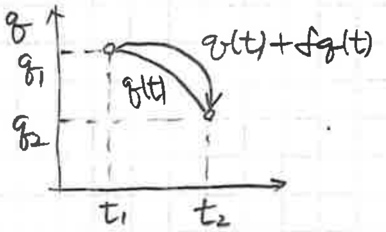
고전역학 2-1 Hamilton's principle

① Hamilton's principle



$$I[q] \equiv \int_{t_1}^{t_2} dt\, L(q, \dot{q}, t)$$

$$\delta I[q] = I[q+\delta q] - I[q] = 0 \quad \text{for actual path}$$

$\longrightarrow$ Hamilton's principle

② Derivation of Lagrange's equations from Hamilton's principle

$$\delta I[q] = \int_{t_1}^{t_2} dt \left[L(q+\delta q, \dot{q}+\delta\dot{q}, t) - L(q, \dot{q}, t) \right] = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q} \cdot \delta q + \underbrace{\frac{\partial L}{\partial \dot{q}} \cdot \delta \dot{q}}_{(*)} \right)$$

(*) $$\int_{t_1}^{t_2} dt\, \frac{\partial L}{\partial \dot{q}_j} \cdot \delta \dot{q}_j = \cancel{\frac{\partial L}{\partial \dot{q}_j} \delta q_j \Big|_{t_1}^{t_2}}^{\,0} - \int_{t_1}^{t_2} dt \left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j$$

$$\therefore \boxed{\delta I[q] = \int_{t_1}^{t_2} dt \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \right) \delta q_j = 0}$$

(1) no constraints

$$\boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \qquad j=1, \cdots, n}$$ Lagrange eqn. of the 2nd kind

(2) with holonomic constraints (1)

$$\boxed{\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_{\alpha=1}^{c} \lambda_\alpha \frac{\partial f_\alpha}{\partial q_j} = 0 \quad \text{for } j=1, \cdots, n}$$ Lagrange eqn of the 1st-kind.

(3) with holonomic constraints (2)

new Lagrangian $\underline{L'(q, \lambda, \dot{q}, \dot{\lambda}; t) \equiv L(q, \dot{q}, t) + \sum_{\alpha=1}^{c} \lambda_\alpha f_\alpha(q, t)}$ ☆

new action $I'[q, \lambda] = \int_{t_1}^{t_2} L'(q, \lambda, \dot{q}, \dot{\lambda}, t)\, dt.$

$$\delta I'[q, \lambda] = \int_{t_1}^{t_2} \left[\left(\frac{\partial L'}{\partial q} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}}\right) \right) \delta q_j + \left(\frac{\partial L'}{\partial \lambda} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\lambda}}\right) \right) \delta \lambda \right] = 0$$

$$\therefore \boxed{\frac{\partial L'}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{q}_j}\right) = 0 \quad \text{for } j=1, \cdots, n} \quad \& \quad \boxed{\frac{\partial L'}{\partial \lambda_\alpha} - \frac{d}{dt}\left(\frac{\partial L'}{\partial \dot{\lambda}_\alpha}\right) = 0 \quad \text{for } \lambda_\alpha = 1, \cdots, c}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \sum_{\alpha=1}^{c} \lambda_\alpha(t) \frac{\partial f}{\partial q_j} = 0 \qquad f_\alpha = 0 \quad ☆$$

No.

③ Advantages of using Hamilton's principle

i) invariant w.r.t the choice of coordinates. (좌표변환해도 E-L eqn 형태변화 X)

ii) extendable to field / continuums.

iii) $L \to L + \dot{G}$ transformation leaves the dynamics invariant.

↳ basis for understanding Noether's theorem

proof) $I = \int L \, dt$

$I' = \int L + \dot{G} \, dt = I + G$

$\left.\begin{array}{l} \delta I = 0 \\ \delta I' = 0 \end{array}\right)$ 두 시스템의 역학동일

2-2 Noether's theorem

① Canonical momentum

(a) $L=\sum_{i=1}^{n}\frac{1}{2}m_i\dot{\underline{r}}_i^{2}-V(\underline{r}_1,\cdots,\underline{r}_n)$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{r}}_i}\right)=\frac{d}{dt}(m_i\dot{\underline{r}}_i)=\frac{d}{dt}\underline{p}_i$, $\frac{\partial L}{\partial \underline{r}_i}=-\frac{\partial V}{\partial \underline{r}_i}=\underline{F}_i$ $\Rightarrow$ $\underline{F}_i=\frac{d}{dt}\underline{p}_i$

