§3. Harmonic oscillators and normal modes

$$\left(\begin{array}{l}\text{Formulation of the problem}\\ \text{Free oscillations and normal modes}\\ \text{Symmetries and normal modes}\\ \text{Damped / forced oscillators}\end{array}\right)$$

$\boxed{1}$ Formulation of the problem

- $\underline{q} = (q_1, \cdots, q_n) \rightarrow \left\{ \begin{array}{l} T = \frac{1}{2} \sum_{ij} m_{ij}(\underline{q}) \dot{q}_i \dot{q}_j \\ V = V(\underline{q}) \end{array} \right\}$

- system in stable mechanical equilibrium at $\underline{q} = \underline{q}_0$

  $\underline{\left(\frac{\partial V}{\partial \underline{q}}\right)_0 = 0}$, Hessian $\underline{\underline{K}}$: $\underline{K_{ij} = \frac{\partial^2 V}{\partial q_i \partial q_j}}$; positive definite

- $\underline{q} = \underline{q}_0 + \underline{\eta}$, $\rightarrow$ $\underline{\underline{L = T - V \simeq \frac{1}{2} \dot{\underline{\eta}}^T \underline{\underline{M}} \dot{\underline{\eta}} - \frac{1}{2} \underline{\eta}^T \underline{\underline{K}} \underline{\eta} - V(\underline{q}_0)}}$

  (where $\underline{\underline{M}}$ and $\underline{\underline{K}}$: real, symmetric, and positive-definite)

- $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\underline{\eta}}}\right) - \frac{\partial L}{\partial \underline{\eta}} = \underline{\underline{\underline{\underline{M}} \ddot{\underline{\eta}} + \underline{\underline{K}} \underline{\eta} = \underline{Q}^{(ex)}}}$ ← Extra force

  Coupled with off-diagonal elements of $\underline{\underline{M}}$ and $\underline{\underline{K}}$.

- Goal : Find a set of generalized coordinates that simplifies the problem by decoupling the equations.

$$\boxed{\therefore\ \eta_\mu(t) = \sum_{\nu=1}^{n} \mathrm{Re}\left[A_{\mu\nu}\,\xi_\nu(0)\,e^{-i\omega_\nu t}\right]}$$

Summary.

① Calculate the n natural frequencies by solving the secular equation $|\underline{\underline{K}} - \omega^2 \underline{\underline{M}}| = 0$

② For each natural frequency ω_μ, obtain the corresponding normal modes by finding non-zero solutions for $(\underline{\underline{K}} - \omega_\mu \underline{\underline{M}})\underline{a}_\mu = \underline{0}$

③ Linearly combine the normal modes to fulfill the initial conditions.

$\underline{q} = \underline{q_0} + \underline{\eta}$, $L = T - V = \frac{1}{2}\dot{\underline{\eta}}^T \underline{\underline{M}}\,\dot{\underline{\eta}} - \frac{1}{2}\underline{\eta}^T \underline{\underline{K}}\,\underline{\eta} - V(\underline{q_0})$

E-L equation : $\underline{\underline{M}}\,\ddot{\underline{\eta}} + \underline{\underline{K}}\,\underline{\eta} = \underline{0}$

Suppose) $\underline{\eta} = A\underline{\xi}$, $(\underline{\xi} = A^{-1}\underline{\eta})$, $\underline{\underline{A}}^T \underline{\underline{M}}\,\underline{\underline{A}} = 1$, $\underline{\underline{A}}^T \underline{\underline{K}}\,\underline{\underline{A}} = \underline{\underline{\Delta}}$

then, $L = \frac{1}{2}\sum_{\mu=1}^{n} \dot{\xi}_\mu^2 - \frac{1}{2}\sum_{\mu=1}^{n} \lambda_\mu \xi_\mu^2$ $(\because\ \underline{\underline{K}}\,\underline{a}_\mu = \lambda_\mu \underline{\underline{M}}\,\underline{a}_\mu)$

$(\underline{\underline{K}}\,\underline{\underline{A}} = \underline{\underline{M}}\,\underline{\underline{A}}\,\underline{\underline{\Delta}})$

$\therefore\ \xi_\mu(t) = \mathrm{Re}\left[\tilde{\xi}_\mu(0)\,e^{-i\omega_\mu t}\right]$

$\eta_\mu(t) = \sum_{\nu=1}^{n} \mathrm{Re}\left[A_{\mu\nu}\,\xi_\nu(0)\,e^{-i\omega_\nu t}\right]$

3 Symmetries and normal modes

- 역학 문제를 어떻게 대칭성으로 쉽게 풀 수 있는가?

