Fusion Plasma Theory I, Lecture 18. Energy Principle

- Plasma displacement in ideal MHD

$$\vec{x_0} \rightarrow \vec{x_0} + \vec{\xi}(\vec{x_0}, t) \quad (\vec{\xi} = \text{plasma displacement})$$

Then, perturbed ideal MHD becomes, (by assuming $\vec{u_0} = 0$, $\vec{u_1} = \frac{d\vec{\xi}}{dt}$)

$$\frac{d\rho}{dt} + \vec{u}\cdot\vec{\nabla}\rho + \rho\vec{\nabla}\cdot\vec{u} = 0 \quad \rightarrow \quad \rho_1 = -\vec{\xi}\cdot\vec{\nabla}\rho_0 - \rho_0\vec{\nabla}\cdot\vec{\xi}$$

$$\frac{dp}{dt} + \vec{u}\cdot\vec{\nabla}p + \gamma p\vec{\nabla}\cdot\vec{u} = 0 \quad \rightarrow \quad p_1 = -\vec{\xi}\cdot\vec{\nabla}p_0 - \gamma p_0\vec{\nabla}\cdot\vec{\xi}$$

$$\frac{d\vec{B}}{dt} = -\vec{\nabla}\times\vec{E} = \vec{\nabla}\times(\vec{u}\times\vec{B}) \quad \rightarrow \quad \vec{B_1} = \vec{\nabla}\times(\vec{\xi}\times\vec{B_0})$$

$$\mu_0\vec{j} = \vec{\nabla}\times\vec{B} \quad \rightarrow \quad \mu_0\vec{j_1} = \vec{\nabla}\times(\vec{\nabla}\times(\vec{\xi}\times\vec{B_0}))$$

$$\rho\left(\frac{d\vec{u}}{dt} + \cancel{\vec{u}\cdot\vec{\nabla}u}^{\,0}\right) = \vec{j}\times\vec{B} - \vec{\nabla}p \quad \rightarrow \quad \rho_0\frac{d^2\vec{\xi}}{dt^2} = \vec{j_1}\times\vec{B_0} + \vec{j_0}\times\vec{B_1} - \vec{\nabla}p_1 = \vec{F}[\vec{\xi}]$$

- (Ideal perturbed) force operator

$$\vec{F}[\vec{\xi}] = \vec{j_1}\times\vec{B_0} + \vec{j_0}\times\vec{B_1} - \vec{\nabla}p_1$$

$$= \frac{1}{\mu_0}\left[\vec{\nabla}\times\vec{\nabla}\times(\vec{\xi}\times\vec{B_0})\right]\times\vec{B_0} + \frac{1}{\mu_0}\left[(\vec{\nabla}\times\vec{B_0})\times\vec{\nabla}\times(\vec{\xi}\times\vec{B_0})\right] + \vec{\nabla}\left[\vec{\xi}\cdot\vec{\nabla}p_0 - \gamma p_0(\vec{\nabla}\cdot\vec{\xi})\right]$$

If we omit the subscript "0" and ansatz $\vec{\xi}(\vec{x},t) = \vec{\xi}(\vec{x})e^{-i\omega t}$, we obtain "normal mode equation"

$$\boxed{\omega^2\rho\vec{\xi} = -\vec{F}[\vec{\xi}]}$$

Property of force operator (self-adjointness)

$$\int \vec{\eta}\cdot\vec{F}[\vec{\xi}]\,dx^3 = \int \vec{\xi}\cdot\vec{F}[\vec{\eta}]\,d^3x$$ ← Accept this!

① w^2 is pure real

$$w^2\int \rho|\vec{\xi}|^2 dx^3 = -\int \vec{\xi}^{*}\,\vec{F}[\vec{\xi}]\,dx^3$$

$$(w^2)^{*}\int \rho|\vec{\xi}|^2 dx^3 = -\int \vec{\xi}\cdot\vec{F}[\vec{\xi}^{*}]\,dx^3$$

$\Rightarrow w^2 = (w^2)^{*} \Rightarrow$ w^2 is real

② Eigenfunction is orthogonal

$$w_n^2\int \rho\,\vec{\xi}_m^{*}\cdot\vec{\xi}_n\,dx^3 = -\int \vec{\xi}_m^{*}\cdot\vec{F}[\vec{\xi}_n]\,dx^3$$

$$w_m^2\int \rho\,\vec{\xi}_n\cdot\vec{\xi}_m^{*}\,dx^3 = -\int \vec{\xi}_n\cdot\vec{F}[\vec{\xi}_m^{*}]\,dx^3$$

