13. Action-angle variable

(Action-angle variables and quasi-periodic motion
Application to the Kepler problem)

1 Action-angle variables and quasi-periodic motion

A. Definition of action-angle variables

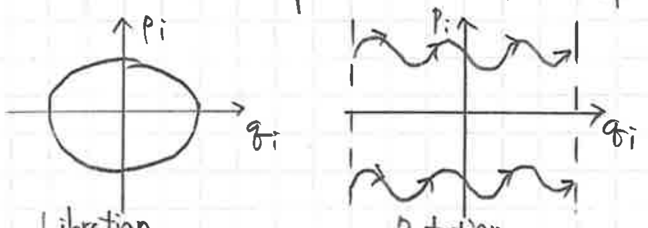
- Assumption

① HJEs of the system is <u>completely separable</u> (integrable)

$$\begin{cases} S(q_1,\cdots,q_n;\alpha_1,\cdots,\alpha_n;t) = \sum_{i=1}^{n} W_i(q_i;\alpha_1,\cdots,\alpha_n) - \alpha_1 t \\ p_i = \dfrac{\partial S}{\partial q_i} = \dfrac{\partial W_i}{\partial q_i} = p_i(q_i;\alpha_1,\cdots,\alpha_n) \end{cases}$$

← momentum depends only on its conjugate position.

② The motion of each pair of (q_i, p_i) is <u>periodic.</u>



Libration　　　　Rotation

- Definitions

← 가정 ①, ② 모두 적용.

action variable $\boxed{J_i \equiv \oint dq_i\, p_i(q_i;\underline{\alpha}) = J_i(\underline{\alpha})}$

$J_i(\underline{\alpha})$ 이므로 , $W(\underline{q};\underline{\alpha}) = W(\underline{q};\underline{J}) = \sum_{i=1}^{n} W_i(q_i;\underline{J})$

angle variable $\boxed{w_i = \dfrac{\partial W}{\partial J_i} = \sum_{j=1}^{n} \dfrac{\partial W_j(q_j;\underline{\alpha})}{\partial J_i}}$, $\partial W/\partial t = 0 \rightarrow K = H$

- Periodicity of motion in angle variables

If each q_j changes by m_j cycles, then angle variable w_i changes by

$$\boxed{\Delta w_i} = \sum_{j=1}^{n} m_j \oint dq_j \frac{\partial w_i}{\partial q_j} = \sum_{j=1}^{n} m_j \oint dq_j \frac{\partial^2 W}{\partial J_i\, \partial q_j} = \frac{\partial}{\partial J_i}\sum_{j=1}^{n} m_j \oint dq_j\, p_j = \frac{\partial}{\partial J_i}\sum_{j=1}^{m} m_j J_j = \boxed{m_i}$$

So q_i completes a single period as w_i increases by 1.

B. Quasiperiodic motion

- Hamilton's equations (w_i, J_i)

  Hamiltonian does not depend on angle variables : $H = \alpha_1(J_1, \cdots, J_n)$

  (용수철 진자의 phase 와 무관하게, conserved 된 system의 에너지는 일정하다.)

$$\boxed{\begin{aligned} &① \quad \dot{J}_i = -\frac{\partial H}{\partial w_i} = 0 \\ &② \quad \dot{w}_i = \frac{\partial H}{\partial J_i} \equiv \nu_i(J_1, \cdots, J_n) \end{aligned}}$$

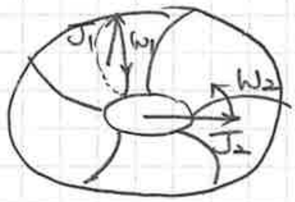
$\Leftarrow$ J_i is constant of motion

$\Leftarrow$ ν_i is constant of motion

$\hookrightarrow$ $\underline{w_i = \nu_i t + \beta_i}$ (ν_i = frequency of periodic motion of q_i)

- Invariant tori

  Given two action-angle pairs (w_1, J_1), (w_2, J_2)