- T : orthogonal transformation $\rightarrow$ $q' = T q$

  $(T^T T = T T^T = I)$

Length preserving $\Rightarrow$ $(q'^T q' = q^T T^T T q = q^T q)$

- Under this orthogonal transformation $(q = T^{-1} q' = T^T q')$

$$L = \frac{1}{2}\dot{q}^T M \dot{q} - \frac{1}{2} q^T K q = \frac{1}{2}\dot{q}'^T T M T^{-1} \dot{q}' - \frac{1}{2} q'^T T K T^{-1} q'$$

L is symmetric / invariant iff

$$T M T^{-1} = M \ \& \ T K T^{-1} = K \Leftrightarrow TM = MT \ \& \ TK = KT$$

Then, $K T A = T K A = T M A \Lambda = M T A \Lambda$

$$A^T (K T A) \overset{①}{=} A^T M T A \Lambda = A^{-1} T A \Lambda$$

$$\overset{②}{=} (K A)^T T A = (M A \Lambda)^T T A$$

$$= [(A^T)^{-1} \Lambda]^T T A = \Lambda A^{-1} T A$$

$\therefore A^{-1} T A \Lambda = \Lambda A^{-1} T A$ $\rightarrow$ $(A^{-1} T A)_{ij} \lambda_j = \lambda_i (A^{-1} T A)_{ij}$

for any i, j.

if $\lambda_i \neq \lambda_j \Rightarrow (A^{-1} T A)_{ij} = 0$ Thus $A^{-1} T A$ can be block-diagonal matrix.

$\lambda_i = \lambda_j \Rightarrow (A^{-1} T A)_{ij} \neq 0$ 가능

↳ 같은 ω를 공유하는 mode끼리는 coupling될 수 있다.

This means, $A^{-1} T A$ block diagonal $\rightarrow$ T only mixes same (ω_n).

$\therefore$ symmetric transformation only mixes same-freq modes.

Since $\underline{\underline{T}}$ is orthogonal matrix, $\rightarrow$ diagonalizable

$\hookrightarrow$ So is $\underline{\underline{A}}^{-1}\underline{\underline{T}}\,\underline{\underline{A}}$. ($\because$ $\underline{\underline{T}}$를 similarity 변환한것)

Suppose, $\underline{\underline{A}}^{-1}\underline{\underline{T}}\,\underline{\underline{A}} = \underline{\underline{R}}\,\underline{\underline{D}}\,\underline{\underline{R}}^{-1}$ where $\underline{\underline{D}}$ is diagonal.
← 대각화

$\Longrightarrow$ $\underline{\underline{T}} = \underline{\underline{A}}\,\underline{\underline{R}}\,\underline{\underline{D}}\,\underline{\underline{R}}^{-1}\underline{\underline{A}}^{-1}$ ← $\underline{\underline{A}}\,\underline{\underline{R}}$ is an eigenvector matrix for $\underline{\underline{T}}$

Since $\underline{\underline{A}}^{-1}\underline{\underline{T}}\,\underline{\underline{A}}$ is block diagonal, ✓ So is $\underline{\underline{R}}$ ($\underline{\underline{R}}\,\underline{\underline{D}}\,\underline{\underline{R}}^{-1}$에서 $\underline{\underline{R}}$은 $\underline{\underline{D}}$를 섞어주는데 Block 밖안 섞어준다!)
with each block coupling the rows and columns corresponding to the same natural frequency.

$\Longrightarrow$ $\underline{\underline{A}}\,\underline{\underline{R}}$ is an eigenvector matrix for $\underline{\underline{T}}$.

Summary

① Identify a length-preserving transformation that keeps the oscillatory motion of the system invariant.

② Find the eigenvectors of the transformation.

③ Using the eigenvectors, diagonalize $\underline{\underline{K}}$ to find the correspondence between the normal modes and the natural frequencies.