$\therefore (w_n^2 - w_m^2)\int \rho\,\vec{\xi}_m^{*}\cdot\vec{\xi}_n\,dx^3 = 0$ $\quad\therefore$ for $w_n \neq w_m$, $\int \rho\,\vec{\xi}_m^{*}\,\vec{\xi}_n\,dx^3 = 0$

Variational Principle

$$w^2 = \frac{-\frac{1}{2}\int \vec{\xi}^{*}\cdot\vec{F}[\vec{\xi}]\,dx^3}{\frac{1}{2}\int \rho\,\vec{\xi}^{*}\cdot\vec{\xi}\,dx^3} \equiv \frac{\delta W[\vec{\xi}^{*},\vec{\xi}]}{\delta K[\vec{\xi}^{*},\vec{\xi}]} = \frac{\text{Perturbed Potential Energy}}{\text{Perturbed Kinetic Energy}}$$

$$\therefore \boxed{\delta W[\vec{\xi}^{*},\vec{\xi}] = -\frac{1}{2}\int \vec{\xi}^{*}\cdot\vec{F}[\vec{\xi}]\,dx^3}$$

The most unstable perturbation would be the one that makes the minimum w^2.

Then, the question is when we minimize w^2, would it be consisten with $\langle w^2\rho\,\vec{\xi} = -F[\vec{\xi}]\rangle$?

(proof)

$$\delta\omega^2 = \frac{\delta(\delta W)}{\delta K} - \delta W \frac{\delta(\delta K)}{(\delta K)^2} = \frac{\delta(\delta W) - \omega^2 \delta(\delta K)}{\delta K}$$

$$= \frac{\delta W[\delta\vec{\xi}^*, \vec{\xi}] + \delta W[\vec{\xi}^*, \delta\vec{\xi}] - \omega^2\left[\delta K[\delta\vec{\xi}^*, \vec{\xi}] + \delta K[\vec{\xi}^*, \delta\vec{\xi}]\right]}{\delta K[\vec{\xi}^*, \vec{\xi}]}$$

using eq of δW, δK and self-adj.

$$= -\frac{\int dx^3 \left\{\delta\vec{\xi}^* \cdot (\vec{F}[\vec{\xi}] + \omega^2\rho\vec{\xi})\right\} + \int dx^3 \left\{\delta\vec{\xi}^* \cdot (F[\vec{\xi}] + \omega^2\rho\vec{\xi})\right\}^*}{\delta K[\vec{\xi}^*, \vec{\xi}]}$$

$$\boxed{\delta\omega^2 = 0 \quad \Leftrightarrow \quad \omega^2\rho\vec{\xi} = -F[\vec{\xi}]}$$

ω^2의 최솟값을 찾는 문제나, normal mode equation을 푸는 문제나 같은 문제를 푸는것이다.

$\Rightarrow$ The stability can be determined by minimizing Eq. (19). ($\delta\omega^2 = 0$) rather than the full normal mode analysis of ($\omega^2\rho\vec{\xi} = -F[\vec{\xi}]$)

- <u>Energy principle</u>

Any possible displacement can be represented by $\vec{\xi} = \sum_n a_n \vec{\xi}_n$ if eigenmodes are discrete.

$$\delta W = -\frac{1}{2}\int dx^3 \left(\sum_m a_m \vec{\xi}_m^*\right) \cdot \vec{F}\Big[\underbrace{\sum_n a_n \vec{\xi}_n}_{}\Big] = \frac{1}{2}\sum_n |a_n|^2 \omega_n^2 \int \rho |\vec{\xi}_n|^2 dx^3$$

$= -\rho\omega^2\vec{\xi}$

$\begin{cases} \text{If } \delta W > 0, \ \omega_n^2 \text{ must be positive against arbitrary } a_n : \text{stable} \\ \text{If } \delta W < 0, \text{ there must be at least one negative } \omega_n^2 : \text{unstable} \end{cases}$

$\Rightarrow$ **Energy Principle** : Equilibrium is stable iff $\delta W > 0$, for all allowable trial function $\vec{\xi}(\vec{x})$

