2.2 Noether's theorem

$\left(\begin{array}{l} \text{Canonical momentum} \\ \text{Cyclic coordinates and conservation laws} \\ \text{Noether's theorem} \end{array} \right)$

I Canonical momentum

- $L = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{\underline{r}}_i^2 - V(\underline{r}_1, \cdots, \underline{r}_n)$

For each particle, $\left. \begin{array}{l} \frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{r}}_i} = \frac{d}{dt} m_i \dot{\underline{r}}_i = \frac{d}{dt}\underline{p}_i \\ \frac{\partial L}{\partial \underline{r}_i} = -\frac{\partial V}{\partial \underline{r}_i} = F_i \end{array} \right) \Rightarrow F_i = \dot{p}_i$

- Generalized Newton's Law

$\left(\begin{array}{ll} \text{canonical momentum} & p_j(q_j, \dot{q}_j, t) = \frac{\partial L}{\partial \dot{\underline{r}}_i} \\ \text{generalized force} & Q_j(q_j, \dot{q}_j, t) = \frac{\partial L}{\partial r_i} \end{array} \right.$

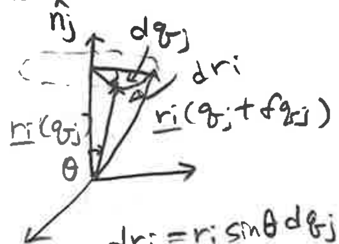
$\rightarrow$ E-L eqn restated as $\dot{p}_j = Q_j$.

<u>Note</u> : canonical momentum $\neq$ mechanical momentum

$$L = \frac{1}{2} m \dot{\underline{r}}^2 - q\phi(\underline{r}) + q\underline{A}(\underline{r}) \cdot \dot{\underline{r}}$$

$$p = \frac{\partial L}{\partial \dot{\underline{r}}} = m\dot{\underline{r}} + \underline{q\underline{A}(\underline{r})}$$ $\leftarrow$ field contribution to momentum

- Application to angular momentum and torque



$L = \sum_{i=1}^{n} \frac{1}{2} m \dot{\underline{r}}_i^2 - V(\underline{r}_1, \cdots, \underline{r}_n)$

$dr_i = r_i \sin\theta \, dq_j$

$dr_i = \hat{\underline{n}}_j \times \underline{r}_i \, dq_j$

$p_j = \frac{\partial L}{\partial \dot{q}_j} = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{r}_i} \cdot \frac{\partial \dot{r}_i}{\partial \dot{q}_j} = \sum_{i=1}^{n} m\dot{\underline{r}}_i \cdot \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{n} m \dot{r}_i \cdot (\hat{n}_j \times \underline{r}_i)$

$= \hat{n}_j \cdot \sum_{i=1}^{n} \underline{r}_i \times m\dot{r}_i = \hat{n}_j \cdot \underline{L}$ $\rightarrow$ p_j = angular momentum component along axis of rotation

$Q_j = \frac{\partial L}{\partial q_j} = \sum_{i=1}^{n} \frac{\partial L}{\partial r_i} \frac{\partial r_i}{\partial q_j} = \sum_{i=1}^{n} F_i \, \hat{n}_j \times \underline{r}_i = \hat{n}_j \cdot \sum_{i=1}^{n} \underline{r}_i \times \underline{F}_i = \hat{n}_j \cdot \underline{\tau}$

$\rightarrow Q_j$ = torque component along the axis of torsion

$\Rightarrow \boxed{\hat{n}_j \cdot \dot{\underline{L}} = \hat{n}_j \cdot \underline{\tau}}$

2 Cyclic coordinates and conservation laws

- conservation of canonical momentum

$\left(p_j = \frac{\partial L}{\partial \dot{q}_j}\right)$: if q_j is cyclic/ignorable coordinate, its conjugate momentum p_j is conserved.

: if L is symmetric/invariant under infinitesimal transformation $q_j \to q_j + dq_j$, its conjugate momentum p_j is conserved.

- Conservation of energy

$\left(\frac{\partial L}{\partial t}=0\right)$

$$\frac{\partial L}{\partial t} = \frac{dL}{dt} - \left(\frac{\partial L}{\partial \underline{q}}\cdot\frac{d\underline{q}}{dt} + \frac{\partial L}{\partial \dot{\underline{q}}}\cdot\frac{d\dot{\underline{q}}}{dt}\right)$$

$$= \frac{dL}{dt} - \left[\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{q}}}\right)\cdot\dot{\underline{q}} + \frac{\partial L}{\partial \dot{\underline{q}}}\cdot\frac{d\dot{\underline{q}}}{dt}\right]$$

$$= \frac{dL}{dt} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{q}}}\cdot\dot{\underline{q}}\right) = -\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{q}}}\cdot\dot{\underline{q}} - L\right)$$

$$= -\frac{d}{dt}\left(\underline{p}\cdot\dot{\underline{q}} - L\right) = -\frac{d}{dt}h(\underline{q},\dot{\underline{q}},t)$$

Energy function : $h(\underline{q},\dot{\underline{q}},t) \equiv \frac{\partial L}{\partial \dot{\underline{q}}}\cdot\dot{\underline{q}} - L$

If L is symmetric/invariant under an infinitesimal transformation $t \to dt$, h is conserved.