④ Linearly combine the normal modes to fulfill the initial conditions.

4 Damped / forced oscillation

(A) Damped oscillator

- Linear drag force $\underline{Q^{(ex)} = -\underline{\underline{G}}\dot{\eta}}$ $\Rightarrow$ $\underline{\underline{\underline{M}}\ddot{\eta} + \underline{\underline{G}}\dot{\eta} + \underline{\underline{K}}\eta = 0}$

$$(\underline{\underline{M}}, \underline{\underline{G}}, \underline{\underline{K}} > 0)$$

use ansatz, of $\eta(t) = \mathrm{Re}[\tilde{\eta}(\nu)e^{\nu t}]$

$$\Rightarrow \underline{(\nu^2\underline{\underline{M}} + \nu\underline{\underline{G}} + \underline{\underline{K}})\tilde{\eta} = \underline{0}}$$

To have non zero solution, $\underline{|\nu^2\underline{\underline{M}} + \nu\underline{\underline{G}} + K| = 0}$

- Since $\underline{\underline{M}}, \underline{\underline{G}}, \underline{\underline{K}}$ are all real, ν & ν^* are roots.

using $\tilde{\eta}^T(\nu^2\underline{\underline{M}} + \nu\underline{\underline{G}} + \underline{\underline{K}})\tilde{\eta} = 0$, then

$$\underline{\mathrm{Re}\,\nu = \frac{\nu+\nu^*}{2} = -\frac{\tilde{\eta}^T\underline{\underline{G}}\tilde{\eta}}{2\tilde{\eta}^T\underline{\underline{M}}\tilde{\eta}} < 0}$$

& oscillate with $\underline{w = \mathrm{Im}(\nu)}$, decays at rate $\underline{\mathrm{Re}\,\nu}$

- For the special case where the normal modes also diagonalize $\underline{\underline{G}}$ (i.e. $\underline{\underline{A}}^T\underline{\underline{G}}\underline{\underline{A}} = \mathrm{diag}(g_1, \cdots, g_n)$), then $\underline{\ddot{\xi}_\mu + g_\mu\dot{\xi}_\mu + w_\mu^2\xi_\mu = 0}$

, which is solved by $\underline{\xi_\mu(t) = \mathrm{Re}\left[\xi_\mu(0)\, e^{-\frac{g_\mu t}{2}} e^{-iw'_\mu t}\right]}$

where $\underline{w'_\mu = \pm\sqrt{w_\mu^2 - g_\mu^2/4} \neq w_\mu}$.

(B) Forced oscillators with damping.

- $\underline{Q}_j^{(ex)} = -\underline{G}\dot{\underline{\eta}} + \mathrm{Re}[\tilde{\underline{F}}_0 e^{-i\omega t}]$

$\rightarrow \underline{M}\ddot{\underline{\eta}} + \underline{G}\dot{\underline{\eta}} + \underline{K}\underline{\eta} = \mathrm{Re}[\tilde{\underline{F}}_0 e^{-i\omega t}]$

Let's divide $\underline{\eta} \to \underline{\eta} + \underline{\eta}^{(0)}$, then $\underline{\eta}^{(0)} \to 0$ when $t \to \infty$
(∵ decaying)

∴ For steady state behavior, use ansatz $\underline{\eta}(t) = \mathrm{Re}[\tilde{\underline{\eta}}(\omega) e^{-i\omega t}]$

which yields, $\underline{(\underline{\underline{K}} - i\omega \underline{\underline{G}} - \omega^2 \underline{\underline{M}})\tilde{\underline{\eta}} = \tilde{\underline{F}}_0}$

- By Cramer's rule, $\tilde{\eta}_j(\omega) = \dfrac{|\underline{\underline{K}} - i\omega\underline{\underline{G}} - \omega^2\underline{\underline{M}}|_j}{|\underline{\underline{K}} - i\omega\underline{\underline{G}} - \omega^2\underline{\underline{M}}|}$ ← (*)

where (*) is the determinant of $\underline{\underline{K}} - i\omega\underline{\underline{G}} - \omega^2\underline{\underline{M}}$ with the j-th column replaced with $\tilde{\underline{F}}_0$.

