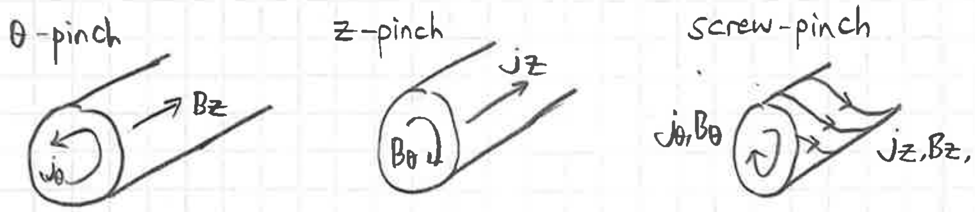
Lecture 19. Kink and Interchange

I Pinches



∘ <u>θ-pinch equilibrium</u> (j_θ, B_z only, poloidal and toroidal symmetry : $\frac{\partial}{\partial\theta} = \frac{\partial}{\partial z} = 0$)
to the magnetic field

i) $\vec{\nabla}\cdot\vec{B} = 0$

ii) $\vec{\nabla}\times\vec{B} = \mu_0\vec{j} \rightarrow \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \cancel{\frac{\partial}{\partial\theta}} & \cancel{\frac{\partial}{\partial z}} \\ 0 & 0 & B_z \end{vmatrix} = -\frac{dB_z}{dr} = \mu_0 j_\theta \qquad \therefore j_\theta = -\frac{1}{\mu_0}\frac{dB_z}{dr}$

iii) $\vec{\nabla}p = \vec{j}\times\vec{B} \rightarrow \frac{dp}{dr} = j_\theta B_z = -\frac{1}{\mu_0}B_z\frac{dB_z}{dr}$

$$\therefore \frac{d}{dr}\left(p + \frac{B_z^2}{2\mu_0}\right) = 0 \longrightarrow \boxed{p + \frac{B_z^2}{2\mu_0} = \frac{B_0^2}{2\mu_0}}$$

In θ-pinch, $j_{\parallel} = j_z = 0$ and $\vec{K} = 0$ ($\because$ only $B_z \neq 0$) $\Rightarrow$ <u>Always stable</u>

(refer the δW; then you'll find all the remaining terms are positive)

∘ <u>Z-pinch equilibrium</u> (B_θ, j_z only, $\frac{\partial}{\partial\theta} = \frac{\partial}{\partial z} = 0$)

i) $\vec{\nabla}\cdot\vec{B} = 0$

ii) $\vec{\nabla}\times\vec{B} = \mu_0\vec{j} \rightarrow \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ 0 & rB_\theta & 0 \end{vmatrix} = -\frac{1}{r}\frac{\partial B_\theta}{\partial z}\hat{r} + \frac{1}{r}\frac{d(rB_\theta)}{dr}\hat{z} = \mu_0\vec{j} \qquad \therefore j_z = \frac{1}{\mu_0 r}\frac{d(rB_\theta)}{dr}$

iii) $\vec{\nabla}p = \vec{j}\times\vec{B} \rightarrow \frac{dp}{dr} = -j_z B_\theta = -\frac{B_\theta}{\mu_0 r}\frac{d(rB_\theta)}{dr} = -\frac{B_\theta}{\mu_0}\frac{dB_\theta}{dr} - \frac{B_\theta^2}{\mu_0 r}$

$$\therefore \boxed{\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0}$$

tension by curved magnetic field

- <u>screw pinch equilibrium</u> (both $j_\theta, j_z, B_\theta, B_z$, $\frac{\partial}{\partial\theta} = \frac{\partial}{\partial z} = 0$)

i) $\vec{\nabla}\cdot\vec{B} = 0$

ii) $\vec{\nabla}\times\vec{B} = \mu_0\vec{j} \rightarrow \frac{1}{r}\begin{vmatrix} \hat{r} & r\hat{\theta} & \hat{z} \\ \frac{\partial}{\partial r} & 0 & 0 \\ 0 & rB_\theta & B_z \end{vmatrix} = -\frac{1}{r}\frac{\partial B_z}{\partial r} r\hat{\theta} + \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta)\hat{r} = \mu_0\vec{j}$

