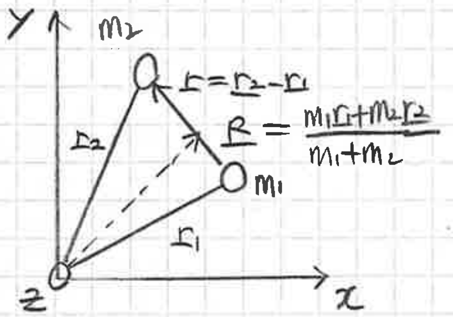
4. Central-force problem

① Central force problem and its constants of motion



(A) $L = \frac{1}{2}m_1\dot{\underline{r_1}}^2 + \frac{1}{2}m_2\dot{\underline{r_2}}^2 - V(|\underline{r_1} - \underline{r_2}|)$

$\downarrow$ $\left(\underline{R} \equiv \dfrac{m_1\underline{r_1} + m_2\underline{r_2}}{m_1+m_2},\ \underline{r} = \underline{r_2} - \underline{r_1}\right)$

$$\boxed{L = \frac{1}{2}(m_1+m_2)\dot{\underline{R}}^2 + \frac{1}{2}\mu\dot{\underline{r}}^2 - V(r)} \quad \left(\mu = \frac{m_1m_2}{m_1+m_2}\right)$$

(B) constants of motion

(i) $\dfrac{\partial L}{\partial \underline{R}} = 0 \rightarrow \dfrac{\partial L}{\partial \dot{\underline{R}}} = (m_1+m_2)\dot{\underline{R}} = \text{const} \xRightarrow[\underline{R}\text{은 이제 무시}]{} \boxed{L = \frac{1}{2}\mu\dot{\underline{r}}^2 - V(r)}$ (one-body problem)

(ii) spherical symmetry : $L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - V(r) \rightarrow \dfrac{\partial L}{\partial \theta} = 0 \rightarrow \dfrac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta} = l$

$\therefore$ angular momentum conservation : $\boxed{l = \mu r^2\dot{\theta}}$

(iii) time-symmetry : $h = \sum \underline{p}\dot{\underline{q}} - L = \dfrac{\partial L}{\partial \dot{\theta}}\dot{\theta} + \dfrac{\partial L}{\partial \dot{r}}\dot{r} - L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$

$$= \frac{1}{2}\mu\dot{r}^2 + V(r) + \frac{l^2}{2\mu r^2} = E.$$

$\therefore$ energy conservation : $\boxed{E = \frac{1}{2}\mu\dot{r}^2 + V(r) + \frac{l^2}{2\mu r^2}}$ $\left(V_{eff} = V(r) + \dfrac{l^2}{2\mu r^2}\right)$

② Formal solution for the orbit geometry (dependence of r on θ)

$$l = \mu r^2\dot{\theta} \rightarrow dt = \frac{\mu r^2}{l}d\theta = \frac{\mu}{lu^2}d\theta$$

$$E = \frac{1}{2}\mu\dot{r}^2 + V(r) + \frac{l^2}{2\mu r^2} \underset{*}{=} \frac{l^2}{2\mu}\left(\frac{du}{d\theta}\right)^2 + V(u^{-1}) + \frac{l^2}{2\mu r^2}$$

$$\left(\frac{dr}{dt} = \left(-\frac{1}{u^2}\right)\frac{du}{dt} = \left(-\frac{1}{u^2}\right)\frac{lu^2du}{\mu d\theta} = -\frac{l}{\mu}\frac{du}{d\theta}\right)$$

$$\boxed{\left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{l^2}\left[E - V(u^{-1})\right] - u^2}$$

$\theta = \theta^*$; $\dfrac{du}{d\theta^*} = 0$: apsis $\begin{cases} \text{periapsis (근점)} \\ \text{apoapsis (원점)} \end{cases}$

$(r = r_{min}, \theta = 0) \rightarrow (r = r_{max}; \theta = \theta_a')$

$$\frac{du}{d\theta} = -\left\{\frac{2\mu}{l^2}\left[E - V(u^{-1})\right] - u^2\right\}^{1/2} \Rightarrow \boxed{\theta = -\int_{r_{min}^{-1}}^{r^{-1}} \frac{du}{\left\{\frac{2\mu}{l^2}\left[E - V(u^{-1})\right] - u^2\right\}^{1/2}}}$$

