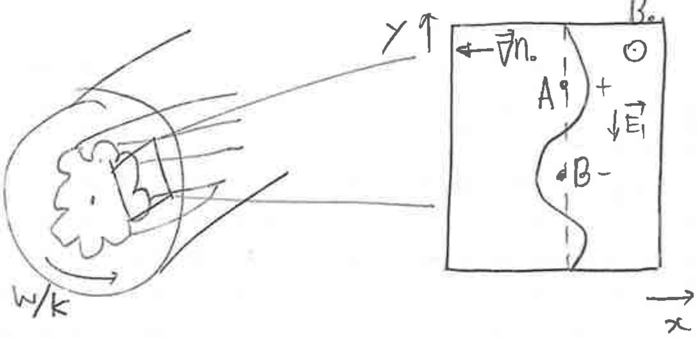
Fusion Plasma Theory 2

Lecture 16 Collisionless Drift Wave Instability

① Importance of drift wave instability in plasma physics.

- $\vec{\nabla} n$, $\vec{\nabla} T$ (w/o a particular driving force) can drive drift wave instability.
  ↳ unavoidable, universal., electrostatic (← no need of perturbations)
- Lead to fluctuations in n_1, T_1, ϕ_1 ⟶ source of turbulence.

② Mechanism for electron drift waves



n_1 under $\vec{\nabla} n_0$ → charge separation $\vec{E_1} = -\vec{\nabla}\phi_1$

→ $\vec{E_1}\times\vec{B_0}$ drift.

(it can be unstable when n_1 and ϕ_1 has a phase difference)

↳ Let's see how it works.

③ Representations of drift waves

Typical wave analysis + { (a) Background non-uniformity. (b) Dissipation mechanism

(a) Slab representation

inhomogenity in x-direction. $k_z = k_\parallel > 0$ w/o loss of generality.

$$\rightarrow \phi_1(x,y,z) = \sum_{k,w} \hat{\phi}_1(x) \exp\left[i\left(k_y y + k_z z - wt\right)\right]$$

$$n_1(x,y,z) = \sum_{k,w} \hat{n}_1(x) \exp\left[i\left(k_y y + k_z z - wt\right)\right]$$

$B = B_0 + \cancel{B_1}$ (electrostatic, constant $B_0 = B_0 \hat{z}$)

(b) Dissipation mechanism

Use kinetic-MHD, or drift-kinetic approach.

④ Basic electron and ion response in electron drift waves

Basic drift waves emerge when $V_{th,i} \ll \omega / k_z \ll V_{th,e}$

- electron obeys Boltzmann response (adiabatic electron response)

$$n_{e1} = n_0 \exp\left(\frac{e\phi_1}{T_{e0}}\right) - n_{e0} = n_{e0}\frac{e\phi_1}{T_{e0}}$$

← n_{e1} & ϕ_1 has same phase

It's the result of parallel momentum balance of electrons under $T_{e1} = 0$

$$\left(-en_{e0}E_{\parallel} = \nabla_{\parallel} p_{e1} \rightarrow en_{e0}\nabla_{\parallel}\phi_1 = T_{e0}\nabla_{\parallel} n_e\right)$$

*note
$mn\frac{dv}{dt} = ne[\vec{E} + \vec{v}\times\vec{B}] - \vec{\nabla}p$

- ion response in electron drift wave

Assume
- quasi-neutrality $n_0 = n_{e0} = n_{i0}$
- cold ions $k_\perp \rho_i \ll 1$ (ignoring FLR effect)
- $k_\perp \rho_s \sim O(1)$ (ρ_s = ion Larmor-radius at T_e)

여기서 시작!

$$m_i n_i \frac{d\vec{u}_{i1}}{dt} = en_i\left(\vec{E}_1 + \vec{u}_{i1}\times\vec{B}_0\right)$$

⟨perturbed ion equation of motion⟩

LHS ≪ RHS 라고 생각할 수 있음 (ordering)

