8 Rolling Contact and Nonholonomic Constraints

The motion of mechanical systems having rolling contact, as in road vehicles and track-guided vehicles, has intrigued the author from early childhood on. In broadening our horizons we make use of different means of transport, evolving from tricycles, skates, scooters, bicycles, motor scooters and cars to trains, ships and planes. The motion of these type of systems can be investigated in an approximate way by a mechanical model having nonholonomic constraints. These constraints are formulated on the velocity level, and express the conditions of vanishing slips at the contact points. A mechanical system with nonholonomic constraints is called nonholonomic. Whereas the dynamics of mechanical systems with ideal holonomic constraints was almost completed by the publication of Lagrange's monumental Méchanique Analitique [5], Heinrich Hertz [14] was the first to describe and name systems with nonholonomic constraints. Although the principle of minimal action fails for these systems, the principle of virtual power and the principle of D'Alembert can be applied. In their excellent book [15] Neĭmark and Fufaev treat the kinematics and dynamics of nonholonomic mechanical systems in great detail. They illustrate the presented theory with worked-out examples and give an elaborate reference list with over 500 items.

In this chapter we will derive the equations of motion for a 2D mechanical system which consists of rigid bodies with constraints, in particular for velocity or nonholonomic constraints. We will also demonstrate how to obtain the motion in time by numerical integration, and get rid of any drift in the solutions by applying a Coordinate Projection Method or Gauß-Newton proces on the constraints.

8.1 Rolling Without Slip in 2D

We will investigate the rolling constraints for of a wheel which moves over the xy-plane, and where the plane of the wheel will always remain perpendicular to the the xy-plane; the straight upright configuration, see Figure 8.1. Such a wheel fits well into a 2D MBD system analysis.

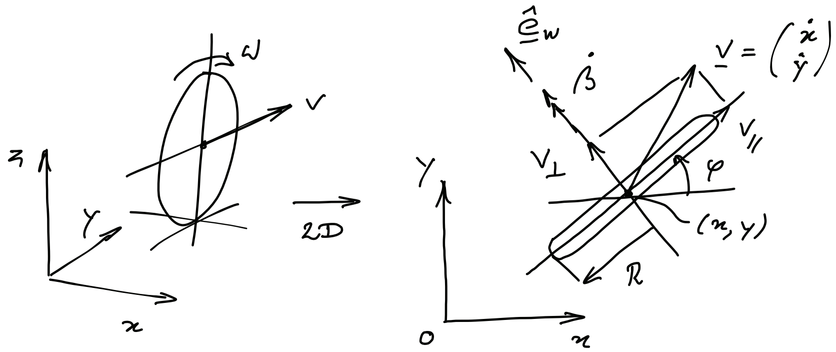

Figure 8.1: Rolling wheel in 3D and straight upright non-leaning wheel in 2D.

The position and orientation of the wheel can be described by the cm coordinates (x, y) and the angle φ, the orientation of the wheel or to be more specific, the rotation of the wheel about the z-axis. But for a rolling wheel there is also a second angle in play, the rotation about the wheel axis, which we will denote by the angle β. Thus the coordinates which describe the position and orientations of the wheel are,

$$x_i = (x,\, y,\, \varphi,\, \beta)^{\mathrm{T}} \tag{8.1}$$

As a rigid body, this wheel can be positioned anywhere on the xy-plane at any orientation angle φ and at any angle β, thus as a rigid body, it has four degrees of freedom. However, for a pure rolling wheel the velocity of the cm of the wheel, $\boldsymbol{v} = (\dot{x}, \dot{y})$, should alway be in the plane of the wheel. Or in other words, there should be no side slip, $v_{\perp} = 0$. A second condition for pure rolling, is no slip in the longitudinal direction of the wheel, or in other words, that the forward velocity complies with the rotational speed times the radius, $v_{\shortparallel} = \dot{\beta} R$. These two conditions are constraints on the level of the velocities, for which we use the terminology *nonholonomic constraints*, as introduced by Heinrich Hertz [16]. These nonholonomic constraints are,

$$v_{\perp} = 0 \qquad \Rightarrow \qquad v_{\perp} = \boldsymbol{v} \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix} = -\dot{x}\sin\varphi + \dot{y}\cos\varphi, \tag{8.2}$$

$$v_{\shortparallel} = \dot{\beta} R \qquad \Rightarrow \qquad v_{\shortparallel} = \boldsymbol{v} \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} = \dot{x}\cos\varphi + \dot{y}\sin\varphi = \dot{\beta} R, \tag{8.3}$$

where we have used the wheel axis direction, $\hat{\boldsymbol{e}}_{\mathrm{w}} = (-\sin\varphi, \cos\varphi)$ and it's perpendicular direction $(\cos\varphi, \sin\varphi)$, which is the driving direction of the wheel. And although we have four degrees of freedom in the coordinate space, due to the two nonholonomic constraints we only have two degrees of freedom in the velocity space. These velocities degree of freedom are sometimes also called the mobility, that is this system has a mobility of two.

8.1.1 No Sideslip, a Skate

A very convenient way to think about pure rolling is to express the constraints in terms of zero slip, where slip is a velocity. This gives us the opportunity to either introduce this as a constraint or allow slip and then add a constitutive equation for the behaviour. Such a constitutive behaviour can be for instance a tire model, where sideslip generates a side force, the so-called cornering stiffness of a tire. Then side slip in accordance with (8.2), can be defined as

$$S_1 = -\dot{x}\sin\varphi + \dot{y}\cos\varphi \tag{8.4}$$

For now we focus on side slip only and forget about the slip in longitudinal direction. You can compare this to the behaviour of a skate on the ice, see Figure 8.2. The coordinates for such a skate as a rigid body are $x_i = (x,\, y,\, \varphi)^{\mathrm{T}}$.

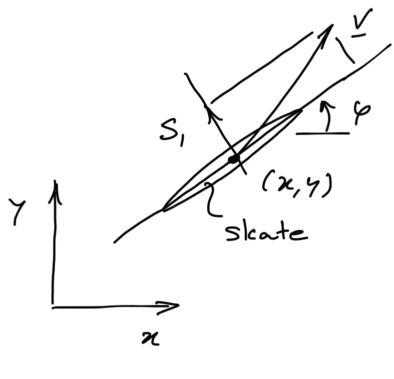

Figure 8.2: Skate with possible sideslip S.

