Lecture 12 - Curvilinear Coordinates

[1] Dual basis vector (dual or reciprocal)

- $\vec{W} = W_a \vec{a} + W_b \vec{b} + W_c \vec{c}$ where $W_a = \vec{W}\cdot\vec{A}$, $W_b = \vec{W}\cdot\vec{B}$, $W_c = \vec{W}\cdot\vec{C}$

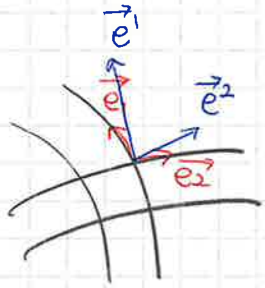
$$\Rightarrow \vec{a}\cdot\vec{A} = \vec{b}\cdot\vec{B} = \vec{c}\cdot\vec{C} = 1, \quad \vec{b}\cdot\vec{A} = \vec{C}\cdot\vec{A} = \vec{a}\cdot\vec{B} = \cdots = 0$$

$$\Rightarrow \vec{A} = \frac{\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}}, \quad \vec{B} = \frac{\vec{c}\times\vec{a}}{\vec{a}\cdot\vec{b}\times\vec{c}}, \quad \vec{C} = \frac{\vec{a}\times\vec{b}}{\vec{a}\cdot\vec{b}\times\vec{c}}$$

- $\vec{W} = W_A \vec{A} + W_B \vec{B} + W_C \vec{C}$ where $W_A = \vec{W}\cdot\vec{a}$, $W_B = \vec{W}\cdot\vec{B}$, $W_C = \vec{W}\cdot\vec{c}$

$$\Rightarrow \vec{a} = \frac{\vec{B}\times\vec{C}}{\vec{A}\cdot\vec{B}\times\vec{C}}, \quad \vec{b} = \frac{\vec{C}\times\vec{A}}{\vec{A}\cdot\vec{B}\times\vec{C}}, \quad \vec{c} = \frac{\vec{A}\times\vec{B}}{\vec{A}\cdot\vec{B}\times\vec{C}}$$

[2] Covariant vs. Contravariant representation



$\vec{e}_i = \frac{d\vec{x}}{du^i}$: tangent basis vector

$\vec{e}^i = \vec{\nabla} u_i \left(= \frac{du_i}{d\vec{x}}\right)$: gradient basis vector

$$\vec{e}_i \cdot \vec{e}^j = \delta_i^j$$

(these basis vector are not unit vector) $\rightarrow$ $h_i \equiv |\vec{e}_i|$ (scale factor)

기준: tangent-basis vector : $\vec{e}_i = h_i \hat{e}_i$, $\vec{e}^i = \frac{\hat{e}^i}{h_i}$

$$\therefore \vec{e}^1 = \vec{\nabla} u^1 = \frac{\vec{e}_2\times\vec{e}_3}{\vec{e}_1\cdot(\vec{e}_2\times\vec{e}_3)}, \quad \vec{e}_1 = \frac{d\vec{x}}{du} = \frac{\vec{e}^2\times\vec{e}^3}{\vec{e}^1\cdot(\vec{e}^2\times\vec{e}^3)}$$

* $\vec{e}_i$ 기준으로 component가 졸이 커지는지, 작아지는지

Covariant representation

$$\vec{W} = (\vec{W}\cdot\vec{e}_1)\vec{e}^1 + (\vec{W}\cdot\vec{e}_2)\vec{e}^2 + (\vec{W}\cdot\vec{e}_3)\vec{e}^3 = W_1\vec{\nabla}u^1 + W_2\vec{\nabla}u^2 + W_3\vec{\nabla}u^3$$

