5. Guiding Center Lagrangian

p.2 Lagrangian vs. Hamiltonian

- Euler-Lagrange equation by Lagrangian $L(\vec{q}, \dot{\vec{q}}, t)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

  - covariant feature : E-L equation do not change for any generalized coordinates

    cf) In Newton's, $F_i = m\ddot{x}_i \xrightarrow[x_i = x_i(q_j, t)]{} F_j \neq m\ddot{q}_j$

- Hamiltonian (by Legendre transformation : $(q_i, \dot{q}_i) \to (q_i, p_i)$)

$$L(q_i, \dot{q}_i, t) \to p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

$$H(q_i, p_i, t) = p_i \dot{q}_i - L \quad \leftarrow \text{Hamilton E.o.M}$$

  $\therefore$ Law of motion by Hamiltonian $H(q_i, p_i, t)$:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

  - Liouville theorem : phase-space volume is conserved.

p.3 Lagrangian and Hamiltonian with EM fields

If one define, $L(\vec{x}, \dot{\vec{x}}, t) = \frac{1}{2} m |\dot{\vec{x}}|^2 + e\vec{A}(\vec{x}, t)\cdot\dot{\vec{x}} - e\phi(\vec{x}, t)$

$$\left(T = \frac{1}{2} m |\dot{\vec{x}}|^2, \ V = e\phi - e\vec{A}\cdot\dot{\vec{x}} \Rightarrow L = T - V\right)$$

(E-L eq.)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{x}}}\right) = \frac{d}{dt}\left(m\dot{\vec{x}} + e\vec{A}\right) = m\ddot{\vec{x}} + e\frac{\partial \vec{A}}{\partial t} + e(\dot{\vec{x}}\cdot\vec{\nabla})\vec{A} = \frac{\partial L}{\partial \vec{x}} = e\vec{\nabla}\vec{A}\cdot\dot{\vec{x}} - e\vec{\nabla}\phi$$

$$m\ddot{\vec{x}} = -e\left(\vec{\nabla}\phi + \frac{\partial \vec{A}}{\partial t}\right) + e\vec{\nabla}\vec{A}\cdot\dot{\vec{x}} - (\dot{\vec{x}}\cdot\vec{\nabla})\vec{A} = e\vec{E} + e\left(\dot{\vec{x}} \times (\vec{\nabla}\times\vec{A})\right)$$

(Newton) $\quad = e(\vec{E} + \dot{\vec{x}} \times \vec{B})$

Hamiltonian,

$$H(\vec{x},\vec{p},t) = \vec{p}\cdot\dot{\vec{x}} - L \quad \left(\text{where } \vec{p} = \frac{\partial L}{\partial \dot{\vec{x}}} = m\dot{\vec{x}} + e\vec{A}(\vec{x},t)\right)$$

$$= m\dot{\vec{x}}^2 + e\vec{A}(\vec{x},t)\cdot\dot{\vec{x}} - \left(\frac{1}{2}m|\dot{\vec{x}}|^2 + e\vec{A}(\vec{x},t)\cdot\dot{\vec{x}} - e\phi(\vec{x},t)\right)$$

$$= \frac{1}{2}m|\dot{\vec{x}}|^2 + e\phi(\vec{x},t)$$

$$\therefore H(\vec{x},\vec{p},t) = \frac{1}{2m}\left|\vec{p} - e\vec{A}(\vec{x},t)\right|^2 + e\phi(\vec{x},t)$$

Here, $(\vec{x},\vec{p})$ are canonical variables : they satisfy Hamiltonian E.o.M.

p.4 Phase-space Lagrangian

Issue in L : It does not provide energy or phase space volume conservation explicitly

Issue in H : It requires canonical transformation and canonical variables are often unphysical.