And the no sideslip constraint can now be expressed as zero sideslip,

$$S_1 = S_{1i}\dot{x}_i = 0 \tag{8.5}$$

$$= \begin{pmatrix} -\sin\varphi & \cos\varphi & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\varphi} \end{pmatrix} = 0. \tag{8.6}$$

Next we differentiate with respect to time, to get the constraint on the accelerations $\ddot{x}_i$,

$$\dot{S}_1 = S_{1i}\ddot{x}_i + S_{1i,j}\dot{x}_i\dot{x}_j = 0 \tag{8.7}$$

$$= \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\varphi} \end{pmatrix} + \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\varphi} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} 0 & 0 & -\cos\varphi \\ 0 & 0 & -\sin\varphi \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\varphi} \end{pmatrix} = 0. \tag{8.8}$$

From which we see that the convective terms, $h_1 = S_{1i,j}\dot{x}_i\dot{x}_j$ in $\dot{S}_1 = S_{1i}\ddot{x}_i + h_1$, are,

$$h_1 = -\dot{x}\dot{\varphi}\cos\varphi - \dot{y}\dot{\varphi}\sin\varphi. \tag{8.9}$$

It's interesting to see that the the matrix $S_{1i,j}$ is non-symmetric in i and j. This is actually a hallmark of a nonholonomic constraint. In mathematical terms it means that there exists no primitive of S_1 or in other words, S_1 is not a derivative of a function in terms of the coordinates. To proof this, we assume we have a function for $\int S_1 \mathrm{d}t = f(x_i)$ then

$$S_1 = f_{,i}\dot{x}_i, \tag{8.10}$$

$$\dot{S}_1 = f_{,i}\ddot{x}_i + f_{,ij}\dot{x}_i\dot{x}_j, \tag{8.11}$$

with

$$f_{,ij} = \frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i} = f_{,ji}. \tag{8.12}$$

Ergo, $f_{,ij}$ is symmetric, but clearly $S_{1i,j}$ is not. Therefore, S_1 has no primitive function (is non integrable), and is a nonholonomic constraint.

8.1.2 Pure Rolling in Forward Direction

Next, we look at the pure rolling in forward or longitudinal direction. The constraint for that is, $v_{\parallel} = \dot{\beta}R$. And again we will rewrite this constraint as a zero slip condition, by introducing the longitudinal slip,

$$S_2 = \dot{x}\cos\varphi + \dot{y}\sin\varphi - \dot{\beta}R. \tag{8.13}$$

Pure rolling in the longitudinal direction is then expressed by $S_2 = 0$. Note that we now deal with a wheel, and that there are four coordinates for the rigid body, $x_i = (x, y, \varphi, \beta)^{\mathrm{T}}$. Then the longitudinal slip constraint in matrix-vector form is,

$$S_2 = S_{2i}\dot{x}_i = 0 \tag{8.14}$$

$$= \begin{pmatrix}\cos\varphi & \sin\varphi & 0 & -R\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\varphi}\\ \dot{\beta}\end{pmatrix} = 0. \tag{8.15}$$

Next we differentiate with respect to time, to get the constraint on the accelerations $\ddot{x}_i$,

$$\dot{S}_2 = S_{2i}\ddot{x}_i + S_{2i,j}\dot{x}_i\dot{x}_j = 0 \tag{8.16}$$

$$= \begin{pmatrix}\cos\varphi\\ \sin\varphi\\ 0\\ -R\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{\varphi}\\ \ddot{\beta}\end{pmatrix} + \begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\varphi}\\ \dot{\beta}\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}0 & 0 & -\sin\varphi & 0\\ 0 & 0 & \cos\varphi & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{\varphi}\\ \dot{\beta}\end{pmatrix} = 0. \tag{8.17}$$

And the convective terms, $h_2 = S_{2i,j}\dot{x}_i\dot{x}_j$ in $\dot{S}_2 = S_{2i}\ddot{x}_i + h_2$, are

$$h_2 = -\dot{x}\dot{\varphi}\sin\varphi + \dot{y}\dot{\varphi}\cos\varphi. \tag{8.18}$$

Again we see that zero longitudinal slip, $S_2 = 0$, is clearly a nonholonomic constraint for which no primitive (function in terms of the coordinates) exists. The condition for zero longitudinal slip can only be expressed in terms of the velocities (and the coordinates).

8.2 Nonholonomic System Constraint EOM

Next we derive the constraint equations of motion for a system with nonholonomic constraints. This is completely along the lines of previous derivation of DAE with regular or holonomic constraints.

We start with the virtual power of the applied and inertia forces for the unconstrained system,

$$\delta P = (F_i - M_{ij}\ddot{x}_j)\,\delta\dot{x}_i. \tag{8.19}$$

Next we add that the virtual velocities should fulfil the nonholonomic constraints,

$$S_k = S_{ki}\dot{x}_i = 0, \qquad k = 1...\#\text{nonholonomic constraints.} \tag{8.20}$$

Instead of real velocities $\dot{x}_i$ we use virtual velocities $\delta\dot{x}_i$:

$$\delta S_k = S_{ki}\delta\dot{x}_i = 0. \tag{8.21}$$

Add these constraints by means of the Lagrange multipliers λ_k to the virtual power expression,

$$\begin{aligned}\delta P^* = (F_i - M_{ij}\ddot{x}_j)\delta\dot{x}_i - \lambda_k\delta S_k = 0 \\ \forall\,\{\delta S_k \,|\, \delta S_k = S_{ki}\delta\dot{x}_i \text{ and } \delta\dot{x}_i \neq 0\}\end{aligned} \tag{8.22}$$

$$= (F_i - M_{ij}\ddot{x}_j)\delta\dot{x}_i - \lambda_k S_{ki}\delta\dot{x}_i = 0 \qquad \forall\,\delta\dot{x}_i \neq 0. \tag{8.23}$$

Note that the term $\lambda_k\delta S_k$ is power and since δS_k is a slip or velocity, clearly λ_k is the constraint force. For arbitrary $\delta\dot{x}_i \neq 0$ we get i equations of motion,

$$F_i - M_{ij}\ddot{x}_j - \lambda_k S_{ki} = 0, \tag{8.24}$$

together with k constraints,

$$S_{ki}\dot{x}_i = 0. \tag{8.25}$$

Differentiate the constraints with respect to time to get them on the level of acceleration constraints, and add them to the equations of motion,

$$M_{ij}\ddot{x}_j + S_{ki}\lambda_k = F_i, \tag{8.26}$$

$$S_{kj}\ddot{x}_j = -S_{kj,l}\dot{x}_j\dot{x}_l. \tag{8.27}$$

These DAE for a nonholonomic constrained system can be written in mixed index matrix-vector form as,

$$\begin{pmatrix} M_{ij} & S_{ki} \\ S_{kj} & \mathbf{0} \end{pmatrix}\begin{pmatrix} \ddot{x}_j \\ \lambda_k \end{pmatrix} = \begin{pmatrix} F_i \\ -S_{kj,l}\dot{x}_j\dot{x}_l \end{pmatrix}. \tag{8.28}$$

Given the state x_i, $\dot{x}_i$ at time t you can solve for the accelerations $\ddot{x}_i$ (and the constraint forces λ_k) at time t. Note that the velocities $\dot{x}_i$ must fulfil the constraints $S_{ki}\dot{x}_i = 0$.

