6. Kinematics of rigid-body rotation

① Uniqueness of the angular velocity

- The angular velocity of a rigid body is independent of the choice of the origin of the body axes

proof) For $\underline{r} = \underline{R}_1 + \underline{r}_1 = \underline{R}_2 + \underline{r}_2$,

i) $$\left(\frac{d\underline{r}}{dt}\right)_{lab} = \left(\frac{d\underline{R}_1}{dt}\right)_{lab} + \left(\frac{d\underline{r}_1}{dt}\right)_{lab} = \left(\frac{d\underline{R}_1}{dt}\right)_{lab} + \underline{\omega}_1 \times \underline{r}_1$$

$$= \left(\frac{d\underline{R}_2}{dt}\right)_{lab} + \left(\frac{d\underline{r}_2}{dt}\right)_{lab} = \left(\frac{d\underline{R}_2}{dt}\right)_{lab} + \underline{\omega}_2 \times \underline{r}_2$$

ii) $\underline{R} = \underline{R}_2 - \underline{R}_1$, $$\left(\frac{d\underline{R}}{dt}\right)_{lab} = \left(\frac{d\underline{R}_2}{dt}\right)_{lab} - \left(\frac{d\underline{R}_1}{dt}\right)_{lab} = \underline{\omega}_1 \times \underline{r}_1 - \underline{\omega}_2 \times \underline{r}_2$$

$$= \underbrace{\cancel{\left(\frac{d\underline{R}}{dt}\right)_{body}}}_{0} + \underline{\omega}_1 \times \underline{R} = \underline{\omega}_1 \times (\underline{r}_1 - \underline{r}_2)$$

$$\therefore -\underline{\omega}_2 \times \underline{r}_2 = -\underline{\omega}_1 \times \underline{r}_2 \rightarrow (\underline{\omega}_2 - \underline{\omega}_1) \times \underline{r}_2 = 0 \rightarrow \boxed{\underline{\omega}_1 = \underline{\omega}_2} \quad ▧$$

② Inertia tensor

(A) Angular momentum of a rigid body

- $$\underline{L} = \sum_i m_i \underline{r}_i \times \dot{\underline{r}}_i = \sum_i m_i \left[\underline{r}_i \times (\underline{\omega} \times \underline{r}_i)\right] = \sum_i m_i \left[\underline{\omega} r_i^2 - \underline{r}_i (\underline{r}_i \cdot \underline{\omega})\right]$$

$$\left(L_\alpha = \sum_i m_i \sum_\beta (\delta_{\alpha\beta} r_i^2 - r_{i,\alpha} r_{i,\beta}) \omega_\beta \right)$$

$$(\underline{L} = \underline{\underline{I}}\,\underline{\omega}) \rightarrow \boxed{I_{\alpha\beta} = \sum_i m_i (r_i^2 \delta_{\alpha\beta} - r_{i,\alpha} r_{i,\beta})} \quad (\underline{\underline{I}} = \text{inertia tensor})$$

- $$\underline{L} = \int d^3\underline{r}\, \rho(\underline{r})\, \underline{r} \times \dot{\underline{r}} = \int d^3 r \rho(\underline{r})\, \underline{r} \times (\underline{\omega} \times \underline{r}) \rightarrow \boxed{I_{\alpha\beta} = \int d^3 r \rho(\underline{r}) (r^2 \delta_{\alpha\beta} - r_\alpha r_\beta)}$$

- Change of the origin

$$I_{\alpha\beta} = \int d^3\underline{r}\, \rho(\underline{r}) \left[(\underline{r} + \underline{R}^{(cm)})^2 \delta_{\alpha\beta} - (r_\alpha + R_\alpha^{(cm)})(r_\beta + R_\beta^{(cm)}) \right]$$

$$= \int d^3\underline{r}\, \rho(\underline{r}) (r^2 \delta_{\alpha\beta} - r_\alpha r_\beta) + 2\delta_{\alpha\beta} \underline{R}^{(cm)} \cdot \underbrace{\cancel{\int d^3 r \rho(\underline{r}) \underline{r}}}_{0} - R_\alpha^{(cm)} \underbrace{\cancel{\int d^3 r \rho(\underline{r}) r_\beta}}_{0} - R_\beta^{(cm)} \underbrace{\cancel{\int d^3 r \rho(\underline{r}) r_\alpha}}_{0}$$
$$+ \int d^3\underline{r}\, \rho(\underline{r}) \left[R^{(cm)2} \delta_{\alpha\beta} - R_\alpha^{(cm)} R_\beta^{(cm)} \right]$$

$$= \boxed{I_{\alpha\beta}^{(cm)} + M \left[R^{(cm)2} \delta_{\alpha\beta} - R_\alpha^{(cm)} R_\beta^{(cm)} \right]}$$