(i) Parallel component :

$$m_i n_0 \frac{du_{i\parallel}}{dt} = en_0 E_{\parallel} = -en_0\nabla_{\parallel}\phi_1 \rightarrow m_i n_0(+i\omega)u_{i\parallel} = +en_0 i k_{\parallel}\phi_1 \rightarrow u_{i\parallel} = \frac{k_{\parallel}e\phi_1}{m_i\omega}$$

(ii) perpendicular component

(First order) $$\vec{u}_{i\perp}^{(1)} = \frac{\vec{E}_1\times\vec{B}_0}{B_0^2} = \frac{\hat{b}\times\vec{\nabla}\phi}{B_0} \equiv \vec{u}_E$$ (assuming $\omega/\omega_{ci} \ll 1$)

(Next order)

$$m_i n_i \frac{d\vec{u}_{i\perp}^{(1)}}{dt} = en_i \vec{u}_{i\perp}^{(2)}\times\vec{B}_0 \qquad (\vec{E}_1 \ \text{0-th order})$$

$$m_i n_i \frac{d}{dt}\left(\frac{\vec{E}_1\times\vec{B}_0}{B_0^2}\right) = m_i n_i \frac{d\vec{E}_1}{dt}\times\frac{\vec{B}}{B_0^2} = en_i\vec{u}_{i\perp}^{(2)}\times\vec{B}_0$$

$$\Rightarrow \vec{u}_{i\perp}^{(2)} = \frac{m_i}{eB_0^2}\frac{d}{dt}\vec{E}_\perp = -\frac{m_i}{eB_0^2}\frac{d}{dt}\nabla_\perp\phi_1 \equiv \vec{u}_{pol}$$

⑤ Linear dispersion relation for electron drift waves.

(ion의 density response 를 구하기 위해 ion의 motion을 먼저 알아보자)

$\boxed{\text{ion continuity equation}}$: $\frac{\partial n_{i1}}{\partial t} + \vec{\nabla}\cdot(n_0\vec{u_{i1}}) = 0$ where $\vec{u_{i1}} = \vec{u_{\parallel}} + \vec{u_E} + \vec{u_{pol}}$

$$\Downarrow \qquad \vec{u_E} \gg \vec{u_{pol}}$$

$$\frac{\partial n_{i1}}{\partial t} + n_0\left(\vec{\nabla}\cdot\vec{u_{i\parallel}} + \cancel{\vec{\nabla}\cdot\vec{u_E}}^{\,0} + \vec{\nabla}\cdot\vec{u_{pol}}\right) + \left(\cancel{\vec{u_{\parallel}}} + \vec{u_E} + \cancel{\vec{u_{pol}}}\right)\cdot\vec{\nabla}n_0 = 0$$

$\vec{u_{\parallel}}\cdot\vec{\nabla}n_0 = 0$ ($\because$ flux-surface)

$$\Rightarrow \underline{\frac{\partial n_{i1}}{\partial t} + n_0\nabla_{\parallel}u_{i\parallel} + n_0\vec{\nabla}\cdot\vec{u_{pol}} + \vec{u_E}\cdot\vec{\nabla}n_0 = 0}$$

$\boxed{\text{Fourier decomposition}}$: $\frac{\partial}{\partial t}\to -i\omega$, $\nabla_{\parallel}\to ik_{\parallel}$, $\vec{\nabla}_{\perp}\to i\vec{k_{\perp}}$, $\vec{\nabla}n_0 = -\hat{x}\frac{n_0}{L_n}$

(Assume $\rho_i < k_{\perp} < L_n$: slow varying background distribution)

$$\underbrace{-i\omega n_{i1}}_{(1)} + \underbrace{n_0 ik_{\parallel}\left(\frac{k_{\parallel}e\phi_1}{\omega m_i}\right)}_{(2)} + \underbrace{n_0\left(-\frac{m_i}{eB_0^2}\right)(-i\omega)(-k_{\perp}^2)\phi_1}_{(3)} + \underbrace{\left(\left(\frac{-ik_y\phi_1\hat{x}}{B_0}\right)\cdot\left(-\frac{n_0\hat{x}}{L_n}\right)\right)}_{(4)} = 0$$

