Fusion Plasma Theory 2

# Lecture 13 : Causality, Dispersion, Nyquist

## ① Dielectric response for normal mode

$$\vec{\nabla}\times\vec{B}_1 = \mu_0\vec{j}_1 + \frac{1}{c^2}\frac{\partial \vec{E}_1}{\partial t} = \frac{1}{c^2}\frac{\partial \vec{D}}{\partial t}$$

$$i\vec{k}\times\vec{B}_1 = \mu_0\vec{j}_1 - \frac{i\omega}{c^2}\vec{E}_1 = \mu_0\overleftrightarrow{\sigma}\cdot\vec{E}_1 - \frac{i\omega}{c^2}\vec{E}_1 = -\frac{i\omega}{c^2}\vec{D}_1$$

$$\vec{D}_1(\vec{k},\omega) = \left(\overleftrightarrow{I} + \frac{i}{\epsilon_0\omega}\overleftrightarrow{\sigma}(\vec{k},\omega)\right)\cdot\vec{E}_1(\vec{k},\omega) = \overleftrightarrow{\epsilon}(\vec{k},\omega)\cdot\vec{E}_1(\vec{k},\omega)$$

$$D_1(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}dk\, e^{i(kz-\omega t)}D_1(k,\omega)$$ Inverse Fourier Transform

## ② Non-locality of Fourier modes

$$\vec{D}_1(k,\omega) = \overleftrightarrow{\epsilon}(k,\omega)\cdot\vec{E}_1(k,\omega)$$

$$\vec{D}_1(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\int_{-\infty}^{\infty}dk\, e^{i(kz-\omega t)}\times$$

$$\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}dz''\int_{-\infty}^{\infty}dt''\, e^{-i(kz''-\omega t'')}\overleftrightarrow{\epsilon}(z'',t'')\right)\cdot\left(\frac{1}{2\pi}\int_{-\infty}^{\infty}dz'\int_{-\infty}^{\infty}dt'\, e^{-i(kz'-\omega t')}\vec{E}_1(z',t')\right)$$

$$\left(\delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{\infty}dk\, e^{ik(z-z')}\quad,\quad \delta(t-t') = \frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega\, e^{-i\omega(t-t')}\right)$$

$$\Rightarrow \vec{D}_1(z,t) = \frac{1}{2\pi}\int dz'\int dz''\int dt'\int dt''\,\overleftrightarrow{\epsilon}(z'',t'')\,\vec{E}_1(z',t')\times$$

$$\underbrace{\left(\frac{1}{2\pi}\int dk\, e^{ik(z-z'-z'')}\right)}_{=\delta(z-z'-z'')}\times\underbrace{\left(\frac{1}{2\pi}\int d\omega\, e^{-i\omega(t-t'-t'')}\right)}_{=\delta(t-t'-t'')}$$

$$\underline{\vec{D}_1(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}dz'\int_{-\infty}^{\infty}dt'\,\overleftrightarrow{\epsilon}(z-z',t-t')\cdot\vec{E}(z',t')}$$ Displacement field in configuration space.

similarly

$$\underline{\vec{j}_1(z,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}dz'\int_{-\infty}^{\infty}dt'\,\overleftrightarrow{\sigma}(z-z',t-t')\cdot\vec{E}(z',t')}$$

③ Causality in normal modes

The non-local response immediately raises the question of the causality in normal mode approach. There must be no response for $t<0$ if the initial kick occurs at $t=0$. (Q. Does response occur always after the impulse that stimulates it?)

Suppose a kick $E_1(t) = E_0\,\delta(t) \rightarrow E_1(\omega) = \dfrac{E_0}{\sqrt{2\pi}}$.

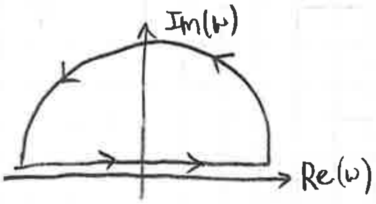
$$\Rightarrow j_1(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, \underset{\substack{\uparrow \\ j_1(\omega) = \sigma(\omega)E_1(\omega)}}{j_1(\omega)}\, e^{-i\omega t} = \frac{E_0}{2\pi}\int_{-\infty}^{\infty} d\omega\, \sigma(\omega) e^{-i\omega t} \underset{\substack{\uparrow \\ \chi = \frac{i\sigma}{\omega\epsilon_0} \rightarrow \sigma = -i\chi\omega\epsilon_0}}{=} -i\frac{\epsilon_0 E_0}{2\pi}\int_{-\infty}^{\infty} d\omega\, \omega\,\chi(\omega) e^{-i\omega t}$$

(This integral vanishes for $t<0$ if the integrand is analytic in the upper half of the complex $\omega$-plane. This is the causality condition for normal mode.)

