Fusion Plasma Theory 2

Lecture 17. ETG and ITG.

① Electron Temperature Gradient (ETG)

$\vec{\nabla} T$ can also affects electron drift waves. Let's consider non-uniform temperature

$$f_{eM} = n_o(x)\left(\frac{m}{2\pi T_e(x)}\right)^{3/2} \exp\left[-\frac{m}{2T_e(x)}(v_\perp^2 + v_\parallel^2)\right]$$

$$\frac{df_{eM}}{dx} = \frac{df_{eM}}{dn_o}\frac{dn_o}{dx} + \frac{df_{eM}}{dT_e}\frac{dT_e}{dx} = \frac{f_{eM}}{n_o}\frac{dn_o}{dx} + \frac{dT_e}{dx}\left(-\frac{3}{2}\frac{f_{eM}}{T_{eo}} + \frac{mv^2}{2T_e^2(x)}\right) \quad (T_e = mv_{te}^2 \text{ 가정})$$

$$= \frac{f_{eM}}{n_o}\frac{dn_o}{dx} - \frac{f_{eM}}{T_e}\frac{dT_e}{dx}\left(\frac{3}{2} - \frac{v^2}{2v_{t,e}^2}\right)$$

Scale length ratio of L_n/L_T

$$= \frac{f_{eM}}{n_o}\frac{dn_o}{dx}\left[1 - \eta_e\left(\frac{3}{2} - \frac{v^2}{v_{t,e}^2}\right)\right] \quad \text{where } \eta_e = \frac{L_n}{L_T},\ L_n = -\frac{n_o}{dn_o/dx},\ L_T = -\frac{T_o}{dT_o/dx}$$

② Electron response with ETG

(same procedure in Lec16)

$$\frac{\partial f}{\partial t} + (\vec{v_\parallel} + \vec{v_E})\cdot\vec{\nabla} f + \frac{qE_\parallel}{m}\frac{df}{dv_\parallel} = 0 \quad \Rightarrow \quad \frac{df_e}{dt} + \left(v_\parallel\hat{z} + \frac{\hat{z}\times\vec{\nabla}\phi_1}{B_o}\right)\cdot\vec{\nabla} f + \frac{e}{m}\frac{d\phi_1}{dz}\frac{df}{dv_\parallel} = 0$$

⟨Drift-kinetic⟩ ⟨Electron drift kinetic⟩

$$\Rightarrow \frac{df_{e1}}{dt} + v_\parallel\frac{df_{e1}}{dz} + \frac{\hat{b}\times\vec{\nabla}\phi}{B_o}\cdot\vec{\nabla} f_{eM} + \frac{e}{m}\frac{d\phi_1}{dz}\frac{df_{eM}}{dv_\parallel} = 0$$

⟨Linearized electron drift kinetic eqn⟩

$$\Rightarrow -i(\omega - k_\parallel v_\parallel) f_{e1} = -\frac{\hat{z}\times\left(\frac{d\phi_1}{dy}\right)\hat{y}}{B_o}\cdot\left(\frac{df_{eM}}{dx}\right)\hat{x} - i\frac{e}{m}k_\parallel\phi_1\frac{df_{eM}}{dv_\parallel}$$

$$= i\left[\frac{k_y}{B_o}\phi_1\left(\frac{f_{eM}}{n_o}\frac{dn_o}{dx}\left(1-\eta_e\left(\frac{3}{2}-\frac{v^2}{2v_{t,e}^2}\right)\right)\right) + \frac{e}{m}k_\parallel\phi_1\left(\frac{v_\parallel}{v_{t,e}^2}\right)f_{Me}\right]$$

note $\omega_{*e} = -\frac{k_y}{eB_o}\frac{1}{n_o}\left(\frac{dn_o}{dx}\right)$

$$= i\left[-\phi_1\frac{e}{T_e}\omega_{*e}\left(1-\eta_e\left(\frac{3}{2}-\frac{v^2}{2v_{te}^2}\right)\right) + \frac{e\phi_1}{T_e}k_\parallel v_\parallel\right]f_{Me}$$

