12. Hamilton Jacobi theory

- Hamilton-Jacobi equations
- Separation of variables
- 1-D harmonic oscillator
- Kepler problem

1 Hamilton-Jacobi equations (HJEs)

* Hamilton equation을 만족하는 시스템의 phase-space 경로를 찾는 방법

① $\dot{q} = \frac{\partial H}{\partial p}$, $\dot{p} = -\frac{\partial H}{\partial q}$ ⇒ Integrate

② Find the canonical transformation (and its inverse) joining the given pair of initial and final states.

⇒ HJEs yield the generating function for this canonical transformation.

* Goal

① Find generating function F of canonical transformation $(q, p) \rightarrow (Q, P)$

satisfying $\underline{K(Q, P) = H(q, p, t) + \frac{\partial F}{\partial t} = 0}$

so that $\dot{Q} = \partial K / \partial P = 0$, $\dot{P} = -\partial K / \partial Q = 0$

② Fixing $Q = q_0$, $P = p_0$, F generates the canonical transformation from (q, p) at arbitrary time t to (q_0, p_0), which specifies initial state

③ The trajectory of (q, p) is obtained by inverting the canonical transformation.

* Derivation of the HJEs.

- Choose $F = F_2(\underline{q}, \underline{P}, t) - \underline{Q}\cdot\underline{P}$ $\rightarrow$ $\underline{p} = \frac{\partial F_2}{\partial \underline{q}}$, $\underline{Q} = \frac{\partial F_2}{\partial \underline{P}}$, $K = H + \frac{\partial F_2}{\partial t}$

  to include identity transformation

- $H(\underline{q}, \underline{p}, t) = H(\underline{q}, \frac{\partial F_2}{\partial \underline{q}}, t)$ $\rightarrow$ $\underline{H(\underline{q}, \frac{\partial F_2}{\partial \underline{q}}, t) + \frac{\partial F_2}{\partial t} = 0}$

  (1st-order PDE of (n+1) variable)

- Solution of the above equation has the form $F_2 = S(q_1, \cdots, q_n ; \alpha_1 \cdots \alpha_{n+1} ; t)$

  but α_{n+1} = constant do not change the dynamics. ($\because F_2 \rightarrow F_2 + \text{const}$)

- Hamilton-Jacobi equation : $\boxed{H(\underline{q}, \frac{\partial S}{\partial \underline{q}}, t) + \frac{\partial S}{\partial t} = 0}$ $(S = S(\underline{q}; \underline{\alpha}; t))$

  Hamilton's principal function

* Calculation of phase-space trajectories from the HJEs.

- $S(\underline{q}; \underline{\alpha}; t) = F_2(\underline{q}, \underline{P}, t)$ $\rightarrow$ $\underline{\alpha} = \underline{\alpha}(\underline{P})$

  $\Rightarrow$ $\underline{p} = \frac{\partial S}{\partial \underline{q}} = \underline{p}(\underline{q}; \underline{\alpha}; t)$, $\underline{Q} = \frac{\partial S}{\partial \underline{P}} = \sum_{i=1}^{n} \frac{\partial S}{\partial \alpha_i} \frac{\partial \alpha_i}{\partial \underline{P}} = \sum_{i=1}^{n} \beta_i \frac{\partial \alpha_i}{\partial \underline{P}}$.

  $\Leftrightarrow$ $\underline{\beta} \equiv \frac{\partial S}{\partial \underline{\alpha}} = \underline{\beta}(\underline{q}; \underline{\alpha}; t)$

  Note that $\underline{\alpha} = \underline{\alpha}(\underline{P})$, $\underline{\beta} = \frac{\partial S}{\partial \underline{\alpha}} = \underline{\beta}(\underline{q}; \underline{\alpha}; t)$ are constants of motion

  (운동 전체에 걸쳐 성질변화 x)

  Inverting the above equation for $\underline{\beta}$, we obtain phase space trajectory

  $\underline{\beta}(\underline{q}, \underline{\alpha}, t) \longrightarrow \underline{q}(\underline{\alpha}, \underline{\beta}, t)$

  $\underline{p}(\underline{q}, \underline{\alpha}, t) \longrightarrow \underline{p}(\underline{\alpha}, \underline{\beta}, t)$

  (* $\underline{\alpha}$ and $\underline{\beta}$ can be any constants specifying the initial state.)

