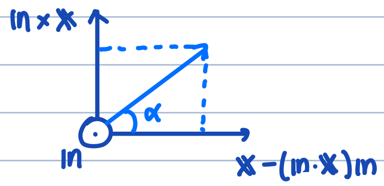
Ch6. Motion and deformation

(How a particle moves in relation to its neighbouring particles)

6.1 Rigid body motion

$x_i = x_i(X_R, t)$, or, $\mathbf{x} = \mathbf{x}(\mathbf{X}, t)$

- Translation : $\boxed{x_i = X_R + C_i(t) \text{ , or. } \mathbf{x} = \mathbf{X} + \mathbf{C}(t)}$
- Rotation : axis of rotation $\mathbf{n}$, by α.



$$\boxed{\begin{aligned} \mathbf{x} &= (\mathbf{n}\cdot\mathbf{X})\mathbf{n} + (\mathbf{X} - (\mathbf{n}\cdot\mathbf{X})\mathbf{n})\cos\alpha + (\mathbf{n}\times\mathbf{X})\sin\alpha \\ x_i &= X_i\cos\alpha + e_{ijR}n_jX_R\sin\alpha + (1-\cos\alpha)X_Rn_Rn_i \end{aligned}}$$

as, $\underline{x_i = Q_{iR}X_R}$

$$\boxed{Q_{iR} = \delta_{iR}\cos\alpha + e_{ijR}n_j\sin\alpha + (1-\cos\alpha)n_in_R}$$

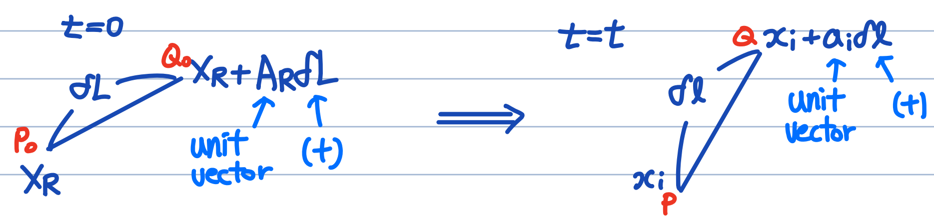
$\therefore\ \mathbf{x} = \mathbf{Q}\cdot\mathbf{X} \leftrightarrow \mathbf{X} = \mathbf{Q}^T\cdot\mathbf{x}$

→ Any rigid body motion :
$$\boxed{\begin{cases} \mathbf{x} = \mathbf{Q}(t)\cdot\mathbf{X} + \mathbf{C}(t) \\ \mathbf{X} = \mathbf{Q}^T(t)\cdot\mathbf{x} - \mathbf{Q}^T(t)\mathbf{C}(t) \end{cases}}$$

6.2 Extension of a material line element

Deformation = change of shape $\Leftrightarrow$ change of position & orientation

change of distance between particles



$P:\ x_i^{(0)} = x_i(X_R^{(0)})$

$Q:\ x_i^{(0)} + a_i\delta l = x_i(X_R^{(0)} + A_R\delta L)$

(by tailor theorem)

$$\underline{x_i^{(0)} + a_i\delta l} = x_i(X_R^{(0)}) + A_S\delta L\frac{\partial x_i(X_R^{(0)})}{\partial X_S} + O\{(\delta L)^2\} = \underline{x_i^{(0)} + A_S\delta L\frac{\partial x_i(X_R^{(0)})}{\partial X_S}}$$

$\Rightarrow a_i\delta l = A_S\delta L\frac{\partial x_i}{\partial X_S} \Rightarrow \boxed{a_i\frac{\delta l}{\delta L} = A_S\frac{\partial x_i}{\partial X_S}} \Rightarrow \underline{\lambda a_i = A_S\frac{\partial x_i}{\partial X_S}}$

$\lambda = \frac{\delta l}{\delta L}$ (extension ratio, stretch ratio)

Using $a_ia_i = 1$,

$(\lambda a_i)(\lambda a_i) = A_S \frac{dx_i}{dX_S} A_T \frac{dx_i}{dX_T} \Rightarrow \lambda^2 = A_S A_T \frac{dx_i}{dX_S}\frac{dx_i}{dX_T}$ (λ정하면 a_i 정할수 있음)

Determine $\lambda\,(>0)$ and then a_i's.

Likewise, $A_S\, dL = a_i\, dl \frac{dX_S}{dx_i}$, $A_S = \frac{dl}{dL} a_i \frac{dX_S}{dx_i} = \lambda a_i \frac{dX_S}{dx_i}$

using $A_S A_S = 1$, $\lambda^{-2} = a_i a_j \frac{dX_S}{dx_i}\frac{dX_S}{dx_j}$ (λ 정하면, A_S 정할수 있음)

두 가지 방법 $x_i(X_S,t)$ or $X_S(x_i,t)$ 중 편한 것으로 하면 된다.