$$\Rightarrow f_{e1} = \frac{e\phi_1}{T_e}f_{Me}\left[\frac{\omega_{*e}(1-\eta\cdots) + k_\parallel v_\parallel}{\omega - k_\parallel v_\parallel}\right] = \frac{e\phi_1}{T_e}f_{Me}\left[1 - \frac{\omega - \omega_{*e}\left(1-\eta\left(\frac{3}{2}-\frac{v^2}{2v_{te}^2}\right)\right)}{\omega - k_\parallel v_\parallel}\right]$$

$$\Rightarrow n_{e1} = \int f_{e1}\,d\vec{v} = \frac{e\phi_1}{T_e}\left[n_o - \int_{-\infty}^{\infty}\frac{f_o(v_\parallel)dv_\parallel}{\omega - k_\parallel v_\parallel}\left\{\omega - \omega_{*e}\left[1-\eta_e\left(\frac{3}{2} - 1 - \frac{v_\parallel^2}{2v_{te}^2}\right)\right]\right\}\right]$$

$$f_o(v_\parallel) = \int 2\pi v_\perp\, dv_\perp\, f_{eM}$$

* note

$$\left(\begin{aligned} \int 2\pi v_\perp dv_\perp dv_\parallel f_{eM} &= \int_{-\infty}^{\infty} dv_\parallel \int (2\pi v_\perp)\, dv_\perp\, n_0 \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{v_\perp^2}{2v_{te}^2}\right) \exp\left(-\frac{v_\parallel^2}{2v_{te}^2}\right) \\ &= n_0 \left(\frac{m}{2\pi T}\right)^{3/2} \exp\left(-\frac{v_\parallel^2}{2v_{te}^2}\right) \underbrace{\int (2\pi v_\perp)\, dv_\perp \exp\left(-\frac{m v_\perp^2}{2T}\right)}_{= \left(-\frac{2\pi T}{m}\right) \exp\left(-\frac{m v_\perp^2}{2T}\right)\Big|_0^\infty = \frac{2\pi T}{m}} = n_0 \left(\frac{m}{2\pi T}\right)^{1/2} \exp\left(-\frac{m v_\parallel^2}{2T}\right) \equiv f_0 \end{aligned} \right)$$

$$\int_{-\infty}^{\infty} \frac{f_0(v_\parallel)\, dv_\parallel}{\omega - k_\parallel v_\parallel} = \cancelto{0}{P\int_{-\infty}^{\infty} \frac{f_0(v_\parallel)\, dv_\parallel}{\omega - k_\parallel v_\parallel}} - i\frac{\pi}{k_\parallel} f_0\left(\frac{\omega}{k_\parallel}\right)$$

(odd function)

$\Downarrow$

$$\begin{aligned} n_{e1} &= \frac{e\phi_1}{T_{e0}} \left[n_0 + i\frac{\pi}{k_\parallel} f_0\left(\frac{\omega}{k_\parallel}\right) \cdot \left(\omega - \omega_{*e}\left(1 - \eta_e\left(\frac{1}{2} - \frac{v_\parallel^2}{2v_{te}^2}\right)\right)\right)\right] \\ &= \frac{e\phi_1}{T_{e0}} \left[n_0 + i\frac{\pi}{k_\parallel} n_0 \left(\frac{m}{2\pi T}\right)^{1/2} \exp\left(\cancelto{0}{-\frac{\omega^2/k_\parallel^2}{2v_{te}^2}}\right) \left(\omega - \omega_{*e}\left(1 - \frac{\eta_e}{2}\right) + \cancelto{0}{\omega_{*e}\eta_e \frac{\omega^2/k_\parallel^2}{2v_{te}^2}} \right)\right] \\ &= \frac{e\phi_1}{T_{e0}} \left[n_0 + n_0\, i \left(\frac{\pi}{2}\right)^{1/2} \frac{\omega - \omega_{*e}(1 - \eta_e/2)}{k_\parallel v_{te}} \right] \end{aligned}$$

$$\therefore \frac{n_{e1}}{n_0} = \left[1 + i\left(\frac{\pi}{2}\right)^{1/2} \frac{\omega - \omega_{*e}(1-\eta_e/2)}{k_\parallel v_{te}} \right] \quad \overset{\text{원래}}{\Longleftrightarrow} \quad \frac{n_{e1}}{n_0} = \left[1 + i\left(\frac{\pi}{2}\right)^{1/2} \frac{\omega - \omega_{*e}}{k_\parallel v_{te}} \right]$$