8.3 Numerical Integration of Nonholonomic Constrained Systems

To obtain the motion in time of such a nonholonomic constrained system we have to use numerical integration. Since we have constraints, this will introduce drift in the solution, which we will solve by a Gauß-Newton minimisation process, after each numerical integration step. Just like we used it in the coordinate projection method for holonomic constrained systems.

After one numerical integration step we obtain a predicted solution for for the coordinates $\bar{x}_i$ and the speeds $\dot{\bar{x}}_i$,

$$\bar{x}_i(t + \mathrm{d}t) = x_i(t) + \int \dot{x}_i(t)\,\mathrm{d}t, \tag{8.29}$$

$$\dot{\bar{x}}_i(t + \mathrm{d}t) = \dot{x}_i(t) + \int \ddot{x}_i(t)\,\mathrm{d}t. \tag{8.30}$$

If there are *no* holonomic constraints, then the predicted coordinates are correct, but the predicted velocities $\dot{\bar{x}}_i$ will in general not be on the constraint surface. In other words

$$C_{ki}\dot{\bar{x}}_i \neq 0. \tag{8.31}$$

This can easily be solved by a one-step Gauß-Newton proces. One-step, because we correct the speeds $\dot{x}_i$ and this minimisation problem is *linear* in the speeds. The one-step Gauß-Newton method applied on the speeds $\dot{\bar{x}}_i$ is,

$$\begin{aligned} C_{ki}\Delta\dot{x}_i &= -C_{ki}\dot{\bar{x}}_i, \\ \Delta\dot{x}_i &= -C_{ki}^{\mathrm{T}}\left(C_{ki}C_{ki}^{\mathrm{T}}\right)^{-1}C_{ki}\dot{\bar{x}}_i, \\ \dot{x}_i &= \dot{\bar{x}}_i + \Delta\dot{x}_i. \end{aligned}$$

8.4 2D Wheel Inertia Forces

To complete the treatment of the 2D upright wheel we have to supply the expressions for the inertia forces, $\boldsymbol{f}_{in} = -\boldsymbol{M}\ddot{\boldsymbol{x}}$. Or in other words, we have to define the mass matrix $\boldsymbol{M}$ for such a rigid body.

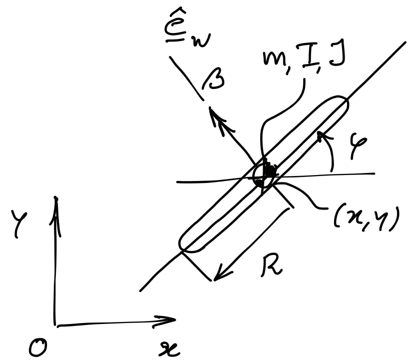

Figure 8.3: 2D wheel, top view with parameters and coordinates.

The Newton-Euler equations of motion for such a wheel, as depicted in Figure 8.3, are,

$$\left.\begin{aligned} m\ddot{x} &= \sum F_x \\ m\ddot{y} &= \sum F_y \\ I\ddot{\varphi} &= \sum M_\varphi \\ J\ddot{\beta} &= \sum M_\beta \end{aligned}\right\} \quad \Rightarrow \quad \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & J \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\varphi} \\ \ddot{\beta} \end{pmatrix} = \begin{pmatrix} \sum F_x \\ \sum F_y \\ \sum M_\varphi \\ \sum M_\beta \end{pmatrix},$$

with the total mass m, the mass moment of inertia I at the cm about the z-axis, and the mass moment of inertia J about the axis of the β-rotation of the wheel. For a thin homogeneous solid disk with mass m and radius R we have $I = \frac{1}{2}mR^2$, $J = mR^2$.

8.5 Holonomic and Nonholonomic System Constraint EOM

The derivation of the equations of motion for a system with both holonomic and nonholonomic constraints is now trivial, and the resulting equations are,

$$\begin{pmatrix} M_{ij} & C_{k,i} & S_{mi} \\ C_{k,j} & \mathbf{0} & \mathbf{0} \\ S_{mj} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \ddot{x}_j \\ \lambda_k \\ \lambda_m \end{pmatrix} = \begin{pmatrix} F_i \\ -C_{k,jl}\dot{x}_j\dot{x}_l \\ -S_{mj,l}\dot{x}_j\dot{x}_l \end{pmatrix}, \tag{8.32}$$

together with the holonomic constraints $C_k(x_i) = 0$, and the nonholonomic constraints $S_{mj}(x_i)\dot{x}_j = 0$.

8.6 Degrees of Freedom in Holonomic and Nonholonomic Constrained Systems

For nonholonomic constrained systems the term degree of freedom becomes confusing. Imagine we have a mechanical systems with n cm coordinates, nc holonomic constraints, and nh nonholonomic constraints. Then the number of free or unconstrained coordinates is equal to $n-nc$, and the number of free or unconstrained velocities is $n-nc-nh$, because all holonomic constraints will impose the same number of constraints on the velocities. Clearly we have more freedom in our coordinates than in our velocities. The degrees of freedom in the coordinates we will denote the kinematic degrees of freedom, whereas the degrees of freedom in the velocities we will denote by the velocity degrees of freedom, or in short, the mobility number.

Example 8.1 A classic example of a nonholonomic system is the so-called Chaplygin sleigh. This example was introduced by Sergey Alexeyevich Chaplygin (5 April 1869 – 8 October 1942), teacher of Leonid Ivanovitch Sedov (14 November 1907 – 5 September 1999). The Chaplygin sleigh is a rigid body which moves on a 2D plane and has *one* nonholonomic constraint in the form of a skate rigidly attached to the rigid body, as shown in Figure 8.4.