(Example 1, 보존장) $L = \sum_{i=1}^{n}\frac{1}{2}m_i\dot{\underline{r}}_i^2 - V(\underline{r}_1,\cdots,\underline{r}_n)$

$$h = \sum_{i=1}^{n}\frac{\partial L}{\partial \dot{\underline{r}}_i}\cdot\dot{\underline{r}}_i - L = \sum_{i=1}^{n}\frac{1}{2}m_i\dot{\underline{r}}_i^2 + V$$

(= mechanical energy)

(Example 2, monogenic) $L = \frac{1}{2}m\dot{\underline{r}}^2 - [q\phi(\underline{r}) - q\underline{A}(\underline{r})\cdot\dot{\underline{r}}]$

$$h = \frac{\partial L}{\partial \dot{\underline{r}}}\cdot\dot{\underline{r}} - L = m\dot{\underline{r}}^2 + \cancel{q\underline{A}(\underline{r})\cdot\dot{\underline{r}}} - \left(\frac{1}{2}m\dot{\underline{r}}^2 - q\phi(\underline{r}) + \cancel{q\underline{A}(\underline{r})\cdot\dot{\underline{r}}}\right)$$

$$= \frac{1}{2}m\dot{\underline{r}}^2 + q\phi(\underline{r})$$

(Example 3, Rayleigh dissipation)

System : $L(\underline{q}, \dot{\underline{q}})$, $G(\dot{\underline{q}}) = \frac{1}{2}\sum_{j,k} r_{jk} \dot{q}_j \dot{q}_k$

- the energy function monotonically decreases in time according to

$$\frac{dh}{dt} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{q}}} \cdot \dot{\underline{q}} - L\right) = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{q}}}\right) \cdot \dot{\underline{q}} + \cancel{\frac{\partial L}{\partial \dot{\underline{q}}} \frac{d\dot{\underline{q}}}{dt}} - \left(\frac{\partial L}{\partial \underline{q}} \cdot \dot{\underline{q}} + \cancel{\frac{\partial L}{\partial \dot{\underline{q}}} \cdot \ddot{\underline{q}}}\right)$$

$$= \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{q}}} - \frac{\partial L}{\partial \underline{q}}\right) \cdot \dot{\underline{q}} = -\frac{\partial G}{\partial \dot{\underline{q}}} \cdot \dot{\underline{q}} = -2G(\dot{\underline{q}})$$

- Thus the Rayleigh dissipation function quantifies the rate at which the system dissipates energy.

(Example 4, moving observer $\underline{v}$)

$$L = \sum_{i=1}^{n} \frac{1}{2} m_i (\dot{\underline{r}}_i + \underline{v})^2 - \sum_{i,j} V(\underline{r}_i - \underline{r}_j)$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\underline{r}}_i} - \frac{\partial L}{\partial \underline{r}_i} = m_i \ddot{r}_i + \frac{\partial}{\partial r_i} \sum_{i \neq j} V(\underline{r}_i - \underline{r}_j) = 0$$

$$h = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{\underline{r}}_i} \cdot \dot{\underline{r}}_i - L = \sum_{i=1}^{n}\left[\left(m_i(\dot{r}_i + \underline{v})\right) \cdot \dot{\underline{r}}_i - \frac{1}{2} m_i (\dot{r}_i + \underline{v})^2\right] + \sum_{i,j} V(r_i - r_j)$$

$$= \underbrace{\sum_{i=1}^{n} \frac{1}{2} m_i \dot{r}_i^2 + \sum_{i,j} V(r_i - r_j)}_{\text{Mechanical energy as seen by the observer.}} - \sum_{i=1}^{n} \frac{1}{2} m v^2$$

$\therefore$ h is not always the total energy of the system.

3 Noether's theorem for particles

If the mechanics is symmetric/invariant under an infinitesimal spatiotemporal transformation, there exists a corresponding conservation law.

(Proof) · Consider $(q_I(t), t) \to (q'_I(t'), t')$

given by $q'_I(t') = q_I(t) + \delta q_I(t)$, $t' = t + \delta t(t)$

◦ The mechanics (Lagrange's equations) is invariant under the transformation if the change of action satisfies

$$I'[q'(t')] = \int_{t'(t_1)}^{t'(t_2)} ds\, L(q'(s), \dot{q}'(s), s)$$

$$= \int_{t_1}^{t_2} ds\, L(q(s), \dot{q}(s), s) + \delta\Lambda(q(t_2), t_2) - \delta\Lambda(q(t_1), t_1)$$

$$= I[q(t)] + \delta\Lambda(q(t_2), t_2) - \delta\Lambda(q(t_1), t_1)$$

for some infinitesimal $\delta\Lambda$ and arbitrary t_1 and t_2.