Contravariant representation

$$\vec{W} = (\vec{W}\cdot\vec{e}^1)\vec{e}_1 + (\vec{W}\cdot\vec{e}^2)\vec{e}_2 + (\vec{W}\cdot\vec{e}^3)\vec{e}_3 = W^1\vec{e}_1 + W^2\vec{e}_2 + W^3\vec{e}_3$$

$$= W^1 J(\vec{\nabla}u^2\times\vec{\nabla}u^3) + W^2 J(\vec{\nabla}u^3\times\vec{\nabla}u^1) + W^3 J(\vec{\nabla}u^1\times\vec{\nabla}u^2)$$

$$\left(J = \vec{e}_1\cdot\vec{e}_2\times\vec{e}_3 = \frac{1}{\vec{\nabla}u_1\cdot(\vec{\nabla}u_2\times\vec{\nabla}u_3)}\right)$$

[3] Metric tensors and Jacobian

$$g_{ij} = \vec{e_i}\cdot\vec{e_j} \qquad (g_{ii} = |h_i|^2) \quad \Rightarrow \left(\begin{array}{l} \vec{e_i} = g_{ij}\vec{e^j}, \quad \vec{e^i} = g^{ij}\vec{e_j} \\ w_i = g_{ij} w^j, \quad w^i = g^{ij} w_j \end{array} \right)$$

$$(g^{ij} = \vec{e^i}\cdot\vec{e^j}$$

$$\delta_i^k = \vec{e_i}\cdot\vec{e^k} = g_{ij}\,\vec{e^j}\,\vec{e^k} = g_{ij}\,g^{jk} \Rightarrow \underline{\text{metric tensor are inverse each other.}}$$

$$g \equiv \det[g_{ij}] \quad , \quad g^{-1} \equiv \det[g^{ij}]$$

* Jacobian of a coordinate system

$$J \equiv J(u^1, u^2, u^3) = \frac{\partial(x,y,z)}{\partial(u^1,u^2,u^3)} = \begin{vmatrix} \frac{\partial x}{\partial u^1} & \frac{\partial x}{\partial u^2} & \frac{\partial x}{\partial u^3} \\ \frac{\partial y}{\partial u^1} & \frac{\partial y}{\partial u^2} & \frac{\partial y}{\partial u^3} \\ \frac{\partial z}{\partial u^1} & \frac{\partial z}{\partial u^2} & \frac{\partial z}{\partial u^3} \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial y}{\partial u_1} & \frac{\partial z}{\partial u^1} \\ \frac{\partial x}{\partial u_2} & \frac{\partial y}{\partial u_2} & \frac{\partial z}{\partial u^2} \\ \frac{\partial x}{\partial u_3} & \frac{\partial y}{\partial u_3} & \frac{\partial z}{\partial u^3} \end{vmatrix}$$

$$= \frac{\partial\vec{x}}{\partial u^1}\cdot\left(\frac{\partial\vec{x}}{\partial u^2}\times\frac{\partial\vec{x}}{\partial u^3}\right) = \vec{e_1}\cdot(\vec{e_2}\times\vec{e_3})$$

$$\frac{1}{J} = \vec{e^1}\cdot(\vec{e^2}\times\vec{e^3}) = \frac{1}{J}(\vec{e_2}\times\vec{e_3})\cdot\left(\frac{1}{J}(\vec{e_3}\times\vec{e_1})\times\frac{1}{J}(\vec{e_1}\times\vec{e_2})\right)$$

$$= \frac{1}{J^3}(\vec{e_2}\times\vec{e_3})\cdot\left[(\vec{e_3}\times\vec{e_1})\times(\vec{e_1}\times\vec{e_2})\right]$$

$$= \frac{1}{J^3}(\vec{e_2}\times\vec{e_3})\cdot\left[\vec{e_3}\cdot(\vec{e_1}\times\vec{e_2})\vec{e_1} - \vec{e_1}\cdot\cancelto{0}{(\vec{e_1}\times\vec{e_2})}\vec{e_3}\right]$$

$$= \frac{1}{J^3}(\vec{e_2}\times\vec{e_3})\cdot\vec{e_1}\left(\vec{e_3}\cdot(\vec{e_1}\times\vec{e_2})\right) = \frac{1}{J}$$

$$[g_{ij}] = \left[\frac{\partial\vec{x}}{\partial u^i}\frac{\partial\vec{x}}{\partial u^j}\right] \rightarrow g \equiv \det[g_{ij}] = J^2 \quad \text{or} \quad J = \sqrt{g}$$