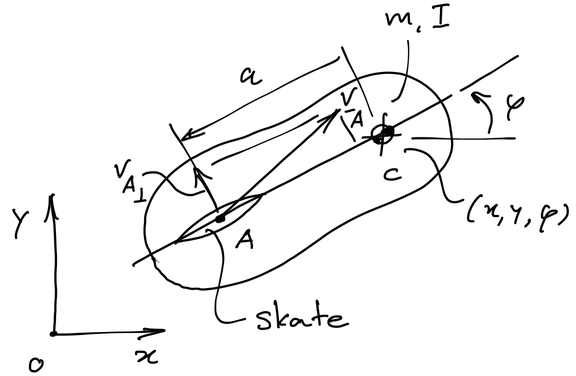

Figure 8.4: Chaplygin sleigh.

The skate in A can have no sideways velocity, and the nonholonomic constraint therefore is, $v_{A\perp} = 0$. The velocity in A can be expressed in terms of the velocities at the cm as,

$$\begin{aligned}
\boldsymbol{v}_A &= \boldsymbol{v}_C + \boldsymbol{\omega} \times \boldsymbol{r}_{A/C} \\
&= \begin{pmatrix} \dot{x} \\ \dot{y} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \dot{\varphi} \end{pmatrix} \times \begin{pmatrix} -a\cos\varphi \\ -a\sin\varphi \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \dot{x} + a\dot{\varphi}\sin\varphi \\ \dot{y} - a\dot{\varphi}\cos\varphi \\ 0 \end{pmatrix}.
\end{aligned}$$

Then, the sideward velocity in A is,

$$\begin{aligned}
v_{A\perp} &= \boldsymbol{v}_A \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} \\
&= -\dot{x}\sin\varphi + \dot{y}\cos\varphi - a\dot{\varphi}.
\end{aligned}$$

The zero sideslip constraint and it's derivative are therefore

$$\begin{aligned}
S_1 &= -\dot{x}\sin\varphi + \dot{y}\cos\varphi - a\dot{\varphi} = 0, \\
\dot{S}_1 &= -\ddot{x}\sin\varphi + \ddot{y}\cos\varphi - a\ddot{\varphi} + (-\dot{x}\dot{\varphi}\cos\varphi - \dot{y}\dot{\varphi}\sin\varphi) = 0.
\end{aligned}$$

Which, according to (8.28), results in the following constrained equations of motion for the Chaplygin sleigh,

$$\begin{pmatrix} m & 0 & 0 & -\sin\varphi \\ 0 & m & 0 & \cos\varphi \\ 0 & 0 & I & a \\ -\sin\varphi & \cos\varphi & -a & 0 \end{pmatrix} \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{\varphi} \\ \lambda \end{pmatrix} = \begin{pmatrix} \sum F_x \\ \sum F_y \\ \sum M_C \\ \dot{x}\dot{\varphi}\cos\varphi + \dot{y}\dot{\varphi}\sin\varphi \end{pmatrix}$$

together with the velocity constraint,

$$-\dot{x}\sin\varphi + \dot{y}\cos\varphi - a\dot{\varphi} = 0.$$

The Lagrange multiplier λ is in this case the lateral constraint force at the skate in A. With this one nonholonomic constraint the Chaplygin sleigh has 3 kinematic degrees of freedom and 2 velocity degrees of freedom, or a mobility of 2.

8.7 Problems

Problem 8.1 The EzyRoller, shown in the top Figure 8.5, is the ultimate riding machine for kids. The machine is propelled by an oscillating motion of the steering assembly, which is operated by the feet. For the proper operation watch the video[a]. A mechanical model for this machine is shown in the bottom figure. The model consists of two rigid bodies connected by a hinge in B. Each body has a rolling contact, for body 1, the wheel in A, and for body 2, the wheel in C. The centre of mass location is denoted by the numbers 1 and 2. With the help of this model we like to demonstrate the operation of this machine in a number of steps.

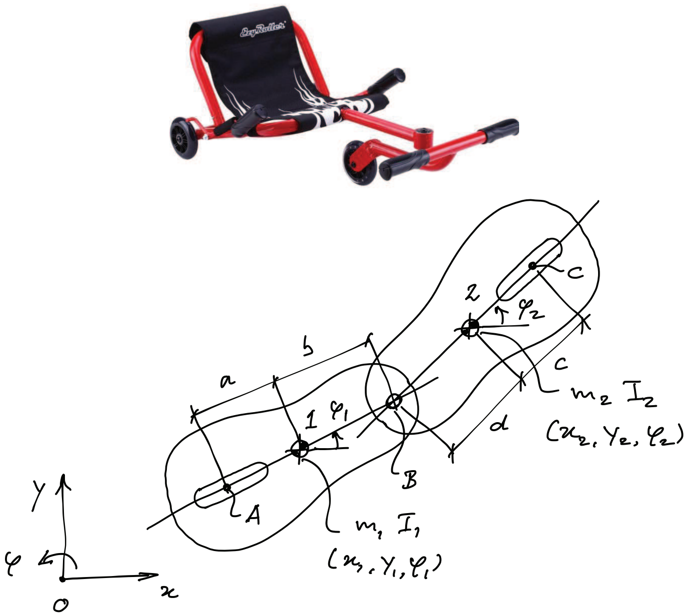

Figure 8.5: Ezyroller, physical system (top) and 2D model (bottom).

The system consists of two rigid bodies and therefore has six coordinates. The system has 2 holonomic constraints, the hinge in B, and two nonholonmic constraints, the rolling contact condition in A and C. Therefore you have two constraints on six coordinates and four constraints on the six velocities. So the number of degrees of freedom in the coordinates space is four, whereas you only have two degrees of freedom in the velocity space. We will solve this problem by setting up the constraint equations of motion in DAE form and stabilise the constraints by means of the Coordinate Projection Method. Note that there is now a difference between the constraints on the coordinates and the constraints on the velocities.