$\Rightarrow$ Elegant way to overcome this issue : Phase-Space Lagrangian

$$\boxed{L(\vec{q},\dot{\vec{q}},\vec{p},\dot{\vec{p}},t) = p_i\dot{q}_i - H(q_i,p_i,t)}$$

It is said to be canonical, in a sense that E-L equation yields Hamiltonian equation.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \rightarrow \dot{p}_i = -\frac{\partial H}{\partial q_i}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{p}_i}\right) - \frac{\partial L}{\partial p_i} = 0 \rightarrow \dot{q}_i = \frac{\partial H}{\partial p_i}$$

p.5 Non-canonical phase space Lagrangian with EM fields

$$L(\vec{x},\dot{\vec{x}},\vec{p},\dot{\vec{p}},t) = \vec{p}\cdot\dot{\vec{x}} - \frac{1}{2m}\left|\vec{p} - e\vec{A}(\vec{x},t)\right|^2 - e\phi(\vec{x},t)$$

Lagrangian works with non-canonical coordinates

$$L(\vec{x},\dot{\vec{x}},\vec{v},\dot{\vec{v}},t) = \left(m\vec{v} + e\vec{A}(\vec{x},t)\right)\cdot\dot{\vec{x}} - \frac{1}{2}m|\vec{v}|^2 - e\phi(\vec{x},t)$$

(again, $\vec{x}$, $\vec{v}$ are independent in terms of L)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{v}}}\right) = \frac{\partial L}{\partial \vec{v}} \rightarrow 0 = m\dot{\vec{x}} - m\vec{v} \qquad \therefore \dot{\vec{x}} = \vec{v}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{x}}}\right) = \frac{\partial L}{\partial \vec{x}} \rightarrow \frac{d}{dt}\left(m\vec{v} + e\vec{A}\right) - e\vec{\nabla}\vec{A}\cdot\dot{\vec{x}} + e\vec{\nabla}\phi = 0$$

$$m\frac{d\vec{v}}{dt} + e\left(\frac{\partial\vec{A}}{\partial t} + (\dot{\vec{x}}\cdot\vec{\nabla})\vec{A}\right) = e\vec{\nabla}\vec{A}\cdot\dot{\vec{x}} - e\vec{\nabla}\phi$$

$$\therefore m\frac{d\vec{v}}{dt} = e(\vec{\nabla}\vec{A}\cdot\dot{\vec{x}} - (\dot{\vec{x}}\cdot\vec{\nabla})\vec{A}) - e(\vec{\nabla}\phi + \frac{d\vec{A}}{dt})$$
$$= e(\dot{\vec{x}}\times(\vec{\nabla}\times\vec{A})) - e(\vec{\nabla}\phi + \frac{d\vec{A}}{dt})$$
$$= e(\vec{E} + \dot{\vec{x}}\times\vec{B})$$

* Remember that E-L is not changed when adding arbitrary total time derivative : $L \to L + \frac{dS}{dt}$ (gauge transformation)

$$I = \int_1^2 L\,dt\,,\quad I' = \int_1^2 (L + \frac{dS}{dt})dt = \int_1^2 L\,dt + S(2) - S(1) = I + S(2) - S(1)$$

$\delta I = 0 \quad \Leftrightarrow \delta(I + S(2) - S(1)) = \delta I = 0 \quad \Rightarrow$ same E-L equation (same actual path)

p.7 Small m/e expansion for guiding center motion

Let $L \to L/m$, $\epsilon = m/e$, then $L = (\vec{v} + \epsilon^{-1}\vec{A})\cdot\dot{\vec{x}} - \frac{1}{2}v^2 - \epsilon^{-1}\phi$

$O(\epsilon^{-1})$ order : $L_{-1} = \epsilon^{-1}(\vec{A}\cdot\dot{\vec{x}} - \phi)$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\vec{x}}}\right) - \frac{\partial L}{\partial \vec{x}} = 0 \rightarrow \frac{d\vec{A}}{dt} = \vec{\nabla}\vec{A}\cdot\dot{\vec{x}} - \vec{\nabla}\phi \rightarrow \frac{d\vec{A}}{dt} + (\dot{\vec{x}}\cdot\vec{\nabla})\vec{A} - \vec{\nabla}\vec{A}\cdot\dot{\vec{x}} + \vec{\nabla}\phi = 0$$

$$\vec{\nabla}\phi + \frac{d\vec{A}}{dt} + \dot{\vec{x}}\times(\vec{\nabla}\times\vec{A}) = 0 \rightarrow \vec{E} + \dot{\vec{x}}\times\vec{B} = 0$$