6.3 The deformation gradient tensor

- Deformation gradient tensor : $\boxed{F_{iR} = \frac{dx_i}{dX_R}}$; 9 quantities
- It is a second order tensor

$X_S = M_{RS}\overline{X_R}$

$$\begin{cases} \overline{x_i} = M_{ij}x_j, & \overline{X_R} = M_{RS}X_S \\ x_j = M_{ij}\overline{x_i}, & X_S = M_{RS}\overline{X_R} \end{cases} \Rightarrow \overline{F_{iR}} = \frac{d\overline{x_i}}{d\overline{X_R}} = \frac{dX_S}{d\overline{X_R}}\frac{d\overline{x_i}}{dX_S} = M_{RS}M_{ij}\frac{dx_j}{dX_S} = M_{ij}M_{RS}F_{jS}$$

$\mathbb{F}^T$ also is a 2^{nd} order tensor ($\neq \mathbb{F}$ in general)

$\mathbb{F}^{-1}$ also is a 2^{nd} order tensor (if $\det\mathbb{F} \neq 0$)

- Component of $\mathbb{F}^{-1}$

using $\frac{dx_i}{dX_R}\frac{dX_R}{dx_j} = \frac{dx_i}{dx_j} = \delta_{ij}$, $\boxed{F^{-1}_{Rj} = \frac{dX_R}{dx_j}}$

- From section 6.2,

$\boxed{\lambda\mathbb{a} = \mathbb{F}\cdot\mathbb{A}}$ $\left(\lambda a_i = \frac{dx_i}{dX_S}A_S = F_{iS}A_S\right)$

$$\begin{cases} \lambda^2 = \mathbb{A}\cdot\mathbb{F}^T\cdot\mathbb{F}\cdot\mathbb{A} & (A_R F^T_{Ri} F_{iS} A_S) \\ \mathbb{F}^{-1}\mathbb{a} = \lambda^{-1}\mathbb{A} \rightarrow \lambda^{-2} = \mathbb{a}\cdot(\mathbb{F}^{-1})^T\cdot(\mathbb{F}^{-1})\cdot\mathbb{a} \end{cases}$$

$F^T_{Ri} = F_{iR}$

(matrix notation)

$$\begin{cases} \mathbb{a} = \lambda^{-1}\mathbb{F}\mathbb{A} \rightarrow \lambda^2 = \mathbb{A}^T\mathbb{F}^T\mathbb{F}\mathbb{A} \\ \mathbb{A} = \lambda\mathbb{F}^{-1}\mathbb{a} \rightarrow \lambda^{-2} = \mathbb{a}^T(\mathbb{F}^{-1})^T\mathbb{F}^{-1}\mathbb{a} \end{cases}$$

- displacement vector : $\mathbb{u} = \mathbb{x} - \mathbb{X}$

displacement gradient : $\frac{du_i}{dX_R} = \frac{dx_i}{dX_R} - \delta_{iR} = F_{iR} - \delta_{iR}$

↳ component of tensor $\mathbb{F} - \mathbb{I}$

If no motion $\frac{d\mathbb{u}}{d\mathbb{X}} = 0$.

But if rigid body motion, $\mathbb{x} = \mathbb{Q}(t)\mathbb{X} + \mathbb{c}(t) \longrightarrow \underline{\mathbb{Q}(t) - \mathbb{I}}$

0이 아니므로 deformation의 good measure 이 아니다.

6.4 Finite deformation and strain tensors

- Define $\boxed{\mathbb{C} = \mathbb{F}^T \cdot \mathbb{F}} \longrightarrow \underline{C_{RS} = F_{iR} F_{iS} = \left(\frac{dx_i}{dX_R}\right)\left(\frac{dx_i}{dX_S}\right)} = C_{SR} \longrightarrow \underline{\mathbb{C} = \mathbb{C}^T}$

  ↑ 2nd order tensor since $\mathbb{F}^T$ and $\mathbb{F}$ are.

$\lambda^2 = \mathbb{A} \cdot \mathbb{F}^T \cdot \mathbb{F} \cdot \mathbb{A} = \mathbb{A} \cdot \mathbb{C} \cdot \mathbb{A} = A_R C_{RS} A_S$

$\rightarrow$ $\mathbb{C}$ determines the local deformation in the neighborhood of a particle.

For a rigid body motion, $\mathbb{F} = \mathbb{Q}(t)$, $\mathbb{C} = \mathbb{Q}^T \mathbb{Q} = \mathbb{I}$

↑ since $\mathbb{Q}$ is symmetric.