Address the following questions:

a. Derive the equations of motion for this system in DAE form.

b. Formulate the Coordinate Projection Method for the constraints on the coordinates and for the constraints on the velocities.

c. Implement the DAE together with the constraint stabilisation in a Matlab program. Try not to form the equations of motion in an explicit form but evaluate your equations in a step-by-step manner.

d. Determine the motion of the unpowered system by numerical integration of the equations of motion (use a fixed step RK4 method). For dimensions of the EzyRoller we take: $a = 0.5, b = 0.5, c = 0.125, d = 0.125, m_1 = 1, I_1 = 0.1, m_2 = 0, I_2 = 0$ (SI units). For initial conditions take some non-zero values for the coordinates and the speeds and show that the system is moving properly.

e. Add and action-reaction torque in the hinge such that it resembles an oscillating torque as applied by the rider on the steering assembly. Take for the torque the following function $M = M_0 \cos(\omega t)$, with $M_0 = 0.1$ and $\omega = \pi$ [rad/s]. Start from rest with body 1 and body 2 aligned along the x-axis, $\phi_1 = 0, \phi_2 = \pi$ (point C between A and B). Determine the motion of the system by numerical integration of the constraint equations of motion for 100 seconds.

f. Plot the path of point A and C.

g. Plot the linear and angular velocities of the CM's of body 1 and body 2.

h. Determine the Kinetic energy of the system and plot it as a function of time.

i. Determine the work done by the torque as a function of time and plot this as a function of time in the same figure as the Kinetic Energy of the system. What do you notice? Can you explain this?

[a] https://youtu.be/GMnJ9q1D4hU

9 From 2D to 3D

As a gentle introduction to the dynamics of three dimensional multibody systems which consists of rigid bodies with constraints, we will start by revisiting the Newton-Euler equations of motion for a rigid body in a two and three dimensional world.

9.1 Newton-Euler in 2D

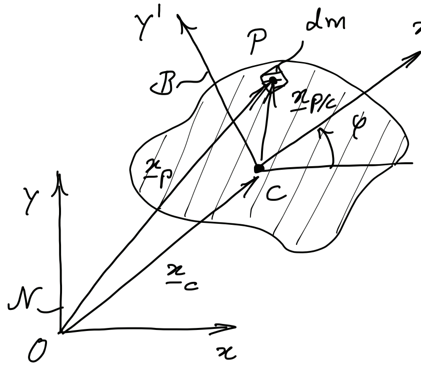

Figure 9.1: A rigid body in 2D.

The Newton-Euler equations of motion for a rigid body which moves in a two dimensional space are,

$$\sum F_{x_i} = m\ddot{x}_C, \tag{9.1}$$

$$\sum F_{y_i} = m\ddot{y}_C, \tag{9.2}$$

$$\sum M_C = I_C\ddot{\varphi}, \tag{9.3}$$

with the centre of mass position (x_C, y_C) and orientation φ of the rigid body with respect to an inertial frame $\mathcal{N}$, the mass m and mass moment of inertia at the centre of mass,

$$I_C = \int_V \left(x'^2 + y'^2\right)\,\mathrm{d}m. \tag{9.4}$$

Here, the integral over the volume is described in terms of a body fixed frame $\mathcal{B}$, with coordinates $x'y'$ and with the origin in C. But of course the resulting mass moment of inertia is invariant for rotation about the z-axis, so any frame centred in C would do. The first two equations are the Newton part, which describe the motion of the centre of mass of the rigid body as if it was a point mass. The third equation is the Euler part, which describes the change in orientation due to the applied torques about the z-axis. These applied torques are usually the result from the applied forces times the distance to the centre of mass. Before we dive into the three dimensional mechanics of rigid bodies, we will trace back the roots of the Euler equation of motion and the concept of mass moment of inertia in 2D.

A way of seeing how the mass moment of inertia turns up in the equations is the following. We look at infinitesimal mass elements $\mathrm{d}m_i$ at postion (x_i, y_i), as shown in Figure 9.2.

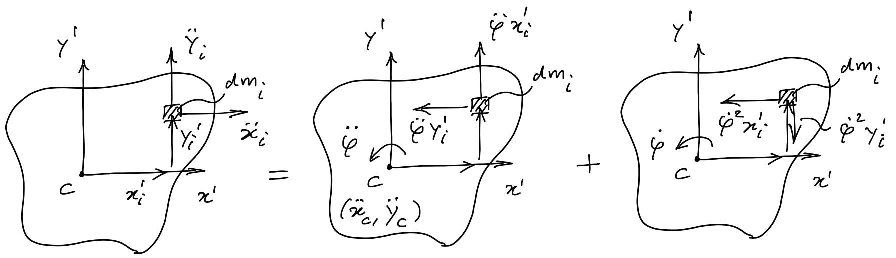

Figure 9.2: Accelerations of an infinitesimal mass element $\mathrm{d}m_i$ of a rigid body.

Because this is a rigid body, we can determine the position of the mass element, it's velocity and acceleration, as a function of the postion of the centre of mass (x_C, y_C) and the orientation of the body φ, and their time derivatives. The accelerations of the mass element, expressed in the inertial frame $\mathcal{N}$ are,

$$\ddot{x}_i = \ddot{x}_C - y'_i\ddot{\varphi} - x'_i\dot{\varphi}^2,$$
$$\ddot{y}_i = \ddot{y}_C + x'_i\ddot{\varphi} - y'_i\dot{\varphi}^2.$$

Then the Newton equations of motion become,

$$\sum F_x = \int \ddot{x}_i \,\mathrm{d}m_i = \int_V \left(\ddot{x}_C - y'_i\ddot{\varphi} - x'_i\dot{\varphi}^2\right)\,\mathrm{d}m_i$$
$$\sum F_y = \int \ddot{y}_i \,\mathrm{d}m = \int_V \left(\ddot{y}_C - x'_i\ddot{\varphi} - y'_i\dot{\varphi}^2\right)\,\mathrm{d}m_i.$$

Since C is the center of mass we know that static moments about the x' and y' axes are zero, that is $\int y_i' \,\mathrm{d}m = 0$ and $\int x_i' \,\mathrm{d}m = 0$, which is actually the definition of the centre mass. Therefore, the Newton equations of motion for a rigid body are,

$$\sum F_x = m\ddot{x}_C \quad \text{and} \quad \sum F_y = m\ddot{y}_C.$$

Next we equate the sum of the applied torques to the inertial torques which arise from the torques from the inertia forces $\mathrm{d}F_x = \ddot{x}_i \mathrm{d}m_i$ and $\mathrm{d}F_y = \ddot{y}_i \mathrm{d}m_i$ at a distance (x_i', y_i') from the centre of mass,

$$\begin{aligned}
\sum M_C &= \int -\left(\ddot{x}_C - y_i'\ddot{\varphi} - x_i'\dot{\varphi}^2\right) y_i' + \left(\ddot{y}_C - x_i'\ddot{\varphi} - y_i'\dot{\varphi}^2\right) x_i' \,\mathrm{d}m_i \\
&= \int -\ddot{x}_C y_i' - \ddot{y}_c x_i' + \left({y_i'}^2 + {x_i'}^2\right)\ddot{\varphi} + \left(x_i' y_i' - y_i' x_i'\right)\dot{\varphi}^2 \,\mathrm{d}m_i \\
&= \int \left({x_i'}^2 + {y_i'}^2\right) \mathrm{d}m_i \,\ddot{\varphi} \\
&= I_C \ddot{\varphi}
\end{aligned}$$

and we see that we end up with the Euler equation of motion for a rigid body, where the mass moment of inertia I_C (9.4) shows up as an invariant quantity of the rigid body.