This determines the perpendicular drift to the leading order becomes $\vec{E}\times\vec{B}$ drift

$$\dot{\vec{x}}_\perp = \vec{v}_E = \frac{\vec{E}\times\vec{B}}{B^2}\,,$$

$E_\parallel$ must vanish $O(\epsilon^0)$ is consistent with this result

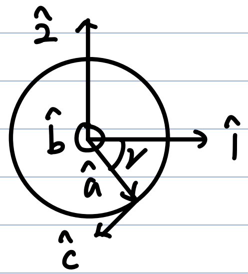
$O(\epsilon^0)$ order

In higher order expansion of Lagrangian, our actual goal is to find **new phase space coordinates** where the rapid oscillatory motion becomes ignorable.

$(\vec{x}, \vec{v}) \longrightarrow (\vec{X}, u, w, \gamma)$ variable수 6→6, ρ는 u,w,γ로 표현가능

where $\vec{X}$ = guiding center, $\vec{u}$ = parallel velocity, $\vec{w}$ = perpendicular velocity, γ = gyrophase (← which should be ignorable)

p.9 Phase space coordinates



$$\vec{v} = u\hat{b} + \vec{w} + \vec{v}_E = u\hat{b} + w\hat{c} + \vec{v}_E$$

let rotating perpendicular velocity as, $\hat{c} = -\hat{1}\sin\gamma - \hat{2}\cos\gamma$

then, direction of gyrovector becomes, $\vec{\rho} = \rho\hat{a}$

$$\hat{a} = \hat{b}\times\hat{c} = \hat{b}\times(-\hat{1}\sin\gamma - \hat{2}\cos\gamma) = \hat{1}\cos\gamma - \hat{2}\sin\gamma = -\frac{d\hat{c}}{d\gamma}$$

However, $\vec{\rho}$ is the gyrovector in the frame drifting with $\vec{E}\times\vec{B}$ velocity,

thus, let's make the reference frame flexible upon $\vec{E}\times\vec{B}$ by using $\vec{d} = \vec{d}(\vec{X}, t)$

- $\vec{\rho} = \vec{\rho}_0 + \vec{d} = \dfrac{\hat{b}\times\vec{w}}{B} + \vec{d} \quad \rightarrow \quad \left(\dfrac{\hat{b}\times w\hat{c}}{B} = \dfrac{w}{B}\hat{a}, \quad \dfrac{w}{B}\hat{a}\times\left(\dfrac{m}{e}\right) = \dfrac{mw}{eB}\hat{a} \right)$

  $\therefore\ \epsilon\vec{\rho}_0$ is the gyrovector in const $\vec{B}$ and $\vec{E} = 0$ to the leading order

$\vec{x} = \vec{X} + \epsilon\vec{\rho}\ (o)$

$\dot{\vec{x}} = \dot{\vec{X}} + \dot{\vec{\rho}}\ (o)$

- $\dot{\vec{x}} = \dot{\vec{X}} + \dot{\vec{\rho}}$ where $\vec{\rho} = \vec{\rho}(\vec{X}, \gamma, t)$

  $$\dot{\vec{\rho}} = \dot{\gamma}\frac{d\vec{\rho}_0}{dt} + \epsilon\left(\frac{d\vec{\rho}_0}{dt} + \vec{v}\cdot\vec{\nabla}\vec{\rho}_0 + \dot{\vec{d}}\right) = \frac{\vec{w}}{B}\dot{\gamma} + O(\epsilon)$$