③ Electron drift wave dispersion under ETG

$$1 + k_\perp^2 r_s^2 - \frac{\omega_{*e}}{\omega} - \frac{k_\parallel^2 C_s^2}{\omega^2} = -i\left(\frac{\pi}{2}\right)^{1/2} \frac{\omega - \omega_{*e}(1-\eta_e/2)}{k_\parallel v_{t,e}} \qquad (\omega_{*e} \rightarrow \omega_{*e}(1-\eta_e/2))$$

$$\frac{\gamma}{\omega_{*e}} = \frac{\mathrm{Im}(\omega)}{\omega_{*e}} \simeq \left(\frac{\pi}{2}\right)^{1/2} \frac{\omega_{*e}}{k_\parallel v_{te}} \left(k_\perp^2 r_s^2 - \frac{k_\parallel C_s^2}{\omega_{*e}^2} - \frac{\eta_e}{2} \right)$$

$\therefore$ Electron drift wave condition with ETG

$$\Rightarrow \boxed{k_\perp^2 r_s^2 > \frac{k_\parallel C_s^2}{\omega_{*e}^2} + \frac{\eta_e}{2}}$$ ← stabilizing effect due to ETG!

④ Shifted Maxwellian with electron currents.

$j_{\parallel} = -e n_0 u_{e0}$ can also affect the electron drift waves.

$$f_{eM} = n_0(x)\left(\frac{m}{2\pi T_e}\right)^{3/2}\exp\left(-\frac{m}{2T_e}\left(v_\perp^2+(v_\parallel-u_{e0})^2\right)\right)$$

$$\frac{\partial f_{eM}}{\partial v_\parallel} = -\frac{v_\parallel-u_{e0}}{v_{te}^2} f_{eM} \qquad (T_e = m v_{te}^2 \text{ 가정}) \quad / \quad \frac{\partial f_{eM}}{\partial x} = \frac{f_{eM}}{n_0}\frac{dn}{dx}$$

<Drift kinetic, again>

$$\frac{\partial f}{\partial t} + \left(v_\parallel\hat b + \frac{\hat b\times\vec\nabla\phi}{B_0}\right)\cdot\vec\nabla f + \frac{qE_\parallel}{m}\frac{\partial f}{\partial v_\parallel}=0 \rightarrow \frac{\partial f}{\partial t}+\left(v_\parallel \hat b + \frac{\hat b\times\vec\nabla\phi}{B_0}\right)\cdot\vec\nabla f + \frac{e}{m}\frac{\partial\phi_1}{\partial z}\frac{\partial f}{\partial v_\parallel}=0$$

$$\frac{\partial f_{e1}}{\partial t} + v_\parallel\frac{\partial f_{e1}}{\partial z} + \frac{\hat z\times\frac{\partial\phi_1}{\partial y}\hat y}{B_0}\cdot\frac{\partial f_M}{\partial x}\hat x + \frac{e}{m}\frac{\partial\phi_1}{\partial z}\frac{\partial f_M}{\partial v_\parallel}=0$$

$$-i(\omega-k_\parallel v_\parallel)f_{e1} - ik_y\frac{\phi_1}{B_0}\frac{\partial f_M}{\partial x} + i\frac{e}{m}k_\parallel\phi_1\frac{\partial f_M}{\partial v_\parallel}=0$$

$$(\omega-k_\parallel v_\parallel)f_{e1} + k_y\frac{\phi}{B_0}\frac{f_M}{n_0}\frac{dn}{dx} - \frac{e}{m}k_\parallel\phi_1\left(-\frac{v_\parallel-u_{e0}}{v_{te}^2}\right)f_{eM}=0$$

$$\left(\text{*note } \omega_{*e} = -\frac{k_y T_e}{eB}\frac{1}{n_0}\frac{dn}{dx}\right)$$

$$(\omega-k_\parallel v_\parallel)f_{e1} - \frac{e\phi_1}{T_e}\omega_{*e}f_{eM} + \frac{e\phi_1}{T_e}k_\parallel(v_\parallel-u_{e0})f_{eM}=0$$

$$\therefore f_{e1} = \frac{e\phi_1}{T_e}f_{eM}\left(\frac{\omega_{*e}-k_\parallel(v_\parallel-u_{e0})}{\omega-k_\parallel v_\parallel}\right) = \left(1-\frac{\omega-\omega_{*e}-k_\parallel u_{e0}}{\omega-k_\parallel v_\parallel}\right)\frac{e\phi_1}{T_e}f_{eM}$$

clearly we only see the Dopler shift $\omega\to\omega-k_\parallel u_{e0}$ $\quad (\omega^*\to\omega^*+k_\parallel u_{e0})$