- The magnitude of $|\underline{\underline{K}} - i\omega\underline{\underline{G}} - \omega^2\underline{\underline{M}}|$ satisfies

$$\left\| \underline{\underline{K}} - i\omega\underline{\underline{G}} - \omega^2\underline{\underline{M}} \right\| \propto \prod_{\mu=1}^{n} \left| (\omega + i\gamma_\mu)(\omega - i\gamma_\mu^*) \right|$$

(∵ γ_μ = root of $|\underline{\underline{K}} + \gamma\underline{\underline{G}} + \omega^2\underline{\underline{K}}| = 0$)

$$= \prod_{\mu=1}^{n} \left| \omega^2 - 2i\omega\,\mathrm{Re}\,\gamma_\mu - |\gamma_\mu|^2 \right|$$

$-i\omega_\mu = \gamma_\mu$, $\omega_\mu = +i\gamma_\mu$ | $-i\omega_\mu = \gamma_\mu^*$, $\omega_\mu = +i\gamma_\mu^*$

$$= \prod_{\mu=1}^{n} \left[(\omega^2 - |\gamma_\mu|^2)^2 + 4\omega^2 (\mathrm{Re}\,\gamma_\mu)^2 \right]^{1/2}$$

$$\left(\because \prod_{\mu=1}^{n} \left[(\omega^2 - |\gamma_\mu|^2 - 2i\omega\,\mathrm{Re}\,\gamma_\mu)(\omega^2 - |\gamma_\mu|^2 + 2i\omega\,\mathrm{Re}\,\gamma_\mu) \right]^{1/2} = \prod_{\mu=1}^{n} \left[(\omega^2 - |\gamma_\mu|^2)^2 + 4\omega^2 (\mathrm{Re}\,\gamma_\mu)^2 \right]^{1/2}\right.$$

Suppose) Drag force $\underline{\underline{G}}$ is much smaller than $\underline{\underline{M}}$, $\underline{\underline{K}}$, then,

$$\gamma_\mu = -i\omega_\mu$$ (ω_μ is natural frequency of normal mode μ in the absence of damping).

$\therefore$ For ω in the vicinity of ω_μ,

$$\left[(\omega^2 - |\gamma_\mu|^2)^2 + 4\omega^2 (\mathrm{Re}\,\gamma_\mu)^2 \right]^{1/2}$$

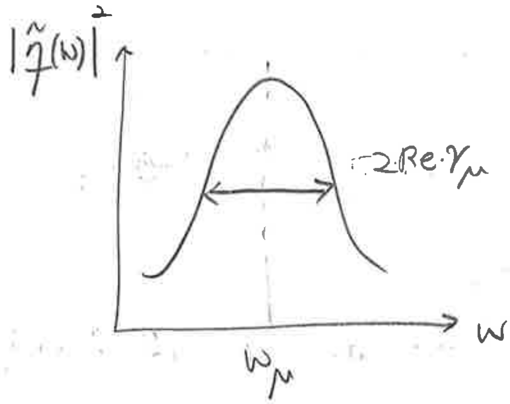
$$\simeq \left[(\omega - |\gamma_\mu|)^2 \underbrace{(\omega + |\gamma_\mu|)^2}_{4\omega^2} + 4\omega^2 (\mathrm{Re}\,\gamma_\mu)^2 \right]^{1/2}$$

$$= 2|\omega_\mu| \left[(\omega - \omega_\mu)^2 + (\mathrm{Re}\,\gamma_\mu)^2 \right]^{1/2}$$

$\therefore$ amplitude exhibits a resonance peak at a natural f of free oscillation, with the width of the peak proportional to the decay rate $|\mathrm{Re}\,\gamma_\mu|$.

(+) 저항 때문에 진폭이 무한대로 가는 것은 아니다.

(γ_μ는 homogeneous solution)



$|\tilde{\eta}(\omega)|^2$이 $\frac{1}{2}$배 ($\tilde{\eta}(\omega)$는 $\frac{1}{\sqrt{2}}$배) 가 되는 지점은

$\omega = \omega_\mu + \mathrm{Re}\,\gamma_\mu$ 이다

결론 : $\omega = \omega_\mu$ 에서 공명이 발생한다.