$$\left(\begin{aligned} &\dot{\vec{\rho}} = \dot{\gamma}\frac{d\vec{\rho}}{d\gamma} + \epsilon\frac{d\vec{\rho}}{dt} = \dot{\gamma}\frac{d\vec{\rho}}{d\gamma} + O(\epsilon) \\ &\vec{\rho} = \frac{\hat{b}\times\hat{c}}{B}w = \frac{w}{B}\hat{a} = \frac{w}{B}(\hat{1}\cos\gamma - \hat{2}\sin\gamma) \\ &\frac{d\vec{\rho}}{d\gamma} = \frac{w}{B}(-\hat{1}\sin\gamma - \hat{2}\cos\gamma) = \frac{w}{B}\hat{c} \end{aligned} \right)$$

$$\dot{\vec{\rho}} = \left(\frac{\partial}{\partial\gamma}\dot{\gamma} + \frac{d}{dt}\right)\epsilon\vec{\rho} = \left(\frac{d\vec{\rho}}{d\gamma} + \epsilon\frac{d\vec{\rho}}{dt}\right)$$

p.11 Expansion for guiding center Lagrangian

$$L = (\vec{v} + \epsilon^{-1}\vec{A})\cdot\dot{\vec{x}} - \tfrac{1}{2}v^2 - \epsilon^{-1}\phi \qquad \text{(same as above)}$$

$$L = \epsilon^{-1}\left(\vec{A} + \epsilon(\vec{\rho}\cdot\vec{\nabla})\vec{A}\right)\cdot(\dot{\vec{X}} + \dot{\vec{\rho}}) - \epsilon^{-1}(\phi + \epsilon(\vec{\rho}\cdot\vec{\nabla})\phi)$$
$$+ \vec{v}\cdot(\dot{\vec{X}} + \dot{\vec{\rho}}) - \tfrac{1}{2}v^2 + O(\epsilon)$$
$$= \epsilon^{-1}\vec{A}\cdot\dot{\vec{X}} + \underbrace{\epsilon^{-1}\vec{A}\cdot\dot{\vec{\rho}}}_{(a)} + \underbrace{(\vec{\rho}\cdot\vec{\nabla})\vec{A}\cdot\dot{\vec{X}}}_{(b)} + \underbrace{(\vec{\rho}\cdot\vec{\nabla})\vec{A}\cdot\dot{\vec{\rho}}}_{(c)}$$
$$- \epsilon^{-1}\phi - \underbrace{(\vec{\rho}\cdot\vec{\nabla})\phi}_{(d)} + \vec{v}\cdot(\dot{\vec{X}} + \dot{\vec{\rho}}) - \tfrac{1}{2}v^2 + O(\epsilon)$$

Note that $L \rightarrow L + \frac{dS}{dt}$ removing $\frac{dS}{dt}$ is possible.

- (a) $\epsilon^{-1}\vec{A}\cdot\dot{\vec{\rho}} = \cancel{\frac{d}{dt}(\vec{A}\cdot\vec{\rho})} - \left(\frac{d\vec{A}}{dt} + \dot{\vec{X}}\cdot\vec{\nabla}\vec{A}\right)\cdot\vec{\rho}$

  $\frac{d}{dt}(\vec{A}\cdot\vec{\rho}) = \dot{\vec{A}}\cdot\vec{\rho} + \vec{A}\cdot\dot{\vec{\rho}} \rightarrow \frac{d}{dt}(\vec{A}\cdot\vec{\rho}) = \dot{\vec{A}}\cdot\vec{\rho} + \epsilon^{-1}\vec{A}\cdot\dot{\vec{\rho}}$

- $(a)+(b)+(d) = -\frac{d\vec{A}}{dt}\cdot\vec{\rho} - \dot{\vec{X}}\cdot\vec{\nabla}\vec{A}\cdot\vec{\rho} + (\vec{\rho}\cdot\vec{\nabla})\vec{A}\cdot\dot{\vec{X}} - (\vec{\rho}\cdot\vec{\nabla})\phi$

$$= \vec{\rho}\cdot\vec{E} + \vec{\rho}\cdot(\vec{\nabla}\vec{A}\cdot\dot{\vec{X}} - \dot{\vec{X}}\cdot\vec{\nabla}\vec{A}) = \vec{\rho}\cdot(\vec{E} + \dot{\vec{X}}\times\vec{B})$$