⑤ Drift wave dispersion with electron currents.

$$1+k_\perp^2\rho_s^2 - \frac{\omega_{*e}}{\omega} - \frac{k_\parallel c_s^2}{\omega^2} = -i\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega-\omega^*-k_\parallel u_{e0}}{k_\parallel v_{te}} \quad \leftarrow \quad \omega-\omega^*\left(1+\frac{k_\parallel u_{e0}}{\omega^*}\right)$$

그대로 추가

$$\frac{\gamma}{\omega_{*e}} = \frac{\mathrm{Im}(\omega)}{\omega_{*e}} \simeq \left(\frac{\pi}{2}\right)^{1/2}\frac{\omega_{*e}}{k_\parallel v_{te}}\left(k_\perp^2\rho_s^2 - \frac{k_\parallel c_s^2}{\omega_{*e}^2} - \frac{k_\parallel}{\omega^*}\frac{j_\parallel}{en_0}\right)$$

∴ Electron drift wave instability condition with the electron currents:

$$\boxed{k_\perp^2\rho_s^2 > \frac{k_\parallel c_s^2}{\omega_{*e}^2} + \frac{k_\parallel}{\omega^*}\frac{j_\parallel}{en_0}}$$

$j_\parallel > 0$: stabilizing

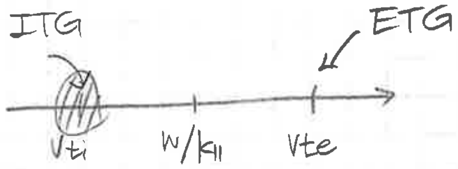
$j_\parallel < 0$: destabilizing.

⑥ Drift waves with Ion Temperature Gradient (ITG)

- $\vec{\nabla} T_i$ can excite an instability by driving the ion accoustic wave unstable, as called ITG mode.

- It can be driven collisionlessly, even without kinetic wave particle interaction

⑦ Electron and ion responses by ITG



In this limit, treat adiabatic electron (Boltzmann relation)

$$n_{e1} = \frac{e n_0}{T_e}\phi_1$$

But restore ion thermal effects (which we ignored by "cold" ion assumption)

$$\tau \equiv \frac{T_e}{T_i}.$$

Linearized drift-kinetic equation for ions:

$$\frac{\partial f_{i1}}{\partial t} + v_\parallel \frac{\partial f_{i1}}{\partial z} + \frac{\hat{z}\times\vec{\nabla}\phi_1}{B_0}\cdot\vec{\nabla} f_{iM} - \frac{e}{M}\frac{\partial \phi_1}{\partial z}\frac{\partial f_{iM}}{\partial v_\parallel} = 0$$

↳ ignores ion polarization or any order $O(k_\perp r_{Li})$ ($\because k_\perp$ is small in ion)

Analogous to that carried out for electrons

$$n_{i1} = \int f_{i1} d\vec{v} = -\frac{e\phi_1}{T_i}\left[n_0 - \int_{-\infty}^{\infty}\frac{f_0(v_\parallel)\,dv_\parallel}{\omega - k_\parallel v_\parallel}\left\{\omega - \omega_{*i}\left[1-\eta_i\left(\frac{1}{2}-\frac{v_\parallel^2}{v_{ti}^2}\right)\right]\right\}\right]$$

Define: $\left(\omega_{*pi} \equiv \omega_{*i} + \omega_{*T_i} = k_y \frac{1}{neB}\frac{dp_i}{dx} < 0 \quad \left(\omega_{*i} \equiv k_y \frac{T}{neB}\frac{dn_i}{dx} < 0 \quad \omega_{*T_i} \equiv k_y \frac{1}{eB}\frac{dT_i}{dx} < 0\right)\right)$ / $\left(\eta_i \equiv \frac{L_n}{L_T} \quad \left(L_n = -\frac{n}{dn/dx},\ L_T = -\frac{T}{dT/dx}\right)\right)$

