Fusion Plasma Theory II

Lecture 11 - oblique, Cyclotron, Drift waves

① Electron wave in oblique propagation

Assuming $k_\parallel$ and $k_\perp$ but ignoring ion motion in high frequency,

$$An^4 - Bn^2 + C = 0 \longrightarrow n^2 = 1 - \frac{2\omega_{pe}^2(\omega^2-\omega_{pe}^2)/\omega^2}{2(\omega^2-\omega_{pe}^2) - \omega_{ce}^2\sin^2\theta \pm \omega_{ce}\Delta}$$

where $\Delta = \left[\omega_{ce}^2\sin^4\theta + 4\omega^{-2}(\omega^2-\omega_{pe}^2)^2\cos^2\theta\right]^{1/2}$

$\theta \simeq 90^\circ$, $\Delta = \omega_{ce}\sin^2\theta$

(−) 부호 : $n^2 = 1 - \dfrac{\cancel{2}\omega_{pe}^2(\omega^2-\omega_{pe}^2)/\omega^2}{\cancel{2}(\omega^2-\omega_{pe}^2-\omega_{ce}^2\sin^2\theta)} = \dfrac{\omega^2(\omega^2-\omega_{pe}^2-\omega_{ce}^2\sin^2\theta)-\omega_{pe}^2(\omega^2-\omega_{pe}^2)}{\omega^2(\omega^2-\omega_{pe}^2)-\omega^2\omega_{ce}^2\sin^2\theta}$

$$= \boxed{\frac{(\omega^2-\omega_{pe}^2)^2 - \omega^2\omega_{ce}^2\sin^2\theta}{\omega^2(\omega^2-\omega_{pe}^2)-\omega^2\omega_{ce}^2\sin^2\theta}} \quad : \boxed{\text{QT-X (Quasi-Transverse X-wave)}}$$

$$\Delta = \omega_{ce}\sin^2\theta\left[1 + \frac{4\omega^{-2}(\omega^2-\omega_{pe}^2)^2\cos^2\theta}{\omega_{ce}^2\sin^4\theta}\right]^{1/2} \underset{\text{이항정리}}{\simeq} \omega_{ce}\sin^2\theta + \frac{2(\omega^2-\omega_{pe}^2)^2}{\omega^2\omega_{ce}\sin^2\theta}\cos^2\theta$$

(+) 부호 : (분모) : $2(\omega^2-\omega_{pe}^2) - \cancel{\omega_{ce}^2\sin^2\theta} + \omega_{ce}\left(\cancel{\omega_{ce}\sin^2\theta} + \dfrac{2(\omega^2-\omega_{pe}^2)^2}{\omega^2\omega_{ce}\sin^2\theta}\cos^2\theta\right)$

$$= 2(\omega^2-\omega_{pe}^2)\left[1 + \frac{\omega^2-\omega_{pe}^2}{\omega^2\sin^2\theta}\cos^2\theta\right]$$

$$n^2 = 1 - \frac{\omega_{pe}^2/\omega^2}{1+\dfrac{\omega^2-\omega_{pe}^2}{\omega^2\sin^2\theta}\cos^2\theta} = 1 - \frac{\omega_{pe}^2\sin^2\theta}{\omega^2\sin^2\theta + (\omega^2-\omega_{pe}^2)\cos^2\theta}$$

$$= 1 - \frac{\omega_{pe}^2\sin^2\theta}{\omega^2 - \omega_{pe}^2\cos^2\theta} = \boxed{\frac{\omega^2-\omega_{pe}^2}{\omega^2-\omega_{pe}^2\cos^2\theta}} \quad : \boxed{\text{QT-O (Quasi-Transverse O-wave)}}$$

$\theta \simeq 0^\circ$, $\Delta = \dfrac{\omega^2-\omega_{pe}^2}{\omega}$ (Quasi-longitudinal)

$$n^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega\pm\omega_{ce})} \ (R/L) \longrightarrow n^2 = 1 - \frac{\omega_{pe}^2}{\omega(\omega\pm\omega_{ce}\cos\theta)} \ (QL-R/L)$$

② Ion wave in oblique propagation

For ion waves, one can generally assume $P = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2} \gg R, L, S, D$