$\mathbb{C}$: right cauchy-green deformation tensor ($\mathbb{F} = \mathbb{R}\mathbb{U} \rightarrow \mathbb{C} = \mathbb{F}^T\mathbb{F} = \mathbb{U}\mathbb{R}^T\mathbb{R}\mathbb{U} = \mathbb{U}^2$)

↳ Not a unique measure of deformation. ($\mathbb{C}^2$ or $\mathbb{C}^{-1}$ can be employed

$\mathbb{C}^{-1} = \mathbb{F}^{-1} \cdot (\mathbb{F}^{-1})^T \Rightarrow \underline{C^{-1}_{RS} = F^{-1}_{Ri} F^{-1}_{Si} = \frac{dX_R}{dx_i}\frac{dX_S}{dx_i}}$

- $\boxed{\mathbb{B} = \mathbb{F} \cdot \mathbb{F}^T}$, $\mathbb{B}^{-1} = (\mathbb{F}^{-1})^T \mathbb{F}^{-1}$.

$\mathbb{B}$: left cauchy-green deformation tensor ($\mathbb{F} = \mathbb{V}\mathbb{R} \rightarrow \mathbb{B} = \mathbb{V}\mathbb{R}\mathbb{R}^T\mathbb{V} = \mathbb{V}^2$)

$\underline{B_{ij} = F_{iR} F_{jR} = \frac{dx_i}{dX_R}\frac{dx_j}{dX_R}}$, $\underline{B^{-1}_{ij} = F^{-1}_{Ri} F^{-1}_{Rj} = \frac{dX_R}{dx_i}\frac{dX_R}{dx_j}}$

$\lambda^{-2} = \mathbb{a} \cdot (\mathbb{F}^{-1})^T \mathbb{F} \mathbb{a} = \mathbb{a} \cdot \mathbb{B}^{-1} \cdot \mathbb{a}$

- $\boxed{\begin{aligned} &\gamma = \tfrac{1}{2}(\mathbb{C} - \mathbb{I}) \ : \text{Lagrangian strain tensor} \\ &\eta = \tfrac{1}{2}(\mathbb{I} - \mathbb{B}^{-1}) \ : \text{Eulerian strain tensor} \end{aligned}}$

Suitable measures for deformation
$\because \gamma = 0$, $\eta = 0$ for a rigid-body motion

if $\mathbb{x}(\mathbb{X})$ is given $\longrightarrow \mathbb{F} \longrightarrow \mathbb{C} \longrightarrow \gamma$: deformation near a particle

if $\mathbb{X}(\mathbb{x})$ is given $\longrightarrow \mathbb{F}^{-1} \longrightarrow \mathbb{B}^{-1} \longrightarrow \eta$: deformation near a point.

- γ_{RS} & η_{ij} are often given in terms of the displacement gradient $\mathbb{u} = \mathbb{x} - \mathbb{X}$,

(i) $F_{iR} = \frac{dx_i}{dX_R} = \frac{du_i}{dX_R} + \delta_{iR}$.

$\gamma_{RS} = \frac{1}{2}\left\{\left(\frac{du_i}{dX_R} + \delta_{iR}\right)\left(\frac{du_i}{dX_S} + \delta_{iS}\right) - \delta_{RS}\right\} = \frac{1}{2}\left(\frac{du_R}{dX_S} + \frac{du_S}{dX_R} + \frac{du_i}{dX_R}\frac{du_i}{dX_S}\right)$

(ex) $\gamma_{11} = \dfrac{\partial u_1}{\partial X_1} + \dfrac{1}{2}\left\{ \left(\dfrac{\partial u_1}{\partial X_1}\right)^2 + \left(\dfrac{\partial u_2}{\partial X_1}\right)^2 + \left(\dfrac{\partial u_3}{\partial X_2}\right)^2 \right\}$

$$\gamma_{23} = \frac{1}{2}\left\{ \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} + \frac{\partial u_1}{\partial X_2}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_2}{\partial X_2}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\frac{\partial u_3}{\partial X_3} \right\}$$

(ii) $F^{-1}_{Ri} = \dfrac{\partial X_R}{\partial x_i} = \dfrac{\partial}{\partial x_i}(x_R - u_R) = \delta_{Ri} - \dfrac{\partial u_R}{\partial x_i}$

$$\eta_{ij} = \frac{1}{2}(\delta_{ij} - F^{-1}_{Ri}F^{-1}_{Rj}) = \frac{1}{2}\left\{\delta_{ij} - \left(\delta_{Ri} - \frac{\partial u_R}{\partial x_i}\right)\left(\delta_{Rj} - \frac{\partial u_R}{\partial x_j}\right)\right\} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_R}{\partial x_i}\frac{\partial u_R}{\partial x_j}\right)$$

