Chap. 8 Linear Constitutive equations

8.1 Constitutive equation & ideal materials

The mechanical constitutive equation of a material specifies the dependence of the stress in a body on kinematic variables such as strain tensor, or, rate of deformation tensor.

* Constitutive equation for the stress $\leftrightarrow$ strain tensor
$\leftrightarrow$ rate of deformation tensor

"Ideal materials" : linear elastic solids
Newtonian viscous fluids, ...

→ models should be selected based not only on the material, but also on the application.

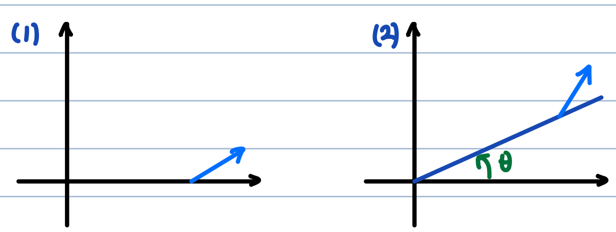
ex) water flow through pipes : incompressible
sound wave in water : compressible!

① dimensional homogeneity
② independence on the choice of coordinate system
*③ rigid body should not affect the stress.

8.2 Material symmetry

isotropic : no preferred direction
transversely isotropic : a single preferred direction

* rotational symmetry



① $\mathbb{x} = \mathbb{F} \cdot \mathbb{X}$
② $\mathbb{Q} \cdot \mathbb{x} = \mathbb{F} \cdot \mathbb{Q} \cdot \mathbb{X}$
$\Rightarrow \mathbb{x} = (\mathbb{Q}^T \cdot \mathbb{F} \cdot \mathbb{Q}) \mathbb{X}$

in here, if $\mathbb{T} \rightarrow \mathbb{Q}^T \cdot \mathbb{T} \cdot \mathbb{Q}$, then the material has symmetry for this rotation.

or, if reference frame rotates, and if

$\mathbb{T} = \mathbb{A}(\mathbb{F}) \Rightarrow \bar{\mathbb{T}} = \mathbb{A}(\bar{\mathbb{F}})$ ⟶ 똑같은 함수 $\mathbb{A}$.

$(\mathbb{M} = \mathbb{Q}^T, \ \bar{\mathbb{T}} = \mathbb{Q}^T \mathbb{T} \mathbb{Q}, \ \bar{\mathbb{F}} = \mathbb{Q}^T \mathbb{F} \mathbb{Q})$

If $\mathbb{Q} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow \theta = \frac{\pi}{2}$ 반시계, e_3 기준, 회전

* Reflection symmetry

$\mathbb{R}\mathbb{x} = \mathbb{F}\mathbb{R}\mathbb{X} \rightarrow \mathbb{x} = \mathbb{R}^T\mathbb{F}\mathbb{R}\mathbb{X}$, $\mathbb{T} \rightarrow \mathbb{R}^T\mathbb{T}\mathbb{R}$

$\boxed{\mathbb{R} = \mathbb{I} - 2\,\mathbb{n}\otimes\mathbb{n}} = \mathbb{I} - 2n_i n_j\, \mathbb{e}_i \otimes \mathbb{e}_j \rightarrow R_{ij} = \delta_{ij} - 2n_i n_j$

($\mathbb{n}$: normal vector for reflection plane)

$\mathbb{R}\cdot\mathbb{n} = -\mathbb{n}$, $\mathbb{R}\cdot\mathbb{n}_i = +\mathbb{n}_i$ $(i=2,3)$

$\mathbb{R}$ is an orthogonal tensor, and $\det\mathbb{R} = -1$. (check!)

* Symmetry groups

isotropic material → full orthogonal group.

transeverse isotropic → all rotations about a special axis.

orthotropic : 3 mutually orthogonal reflection planes.

8.3 Linear elasticity (stress ~ deformation)

- Many solid materials undergo very small changes of shape.

Define : 'linear elastic solid' for which internal energy $\rho_0 e$ per unit volume in the reference configuration.