증명 : $\omega = \omega_r + i\omega_i \rightarrow \exp(-i\omega t) = \exp(-i\omega_r t + \omega_i t) = \exp(-i\omega_r t)\exp(\omega_i t)$

($t<0$) 인 상황에서 $\omega_i$가 매우 크다면 (upper half-plane) $j_1(t)$의 integrand는 항상 0이다.

따라서 적분을 할때 $\omega : (-\infty, \infty)$ 적분과 upper half-plane 적분과 그 값은 동일하다.

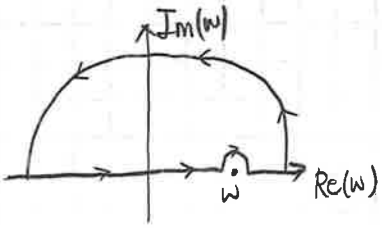
즉, 


적분을 의미한다. ($t<0$) 에서 $j_1(t) = 0$ 이 되기 위해서는 Cauchy's integral theorem 에 의해 UHP가 analytic 해야한다

Again, Causality condition is that the susceptibility should be analytic for $\omega_i \geq 0$ when it is calculated hypothetically by $\omega_r + i\omega_i$, even if $\omega$ is actually real in normal mode

Suppose an integral of

$$I(\omega) = \int_c d\omega' \frac{\omega'\chi(\omega')}{\omega' - \omega}$$




This indicates

$$\underset{\text{(residue)}}{-i\pi\omega\chi(\omega)} + \underset{\text{(real integral)}}{P\int_{-\infty}^{\infty} d\omega' \frac{\omega'\chi(\omega')}{\omega' - \omega}} = 0$$

Dividing $\chi(\omega) = \chi_r(\omega) + i\chi_i(\omega)$ on the real $\omega$-axis, we obtain ⟨Kramers - Kronig (KK) relations for the susceptibility⟩

$$\Rightarrow \omega\chi_r(\omega) = \frac{1}{\pi}P\int_{-\infty}^{\infty} d\omega' \frac{\omega'\chi_i(\omega')}{\omega' - \omega}, \quad \omega\chi_i(\omega) = -\frac{1}{\pi}P\int_{-\infty}^{\infty} d\omega' \frac{\omega'\chi_r(\omega')}{\omega' - \omega}$$

④ Plemelj correction for causality.

Recall cold plasma response. For example L-wave and its susceptibility is

$$\omega \chi_s(\omega) = -\frac{\omega_{ps}^2}{\omega - \omega_{cs}}$$ ← This does not hold the causality due to the pole $\omega = \omega_{cs}$.

To satisfy causality, one can make a slight alteration

$$\omega \chi_s(\omega) = -\lim_{\nu \to 0+} \frac{\omega_{ps}^2}{\omega - (\omega_{cs} - i\nu)}$$ ← same as one introduces a damping term to meet the causality.

Common practice to write an identity due to J. Plemelj.

$$\lim_{\nu \to 0+} \frac{1}{\omega - \omega_0 + i\nu} = P\left(\frac{1}{\omega - \omega_0}\right) - i\pi \delta(\omega - \omega_0).$$ ← meaningful only when integration over $\omega$ is operated.

⑤ Modified dispersion for causality.

Using Plemelj's form, the cold plasma dispersion can be modified.

$$R = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega + \omega_{cs})} \longrightarrow R \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega}\left[P\left(\frac{1}{\omega + \omega_{cs}}\right) - i\pi \delta(\omega + \omega_{cs})\right]$$

$$L = 1 - \sum_s \frac{\omega_{ps}^2}{\omega(\omega - \omega_{cs})} \longrightarrow L \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega}\left[P\left(\frac{1}{\omega - \omega_{cs}}\right) - i\pi \delta(\omega - \omega_{cs})\right]$$

$$P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \longrightarrow P \equiv 1 - \sum_s \frac{\omega_{ps}^2}{\omega}\left[P\left(\frac{1}{\omega}\right) - i\pi \delta(\omega)\right]$$

(and $S = (R+L)/2$, $D = (R-L)/2$ accordingly.)

