- # Diagonalization (22)

Suppose n indep. eigenvectors of $\underline{\underline{A}}$

put them in columns of $\underline{\underline{S}}$.

$$\rightarrow \underline{\underline{A}}\,\underline{\underline{S}} = \underline{\underline{A}}\begin{bmatrix} \underline{x_1} & \underline{x_2} & \underline{x_3} \end{bmatrix} = \begin{bmatrix} \lambda_1 \underline{x_1} & \lambda_2 \underline{x_2} & \lambda_3 \underline{x_3} \end{bmatrix} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \underline{\underline{S}}\,\underline{\underline{\Lambda}}$$

$$\therefore \underline{\underline{A}}\,\underline{\underline{S}} = \underline{\underline{S}}\,\underline{\underline{\Lambda}} \rightarrow \boxed{\underline{\underline{\Lambda}} = \underline{\underline{S}}^{-1}\underline{\underline{A}}\,\underline{\underline{S}}, \quad \underline{\underline{A}} = \underline{\underline{S}}\,\underline{\underline{\Lambda}}\,\underline{\underline{S}}^{-1}}$$

rotation span rotation

$\underline{A}\underline{x} = \lambda \underline{x}$, $\underline{A}^2\underline{x} = \lambda \underline{A}x = \lambda^2 \underline{x}$

$$\underline{A}^2 = \underline{S}\,\underline{\Lambda}\,\underline{S}^{-1}\underline{S}\,\underline{\Lambda}\,\underline{S}^{-1} = \underline{S}\,\underline{\Lambda}^2\underline{S}^{-1}$$

← eigenvectors are same as $\underline{\underline{A}}$, but eigenvalues are quadratic form ($\underline{\underline{\Lambda}}^2$)

$\vdots$

$$\underline{\underline{A}}^k = \underline{\underline{S}}\,\underline{\underline{\Lambda}}^k\,\underline{\underline{S}}^{-1}$$

← eigenvalue and eigenvectors gives a great power to $\underline{\underline{A}}^k$

- # Symmetric matrices and positive definiteness (25)

$$A = A^T$$

① The eigenvalues are REAL

② The eigenvectors are PERPENDICULAR

$\Rightarrow$ S can be represented as orthonormal matrix

USUAL : $A = S\Lambda S^{-1}$ → Symmetric case : $A = Q\Lambda Q^{-1} = Q\Lambda Q^T$

$$\boxed{\therefore A = Q\Lambda Q^T}$$

($\underline{A = Q\Lambda Q^{-1}} \rightarrow \underline{A^T = (Q^{-1})^T \Lambda Q^T}$
$\therefore (Q^{-1})^T = Q \rightarrow Q^T = Q^{-1}$)

cf.) orthogonal matrix : $\underline{Q^TQ = QQ^T = I}$ (by definition) $\leftrightarrow$ $\underline{Q^T = Q^{-1}}$ for orthonormal matrix

$$A = Q\Lambda Q^T = \begin{bmatrix} q_1 & \cdots & q_n \end{bmatrix}\begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}\begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix} = \lambda_1 q_1 q_1^T + \cdots + \lambda_n q_n q_n^T$$

positive definite matrix (symmetric)

$$\left\{ \begin{array}{l} ① \ \underline{x}^T \underline{A} \underline{x} > 0 \ \text{ for all arbitrary } \underline{x}. \\ ② \ \text{all eigenvalues are positive (= all pivots are positive)} \\ ③ \ \text{all sub determinants are positive. (c.e. } \begin{bmatrix} -1 & a \\ a & -3 \end{bmatrix} \to (x)) \end{array} \right\}$$

(*) $B = A^T A$ matrix is always <u>square</u>, <u>symmetric</u>, and <u>positive definite</u>
(rank(A)=n)

proof) $x^T B x = x^T A^T A x = (Ax)^T (Ax) = \|Ax\|^2 \geq 0$

· Singular Value Decomposition (29)

$\boxed{A = U\Sigma V^T}$ (where Σ: diagonal, U & V : orthogonal)

$$A = U\Sigma V^{-1} = U\Sigma V^T$$

1) $\underline{A^T A} = V\Sigma^T U^T U \Sigma V^T = V\Sigma^2 V^T = V\begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix} V^T$

V is a matrix of an eigenvector of $A^T A$.

2) $\underline{AA^T} = U\Sigma V^T V \Sigma U^T = U\Sigma^2 U^T = U\begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_n^2 \end{bmatrix} U^T$

U is a matrix of an eigenvector of AA^T.

(summary) $\underline{A = U\Sigma V^T \quad (AA^T \to U,\ A^T A \to V)}$
(U, V: orthonormal , Σ: diagonal)

$V = \begin{bmatrix} v_1 & v_2 \end{bmatrix}$ $\quad A \quad$ $\begin{bmatrix} \sigma_1 & \\ & \sigma_2 \end{bmatrix}$

의미 : 직교하는 벡터 집합에 대하여, 선형변환 후에 그 크기는 변하지만,
여전히 직교할수 있게 되는 그 직교집합은 무엇인가? 그리고 선형변환 후의 결과는 무엇인가?

$U = \begin{bmatrix} u_1 & u_2 \end{bmatrix}$

$\therefore AV = U\Sigma \ \to \ \underline{A = U\Sigma V^T}$

목적 : $A = U\Sigma V^T = \begin{bmatrix} u_1 \cdots u_n \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} \begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} = \underbrace{\sigma_1}_{\text{정보량의 크기}} \overbrace{u_1 v_1^T}^{\text{matrix}} + \cdots + \underbrace{\sigma_n}_{\text{정보량의 크기}} \overbrace{u_n v_n^T}^{\text{matrix}}$

$\Rightarrow$ A를 σ에 따라 여러 matrix로 쪼갠다.