(느낌) : Coordinate transformation + gauge transformation

$$L \to L' + \delta\dot{S} \quad , \int L\, dt \to \int L' dt + S$$

$$I'[q'(t')] - \left\{ I[q(t)] + \delta\Lambda(q_2(t), t) - \delta\Lambda(q_1(t), t) \right\}$$

$$= \int_{t_1 + \delta t(t_1)}^{t_2 + \delta t(t_2)} ds\, L(q', \dot{q}', s) - \int_{t_1}^{t_2} ds\, L(q, \dot{q}, s) - \delta\Lambda(q_2(t), t) + \delta\Lambda(q_1(t), t)$$

$$= \int_{t_1}^{t_2} ds \left[L(q'(s), \dot{q}'(s), s) - L(q(s), \dot{q}(s), s) \right] \leftarrow (1)$$

$$\left. \begin{aligned} &+ L(q(t_2), \dot{q}(t_2), t)\, \delta t(t_2) - \delta\Lambda(q_2(t), t) \\ &- L(q(t_1), \dot{q}(t_1), t)\, \delta t(t_1) + \delta\Lambda(q_1(t), t) \end{aligned} \right\} \leftarrow (2)$$

$$(1) = \int_{t_1}^{t_2} ds \left[L(\underline{q}'(s), \dot{\underline{q}}'(s), s) - L(\underline{q}(s), \dot{\underline{q}}(s), s) \right]$$

$$= \int_{t_1}^{t_2} ds \left(\frac{\partial L}{\partial \underline{q}} \cdot \bar{\delta} \underline{q} + \frac{\partial L}{\partial \dot{\underline{q}}} \cdot \bar{\delta} \dot{\underline{q}} \right) \quad \left(\text{where } \underline{\bar{\delta} \underline{q}(t) \equiv \underline{q}'(t) - \underline{q}(t)} \right),$$

$$\left(\begin{aligned} \delta \underline{q}(t) &\equiv \underline{q}'(t'(t)) - \underline{q}(t) = \underline{q}'(t + \delta t(t)) - \underline{q}(t) \\ &\simeq \underline{q}'(t) + \dot{\underline{q}}'(t)\, \delta t(t) - \underline{q}(t) = \bar{\delta} \underline{q}(t) + \dot{\underline{q}}(t) \cdot \delta t(t) \end{aligned} \right)$$

$$\left(\text{Noting that } \bar{\delta} \dot{\underline{q}}(t) = \dot{\underline{q}}'(t) - \dot{\underline{q}}(t) = \frac{d}{dt}\left[\underline{q}'(t) - \underline{q}(t) \right] = \frac{d}{dt} \bar{\delta} \underline{q}(t) \right)$$

Lagrange's equation implies,

$$(1) = \int_{t_1}^{t_2} ds \left(\frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\underline{q}}} \right) \cdot \bar{\delta} \underline{q} + \frac{\partial L}{\partial \dot{\underline{q}}} \frac{d}{ds}(\bar{\delta} \underline{q}) \right)$$

$$= \int_{t_1}^{t_2} ds \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\underline{q}}} \bar{\delta} \underline{q} \right) \simeq \int_{t_1}^{t_2} ds \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\underline{q}}} (\delta \underline{q} - \dot{\underline{q}} \delta t) \right)$$

$$= \int_{t_1}^{t_2} ds \frac{d}{ds}\left(\frac{\partial L}{\partial \dot{\underline{q}}} \cdot \delta \underline{q} - \frac{\partial L}{\partial \dot{\underline{q}}} \dot{\underline{q}}\, \delta t \right)$$

$$(2) = \int_{t_1}^{t_2} ds \frac{d}{ds}\left(L(\underline{q}, \dot{\underline{q}}, s) \delta t - \delta \Lambda(\underline{q}, s) \right)$$

$$(1) + (2) = \int_{t_1}^{t_2} ds \frac{d}{ds}\left(\underbrace{\frac{\partial L}{\partial \dot{\underline{q}}}}_{= \underline{p}} \cdot \delta \underline{q} - \underbrace{\left(\frac{\partial L}{\partial \dot{\underline{q}}} \cdot \dot{\underline{q}} - L \right)}_{= h} \delta t - \delta \Lambda \right) = 0$$

$$\boxed{\frac{d}{dt}\left(\underline{p}(\underline{q}, \dot{\underline{q}}, t) \cdot \delta \underline{q}(t) - h(\underline{q}, \dot{\underline{q}}, t) \cdot \delta t(t) - \delta \Lambda(t) \right] = 0}$$

Remark

① Inverse Noether's theorem exist.

② Noether's theorem applies only to monogenic systems with continuous and smooth symmetries.