Also Plemelj's form led exactly to Landau damping correction.

$$\hat{f}_1(k,\omega) = -i\frac{q\hat{E}_1}{m}\frac{df_0/dv}{\omega - kv} \longrightarrow \hat{f}_1(k,\omega) \equiv -i\frac{q\hat{E}_1}{m}\frac{df_0}{dv}\left[P(\omega - kv) - i\pi\delta(\omega - kv)\right]$$

⑯ Plasma dispersion function

Dispersion function by Vlasov (Vlasov + Poisson equation)

with Maxwellian $f_{s0} = f_{M0} = (n_{s0}/v_{ts}\pi^{1/2}) e^{-v^2/v_{ts}^2}$, $t \equiv v/v_{ts}$, $dt = dv/v_{ts}$

$$0 = D(k,\omega) = 1 + \sum_s \frac{q_s^2}{m_s k \epsilon_0} \int_{-\infty}^{\infty} \frac{df_{s0}/dv}{\omega - kv} dv = 1 - \sum_s \frac{\omega_{ps}^2}{k^2 v_{ts}^2} \left[ \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \frac{(d/dt) e^{-t^2}}{t - \xi_s} dt \right]$$

$(\xi_s \equiv \omega / k v_{ts})$

Here, we define plasma dispersion function, contained in NRL formula,

$$\boxed{Z(\xi) \equiv \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-\xi} dt, \quad \mathrm{Im}(\xi) > 0.}$$ ← spirit of Landau integral

*note

$$Z'(\xi) = \frac{dZ(\xi)}{d\xi} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{e^{-t^2}}{(t-\xi)^2} dt = \cancel{-\frac{1}{\sqrt{\pi}} \frac{e^{-t^2}}{t-\xi} \Big|_{t=-\infty}^{t=\infty}}^{\,0} + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\frac{d}{dt}(e^{-t^2})}{t-\xi} dt$$

Thus, the dispersion becomes $\boxed{k^2 = \sum_s \frac{\omega_{ps}^2}{v_{ts}^2} Z'(\xi_s) = \frac{1}{2} \sum_s k_{ps}^2 Z'(\xi_s)}$

$k_{ps} \equiv \lambda_{ps}^{-1}$ (Debye length)

⑰ Characteristics of plasma dispersion function

- Various representations of dispersion function.

$$Z'(\xi) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{\infty} \frac{(d/dt) e^{-t^2}}{t-\xi} dt = -\frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{t e^{-t^2}}{t-\xi} dt$$

$$= -\frac{2}{\sqrt{\pi}} \left[ \int_{-\infty}^{\infty} \frac{(t-\xi) e^{-t^2}}{t-\xi} dt + \xi \int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-\xi} dt \right] = -2(1 + \xi Z(\xi))$$

$$\Rightarrow \underline{Z' + 2\xi Z + 2 = 0}$$

$(-\infty, 0) + (0, i\xi)$

Let the solution becomes $\underline{Z(\xi) = 2i e^{-\xi^2} \int_{-\infty}^{i\xi} e^{-t^2} dt = i\pi^{1/2} e^{-\xi^2} (1 + \mathrm{erf}(i\xi))}$

then, $Z'(\xi) = -2i\xi e^{-\xi^2} \int_{-\infty}^{i\xi} e^{-t^2} dt + 2i \cancel{e^{-\xi^2}}\, i \cancel{e^{\xi^2}} = -2\xi Z(\xi) - 2$

it satisfies $Z'(\xi) = -2(1 + \xi Z(\xi))$. Thus it is another expression of $Z(\xi)$ by analytic continuation.

Next, $u = i\xi - \frac{t}{2}$, $du = -\frac{1}{2} dt$ → $u: (-\infty, i\xi) \Rightarrow t: (\infty, 0)$

$u$ ↑ original, $t$ ↑ new

$$Z(\xi) = 2i e^{-\xi^2} \int_{-\infty}^{i\xi} e^{-u^2} du = 2i e^{-\xi^2} \int_{\infty}^{0} e^{-(i\xi - t/2)^2} \left(-\frac{dt}{2}\right) = \underline{i \int_0^{\infty} e^{i\xi t - \frac{t^2}{4}} dt}$$