- (c) $(\vec{\rho}\cdot\vec{\nabla})\vec{A}\cdot\dot{\vec{\rho}} = \epsilon\frac{d}{dt}\cancel{(\vec{\rho}\cdot\vec{\nabla}\vec{A}\cdot\vec{\rho})} - \dot{\vec{\rho}}\cdot\vec{\nabla}\vec{A}\cdot\vec{\rho} - \cancel{\epsilon\vec{\rho}\cdot\frac{d\vec{A}}{dt}\cdot\vec{\rho}}$ high order

$$\epsilon^{\cancel{2}}\frac{d}{dt}(\vec{\rho}\cdot\vec{\nabla}\vec{A}\cdot\vec{\rho}) = \cancel{\epsilon}\,\dot{\vec{\rho}}\cdot\vec{\nabla}\vec{A}\cdot\vec{\rho} + \epsilon^{2}\vec{\rho}\cdot\frac{d}{dt}(\vec{\nabla}\vec{A})\cdot\vec{\rho} + \cancel{\epsilon}(\vec{\rho}\cdot\vec{\nabla})\vec{A}\cdot\dot{\vec{\rho}}$$

$$= \frac{1}{2}\vec{\rho}\cdot(\vec{\nabla}\vec{A}\cdot\dot{\vec{\rho}} - \dot{\vec{\rho}}\cdot\vec{\nabla}\vec{A}) = \frac{1}{2}\vec{\rho}\cdot(\dot{\vec{\rho}}\times(\vec{\nabla}\times\vec{A})) = \frac{1}{2}\vec{\rho}\cdot(\dot{\vec{\rho}}\times\vec{B})$$

$$\Rightarrow \boxed{L = \epsilon^{-1}(\vec{A}\cdot\dot{\vec{X}} - \phi) + \vec{\rho}\cdot(\vec{E} + \dot{\vec{X}}\times\vec{B}) + \frac{1}{2}\vec{\rho}\cdot(\dot{\vec{\rho}}\times\vec{B}) + \vec{v}\cdot(\dot{\vec{X}} + \dot{\vec{\rho}}) - \frac{1}{2}v^2 + O(\epsilon)}$$

One can obtain (from $\vec{\rho} = \frac{\hat{b}\times w\hat{c}}{B} + \vec{d}$, $\dot{\vec{\rho}} = \frac{\vec{w}}{B}\dot{\gamma}$)

- $\vec{\rho}\cdot\vec{E} = \frac{\hat{b}\times\vec{w}}{B}\cdot\vec{E} + \vec{d}\cdot\vec{E} = \vec{w}\cdot\frac{\vec{E}\times\hat{b}}{B} + \vec{d}\cdot\vec{E} = \underline{\vec{w}\cdot\vec{v}_E + \vec{d}\cdot\vec{E}}$

- $\vec{\rho}\cdot(\dot{\vec{X}}\times\vec{B}) = \frac{w\hat{a}}{B}\cdot(\dot{\vec{X}}\times B\hat{b}) + \vec{d}\cdot(\dot{\vec{X}}\times\vec{B}) = \dot{\vec{X}}(w\hat{b}\times\hat{a}) + \dot{\vec{X}}(\vec{B}\times\vec{d})$

$$= \underline{-\dot{\vec{X}}\cdot\vec{w} + \dot{\vec{X}}(\vec{B}\times\vec{d})}$$