* Interpretation of Hamilton's principal function $S(\underline{q};\underline{\alpha};t)$

∘ $$\frac{dS}{dt} = \frac{\partial S}{\partial \underline{q}}\cdot\dot{\underline{q}} + \frac{\partial S}{\partial t} = \underline{p}\cdot\dot{\underline{q}} - H = L \;\rightarrow\; S = \underbrace{\int_{t_0}^{t} dt\, L}_{I[\underline{q}(\underline{\alpha},\underline{\beta},t)]} + \text{const}$$

• $S(\underline{q};\underline{\alpha};t)$ = Action accummulated by the actual dynamical trajectory from a given initial condition (constrained by $\underline{\alpha}$) to the position $\underline{q}$ at time t.

$\boxed{2}$ Separation of variables

HJEs can be simplified to lower dimensional problems in the following cases.

∘ Case $\partial H/\partial t = 0$

Solution of HJE : $S(\underline{q};\underline{\alpha};t) = W(\underline{q};\underline{\alpha}) - \alpha_1 t$, $H\left(\underline{q}, \frac{\partial W}{\partial \underline{q}}\right) = \alpha_1$

Hamilton's characteristic function $\left(H\left(\underline{q}, \frac{\partial S}{\partial \underline{q}}, t\right) + \frac{\partial S}{\partial t} = 0\right)$

∘ Case $\partial H/\partial t = \partial H/\partial q_{n-k+1} = \cdots = \partial H/\partial q_n = 0$ (k개의 cyclic coordinates)

Solution of HJE : $S(\underline{q};\alpha;t) = W(\underline{q};\alpha) - \alpha_1 t$,

Hamilton's characteristic function $W(\underline{q};\underline{\alpha}) = W_{[n-k+1,\cdots,n]}(q_1,\cdots,q_{n-k};\underline{\alpha}) + \sum_{i=n-k+1}^{n} \alpha_i q_i$

where $W_{[n-q+1,\cdots,n]}$ satisfies $H\left(q_1,\cdots,q_{n-k}, \frac{\partial W[\,]}{\partial q_1}, \cdots, \frac{\partial W[\,]}{\partial q_{n-k}}, \alpha_{n-k+1},\cdots,\alpha_n\right) = \alpha_1$

∘ Case $H = H(\underline{q},\underline{p}; f(\tilde{\underline{q}},\tilde{\underline{p}}))$

Hamilton's characteristic function $W(\underline{q},\tilde{\underline{q}};\underline{\alpha}) = W_{\underline{q}}(\underline{q};\underline{\alpha}) + W_{\tilde{\underline{q}}}(\tilde{\underline{q}};\underline{\alpha})$

$$H\left(\underline{q}, \frac{\partial W_{\underline{q}}}{\partial \underline{q}}; f\left(\tilde{\underline{q}}, \frac{\partial W_{\tilde{\underline{q}}}}{\partial \tilde{\underline{q}}}\right)\right) = \alpha_1 \implies f\left(\tilde{\underline{q}}, \frac{\partial W_{\tilde{\underline{q}}}}{\partial \tilde{\underline{q}}}\right) = g\left(\underline{q}, \frac{\partial W_{\underline{q}}}{\partial \underline{q}}\right)$$

$$\therefore f\left(\tilde{\underline{q}}, \frac{\partial W_{\tilde{\underline{q}}}}{\partial \tilde{\underline{q}}}\right) = \tilde{\alpha}, \quad g\left(\underline{q}, \frac{\partial W_{\underline{q}}}{\partial \underline{q}}\right) = \tilde{\alpha}$$

※ HJEs can be solved in a closed form when it is completely separable.

