BIOS 791 Miniproject 2

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The paper chosen for this miniproject was Maximum Likelihood Estimation for Semiparametric Regression Models with Panel Count Data[1]. The paper proposed and proved two theorems under their imposed regularity conditions. In this miniproject, one of the key steps in the first theorem proof would be verified, and the required conditions were given below.

Theorem 1. Under Conditions 1–3, $\|\hat{\theta}-\theta_0\| \to 0$ almost surely, where $\|\cdot\|$ is the Euclidean norm. In addition, $\sum_{k=1}^{K} \sup_{t\in[0,\tau_k]} \left|\hat{\Lambda}_k(t) - \Lambda_{k0}(t)\right| \to 0$ almost surely.

Condition 1. The parameter θ_0 belongs to a compact set in the interior of its domain. In addition, $\Lambda_{k0}(\cdot)$ is continuously differentiable and strictly increasing in $\mathcal{U} \cap [0, \tau_k]$.

Condition 3. The number of examination times, m_{ki}, is at least 1, with $E(m_{ki}^2) < \infty$. In addition, there exists a positive constant η such that $\text{pr}\{\min_{j=1,\ldots,m_{ki}} (U_{kij} - U_{ki,j-1}) > \eta \Big| m_{ki}, X_i\} = 1$, and the conditional density of $(U_{ki,j-1} = u, U_{kij} = v)$, given m_{ki} and and X_i exists, and is twice continuously differentiable with respect to u and v in their support. Finally, $\text{pr}(C_{ki} = \tau_k \Big| X_i) > c_0$ for some $c_0 > 0$.

We want to show $\limsup_n \hat{\Lambda}_k(\tau_k - \epsilon) < \infty$ with probability 1 for any $\epsilon > 0$. They noted the cumulative counts over time should be considered in the proof and that they can be unbounded. A log-average likelihood was then proposed such that $g(\theta, \Lambda) = \log[\{L(\theta, \Lambda) + L(\theta_0, \Lambda_0)\}/2]$ with the class of functions $\mathcal{F}^* = \{g(\theta, \Lambda) : \theta \in \Theta, \Lambda_k \in \mathcal{A}^*,\ k = 1, \ldots, K\}$, where Θ denotes the parameter space for θ_0, and $\mathcal{A}^*$ is the set of nondecreasing functions $\Lambda(\cdot)$ with $\Lambda(0) = 0$. Followed by Condition 1 which requires θ_0 to be bounded, we know $L(\theta_0, \Lambda_0)$ is bounded. Suppose C denotes a positive constant such that $L(\theta_0, \Lambda_0) \geq C$. Then, $L(\theta, \Lambda) + L(\theta_0, \Lambda_0) \geq L(\theta_0, \Lambda_0) \geq C$, implying the boundedness of $g(\theta, \Lambda)$ and that $\mathcal{F}^*$ is a Glivenko-Cantelli class. Given that $\hat{\theta}$ and $\hat{\Lambda}$ are the maximum likelihood estimators,

$$l(\hat{\theta}, \hat{\Lambda}) \geq l(\theta_0, \Lambda_0) = \log[L(\theta_0, \Lambda_0) + L(\theta_0, \Lambda_0)/2] = g(\theta_0, \Lambda_0).$$

Then, $\mathbb{P}_n l(\hat{\theta}, \hat{\Lambda}) \geq \mathbb{P}_n l(\theta_0, \Lambda_0) = \mathbb{P}_n g(\theta_0, \Lambda_0)$. Taking $\liminf_n$ on both sides of the inequality, we get $\liminf_n \mathbb{P}_n l(\hat{\theta}, \hat{\Lambda}) \geq \liminf_n \mathbb{P}_n g(\theta_0, \Lambda_0) = \mathbb{P} g(\theta_0, \Lambda_0)$, where $\mathbb{P}$ denotes its expectation, since $\mathcal{F}^*$ is a Glivenko-Cantelli class. Using $\prod_{i=1}^{m} x_i \leq y^m \leq (m+1)! \exp(y)/y$ for any $0 < x_i \leq y$ and any positive number m from the paper, and we know $\Delta_{kj} = N_k(U_{kj}) - N_k(U_{k,j-1})$, where $0 < U_{k1} < \ldots < U_{km_k} = C_k$,

$$\prod_{j=1}^{m_k} \left\{ \int_{U_{k,j-1}}^{U_{kj}} e^{\beta_k^\top X(t) + b_k^\top Z(t) + \xi^\top \tilde{Z}(t)}\, d\hat{\Lambda}_k(t) \right\}^{\Delta_{kj}} \leq \left\{ \int_0^{C_k} e^{\beta_k^\top X(t) + b_k^\top Z(t) + \xi^\top \tilde{Z}(t)} d\hat{\Lambda}_k(s) \right\}^{N_k(C_k)}.$$

The above is true because $0 < \int_{U_{k,j-1}}^{U_{kj}} e^{\beta_k^\top X(t)+b_k^\top Z(t)+\xi^\top \tilde{Z}(t)}\, d\hat{\Lambda}_k(t) \le \int_0^{C_k} e^{\beta_k^\top X(t)+b_k^\top Z(t)+\xi^\top \tilde{Z}(t)} d\hat{\Lambda}_k(s)$. Then,

$$\left\{\int_0^{C_k} e^{\beta_k^\top X(t)+b_k^\top Z(t)+\xi^\top \tilde{Z}(t)} d\hat{\Lambda}_k(s)\right\}^{N_K(C_k)} \le \frac{\{N_k(C_k)+1\}!\exp\left\{\int_0^{C_k} e^{\beta_k^\top X(t)+b_k^\top Z(t)+\xi^\top \tilde{Z}(t)} d\hat{\Lambda}_k(s)\right\}}{\int_0^{C_k}\exp\left\{\beta_k^\top X(t)+b_k^\top Z(t)+\xi^\top \tilde{Z}(t)\right\} d\hat{\Lambda}_k(t)}.$$