9.2 Newton-Euler in 3D

Transforming the Newton part from 2D to 3D is easy, because it is just like a point mass and we simply add the Newton equation of motion for the third direction,

$$\left.\begin{aligned}
\sum F_{x_i} &= m\ddot{x}_C \\
\sum F_{y_i} &= m\ddot{y}_C \\
\sum F_{z_i} &= m\ddot{z}_C
\end{aligned}\right\} \quad \Rightarrow \quad \sum \boldsymbol{F}_i = m\ddot{\boldsymbol{x}}_C. \tag{9.5}$$

For the Euler equations of motion, which describes the rotational motions of the rigid body under the action of applied forces and moments, we have to look at the rigid body in 3D, as shown in Figure 9.3. We can proceed exactly along similar lines as in the 2D case, but since we now have three angular speeds and three angular accelerations, the work becomes somewhat more complex.

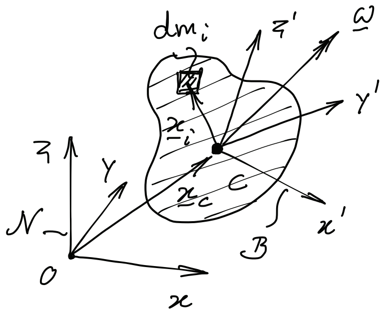

Figure 9.3: A rigid body in 3D.

We start with the kinematics in 3D. The speed and acceleration of an infinitesimal mass element $\mathrm{d}m_i$ of the rigid body at the position $\boldsymbol{x}_i'$, are,

$$\dot{\boldsymbol{x}}_i' = \dot{\boldsymbol{x}}_C + \boldsymbol{\omega} \times \boldsymbol{x}_i', \tag{9.6}$$
$$\ddot{\boldsymbol{x}}_i' = \ddot{\boldsymbol{x}}_C + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \boldsymbol{x}_i'\right) + \dot{\boldsymbol{\omega}} \times \boldsymbol{x}_i', \tag{9.7}$$

with the centre of mass coordinates $\boldsymbol{x}_C$, the angular speeds $\boldsymbol{\omega}$, and angular accelerations $\dot{\boldsymbol{\omega}}$. The torques of the applied forces and moments at the centre of mass are now equal to torques from the inertia forces $\mathrm{d}\boldsymbol{F}_i = \ddot{\boldsymbol{x}}_i'\mathrm{d}m_i$ at a distance $\boldsymbol{x}_i$ from the centre of mass, as in,

$$\sum \boldsymbol{M}_C = \int \boldsymbol{x}_i \times \mathrm{d}\boldsymbol{F}_i = \int \boldsymbol{x}_i \times \ddot{\boldsymbol{x}}_i' \,\mathrm{d}m_i. \tag{9.8}$$

Substitution of the accelerations from (9.2) and expanding of terms, where all static moments at the cm cancel out, results finally in the Euler equations of motion for a rigid body,

$$\sum \boldsymbol{M}_C = \mathbf{I}_C\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \left(\mathbf{I}_C\boldsymbol{\omega}\right), \tag{9.9}$$

with the angular velocity vector $\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)^{\mathrm{T}}$ and the corresponding angular accelerations $\dot{\boldsymbol{\omega}} = (\dot{\omega}_x, \dot{\omega}_y, \dot{\omega}_z)^{\mathrm{T}}$, and the mass moment of inertia matrix

$$\mathbf{I}_C = \begin{pmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{pmatrix}. \tag{9.10}$$

The individual entries in the mass moment of inertia matrix, where the off-diagonal terms are often called products of inertia, are now defined as,

$$I_{xx} = \int \left(y^2 + z^2\right) \, \mathrm{d}m, \tag{9.11}$$

$$I_{yy} = \int \left(x^2 + z^2\right) \, \mathrm{d}m, \tag{9.12}$$

$$I_{zz} = \int \left(x^2 + y^2\right) \, \mathrm{d}m, \tag{9.13}$$

$$I_{xy} = I_{yx} = \int xy \, \mathrm{d}m, \tag{9.14}$$

$$I_{xz} = I_{zx} = \int xz \, \mathrm{d}m, \tag{9.15}$$

$$I_{yz} = I_{zy} = \int yz \, \mathrm{d}m. \tag{9.16}$$

Clearly the mass moments of inertia matrix is symmetric. We can transform this matrix to a diagonal matrix by means of the eigenvectors. The eigenvectors are the principal axes along which we have the eigenvalues as principal values of mass moments of inertia. The transformed mass moment of inertia matrix then takes on the form,

$$\bar{\mathbf{I}}_C = \begin{pmatrix} \bar{I}_{xx} & 0 & 0 \\ 0 & \bar{I}_{yy} & 0 \\ 0 & 0 & \bar{I}_{zz} \end{pmatrix}. \tag{9.17}$$

The principal axes and the principal values $\bar{I}_{xx}$, $\bar{I}_{yy}$, and $\bar{I}_{zz}$ describe the mass moment of inertia properties of the rigid body completely.

○ **Problem 9.1** Derive the Euler equations of motion for a 3D rigid body (9.9) from the expressions (9.8). It may be useful to use the tilde matrix notation for the cross product, as described in Appendix B, and or use symbolic computing (Maple or Matlab symbolic toolbox).

Let us now have a closer look at the Euler equations (9.9). If the body rotates then $\mathbf{I}_C$ is generally *not* constant. Therefore, it is often better to express Euler in a body-fixed frame $\mathcal{B}$:

$$\sum {}^{\mathcal{B}}\boldsymbol{M}_C = {}^{\mathcal{B}}\mathbf{I}_C \, {}^{\mathcal{B}}\dot{\boldsymbol{\omega}} + {}^{\mathcal{B}}\boldsymbol{\omega} \times \left({}^{\mathcal{B}}\mathbf{I}_C \, {}^{\mathcal{B}}\boldsymbol{\omega}\right), \tag{9.18}$$

with ${}^{\mathcal{B}}\boldsymbol{\omega}$ the angular velocity in the (local) body-fixed frame, ${}^{\mathcal{B}}\boldsymbol{M}_C$ the moment in the local body-fixed frame, and ${}^{\mathcal{B}}\mathbf{I}_C$ a new constant matrix.

