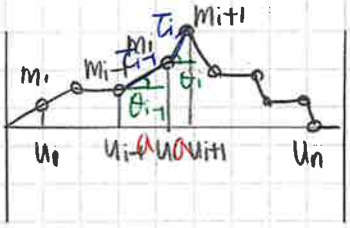
7. Lagrangian formalism in the continuum

① Example : 1-d string

(A) Discrete model



(i) $m_i \ddot{u}_i = \tau_i \sin\theta_i - \tau_{i-1} \sin\theta_{i-1}$ (for $i = 1, 2, \cdots, n$)

$u_0 = u_{n+1} = 0$

(ii) assume $|\theta| \ll 1$.

$$\boxed{m_i \ddot{u}_i = \tau_i \tan\theta_i - \tau_{i-1} \tan\theta_{i-1} = \tau_i \frac{u_{i+1} - u_i}{a} - \tau_{i-1} \frac{u_i - u_{i-1}}{a}}$$

(iii) $$\boxed{L = \sum_{i=1}^{n} \left[\frac{1}{2} m_i \dot{u}_i^2 - \frac{1}{2} \frac{\tau_i}{a} (u_{i+1} - u_i)^2 \right]}$$

$$\frac{\partial L}{\partial u_i} = \sum_{i=1}^{n} \frac{\tau_i}{a} (u_{i+1} - u_i) \quad , \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{u}_i}\right) = \sum_{i=1}^{n} m \ddot{u}_i$$

(B) Continuum limit

$$\left(\begin{array}{l} \text{limit} : a \to 0, n \to \infty \\ \text{constraint} : \ell = (n+1) a \end{array} \right) \to \text{변수변환 } x = ia \left(\begin{array}{l} u_i(t) = u(ia, t) = u(x, t) \\ \tau_i(t) = \tau(ia, t) = \tau(x, t) \\ m_i / a = \lambda(ia) = \lambda(x) \end{array} \right)$$

(경계조건 : $u(0, t) = u(\ell, t) = 0$)

$$\frac{m_i \ddot{u}_i}{a} = \frac{1}{a} \left[\tau_i \frac{u_{i+1} - u_i}{a} - \tau_{i-1} \frac{u_i - u_{i-1}}{a} \right] \to \boxed{\lambda(x) \frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}\left[\tau(x) \frac{\partial u}{\partial x} \right]}$$

for constant λ and τ, $\frac{\partial^2 u}{\partial t^2} = \frac{\tau}{\lambda} \frac{\partial^2 u}{\partial x^2} = C_s^2 \frac{\partial^2 u}{\partial x^2} \to u_n^{\pm}(x, t) = \sin\left(\frac{2\pi x}{\ell}\right) e^{\pm i w_n t}$

$\left(w_n = \frac{n \pi C_s}{\ell} \right)$

- Lagrangian density

$$L = \sum_{i=1}^{n} a \left[\frac{1}{2} \frac{m_i}{a} \dot{u}_i^2 - \tau_i \left(\frac{u_{i+1} - u_i}{a} \right)^2 \right] \to \boxed{\int_0^{\ell} dx \left[\underline{\frac{1}{2} \lambda(x) \left(\frac{\partial u}{\partial t} \right)^2 - \tau_i \left(\frac{\partial u}{\partial x} \right)^2} \right]}$$

$\mathcal{L}\left(u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x} ; x, t\right)$: Lagrangian density

(변위, $\dot{\text{변위}}$, $\vec{\nabla}$변위) 의 범함수

② Hamilton's principle in the continuum

(A) Hamilton's principle

- $x_\nu = (x_0, x_1, \cdots, x_d) = (t, \underline{x})$, / $u_1, u_2, \cdots$: u_p = p번째 field

  (treating time and space equally)

notation $\underline{u_{p,\nu} \equiv du_p / dx_\nu}$ (p번째 field를 ν방향으로 미분)

- Hamilton's principle : $\boxed{\delta \int_\Omega d^{d+1}\underline{x}\, \mathcal{L}(u_p, u_{p,\nu} : x_\nu) = 0}$ (시공간적분)

  for <u>variations</u>, with $\delta u_p(x_\nu) = 0$ at $x_\nu \in d\Omega$

  $(u_p(x_\nu) \rightarrow u_p(x_\nu) + \delta u_p(x_\nu))$

(B) Derivation of Lagrange's equations

$$\delta \int_\Omega d^{d+1}\underline{x}\, L(u_p, u_{p,\nu}; x_\nu) = \int_\Omega d^{d+1}x \left(\frac{dL}{du_p} \cdot \delta u_p + \frac{dL}{du_{p,\nu}} \overset{\frac{d}{dx_\nu}\delta u_p}{\overset{\shortparallel}{\delta u_{p,\nu}}} \right)$$

$$= \int_\Omega d^{d+1}\underline{x} \left(\frac{dL}{du_p} + \frac{d}{dx_\nu}\left(\frac{dL}{du_{p,\nu}}\right)\right) \delta u_p = 0$$

$$\therefore \boxed{\frac{dL}{du_p} - \frac{d}{dx_\nu}\left(\frac{dL}{du_{p,\nu}}\right) = 0}$$

