \vec{\nabla}\theta + \vec{\nabla}\rho \times \vec{\nabla}\tilde{\nu}$$

* Flux Function, $\chi = \frac{\psi_p}{2\pi}$, $\psi = \frac{\psi_t}{2\pi}$, then

$$\underline{\underline{\vec{B} = \vec{\nabla}\varphi \times \vec{\nabla}\chi + \vec{\nabla}\psi \times \vec{\nabla}\theta + \vec{\nabla}\rho \times \vec{\nabla}\tilde{\nu}}}$$

4 Straight-Field-Line (SFL) coordinates

$$\nu = \frac{d\psi}{d\rho}\theta - \frac{d\chi}{d\rho}\varphi + \tilde{\nu}(\rho,\theta,\varphi)$$

☆ If one choose $\tilde{\nu}(\rho,\theta,\varphi) = \tilde{\nu}(\rho)$, then $\nu = \frac{d\psi}{d\rho}\theta_f - \frac{d\chi}{d\rho}\varphi_f + \tilde{\nu}(\rho)$.
$\underline{(\theta,\varphi)} \rightarrow \underline{(\theta^f,\varphi^f)}$

$$\Rightarrow J B^{\theta_f} = -\frac{d\nu}{d\varphi_f} = \frac{d\chi}{d\rho}, \quad J B^{\varphi_f} = \frac{d\nu}{d\theta_f} = \frac{d\psi}{d\rho}$$

$$\Rightarrow \underline{\underline{\frac{d\theta_f}{d\varphi_f} = \frac{B^{\theta_f}}{B^{\varphi_f}} = \frac{d\chi}{d\psi}}}, \quad \underline{\underline{\iota(\rho) = \frac{1}{q(\rho)} = \frac{d\theta_f}{d\varphi_f}}}$$

$$\boxed{\vec{B} = \vec{\nabla}\psi \times \vec{\nabla}\theta_f + \vec{\nabla}\varphi_f \times \vec{\nabla}\chi = \vec{\nabla}\psi \times \vec{\nabla}(\theta_f - \iota\varphi_f) = \vec{\nabla}\psi \times \vec{\nabla}\alpha}$$

$$= \vec{B_T} \qquad \vec{B_p} \; (\vec{\nabla}\chi = \iota(\rho)\vec{\nabla}\psi)$$

(+) $\left(\vec{B}\cdot\vec{\nabla}\varphi_f = \frac{1}{J}, \; \vec{B}\cdot\vec{\nabla}\theta_f = \frac{\iota}{J}\right) \Rightarrow \boxed{\vec{B} = \frac{1}{J}(\iota\vec{e_{\theta_f}} + \vec{e_{\varphi_f}})}$ ☆

But still we have one more freedom, $\theta_f' = \theta_f - \iota w$, $\varphi_f' = \varphi_f - w$.

5 Covariant representation.

(same logic for $\vec{j}$) $\vec{j}\cdot\vec{\nabla}p = 0$, $\vec{\nabla}\cdot\vec{j} = 0$ in ideal equilibria.

but replacing $\psi = I/\mu_0$, $\chi \rightarrow -G/\mu_0$, $\tilde{\nu} = -K/\mu_0$

$$\boxed{\mu_0\vec{j} = \vec{\nabla}I \times \vec{\nabla}\theta + \vec{\nabla}G \times \vec{\nabla}\varphi + \vec{\nabla}K \times \vec{\nabla}\psi} \quad \left(I_t = \frac{2\pi}{\mu_0}I, \; I_p^d = \frac{2\pi}{\mu_0}G\right)$$

$$\downarrow$$

$$\mu_0\vec{j} = \vec{\nabla}\times(I\vec{\nabla}\theta + G\vec{\nabla}\varphi + K\vec{\nabla}\psi) = \vec{\nabla}\times\vec{B}$$

$$\downarrow$$

$$\boxed{\vec{B} = I\vec{\nabla}\theta + G\vec{\nabla}\varphi + K\vec{\nabla}\psi + \vec{\nabla}H}$$

No.

[6] Boozer coordinates (use freedom of $\theta_f = \theta_f' + \iota w,\ \varphi_f = \varphi_f' + w$)

$$\vec{B} = I\vec{\nabla}\theta_f' + G\vec{\nabla}\varphi_f' + K'\vec{\nabla}\psi + \vec{\nabla}H'$$

with $\quad K' = K - w\frac{d}{d\psi}(G+\iota I)$

$\qquad\ \ H' = H + (G+\iota I)w$ chain rule in $\vec{\nabla}H$ 생각하면 가능

* Boozer coordinates ($H'=0$)

$$\boxed{\vec{B} = I_B\vec{\nabla}\theta_B + G\vec{\nabla}\varphi_B + K'\vec{\nabla}\psi} \qquad \left(H'=0 \Leftrightarrow w = -\frac{H}{G+\iota I}\right)$$

$$B^2 = \vec{B}\cdot\vec{B} = \frac{1}{J}(\iota\vec{e_\theta} + \vec{e_\varphi})\cdot(I\vec{\nabla}\theta_B + G\vec{\nabla}\varphi_B + K'\vec{\nabla}\psi) = \frac{G+\iota I}{J}$$

$$\therefore \boxed{J = \frac{G+\iota I}{B^2}}$$

[7] Hamada coordinate

$$K'=0 \Leftrightarrow w = \frac{\frac{d}{d\psi}(G+\iota I)}{K} \Rightarrow \boxed{\vec{B} = I\vec{\nabla}\theta_H + G\vec{\nabla}\varphi_H + \vec{\nabla}H}$$

$$\mu_0\vec{j} = \vec{\nabla}\times\vec{B} = \vec{\nabla}I\times\vec{\nabla}\theta_H + \vec{\nabla}G\times\vec{\nabla}\varphi_H = \frac{dI}{d\psi}\vec{\nabla}\psi\times\vec{\nabla}\theta_H + \frac{dG}{d\psi}\vec{\nabla}\psi\times\vec{\nabla}\varphi_H$$

$$\Rightarrow j^{\theta_H} = \vec{j}\cdot\vec{\nabla}\theta_H = -\frac{1}{\mu_0 J}\frac{dG}{d\psi}, \quad j^{\varphi_H} = \vec{j}\cdot\vec{\nabla}\varphi_H = \frac{1}{\mu_0 J}\frac{dI}{d\psi}$$

$$\therefore \frac{d\theta_H}{d\varphi_H} = \frac{j^{\theta_H}}{j^{\varphi_H}} = -\frac{dG(\psi)}{dI(\psi)} \Rightarrow \text{Both } \vec{B} \text{ and } \vec{J} \text{ are straight in Hamada.}$$

$$\boxed{\mu_0\vec{j} = \frac{1}{J}(-G'\vec{e}_{\theta_H} + I'\vec{e}_{\varphi_H})}$$

$$\mu_0\vec{j}\times\vec{B} = \frac{1}{J^2}(-G'\vec{e}_{\theta_H} + I'\vec{e}_{\varphi_H})\times(\iota\vec{e}_{\theta_H} + \vec{e}_{\varphi_H}) = -\frac{1}{J^2}(G'+\iota I')\,\vec{e}_{\theta_H}\times\vec{e}_{\varphi_H}$$

$(\ )\cdot\vec{e_\psi}$

$$\mu_0 p' = -\frac{1}{J}(G'+\iota I') \Rightarrow \boxed{J = -\frac{G'+\iota I'}{\mu_0 p'}}$$