③ Kepler problem

(A) orbit geometry. $\left(V(r) = -\frac{k}{r} = -ku\right)$

$$\theta = -\int_{r_{min}^{-1}}^{r^{-1}} \frac{du}{\sqrt{\frac{2\mu E}{l^2} + \frac{2\mu k}{l^2}u - u^2}} \underset{\uparrow}{=} \arccos \left. \frac{2u - \frac{2\mu k}{l^2}}{\sqrt{\frac{4\mu^2 k^2}{l^4} + \frac{8\mu E}{l^2}}} \right|_{u=r_{min}^{-1}}^{u=r^{-1}}$$

$$\int \frac{dx}{\sqrt{\alpha + \beta x - x^2}} = -\arccos \frac{2x - \beta}{\sqrt{\beta^2 + 4\alpha}} + \text{cost}$$

$$\theta = \arccos \frac{\frac{l^2}{\mu k r} - 1}{\sqrt{1 + \frac{2El^2}{\mu k^2}}}$$

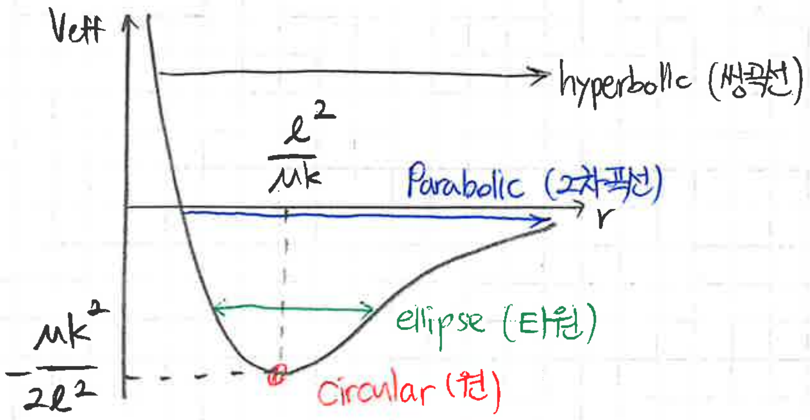
$(\because \theta' = 0 \text{ at } r = r_{min})$ ← 우리가 이렇게 정의함.

↳ $\cos\theta' = \cos 0 = 1$

Define $\boxed{e = \sqrt{1 + \frac{2El^2}{\mu k^2}}}$ → $e\cos\theta + 1 = \frac{l^2}{\mu k r}$ ∴ $\boxed{r = \frac{l^2}{\mu k}(1 + e\cos\theta)}$

eccentricity (이심률)

- 4 categories $\left(V_{eff} = -\frac{k}{r} + \frac{l^2}{2\mu r^2},\ V_{eff}' = +\frac{k}{r^2} - \frac{l^2}{\mu r^3}\right)$



No.

(i) Hyperbolic (쌍곡선) $(E>0,\ e>1)$

$$r=\frac{\ell^2}{\mu k}\frac{1}{(1+e\cos\theta)}=\frac{\mu k^2}{2E\ell^2}\cdot\frac{\ell^2}{\mu k}\frac{(e^2-1)}{1+e\cos\theta}=\frac{k}{2E}\frac{e^2-1}{1+e\cos\theta}$$

$$\left(\text{semi-major axis}: a\equiv\frac{k}{2E}\right)=\frac{a(e^2-1)}{1+e\cos\theta}>r_{min}=a(e-1)$$

$$\therefore \boxed{r=\frac{a(e^2-1)}{1+e\cos\theta}>r_{min}=a(e-1)} \quad \text{where } |\cos\theta|<e^{-1}\Leftrightarrow|\theta|<\cos^{-1}e^{-1}$$

(ii) Parabolic (2차곡선) $(E=0,\ e=1)$

$$r=\frac{\ell^2}{\mu k}\frac{1}{(1+e\cos\theta)}=\frac{\ell^2}{\mu k}\frac{1}{1+\cos\theta}>r_{min}=\frac{\ell^2}{2\mu k}$$

$$r_{min}=\frac{\ell^2}{2\mu k}\rightarrow\boxed{r=\frac{2r_{min}}{1+\cos\theta}}$$

(iii) Ellipse (타원) $\left(-\frac{\mu k^2}{2\ell^2}<E<0,\ 0<e<1\right)$

$$r=\frac{\ell^2}{\mu k}\frac{1}{1+e\cos\theta}=\frac{k}{2E}\frac{e^2-1}{1+e\cos\theta}=-\frac{k}{2E}\frac{1-e^2}{1+e\cos\theta}$$

$$\left(\text{semi-major axis}: a\equiv-\frac{k}{2E}\right)=\frac{a(1-e^2)}{1+e\cos\theta}\rightarrow\boxed{r=\frac{a(1-e^2)}{1+e\cos\theta}}$$

$$\boxed{r=\frac{a(1-e^2)}{1+e\cos\theta}},\quad \underline{r_{min}=a(1-e)<r<r_{max}=a(1+e)}$$

(iv) circular (원) $\left(E=-\frac{\mu k}{2\ell^2},\ e=0\right)$

$$\boxed{\therefore r=\frac{\ell^2}{\mu k}}$$

(B) Kepler's Law of planetary motion

① 행성은 타원운동

② 행성과 태양이 휩쓰는 단위시간 당 면적은 항상 같다.