< Invariant tori >

ν_1/ν_2 = rational $\rightarrow$ w_1 & w_2 are <u>commensurate</u>

ν_1/ν_2 = irrational $\rightarrow$ w_1 & w_2 are <u>incommensurate</u>

- Degeneracies of frequencies

  Given integrable system with n frequencies,

  its motion is said to be <u>m-fold degenerate</u> (or $(n-m)$ fold periodic)

  if there exists integers <u>j_{ki}</u> such that $\underline{\sum_{i=1}^{n} j_{ki}\nu_i = 0 \quad \text{for } k = 1, \cdots, m}$

  In this case, point transformation $(\underline{w}, \underline{J}) \rightarrow (\underline{w}', \underline{J}')$

  is generated by $\underline{F_2(\underline{w}, \underline{J}') = \sum_{k=1}^{m} J_k' \sum_{i=1}^{n} j_{ki} w_i + \sum_{k=m+1}^{n} J_k' w_k}$.

$$w_k' = \frac{\partial F_2}{\partial J_k'} = \begin{cases} \sum_{i=1}^{n} j_{ki} w_i & (k = 1, \cdots, m) \\ w_k & (k = m+1, \cdots, n) \end{cases}, \quad J_i = \frac{\partial F_2}{\partial w_i} = \sum_{k=1}^{m} J_k' j_{ki} + \sum_{k=m+1}^{n} J_k' \delta_{ki} \quad (i = 1, \cdots, n)$$

$$\nu_k' = \dot{w}_k' = \begin{cases} \sum_{i=1}^{n} j_{ki}\nu_i = 0 & (k = 1, \cdots, m) \\ \nu_k & (k = m+1, \cdots, n) \end{cases}$$

$\Leftarrow$ (w_i', J_i') coordinate 에서 m-fold degenerate는 진동 X, 나머지는 진동 O

C. Simple example : 1-d harmonic oscillator

$$H = \frac{1}{2m}(p^2 + m^2w^2q^2) = E \quad \longrightarrow \quad p = \pm(2mE - m^2w^2q^2)^{1/2}$$

$$J \equiv \oint p\,dq = 2\int_{-\sqrt{\frac{2E}{mw^2}}}^{\sqrt{\frac{2E}{mw^2}}} dq\,(2mE - m^2w^2q^2)^{1/2} = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\frac{2E}{mw^2}}\cos\theta\,(2mE - 2mE\sin^2\theta)^{1/2}\,d\theta$$

$$\left(q = \sqrt{\frac{2E}{mw^2}}\sin\theta\right)$$

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2\frac{E}{w}\cos^2\theta = \frac{2\pi E}{w} \qquad \therefore \underline{J = \frac{2\pi E}{w}} \rightarrow \underline{H = E = \frac{wJ}{2\pi}}$$

$$\underline{J = \frac{2\pi E}{w} \rightarrow H = E = \frac{wJ}{2\pi} \longrightarrow \dot{w} = \nu = \frac{\partial H}{\partial J} = \frac{w}{2\pi}} \quad : \omega = \frac{w}{2\pi}(t - t_0),$$

From $q = \sqrt{\frac{2E}{mw^2}}\sin w(t-t_0)$, $p = \sqrt{2mE}\cos w(t-t_0)$.

we know $\underline{q = \left(\frac{J}{\pi m w}\right)^{1/2}\sin(2\pi\omega), \quad p = \left(\frac{mJw}{\pi}\right)^{1/2}\cos(2\pi\omega)}$

$\boxed{2}$ Application to the Kepler problem (3-d space)

- H in spherical coordinate system

$$L = \frac{m}{2}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2) + \frac{k}{r}$$

$$\left(p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}, \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}, \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\sin^2\theta\,\dot{\phi}\right)$$

$$H = p_r\dot{r} + p_\theta\dot{\theta} + p_\phi\dot{\phi} - L = \frac{1}{2m}\left[p_r^2 + \frac{1}{r^2}\left(p_\theta^2 + \frac{p_\phi^2}{\sin^2\theta}\right)\right] - \frac{k}{r}$$

- HJEs and the separation of variables

$$\frac{\partial H}{\partial \phi} = 0, \ \frac{\partial H}{\partial t} = 0 \ \longrightarrow \ S = W_r(r) + W_\theta(\theta) + L_z\phi - Et$$

$$H\left(r,\theta,\phi : \frac{\partial S}{\partial r}, \frac{\partial S}{\partial \theta}, \frac{\partial S}{\partial \phi}\right) + \frac{\partial S}{\partial t} = 0 \ \longrightarrow \ p_r = \frac{dW_r}{dr}, \ p_\theta = \frac{dW_\theta}{d\theta}, \ p_\phi = L_z.$$