- $\frac{1}{2}\vec{\rho}\cdot(\dot{\vec{\rho}}\times\vec{B}) = \frac{1}{2}\frac{w\hat{a}}{B}\cdot(\frac{w\hat{c}}{B}\dot{\gamma}\times B\hat{b}) + \frac{1}{2}\vec{d}\cdot(\frac{w\hat{c}}{B}\dot{\gamma}\times B\hat{b}) = -\frac{w^2}{2B}\dot{\gamma} - \frac{w\dot{\gamma}}{2}\hat{a}\cdot\vec{d}$

$$= \underline{-\frac{w^2}{2B}\dot{\gamma} - \frac{B}{2}\vec{d}\cdot\vec{\rho}_0\dot{\gamma}} \quad (\vec{\rho}_0 = \frac{\hat{b}\times w\hat{c}}{B} = \frac{w}{B}\hat{a} \rightarrow w\hat{a} = \vec{\rho}_0 B)$$

- $\vec{v}\cdot\dot{\vec{\rho}} = (\vec{u} + \vec{w} + \vec{v}_E)\cdot\dot{\vec{\rho}} = \underline{\frac{w^2}{B}\dot{\gamma} + \frac{\vec{v}_E\cdot\vec{w}}{B}\dot{\gamma}}$

- $\frac{1}{2}v^2 = \underline{\frac{1}{2}u^2 + \frac{1}{2}w^2 + \frac{1}{2}v_E^2 + \vec{w}\cdot\vec{v}_E}$

Altogether one obtains:

$$L = \epsilon^{-1}(\underline{\vec{A}\cdot\dot{\vec{X}}} - \underline{\phi}) + \cancel{\vec{w}\cdot\vec{v}_E} + \underline{\vec{d}\cdot\vec{E}} - \dot{\vec{X}}(\underline{\cancel{\vec{w}}} - \underline{\vec{B}\times\vec{d}}) - \underline{\dot{\gamma}\left(\frac{w^2}{2B} + \frac{B\vec{d}\cdot\vec{\rho}_0}{2} - \frac{w^2}{B} - \frac{\vec{v}_E\cdot\vec{w}}{B}\right)}$$

$$+ \underline{(\vec{u} + \vec{w} + \vec{v}_E)\cdot(\dot{\vec{X}} + \dot{\vec{\rho}})} - \underline{\frac{1}{2}u^2 - \frac{1}{2}w^2} - \underline{\frac{1}{2}v_E^2} - \cancel{\vec{w}\cdot\vec{v}_E}$$

$$= \underline{u\hat{b}\cdot\dot{\vec{X}}} + \cancel{\vec{w}\cdot\dot{\vec{X}}} + \underline{\vec{v}_E\cdot\dot{\vec{X}}} + (\vec{w}\cdot\dot{\vec{\rho}} + \vec{v}_E\cdot\dot{\vec{\rho}}) \Longrightarrow \text{얘는 어디로?}$$

$$L = (\epsilon^{-1}\vec{A} + u\hat{b})\cdot\dot{\vec{X}} + \left(\frac{w^2}{2B} + \frac{\vec{v}_E\cdot\vec{w}}{B} - \frac{B\vec{d}\cdot\vec{\rho}_0}{2}\right)\dot{\gamma}$$
$$+\ \dot{\vec{X}}\cdot(\vec{v}_E + \vec{B}\times\vec{d}) + \left(\vec{d}\cdot\vec{E} - \frac{v_E^2}{2}\right) - \left(\epsilon^{-1}\phi + \frac{u^2}{2} - \frac{w^2}{2}\right) + O(\epsilon) \quad \cdots (38)$$

p.14 Guiding center Lagrangian by Littlejohn and Cary

let's choose $\vec{d} = \dfrac{\vec{B}\times\vec{v}_E}{B^2}$: guiding-center polarization displacement in the rotating $\vec{E}\times\vec{B}$ frame