$\therefore j_\theta = -\frac{1}{\mu_0}\frac{\partial B_z}{\partial r}, \ j_z = \frac{1}{\mu_0 r}\frac{\partial}{\partial r}(rB_\theta)$

iii) $\vec{\nabla}p = \vec{j}\times\vec{B} = j_\theta B_z - j_z B_\theta \rightarrow \frac{dp}{dr} = -\frac{B_z}{\mu_0}\frac{\partial B_z}{\partial r} - \frac{B_\theta^2}{\mu_0 r} - \frac{B_\theta}{\mu_0}\frac{\partial B_\theta}{\partial r}$

$$\therefore \boxed{\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_0} + \frac{B_z^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0}$$

- θ-pinch, z-pinch는 $p, \vec{j}, \vec{B}$ 중 하나만 알아도, 나머지 2개 결정 가능

screw pinch는 $p, \vec{j}, \vec{B}$ 중 2개를 알아야, 나머지 1개 결정가능.

<u>(핵융합기초 과제4, 문제 4 다시 복습하기)</u>

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2 Stability analysis using δW

$$\boxed{\begin{aligned} \delta W_F = \frac{1}{2}\int\Big[&\underbrace{\frac{1}{\mu_0}|\vec{B}_{1\perp}|^2}_{(a)} + \underbrace{\frac{1}{\mu_0}\left|\vec{\nabla}\cdot\vec{\xi}_\perp + 2\vec{\xi}_\perp\cdot\vec{\kappa}\right|^2 B_0^2}_{(b)} + \underbrace{\gamma p|\vec{\nabla}\cdot\vec{\xi}|^2}_{(c)}\Big]dx^3 \\ - \frac{1}{2}\int\Big[&\underbrace{2(\vec{\xi}_\perp^*\cdot\vec{\kappa})(\vec{\xi}_\perp\cdot\vec{\nabla}p)}_{(d)} + \underbrace{(j_{0\parallel}/B_0)\vec{B}_{1\perp}\cdot(\vec{\xi}_\perp^*\times\vec{B}_0)}_{(e)}\Big]dx^3 \end{aligned}}$$

2① δW for z-pinch

- $j_\parallel = j_\theta = 0 \rightarrow$ <u>(e) = 0</u>, (d) $\neq 0$ $\left(\vec{\kappa} = -\frac{\hat{r}}{r}\right)$

Expand $\vec{\xi}$ in Fourier series :

$\vec{\xi} = \vec{\xi}_{mk}(r)e^{i(m\theta+kz)} = (\xi_r(r)\hat{r} + \xi_\theta(r)\hat{\theta} + \xi_z(r)\hat{z})e^{i(m\theta+kz)}$

$\left(\frac{\partial}{\partial\theta} = im, \ \frac{\partial}{\partial z} = ik, \ \frac{\partial}{\partial r} = {}'\right)$

- <u>(c)</u> : <u>$\vec{\nabla}\cdot\vec{\xi} = \frac{1}{r}(r\xi_r)' + \frac{im}{r}\xi_\theta + ik\xi_z$</u>

note that $\vec{\nabla}\cdot\vec{\xi}$ is the only term containing $\xi_{\parallel}=\xi_\theta$ in δW.

Thus, ξ_θ can be determined always in such a way to make $r|\vec{\nabla}\cdot\vec{\xi}|^2=0$ in the $\delta W=0$ minimization, unless $m=0$

For $m\neq 0$, $\underline{\vec{\nabla}\cdot\vec{\xi}=0}$ with $\xi_\theta=\frac{i}{m}(r\xi_r)'-\frac{k}{m}r\xi_z$

For $m=0$, $\underline{\vec{\nabla}\cdot\vec{\xi}=\vec{\nabla}\cdot\vec{\xi}_\perp=\frac{1}{r}(r\xi_r)'+ik\xi_z}$

perturbed vector potential

∘ $\underline{(a):\ \vec{B}_{1\perp}}=(\vec{\nabla}\times(\vec{\xi}\times B_\theta\hat{\theta}))_\perp=\left(\vec{\nabla}\times(\xi_r B_\theta\hat{z}-\xi_z B_\theta\hat{r})\right)_\perp=(\vec{\nabla}\times\vec{A}_1)_\perp$

$$=\frac{1}{r}\begin{vmatrix}\hat{r} & r\hat{\theta} & \hat{z}\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z}\\ A_r & \cancel{A_\theta} & A_z\end{vmatrix}=\frac{1}{r}\frac{\partial A_z}{\partial\theta}\hat{r}+\cancel{(\ )}\hat{\theta}-\frac{1}{r}\frac{\partial A_r}{\partial\theta}\hat{z}=\underline{\frac{im}{r}B_\theta(\xi_r\hat{r}+\xi_z\hat{z})}$$