- <u>Standard form of δW</u>

$$\delta W = -\frac{1}{2}\int \vec{\eta}\cdot\Big[\underbrace{\frac{1}{\mu_0}(\vec{\nabla}\times\vec{B_1})\times\vec{B_0}}_{(a)} \underbrace{+\vec{j_0}\times\vec{B_1} + \vec{\nabla}(\vec{\xi}\cdot\vec{\nabla}p)}_{(b)} \underbrace{+\gamma\vec{\nabla}(p(\vec{\nabla}\cdot\vec{\xi}))}_{(c)}\Big]dx^3$$

$$\left(\vec{B_1} = \vec{B_1}(\vec{\xi_\perp}) = \vec{\nabla}\times(\vec{\xi_\perp}\times\vec{B_0}) \rightsquigarrow \vec{B_1}(\vec{\eta_\perp}) = \vec{\nabla}\times(\vec{\eta_\perp}\times\vec{B_0})\right)$$

$$(a) = \frac{1}{\mu_0}\int(\vec{\nabla}\times\vec{B_1})\cdot(\vec{B_0}\times\vec{\eta_\perp})\,dx^3 \qquad \vec{\nabla}\cdot(\vec{A}\times\vec{B}) = (\vec{\nabla}\times\vec{A})\cdot\vec{B} - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$$

$$= \frac{1}{\mu_0}\int\vec{\nabla}\cdot(\vec{B_1}\times(\vec{B_0}\times\vec{\eta_\perp}))dx^3 + \frac{1}{\mu_0}\int\vec{B_1}\cdot\vec{\nabla}\times(\vec{B_0}\times\vec{\eta_\perp})\,dx^3$$

$$= \underbrace{-\frac{1}{\mu_0}\int\vec{B_0}\cdot\vec{B_1}(\vec{\xi_\perp})(\vec{\eta_\perp}\cdot d\vec{a})}_{S_1} \underbrace{-\frac{1}{\mu_0}\int\vec{B_1}(\vec{\eta_\perp})\cdot\vec{B_1}(\vec{\xi_\perp})\,dx^3}_{F_1}$$

(추가 노트) $\int\vec{\nabla}\cdot(\vec{B_1}\times(\vec{B_0}\times\vec{\eta_\perp}))\,dx^3 = \int\vec{B_1}\times(\vec{B_0}\times\vec{\eta_\perp})\cdot d\vec{a} = \int\vec{B_1}\cdot(\vec{B_0}\times\vec{\eta_\perp})\times d\vec{a}$

$$= \int\vec{B_1}\cdot\left(\vec{\eta_\perp}\cancelto{0}{(\vec{B_0}\cdot d\vec{a})} - \vec{B_0}(\vec{\eta_\perp}\cdot d\vec{a})\right) = \int\vec{B_0}\cdot\vec{B_1}(\vec{\eta_\perp}\cdot d\vec{a})$$

(use $\vec{\nabla}\cdot(\vec{A}\times\vec{B}) = (\vec{\nabla}\times\vec{A})\cdot\vec{B} - \vec{A}\cdot(\vec{\nabla}\times\vec{B})$)

(b); first note that

$$\vec{B_0}\cdot(\vec{j_0}\times\vec{B_1}) = -\vec{B_1}\cdot\vec{\nabla}p = -\vec{\nabla}\times(\vec{\xi_\perp}\times\vec{B_0})\cdot\vec{\nabla}p = \vec{\nabla}\cdot(\vec{\nabla}p\times(\vec{\xi}\times\vec{B_0}))$$

$$= -\vec{\nabla}\cdot(\vec{B_0}(\vec{\xi}\cdot\vec{\nabla}p)) = -\vec{B_0}\cdot\vec{\nabla}(\vec{\xi}\cdot\vec{\nabla}p)$$

it gives the relation of $\vec{\eta_\parallel}\cdot(\vec{j_0}\times\vec{B_1} + \vec{\nabla}(\vec{\xi}\cdot\vec{\nabla}p)) = 0$.