* note : $\frac{d}{dx_\nu}$ 할때, x_ν 에 의존적인 u 등은 그대로 고려해야함

Application for $\mathcal{L} = \frac{1}{2}\lambda(x)\left(\frac{du}{dt}\right)^2 - \frac{1}{2}\tau(x)\left(\frac{du}{dx}\right)^2$

$$\underline{\frac{dL}{du} - \frac{d}{dt}\left(\frac{dL}{d(du/dt)}\right) - \frac{d}{dx}\left(\frac{dL}{d(du/dx)}\right)} = -\frac{d}{dt}\left(\lambda(x)\frac{du}{dt}\right) - \frac{d}{dx}\left(-\tau(x)\frac{du}{dx}\right) = 0$$

$$\Rightarrow \lambda(x)\frac{d^2u}{dt^2} - \frac{d}{dx}\left(\tau(x)\frac{du}{dx}\right) = 0 \quad \leftarrow \text{1-D string eqn}$$

③ Symmetries and conservation laws

(A) Noether's theorem

. infinitesimal coordinate and field transformation

$$x_\mu \to \underline{x_\mu' = x_\mu + \delta x_\mu} \,, \quad u_\rho(x_\mu) \to \underline{u_\rho'(x_\mu') = u_\rho(x_\mu) + \delta u_\rho(x_\mu)}$$

. symmetry condition

$$\boxed{\int_{\Omega'} d^{d+1}x_\mu' \, L(u_\rho', u_{\rho,\mu}' ; x_\mu') = \int_\Omega d^{d+1}x_\mu \left[L(u_\rho, u_{\rho,\mu} ; x_\mu) + \frac{d}{dx_\nu}\delta\Lambda_\nu \right]}$$

(만약 transformation이 action을 위처럼 바꾸면, 시스템의 역학은 transformation에 대해 cyclic/invariant 하다)

. Derivation of the conservation law

$$\int_{\Omega'} d^{d+1}x_\mu' \, L(u_\rho', u_{\rho,\mu}' ; x_\mu') = \int_\Omega d^{d+1}x_\mu \, L(u_\rho', u_{\rho,\nu}' ; x_\mu) + \int_{\partial\Omega} dS_\nu \, \delta x_\nu \, L(u_\rho, u_{\rho,\nu} ; x_\mu)$$

(발산정리) $\Rightarrow$

$$= \int_\Omega d^{d+1}x_\mu \left[L(u_\rho', u_{\rho,\nu}' ; x_\mu) + \frac{d}{dx_\nu}\delta x_\nu \, L(u_\rho, u_{\rho,\nu} ; x_\mu) \right]$$

* writing $u_\rho'(x_\mu) = u_\rho(x_\mu) + \bar{\delta} u_\rho(x_\mu)$ $\quad [\bar{\delta} u_\rho \neq \delta u_\rho]$

$$L(u_\rho', u_{\rho,\nu}' ; x_\mu) = L(u_\rho, u_{\rho,\nu} ; x_\mu) + \frac{\partial L}{\partial u_\rho}\bar{\delta} u_\rho + \frac{\partial L}{\partial u_{\rho,\nu}}\bar{\delta} u_{\rho,\nu}$$

$$= L(u_\rho, u_{\rho,\nu} ; x_\mu) + \frac{d}{dx_\nu}\left(\frac{\partial L}{\partial u_{\rho,\nu}} \cdot \bar{\delta} u_\rho \right)$$

$$\therefore \int_\Omega d^{d+1}x_\mu \frac{d}{dx_\nu}\left(\frac{\partial L}{\partial u_{\rho,\nu}} \cdot \bar{\delta} u_\rho + L\,\delta x_\nu - \delta\Lambda_\nu \right) = 0$$

$$\Rightarrow \boxed{\frac{d}{dx_\nu}\left(\frac{\partial L}{\partial u_{\rho,\nu}} \cdot \bar{\delta} u_\rho + L\,\delta x_\nu - \delta\Lambda_\nu \right) = 0}$$

· parametrization of infinitesimal transformation

$$\delta x_\mu = \epsilon_r X_{r\mu}, \quad \delta u_\rho = \epsilon_r U_{r\rho}, \quad \delta \Lambda_\mu = \epsilon_r G_{r\mu} \quad (\epsilon_r,\ r = 1, 2, \cdots, R)$$

* note : $\left(\underline{\delta u_\rho} = u_\rho'(x_\mu') - u_\rho(x_\mu) = u_\rho'(x_\mu') - u_\rho'(x_\mu) + \bar{\delta} u_\rho \simeq \underline{\delta x_\mu u_{\rho,\mu} + \bar{\delta} u_\rho}\right)$

$$\bar{\delta} u_\rho = \epsilon_r \left(U_{r\rho} - u_{\rho,\mu} X_{r\mu}\right)$$

$$\therefore \frac{\partial}{\partial x_\nu}\left[\left(L\delta_{\mu\nu} - \frac{\partial L}{\partial u_{\rho,\nu}} u_{\rho,\mu}\right) X_{r\mu} + \frac{\partial L}{\partial u_{\rho,\nu}} U_{r\rho} - G_{r\nu}\right] = 0$$

$$\boxed{\begin{aligned}\frac{\partial}{\partial t}&\left(L X_{r0} - \frac{\partial L}{\partial u_{\rho,0}} u_{\rho,\mu} X_{r\mu} + \frac{\partial L}{\partial u_{\rho,0}} U_{r\rho} - G_{r0}\right) \\ &= -\sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left[\left(L\delta_{i\mu} - \frac{\partial L}{\partial u_{\rho,i}} u_{\rho,\mu}\right) X_{r\mu} + \frac{\partial L}{\partial u_{\rho,i}} U_{r\rho} - G_{ri}\right]\end{aligned}}$$

Noether charge is locally conserved.