It is usually convenient to align this frame with the principal axes of the body such that we get a diagonal mass moment of inertia matrix,

$$
{}^{\mathcal{B}}\mathbf{I}_C = \begin{pmatrix} {}^{\mathcal{B}}I_{xx} & 0 & 0 \\ 0 & {}^{\mathcal{B}}I_{yy} & 0 \\ 0 & 0 & {}^{\mathcal{B}}I_{zz} \end{pmatrix}. \tag{9.19}
$$

Summarising, the Newton-Euler equations of motion for a rigid body as shown in Figure 9.3, where the Euler part is expressed in a body fixed frame $\mathcal{B}$, which are assumed to be the principal axes of the body, are,

$$
\sum F_x = m\ddot{x}_C, \tag{9.20}
$$

$$
\sum F_y = m\ddot{y}_C, \tag{9.21}
$$

$$
\sum F_z = m\ddot{z}_C, \tag{9.22}
$$

$$
\sum {}^{\mathcal{B}}M_x = {}^{\mathcal{B}}I_{xx}{}^{\mathcal{B}}\dot{\omega}_x - \left({}^{\mathcal{B}}I_{yy} - {}^{\mathcal{B}}I_{zz}\right){}^{\mathcal{B}}\omega_y{}^{\mathcal{B}}\omega_z, \tag{9.23}
$$

$$
\sum {}^{\mathcal{B}}M_y = {}^{\mathcal{B}}I_{yy}{}^{\mathcal{B}}\dot{\omega}_y - \left({}^{\mathcal{B}}I_{zz} - {}^{\mathcal{B}}I_{xx}\right){}^{\mathcal{B}}\omega_z{}^{\mathcal{B}}\omega_x, \tag{9.24}
$$

$$
\sum {}^{\mathcal{B}}M_z = {}^{\mathcal{B}}I_{zz}{}^{\mathcal{B}}\dot{\omega}_z - \left({}^{\mathcal{B}}I_{xx} - {}^{\mathcal{B}}I_{yy}\right){}^{\mathcal{B}}\omega_x{}^{\mathcal{B}}\omega_y. \tag{9.25}
$$

Example 9.1 The 3D Euler equations look strange and complex, but 3D motion of rigid bodies can be strange and complex. For example think of the following question: if you throw a rotating 3D rigid body in space will it keep rotating about the initial axis or will it start wobbling? Think of a satellite, a tennis racket or a remote control thrown up in the air.

Assume a 3D rigid body with an initial angular velocity about one of the principal axis, say the first, $\boldsymbol{\omega} = (\omega,\, 0,\, 0)^{\mathrm{T}}$. Will this stay this way or will the body start rotating along one of the other axes and wobble? In other words, is the rotation about this axis stable?

We will analyse everything in a body-fixed frame $\mathcal{B}$ with principal axes at the center of mass. Assume an initial angular velocity ${}^{\mathcal{B}}\boldsymbol{\omega}_0 = (\omega,\, 0,\, 0)^{\mathrm{T}}$. The inertia matrix is ${}^{\mathcal{B}}\mathbf{I}_C = \mathrm{diag}(A,\, B,\, C)$ and there are no applied torques, $\sum \boldsymbol{M} = \mathbf{0}$. The Euler equation of motion, expressed in the body fixed frame is,

$$
\sum {}^{\mathcal{B}}\boldsymbol{M}_C = {}^{\mathcal{B}}\mathbf{I}_C\,{}^{\mathcal{B}}\dot{\boldsymbol{\omega}} + {}^{\mathcal{B}}\boldsymbol{\omega} \times \left({}^{\mathcal{B}}\mathbf{I}_C\,{}^{\mathcal{B}}\boldsymbol{\omega}\right).
$$

We assume small changes from the steady motion, $\sum \boldsymbol{M} = \boldsymbol{M}_0 + \Delta\boldsymbol{M}$ and $\boldsymbol{\omega} = \boldsymbol{\omega}_0 + \Delta\boldsymbol{\omega} \;\Rightarrow\; \dot{\boldsymbol{\omega}} = \dot{\boldsymbol{\omega}}_0 + \Delta\dot{\boldsymbol{\omega}}$. Substitution in the Euler equations of motion results in,

$$\begin{aligned}\boldsymbol{M}_0 + \Delta\boldsymbol{M} &= \mathbf{I}_C\left(\dot{\boldsymbol{\omega}}_0 + \Delta\dot{\boldsymbol{\omega}}\right) && + \left(\boldsymbol{\omega}_0 + \Delta\boldsymbol{\omega}\right) \times \left(\mathbf{I}_C\left(\boldsymbol{\omega}_0 + \Delta\boldsymbol{\omega}\right)\right) \\ &= \mathbf{I}_C\dot{\boldsymbol{\omega}}_0 + \mathbf{I}_C\Delta\dot{\boldsymbol{\omega}} && + \boldsymbol{\omega}_0 \times \mathbf{I}_C\boldsymbol{\omega}_0 + \boldsymbol{\omega}_0 \times \mathbf{I}_C\Delta\boldsymbol{\omega} \\ & && + \Delta\boldsymbol{\omega} \times \mathbf{I}_C\boldsymbol{\omega}_0 + \cancelto{\boldsymbol{O}(\Delta^2)}{\Delta\boldsymbol{\omega} \times \mathbf{I}_C\Delta\boldsymbol{\omega}}.\end{aligned}$$

The applied moments are zero and the cross product of this angular velocity vector (with only one nonzero element) with an inertia scaled version of itself is also zero. Therefore the zero order equations are,

$$\cancelto{0}{\boldsymbol{M}_0} = \mathbf{I}_C\dot{\boldsymbol{\omega}}_0 + \cancelto{0}{\boldsymbol{\omega}_0 \times \mathbf{I}_C\boldsymbol{\omega}_0} \quad \Rightarrow \quad \dot{\boldsymbol{\omega}}_0 = \mathbf{0} \quad (\boldsymbol{\omega}_0 = \text{constant}),$$

which are what we expected. The first order perturbation equations are now,

$$\Delta\boldsymbol{M} = \mathbf{I}_C\Delta\dot{\boldsymbol{\omega}} + \boldsymbol{\omega}_0 \times \mathbf{I}_C\Delta\boldsymbol{\omega} + \Delta\boldsymbol{\omega} \times \mathbf{I}_C\boldsymbol{\omega}_0.$$