∘ $\underline{(b):\ \vec{\nabla}\cdot\vec{\xi}_\perp+2\vec{\xi}_\perp\cdot\vec{K}}=\frac{1}{r}(r\xi_r)'+ik\xi_z-2\frac{\xi_r}{r}=\xi_r'-\frac{\xi_r}{r}+ik\xi_z=\underline{r\left(\frac{\xi_r}{r}\right)'+ik\xi_z}$

∘ $\underline{(d):\ -2(\vec{\xi}_\perp^*\cdot\vec{K})(\vec{\xi}_\perp\cdot\vec{\nabla}p)}=-2\left(-\frac{\xi_r}{r}\right)(\xi_r p')=\underline{\frac{2p'}{r}|\xi_r|^2}$

∘ In cylindrical geometry, the integration become

$$\underline{\delta W_F=\frac{1}{2}\int_0^{2\pi R_0}dz\int_0^{2\pi}d\theta\int_0^a rW(r)\,dr=2\pi^2R_0\int_0^a rW(r)\,dr}$$

∘ Let's assume $m\neq 0$ ($\vec{\nabla}\cdot\vec{\xi}=0$; (c)=0)

$$W(r)=+\frac{m^2B_\theta^2}{\mu_0 r^2}(\xi_r^2+\xi_z^2)+\frac{B_\theta^2}{\mu_0}\left|r\left(\frac{\xi_r}{r}\right)'+ik\xi_z\right|^2+\frac{2p'}{r}|\xi_r|^2$$

$$=\frac{B_\theta^2}{\mu_0}\left(\frac{m^2}{r^2}+k^2\right)\xi_z^2+\frac{B_\theta^2}{\mu_0}\left(ikr\left(\frac{\xi_r}{r}\right)'\xi_z-ikr\left(\frac{\xi}{r}\right)'\xi_z^*\right)$$

$$+\frac{B_\theta^2}{\mu_0}r^2\left|\left(\frac{\xi_r}{r}\right)'\right|^2+\left(\frac{m^2B_\theta^2}{\mu_0 r^2}+\frac{2p'}{r}\right)|\xi_r|^2$$ ← quadratic form for ξ_z.

$\therefore\ \xi_z=\frac{ikr^3}{m^2+k^2r^2}\left(\frac{\xi_r}{r}\right)'$ will minimize $W(r)$, then we obtain

$$\underline{W(r)=\left(2p'r+\frac{m^2B_\theta^2}{\mu_0}\right)\left|\frac{\xi_r}{r}\right|^2+\frac{m^2r^2B_\theta^2}{\mu_0(m^2+k^2r^2)}\left|\left(\frac{\xi_r}{r}\right)'\right|^2}$$

$W(r)$ is functional ; function of function of $\xi(r)$.

What we want to do is to find a function $\xi(r)$ that minimizes $rW(r)$.

thus we can use Hamilton's principle to get Euler-Lagrange equation.

$$J = \int_a^b f(y, y'; r)\,dr \quad \rightarrow \quad \frac{\partial f}{\partial y} - \frac{d}{dr}\left(\frac{\partial f}{\partial y'}\right) = 0 \qquad \begin{pmatrix} f = rW(r) \\ y = \xi/r \end{pmatrix}$$

$\therefore$ E-L equation $\Rightarrow$ $$\boxed{\left(\frac{m^2 r^3 B_\theta^2}{\mu_0 (m^2 + k^2 r^2)}\left(\frac{\xi_r}{r}\right)'\right)' - \left(2p' r + \frac{m^2 B_\theta^2}{\mu_0}\right)\frac{\xi_r}{r} = 0}$$

최종 ξ_r에 대한 미분방정식.