Then, eq. (38) simplifies to $\left(\frac{B}{2}\frac{\vec{B}\times\vec{v}_E}{B^2}\cdot\vec{\rho}_0 = \frac{B\hat{b}\times\vec{v}_E}{2B}\cdot\frac{w}{B}\hat{a} = \frac{\vec{v}_E\cdot\vec{w}}{2B}\right)$

$$L = (\epsilon^{-1}\vec{A} + u\hat{b})\cdot\dot{\vec{X}} + \left(\frac{w^2}{2B} + \frac{\vec{v}_E\cdot\vec{w}}{2B}\right)\dot{\gamma} - \left(\epsilon^{-1}\phi + \frac{u^2}{2} + \frac{w^2}{2} - \frac{v_E^2}{2}\right) + O(\epsilon)$$

$\left(\vec{d}\cdot\vec{E} = \frac{\vec{B}\times\vec{v}_E}{B^2}\cdot\vec{E} = \frac{\vec{v}_E}{B^2}\cdot(\vec{E}\times\vec{B}) = v_E^2\right)$

(*)

(*) $\left(\vec{v}_E\cdot\frac{\vec{w}}{2B}\right)\dot{\gamma} = \cancel{\frac{\epsilon}{2}\frac{d}{dt}(\vec{v}_E\cdot\vec{\rho}_0)} - \cancel{\frac{\epsilon}{2}\left(\frac{d\vec{v}_E}{dt}\right)\cdot\vec{\rho}_0}$ $\quad \frac{d}{dt} \quad \epsilon$

$\epsilon\frac{d}{dt}(\vec{v}_E\cdot\vec{\rho}_0) = \epsilon\frac{d}{dt}\left(\frac{d\vec{v}_E}{dt}\right)\vec{\rho}_0 + \vec{v}_E\cdot\frac{d\vec{\rho}_0}{dt} \quad \left(\frac{d\vec{\rho}_0}{dt} = \frac{\vec{w}}{B}\dot{\gamma}\right)$

$$\therefore L = (\epsilon^{-1}\vec{A} + u\hat{b})\cdot\dot{\vec{X}} + \frac{w^2}{2B}\dot{\gamma} - \left(\epsilon^{-1}\phi + \frac{u^2}{2} + \frac{w^2}{2} - \frac{v_E^2}{2}\right) + O(\epsilon)$$

---

put back $\epsilon = \frac{m}{e}$, $L \to mL$, $\dot{\gamma} \to \epsilon\dot{\gamma}$, with definition of $\mu \equiv \dfrac{mw^2}{2B}$

$$L = m\left(\frac{e}{m}\vec{A} + u\hat{b}\right)\cdot\dot{\vec{X}} + \frac{mw^2}{2B}\epsilon\dot{\gamma} - m\left(\frac{e}{m}\phi + \frac{u^2}{2} + \frac{w^2}{2} - \frac{v_E^2}{2}\right)$$

$$\therefore \boxed{\begin{aligned} L_{gc} &\equiv (mu\hat{b} + e\vec{A})\cdot\dot{\vec{X}} + \frac{m}{e}\mu\dot{\gamma} - H_{gc} \\ H_{gc} &\equiv \frac{1}{2}mu^2 + \mu B + e\phi - \frac{1}{2}mv_E^2 \end{aligned}}$$

note that 'γ' is a cyclic coordinate

Littlejohn's G.C. Lagrangian can be obtained by $v_E = 0$

The addition of v_E was done later by Cary.

p.16 Modified potentials and EM fields

Littlejohn's L_{gc} is often written:

$$L_{gc} = e\vec{A}^* \cdot \dot{\vec{X}} + \epsilon\mu\dot{\gamma} - \left(\frac{1}{2}mu^2 + \mu B + e\phi^*\right)$$

with
- modified vector potential $\vec{A}^* \equiv \vec{A} + \epsilon u \hat{b}$
- modified electric potential $\vec{\phi}^* \equiv \phi - \frac{1}{2}\epsilon v_E^2$

Accordingly modifield EM fields, seen by guiding center becomes

$$\vec{B}^* = \vec{\nabla} \times \vec{A}^*$$

$$\vec{E}^* = -\vec{\nabla}\phi^* - \frac{d\vec{A}^*}{dt}$$