(2) : $C_s^2 = \frac{T_{e0}}{m_i} \to in_0 ek_{\parallel}^2\phi_1\frac{C_s^2}{T_{e0}}\cdot\frac{1}{\omega}$

(3) : $r_L^2 = \frac{m^2v_L^2}{e^2B^2} = \frac{mT_{e0}}{e^2B^2} \to -i\omega n_0 r_L^2\frac{e}{T_{e0}}k_{\perp}^2\phi_1 = -i\omega n_0\frac{e\phi_1}{T_{e0}}k_{\perp}^2r_L^2$

(4) : $\omega_{*e} = k_y\frac{T_{e0}}{eB_0L_n} \to i\omega_{*e}\frac{e}{T_{e0}}n_0\phi_1 = i\frac{e\phi_1}{T_{e0}}n_0\omega_{*e}$

$$\Rightarrow -i\omega n_{i1} + in_0\frac{e\phi_1}{T_{e0}}\omega_{*e} + in_0\frac{e\phi_1}{T_{e0}}\frac{k_{\parallel}^2C_s^2}{\omega} - i\omega n_0\frac{e\phi_1}{T_{e0}}k_{\perp}^2r_s^2 = 0$$

(divide by $i\omega n_0$)

$$-\frac{n_{i1}}{n_0} + \frac{e\phi_1}{\omega T_{e0}}\omega_{*e} + \frac{e\phi_1k_{\parallel}^2C_s^2}{T_{e0}\omega^2} - \frac{e\phi_1}{T_{e0}}k_{\perp}^2r_s^2 = 0$$

$$\therefore \boxed{\frac{n_{i1}}{n_0} = \Big(\underbrace{\frac{\omega_{*e}}{\omega}}_{\vec{u_E}} + \underbrace{\frac{k_{\parallel}^2C_s^2}{\omega^2}}_{\vec{u_{\parallel}}} \underbrace{-k_{\perp}^2r_s^2}_{\vec{u_{pol}}}\Big)\frac{e\phi_1}{T_{e0}}}$$

(
✡ $V_{*e} = k_y\left(-\frac{T_{e0}}{en_0B}\right)\left(\frac{dn}{dx}\right) = k_y\frac{T_{e0}}{eBL_n}$

✡ $\omega_{*e} = k_yV_{*e} = k_y^2\frac{T_{e0}}{eBL_n}$
)

$$\varepsilon_0 \vec{\nabla}\cdot\vec{E} = \rho = n_1 e \ , \quad -\varepsilon_0 \nabla^2 \phi_1 = e(n_{i1} - n_{e1}) \ , \quad \lambda_D^2 = \frac{\varepsilon_0 T}{ne^2}$$

$$\Rightarrow -\left(\frac{\varepsilon_0 T}{ne^2}\right)\nabla^2\left(\frac{e\phi_1}{T}\right) = \frac{n_{i1} - n_{e1}}{n} \quad \Rightarrow \quad \underline{-\lambda_D^2 \nabla^2\left(\frac{e\phi_1}{T_{e0}}\right) = e(n_{i1} - n_{e1}) \simeq 0}$$

∴ Linear dispersion relation for pure electron drift wave $\left(n_{i1} \simeq n_{e1} \ \& \ \frac{n_1}{n_0} = \frac{e\phi_1}{T_{e0}}\right)$

$$\boxed{1 + k_\perp^2 r_s^2 - \frac{\omega_{*e}}{\omega} - \frac{k_\parallel^2 C_s^2}{\omega^2} = 0}$$

polarization drift correction

⑥ Implication of pure electron drift wave

- ω are always real & <u>no instability</u>
- <u>No particle flux.</u> $\Gamma = \langle n_{e1} V_x \rangle$