∴ Summary of various representations of dispersion function

$$\begin{cases} Z(\zeta) = \dfrac{1}{\sqrt{\pi}}\displaystyle\int_{-\infty}^{\infty} \frac{e^{-t^2}}{t-\zeta}\,dt\,, \quad \mathrm{Im}(\zeta)>0 \\ \quad = 2i\, e^{-\zeta^2}\displaystyle\int_{-\infty}^{i\zeta} e^{-t^2}dt = i\pi^{1/2} e^{-\zeta^2}(1+\mathrm{erf}(i\zeta)) \quad \leftarrow \text{from } Z'+2\zeta Z+2=0 \\ \quad = i\displaystyle\int_0^{\infty} e^{i\zeta t - t^2/4}dt \quad \leftarrow \text{from 2nd expression, use change of variables } u = i\zeta - \frac{t}{2}. \end{cases}$$

The power series of plasma dispersion function,

$$Z(\zeta) = i\sqrt{\pi}\, e^{-\zeta^2}\left(1+\frac{2}{\sqrt{\pi}}\int_0^{i\zeta} e^{-t^2}dt\right)$$

$$= i\sqrt{\pi}\, e^{-\zeta^2} - 2\zeta\left(1-\frac{2}{3}\zeta^2+\frac{4}{15}\zeta^4 \cdots\right) \quad \leftarrow \text{for small } \zeta$$

✲ $$Z(\zeta) = i\pi^{1/2}\sigma e^{-\zeta^2} - \frac{1}{\zeta}\left(1+\frac{1}{2\zeta^2}+\frac{3}{4\zeta^4}+\frac{15}{8\zeta^6}\cdots\right) \quad \leftarrow \text{for large } \zeta.$$

where $\sigma=0$ for $\mathrm{Im}(\zeta)>0$, $\sigma=1$ for $\mathrm{Im}(\zeta)=0$, $\sigma=2$ for $\mathrm{Im}(\zeta)<0$.

← 무슨 말이지?

## ⑧ Ion Landau damping.

low-frequency ion waves $kv_{ti} \ll \omega \ll kv_{t,e}$ $\leftrightarrow$ $\zeta_e \ll 1$ ; small $\zeta$ expansion for electron

$\zeta_i \gg 1$ : large $\zeta$ expansion for ion

$$\Rightarrow k^2 = \frac{\omega_{pe}^2}{v_{te}^2}\left(-2i(\pi)^{1/2}\zeta_e e^{-\zeta_e^2} - 2 - \cdots\right) + \frac{\omega_{pi}^2}{v_{ti}^2}\left(-2i(\pi)^{1/2}\zeta_i e^{-\zeta_i^2} + \frac{1}{\zeta_i^2} + \cdots\right)$$

Using $\zeta_s = \omega/kv_{ts}$ and $v_{ts}^2 = 2v_{t,s}^2$ (consistent w/ Goldston)

$$\Rightarrow D(k,\omega) = 1 - \frac{\omega_{pe}^2}{k^2 v_{te}^2}Z'(\zeta_e) - \frac{\omega_{pi}^2}{k^2 v_{ti}^2}Z'(\zeta_i) = 0$$

$v_{ti}^2 = 2v_{t,i}^2$

$$\begin{cases} \text{Real term : } 1 - \dfrac{\omega_{pe}^2}{k^2 v_{te}^2}(-2) + \dfrac{\omega_{pi}^2}{k^2 v_{ti}^2}\left(\dfrac{1}{\zeta_i^2}\right) = 1 + \dfrac{2\omega_{pe}^2}{k^2 v_{te}^2} - \dfrac{\omega_{pi}^2}{\cancel{k^2 v_{ti}^2}}\dfrac{\cancel{k^2 v_{ti}^2}}{\omega^2} = 1 + \dfrac{\omega_{pe}^2}{k^2 v_{t,e}^2} - \dfrac{\omega_{pi}^2}{\omega^2} \\ \text{Imaginary term : } -\dfrac{\omega_{pe}^2}{k^2 v_{te}^2}\left(-2i\pi^{1/2}\zeta_e e^{-\zeta_e^2}\right) - \dfrac{\omega_{pi}^2}{k^2 v_{ti}^2}\left(-2i\pi^{1/2}\zeta_i e^{-\zeta_i^2}\right) \end{cases}$$