$(\because \omega \sim \omega_{ci} \ll \omega_{ce}, \omega_{pe})$

This gives $A = S\sin^2\theta + P\cos^2\theta \simeq P\cos^2\theta$

$B = RL\sin^2\theta + PS(1+\cos^2\theta) \simeq PS(1+\cos^2\theta)$

$C = PRL$

Using $n_\parallel = n\cos\theta$, $n_\perp = n\sin\theta$,

$$An^4 - Bn^2 + C = 0 \longrightarrow n^2 n_\parallel^2 - S(n^2 + n_\parallel^2) + RL = 0$$

This can be factored out for $n_\perp$.

$$(n_\parallel^2 + n_\perp^2)n_\parallel^2 - S(n_\perp^2 + 2n_\parallel^2) + RL = 0$$

$$n_\perp^2(n_\parallel^2 - S) - 2Sn_\parallel^2 + RL = 0 \longrightarrow \boxed{n_\perp^2 = \frac{(R-n_\parallel)(L-n_\parallel)}{S-n_\parallel^2}}$$

$(2S = R+L)$

Implication : ① wave 쏠 때 정해진 ω, $k_\parallel$을 통해 $k_\perp$은 계산할수 있다

② resonance는 항상 두 cutoff R, L 사이 $S = \frac{R+L}{2}$에 있다.

③ Alfven resonances ($\omega_{ci} > \omega$ 이지만 Alfvén 파으로 낮지는 않은 상황)

Near the resonance $\omega \approx \omega_{ci}$,

$$R, L \simeq \pm \frac{\omega_{pi}^2}{\omega_{ci}(\omega \pm \omega_{ci})} = \frac{\gamma_i \omega_{ci}}{\omega_{ci} \pm \omega} \quad \left(\gamma_i \equiv \frac{\omega_{pi}^2}{\omega_{ci}^2},\ S = \frac{\gamma_i \omega_{ci}^2}{\omega_{ci}^2 - \omega^2} = \frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2}\right)$$

New resonances at $S = n_\parallel^2$ is called Alfvén resonances

$$S = \frac{c^2 k_\parallel^2}{\omega^2} = \frac{\omega_{pi}^2}{\omega_{ci}^2 - \omega^2} \longrightarrow \omega^2(\omega_{pi}^2 + c^2 k_\parallel^2) = \omega_{ci}^2 c^2 k_\parallel^2$$

$$\boxed{\omega^2 = \omega_{ci}^2 \frac{k_\parallel^2 c^2}{k_\parallel^2 c^2 + \omega_{pi}^2}} \leftarrow \text{Alfven resonance at } \omega < \omega_{ci}$$

④ Ion cyclotron waves

$$An^4 - Bn^2 + C = 0 \quad \text{, near } \omega \simeq \omega_{ci}, \quad R = 1 - \frac{\omega_{pi}^2}{\omega(\omega+\omega_{ci})} \ll L = 1 - \frac{\omega_{pi}^2}{\omega(\omega-\omega_{ci})}$$

(1) Fast Branch $(v_p \uparrow, n \downarrow)$

$$n^2 \simeq \frac{C}{B} = \frac{PRL}{PS(1+\cos^2\theta)} = \frac{RL}{S(1+\cos^2\theta)} = \frac{R\cdot \cancel{L}}{(\cancel{L}/2)(1+\cos^2\theta)} = \frac{2R}{1+\cos^2\theta}$$

(near $\omega \simeq \omega_{ci}$) $\quad R \simeq \dfrac{\omega_{pi}^2}{\omega_{ci}(\omega_{ci}+\omega)} \simeq \dfrac{\omega_{pi}^2}{2\omega_{ci}^2} \rightarrow \boxed{n^2 = \dfrac{\omega_{pi}^2/\omega_{ci}^2}{1+\cos^2\theta} = \dfrac{c^2/v_A^2}{1+\cos^2\theta}}$

<low-frequency Alfvén waves>

(2) Slow Branch $(v_p \downarrow, n \uparrow)$

$$n^2 \simeq \frac{B}{A} = \frac{PS(1+\cos^2\theta)}{P\cos^2\theta} = S\,\frac{1+\cos^2\theta}{\cos^2\theta} \rightarrow \boxed{n^2\cos^2\theta = \frac{\omega_{pi}^2}{\omega_{ci}^2-\omega^2}(1+\cos^2\theta)}$$

<Ion cyclotron wave>

⑤ Ion-ion hybrid resonance with minority

Unfortunately, $S = n_\parallel^2$ resonant layer is usually between $(R, L) = n_\parallel^2$ cutoff layer.