$$V_x = \frac{E_y B_0}{B_0^2} = -\frac{1}{B_0}\frac{d\phi_1}{dy} \ ,$$

$$\phi_1^2(L) = \phi_1^2(0)$$

$$\frac{1}{L}\int_0^L \frac{d\phi_1^2}{dy}\,dy = \frac{1}{L}\phi_1^2\Big|_0^L = 0$$

$$\Gamma = \langle n_{e1} V_x \rangle = \left\langle \frac{n_0 e\phi_1}{T_{e0}}\cdot\left(-\frac{1}{B_0}\frac{d\phi_1}{dy}\right)\right\rangle = -\frac{en_0}{B_0 T_{e0}}\left\langle \phi_1 \frac{d\phi_1}{dy}\right\rangle = \frac{en_0}{2B_0 T_{e0}}\left\langle \frac{d\phi_1^2}{dy}\right\rangle = 0$$

No flux or instability is that n_{e1} and ϕ_1 are in phase (Boltzmann relation)
adiabatic response.

⑦ Mechanisms driving instabilities in electron drift waves.

For an instability and net particle flux, there should be a phase shift between n_{e1} and ϕ_1.

$$\underline{\frac{n_{e1}}{n_0} = (1 - i\,\delta_{k,\omega})\frac{e\phi_1}{T_{e0}}} \Rightarrow \underline{1 - i\,\delta_{k,\omega} + k_\perp^2 \rho_s^2 - \frac{\omega_*}{\omega} - \frac{k_\parallel^2 C_s^2}{\omega^2} = 0}.$$

This will drive an instability and net particle flux $\underline{\gamma = \mathrm{Im}(\omega) \simeq \delta_{k,\omega}\,\omega_{*e}}$

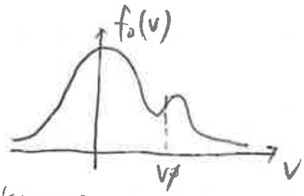
$$\left(1 - i\delta - \frac{\omega_*}{\omega} = 0 \rightarrow \frac{\omega_*}{\omega} = 1 - i\delta \rightarrow \omega = \frac{\omega_*}{1 - i\delta} \simeq \omega_*(1 + i\delta) \Rightarrow \omega_i = \omega_*\,\delta_{k,\omega}\right)$$

There are various mechanisms that create the phase shift

- Collisionless landau damping (ETG, ITG)
- Trapped Electron Mode (TEM)
- Magnetic shear (stabilizing mechanism)

⑧ Drift kinetics of inverse Landau damping.

- Landau damping : $df_0/dv < 0$ damps wave.
- Inverse Landau damping : $df_0/dv > 0$



But we're assuming Maxwellian for drift waves.

↳ $\vec{\nabla} n_0$ & ion polarization effect ⇒ Drift wave instability.

Full Vlasov equation complicates the problem due to FLR effects.

↳ can be removed by the drift kinetic equation

⑨ Drift kinetic equation

$\omega \ll \omega_{cs}$, $k_\perp r_{Ls} \ll 1$, $k_\perp L_n \gg 1$ & $\underline{f(\vec{x}, v_\parallel, \mu, \gamma, t)}$

$$\Rightarrow \underline{\frac{\partial f}{\partial t} + \dot{\vec{x}} \cdot \vec{\nabla} f + \dot{v}_\parallel \frac{\partial f}{\partial v_\parallel} + \cancel{\dot{\mu} \frac{\partial f}{\partial \mu}}^{\,0 \;\; \dot{\mu}=0} = 0} \qquad \left(\text{for } f(\vec{x}, v_\parallel, \mu) = \frac{1}{2\pi} \oint f d\gamma \right)$$

For electrostatic drift waves, it is sufficient to include only the leading order as:

$$\Rightarrow \underline{\frac{\partial f}{\partial t} + (v_\parallel \hat{b} + \vec{v}_E) \cdot \vec{\nabla} f + \frac{q E_\parallel}{m} \frac{\partial f}{\partial v_\parallel} = 0}$$

