15. Logistic map and chaos

- Basic concepts of maps
- Logistic map
- Onset of chaotic dynamics
- Hamiltonian chaos

1 Basic concepts of maps

* Map : 시간을 불연속으로 생각할 수 있는 동역학

- 1-D map : $x_{n+1} = f(x_n)$
- orbit : $x_0, x_1, x_2, \cdots, x_n$ 의 sequence
- continuous 한 궤적을 분석하기 위한 snapshot
- 예시) Poincare map , Lorenz map

* Fixed point and linear stability

- $x^* = f(x^*)$: Fixed point
- $x_n = x^* + \eta_n \Rightarrow x^* + \eta_{n+1} = f(x^* + \eta_n) = f(x^*) + \eta_n f'(x^*) + O(\eta_n^2)$

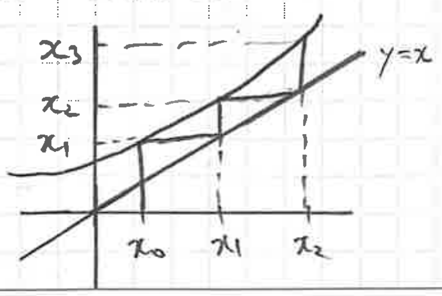
$$\hookrightarrow \eta_{n+1} = \eta_n f'(x^*) \equiv \lambda \eta_n$$

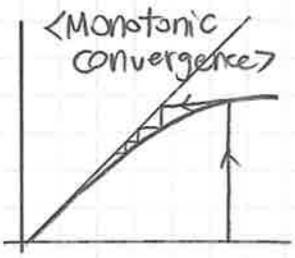
$|\lambda| = |f'(x^*)| < 1$: $\eta_n \to 0$: stable

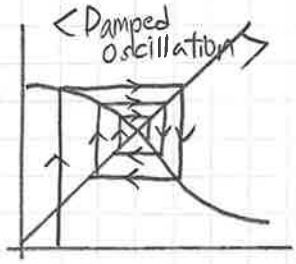
$|\lambda| = |f'(x)| > 1$: $\eta_n \to \infty$: un-stable

$|\lambda| = |f'(x)| = 1$: $\eta_n \to ?$: marginal ($O(\eta_n^2)$ determines stability)

* Cobweb construction (visualizing the behavior of a map)



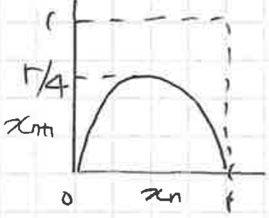




[2] Logistic map

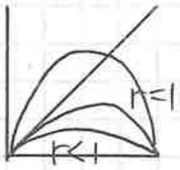
* $x_{n+1} = f(x_n) = r x_n (1-x_n)$ $\quad (0 \le r \le 4,\ 0 \le x \le 1)$

$$\Delta x = x_{n+1} - x_n = \underset{\text{Reproduction}}{r x_n} - \underset{\text{death}}{x_n} - \underset{\text{competition}}{r x_n^2}$$



* Fixed points and bifurcations

Bifurcation : Change of dynamics (creation, destruction, stability)

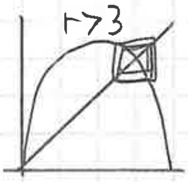


$\underline{x_1^*=0}$, $\underline{x_2^* = 1-\frac{1}{r}}$

i) $0 \le r < 1$: x_1^* stable, x_2^* : unstable

ii) $1 < r \le 3$: x_1^* unstable, x_2^* : stable

① ⇒ At $r=1$, two points exchange their stability : $\boxed{\text{transcritical bifurcation point}}$



$\underline{x_1^* = 1-\frac{1}{r}}$ i) $r>3$: x_1^* unstable ⇒ 2-cycle.

② ⇒ At $r=3$, a point change its stability : $\boxed{\text{flip bifurcation point}}$

∘ 2-cycle : $f^2(x) = f(f(x)) = x$

$$f^2(x) - x = r(rx(1-x))(1-rx(1-x)) - x$$
$$= r(rx - rx^2)(1 - rx + rx^2) - x$$
$$= \underbrace{-rx\left(x - 1 + \frac{1}{r}\right)}_{\text{Fixed point}} \underbrace{\left[r^2x^2 - r(1+r)x + 1 + r\right]}_{\text{2-cycle}} = 0$$

$$x = \frac{1}{2r^2}\left[r(1+r) \pm \sqrt{r^2(1+r)^2 - 4r^2(1+r)}\right]$$
$$= \frac{1}{2r}\left[(r+1) \pm \sqrt{(r-3)(r+1)}\right] \Leftarrow \text{2-real roots for } 3>1$$

For $r>3$, always 2-cycle, which bifurcates continuously from

$x^* = 1 - \frac{1}{r} = \frac{2}{3}$ at $r=3$.

③ · Stability of 2-cycle.

$$\lambda = \frac{d}{dx} f^2(x) = f'(f(x))\,f'(x) = f'(q)\,f'(p)$$

$f'(x) = r(1-2x) \rightarrow$

$$= r^2(1-2q)(1-2p) = r^2\left[1-2(p+q)+4pq\right] = \underline{4+2r-r^2}$$

$\left(\text{From } f(f(x)).\ \ p+q = \frac{1+r}{r},\ \ pq = \frac{1+r}{r^2}\right)$

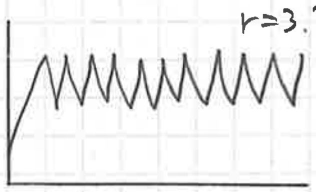
stable if $|4+2r-r^2|<1$, i.e. $\underline{3<r<1+\sqrt{6}}$

- Attractor.