Create another (weak but nontrivial) ion-ion hybrid resonant layer by adding other ion species.

$$S \simeq \frac{\gamma_i \omega_{ci}^2}{\omega_{ci}^2-\omega^2} + \frac{\gamma_j \omega_{cj}^2}{\omega_{cj}^2-\omega^2} = \frac{\gamma_i\omega_{ci}^2\omega_{cj}^2 + \gamma_j\omega_{ci}^2\omega_{cj}^2 - \omega^2(\gamma_i\omega_{ci}^2+\gamma_j\omega_{cj}^2)}{(\omega_{ci}^2-\omega^2)(\omega_{cj}^2-\omega^2)}$$

This gives $S=0$ X-wave condition

$$\omega^2 = \frac{\gamma_i\omega_{ci}^2\omega_{cj}^2 + \gamma_j\omega_{ci}^2\omega_{cj}^2}{\gamma_i\omega_{ci}^2+\gamma_j\omega_{cj}^2} = \frac{\omega_{pi}^2\omega_{cj}^2+\omega_{pj}^2\omega_{ci}^2}{\omega_{pi}^2+\omega_{pj}^2}$$

$$= \frac{x_i\omega_{ci}\omega_{cj}^2 + x_j\omega_{cj}\omega_{ci}^2}{x_i\omega_{ci}+x_j\omega_{cj}}$$

$$= \boxed{\omega_{ci}\omega_{cj}\,\frac{x_i\omega_{cj}+x_j\omega_{ci}}{x_i\omega_{ci}+x_j\omega_{cj}}}$$

<Buchsbaum resonance>

(*note $\omega_p^2 = \dfrac{Z^2e^2n}{m\epsilon_0}$, $\omega_c = \dfrac{ZeB}{m}$)

$\left(\omega_{pk}^2 = \omega_{ck}\dfrac{Zen}{\epsilon_0 B} = C\,x_k\omega_{ck}\right)$

where $x_k \equiv n_k Z_k$

① $\omega \simeq 2\sim3\,\omega_{ci}$ (Buchsbaum resonance)
when He minority ions are used.

⑯ MHD approaches

$$\vec{E}+\vec{u}\times\vec{B}=\begin{cases}\dfrac{\vec{j}\times\vec{B}}{en_e} & : \text{ion cyclotron waves}\\[2mm] -\dfrac{\vec{\nabla}p_e}{en_e} & : \text{Drift waves}\\[2mm] \eta\vec{j}-\dfrac{\vec{\nabla}p_e}{en_e} & : \text{Resistive Alfvén wave + drift wave instabilities.}\end{cases}$$

⑰ Slab 3-field model for resistive drift wave instabilities.

- $\vec{B}=B_0\hat{z}$, $\frac{dn}{dx}=n'$, $\frac{dp}{dx}=p'$, $\frac{d}{dx}=ik_x$
- $\vec{E_1}=-\vec{\nabla}\phi+E_1\hat{z}$ (static $\vec{E}$ along $\vec{B}$)
- plasma moves under perturbation only by $\vec{E_1}\times\vec{B_0}$:

$$\vec{u_1}=\vec{u_\perp}+u_z\hat{z}=\hat{z}\times\vec{\nabla}\phi+u_z\hat{z} \quad (\vec{\nabla}\cdot\vec{u_\perp}=0)$$

- $\vec{\nabla}\cdot(\rho_0\vec{u_\perp})=0$ even if $\frac{dn}{dx}\neq 0$ for simplity.
- Let $\underline{\vec{B_1}=B_x\hat{x}+B_y\hat{y}}$, $\vec{u_1}=u_x\hat{x}+u_y\hat{y}+u_z\hat{z}$

↳ we're ignoring the parallel variation of $\vec{B_1}$. as called 3-field model.