⑩ Electron drift-kinetic equation

With electrostatic fluctuation in our slab geometry is then given by:

$$\underline{\frac{\partial f_e}{\partial t} + \left(v_\parallel \hat{z} + \frac{\hat{b} \times \vec{\nabla} \phi_1}{B_0} \right) \cdot \vec{\nabla} f_e + \frac{e}{m} \frac{\partial \phi_1}{\partial z} \frac{\partial f_e}{\partial v_\parallel} = 0}$$

Linearizing it around a Maxwellian $f_e = f_{Me} + f_{e1}$:

$$\underbrace{\frac{\partial f_{e1}}{\partial t} + v_\parallel \frac{\partial f_{e1}}{\partial z}}_{\text{lower-order} \times f_{e1}} + \underbrace{\frac{\hat{b} \times \vec{\nabla} \phi_1}{B_0}}_{\text{higher order} \times f_M} \cdot \vec{\nabla} f_{Me} + \underbrace{\frac{e}{m} \frac{\partial \phi_1}{\partial z}}_{\rightarrow \text{ same order}} \frac{\partial f_{Me}}{\partial v_\parallel} = 0$$

Assume $\vec{E_1} \times \vec{B_0} \propto \hat{y}$

(E×B drift는 주로 $\hat{y}$ 방향)

$\Downarrow$

$$-i\omega f_{e1} + i k_\parallel v_\parallel f_{e1} = -\frac{\hat{z}}{B_0} \times \hat{y} \frac{\partial \phi_1}{\partial y} \cdot \hat{x} \frac{\partial f_{Me}}{\partial x} - i \frac{e}{m} k_\parallel \phi_1 \frac{\partial f_{Me}}{\partial v_\parallel}$$

$$\frac{\partial f_{Me}}{\partial x} = \frac{\partial f_{Me}}{\partial n_0} \frac{\partial n_0}{\partial x} = \frac{\partial n_0}{\partial x} \frac{f_{Me}}{n_0}$$

$$\frac{\partial f_{Me}}{\partial v_\parallel} = \left(-\frac{v_\parallel}{v_{te}^2} \right) f_{Me}$$

$$-i(\omega - k_\parallel v_\parallel) f_{e1} = \frac{i k_y}{B_0} \phi_1 \frac{\partial n_0}{\partial x} \frac{f_{Me}}{n_0} + i \frac{e}{m} k_\parallel \phi_1 \left(\frac{v_\parallel}{v_{te}^2} \right) f_{Me}$$

$$\Rightarrow \boxed{-i(\omega - k_\parallel v_\parallel) f_{e1} = i \left[\frac{k_y}{B_0} \phi_1 \frac{\partial n_0}{\partial x} \frac{\partial f_{Me}}{\partial n_0} + \frac{e}{m} k_\parallel \phi_1 \left(\frac{v_\parallel}{v_{te}^2} \right) f_{Me} \right]}$$

<Drift kinetic eqn의 정확/시작>

$\partial n_0 / \partial x < 0$ $\Rightarrow$ can potentially cause inverse Landau damping effects

$\downarrow$

$$f_{e1} = \frac{k_y v_{*e} - k_\parallel v_\parallel}{\omega - k_\parallel v_\parallel} \frac{e\phi_1}{T_{e0}} f_{Me} = \left(1 - \frac{\omega - \omega_{*e}}{\omega - k_\parallel v_\parallel} \right) \frac{e\phi_1}{T_{e0}} f_{Me}$$

$\downarrow$

$$n_1 = \int f_{e1} d\vec{v} = \frac{e\phi_1}{T_{e0}} \left[n_0 - (\omega - \omega_{*e}) \int d\vec{v} \frac{f_{eM}}{\omega - k_\parallel v_\parallel} \right] \simeq \frac{e\phi_1}{T_{e0}} \left[n_0 - (\omega - \omega_{*e}) \int dv_\parallel \frac{f_0}{\omega - k_\parallel v_\parallel} \right]$$