4 Vector and del operations

$$\vec{e_k} = \epsilon_{ijk} J\, \vec{e}^{\,i} \times \vec{e}^{\,j} = \epsilon_{ijk}\sqrt{g}\, \vec{e}^{\,i} \times \vec{e}^{\,j}$$

$(\epsilon_{ijk} = \epsilon^{ijk})$

$$\vec{e}^{\,k} = \epsilon^{ijk}\frac{1}{J}\,\vec{e_i}\times\vec{e_j} = \epsilon^{ijk}\frac{1}{\sqrt{g}}\,\vec{e_i}\times\vec{e_j}$$

- Dot product.

$$\vec{A}\cdot\vec{B} = A_i B^i = A^i B_i = g_{ij}A^j B^i = g^{ij}A_j B_i$$

- Cross product

$$\vec{A}\times\vec{B} = \sum_{ij} A^i B^j\, \vec{e_i}\times\vec{e_j} \longrightarrow (\vec{A}\times\vec{B})_k = \epsilon_{ijk} J A^i B^j$$

$$= \sum_{ij} A_i B_j\, \vec{e}^{\,i}\times\vec{e}^{\,j} \longrightarrow (\vec{A}\times\vec{B})^k = \epsilon^{ijk}\frac{1}{J} A_i B_j$$

- Gradient

$$\vec{\nabla}\phi = \frac{\partial\phi}{\partial \vec{x}} = \frac{\partial\phi}{\partial u^i}\frac{\partial u^i}{\partial\vec{x}} = \frac{\partial\phi}{\partial u^i}\vec{\nabla} u^i$$

- Divergence

contravariant

$$\vec{\nabla}\cdot\vec{A} = \vec{\nabla}\cdot(A^i\vec{e_i}) = \vec{\nabla}\cdot\left(JA^i\frac{\vec{e_i}}{J}\right) \underset{(*)}{=} \frac{\vec{e_i}}{J}\cdot\vec{\nabla}(JA^i) = \frac{1}{J}\frac{\partial}{\partial u^i}(JA^i)$$

(*) $\vec{\nabla}\cdot\left(\frac{\vec{e_i}}{J}\right) = \vec{\nabla}\cdot(\vec{\nabla}u^j\times\vec{\nabla}u^k) = \vec{\nabla}\cdot\left(\vec{\nabla}\times(u^j\vec{\nabla}u^k)\right) = 0$

(**) $\vec{\nabla} \equiv \vec{\nabla}u^i\frac{\partial}{\partial u^i} \equiv \vec{e}^{\,i}\frac{\partial}{\partial u^i}$

- Curl

covariant

$$\vec{\nabla}\times\vec{A} = \vec{\nabla}\times(A_j\vec{e}^{\,j}) = \vec{\nabla}A_j\times\vec{e}^{\,j} = \frac{\partial A_j}{\partial u^i}\vec{e}^{\,i}\times\vec{e}^{\,j} = \epsilon^{ijk}\frac{1}{J}\frac{\partial A_j}{\partial u^i}\vec{e_k}$$

5 Vector gradient and Christoffel symbol

$$\vec{\nabla}\vec{A} = \vec{e}^k \frac{d}{du^k}(A^j \vec{e}_j) = \vec{e}^k \vec{e}_j \frac{dA^j}{du^k} + \vec{e}^k A^j \frac{d\vec{e}_j}{du^k}$$

$$= \vec{e}^k \vec{e}_j \left(\frac{dA^j}{du^k} + A^i \frac{d\vec{e}_i}{du^k} \cdot \vec{e}^j \right) = \vec{e}^k \cdot \vec{e}_j \left(\frac{dA^j}{du^k} + A^i \Gamma^j_{ik} \right).$$

* Here <u>Christoffel symbol</u> (3rd order tensor)

$$\Gamma^i_{jk} \equiv \vec{e}^i \cdot \frac{d\vec{e}_j}{du^k} = \vec{e}^i \cdot \frac{d\vec{e}_k}{du^j} = \Gamma^i_{kj}$$