(ex) $\eta_{11} = \dfrac{\partial u_1}{\partial x_1} - \dfrac{1}{2}\left\{ \left(\dfrac{\partial u_1}{\partial x_1}\right)^2 + \left(\dfrac{\partial u_2}{\partial x_1}\right)^2 + \left(\dfrac{\partial u_3}{\partial x_1}\right)^2 \right\}$

$$\eta_{23} = \frac{1}{2}\left\{ \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} - \frac{\partial u_1}{\partial x_2}\frac{\partial u_1}{\partial x_3} - \frac{\partial u_2}{\partial x_2}\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\frac{\partial u_3}{\partial x_3} \right\}$$

<6.4 Summary>

matrices

$\mathbb{F} = (F_{iR}) = (\partial x_i / \partial X_R)$, $\mathbb{F}^{-1} = (F^{-1}_{Ri}) = (\partial X_R / \partial x_i)$

$\mathbb{C} = (C_{RS})$, $\mathbb{B} = (B_{ij})$, $\mathbb{C}^{-1} = (C^{-1}_{RS})$, $\mathbb{B}^{-1} = (B^{-1}_{ij})$

$a = (a_1\ a_2\ a_3)^T$, $A = (A_1\ A_2\ A_3)^T$

$\mathbb{C} = \mathbb{F}^T\mathbb{F}$, $\mathbb{C}^{-1} = \mathbb{F}^{-1}(\mathbb{F}^{-1})^T$, $\mathbb{B} = \mathbb{F}\mathbb{F}^T$, $\mathbb{B}^{-1} = (\mathbb{F}^{-1})^T\mathbb{F}^{-1}$

$\lambda^2 = A^T\mathbb{C}A$, $\lambda^{-2} = a^T\mathbb{B}^{-1}a$, $\gamma = \dfrac{\mathbb{C} - \mathbb{I}}{2}$, $\eta = \dfrac{\mathbb{I} - \mathbb{B}^{-1}}{2}$

$\mathbb{C}, \mathbb{C}^{-1}, \mathbb{B}, \mathbb{B}^{-1}, \gamma, \eta$: symmetric, 2nd-order tensor

$\rightarrow$ real principal components & orthogonal principal direction.

6.5 Some simple finite deformations

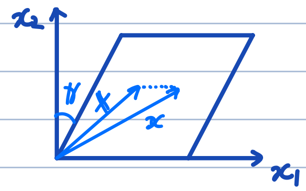
(a) Uniform extensions : $x_1 = \lambda_1 X_1$, $x_2 = \lambda_2 X_2$, $x_3 = \lambda_3 X_3$

$$\boxed{\mathbb{F} = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}}, \quad \mathbb{B} = \mathbb{C} = \begin{pmatrix} \lambda_1^2 & & \\ & \lambda_2^2 & \\ & & \lambda_3^2 \end{pmatrix}$$

$$\gamma = \frac{1}{2}\begin{pmatrix} \lambda_1^2 - 1 & & \\ & \lambda_2^2 - 1 & \\ & & \lambda_3^2 - 1 \end{pmatrix}, \quad \eta = \frac{1}{2}\begin{pmatrix} 1 - \lambda_1^{-2} & & \\ & 1 - \lambda_2^{-2} & \\ & & 1 - \lambda_3^{-2} \end{pmatrix}$$

for $\lambda_i = 1 + \varepsilon_i$, $|\varepsilon_i| \ll 1$ $\quad \gamma = \eta = \begin{pmatrix} \varepsilon_1 & & \\ & \varepsilon_2 & \\ & & \varepsilon_3 \end{pmatrix}$

(b) Simple shear



$x_1 = X_1 + X_2 \tan\gamma \;,\; x_2 = X_2 \;,\; x_3 = X_3$

$$\boxed{\mathbb{F} = \begin{pmatrix} 1 & \tan\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} \qquad \mathbb{F}^{-1} = \begin{pmatrix} 1 & -\tan\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{B} = \mathbb{F}\mathbb{F}^T = \begin{pmatrix} 1 & \tan\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ \tan\gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1+\tan^2\gamma & \tan\gamma & 0 \\ \tan\gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{C} = \mathbb{F}^T\mathbb{F} = \begin{pmatrix} 1 & 0 & 0 \\ \tan\gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \tan\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \tan\gamma & 0 \\ \tan\gamma & 1+\tan^2\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbb{B}^{-1} = \begin{pmatrix} 1 & -\tan\gamma & 0 \\ -\tan\gamma & 1+\tan^2\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad \mathbb{C}^{-1} = \begin{pmatrix} 1+\tan^2\gamma & -\tan\gamma & 0 \\ -\tan\gamma & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\gamma = \frac{1}{2}(\mathbb{C}-\mathbb{I}) = \frac{1}{2}\begin{pmatrix} 0 & \tan\gamma & 0 \\ \tan\gamma & \tan^2\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \eta = \frac{1}{2}(\mathbb{I}-\mathbb{B}^{-1}) = \frac{1}{2}\begin{pmatrix} 0 & \tan\gamma & 0 \\ \tan\gamma & -\tan^2\gamma & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) Homogeneous deformation