✱ $\underline{\vec{B_1}=\hat{z}\times\vec{\nabla}\psi,\ \vec{u_1}=\hat{z}\times\vec{\nabla}\phi+u_{z1}\hat{z} \implies (\psi,\phi,u_{z1}) : \text{3-field model}}$

⑱ Vorticity (momentum) equation

$$\rho_0\frac{d\vec{u_1}}{dt}=\vec{j}\times\vec{B}-\vec{\nabla}p \ \xRightarrow[\text{Take } \hat{z}\cdot\vec{\nabla}\times]{} \ \underbrace{\hat{z}\cdot\vec{\nabla}\times\left(\rho_0\frac{d\vec{u_1}}{dt}\right)}_{\text{(LHS)}}=\hat{z}\cdot\vec{\nabla}\times(\vec{j}\times\vec{B})\simeq\underbrace{\vec{B_0}\cdot\vec{\nabla}j_z}_{\text{(RHS)}}$$

(RHS) $j_z=\frac{1}{\mu_0}\vec{\nabla}\times\vec{B}=\frac{1}{\mu}(ik_xB_y-ik_yB_x)=-i\frac{k_\perp^2}{\mu_0k_y}B_x \quad (\because k_xB_x+k_yB_y=0,\ k_\perp^2=k_x^2+k_y^2)$

$$\vec{B_0}\cdot\vec{\nabla}j_z=B_0ik_z\left(-i\frac{k_\perp^2}{\mu_0k_y}B_x\right)=B_0k_z\frac{k_\perp^2}{\mu_0k_y}B_x$$

(LHS) $\hat{z}\cdot\vec{\nabla}\times\left(\rho_0\frac{d\vec{u_1}}{dt}\right)=-i\omega(ik_x\rho_0u_y-ik_y\rho_0u_x)=-\frac{\omega\rho_0k_\perp^2}{k_y}u_x \quad (\because \vec{\nabla}\cdot(\rho_0\vec{u_\perp})=k_xu_x+k_yu_y=0)$

$\therefore$ (LHS) = (RHS) gives $\left(\text{note } v_A^2 = \frac{B^2}{\mu_0 \rho}\right)$. $\left(\rho = \frac{B}{\mu_0 v_A^2}\right)$

$$B_0 k_z \frac{\cancel{k_\perp}^2}{\mu_0 \cancel{k_y}} B_x = -\frac{\omega \rho_0 \cancel{k_\perp}^2}{\cancel{k_y}} u_x \rightarrow \boxed{\omega u_x = -\frac{k_z v_A^2}{B_0} B_x}$$

$u_x \leftrightarrow B_x$ relation from (perturbed EOM)$_\perp$.

⑨ Faraday's law on Generalized Ohm's law.

$$\vec{E_1} + \vec{u_1} \times \vec{B_0} \simeq \eta \vec{j_1} - \left(\frac{\vec{\nabla} p}{en}\right)_1 \quad \cdots (26)$$

i) $\hat{z} \times$ Eq. (26) $\rightarrow$ $\vec{E_\perp} = -\vec{u_1} \times \vec{B_0}$ $\quad (E_y = u_x B_0)$

ii) $\hat{z} \cdot$ Eq. (26) $\rightarrow$ $E_z = \eta j_z - \frac{1}{en_0}(\hat{b_0} \cdot \vec{\nabla} p_1 + \hat{b_1} \cdot \vec{\nabla} p_0) = \eta j_z - \frac{1}{en_0}\left(ik_z p_1 + \frac{B_x}{B_0} p_0'\right)$

Faraday's law : $\frac{\partial \vec{B_1}}{\partial t} = -\vec{\nabla} \times (\vec{E_\perp} + E_z \hat{z}) \rightarrow -i\omega B_x = -(ik_y E_z - ik_z E_y)$

$$\rightarrow \omega B_x = k_y E_z - k_z E_y$$

$\Rightarrow \omega B_x = k_y \left[\eta j_z - \frac{1}{en_0}\left(ik_z p_1 + \frac{B_x}{B_0} p_0'\right)\right] - k_z u_x B_0$ (note j_z in ⑧ momentum eqn)

$$\therefore \boxed{\left(\omega + i\frac{\eta k_\perp^2}{\mu_0}\right) B_x + k_z B_0 u_x = -\frac{k_y}{en_0}\left(ik_z p_1 + \frac{B_x}{B_0} p_0'\right)}$$

(* 3-field variable 간의 관계를 찾는 과정)

⑩ Parallel momentum and continuity

$\rho_0 \frac{d\vec{u_1}}{dt} = \vec{j} \times \vec{B} - \vec{\nabla} p$ $\underset{\text{Take } \hat{z}.}{\Rightarrow}$ $\rho_0 \frac{du_z}{dt} = -(\nabla_\parallel p)_1$ $\underset{\text{gives}}{\Rightarrow}$ $i\omega \rho_0 u_z = \left(ik_z p_1 + \frac{B_x}{B_0} p_0'\right)$