$\boxed{W \equiv \rho_0 e}$: strain-energy function

(a) $W = \frac{1}{2} C_{ijkl} E_{ij} E_{kl}$

(b) $\rho \frac{De}{Dt} = \frac{\rho}{\rho_0} \frac{D}{Dt} W = T_{ij} D_{ij} - \cancel{\frac{dq_i}{dx_i}}$ ⇒ 무시하는 상황만 고려

- $W = \frac{1}{2} C_{ijkl} E_{ij} E_{kl} = \frac{1}{2} \overline{C_{pqrs}}\, \overline{E_{pq}}\, \overline{E_{rs}}$

If $\overline{e_i} = M_{ij} e_j$, $\overline{E_{rs}} = M_{ri} M_{sj} E_{ij}$ and $E_{ij} = M_{ri} M_{sj} \overline{E_{rs}}$

$W = \frac{1}{2} \overline{C_{pqrs}}\, \overline{E_{pq}}\, \overline{E_{rs}} = \frac{1}{2} C_{ijkl} E_{ij} E_{kl}$

W is a scalar

$C_{ijkl} M_{pi} M_{qj} \overline{E_{pq}} M_{rk} M_{sl} \overline{E_{rs}} = \overline{C_{pqrs}}\, \overline{E_{pq}}\, \overline{E_{rs}}$

$\therefore \overline{C_{pqrs}} = M_{pi} M_{qj} M_{rk} M_{sl} C_{ijkl} \rightarrow$ 4-th order tensor

∘ How many constants in C_{ijkl}..?

$W = \frac{1}{2} C_{ijkl} E_{ij} E_{kl} \longrightarrow \underline{3^4}$ component

(4-th order tensor)

Since $E_{ij} = E_{ji}$, C_{ijkl} may be assumed to have the symmetries

$C_{ijkl} = C_{jikl} = C_{ijlk} \longrightarrow (i,j)$ pair 6 kinds $\longrightarrow 6\times 6 = \underline{36}$

Also $E_{ij} \leftrightarrow E_{kl}$: $6 + {}_6C_2 = {}_6H_2 = \underline{21}$

($ij \leftrightarrow kl$)

Also, W should be positive $\longrightarrow$ E_{ij} : positive-definite.

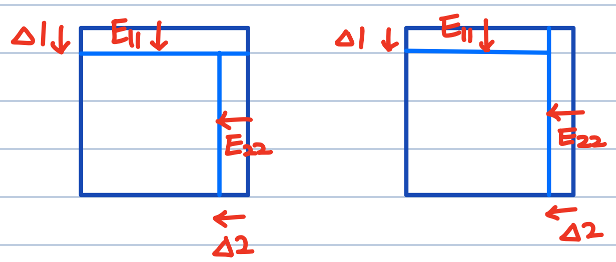
• From the equation of Conservation of energy ($\rho \frac{De}{Dt} = T_{ij}D_{ij} - \frac{dq_i}{dx_i}$)

$T_{ij}D_{ij} = \frac{\rho}{\rho_0}\frac{DW}{Dt} \left(= \rho \frac{De}{Dt} \right)$

$\frac{\rho}{\rho_0} \approx 1$, $T_{ij}D_{ij} \simeq \frac{D}{Dt}W = \frac{dW}{dE_{ij}} \frac{D}{Dt}E_{ij} \approx \frac{dW}{dE_{ij}} D_{ij}$

for all D_{ij} , $T_{ij} \approx \frac{dW}{dE_{ij}} = C_{ijrs}E_{rs}$ $\boxed{\therefore T_{ij} = C_{ijrs}E_{rs}}$ stress $\xleftrightarrow{\text{linear}}$ strain

✩ If $T_{ij} = C_{ijrs}E_{rs}$ is taken as the starting point,



$W = T_{11}\Delta 1 + T_{22}\Delta 2$

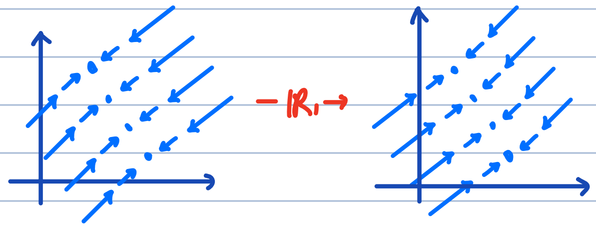
$= (C_{1111}E_{11} + C_{1122}E_{22})\Delta 1 + \left[C_{2211}(E_{11}+\Delta 1) + C_{2222}E_{22} \right]\Delta 2$

$= (C_{2211}E_{11} + C_{2222}E_{22})\Delta 2 + \left[C_{1111}E_{11} + C_{1122}(E_{22}+\Delta 2) \right]\Delta 1$

$\longrightarrow C_{1122}\Delta 2 \Delta 1 = C_{2211}\Delta 1 \Delta 2$ $\boxed{\therefore C_{1122} = C_{2211}}$

• Material symmetry → further reduction in # of C's.