$x_i = \underbrace{C_i + A_{iR}}_{} X_R$ or $\mathbb{x} = \mathbb{C} + \mathbb{A}\cdot\mathbb{X}$

↳ constants, or, a function of time only. (not X_R)

$\boxed{\mathbb{F} = \mathbb{A}}$. All the strain or deformation terms are indep. of x_i or X_R.

(i) plane in the reference config $\longrightarrow$ plane

two parallel planes $\longrightarrow$ parallel plane

(ii) straight line $\longrightarrow$ straight line

parallel line $\longrightarrow$ parallel line

(iii) spherical surface $\longrightarrow$ ellipsoidal surface

pf) $\mathbb{n}\cdot\mathbb{X} = p$

Since $\mathbb{X} = \mathbb{A}^{-1}(x-c) \longrightarrow \underbrace{\mathbb{n}\cdot\mathbb{A}^{-1}}_{\text{new normal vector}}\cdot(x-c) = p$

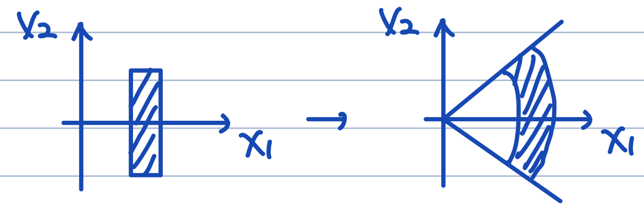
(d) Plane strain : $x_1 = x_1(X_1, X_2)$ $\quad x_2 = x_2(X_1, X_2)$, $x_3 = X_3$

(e) Pure torsion : $r = R$, $\phi = \Phi + \psi Z$, $z = Z$

$$\left(\begin{array}{l} R=\sqrt{X_1^2+X_2^2}\ ,\ \Phi=\tan^{-1}(X_2/X_1)\ ,\ Z=X_3 \\ r=\sqrt{x_1^2+x_2^2}\ ,\ \phi=\tan^{-1}(x_2/x_1)\ ,\ z=x_3 \end{array} \right)$$

No volume change but not homogeneous.

(f) Pure flex : $r=f(\underline{X_1})\ ,\ \phi=g(\underline{X_2})$



6.6 Infinitesimal strain

Many cases, $|du_i/dX_k| \ll 1$, $\dfrac{du_i}{dx_j}=(\delta_{ij}-\dfrac{dX_i}{dx_j})=\mathbb{I}-\mathbb{F}^{-1}$

But, $\mathbb{F}^{-1}=(\mathbb{I}+(\mathbb{F}-\mathbb{I}))^{-1}\simeq\mathbb{I}-(\mathbb{F}-\mathbb{I})=2\mathbb{I}-\mathbb{F}$

$\therefore \dfrac{du_i}{dx_j}\simeq(\mathbb{F}-\mathbb{I})_{ij}=\dfrac{dx_i}{dX_j}-\delta_{ij}=\dfrac{du_i}{dX_j} \implies \boxed{\dfrac{du_i}{dx_j}\simeq\dfrac{du_i}{dX_j}}$

$\therefore \gamma_{ij}\simeq\eta_{ij}\simeq\dfrac{1}{2}\left(\dfrac{du_i}{dX_j}+\dfrac{du_j}{dX_i}\right)\simeq\underline{\dfrac{1}{2}\left(\dfrac{du_i}{dx_j}+\dfrac{du_j}{dx_i}\right)}$

$\implies \boxed{\mathbb{E} \equiv \text{infinitesimal strain tensor} \ ;\ E_{ij}=\frac{1}{2}\left(\frac{du_i}{dx_j}+\frac{du_j}{dx_i}\right)}$

$$(E_{ij})=\begin{pmatrix} \dfrac{\partial u_1}{\partial X_1} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_2}+\dfrac{\partial u_2}{\partial X_1}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial X_3}+\dfrac{\partial u_3}{\partial X_1}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_1}+\dfrac{\partial u_1}{\partial X_2}\right) & \dfrac{\partial u_2}{\partial X_2} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial X_3}+\dfrac{\partial u_3}{\partial X_2}\right) \\ \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial X_1}+\dfrac{\partial u_1}{\partial X_3}\right) & \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial X_2}+\dfrac{\partial u_2}{\partial X_3}\right) & \dfrac{\partial u_3}{\partial X_3} \end{pmatrix}$$