3 1-d harmonic oscillator

$$H(q,p) = \frac{1}{2m}(p^2 + m w^2 q^2)$$

$dH/dt = 0 \longrightarrow$ Hamilton's principal function $S(q;E;t) = W(q;E) - Et$

$$H(q, \frac{dW}{dq}) = \frac{1}{2m}\left(\left(\frac{dW}{dq}\right)^2 + m w^2 q^2 \right) = E \Rightarrow \left(\frac{dW}{dq}\right) = \pm (2mE - m w^2 q^2)^{1/2}$$

$$S(q;E;t) = \pm \int dq\, (2mE - m^2 w^2 q^2)^{1/2} - Et$$

$$\Rightarrow t_0 \equiv -\frac{dS}{dE} = \mp \int dq \frac{m}{(2mE - m^2 w^2 q^2)^{1/2}} + t = \mp \frac{1}{w} \arcsin\left(q \sqrt{\frac{m w^2}{2E}}\right) + t$$

$$\therefore q = \pm \sqrt{\frac{2E}{m w^2}} \sin(w(t - t_0)) \quad \left(-\frac{\pi}{2} \le w(t-t_0) \le \frac{\pi}{2}\right)$$

$$p = \frac{dS}{dq} = (2mE - m^2 w^2 q^2)^{1/2} = \pm \sqrt{2mE} \cos(w(t-t_0))$$

$$\Rightarrow \boxed{q = \sqrt{\frac{2E}{m w^2}} \sin[w(t-t_0)], \quad p = \sqrt{2mE} \cos[w(t-t_0)]}$$

4 Kepler problem (in the plane of rotation)

$$L(r, \dot r, \dot\theta) = \frac{1}{2} m (\dot r^2 + r^2 \dot\theta^2) + \frac{k}{r} \longrightarrow p_r = \frac{dL}{d\dot r} = m\dot r, \quad p_\theta = \frac{dL}{d\dot\theta} = m r^2 \dot\theta$$

$$H(r, p_r, p_\theta) = p_r \dot r + p_\theta \dot\theta - L = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) - \frac{k}{r}$$

$dH/dt = 0$, $dH/d\theta = 0 \longrightarrow S(r, \theta; E, L; t) = W(r) + L\theta - Et$

$$H(r, \frac{dW_r}{dr}) = \frac{1}{2m}\left[\left(\frac{dW_r}{dr}\right)^2 + \frac{L^2}{r^2} \right] - \frac{k}{r} = E \Rightarrow \frac{dW_r}{dr} = \pm \left[2m\left(E + \frac{k}{r}\right) - \frac{L^2}{r^2} \right]^{1/2}$$

$$\therefore S(r, \theta; E, L; t) = \pm \int dr \left[2m\left(E + \frac{k}{r}\right) - \frac{L^2}{r^2} \right]^{1/2} + L\theta - Et$$

$$\theta_0 \equiv \frac{dS}{dL} = \theta \mp \int \frac{L\, dr / r^2}{[2m(E + k/r) - L^2/r^2]^{1/2}} \underset{\left(\frac{1}{r} = u\right)}{=} \theta \pm \int \frac{\mathrm{sign}(L)\, du}{[(2m/L^2)(E + ku) - u^2]^{1/2}}$$

$$= \theta \pm \int \frac{\mathrm{sign}(L)\, du}{[2mE/L^2 + 2mk/L^2 \cdot u - u^2]^{1/2}} \underset{\uparrow}{=} \theta \mp \mathrm{sign}(L) \arccos \frac{L^2/mkr - 1}{[1 + 2EL^2/mk^2]^{1/2}}$$

$$\int \frac{dx}{(\alpha + \beta x - x^2)^{1/2}} = -\arccos \frac{2x - \beta}{(\beta^2 + 4\alpha)^{1/2}} = -\arccos \frac{\frac{2}{\beta}x - 1}{\left(1 + \frac{4\alpha}{\beta^2}\right)^{1/2}}$$

$$\therefore \theta_0 = \theta \mp \mathrm{sign}(L)\arccos\frac{L^2/mkr - 1}{e} \Rightarrow \boxed{r = \frac{L^2/mk}{1+e\cos(\theta-\theta_0)}}$$

<Summary>

① Hamiltonian 정의

② cyclic coordinate 찾기

③ $S(q;E;t) = W(q;E) - Et$ / $S(r,\theta;L,E;t) = W(r) + L\theta - Et$

④ $H(q, \frac{dW}{dq})$ $(= H(r, \frac{dW_r}{dr}))$ $\longrightarrow \left(\frac{dW}{dq} = \sim\right) \rightarrow \left(S = \int dq \left(\frac{dW}{dq}\right) (+L\theta) - Et\right)$

⑤ $\theta_0 = \frac{dS}{dL}$ / $t_0 = -\frac{dS}{dE}$