③ 행성주기 (T) 와 단반경(a)는 $T^2 \propto a^3$ 관계가 있다.

proof) ②

ΔA, $\Delta\theta$, O

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{l}{2\mu} = const$$

③

$$A = \pi a^2 \sqrt{1-e^2} = \pi a^2 \sqrt{-\frac{2El^2}{\mu k^2}} = \pi a^2 \sqrt{\frac{l^2}{a\mu k}} = \frac{\pi a^{3/2} l}{\sqrt{\mu k}}$$

$\left(a = -\frac{\mu}{2E}\right)$

$\left(e = \sqrt{1 + \frac{2El^2}{\mu k^2}}\right)$

$$\tau = \frac{A}{dA/dt} = \frac{\pi a^{3/2} l}{\sqrt{\mu k}} \cdot \frac{2\mu}{l} = 2\pi a^{3/2} \sqrt{\frac{\mu}{k}} = \frac{2\pi a^{3/2}}{\sqrt{G(m_1+m_2)}} \simeq \frac{2\pi a^{3/2}}{\sqrt{GM_\odot}}$$

from (1)

(C) Laplace-Runge-Lenz vector

(1) conservation law

$$\frac{1}{2}\frac{d}{dt}(\mathbf{r}\cdot\mathbf{r}) = r\dot{r}$$

$$\dot{\mathbf{p}} = -\frac{k}{r^3}\mathbf{r} \rightarrow \dot{\mathbf{p}} \times \mathbf{L} = -\frac{\mu k}{r^3}\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = -\frac{\mu k}{r^3}\left[\mathbf{r}(\mathbf{r}\cdot\dot{\mathbf{r}}) - r^2 \dot{\mathbf{r}}\right]$$

$$= \mu k \left(-\frac{\dot{r}}{r^2}\mathbf{r} + \frac{1}{r}\dot{\mathbf{r}}\right) = \frac{d}{dt}(\mu k \hat{\mathbf{r}})$$

$$\therefore \frac{d\mathbf{A}}{dt} = \frac{d}{dt}(\mathbf{p} \times \mathbf{L} - \mu k \hat{\mathbf{r}}) = 0 \rightarrow \boxed{\mathbf{A} = \mathbf{p} \times \mathbf{L} - \mu k \hat{\mathbf{r}}}$$ is conserved

(Laplace-Runge-Lenz vector)

(2) Derivation of orbit equation

$\underline{L} = l\hat{z} \rightarrow \underline{L}\cdot\underline{A} = 0 \rightarrow \underline{A}$ is in x-y plane.

$$\underline{r}\cdot\underline{A} = \underline{r}\cdot(\underline{p}\times\underline{L}) - \mu k r = \underline{L}\cdot(\underline{r}\times\underline{p}) - \mu k r = l^2 - \mu k r$$

$$= A r\cos\theta$$

$$\therefore (A\cos\theta + \mu k) r = l^2 \qquad \therefore \underline{\underline{r = \frac{l^2}{\mu k + A\cos\theta} \geq r_{min} = \frac{l^2}{\mu k + A}}}$$

$$\text{since } E = -\frac{k}{r_{min}} + \frac{l^2}{2\mu r_{min}^2} = \frac{A^2 - \mu^2 k^2}{2\mu l^2} \rightarrow A = \mu k\sqrt{1 + \frac{2El^2}{\mu k^2}} = \mu k e.$$

$$\boxed{\therefore r = \frac{l^2}{\mu k}\frac{1}{1 + e\cos\theta}}$$

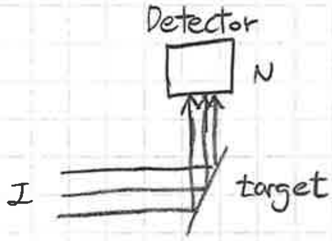
④ Conditions for closed orbits

모든 bound orbit이 닫힌 orbit이 되기 위해서는, 아래 두 종류의 퍼텐셜만 가능하다.

