7. Lagrangian formalism in the continuum

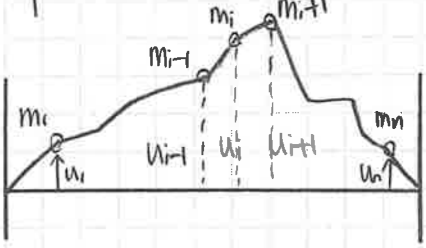
Example 1-d string

Hamilton's principle in the continuum

Symmetries and conservation laws

Example : Homogeneous 1-d string with periodic boundary conditions

① Example : 1-d string



$$m_i \ddot{u}_i = \tau_i \sin\theta_i - \tau_{i-1}\sin\theta_{i-1} \qquad (u_0 = u_{n+1} = 0)$$

(Assume small $|\theta| \ll 1$)

$$m\ddot{u}_i = \tau_i \tan\theta_i - \tau_{i-1}\tan\theta_{i-1} = \tau_i \frac{u_{i+1}-u_i}{a} - \tau_{i-1}\frac{u_i - u_{i-1}}{a}$$

which gives, $L = \sum_{i=1}^{n}\left[\frac{1}{2}m_i \dot{u}_i^2 - \frac{1}{2}\frac{\tau_i}{a}(u_{i+1}-u_i)^2\right]$

* continuum limit $a \to 0,\ n \to \infty$, while $l = (n+1)a = \text{const}$

* continuous spatial coordinate $x = ia$, $u_i(t) \to u(x,t)$, $\tau_i \to \tau(x)$, $\frac{m_i}{a} = \lambda(x)$

$$\frac{m_i \ddot{u}_i}{a} = \frac{1}{a}\left(\tau_i \frac{u_{i+1}-u_i}{a} - \tau_{i-1}\frac{u_i - u_{i-1}}{a}\right) \to \boxed{\lambda(x)\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left(\tau(x)\frac{\partial u}{\partial x}\right)}$$ General 1-D spring equation.

For const λ and τ, we obtain wave equation $\frac{\partial^2 u}{\partial t^2} = \frac{\tau}{\lambda}\frac{\partial^2 u}{\partial x^2}$

$$L = \sum_{i=1}^{n} a\left[\frac{1}{2}\frac{m_i}{a}\dot{u}_i^2 - \frac{1}{2}\tau_i\left(\frac{u_{i+1}-u_i}{a}\right)^2\right] \to \int_0^l dx \underbrace{\left[\frac{1}{2}\lambda(x)\left(\frac{\partial u}{\partial t}\right)^2 - \frac{1}{2}\tau(x)\left(\frac{\partial u}{\partial x}\right)^2\right]}_{\equiv \mathcal{L}\text{agrangian density}}$$

$\therefore$ Lagrangian density $\boxed{\mathcal{L} \equiv \mathcal{L}\left(u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}; x, t\right)}$

② Hamilton's principle in the continuum

Definition : vector $x_\nu : (x_0, x_1, \cdots, x_d) = (t, \underline{x})$

field u_ρ

(notation) $u_{\rho,\nu} \equiv \partial u_\rho / \partial x_\nu$

Hamilton's principle in the continuum.

$$\boxed{\delta \int_\Omega d^{d+1}\underline{x}\, \mathcal{L}(u_\rho, u_{\rho,\nu}; x_\nu) = 0}$$

← 시공간적분. (∵ Lagrangian density) Ω

Derivation of Lagrange's equation

$$\delta \int_\Omega d^{d+1}\underline{x}\, \mathcal{L}(u_\rho, u_{\rho,\nu}; x_\nu) = \int_\Omega d^{d+1}\left(\frac{\partial L}{\partial u_\rho}\delta u_\rho + \frac{\partial L}{\partial u_{\rho,\nu}}\delta u_{\rho,\nu}\right) = \int_\Omega d^{d+1}\underline{x}\left(\frac{\partial L}{\partial u_\rho} - \frac{\partial}{\partial x_\nu}\frac{\partial L}{\partial u_{\rho,\nu}}\right)\delta u_\rho = 0$$

$$\therefore \boxed{\frac{\partial L}{\partial u_\rho} - \frac{\partial}{\partial x_\nu}\frac{\partial L}{\partial u_{\rho,\nu}} = 0}$$

Applications to 1-d string for $\mathcal{L} = \frac{1}{2}\lambda(x)\left(\frac{du}{dt}\right)^2 - \frac{1}{2}\tau(x)\left(\frac{du}{dx}\right)^2$

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial t}\frac{\partial L}{\partial(du/dt)} - \frac{\partial}{\partial x}\frac{\partial L}{\partial(du/dx)} = -\lambda(x)\frac{\partial^2 u}{\partial t^2} - \frac{\partial}{\partial x}\left[\tau(x)\frac{du}{dx}\right] = 0$$