$$\therefore (b) = \int\vec{\eta_\perp}\cdot(\vec{j_0}\times\vec{B_1} + \vec{\nabla}(\vec{\xi}\cdot\vec{\nabla}p))\,dx^3$$

$$= \int\Big[\underbrace{\vec{\eta_\perp}\cdot(\vec{j_0}\times\vec{B_1})}_{F_3} + \underbrace{\vec{\nabla}\cdot(\vec{\eta_\perp}\vec{\nabla}(\vec{\xi}\cdot\vec{\nabla}p))}_{S} \underbrace{-(\vec{\nabla}\cdot\vec{\eta_\perp})(\vec{\xi}\cdot\vec{\nabla}p)}_{F_4}\Big]dx^3$$

$$(c) = \int\Big[\underbrace{\vec{\nabla}\cdot(\vec{\eta}\gamma p(\vec{\nabla}\cdot\vec{\xi}))}_{S_2} \underbrace{-\gamma p(\vec{\nabla}\cdot\vec{\eta})(\vec{\nabla}\cdot\vec{\xi})}_{F_2}\Big]dx^3$$

let $\vec{\eta}=\vec{\xi}^*$, then we obtain $\delta W = \delta W_F + \delta W_s$

(δW_F : fluid perturbed energy, δW_s : surface energy) are given by

$$\boxed{\begin{aligned}\delta W_F &= \frac{1}{2}\int\left[\frac{|\vec{B_1}|^2}{\mu_0} + \gamma p|\vec{\nabla}\cdot\vec{\xi}|^2 - \vec{\xi}_\perp^*\cdot(\vec{j_0}\times\vec{B_1}) + (\vec{\nabla}\cdot\vec{\xi}_\perp)(\vec{\xi}_\perp\cdot\vec{\nabla}p)\right]dx^3 \\ \delta W_s &= \frac{1}{2}\int\left(\frac{\vec{B_0}\cdot\vec{B_1}}{\mu_0} - \gamma p(\vec{\nabla}\cdot\vec{\xi}) - \vec{\xi}\cdot\vec{\nabla}p\right)(\vec{\xi}_\perp^*\cdot d\vec{a})\end{aligned}}$$

δW
① term : magnetic pressure
② term : plasma compression) → stabilizing
③ term : current
④ term : pressure gradient) → destabilizing.

∘ <u>Intuitive form of δW</u> (by H.P. Furth (1965))

∘ $$\vec{B_0}\cdot\vec{B_1} = \vec{B_0}\cdot\vec{\nabla}\times(\vec{\xi}\times\vec{B_0}) = \vec{\nabla}\cdot((\vec{\xi}\times\vec{B_0})\times\vec{B_0}) + (\vec{\xi}_\perp\times\vec{B_0})\cdot(\underbrace{\vec{\nabla}\times\vec{B_0}}_{=\mu_0\vec{j}})$$

$$= -\vec{\nabla}\cdot(\vec{\xi}_\perp B_0^2) - \mu_0(\vec{\xi}_\perp\cdot\vec{\nabla}p) = -(\vec{\nabla}\cdot\vec{\xi}_\perp)B_0^2 - \vec{\xi}_\perp\cdot\vec{\nabla}B_0^2 - \mu_0\vec{\xi}_\perp\cdot\vec{\nabla}p$$

$$= -(\vec{\nabla}\cdot\vec{\xi}_\perp + 2\vec{\xi}_\perp\cdot\vec{K})B_0^2 + \mu_0\vec{\xi}_\perp\cdot\vec{\nabla}p = -\Delta B_0^2 + \mu_0\vec{\xi}_\perp\cdot\vec{\nabla}p$$

$\vec{\nabla}_\perp(B_0^2+2\mu_0 p) = 2B_0^2\vec{K}$ (where $\Delta = \vec{\nabla}\cdot\vec{\xi}_\perp + 2\vec{\xi}_\perp\cdot\vec{K}$)

∘ $$|\vec{B_{1\parallel}}|^2 = \Delta^2 B_0^2 - 2\mu_0(\vec{\xi}_\perp^*\cdot\vec{\nabla}p)\Delta + \frac{\mu_0^2(\vec{\xi}_\perp\cdot\vec{\nabla}p)^2}{B_0^2}$$

∴ 1st-term in δW_F : $$\underline{\frac{|\vec{B_1}|^2}{\mu_0} = \frac{|\vec{B_{1\perp}}|^2}{\mu_0} + \frac{\Delta^2 B_0^2}{\mu_0} - 2(\vec{\xi}^*\cdot\vec{\nabla}p)\Delta + \frac{\mu_0(\vec{\xi}_\perp\cdot\vec{\nabla}p)^2}{B_0^2}}$$