$\Rightarrow$ $\underline{\underline{I}}$ changes as if the entire mass is concentrated at the center of mass.

(Remarks)

① $\underline{w}$ 가 $\underline{\underline{I}}$ 의 eigenvector가 아니면, $\underline{L}$ 과 $\underline{w}$ 는 평행하지 않다.

② $r \to r' = Rr$, $\quad \underline{\underline{I}}' = \underline{\underline{R}}\,\underline{\underline{I}}\,\underline{\underline{R}}^{-1}$ (증명 : 노트참고)

(B) Moment of inertia

- Rotational kinetic energy

$$T = \frac{1}{2}\sum_i m_i \dot{\underline{r}}_i^2 = \frac{1}{2}\sum_i m_i \dot{\underline{r}}_i \cdot (\underline{w} \times \underline{r}_i) = \frac{1}{2}\sum_i m_i \underline{w} \cdot (\underline{r}_i \times \dot{\underline{r}}_i) = \frac{1}{2}\underline{w} \cdot \underline{L} = \boxed{\frac{1}{2}\underline{w}^T \underline{\underline{I}}\,\underline{w}}$$

- Moment of inertia

If $\underline{w} = w\hat{\underline{n}}$, we define $I \equiv \hat{\underline{n}}^T \underline{\underline{I}}\,\hat{\underline{n}}$ so that $T = \frac{1}{2}Iw^2$

$$I_{\alpha\beta} = \int d^3r\,\rho(\underline{r})\left(r^2\delta_{\alpha\beta} - r_\alpha r_\beta\right) \to I = \int d^3r\,\rho(\underline{r})\left(r^2 - (\underline{r}\cdot\hat{\underline{n}})^2\right)$$
$$= \int d^3r\,\rho(\underline{r})\,(\underline{r}\times\hat{\underline{n}})^2$$

(Remarks)

① Moment of inertia (관성모멘트)는 회전축이 고정되었을 때 유용

② 회전축이 돌아가는 경우 Inertia tensor (관성텐서)는 const 하더라도,

Moment of inertia (관성모멘트)는 변화할수 있음.

(C) Principal axes

- Diagonalization

$\underline{\underline{I}}$: real & symmetric $\to$ $\underline{\underline{I}} = \underline{\underline{R}}\,\underline{\underline{I}}_D\,\underline{\underline{R}}^T$ $\quad (\underline{\underline{I}}_D = \mathrm{diag}(I_1, I_2, I_3))$

$\to$ I_1, I_2, I_3 : principal moments

$\to$ eigenvector of $\underline{\underline{I}}$: principal axes

- Angular momentum and kinetic energy

principal axes : $\hat{\underline{n}}_1, \hat{\underline{n}}_2, \hat{\underline{n}}_3 \rightarrow \omega_\alpha = \underline{\omega} \cdot \hat{\underline{n}}_\alpha$

$\rightarrow \underline{L} = \sum_\alpha \hat{\underline{n}}_\alpha \cdot (\underline{\underline{I}}\,\underline{\omega}) \cdot \hat{\underline{n}}_\alpha = \sum_\alpha (\underline{\underline{I}} \cdot \hat{\underline{n}}_\alpha) \cdot \underline{\omega}\, \hat{\underline{n}}_\alpha = \sum_\alpha I_\alpha \omega_\alpha \hat{\underline{n}}_\alpha$

$\rightarrow T = \frac{1}{2}\underline{\omega} \cdot \underline{L} = \frac{1}{2}\sum_\alpha I_\alpha \omega_\alpha^2$

$$\boxed{\therefore \underline{L} = \sum_\alpha I_\alpha \omega_\alpha \hat{\underline{n}}_\alpha, \quad T = \frac{1}{2}\sum_\alpha I_\alpha \omega_\alpha^2}$$

