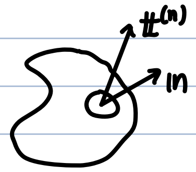
Ch5. Stress

5.1 Surface traction (force per unit area)



$\mathbb{t}^{(n)}$: depends on the surface normal

$$\mathbb{t}^{(n)} = \lim_{\delta S \to 0} \frac{\delta \mathbb{P}}{\delta S} \quad (\text{in general, } \mathbb{t} \text{ does not align with } \mathbb{n})$$

5.2 Components of stress

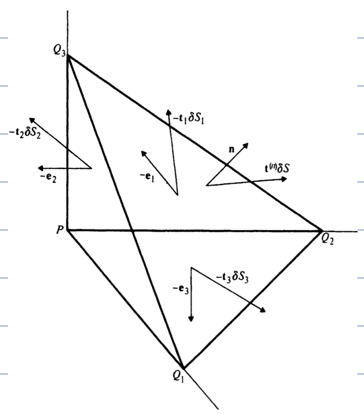
$\mathbb{t}_i$: surface traction if $\mathbb{n} = \mathbb{e}_i$

$\boxed{\mathbb{t}_i \equiv T_{ij}\,\mathbb{e}_j}$, Note : $T_{11} > 0$ if x-positive side pulls the surface.
(정의) $\qquad\qquad\qquad\qquad$ $T_{11} < 0$ if x-positive side compresses the surface.

T_{11}, T_{22}, T_{33} : normal or direct stress components.

T_{12}, T_{13} : shearing stress components.

5.3 The traction on any surface.



body force per unit mass

$$-\mathbb{t}_1\,\delta S_1 - \mathbb{t}_2\,\delta S_2 - \mathbb{t}_3\,\delta S_3 + \mathbb{t}^{(n)}\delta S + \rho\,\mathbb{b}\,dV = \rho\,\mathbb{f}\,dV$$

$$\delta \mathbb{S} = \delta \mathbb{S}\cdot\mathbb{n} \;,\quad \delta S_i = \delta \mathbb{S}\cdot\mathbb{e}_i = \delta S\, n_i$$

$$\therefore\ \mathbb{t}^{(n)} = n_i\,\mathbb{t}_i + \rho\,\underbrace{\frac{\delta V}{\delta S}}_{\to 0 \text{ as } \delta S \to 0\ (\because\ \delta S \sim \ell^2,\ \delta V \sim \ell^3)}(\mathbb{f} - \mathbb{b})$$

$$\boxed{\mathbb{t}^{(n)} = n_i\,\mathbb{t}_i = n_i\,T_{ij}\,\mathbb{e}_j} \;\rightarrow\; \underline{t_j^{(n)} = n_i\,T_{ij}}$$

5.4 Transformation of stress components

$t_j^{(n)} = n_i T_{ij} \longrightarrow \overline{t_j^{(n)}} = \overline{n_i}\,\overline{T_{ij}} \qquad$ ($\Rightarrow$ T_{ij} 와 $\overline{T_{ij}}$ 의 관계를 찾아보자)

$$\overline{t_j^{(n)}} = M_{jk}\,t_k^{(n)} = M_{jk}\,n_\ell\,T_{\ell k} = \underline{n_\ell\,T_{\ell k}\,M_{kj}^T}$$

$$= M_{im}\,n_m\,\overline{T_{ij}} = \underline{n_m\,M_{mi}^T\,\overline{T_{ij}}}$$

$$\therefore\ \underline{\mathbb{n}^T\,\mathbb{T}\,\mathbb{M}^T = \mathbb{n}^T\,\mathbb{M}^T\,\overline{\mathbb{T}}} \;\Rightarrow\; \mathbb{n}^T(\mathbb{T}\,\mathbb{M}^T - \mathbb{M}^T\,\overline{\mathbb{T}}) = 0 \quad \text{for any } \mathbb{n}^T.$$

$$\Rightarrow\ \mathbb{T}\,\mathbb{M}^T = \mathbb{M}^T\,\overline{\mathbb{T}} \;\Rightarrow\; \underline{\overline{\mathbb{T}} = \mathbb{M}\,\mathbb{T}\,\mathbb{M}^T}$$

$$\text{or}\quad \underbrace{\overline{T_{ij}} = M_{ik}\,T_{k\ell}\,M_{\ell j}^T = M_{ik}\,M_{j\ell}\,T_{k\ell}}_{\mathbb{T} \text{ is a tensor}} \;\Rightarrow\; \boxed{\mathbb{t}^{(n)} = \mathbb{n}\cdot\mathbb{T}}$$

5.5 Equation of equilibrium

$$\begin{cases} \iint_S \mathbf{t}^{(n)} dS = \iiint_R \rho\, \mathbf{b}\, dV = 0 & \leftarrow \text{force balance} \\ \iint_S \mathbf{x}\times\mathbf{t}^{(n)} dS + \iiint_R \rho\, \mathbf{x}\times\mathbf{b}\, dV = 0 & \leftarrow \text{couple (torque)} \end{cases}$$

$$\iint_S n_i T_{ij}\, dS + \iiint_R \rho b_j\, dV = 0$$

(발산정리) ↳ $\iiint_R \left(\frac{\partial}{\partial x_i} T_{ij} + \rho b_j\right) dV = 0$ for any R. ∴ $\underline{\frac{\partial}{\partial x_i} T_{ij} + \rho b_j = 0}$

$$\iint_S e_{jpq}\, x_p n_r T_{rq}\, dS + \iiint_R e_{jpq}\, \rho\, x_p b_q\, dV = 0$$