If our system has $\mathbb{R}_1 = \begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ (reflection symmetry)



$F_{21} = \dfrac{dx_2}{dX_1} \longrightarrow -F_{21}$

$F_{12} = \dfrac{dx_1}{dX_2} \longrightarrow -F_{12}$

$\Rightarrow$ $E_{12} \longrightarrow -E_{12}$

$(E_{11} \longrightarrow -E_{11})$

$\therefore W = \frac{1}{2} C_{12aa} E_{12} E_{aa} = \frac{1}{2} C_{12aa}(-E_{12}) E_{aa} \longrightarrow C_{12aa} = 0$

$\longrightarrow C_{1112} = C_{1113} = C_{1222} = C_{1223} = C_{1233} = C_{1322} = C_{1323} = C_{1333} = 0$

↳ $21 - 8 = 13$ indep coef.

$\longrightarrow C_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu \delta_{ik}\delta_{jl} + \nu \delta_{il}\delta_{jk}$

$T_{ij} = C_{ijkl} E_{kl} = \lambda \delta_{ij} E_{kk} + \mu E_{ij} + \nu E_{ji}$

$= \lambda \delta_{ij} E_{kk} + 2\mu E_{ij}$ ($\because$ put $\mu = \nu$, $E_{ij} = E_{ji}$)

$\boxed{\therefore \mathbb{T} = \lambda \mathbb{I}\,\mathrm{tr}\mathbb{E} + 2\mu \mathbb{E}}$ constituitive equation for isotropic linear elastic solid

If instead, we started from $T_{ij} = C_{ijkl} E_{kl}$

$E'_{pq} = Q_{ip} Q_{jq} E_{ij}$, $T_{ij} = Q_{ip} Q_{jq} T'_{pq}$

$T'_{pq} = C_{pqrs} E'_{rs}$

$T_{ij} = Q_{ip} Q_{jq} C_{pqrs} Q_{kr} Q_{ls} E_{kl} = C_{ijkl} E_{kl}$

$\underline{\therefore C_{ijkl} = Q_{ip} Q_{jq} Q_{kr} Q_{ls} C_{pqrs}}$

8.4 Newtonian viscous fluids

Simple shearing flow : $v_1 = S x_2$, $v_2 = v_3 = 0$

→ Shearing stress is proportional to the shearing rate S.

→ Newtonian viscous fluid or linear viscous fluid.

→ water, air , and many other common fluids.

Constitutive equation : $T_{ij} = -p(\rho,\theta)\,\delta_{ij} + B_{ijkl}(\rho,\theta)\,D_{kl}$

If fluid is at rest, $D_{kl} = 0 \rightarrow T_{ij} = -p(\rho,\theta)\,\delta_{ij}$ (hydrostatic)

Additional stress to hydrostatic pressure is linear in $\mathbb{D}$.

If isotropic, $T_{ij} = -\{-p(\rho,\theta) + \lambda(\rho,\theta)\,D_{kk}\} + 2\mu(\rho,\theta)\,D_{ij}$

$\lambda(\rho,\theta)$, $\mu(\rho,\theta)$ = viscosity coeff.

Note that rigid-body motion $\longrightarrow D_{ij} = 0$ & It does not affect T_{ij}.

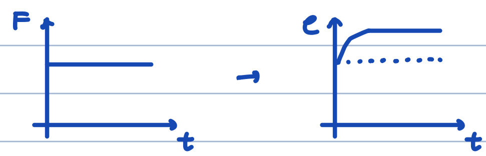
* Inviscid : $\lambda = 0$, $\mu = 0 \longrightarrow$ hydrostatic stress

* incompressible : $\frac{D}{Dt}\rho = 0 \longrightarrow \nabla \cdot \mathbf{v} = 0 \longrightarrow D_{kk} = 0$

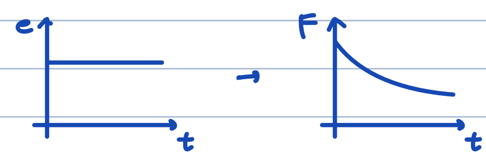
$$T_{ij} = -p\,\delta_{ij} + 2\mu(\theta)\,D_{ij}$$

8.5 Linear viscoelasticity

plastic : both elastic solid like & viscous fluid like $\longrightarrow$ viscoelastic



: creep curve



: Stress relaxation curve

Infinitesimal deformation $\longrightarrow \mathbb{E}$

$$\delta T_{ij}(t) = G_{ijkl}(t-\tau)\,\delta E_{kl}(\tau)$$

$$T_{ij} = \int_{-\infty}^{t} G_{ijkl}(t-\tau)\,\frac{dE_{kl}(\tau)}{d\tau}\,d\tau$$

$\begin{pmatrix} E_{kl} = E_{lk} \\ T_{ij} = T_{ji} \end{pmatrix} \rightarrow G_{ijkl} = G_{jikl} = G_{ijlk}$

Elastic

$G_{ijkl}(t-\tau) = C_{ijkl}$

Viscous

$G_{ijkl} \propto \delta(t-\tau)$