- $m \neq 0$ kink instability in Z-pinch

one can argue that the most unstable mode is always $k \to \infty$.

$$\lim_{k \to \infty} W(r) = \left(2p' r + \frac{m^2 B_\theta^2}{\mu_0}\right)\left|\frac{\xi_r}{r}\right|^2$$

$\therefore$ Z-pinch is stable iff $\boxed{rp' + \frac{m^2 B_\theta^2}{2\mu_0} > 0}$ (derived by Kadomtsev (1966))

(less stable in higher pressure $p' < 0$)

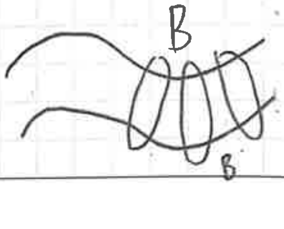
From Z-pinch equilibrium : $\frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$

we know that $p' = -\frac{B_\theta^2}{\mu_0 r} - \frac{1}{2\mu_0}(B_\theta^2)'$, which gives the stability condition

$$-r\left(\frac{B_\theta^2}{\mu_0 r} + \frac{1}{2\mu_0}(B_\theta^2)'\right) + \frac{m^2}{2\mu_0} B_\theta^2 = \frac{1}{2\mu_0}\left(-2B_\theta^2 - r(B_\theta^2)' + m^2 B_\theta^2\right)$$

$$= \frac{1}{2\mu_0}\left((m^2 - 1) B_\theta^2 + (rB_\theta^2)'\right) < 0 \qquad \boxed{m^2 - 1 > \frac{(rB_\theta^2)'}{B_\theta^2}}$$

$\therefore$ In the core region, $m=1$ mode is always unstable.



Kink instability is generic in a device having toroidal current.

- $m=0$ sausage instability

$$W(r) = \gamma p \left|\frac{1}{r}(r\xi_r)' + ik\xi_z\right|^2 + \frac{B_\theta^2}{\mu_0}\left|r\left(\frac{\xi_r}{r}\right)' + ik\xi_z\right|^2 + \frac{2p'}{r}|\xi_r|^2 \Leftarrow \text{quadratic to } \xi_z.$$

$$\Rightarrow \xi_z = i\,\frac{B_\theta^2[r(\xi_r/r)'] + \mu_0\gamma p[(r\xi_r)'/r]}{k(B_\theta^2 + \mu_0\gamma p)}, \quad \boxed{W(r) = \left[\left(\frac{4\gamma B_\theta^2}{B_\theta^2 + \mu_0\gamma p}\right)p + 2\gamma p'\right]\left(\frac{\xi_r}{r}\right)^2}$$

z-pinch is stable for $m=0$ iff $-\frac{rp'}{p} < \frac{2\gamma B_\theta^2}{B_\theta^2 + \mu_0\gamma p} < 2\gamma = \frac{10}{3}$.

- Interchange instability ($\Leftarrow$ interchange instability)

$\omega^2\delta K = \delta W$, $W(r) \sim 2\gamma p'\left(\frac{\xi_r}{r}\right)^2$ (in the limit of high p')

$$\Rightarrow \frac{1}{\cancel{2}}\rho\omega^2\cancel{\xi_r^2} \sim -\frac{1}{\cancel{2}}(2\gamma p')\left(\frac{\cancel{\xi_r}}{r}\right)^2 \Rightarrow \omega \sim \left(-\frac{2p'}{\rho r}\right)^{1/2}$$

2. ② δW for screw-pinch

- $\vec{\xi} = \xi_r\hat{r} + \xi_\theta\hat{\theta} + \xi_z\hat{z} = \xi_r\hat{r} + \xi_\alpha\hat{\alpha} + \xi_\parallel\hat{b}$

since $\xi_\parallel$ appears only in $\vec{\nabla}\cdot\vec{\xi}$, thus $|\vec{\nabla}\cdot\vec{\xi}|^2 = 0$ is used here.

- $\hat{b} = \frac{1}{B}(B_\theta\hat{\theta} + B_z\hat{z})$, $\xi_\parallel = \frac{1}{B}(\xi_\theta B_\theta + \xi_z B_z)$,

$\hat{\alpha} = \frac{1}{B}(B_z\hat{\theta} - B_\theta\hat{z})$, $\xi_\alpha = \frac{1}{B}(\xi_\theta B_z - \xi_z B_\theta)$