- Symmetries and principal axes

$\rightarrow$ If $\underline{\underline{\Sigma}}\,\underline{\underline{I}} = \underline{\underline{I}}\,\underline{\underline{\Sigma}}$ $\quad (\underline{\underline{\Sigma}} = \underline{\underline{I}}\,\underline{\underline{\Sigma}}\,\underline{\underline{I}}^{-1})$ (= T·eigenvector를 섞지 않는 변환),

it is possible to find principal axes $\hat{\underline{n}}_1, \hat{\underline{n}}_2, \hat{\underline{n}}_3$ which are simultaneous eigenvectors of $\underline{\underline{\Sigma}}$ and $\underline{\underline{I}}$.

$\rightarrow$ 회전에 대해 symmetric한 plane을 찾고, 그 plane의 법선벡터가 $\hat{\underline{n}}_i$ 이다.

③ Euler's rotation equations and torque-free rotation

(A) Euler's rotation equations

Euler's rotation equations

$$\boxed{\left(\frac{d\underline{L}}{dt}\right)_{lab}} = \left(\frac{d}{dt}\underline{\underline{I}}\,\underline{\omega}\right)_{lab} = \left(\frac{d}{dt}\underline{\underline{I}}\,\underline{\omega}\right)_{body} + \underline{\omega} \times \underline{\underline{I}}\,\underline{\omega} = \boxed{\underline{\underline{I}}\left(\frac{d\underline{\omega}}{dt}\right)_{body} + \underline{\omega} \times \underline{\underline{I}}\,\underline{\omega} = \underline{\tau}}$$

Using principal axes $\rightarrow$
$$\begin{cases} I_1 \dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 = 0 \\ I_2 \dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 = 0 \\ I_3 \dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 = 0 \end{cases}$$

(B) Torque-free rotation

- Stability of a rotational motion $(\omega_1 \neq 0,\ \omega_2 = \delta\omega_2,\ \omega_3 = \delta\omega_3)$

$$\frac{d}{dt}\begin{bmatrix} \delta\omega_2 \\ \delta\omega_3 \end{bmatrix} = +\begin{bmatrix} 0 & \omega_1 (I_3 - I_1)/I_2 \\ \omega_1 (I_1 - I_2)/I_3 & 0 \end{bmatrix}\begin{bmatrix} \delta\omega_2 \\ \delta\omega_3 \end{bmatrix}$$

Using an ansatz,

$$\begin{pmatrix} \delta\omega_2(t) \\ \delta\omega_3(t) \end{pmatrix} = \mathrm{Re}\left(\begin{bmatrix} \delta\omega_2(0) \\ \delta\omega_3(0) \end{bmatrix} e^{\lambda t}\right) \rightarrow \begin{vmatrix} \lambda & \omega_1 (I_3 - I_1)/I_2 \\ \omega_1 (I_1 - I_2)/I_3 & \lambda \end{vmatrix} = 0$$

$$\boxed{\therefore \lambda^2 = \frac{\omega_1^2}{I_2 I_3}(I_1 - I_2)(I_3 - I_1)}$$

No.

↳ case 1 : If I_1 is max (I_1, I_2, I_3) or min (I_1, I_2, I_3),

$\lambda^2 < 0 \rightarrow \lambda$ is imaginary $\rightarrow$ oscillation (precession)
세차운동

↳ case 2 : If I_1 is intermediate principal moment,

$\lambda^2 > 0 \rightarrow \lambda$ is real-valued $(+/-)$ $\rightarrow$ deviation grow over-time.