③ Symmetries and conservation laws

A. Noether's theorem

$\underline{x_\mu \to x_\mu' = x_\mu + \delta x_\mu}$, $\underline{u_\rho(x_\mu) \to u_\rho'(x_\mu') = u_\rho(x_\mu) + \delta u_\rho(x_\mu)}$

Symmetric condition

$$\boxed{\int_{\Omega'} d^{d+1}x_\mu'\, L(u_\rho', u_{\rho,\mu}'; x_\mu') = \int_\Omega d^{d+1}x_\mu\left[L(u_\rho, u_{\rho,\mu}; x_\mu) + \frac{\partial}{\partial x_\nu}\delta\Lambda_\nu\right]} \quad (1)$$

Derivation of the conservation law

$$\int_{\Omega'} d^{d+1}x'_\mu\, \mathcal{L}(u'_\rho, u'_{\rho,\mu}; x'_\mu) = \int_\Omega d^{d+1} L(u'_\rho, u'_{\rho,\mu}; x_\mu) + \int_{\partial\Omega} dS_\nu\, \delta x_\nu\, L(u_\rho, u_{\rho,\mu}; x_\mu)$$

$$= \int_\Omega d^{d+1}x_\mu \Big[L(u'_\rho, u'_{\rho,\mu}; x_\mu) + \underbrace{\frac{\partial}{\partial x_\nu}\delta x_\nu\, L(u_\rho, u_{\rho,\mu}; x_\mu)}_{(2)} \Big]$$

(writing $u'_\rho(x_\mu) = u_\rho(x_\mu) + \bar{\delta} u_\rho(x_\mu)$)

$$L(u'_\rho, u'_{\rho,\mu}; x_\mu) = L(u_\rho, u_{\rho,\mu}; x_\mu) + \frac{\partial L}{\partial u_\rho}\bar{\delta} u_\rho + \frac{\partial L}{\partial u_{\rho,\mu}}\bar{\delta} u_{\rho,\nu}$$

↙ Euler-Lagrange equation

$$= L(u_\rho, u_{\rho,\mu}; x_\mu) + \underbrace{\frac{\partial}{\partial x_\nu}\left(\frac{\partial L}{\partial u_{\rho,\nu}}\bar{\delta} u_\rho\right)}_{(3)}$$

(1)+(2)+(3)

$$\int_\Omega d^{d+1}x_\mu \frac{\partial}{\partial x_\nu}\left(\frac{\partial L}{\partial u_{\rho,\nu}}\bar{\delta} u_\rho + L\,\delta x_\nu - \delta\Lambda_\nu\right) = 0$$

$$\therefore \boxed{\frac{\partial}{\partial x_\nu}\left(\frac{\partial L}{\partial u_{\rho,\nu}}\bar{\delta} u_\rho - L\,\delta x_\nu - \delta\Lambda_\nu\right) = 0}$$

parametrized infinitesimal transformation

$$\delta x_\mu = \epsilon_r X_{r\mu}, \quad \delta u_\rho = \epsilon_r U_{r\rho}, \quad \delta\Lambda_\mu = \epsilon_r G_{r\mu}$$

(note: $\delta u_\rho = u'_\rho(x'_\mu) - u_\rho(x_\mu) = u'_\rho(x'_\mu) - u_\rho(x_\mu) + \bar{\delta} u_\rho = \delta x_\mu u_{\rho,\nu} + \bar{\delta} u_\rho$)

$$\therefore \rightarrow \bar{\delta} u_\rho = \epsilon_r (U_{r\rho} - u_{\rho,\mu} X_{r\mu})$$

$$\therefore \frac{\partial}{\partial x_\nu}\left[\left(L\delta_{\mu\nu} - \frac{\partial L}{\partial u_{\rho,\nu}} u_{\rho,\nu}\right) X_{r\mu} + \frac{\partial L}{\partial u_{\rho,\nu}} U_{r\rho} - G_{r\nu}\right] = 0$$

Continuity equation

↙ Noether charge

$$\boxed{\begin{aligned} &\frac{\partial}{\partial t}\left(L X_{r0} - \frac{\partial L}{\partial u_{\rho,0}} u_{\rho,\mu} X_{r\mu} + \frac{\partial L}{\partial u_{\rho,0}} U_{r\rho} - G_{r0}\right) \\ &= -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left[\left(L\delta_{i\mu} - \frac{\partial L}{\partial u_{\rho,i}} u_{\rho,\mu}\right) X_{r\mu} + \frac{\partial L}{\partial u_{\rho,i}} U_{r\rho} - G_{ri}\right] \end{aligned}}$$

↖ Noether current

∴ For an infinitesimal coordinate or field transformation, there corresponds a local conservation law.