(1) $\underline{V(r) = -k/r}$, (2) $\underline{V(r) = \frac{1}{2}kr^2}$

⑤ Scattering in a central-force field

(A) Scattering cross-section

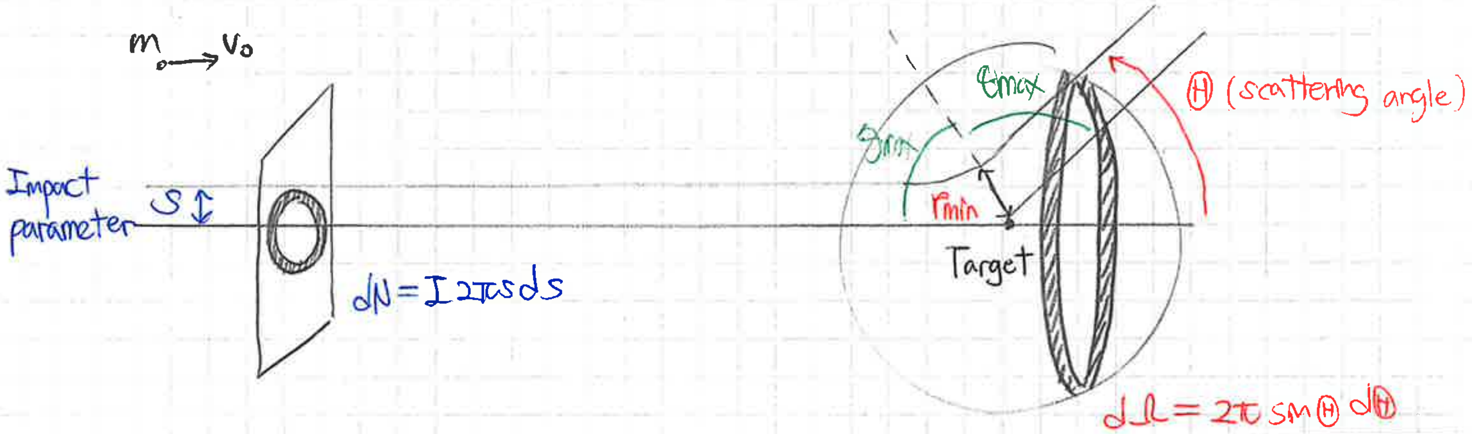


I [unit area·time]

N [unit time]

$$\sigma \equiv N/I \quad \text{(area)}$$

(b) Differential scattering cross-section in a central-force field



$$d\Omega = 2\pi \sin\Theta \, d\Theta$$

$$dN = I\,d\sigma = I \frac{d\sigma}{d\Omega} d\Omega \Rightarrow \frac{d\sigma}{d\Omega} = \frac{dN}{I\,d\Omega} = \frac{2\pi s\,ds}{2\pi \sin\Theta\, d\Theta} \quad \therefore \boxed{\frac{d\sigma}{d\Omega} = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right|}$$

(c) Rutherford Scattering

$$\theta = -\int_{r_{min}^{-1}}^{r^{-1}} \frac{du}{\sqrt{\frac{2\mu E}{\ell^2} + \frac{2\mu k}{\ell^2}u - u^2}} = \arccos \frac{2u + \frac{2\mu k}{\ell^2}}{\sqrt{\frac{4\mu^2k^2}{\ell^4} + \frac{8\mu E}{\ell^2}}}\Bigg|_{u=r_m^{-1}}^{u=r^{-1}}$$

$$\theta = \arccos \frac{\frac{\ell^2}{\mu k r} + 1}{\sqrt{1 + \frac{2E\ell^2}{\mu k^2}}} \quad \to \quad 1 + \frac{\ell^2}{\mu k r} = e\cos\theta \underset{(\mu = m)}{\to} \boxed{r = \frac{\ell^2}{mk}\frac{1}{e\cos\theta - 1}}$$

since $E > 0$, $e > 1$, $|\theta| < \theta_{max} = \arccos e^{-1}$ $\left(\cos\theta_{max} = \frac{1}{e}\right)$

$$\Theta = \pi - 2\theta_{max}$$

i) $\sin\frac{\Theta}{2} = \sin\left(\frac{\pi}{2} - \theta_{max}\right) = \cos\theta_{max} = \frac{1}{e}$

ii) $\cot^2\frac{\Theta}{2} = \csc^2\frac{\Theta}{2} - 1 = e^2 - 1 = \frac{2E\ell^2}{mk^2} \underset{\uparrow}{=} \frac{2Es^2 2mE}{mk^2} \rightarrow \cot\frac{\Theta}{2} = \frac{2Es}{k}$

$\ell = smv_0 = s\sqrt{2mE}$

iii) $-\csc^2\frac{\Theta}{2} \cdot \frac{d\Theta}{2} = \frac{2E}{k}ds \rightarrow \frac{ds}{d\Theta} = -\frac{k}{4E}\csc^2\frac{\Theta}{2}$

iv) $\sin\Theta = 2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}$

$$\therefore \frac{d\sigma}{d\Omega} = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| = \left(\frac{k}{2E}\cot\frac{\Theta}{2}\right)\cdot\left(\frac{1}{2\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}}\right)\cdot\left(\frac{k}{4E}\csc^2\frac{\Theta}{2}\right)$$

$$= \frac{k^2}{16E^2}\csc^4\frac{\Theta}{2}$$

$$\boxed{\therefore \frac{d\sigma}{d\Omega} = \frac{s}{\sin\Theta}\left|\frac{ds}{d\Theta}\right| = \frac{k^2}{16E^2}\csc^4\frac{\Theta}{2}}$$