- $\vec{\xi} = \vec{\xi}(r)e^{i(m\theta + kz)} = \vec{\xi}(r)e^{i(m\theta - n\phi)}$

$$\Rightarrow \delta W_F = \frac{2\pi R_0^2}{\mu_0}\left(\int_0^a dr\left[f\left(\frac{d\xi}{dr}\right)^2 + g\xi^2\right] + \left(\frac{FF^+}{k_0^2}\right)_a\xi^2(a)\right)$$

where, $f = \frac{rF^2}{k_0^2}$, $g = 2\mu_0\frac{k^2}{k_0^2}p' + \frac{k_0^2r^2 - 1}{k_0^2 r}F^2 + 2\frac{k^2}{rk_0^4}FF^+$

$F \equiv \vec{k}\cdot\vec{B} = mB_\theta/r - nB_\phi/R_0 = (B_\theta/r)(m - nq)$

$F^+ = mB_\theta/r + nB_\phi/R_0 = (B_\theta/r)(m + nq)$

The Euler-Lagrange equation that minimizes δW_F becomes :

$$\boxed{\frac{d}{dr}\left(f\frac{d\xi}{dr}\right)-g\xi=0}$$

which was derived first by W. Newcomb (1962)

The Newcomb equation determines the radial profile of radial displacement for the least stable mode.

- Local interchange
  - $F\to 0$ when $q=m/n$ $\to f=0$ $\Rightarrow$ Singular and special newcomb equation
  - consider $r=r_s$ where $q(r_s)=m/n$.

    expand $r=r_s+x$ ($x\ll r_s$), and use taylor expansion

$$m-nq\sim -nq'x \qquad \left(q'=\frac{dq}{dr}\right) \Rightarrow f\sim\frac{rB_\theta^2}{k_0^2}n^2(q')^2x^2, \quad g\sim\frac{2\mu_0n^2r^2}{R_0^2k_0^2}p'$$

- $$\delta W_F\sim\frac{2\pi^2R_0}{\mu_0}\frac{rB_\theta^2n^2(q')^2}{k_0^2}\int_{r_s-\epsilon}^{r_s+\epsilon}\left[x^2\left(\frac{d\xi}{dx}\right)^2-D_s\xi^2\right]dx$$

  where $$D_s\equiv-\frac{2\mu_0p'q^2}{r_sB_z^2(q')^2}$$

- Newcomb equation becomes $$\frac{d}{dx}\left(x^2\frac{d\xi}{dx}+D_s\xi\right)=0$$ / with the solution $$\xi=C_-|x|^{P_-}+C_+|x|^{P_+}$$ / where $$P_\pm=-\frac{1}{2}\pm\frac{1}{2}(1-4D_s)^{1/2}$$

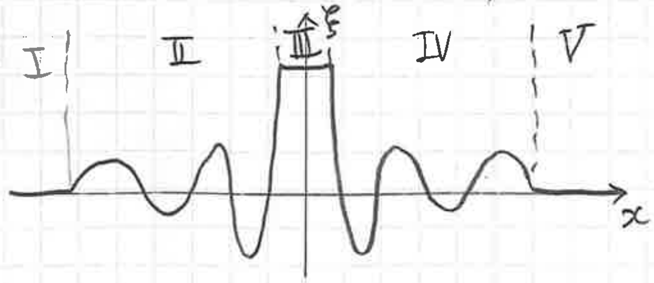
- When $D_s\le\frac{1}{4}$, $P_\pm=$ real

  $P_-\le-\frac{1}{2}$ → big solution (diverge at $x\to0$)

  $P_+\ge-\frac{1}{2}$ → small solution

  when $D_s>\frac{1}{4}$, $P_\pm=$ imaginary → $\xi=$ oscillatory. (앞의 실수부 $-\frac{1}{2}$ 제곱으로

diverge at $x \to 0$ with oscillatory motion 이 나타남. ⇒ local interchange



Due to diverging oscillatory behavior, it is always possible to construct a $\vec{\xi}$ that makes $\delta W_F < 0$.

(임의의 Ⅱ, Ⅳ를 정하면 Ⅲ을 마음대로 정하기가능)

∴ To avoid local interchange, $\boxed{D_s < \frac{1}{4}}$ → $\boxed{r_s B_z^2 \left(\frac{q'}{q}\right)^2 > -8\mu_0 p'}$

☆ Suydam criterion (screw pinch)

◦ Tokamak에서 D_s를 이용하면 → $\boxed{r_s B_z^2 \left(\frac{q'}{q}\right)^2 > -8\mu_0 p (1-q^2)}$

☆ Mercier criterion (tokamak)