$\gamma=\eta=\mathbb{E}$

$\boxed{\mathbb{E}=\frac{1}{2}(\mathbb{F}+\mathbb{F}^T)-\mathbb{I}}$ $\Rightarrow$ 2nd order symmetric tensor

$\left(\dfrac{du_i}{dX_k}=\dfrac{dx_i}{dX_k}-\delta_{ij}=\mathbb{F}-\mathbb{I}.\right)$

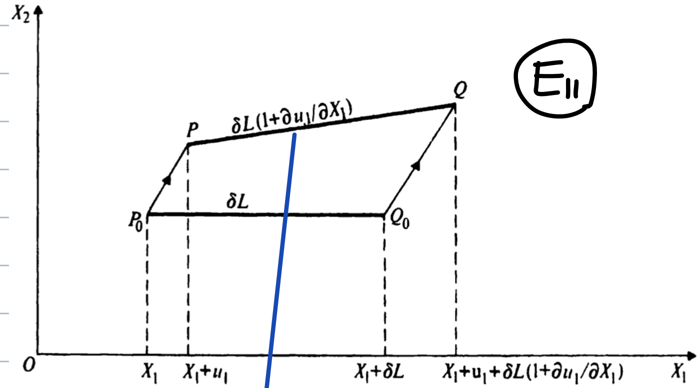
But $\mathbb{E}$ is not a good measure of deformation.

For rotation by α w.r.t x_3 axis,

$$\mathbb{F}=\begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathbb{E}=\begin{pmatrix} \cos\alpha-1 & 0 & 0 \\ 0 & \cos\alpha-1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\neq 0$$

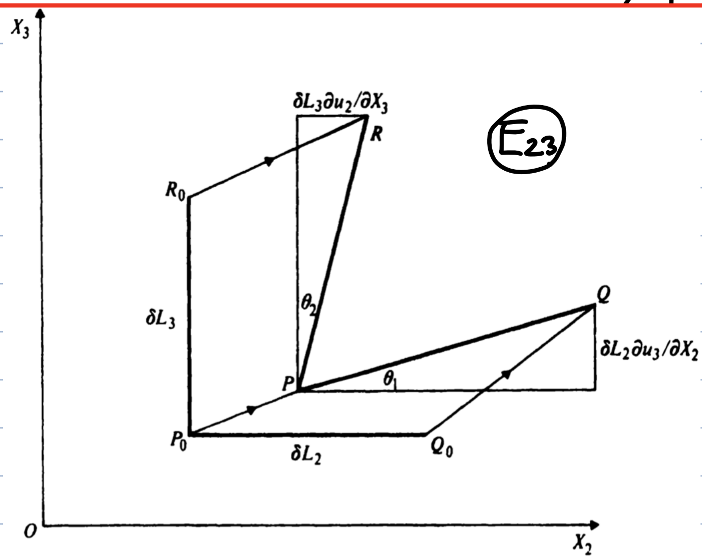
↳ $\mathbb{E}\simeq 0$ up to 1st order in α.
($\cos\alpha$의 테일러전개 ⟶ 2차항부터 존재)

※ Geometrical interpretation of E_{11} & E_{23}.



$$\delta l = \sqrt{(\delta L(1+\partial u_1/\partial X_1))^2 + \cancel{(\partial u_2/\partial X_1)^2} + \cancel{(\partial u_3/\partial X_1)^2}} = \delta L(1+\underbrace{\partial u_1/\partial X_1}_{=E_{11}})$$

$\boxed{\therefore E_{11} = \text{(to first order) the extension per unit length of a line element initially parallel to } X_1.}$



$$E_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\right)$$

$$\theta_1 = \frac{\partial u_3}{\partial X_2}, \quad \theta_2 = \frac{\partial u_2}{\partial X_3}$$

↳ X_2가 조금 갈 때, u_3 방향의 이동량

$\boxed{\therefore 2E_{23} \text{ is (to first order), the decrease during the deformation in the angle between the initially orthogonal line element } P_0Q_0 \text{ \& } P_0R_0.}$

6.7 Infinitesimal rotation

Finite rigid body rotation by $\underline{\alpha}$ about $\underline{n}$.