$$= i\pi^{1/2}\left[\frac{2\omega_{pe}^2\omega}{k^3 v_{te}^3}e^{-\zeta_e^2} + \frac{2\omega_{pi}^2\omega}{k^3 v_{ti}^3}e^{-\zeta_i^2}\right] = i\left(\frac{\pi}{2}\right)^{1/2}\left[\frac{\omega_{pe}^2\omega}{k^3 v_{t,e}^3}\cancelto{1}{e^{-\zeta_e^2}} + \frac{\omega_{pi}^2\omega}{k^3 v_{t,i}^3}e^{-\zeta_i^2}\right]$$

$$= i\left(\frac{\pi}{2}\right)^{1/2}\left[\frac{\omega_{pe}^2\omega}{k^3 v_{t,e}^3} + \frac{\omega_{pi}^2\omega}{k^3 v_{t,i}^3}\exp\left(-\frac{\omega^2}{2k^2 v_{t,i}^2}\right)\right]$$

$$\Rightarrow D(k,\omega) = 1 + \frac{\omega_{pe}^2}{k^2 v_{t,e}^2} - \frac{\omega_{pi}^2}{\omega^2} + i\left(\frac{\pi}{2}\right)^{1/2}\left[\frac{\omega_{pe}^2\omega}{k^3 v_{t,e}^3} + \frac{\omega_{pi}^2\omega}{k^3 v_{ti}^3}\exp\left(-\frac{\omega^2}{2k^2 v_{ti}^2}\right)\right] = 0$$

For wavelength longer than Debye length $k\lambda_{De} \ll 1$. we see $\omega \approx kC_s$

Our interest is to find the small correction $\omega = kC_s - i\gamma$. $(= \omega_r - i\gamma)$

Let's find $\gamma$ !!

$$D(k,\omega) \simeq D(k,\omega_r) + (\omega - \omega_r)\frac{dD}{d\omega}\Big|_{\omega=\omega_r} = 0$$

$$D(k,\omega_r) - i\gamma \frac{dD}{d\omega}\Big|_{\omega=\omega_r} = 0 \xrightarrow[\text{balance}]{\text{1st-order}} i\,Im[D(\omega_r)] - i\gamma \frac{d Re[D(\omega_r)]}{d\omega}\Big|_{\omega=\omega_r} = 0$$

$$\Rightarrow \gamma = \frac{Im[D(\omega_r)]}{dRe[D(\omega)]/d\omega|_{\omega=\omega_r}}$$

$$\frac{dRe[D]}{d\omega} = \frac{d}{d\omega}\left(-\frac{\omega_{pi}^2}{\omega^2}\right) = 2\frac{\omega_{pi}^2}{\omega^3} \simeq 2\frac{\omega_{pi}^2}{(kC_s)^3}$$

$$Im(D) = \left(\frac{\pi}{2}\right)^{1/2}\left[\frac{\omega_{pe}^2(kC_s)}{k^3 V_{te}^3} + \frac{\omega_{pi}^2(kC_s)}{k^3 V_{ti}^3}e^{-\frac{(kC_s)^2}{2k^2V_{ti}^2}}\right] \qquad C_s = \sqrt{\frac{T_e}{M}},\ V_{Te} = \sqrt{\frac{T_e}{m}}$$

$$\therefore \gamma = \frac{\left(\frac{\pi}{2}\right)^{1/2}[A+B]}{2\omega_{pi}^2/(kC_s)^3} \rightarrow A = \frac{(kC_s)^3}{2\omega_{pi}^2}\cdot\frac{\omega_{pe}^2 kC_s}{k^3V_{te}^3} = \frac{1}{2}(kC_s)\frac{\omega_{pe}^2}{\omega_{pi}^2}\left(\frac{C_s}{V_{te}}\right)^3 = \frac{1}{2}kC_s\left(\frac{M}{m}\right)\left(\frac{m}{M}\right)^{3/2}$$

$$= \frac{1}{2}kC_s\left(\frac{m}{M}\right)^{1/2}$$

$$B = \frac{(kC_s)^3}{2\omega_{pi}^2}\frac{\omega_{pi}^2(kC_s)}{k^3V_{ti}^3}e^{-\frac{T_e}{2T_i}} = \frac{1}{2}kC_s\left(\frac{C_s}{V_{ti}}\right)^3 e^{-\frac{T_e}{2T_i}}$$

$$= \frac{1}{2}kC_s\left(\frac{T_e}{T_i}\right)^{3/2}e^{-\frac{T_e}{2T_i}}$$

$$\boxed{\therefore \gamma = \frac{1}{2}\left(\frac{\pi}{2}\right)^{1/2}kC_s\left[\left(\frac{m}{M}\right)^{1/2} + \left(\frac{T_e}{T_i}\right)e^{-\frac{T_e}{2T}}\right]}$$

when $T_i \sim T_e$, Landau damping can be rapidly increasing.