Suppose $\tilde{M} = \sup\left\{\sup|\beta^\top X(t)| + \sup\|\tilde{Z}(t)\| + \|\tilde{Z}(t)\|\right\}$, and by Condition 1 they imposed, $\tilde{M}$ is finite. Using the loglikelihood and $\tilde{M}$,

$$\begin{aligned}
\liminf_n \mathbb{P}_n l(\hat{\theta},\hat{\Lambda}) \le \liminf_n \mathbb{P}_n \Bigg[\log \int_\xi \phi(\xi;\hat{\Psi}) \prod_{k=1}^K \int_{b_k} \phi(b_k;\hat{\Sigma}_k) \\
\prod_{j=1}^{m_k} \frac{\exp\left\{-\int_{U_{k,j-1}}^{U_{kj}} \exp\left(\beta_k^\top X(t)+b_k^\top Z(t)+\xi^\top \tilde{Z}(t)\right) d\hat{\Lambda}_k(t)\right\}}{\Delta_{kj}!} \\
\times \frac{\{N_k(C_k)+1\}!\exp\left\{\int_0^{C_k}\exp\left(\beta_k^\top X(t)+b_k^\top Z(t)+\xi^\top \tilde{Z}(t)\right) d\hat{\Lambda}_k(s)\right\}}{\int_0^{C_k}\exp\left(\beta_k^\top X(t)+b_k^\top Z(t)+\xi^\top \tilde{Z}(t)\right) d\hat{\Lambda}_k(t)} db_k\, d\xi\Bigg] \\
\le \limsup_n \mathbb{P}_n\left[\log\int_\xi \phi(\xi;\hat{\Psi})\prod_{k=1}^K\int_{b_k}\frac{\{N_k(C_k)+1\}!\,\phi(b_k;\hat{\Sigma}_k)}{\int_0^{C_k} e^{-\tilde{M}-\tilde{M}\|b_k\|-\tilde{M}\|\xi\|}\, d\hat{\Lambda}_k(t)}\, db_k d\xi\right]
\end{aligned}$$

$$\begin{aligned}
\le \limsup_n \mathbb{P}_n\left[\log\left\{e^{\tilde{M}}\int_\xi e^{\tilde{M}\|\xi\|}\phi(\xi;\hat{\Psi})\prod_{k=1}^K \frac{1}{\int_0^{C_k} d\hat{\Lambda}_k(t)}\int_{b_k}\{N_k(C_k)+1\}!\, e^{\tilde{M}\|b_k\|}\phi(b_k;\hat{\Sigma}_k)\, db_k d\xi\right\}\right] \\
\le \limsup_n \mathbb{P}_n\left[\log\left\{e^{\tilde{M}}\int_\xi e^{\tilde{M}\|\xi\|}\phi(\xi;\hat{\Psi})\prod_{k=1}^K \frac{1}{\hat{\Lambda}_k(t)}\int_{b_k}\{N_k(C_k)+1\}!\, e^{\tilde{M}\|b_k\|}\phi(b_k;\hat{\Sigma}_k)\, db_k d\xi\right\}\right] \\
\le O(1) - \sum_{k=1}^K \limsup_n \mathbb{P}_n\left\{I(C_k \ge \tau-\epsilon)\log\hat{\Lambda}_k(\tau_k-\epsilon)\right\}.
\end{aligned}$$

Since $e^{\tilde{M}}$ and $\{N_k(C_k)+1\}!$ are bounded by constant, they are absorbed into $O(1)$. Given ξ and b_k are mutually independent and are normal, the integral of $e^{\tilde{M}\|\xi\|}\,\phi(\xi;\hat{\Psi})$ and $e^{\tilde{M}\|b_k\|}\,\phi(b_k;\hat{\Sigma}_k)$ with respect to ξ and b_k can be understood as the moment generating functions of Gaussian distributions, which would be finite. We also have the bounded MLEs. Note we assumed τ_k denotes the least upper bound of finite C_k with $\tau = \max_k \tau_k$. Clearly, for any $\epsilon > 0$, $\limsup_n \mathbb{P}_n\{I(C_k \ge \tau-\epsilon)\log\hat{\Lambda}_k(\tau_k-\epsilon) = O(1)$, and $\mathbb{P}_n I(C_k \ge \tau-\epsilon) \to \mathrm{pr}(C_k \ge \tau-\epsilon)$. Condition 3 ensures $\mathrm{pr}(C_k \ge \tau-\epsilon)$ is positive. Thus, we showed $\limsup_n \hat{\Lambda}_k(\tau_k-\epsilon) < \infty$ with probability 1 for any $\epsilon > 0$. One of the interesting technical aspects of the paper I found was to use $\tilde{M} = \sup\left\{\sup|\beta^\top X(t)| + \sup\|\tilde{Z}(t)\| + \|\tilde{Z}(t)\|\right\}$ to make some terms bounded and absorbed into $O(1)$. The results consisted of quite simple methods that seemed sufficient for the proof.

References

[1] Zeng D. and Lin D. Maximum likelihood estimation for semiparametric regression models with panel count data. *Biometrika*, 108(4):947–960, 2020.