$\boxed{\text{Tennis racket theorem}}$

- rotation of a rigid body about its 1st, 3rd principal axes is stable, while rotation about its second principal axis is not.

$$\Rightarrow \begin{pmatrix} \text{stable} : & \text{1st and 3rd} \\ \text{unstable} : & \text{2nd} \end{pmatrix}$$

④ Motion of symmetric top (팽이)

(1) Choice of generalized coordinates

- Principal axes : $\hat{\underline{r_1}} = \hat{\underline{x}}'$, $\hat{\underline{r_2}} = \hat{\underline{y}}'$, $\hat{\underline{r_3}} = \hat{\underline{z}}'$
- Principal moments : $I_1 = I_2 \neq I_3$
- Angular velocity : $\underline{w} = \underline{w_\phi} + \underline{w_\theta} + \underline{w_\psi}$

i) $\underline{w_\phi} = R_z(\psi) R_x(\theta) \begin{bmatrix} 0 \\ 0 \\ \dot{\phi} \end{bmatrix} = \begin{bmatrix} \dot{\phi} \sin\theta \sin\psi \\ \dot{\phi} \sin\theta \cos\psi \\ \dot{\phi} \cos\theta \end{bmatrix}$

ii) $\underline{w_\theta} = R_z(\psi) \begin{bmatrix} \dot{\theta} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \dot{\theta} \cos\psi \\ -\dot{\theta} \sin\psi \\ 0 \end{bmatrix}$

iii) $\underline{w_\psi} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix}$

$$\therefore \underline{w} = \underline{w_\phi} + \underline{w_\theta} + \underline{w_\psi} = \begin{bmatrix} \dot{\phi} \sin\theta \sin\psi + \dot{\theta} \cos\psi \\ \dot{\phi} \sin\theta \cos\psi - \dot{\theta} \sin\psi \\ \dot{\phi} \cos\theta + \dot{\psi} \end{bmatrix} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$$

No.

(2) construction of Lagrangian.

- $T = \frac{1}{2} I_1 (w_1^2 + w_2^2) + \frac{1}{2} I_3 w_3^2 = \frac{1}{2} I_1 (\dot\theta^2 + \dot\phi^2 \sin^2\theta) + \frac{1}{2} I_3 (\dot\psi + \dot\phi \cos\theta)^2$

- $V = Mg\ell\cos\theta$.

(3) conservation laws

- $\frac{\partial L}{\partial \psi} = 0 \to P_\psi = \frac{\partial L}{\partial \dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta) \equiv I_1 a$

- $\frac{\partial L}{\partial \phi} = 0 \to P_\phi = \frac{\partial L}{\partial \dot\phi} = (I_1 \sin^2\theta + I_3\cos^2\theta)\dot\phi + I_3\dot\psi\cos\theta = I_1 b$

- $\frac{\partial L}{\partial t} = 0 \to h = \dot\phi \frac{\partial L}{\partial \dot\phi} + \dot\theta\frac{\partial L}{\partial \dot\theta} + \dot\psi \frac{\partial L}{\partial \dot\psi} - L = T + V \equiv E$.

$$\boxed{\dot\psi = \frac{I_1 a}{I_3} - \frac{b - a\cos\theta}{\sin^2\theta}\cos\theta, \quad \dot\phi = \frac{b - a\cos\theta}{\sin^2\theta}}$$

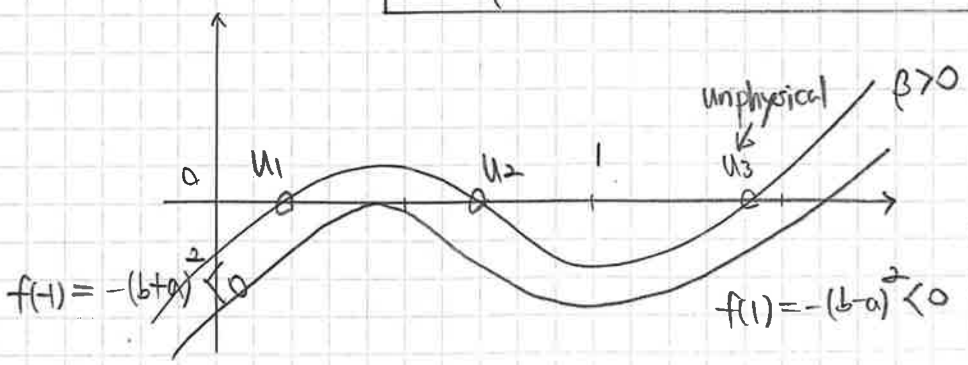
$$\Rightarrow \boxed{E = \frac{I_1^2 a^2}{2 I_3} + \frac{I_1\dot\theta^2}{2} + \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mg\ell\cos\theta}$$