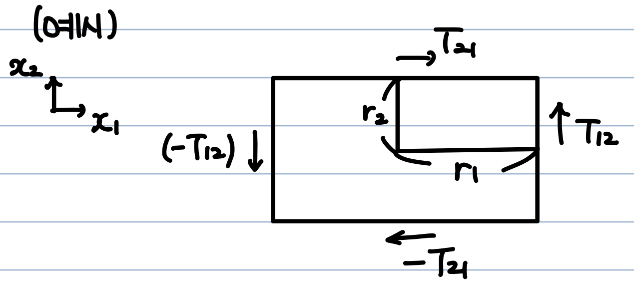
(발산정리) ↳ $\iiint_R e_{jpq}\left(\frac{\partial}{\partial x_r}(x_p T_{rq}) + \rho x_p b_q\right) dV = 0$ for any R.

∴ $e_{jpq}\left(\underline{\frac{\partial}{\partial x_r}(x_p T_{rq})} + \rho x_p b_q\right) = 0$

$e_{jpq}\left(T_{pq} + \cancel{x_p \frac{\partial}{\partial x_r} T_{rq} + \rho x_p b_q}\right) = e_{jpq} T_{pq} = 0$

∴ $\underline{e_{jpq} T_{pq} = 0} \Rightarrow \boxed{T_{pq} = T_{qp}}$: stress tensor is a symmetric tensor.

(예시)



↻ : $r_1 r_2 (T_{12} - T_{21}) = I\ddot{\theta} \sim \rho r_1 r_2 (r_1^2 + r_2^2)$

매우 작은 r_1, r_2에 대해,

$T_{12} \neq T_{21}$ 이면 $\ddot{\theta} = \infty$ 이므로, $T_{12} = T_{21}$ 이다.

5.6 Principal Stress components, principal axes of stress and stress invariants.

$\mathbf{t}^{(n)} \not\parallel \mathbf{n}$ in general. but,

If $\mathbf{t}^{(n)} \parallel \mathbf{n}$, then $\mathbf{t}^{(n)} = T\mathbf{n}$. ($\mathbf{t}^{(n)} = \mathbf{n}\mathbb{T} = T\mathbf{n}$)

→ $n_i T_{ij} = T n_j$ → $T_{ji} n_i = T n_j$ (using $T_{ij} = T_{ji}$)

or $(T_{ij} n_j - T n_j) = (T_{ij} - T\delta_{ij}) n_j = 0$, $\mathbb{T}\cdot\mathbf{n} = T\mathbf{n}$

$T \to T_1, T_2, T_3$: principal components $T_1 \geq T_2 \geq T_3$

$\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$: principal axes

↳ If taken a basis vectors <u>at P</u>, T_{ij} → diagonalized.

(In general, principal directions vary from point to point)

* (예제 5.1, 5.2) 확인하기

Invariants : $J_1 = T_1 + T_2 + T_3 = \mathrm{tr}\,\mathbb{T} = T_{ii}$

$J_2 = -(T_2T_3 + T_3T_1 + T_1T_2) = \frac{1}{2}\{\mathrm{tr}\,\mathbb{I} - (\mathrm{tr}\,\mathbb{I})\} = \frac{1}{2}(T_{ij}T_{ij} - T_{ii}T_{jj})$

$J_3 = T_1T_2T_3 = \det \mathbb{T}$

5.7 The stress deviator tensor

$\boxed{\mathbb{T} = \mathbb{S} + \frac{1}{3}J_1\mathbb{I}}$ $\left(\mathbb{S} = \mathbb{T} - \frac{1}{3}J_1\mathbb{I}\right)$ or $T_{ij} = S_{ij} + \frac{1}{3}J_1\delta_{ij} \longrightarrow \underline{S_{ii} = 0}$

(If $T_{ij} = -p\delta_{ij} + S_{ij}$ where $p = -\frac{1}{3}T_{kk} = -\frac{1}{3}J_1 = -\frac{1}{3}\mathrm{tr}\,\mathbb{T}$)

If $S_{ij} = 0 \longrightarrow \underline{T_{ij} = -p\delta_{ij}}$: pure hydrostatic state of stress

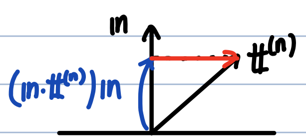
The principal axes of $\mathbb{S}$ = same of those $\mathbb{T}$.