Assume isothermal $T_1 = 0$, then $\boxed{i\omega \rho_0 u_z = T_0\left(ik_z n_1 + \frac{B_x}{B_0} n_0'\right)}$

$$\frac{dn_1}{dt} + \vec{u_\perp} \cdot \vec{\nabla} n_0 + \nabla_\parallel (n_0 u_z) = 0 \rightarrow \boxed{-i\omega n_1 + u_x n_0' + ik_z n_0 u_z = 0}$$

Now 4-equations & 4-unknowns (B_x, u_x, u_z, n_1)

⑪ Dispersion relation for drift waves

$$\left\{\begin{array}{lll} \text{vorticity equation} & : & \omega u_x = -\dfrac{k_z V_A^2}{B_0} B_x \\ \text{Generalized Ohm's law} & : & \left(\omega + i\dfrac{\eta k_\perp^2}{\mu_0}\right) B_x + k_z B_0 u_x = -\dfrac{T_0 k_y}{e n_0}\left(i k_z n_1 + \dfrac{B_x}{B_0} n_0'\right) \\ \text{parallel momentum} & : & i\omega \rho_0 u_z = T_0 \left(i k_z n_1 + \dfrac{B_x}{B_0} n_0'\right) \\ \text{continuity} & : & i\omega n_1 = u_x n_0' + i k_z n_0 u_z \end{array}\right\}$$

$$\Rightarrow \boxed{(\omega^2 - k_z^2 V_A^2)(\omega^2 - \omega_{*e}\omega - k_z^2 C_s^2) = -i\frac{\eta}{\mu_0} k_\perp^2 \omega(\omega^2 - k^2 C_s^2)}$$

where $V_A = \dfrac{B_0}{\sqrt{\mu_0 \rho}}$, $C_s = \sqrt{\dfrac{T_e}{M}}$, $\omega_{*e} = k_y V_{*e}$, $V_{*e} = -\dfrac{T_{e0}}{e n_{e0} B_0}\dfrac{dn_{e0}}{dx}$

electron diamagnetic frequency

⑫ Drift waves.

- With zero resistivity ($\eta = 0$): $\underbrace{(\omega^2 - k_z^2 V_A^2)}_{\text{Alfvén wave}}\underbrace{(\omega^2 - \omega_{*e}\omega - k_z^2 C_s^2)}_{\text{Drift wave}} = 0$

- With finite resistivity ($\eta \neq 0$), $\omega_{*e} = k_y V_{*e} \ll k_z C_s \ll k_z V_A$

one can see $\dfrac{V_{*e}}{C_s} = \dfrac{T_{e0}}{e B_0 L_n}\left(\dfrac{M}{T_{e0}}\right)^{\frac{1}{2}} = \dfrac{(M T_{e0})^{1/2}}{e B_0 L_n} \simeq \dfrac{r_{LS}}{L_n}$.

($r_{LS} \equiv$ ion Larmor-radius evaluated at electron temperature)

In high frequency branch, ($V_A \gg 0$)

$$\omega^2 - k_z^2 V_A^2 \simeq -i\frac{\eta}{\mu_0} k_\perp^2 \omega \quad \text{with grow } \gamma = -\eta k_\perp^2 / 2\mu_0 < 0 \ : \ \underline{\text{damping}}.$$

In low frequency branch

$$\omega^2 - \omega_{*e}\omega - k_z^2 C_s^2 \simeq i\frac{\eta k_\perp^2}{\mu_0 k_z^2 V_A^2}\omega(\omega^2 - k_z^2 C_s^2)$$

when $\omega_{*e} \lesssim k_z C_s$, $\gamma \approx \dfrac{\eta k_\perp^2}{\mu_0}\dfrac{\omega^2(\omega^2 - k_z^2 C_s^2)}{k_z^2 V_A^2(\omega^2 + k_z^2 C_s^2)}$ / when $k_z C_s \ll \omega_{*e}$ $\gamma \simeq \dfrac{\eta k_\perp^2 \omega_{*e}^2}{\mu_0 k_z^2 V_A^2}$

$\therefore$ resistive electron drift wave is always <u>unstable</u>

Summary : Drift wave는 density gradient로 인해 발생하는 입자의 drift velocity로 전파되며 $\vec{E}\times\vec{B}$ drift motion을 통해 에너지를 전달하는 파동.

Drift wave : Universal instability. (거의 항상 불안정하게 자람)