Then, $n_{e1} = n_{i1}$ gives

$$\frac{e n_0}{T_e}\phi_1 = -\frac{e n_0}{T_i}\phi_1\left[1 - \frac{1}{n_0}\int_{-\infty}^{\infty}\cdots\right]$$, divide by $\frac{e n_0}{T_i}\phi_1$

$$\frac{T_i}{T_e} + 1 = 1 + \frac{1}{\tau} = \frac{1}{n_0}\int_{-\infty}^{\infty}\frac{f_0(v_\parallel)\,dv_\parallel}{\omega - k_\parallel v_\parallel}\left\{\omega - \omega_{*i}\left[1-\eta_i\left(\frac{1}{2}-\frac{v_\parallel^2}{2v_{ti}^2}\right)\right]\right\}$$

⑧ Dispersion relation for ITG mode

$$\left(\omega \geq k_\parallel v_\parallel\right) : \frac{1}{\omega - k_\parallel v_\parallel} = \frac{1}{\omega}\frac{1}{(1-k_\parallel v_\parallel/\omega)} = \frac{1}{\omega}\left(1+\frac{k_\parallel v_\parallel}{\omega}+\left(\frac{k_\parallel v_\parallel}{\omega}\right)^2+\cancel{\left(\frac{k_\parallel v_\parallel}{\omega}\right)^3}+\cdots\right) = \frac{1}{\omega}+\frac{k_\parallel v_\parallel}{\omega^2}+\frac{k_\parallel^2 v_\parallel^2}{\omega^3}+\cdots$$

The integrals we need:

$$\left(\begin{array}{l} \displaystyle\int_{-\infty}^{\infty}\frac{f_0(v_\parallel)}{\omega-k_\parallel v_\parallel}dv_\parallel \simeq \frac{n_0}{\omega}\left(1+\frac{k_\parallel^2 v_\parallel^2}{\omega^2}+\cdots\right) - i\pi\frac{1}{k_\parallel}f(\omega/k_\parallel) \\ \displaystyle\int_{-\infty}^{\infty}\frac{f_0(v_\parallel)v_\parallel^2}{\omega-k_\parallel v_\parallel}dv_\parallel = \frac{n_0 v_{ti}^2}{\omega}\left(1+\frac{3k_\parallel^2 v_\parallel^2}{\omega^2}+\cdots\right) - i\pi\frac{1}{k_\parallel}\left(\frac{\omega}{k_\parallel}\right)^2 f(\omega/k) \end{array}\right)$$

Putting together:

$$1+\frac{1}{\tau} = \frac{1}{n_0}\int_{-\infty}^{\infty}\frac{f_0(v_\parallel)dv_\parallel}{\omega-k_\parallel v_\parallel}\left\{\omega-\omega_{*i}\left(1-\frac{\eta_i}{2}\right)\right\} + \frac{1}{n_0}\int_{-\infty}^{\infty}\frac{f_0(v_\parallel)v_\parallel^2 dv_\parallel}{\omega-k_\parallel v_\parallel}\left\{+\frac{\omega_{*i}\eta_i}{2v_{ti}^2}\right\}$$

$$= \frac{1}{\cancel{n_0}}\frac{\cancel{n_0}}{\omega}\left(1+\frac{k_\parallel^2 v_\parallel^2}{\omega^2}\right)\left(\omega-\omega_{*i}\left(1-\frac{\eta_i}{2}\right)\right) + \frac{1}{\cancel{n_0}}\frac{\cancel{n_0}\cancel{v_{ti}^2}}{\omega}\left(1+\cancel{\frac{3k_\parallel^2 v_{ti}^2}{\omega^2}}\right)\left(+\frac{\omega_{*i}\eta_i}{2\cancel{v_{ti}^2}}\right)$$

$$+\frac{1}{n_0}\left(-i\pi\frac{1}{k_\parallel}f(\omega/k_\parallel)\right) + \frac{1}{n_0}\left(-i\pi\frac{\omega^2}{k_\parallel^3}f(\omega/k)\right)$$