(4) Dynamics of θ (nutation)

- Introduce the constraints $\alpha \equiv \frac{2E}{I_1} - \frac{I_1 a^2}{I_3}$, $\beta = \frac{2Mg\ell}{I_1}$

then, $\boxed{\alpha = \dot\theta^2 + \frac{(b - a\cos\theta)^2}{\sin^2\theta} + \beta\cos\theta}$

- $(u = \cos\theta)$
$$\boxed{\begin{aligned} \dot u^2 &= (1-u^2)(\alpha - \beta u) - (b - au)^2 \\ &= \beta u^3 - (\alpha + a^2)u^2 + (2ab - \beta)u + \alpha - b^2 \equiv f(u) \geq 0 \end{aligned}}$$



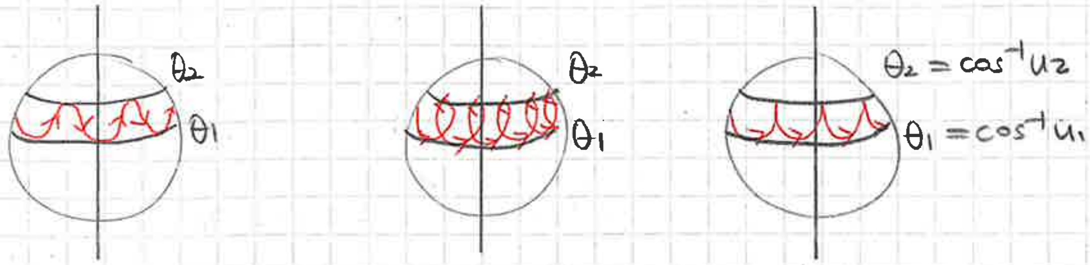
(precession : 세차운동
nutation : 장동운동)

- If $f(u_1) = f(u_2) = 0$ with $-1 < u_1, u_2 < 1$, θ oscillates between $\cos^{-1} u_2 < \theta < \cos^{-1} u_1$

(5) Nutation (θ-oscillation) and Precession (ϕ-oscillation)

$$\dot{\phi} = \frac{b - a\cos\theta}{\sin^2\theta} = \frac{b - au}{1-u^2}$$

If $-1 < u_1, u_2 < 1$, $b/a = P_\phi / P_\psi$



$\frac{b}{a} > u_2$ $\quad$ $u_1 < \frac{b}{a} < u_2$ $\quad$ $\frac{b}{a} = u_2$

∘ Fast-top approximation $\left(\frac{1}{2} I_3 \omega_3^2 \gg Mg\ell \right)$

– Initial condition : $\theta = \theta_0 = \arccos u_0$, $\dot{\theta} = \dot{\phi} = 0$, $\dot{\psi} = \omega_3 = \frac{I_1}{I_3} a = \frac{I_1 b}{I_3 u_0}$ $(u_0 = b/a)$

$$E = \frac{I_1^2 a^2}{2I_3} + Mg\ell u_0 \Rightarrow \boxed{\alpha \equiv \frac{2E}{I_1} - \frac{I_1 a^2}{I_3} = \frac{2Mg\ell u_0}{I_1} = \beta u_0} , \boxed{b = a u_0}$$

$$\Rightarrow \boxed{f(u) = (u_0 - u)\left[\beta(1-u^2) - a^2(u_0 - u)\right]}$$

– maximum value $\theta_1 = \cos^{-1} u \rightarrow \beta(1-u_1^2) - a^2(u_0 - u_1)$

→ Denote amplitude of nutation $A_n \equiv u_0 - u_1$, then

$$\underline{A_n^2 + \left(\frac{a^2}{\beta} - 2u_0\right) A_n + u_0^2 - 1 = 0}$$