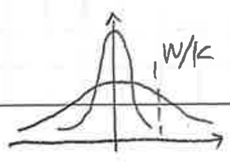
⑨ Physical meaning of Landau damping

It occurs by resonant particles $v = \omega/k$

입자($v \le \omega/k$ 가속 / $v \ge \omega/k$ 감속) by interaction

In Maxwellian $df/dv < 0$ ∴ 항상 $v \le \omega/k$ 가 더 많음 ⇒ wave damping

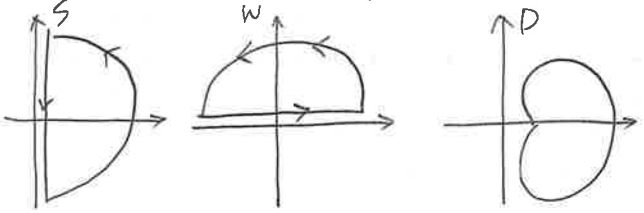
$T_i \ll T_e$ 에서 Landau damping ⇓




( ① few resonant ions
② $df_{Me}/dv \ll 1$ for electrons )

## ⑩ Nyquist Diagram for instability

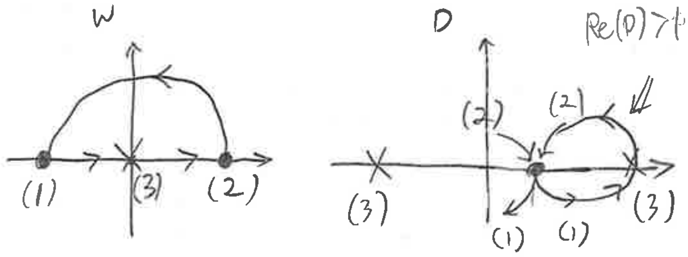
Recall the instability corresponds to zeros of $D(k,s)=0$ with $Re(s)>0$.




If D-contour does not include $D=0$, there is no instability.

If D-contour include $D=0$, or multiple $D=0$, there is unstable mode.

## ⑪ No instability in homogeneous Maxwellian




$$D(k,w) = 1 + \frac{q^2}{mk\epsilon_0}\int_{-\infty}^{\infty} \frac{df_M/dv}{w-kv}$$

Landau correction

(1) : $w \to -\infty$ , $D \to 1$

(2) : $w \to \infty$ , $D \to 1$

$$D(k,w) = 1 + \frac{q^2}{mk\epsilon_0}\left[ P\int_{-\infty}^{\infty} \frac{df_M/dv}{w-kv}dv - \frac{i\pi}{k}\frac{df_M}{dv}\bigg|_{v=w/k}\right]$$

(1) : $w \to -\infty$, $Im(D)<0$ $\Rightarrow$ (1)은 $D=1$에서 아래로 출발

(2) : $w \to \infty$ , $Im(D)>0$ $\Rightarrow$ (2)는 $D=1$에서 위에서 도착.

이제 $w=0$ ((3) point) 에서 D-의 behavior를 확인해보자.

$$w=0 \to v=w/k=0 \to df_M/dv\big|_{w=0}=0 \Rightarrow Im(D)=0.$$

따라서 Re(D)를 체크해보자! (at $w=0$)

$$Re(D) = 1 - \frac{q^2}{mk^2\epsilon_0}\int_{-\infty}^{\infty}\frac{df_M/dv}{v}dv = 1 + \frac{q^2}{mk^2\epsilon_0}\int_{-\infty}^{\infty}\frac{2f_M}{v_t^2}dv = 1 + \frac{1}{k^2v_t^2}\left(\frac{nq^2}{m\epsilon_0}\right)$$

$$= 1 + \frac{w_p^2}{k^2v_t^2} > 1$$

$$\left( * : f_M = \frac{n_0}{\pi^{3/2}v_t^3}\exp\left(-\frac{v^2}{v_t^2}\right),\ \frac{df_M}{dv} = -\frac{2v}{v_t^2}f_M \right)$$

$\therefore Re(D)>1$ 이므로 반드시 오른쪽 path로만 움직인다!