3rd-term in δW_F :

$$\vec{\xi}_\perp^*\cdot(\vec{j_0}\times\vec{B_1}) = \vec{\xi}_\perp^*\cdot(\vec{j_{0\perp}}\times\underbrace{\vec{B_{1\parallel}}}_{=\frac{B_{1\parallel}}{B_0}\vec{B_0}}) + \vec{\xi}_\perp^*\cdot(\vec{j_{0\parallel}}\times\vec{B_{1\perp}}) = \frac{B_{1\parallel}}{B_0}\vec{\xi}_\perp^*\cdot\vec{\nabla}p + \frac{j_{0\parallel}}{B_0}\vec{B_{1\perp}}\cdot(\vec{\xi}_\perp^*\times\vec{B_0})$$

$$\underline{= -(\vec{\xi}_\perp^*\cdot\vec{\nabla}p)\Delta + \frac{\mu_0(\vec{\xi}_\perp\cdot\vec{\nabla}p)^2}{B_0^2} + \frac{j_{0\parallel}}{B_0}\vec{B_{1\perp}}\cdot(\vec{\xi}_\perp^*\times\vec{B_0})}$$

combining modified 1st, 3rd and 4-th term,

$$\frac{|\vec{B_1}|^2}{\mu_0} - \vec{\xi_\perp}\cdot(\vec{j_0}\times\vec{B_1}) + (\vec{\nabla}\cdot\vec{\xi_\perp}^*)(\vec{\xi_\perp}\cdot\vec{\nabla}p)$$

$$= \frac{|\vec{B_{1\perp}}|^2}{\mu_0} + \frac{\Delta^2 B_0^2}{\mu_0} - 2(\vec{\xi}^*\cdot\vec{\nabla}p)\Delta + \cancel{\frac{\mu_0(\vec{\xi}\cdot\vec{\nabla}p)^2}{B_0^2}}$$

$$+ (\vec{\xi_\perp}^*\cdot\vec{\nabla}p)\Delta - \cancel{\frac{\mu_0(\vec{\xi}\cdot\vec{\nabla}p)^2}{B_0^2}} - \frac{j_{0\parallel}}{B_0}\vec{B_{1\perp}}\cdot(\vec{\xi_\perp}^*\times\vec{B_0})$$

$$+ (\vec{\nabla}\cdot\vec{\xi_\perp}^*)(\vec{\xi_\perp}\cdot\vec{\nabla}p)$$

(note $\Delta = \vec{\nabla}\cdot\vec{\xi_\perp} + 2\vec{\xi_\perp}\cdot\vec{K}$)

$$= \underset{(a)}{\frac{|\vec{B_{1\perp}}|^2}{\mu_0}} + \underset{(b)}{\frac{\Delta^2 B_0^2}{\mu_0}} - \underset{(e)}{\frac{j_{0\parallel}}{B_0}\vec{B_{1\perp}}\cdot(\vec{\xi_\perp}^*\times\vec{B_0})} - \underset{(d)}{2(\vec{\xi_\perp}\cdot\vec{K})(\vec{\xi_\perp}\cdot\vec{\nabla}p)}$$

All together, the fluid perturbed potential :

2nd term

$$\boxed{\begin{aligned}\delta W_F = \frac{1}{2}\int\Big[\underset{(a)}{\frac{1}{\mu_0}|\vec{B_{1\perp}}|^2} + \underset{(b)}{\frac{1}{\mu_0}|\vec{\nabla}\cdot\vec{\xi_\perp} + 2\vec{\xi_\perp}\cdot\vec{K}|^2 B_0^2} + \underset{(c)}{\gamma p|\vec{\nabla}\cdot\vec{\xi}|^2}\Big]dx^3 \\ -\frac{1}{2}\int\Big[\underset{(d)}{2(\vec{\xi_\perp}\cdot\vec{K})(\vec{\xi_\perp}\cdot\vec{\nabla}p)} + \underset{(e)}{\frac{j_{0\parallel}}{B_0}\vec{B_{1\perp}}\cdot(\vec{\xi_\perp}^*\times\vec{B_0})}\Big]dx^3\end{aligned}}$$

(a) magnetic pressure (shear Alfvén wave)

(b) magnetic compression (compressional Alfvén wave)

(c) plasma compression (sound wave)

(d) pressure driven instability (interchange & ballooning)
(when $\vec{K}\cdot\vec{\nabla}p > 0$)

(e) current driven instability (kink & tearing)