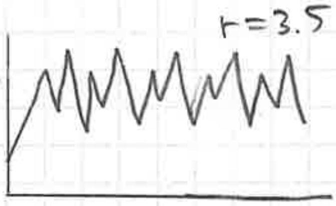
$$\left(\begin{array}{l} 0 \le r \le 1 : \text{system} \rightarrow x^* = 0 \\ 1 < r \le 3 : \text{system} \rightarrow x^* = 1 - 1/r \\ 3 < r \le 1+\sqrt{6} : \text{system} \rightarrow \text{2-cycle.} \end{array}\right)$$

A set of states towards which system tends to evolve for a wide variety of initial conditions are called <u>attractor</u>

④ Period doubling bifurcations



$\Longrightarrow$ Period doubling



r_1	r_2	r_3	...	r_∞	
3,	3.449	3.544		3.5699	2^n-cycle

$$\boxed{\begin{array}{c} 1+\sqrt{6} < r < 3.5699 \\ 2^n\text{-cycle} \end{array}}$$

<u>If $r > 3.5699 \Rightarrow$ Chaotic orbit</u>

3 Onset of chaotic dynamics

* Strange attractor

* Lyapunov exponent

System is chaotic ← exhibits sensitive dependence on initial conditions

- $x_0 \to x_0 + \delta_0$, $\boxed{|\delta_n| \sim |\delta_0| e^{n\lambda} \Leftarrow \cdot\ \lambda \text{ is Lyapunov exponent.}}$

- Formula : $\lambda = \lim_{n\to\infty} \frac{1}{n} \ln \left| \frac{\delta_n}{\delta_0} \right| = \lim_{n\to\infty} \frac{1}{n} \ln \left| \frac{f^n(x_0+\delta_0) - f^n(x_0)}{\delta_0} \right|$

$$= \lim_{n\to\infty} \frac{1}{n} \ln \left| (f^n)'(x_0) \right| = \underline{\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n-1} \ln |f'(x_i)|}$$

- $\lambda > 0$ is a signature of chaos.

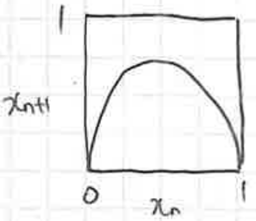
* Lyapunov exponent of the logistic map.

* Orbit diagram (Periodic windows) ⇐ Tangent bifurcations.

① Transcritical ($r=1$) , ② flip ($r=3$) ③ tangent (periodic window)

4 Universality and self-similarity

* unimodal map

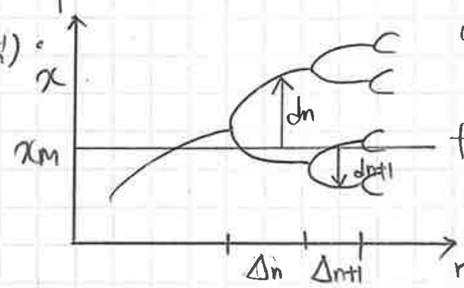


① logistic map : $x_{n+1} = rx_n(1-x_n)$
② sine map : $x_{n+1} = r\sin(\pi x_n)$
$\Rightarrow$ orbit diagrams look remarkably similar!

* Universality

(정성적) · periodic attractors appear in the same universal sequence.

(정량적) ·



d_n: displacement from x_m to the nearest point on a 2^n-cycle

$f(x)$ is maximized at $x = x_m$

Period doubling exhibit universal ratios (Feigenbaum constants)

$\boxed{\delta = \lim_{n\to\infty} \frac{\Delta_n}{\Delta_{n+1}} = 4.669}$, $\boxed{\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}} = -2.5027}$ ← Peak 근처로 오는 2^n-cycle 궤적이 점점 작아진다 (peak 근처로 다가온다)

* Self-similarity of the orbit diagram $\Rightarrow$ Fractal structure.

* Feigenbaum's renormalization theory (재규격화 군이론)

x_m 자체가 x^*가 되는 r을 R_n 이라고 하자. 즉, $x = x_m$ becomes stable fixed point of f^{n+1}

· The map looks almost self-similar after each renormalization.

· complete self-similarity is exhibited by the universal function.

$$\boxed{g(x) = \lim_{i\to\infty} \lim_{n\to\infty} \alpha^n f^{2n}\left(\frac{x}{\alpha^n}, R_{n+i}\right)}$$

which satisfies $g(x) = \alpha g^2(x/\alpha)$

w/o loss of generality, $g(0)=1$, $g'(0)=0$ $\rightarrow$ $g(0) = \alpha g(g(0))$ $\Rightarrow$ $\alpha = 1/g(1)$

Provided that f has quadratic maximum at $r = r_m$,

$g(x) = 1 + c_2x^2 + c_4x^4 + \cdots$ $\xrightarrow[\text{유도}]{}$ $\alpha = -2.5027$

o Empirical observation of Feigenbaum constants

Rayleigh number R (characterizing $\vec{\nabla} T$ across liquid)

As R $\Uparrow$, heat transport occurs through conduction, convection, rings

↳ more complex pattern.

T at fixed point exhibit period - doubling bifurcation $\Rightarrow \delta \approx 4.669$

* Universality

On set of chaos 거동 자체가 self-similarity를 가지고 있으므로 Universality를 갖는다.