→ since $\frac{a^2}{\beta} = \left(\frac{I_3}{I_1}\right) \frac{I_3 \omega_3^2}{2Mg\ell} \gg 1$, $\left(a = \frac{I_3}{I_1}\omega_3 , \beta = \frac{2Mg\ell}{I_1}\right)$

$$\boxed{A_n = \frac{1 - u_0^2 + 2u_0 A_n - A_n^2}{a^2/\beta} \simeq \frac{1-u_0^2}{a^2/\beta} = \frac{I_1}{I_3} \frac{2Mg\ell}{I_3 \omega_3^2} \sin^2\theta_0 \sim \omega_3^{-2}}$$

→ ω_3 커지면 nutation (장동운동) 진폭은 줄어듦

- $\dot{u}^2 = f(u)$, if $x \equiv u_0 - \frac{A_u}{2} - u$ (deviation from midpoint of nutation)

$$\dot{x}^2 = \left(x+\frac{A_u}{2}\right)\left\{\beta\left[1-\left(x-u_0+\frac{A_u}{2}\right)^2\right] - a^2\left(x+\frac{A_u}{2}\right)\right\} \simeq a^2\left(x+\frac{A_u}{2}\right)\left[\underbrace{\frac{\beta}{a^2}(1-u_0)^2}_{=A_u} - \left(x+\frac{A_u}{2}\right)\right]$$

$$\simeq a^2\left(\frac{A_u^2}{4}-x^2\right)$$

since $x(0) = -\frac{A_u}{2}$, $\dot{x}(0) = 0$ $\rightarrow$ $\boxed{x(t) = -\frac{A_u}{2}\cos(at) \quad \text{where } a = \frac{I_3}{I_1}\omega_3 \sim \omega_3^{\,1}}$

$\rightarrow$ ω_3 커지면 진동수도 커짐.

$$- \quad \dot{\phi} = \frac{b-au}{1-u^2} \simeq -\frac{a(u-u_0)}{1-u_0^2} = \frac{a\left(x+\frac{A_u}{2}\right)}{1-u_0^2} = \frac{aA_u}{2(1-u_0^2)}(1-\cos(at)) = \frac{\beta}{2a}[1-\cos(at)]$$

A_u식 대입.

$\boxed{\dot{\phi} = \frac{\beta}{2a}(1-\cos(at))}$, $\boxed{\bar{\dot{\phi}} = \frac{\beta}{2a} = \frac{Mgl}{I_3\omega_3} \sim \omega_3^{-1}}$

- Vertically spinning top ("sleeping top")

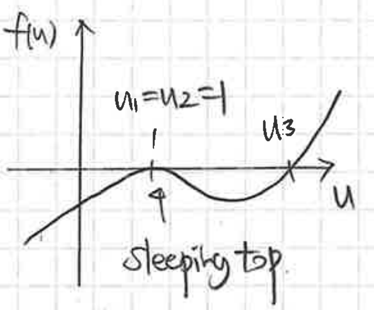
  - If the top is initially spinning vertically, then $a = b = \frac{I_3}{I_1}[\dot{\psi}(0)+\dot{\phi}(0)]$, $\alpha = \beta$

    (∵ (3)의 식에 $\theta = 0$ 대입) (∵ (4)의 식)

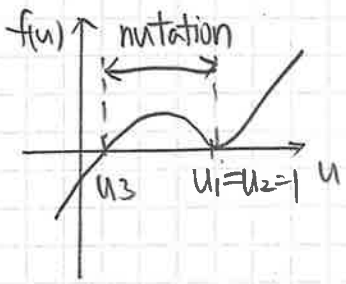
$\rightarrow$ $\dot{u}^2 = f(u) = (1-u^2)\left[\beta(1+u) - a^2\right]$

↳ $f(u)$: double root at $u_1 = u_2 = 1$, 3rd root at $\boxed{u_3 = \frac{a^2}{\beta} - 1 = \frac{I_3^2\omega^2}{2MglI_1} - 1}$

① case 1 ($u_3 \geq 1$)



② case 2 ($u_3 < 1$)



$u_3 \geq 1 \Leftrightarrow \omega_3^2 \geq \frac{4MglI_1}{I_3^2}$ $\Rightarrow$ spinning

$u_3 < 1 \Leftrightarrow \omega_3^2 < \frac{4MglI_1}{I_3}$ $\Rightarrow$ Nutation

Friction makes $\omega_3 \Downarrow$ $\Rightarrow$ starts to nutate.