(b) Canonical momentum : $p_j(\underline{q},\dot{\underline{q}},t)\equiv\frac{\partial L}{\partial \dot{q}_j}$ ($\neq$ mechanical momentum

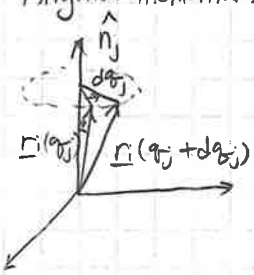
ex) point charge in EM field

$L=\frac{1}{2}m\dot{\underline{r}}^2-q\phi(\underline{r})+q\underline{A}(\underline{r})\cdot\dot{\underline{r}}$

$\rightarrow p=m\dot{r}+q\underline{A}(\underline{r})\neq m\dot{r}$)

generalized force : $Q_j(\underline{q},\dot{\underline{q}},t)\equiv\frac{\partial L}{\partial q_j}$

(c) Angular momentum and torque



$L=\sum_{i=1}^{n}\frac{1}{2}m_i\dot{\underline{r}}_i^{2}-V(\underline{r}_1,\cdots,\underline{r}_n)$

$p_j=\frac{\partial L}{\partial \dot{q}_j}=\sum_{i=1}^{n}\frac{\partial L}{\partial \dot{\underline{r}}_i}\frac{\partial \dot{\underline{r}}_i}{\partial \dot{q}_j}=\sum_{i=1}^{n}m\dot{\underline{r}}_i\frac{\partial \underline{r}_i}{\partial q_j}=\sum_{i=1}^{n}m\dot{\underline{r}}_i\cdot\hat{\underline{n}}_j\times\underline{r}_i=\hat{\underline{n}}_j\cdot\sum_{i=1}^{n}\underline{r}_i\times m\dot{\underline{r}}_i$

$=\hat{\underline{n}}_j\cdot\sum_{i=1}^{n}\underline{r}_i\times\underline{p}_i=\hat{\underline{n}}_j\cdot\underline{L}$ $\quad\therefore\ \boxed{p_j=\hat{\underline{n}}_j\cdot\underline{L}}$

$\left(\begin{array}{l} dr_i=r_i\sin\theta\, dq_j \\ d\underline{r}_i=\hat{\underline{n}}_j\times\underline{r}_i\, dq_j\end{array}\right)$ $Q_j=\frac{\partial L}{\partial q_j}=\sum_{i=1}^{n}\frac{\partial L}{\partial \underline{r}_i}\frac{\partial \underline{r}_i}{\partial q_j}=\sum_{i=1}^{n}\underline{F}_i\cdot\hat{\underline{n}}_j\times\underline{r}_i=\hat{\underline{n}}_j\cdot\sum_{i=1}^{n}\underline{r}_i\times\underline{F}_i=\hat{\underline{n}}_j\cdot\underline{\tau}$

$\therefore\ \boxed{Q_j=\hat{\underline{n}}_j\cdot\underline{\tau}}$

E-L eqn $\Rightarrow$ $\hat{\underline{n}}_j\cdot\dot{\underline{L}}=\hat{\underline{n}}_j\cdot\underline{\tau}$ //

② Cyclic coordinates and conservation laws

(a) $\dot{p}_j=\frac{\partial L}{\partial q_j}$ → if q_j is cyclic/ignorable coordinate, p_j is conserved ($\dot{p}_j=0$)

→ if L is symmetric/invariant under $q_j\rightarrow q_j+dq_j$, p is conserved

(b) $\frac{\partial L}{\partial t}=0$ → $h(\underline{q},\dot{\underline{q}},t)=\underline{p}\cdot\dot{\underline{q}}-L$ is conserved.