$\frac{1}{\omega^3}$ 고차항은 무시

$$= 1+\frac{k_\parallel^2 v_\parallel^2}{\omega^2} - \frac{\omega_{*i}}{\omega}\left(1-\frac{\eta_i}{2}\right) + \frac{\omega_{*i}}{\omega}\frac{\eta_i}{2} - i\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega-\left(\omega_{*i}-\frac{\omega^2}{k^2 v_{ti}^2}\right)(1-\eta_i/2)}{k_\parallel v_{ti}}\exp\left(-\frac{\omega^2}{2k_\parallel^2 v_{ti}^2}\right)$$

$$= 1-\frac{\omega_{*i}}{\omega}+\frac{k_\parallel^2 v_{ti}^2}{\omega^2}\left[1-\frac{\omega_{*i}}{\omega}(1+\eta_i)\right] - i\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega-\left(\omega_{*i}-\frac{\omega^2}{k^2 v_{ti}^2}\right)(1-\eta_i/2)}{k_\parallel v_{ti}}\exp\left(-\frac{\omega^2}{2k_\parallel^2 v_{ti}^2}\right)$$

*note

$\omega_{*i}(1+\eta_i)=\omega_{*pi}$, $\tau\omega_{*i}=-\omega_{*e}$. $\tau v_{ti} = C_s$

Then,

$$\boxed{1-\frac{\omega_{*e}}{\omega}-\frac{k_\parallel^2 C_s^2}{\omega^2}\left(1-\frac{\omega_{*pi}}{\omega}\right) = -i\tau\left(\frac{\pi}{2}\right)^{1/2}\frac{\omega-\left(\omega_{*i}-\frac{\omega^2}{k_\parallel^2 v_{ti}^2}\right)(1-\eta_i/2)}{k_\parallel v_{ti}}\exp\left(-\frac{\omega^2}{2k_\parallel^2 v_{ti}^2}\right)}$$

$\omega_{*pi}\to 0$, $v_{ti}\to 0$ or $\tau\to 0$ gives pure electron driftwave.

⑨ Upper limit of ITG mode growth

$$1-\frac{\omega_{*e}}{\omega}-\frac{k_\parallel^2 C_s^2}{\omega^2}\left(1-\frac{\omega_{*pi}}{\omega}\right)=0$$ (when ignoring kinetic correction)

Consider a strong gradient. $\Rightarrow$ 2nd + 4th term dominate.

$$\frac{\omega_{*e}}{\omega}=\frac{k_\parallel^2 C_s^2}{\omega^3}\omega_{*pi} \rightarrow \omega^2=k_\parallel^2 C_s^2\left(\frac{\omega_{*pi}}{\omega_{*e}}\right)=k_\parallel^2 C_s^2\left(\frac{\omega_{*i}(1+\eta_i)}{-\tau\omega_{*i}}\right)=-\left(\frac{1+\eta_i}{\tau}\right)k_\parallel^2 C_s^2$$

$$\boxed{\gamma=\left(\frac{1+\eta_i}{\tau}\right)^{1/2}k_\parallel C_s}$$ $\Leftarrow$ upper bound of ITG linear growth rate.

(가장 극단적인 상황 (strong gradient) 가정)

⑩ ITG mode in flat density limit

$L_n \rightarrow \infty$, $\omega_{*e}/\omega \ll 1$ $\Rightarrow$ 1st + 4th term dominate

$$1+\frac{k_\parallel^2 C_s^2}{\omega^3}\omega_{*pi}=0 \rightarrow \omega^3=k_\parallel^2 C_s^2(-\omega_{*Ti})$$

$$\boxed{\gamma=|\omega_{*Ti}|^{1/3}(k_\parallel C_s)^{2/3}}$$ one of the 3 roots are unstable

⑪ ITG mode onset condition

ITG mode remains unstable even for very weak ITG $L_{Ti}\rightarrow\infty$ or $\eta_i\rightarrow 0$.

This is unphysical and does not agree with numerical modeling results.

$$1-\frac{\omega_{*e}}{\omega}-\frac{k_\parallel^2 C_s^2}{\omega^2}=-i\tau\left(\frac{\pi}{2}\right)^{1/2}\left[\frac{\omega-\omega_{*i}(1-\eta_i/2)}{k_\parallel V_{ti}}\right]$$

If one assumes $\underline{\omega \approx k_\parallel C_s \ll \omega_{*i}}$ it gives

$$(1-\eta_i/2)<0 \rightarrow \boxed{\eta_i>2}$$ <onset condition for ITG mode>