$\downarrow$ Plemelj correction

$$n_{e1} = \frac{e\phi_1}{T_{e0}} \left[n_0 - (\omega - \omega_{*e}) \left\{ \mathcal{P} \int_{-\infty}^{\infty} dv_\parallel \frac{f_0}{\omega - k_\parallel v_\parallel} - i\pi \frac{f_0(\omega / k_\parallel)}{k_\parallel} \right\} \right]$$

In the limit of $\omega \ll k_\parallel v_\parallel$, only Landau damping term remains meaningful.

$$f_0 = n_0 \left(\frac{m}{2\pi T}\right)^{1/2} \cancel{\exp\left(-\frac{mv^2}{2T_e}\right)} \simeq n_0 \left(\frac{m}{2\pi T}\right)^{1/2}$$ ← 1-D 이므로 계수가 1/2 제곱.

$v = \omega/k_\parallel \gg 1$

$$n_{e1} = \frac{e\phi_1}{T_{e0}}\left[n_0 + (\omega-\omega_{*e})\, i\frac{\pi}{k_\parallel} n_0 \left(\frac{m}{2\pi T}\right)^{1/2}\right] \Rightarrow \boxed{\frac{n_{e1}}{n_0} = \left[1 + i\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega-\omega_{*e}}{k_\parallel v_{te}}\right]\frac{e\phi_1}{T_{e0}}}$$

n_{e1} and ϕ_1 can have phase difference

Dispersion relation for collisionless electron drift wave is then,

$$\boxed{1 + k_\perp^2 r_s^2 - \frac{\omega_{*e}}{\omega} - \frac{k_\parallel^2 C_s^2}{\omega^2} = -i\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega-\omega_{*e}}{k_\parallel v_{te}}}$$

⑪ Physical picture of drift wave instability. (assume $k_\perp^2 r_s^2 \ll 1$, $k_\parallel^2 C_s^2 \ll \omega$)

$$Re(\omega) = \omega_{*e}\left(\frac{1}{1+k_\perp^2 r_s^2 - k_\parallel^2 C_s^2/\omega_{*e}^2}\right) \simeq \boxed{\omega_{*e}\left(1 - k_\perp^2 r_s^2 + k_\parallel^2 C_s^2/\omega_{*e}^2\right)}$$

$$\frac{1}{\omega} = \frac{1}{\omega_r + i\gamma} = \frac{1}{\omega_r}\left(\frac{1}{1+i\gamma/\omega_r}\right) = \frac{1}{\omega_r}(1-i\gamma/\omega_r) = \frac{1}{\omega_r} - i\frac{\gamma}{\omega_r^2}$$

$$Im(LHS) = -\frac{\omega_{*e}}{\omega} = -\omega_{*e}\left(-\frac{\gamma}{\omega_r^2}\right) = \frac{\gamma}{\omega_{*e}}$$

$$Im(RHS) = -\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega-\omega_{*e}}{k_\parallel v_{te}} = -\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega_{*e}}{k_\parallel v_{te}}\left(-k_\perp^2 r_s^2 + k_\parallel^2 C_s^2/\omega_{*e}^2\right) = \frac{(\pi/2)^{1/2}}{k_\parallel v_{te}}\left(k_\perp^2 r_s^2 - k_\parallel^2 C_s^2/\omega_{*e}^2\right)\omega_{*e}$$

$$\therefore \boxed{\frac{\gamma}{\omega_{*e}} = \left(\frac{\pi}{2}\right)^{1/2}\frac{\omega_{*e}}{k_\parallel v_{te}}\left(k_\perp^2 r_s^2 - \frac{k_\parallel^2 C_s^2}{\omega_{*e}^2}\right)} \rightarrow \boxed{k_\perp^2 r_s^2 > \frac{k_\parallel^2 C_s^2}{\omega_{*e}^2}}$$ ← wave-particle interaction

ion polarization <drift wave instability condition>

$$\omega < \omega_{*e} = k_y v_{*e} = k_y\left(-\frac{T_e}{e n_{e0} B_0}\right)\left(\frac{dn_0}{dx}\right)$$ 이랑 consistent.

However, one can see γ is small. This is not the main source of plasma turbulence.