If $|\alpha| \ll 1$, $\sin\alpha \simeq \alpha$, $\cos\alpha \simeq 1$ $\quad (u \simeq \alpha\, n \times X)$

$$u_i = x_i - X_i \simeq \alpha e_{ijk} n_j X_k$$

$$\rightarrow \frac{\partial u_i}{\partial X_k} = \alpha e_{ijk} n_j = \begin{pmatrix} 0 & -\alpha n_3 & \alpha n_2 \\ \alpha n_3 & 0 & -\alpha n_1 \\ -\alpha n_2 & \alpha n_1 & 0 \end{pmatrix}$$

$$\boxed{\Omega = \frac{1}{2}(F - F^T)} \rightarrow \Omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} - \frac{\partial u_j}{\partial X_i}\right) \text{ ; Infinitesimal rotation tensor}$$

↳ 2nd order anti-symmetric tensor → represent an infinitesimal rotation

Displacement gradient tensor $\mathbb{F}-\mathbb{I} = \underbrace{\frac{1}{2}(\mathbb{F}+\mathbb{F}^T)-\mathbb{I}}_{\mathbb{E}} + \underbrace{\frac{1}{2}(\mathbb{F}-\mathbb{F}^T)}_{\Omega} = \mathbb{E}+\Omega$

If $\mathbb{F}$ is very close to $\mathbb{I}$, $\mathbb{F} = \mathbb{I}+\mathbb{E}+\Omega \simeq (\mathbb{I}+\mathbb{E})(\mathbb{I}+\Omega)$

Infinitesimal rotation vector ω

$$\omega = \frac{1}{2}\nabla\times u \quad \text{or} \quad \omega_i = \frac{1}{2}e_{ijk}\frac{\partial u_k}{\partial x_j}$$

$$\left(\begin{array}{l} \Omega_{jk} = -e_{ijk}\omega_i \\ \omega_i = \frac{1}{2}e_{iab}\partial_a u_b \end{array} \right) \Rightarrow \Omega_{jk} = \frac{1}{2}(\partial_k u_j - \partial_j u_k) \Big) \; ...?$$

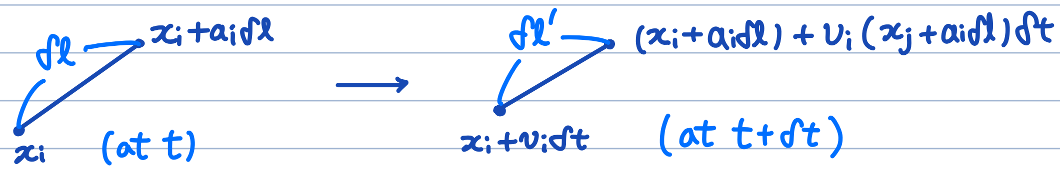
6.8 The rate of deformation tensor

The kinematic property of interest : the rate at which the change of shape is taking place.

Fluid mechanics : Shape of the body at $t=0$ is not relevant.

$\rightarrow$ we want to know $\frac{1}{\lambda}\frac{D\lambda}{Dt}$ at x_i & t.

(현재 λ) (변화 λ)



$$d\ell'^2 = \left(a_i\delta\ell + a_j\frac{\partial v_i}{\partial x_j}\delta\ell\,\delta t\right)\left(a_i\delta\ell + a_k\frac{\partial v_i}{\partial x_k}\delta\ell\,\delta t\right)$$

$$\simeq \delta\ell^2\left(1+a_ia_j\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_i}{\partial x_j}\right)\delta t\right)$$

$$= \delta\ell^2\left(1+a_ia_j\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\delta t\right)$$

} (i,j) dummy variable

$$\frac{\delta\ell'}{\delta\ell} \simeq \frac{\lambda+d\lambda}{\lambda} \longrightarrow \left(\frac{\delta\ell'}{\delta\ell}\right)^2 \simeq 1+\frac{2d\lambda}{\lambda} \longrightarrow \frac{2d\lambda}{\lambda} = a_ia_j\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)\delta t$$

$$\therefore \frac{1}{\delta t}\frac{d\lambda}{\lambda} = \frac{1}{\lambda}\frac{D\lambda}{Dt} = \underline{\frac{1}{2}a_ia_j\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)} \qquad \boxed{D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)}$$

$\boxed{\mathbb{D} : \text{rate of deformation tensor}}$

Unlike $\mathbb{E}$ (correct only up to 1st order),

$\mathbb{D}$ is an exact measure of the deformation rate

6.9 The velocity gradient and spin tensors.

$$\boxed{L_{ij} = \frac{\partial v_i}{\partial x_j}} \;,\; D_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right) \longrightarrow \boxed{W_{ij} \equiv \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j}-\frac{\partial v_j}{\partial x_i}\right)}$$