* Then, one can show $\Gamma^i_{jk} = \frac{1}{2} g^{in} \left[\frac{dg_{nj}}{du^k} + \frac{dg_{nk}}{du^j} - \frac{dg_{jk}}{du^n} \right]$

proof) $\frac{d\vec{e}_i}{du^k} = \underbrace{\left(\frac{d\vec{e}_i}{du^k} \cdot \vec{e}^j \right)}_{(1)} \vec{e}_j = \left(\frac{d\vec{e}_i}{du^k} \right)^j \vec{e}_j$

(1) $$\left(\frac{d\vec{e}_i}{du^k} \cdot \vec{e}^j \right) = g^{jn} \left[\frac{d\vec{e}_i}{du^k} \cdot \vec{e}_n \right] = g^{jn} \frac{1}{2} \left[\vec{e}_n \cdot \frac{d\vec{e}_i}{du^k} + \vec{e}_n \cdot \frac{d\vec{e}_k}{du^i} \right]$$

$$= g^{jn} \frac{1}{2} \left[\frac{d}{du^k}(\vec{e}_n \cdot \vec{e}_i) + \frac{d}{du^i}(\vec{e}_n \cdot \vec{e}_k) - \vec{e}_i \frac{d\vec{e}_n}{du^k} - \vec{e}_k \frac{d\vec{e}_n}{du^i} \right]$$

$$= g^{jn} \frac{1}{2} \left[\frac{dg_{ni}}{du^k} + \frac{dg_{nk}}{du^i} - \frac{dg_{ik}}{du^n} \right] \blacksquare$$

6 Tensorial object

$$\overleftrightarrow{F} = F_{ij} \vec{e}^i \vec{e}^j = F^{ij} \vec{e}_i \vec{e}_j = F_i^{\ j} \vec{e}^i \vec{e}_j$$

dyad: $\overleftrightarrow{F} = \vec{A}\vec{B} \quad (\vec{A}\vec{B} \neq \vec{B}\vec{A})$

[7] Differential elements

$$\underline{d\ell_i} = \sqrt{du^i\vec{e_i}\cdot du^i\vec{e_i}} = \sqrt{g_{ii}}\,du^i = \underline{J\left|\vec{\nabla}u^j\times\vec{\nabla}u^k\right|du^i}$$

$$\underline{dS_i} = \left|\frac{d\vec{x}}{du^j}\times\frac{d\vec{x}}{du^k}\right|du^j du^k = \left|\vec{e_j}\times\vec{e_k}\right|du^j du^k = \underline{J\left|\vec{\nabla}u^i\right|du^j du^k}$$

$$\hookrightarrow \underline{d\vec{S_i} = J\,\vec{\nabla}u_i\,du^j du^k}$$

$$dV = d^3\vec{x} = \frac{d\vec{x}}{du^i}\cdot\left(\frac{d\vec{x}}{du^j}\times\frac{d\vec{x}}{du^k}\right)du^i du^j du^k = \sqrt{g}\,du^i du^j du^k$$

[8] Field line tracing

$$\vec{B} = c\,d\vec{x} \rightarrow B = c\left|d\vec{x}\right| = d\ell \rightarrow \vec{B}\cdot\vec{\nabla}u^i = d\ell\,\vec{\nabla}u^i$$

$$\Rightarrow \frac{B}{d\ell} = \frac{B^1}{du^1} = \frac{B^2}{du^2} = \frac{B^3}{du^3}$$

For example) $$\frac{\vec{B}\cdot\vec{\nabla}R}{dR} = \frac{\vec{B}\cdot\vec{\nabla}z}{dz} = \frac{\vec{B}\cdot\vec{\nabla}\phi}{d\phi}$$

$$\frac{B_R}{dR} = \frac{B_z}{dz} = \frac{B_\phi}{R\,d\phi} \Rightarrow \underline{\frac{dR}{d\phi} = \frac{RB_R}{B_\phi}},\quad \underline{\frac{dz}{d\phi} = \frac{RB_z}{B_\phi}}$$