↳ Noether's theorem

B. Stress-energy tensor

- If system is symmetric in x_μ-direction, Noether's theorem implies

$$\frac{\partial}{\partial x_\nu} T_{\mu\nu} = 0 \quad \text{where} \quad \boxed{T_{\mu\nu} \equiv \frac{\partial L}{\partial u_{\rho,\nu}} u_{\rho,\mu} - L\,\delta_{\mu\nu}} \ : \text{Canonical stress-energy tensor}$$

- Components of $T_{\mu\nu}$

T_{00}	field energy density
T_{0j}	field energy current density in the x_j-direction
$-T_{i0}$	field momentum density, i-th component
$-T_{ij}$	current density in the x_j-direction for i-th component of the field momentum density

$$\frac{d}{dt} P_i = -\frac{d}{dt}\int_V d^d\underline{x}\, T_{i0} \underset{\text{연속방정식}}{=} \sum_{j=1}^{d} \int_V d^d\underline{x} \frac{\partial}{\partial x_j} T_{ij} \underset{\text{발산정리}}{=} \sum_{j=1}^{d} \int_{\partial V} dS\, \hat{\underline{n}}_j T_{ij} = F_i$$

- General case (no symmetry in x_μ-direction)

$$\frac{\partial}{\partial x_\nu} T_{\mu\nu} = \frac{\partial}{\partial x_\nu}\left(\frac{\partial L}{\partial u_{\rho,\nu}} u_{\rho,\mu}\right) - \frac{\partial L}{\partial u_\rho} u_{\rho,\mu} - \frac{\partial L}{\partial u_{\rho,\nu}} u_{\rho,\mu\nu} - \frac{\partial L}{\partial x_\mu}$$

$$= \cancelto{0}{\left(\frac{\partial}{\partial x_\nu}\frac{\partial L}{\partial u_{\rho,\nu}} - \frac{\partial L}{\partial u_\rho}\right)} u_{\rho,\mu} - \frac{\partial L}{\partial x_\mu} = -\frac{\partial L}{\partial x_\mu} \qquad \therefore \boxed{\frac{\partial}{\partial x_\nu} T_{\mu\nu} = -\frac{\partial L}{\partial x_\mu}}$$

($T_{\mu 0}$ is locally conserved if L is not explicitly dependent on x_μ)

④ Example : Homogeneous 1-d string with periodic boundary condition $[u(0,t)=u(\ell,t)]$

$$L=\frac{\lambda}{2}\left(\frac{\partial u}{\partial t}\right)^2-\frac{\tau}{2}\left(\frac{\partial u}{\partial x}\right)^2 \Rightarrow \frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial u/\partial t)}=-\frac{\partial}{\partial x}\frac{\partial L}{\partial(\partial u/\partial x)} \Rightarrow \frac{\partial^2 u}{\partial t^2}=\frac{\tau}{\lambda}\frac{\partial^2 u}{\partial x^2}$$

$$\therefore u_n^{\pm}(x,t)\equiv A\cos\left[k_n(x\pm c_s t)\right] \quad \text{where } c_s\equiv\sqrt{\frac{\tau}{\lambda}},\ k_n=\frac{2\pi n}{\ell}$$

- Vertical motion ~ momentum density

$$P=\frac{\partial L}{\partial(\partial u/\partial t)}=\lambda\frac{\partial u}{\partial t}, \quad \frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}\frac{\partial L}{\partial(\partial u/\partial x)}=\frac{\partial}{\partial x}\tau\left(\frac{\partial u}{\partial x}\right)$$

- Energy density and energy current density

$$T_{00}=\frac{\partial L}{\partial(\partial u/\partial t)}\frac{\partial u}{\partial t}-L=\frac{\lambda}{2}\left(\frac{\partial u}{\partial t}\right)^2+\frac{\tau}{2}\left(\frac{\partial u}{\partial x}\right)^2$$

$$T_{01}=\frac{\partial L}{\partial(\partial u/\partial x)}\frac{\partial u}{\partial t}=-\tau\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}$$

- x-momentum density and x-momentum current density

$$-T_{10}=\frac{\partial L}{\partial(\partial u/\partial t)}\frac{\partial u}{\partial x}=-\lambda\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}$$

$$-T_{11}=\frac{\partial L}{\partial(\partial u/\partial x)}\frac{\partial u}{\partial x}+L=\frac{\lambda}{2}\left(\frac{\partial u}{\partial t}\right)^2+\frac{\tau}{2}\left(\frac{\partial u}{\partial x}\right)^2$$

- For the traveling-wave solution $u=u_n^{\pm}(x,t)=A\cos\left[k_n(x\pm c_s t)\right]$

$$T_{00}=-T_{11}=\tau k_n^2 A^2\sin^2\left[k_n(x\pm c_s t)\right]$$

$$T_{01}=-c_s^2 T_{10}=\mp\tau c_s k_n^2 A^2\sin^2\left[k_n(x\pm c_s t)\right]=\mp c_s T_{00}$$

$$\therefore\ \underbrace{c_s=\frac{|T_{01}|}{T_{00}}}\ :\ \underbrace{T_{10}=\frac{T_{00}}{c_s}}$$

Traveling wave carries energy current at speed c_s in the direction in which it travels.
Traveling wave has a momentum density whose magnitude is given by T_{00}/c_s in the direction in which it travels.