$\therefore$ HJEs reduces to $\underline{\left(\frac{dW_r}{dr}\right)^2} + \frac{1}{r^2}\Big[\underbrace{\left(\frac{dW_\theta}{d\theta}\right)^2 + \frac{L_z^2}{\sin^2\theta}}_{f(\theta)}\Big] \underbrace{- \frac{2mk}{r} = 2mE}_{g(r)}$

$$\underline{\therefore \left(\frac{dW_\theta}{d\theta}\right)^2 + \frac{L_z^2}{\sin^2\theta} = L^2, \quad \left(\frac{dW_r}{dr}\right)^2 + \frac{L^2}{r^2} = 2m\left(E + \frac{k}{r}\right)} \,//$$

- Action variables

① $J_\phi = \oint d\phi\, p_\phi = \boxed{2\pi L_z}$

$\cos i = L_z/L$

② $J_\theta = \oint d\theta\, p_\theta = \oint d\theta \frac{dW_\theta}{d\theta} = \oint d\theta\, L\left(1-\frac{L_z^2}{L^2\sin^2\theta}\right)^{1/2} = 2L\int_{\frac{\pi}{2}-i}^{\frac{\pi}{2}+i} d\theta\left(1-\cos^2 i\csc^2\theta\right)^{1/2}$

$= 2L\int_{\frac{\pi}{2}-i}^{\frac{\pi}{2}+i} d\theta\csc\theta\left(\sin^2\theta-\cos^2 i\right)^{1/2} = 2L\int_{-i}^{i} d\theta\sec\theta\left(\cos^2\theta-\cos^2 i\right)^{1/2}$

$\theta = \theta + \frac{\pi}{2}$

$= 2L\int_{-i}^{i} d\theta\sec\theta\left(\cos^2\theta - 1 + 1 - \cos^2 i\right)^{1/2} = 2L\int_{-i}^{i} d\theta\sec\theta\left(\sin^2 i - \sin^2\theta\right)^{1/2}$

$= 4L\int_0^i d\theta\,\sec\theta\left(\sin^2 i - \sin^2\theta\right)^{1/2} = 4L\int_0^{\frac{\pi}{2}}\left(\sin^2 i\left(1-\sin^2\psi\right)\right)^{1/2}\frac{\cos\theta\, d\theta}{\cos^2\theta}$

$\left(\begin{array}{l}\sin\theta = \sin i\sin\psi \\ \cos\theta\, d\theta = \sin i\cos\psi\, d\psi\end{array}\right)$

$= 4L\int_0^{\frac{\pi}{2}}\sin i\cos\psi\,\frac{\sin i\cos\psi}{1-\sin^2 i\sin^2\psi}\,d\psi = 4L\sin^2 i\int_0^{\frac{\pi}{2}}\frac{\cos^2\psi}{1-\sin^2 i\sin^2\psi}\,d\psi$

$= 4L\sin^2 i\int_0^{\frac{\pi}{2}}\frac{\cos^2\psi}{1-\left(1-\cos^2 i\right)\sin^2\psi}\,d\psi = 4L\sin^2 i\int_0^{\frac{\pi}{2}}\frac{\cos^2\psi}{1+\cos^2 i\sin^2\psi-\sin^2\psi}\,d\psi$

$= 4L\sin^2 i\int_0^{\frac{\pi}{2}}\frac{\cos^2\psi}{\cos^2\psi+\cos^2 i\sin^2\psi}\,d\psi = 4L\sin^2 i\int_0^{\frac{\pi}{2}}\frac{1}{1+\cos^2 i\tan^2\psi}\,d\psi$

$= 4L\sin^2 i\int_0^{\infty}\frac{du}{\left(1+u^2\right)\left(1+u^2\cos^2 i\right)} = 4L\int_0^{\infty}du\left(\frac{1}{1+u^2}-\frac{\cos^2 i}{1+u^2\cos^2 i}\right)$