$\frac{\partial L}{\partial t}=\frac{dL}{dt}-\left(\frac{\partial L}{\partial \underline{q}}\cdot\frac{d\underline{q}}{dt}+\frac{\partial L}{\partial \dot{\underline{q}}}\cdot\frac{d\dot{\underline{q}}}{dt}\right)=\frac{dL}{dt}-\left[\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{q}}}\right)\dot{\underline{q}}+\frac{\partial L}{\partial \dot{\underline{q}}}\frac{d\dot{\underline{q}}}{dt}\right]$

$=\frac{dL}{dt}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\cdot\dot{q}\right)=-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{q}}}\cdot\dot{\underline{q}}-L\right)=0$

Energy function $\boxed{h(\underline{q},\dot{\underline{q}},t)=\frac{\partial L}{\partial \dot{\underline{q}}}\cdot\dot{\underline{q}}-L=\underline{p}\cdot\dot{\underline{q}}-L}$ note that $h=T+V$ is not always true.

No.

↙ see more examples.

example 3) Rayleigh dissipation $G(\dot{q}) = \frac{1}{2}\sum_{j,k} \gamma_{j,k}\, \dot{q}_j\, \dot{q}_k$

$$\frac{dh}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\cdot\dot{q} + \cancel{\frac{\partial L}{\partial \dot{q}}\cdot\ddot{q}} - \left(\frac{\partial L}{\partial q}\frac{dq}{dt} + \cancel{\frac{\partial L}{\partial \dot{q}}\cdot\frac{d\dot{q}}{dt}}\right)$$

$$= \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}\right)\dot{q} = -\frac{\partial G}{\partial \dot{q}}\cdot\dot{q} = -2G(\dot{q})$$

← G ~ system dissipation energy.

③ Noether's theorem for particles

Consider an infinitesimal transformation $(q(t), t) \to (q'(t'), t')$.

L is invariant under transformation iff

$$I'[q'(t')] = I[q(t)] + \delta\Lambda(q(t_2), t_2) - \delta\Lambda(q(t_1), t_1)$$

$$\delta I = I'[q'(t')] - \left(I[q(t)] + \delta\Lambda(q(t_2), t_2) - \delta\Lambda(q(t_1), t_1)\right)$$

$$= \int_{t_1+\delta t(t_1)}^{t_2+\delta t(t_2)} ds\, L(q'(s), \dot{q}'(s), s) - \int_{t_1}^{t_2} ds\, L(q(s), \dot{q}(s), s)$$

$$- \delta\Lambda(q(t_2), t_2) + \delta\Lambda(q(t_1), t_1)$$

(*1) (*2)

$$= \int_{t_1}^{t_2} ds\left[L(q'(s), \dot{q}'(s), s) - L(q(s), \dot{q}(s), s)\right] + L(q(t_2), \dot{q}(t_2), t_2)\,\delta t(t_2)$$

$$- L(q(t_1), \dot{q}(t_1), t_1)\,\delta t(t_1) - \delta\Lambda(q(t_2), t_2) + \delta\Lambda(q(t_1), t_1)$$

$$(*1) = \int_{t_1}^{t_2} ds\left(\frac{\partial L}{\partial q}\cdot\bar{\delta} q + \frac{\partial L}{\partial \dot{q}}\cdot\bar{\delta}\dot{q}\right)$$

(where $\bar{\delta} q = q'(t) - q(t)$,
$\delta q = q'(t'(t)) - q(t) = \bar{\delta} q + \dot{q}(t)\,\delta t(t)$)

$$= \int_{t_1}^{t_2} ds\left(\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{q}}\right)\cdot\bar{\delta} q + \frac{\partial L}{\partial \dot{q}}\cdot\bar{\delta}\dot{q}\right) = \int_{t_1}^{t_2} ds\,\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{q}}\cdot\bar{\delta} q\right)$$

$$= \int_{t_1}^{t_2} ds\,\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{q}}\delta q - \frac{\partial L}{\partial \dot{q}}\dot{q}\,\delta t\right)$$

$$(*2) = \int_{t_1}^{t_2} ds\,\frac{d}{ds}\left[L(q(s), \dot{q}(s), s)\,\delta t(s) - \delta\Lambda(q(s), s)\right]$$

$$(*1) + (*2) = \int_{t_1}^{t_2} ds\,\frac{d}{ds}\left[\frac{\partial L}{\partial \dot{q}}\delta q - \left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right)\delta t - \delta\Lambda\right] = 0.$$

$$\therefore \boxed{\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}}\delta q - \left(\frac{\partial L}{\partial \dot{q}}\dot{q} - L\right)\delta t - \delta\Lambda\right] = 0}$$ ▪

← applies only to (① monogenic
② continuous and smooth symmetry)

← Inverse Noether theorem 존재 O