$$S_1 + S_2 + S_3 = 0 \quad \text{and} \quad \begin{aligned} S_1 &= T_1 - \tfrac{1}{3}(T_1+T_2+T_3) = \tfrac{1}{3}(2T_1 - T_2 - T_3) \\ S_2 &= T_2 - \tfrac{1}{3}(T_1+T_2+T_3) = \tfrac{1}{3}(2T_2 - T_1 - T_3) \\ S_3 &= T_3 - \tfrac{1}{3}(T_1+T_2+T_3) = \tfrac{1}{3}(2T_3 - T_1 - T_2) \end{aligned}$$

5.8 Shear stress

ex) $\mathbf{n} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \longrightarrow t_j^{(n)} = n_iT_{ij} = T_{1j} \Rightarrow \mathbf{t}_1 = \begin{pmatrix} T_{11} \\ T_{12} \\ T_{13} \end{pmatrix}$

"Shear stress" $= \begin{pmatrix} 0 \\ T_{12} \\ T_{13} \end{pmatrix} \longrightarrow$ magnitude $= \sqrt{T_{12}^2 + T_{13}^2}$



$\underline{\mathbf{t}^{(n)} - (\mathbf{n}\cdot\mathbf{t}^{(n)})\mathbf{n} = (n_iT_{ij} - n_kn_\ell T_{\ell k}n_j)\mathbf{e}_j = n_\ell T_{\ell k}(\delta_{kj} - n_kn_j)\mathbf{e}_j}$

When this shear stress will be maximum?

Coord

$T_1 \geq T_2 \geq T_3$ $\quad \mathbb{T} = \begin{pmatrix} T_1 & & 0 \\ & T_2 & \\ 0 & & T_3 \end{pmatrix} \Rightarrow$ when $\mathbf{n}$ bisects $\mathbf{n}_1$ & $\mathbf{n}_3$. $\Rightarrow$ proof..?

principal axes

5.9 Some simple state of stress

(a) Hydrostatic pressure (정유체압력)

$\boxed{T_{ij} = -p\delta_{ij}}$ (compression $\longrightarrow T_{ij} < 0$ but pressure $p > 0$)

$T_{11} = T_{22} = T_{33} = -p$, $T_{12} = T_{13} = T_{32} = 0$

(비점성)

→ Any fluid in equilibrium, or inviscid fluid (whether in equil or not) and $p = p(\mathbf{x})$. (p is, in general, a function of position)

(*) In the remaining examples, body force will be neglected.

$\frac{\partial}{\partial x_i} T_{ij} + \rho b_j = 0$ ⟶ ex.) $\rho g h$ term will be neglected.

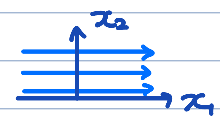
$\underbrace{6\ T_{ij}\text{'s}\ ,\ 3\ \text{eqns}}_{\text{(insufficient)}}$ ⟶ one solution : $T_{ij} = \text{const}$ (*)

↳ stress is homogeneous.

(b) Uniform tension or compressing along x_1

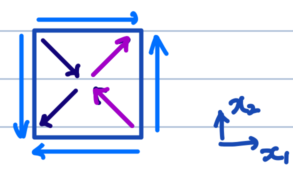
$\boxed{\mathbb{T} = \begin{pmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$, principal axes = $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ & any two orthogonal direction.

(c) Uniform shear stress in x_1, on plane $x_2 = \text{const}$.

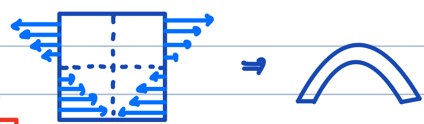


$\boxed{\mathbb{T} = \begin{pmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$, principal axes = $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \underset{\tau}{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}} \underset{-\tau}{\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}}$

(1,1)방향, (1,-1)방향



(d) Pure bending



$\boxed{\mathbb{T} = \begin{pmatrix} Cx_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$, principal axes : $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ & any two orthogonal direction.

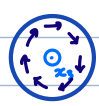
(e) Plane stress

$\boxed{\mathbb{T} = \begin{pmatrix} T_{11} & T_{21} & 0 \\ T_{12} & T_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}}$

$\begin{cases} \frac{\partial}{\partial x_1} T_{11} + \frac{\partial}{\partial x_2} T_{21} = 0 \\ \frac{\partial}{\partial x_1} T_{12} + \frac{\partial}{\partial x_2} T_{22} = 0 \end{cases}$ ← Equilibrium eqn (w/o body force)

principal axes : $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ & some orthogonal direction.

(f) pure torsion



$\mathbb{T} = \underset{r = \sqrt{x_1^2 + x_2^2}}{g(r)} \begin{pmatrix} 0 & 0 & x_2 \\ 0 & 0 & -x_1 \\ x_2 & -x_1 & 0 \end{pmatrix}$

① force acting on surface $\mathbf{n} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ⇒ $g(r) \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix}$

? ② no force on this surface $\mathbf{n} = \frac{1}{r} \begin{pmatrix} x_1 \\ x_2 \\ 0 \end{pmatrix}$ ⇒ 0

③ Shear force $\hat{\theta}$ surface $\mathbf{n} = \frac{1}{r} \begin{pmatrix} -x_2 \\ x_1 \\ 0 \end{pmatrix}$ ⇒ $g(r) \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix}$