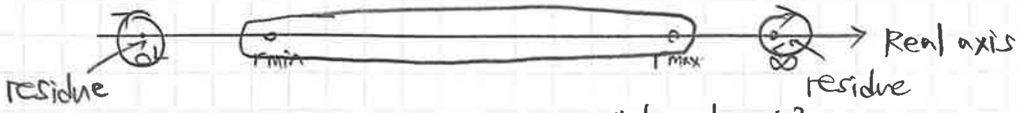
$u = \tan\psi$

$= 4L\left[\arctan u - \cos i\arctan\left(u\cos i\right)\right]_0^{\infty} = 2\pi L\left(1-\cos i\right) = \boxed{2\pi\left(L-L_z\right)}$

③ $J_r = \oint dr\, p_r = \oint dr \frac{dW_r}{dr} = 2\int_{r_{min}}^{r_{max}} dr \left[2m\left(E+\frac{k}{r}\right) - \frac{L^2}{r^2}\right]^{1/2}$ where $\left(\frac{dW_r}{dr}\right)(r=r_{min}, r=r_{max}) = 0$

* convert it to a contour integral in complex plane.

$$2\int_{r_{min}}^{r_{max}} dr \left[2m\left(E+\frac{k}{r}\right) - \frac{L^2}{r^2}\right]^{1/2} = \oint_c dr \left[2m\left(E+\frac{k}{r}\right) - \frac{L^2}{r^2}\right]^{1/2}$$



*branch cut?

* Method of residues

i) $\left[2m\left(E+\frac{k}{r}\right) - \frac{L^2}{r^2}\right]^{1/2} = -\frac{iL}{r} + \frac{imk}{L} + \cdots$ (near $r=0$)

제일 큰 항　　이건?

$\Rightarrow r_1 = -iL$

ii) $\oint dz \left(-\frac{1}{z^2}\left[2m(E+kz) - L^2z^2\right]^{1/2}\right) = -\oint\left(-\frac{i\sqrt{-2mE}}{z^2} + ik\sqrt{\frac{m}{2(-E)}}\frac{1}{z}\right)dz$ (near $z=0$)

$\Rightarrow r_2 = ik\sqrt{m/(-2E)}$

$$\Rightarrow \oint_c dr\left[2m\left(E+\frac{k}{r}\right) - \frac{L^2}{r^2}\right]^{1/2} = (-2\pi i)\left(-iL + ik\sqrt{\frac{m}{-2E}}\right) = \boxed{\pi k\sqrt{\frac{2m}{-E}} - 2\pi L}$$

$$\therefore J_r = \pi k\sqrt{\frac{2m}{-E}} - (J_\theta + J_\phi) \implies J_r + J_\theta + J_\phi = \pi k\sqrt{\frac{2m}{-E}}$$

$$\therefore \boxed{H = E = -\frac{2m\pi^2k^2}{(J_r+J_\theta+J_\phi)^2}}, \quad \boxed{\nu = \frac{\partial H}{\partial J_1} = \frac{\partial H}{\partial J_\theta} = \frac{\partial H}{\partial J_\phi} = \frac{4m\pi^2k^2}{(J_r+J_\theta+J_\phi)^3}}$$

↳ Kepler problem is closed and periodic.

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* Method of residues 구체적인 이항전개.

① $\left[2m\left(E+\frac{k}{r}\right) - \frac{L^2}{r^2}\right]^{1/2} = -\frac{L}{r}i\left[1 - \frac{2mk}{L^2}r + \cdots\right]^{1/2} = -\frac{L}{r}i\left(1 - \frac{mk}{L^2}r + \cdots\right) = -\frac{iL}{r} + \frac{imk}{L} + \cdots$

② $-\frac{1}{z^2}\left[2m(E+kz) - L^2z^2\right]^{1/2} = -\frac{1}{z^2}\sqrt{2mE}\left[1 + \frac{k}{E}z + \cdots\right]^{1/2} = -\frac{i}{z^2}\sqrt{2m(-E)}\left(1 + \frac{k}{2E}z\right)$

$$= -\frac{i}{z^2}\sqrt{2m(-E)} + ik\sqrt{\frac{m}{2(-E)}}\frac{1}{z} + \cdots$$