$\boxed{\mathbb{W} : \text{spin or vorticity tensor}}$

$\mathbb{W}$ is similar to infinitesimal rotation tensor except that there is no approximation here.

$$\underset{\substack{\text{velocity}\\ \text{gradient}}}{\mathbb{L}} = \underset{\substack{\text{deformation}\\ \text{rate}}}{\mathbb{D}} + \underset{\text{spin}}{\mathbb{W}} , \quad \mathbb{D} = \frac{1}{2}(\mathbb{L}+\mathbb{L}^T) , \quad \mathbb{W} = \frac{1}{2}(\mathbb{L}-\mathbb{L}^T)$$

$$\boxed{\begin{array}{l} \mathbf{w} \equiv \nabla \times \mathbf{v} , \quad w_i = e_{ijk}\dfrac{\partial v_k}{\partial x_j} \\ \mathbf{w} : \text{vorticity vector} \end{array}}$$

$$\underline{W_{jk} = -\frac{1}{2} e_{ijk} w_i}$$
$$= -\frac{1}{2} e_{ijk} e_{ilm} \frac{\partial v_m}{\partial x_l}$$
$$= -\frac{1}{2}(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \frac{\partial v_m}{\partial x_l}$$
$$= -\frac{1}{2}\left(\frac{\partial v_k}{\partial x_j} - \frac{\partial v_j}{\partial x_k}\right) = \frac{1}{2}\left(\frac{\partial v_j}{\partial x_k} - \frac{\partial v_k}{\partial x_j}\right)$$

$\leftrightarrow$

$$\underline{w_i = -e_{ijk} W_{jk}}$$
$$= -e_{ijk}\left(-\frac{1}{2} e_{ljk} w_l\right) = \frac{1}{2}\cdot 2\delta_{il} w_l = w_i$$

Rigid-body rotation : $\mathbf{v} = \omega \mathbf{n} \times \mathbf{x}$, $v_i = e_{ijk}\omega n_j x_k$

$$\mathbf{w} = \nabla \times \mathbf{v} = \nabla \times (\omega \mathbf{n} \times \mathbf{x}) = 2\omega \mathbf{n}$$

$$w_i = e_{ijk}\frac{\partial}{\partial x_j}(e_{klm}\omega n_l x_m) = e_{ijk} e_{klj}\omega n_l = 2\delta_{il}\omega n_l = 2\omega n_i$$

$$\mathbf{w} = \nabla \times (\omega \mathbf{n} \times \mathbf{x}) = \omega \mathbf{n}(\nabla \cdot \mathbf{x}) - \mathbf{x}(\omega \mathbf{n} \cdot \nabla) = \omega \mathbf{n} \cdot 3 - \omega \mathbf{n} = 2\omega \mathbf{n}$$

6.10 Some simple flows

(a) Simple shearing flow

$v_1 = S x_2$, $v_2 = v_3 = 0$

$$\mathbb{L} = \begin{pmatrix} 0 & S & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbb{D} = \begin{pmatrix} 0 & S/2 & 0 \\ S/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \mathbb{W} = \begin{pmatrix} 0 & S/2 & 0 \\ -S/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) Rectilinear flow : flows in parallel straight lines

$v_1 = v_2 = 0$, $v_3 = f(x_1, x_2, x_3)$

$$\mathbb{L} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \frac{\partial f}{\partial x_3} \end{pmatrix} , \quad \mathbb{D} = \frac{\mathbb{L}+\mathbb{L}^T}{2} , \quad \mathbb{W} = \frac{\mathbb{L}-\mathbb{L}^T}{2}$$

(c) Vortex flow : vortex line lying along x_3

$$v_1 = -\frac{k x_2}{x_1^2+x_2^2} , \quad v_2 = \frac{k x_1}{x_1^2+x_2^2} , \quad v_3 = 0 \quad (x_1^2+x_2^2 \neq 0)$$

$$\mathbb{L} = \begin{pmatrix} \frac{2k x_1 x_2}{(x_1^2+x_2^2)^2} & \frac{k(x_2^2-x_1^2)}{(x_1^2+x_2^2)^2} & 0 \\ \frac{k(x_2^2-x_1^2)}{(x_1^2+x_2^2)^2} & \frac{-2k x_1 x_2}{(x_1^2+x_2^2)^2} & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad \mathbb{D} = \mathbb{L} , \quad \mathbb{W} = 0$$

(d) Plane flow

$v_1 = v_1(x_1, x_2, t)$, $v_2 = v_2(x_1, x_2, t)$, $v_3 = 0$

$$\mathbb{L} = \left(\begin{matrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial x_1}{\partial x_2} & 0 \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_1} & 0 \\ 0 & 0 & 0 \end{matrix} \right.$$