The applied torques (which are zero) do not chance, and therefore $\Delta\boldsymbol{M} = \mathbf{0}$. Next we write for the change in the angular speed $\Delta\boldsymbol{\omega} = (\alpha, \beta, \gamma)$, and with this notation we expand the terms, resulting in,

$$\begin{aligned}\mathbf{0} &= \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} && + \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} A\alpha \\ B\beta \\ C\gamma \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \times \begin{pmatrix} A\omega \\ 0 \\ 0 \end{pmatrix} \\ & && + \begin{pmatrix} 0 \\ -\omega C\gamma \\ \omega B\beta \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma A\omega \\ -\beta A\omega \end{pmatrix} \\ &= \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}\begin{pmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{pmatrix} && + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & (A-C)\,\omega \\ 0 & (B-A)\,\omega & 0 \end{pmatrix}\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.\end{aligned}$$

This is a set of first order differential equations for the changes in the angular speeds $\Delta\boldsymbol{\omega}$. Assuming exponential solutions of the form $\Delta\boldsymbol{\omega} = \hat{\boldsymbol{a}}e^{\lambda t}$ leads to the following eigenvalue problem,

$$\begin{pmatrix} A\lambda & 0 & 0 \\ 0 & B\lambda & (A-C)\,\omega \\ 0 & (B-A)\,\omega & C\lambda \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} e^{\lambda t} = \mathbf{0}$$

For a nontrivial solution ($\hat{\boldsymbol{a}} \neq \mathbf{0}$) the determinant of the leading matrix should be zero, which leads to the following characteristic equation,

$$\begin{aligned} ABC\lambda^3 + 0 + 0 - (B-A)\,\omega\,(A-C)\,\omega A\lambda - 0 &= 0, \\ A\lambda\left(BC\lambda^2 - (A-C)\,(B-A)\,\omega^2\right) &= 0, \end{aligned}$$

from which we can solve the eigenvalues as,

$$\Rightarrow \qquad \lambda_1 = 0 \qquad \text{and} \qquad \lambda_{2,3} = \sqrt{\frac{(A-C)\,(B-A)}{BC}}\,\omega.$$

If the non-zero eigenvalues are all positive, $\lambda_{2,3} > 0$, then the system is unstable, which is the case when,

$$\begin{array}{lccccc} & A > C & \text{and} & B > A & \Rightarrow & B > A > C, \\ \text{or} & C > A & \text{and} & A > B & \Rightarrow & C > A > B. \end{array}$$

And we conclude that rotation about the axis which has the intermediate mass moment of inertia is unstable! Or in other words, stable torque free rotation of a rigid body in space is only possible around the axis which has either the smallest or the largest mass moment of inertia.

As an example of this we look at a remote control (or a smartphone if you want) of mass m and dimensions as drawn in Figure 9.4.

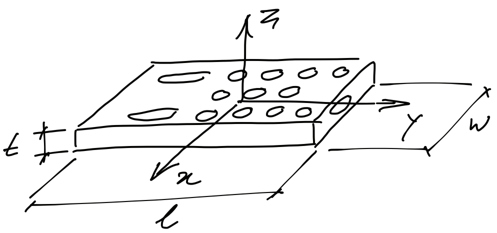

Figure 9.4: Remote control with its three principal axes.

The mass moments of inertia around the three principal axes are,

$$\begin{aligned} I_{xx} &= m\left(l^2 + t^2\right), \\ I_{yy} &= m\left(w^2 + t^2\right), \\ I_{zz} &= m\left(l^2 + w^2\right). \end{aligned}$$

With the length ratios as depicted in the figure we see that from large to small we have $I_{zz} > I_{xx} > I_{yy}$. Rotation around the remote control's y- and z-axis is thus stable. However, rotation about the x-axis is unstable (do try this at home by flipping up the remote in the air with a rotation about the x-axis and observe it wobbling, irrespectively of how hard you try to initiate a pure rotation about the x-axis). A similar analysis can be conducted for other objects with easy distinctable mass moments of inertia around their principal axes, e.g. the tennis racket shown in Figure 9.5.

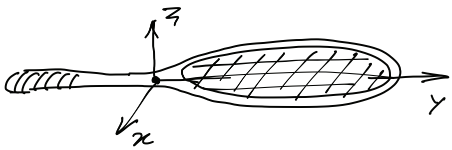

Figure 9.5: Tennis racket with its three principal axes, torque free rotation about the x-axis is unstable.

Appendices

B Cross Product, Tilde Notation

The cross product or vector product in 3D of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ results in a vector, which is perpendicular to the two vectors, and is defined as,

$$\boldsymbol{w} = \boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \times \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} yc - zb \\ za - xc \\ xb - ya \end{pmatrix} \tag{B.1}$$

A convenient way to write and work with this cross product is by the use of the so-called tilde notation, where we write the cross product as the result of a matrix-vector multiplication. For the cross product we can write,

$$\boldsymbol{u} \times \boldsymbol{v} = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \tilde{\boldsymbol{u}}\boldsymbol{v}, \tag{B.2}$$

where we have introduced the matrix $\tilde{\boldsymbol{u}}$ (pronounce as u-tilde) from the vector $\boldsymbol{u}$, which is defined by,

$$\tilde{\boldsymbol{u}} = \begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}. \tag{B.3}$$

Note that this matrix is skew symmetric.

An example of the usage of this notation is for instance in kinematics in the expression for the centrifugal acceleration,

$$\boldsymbol{a}_c = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x}). \tag{B.4}$$

With the tilde notation this becomes simply the product of two matrices and a vector,

$$\boldsymbol{a}_c = \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{x}) = \tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\boldsymbol{x}. \tag{B.5}$$

Expanding the product of these two tilde matrices results in the following expressions,

$$\tilde{\boldsymbol{\omega}}\tilde{\boldsymbol{\omega}}\boldsymbol{x} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{B.6}$$

$$= \begin{pmatrix} -(\omega_y^2 + \omega_z^2) & \omega_y\omega_x & \omega_z\omega_x \\ \omega_x\omega_y & -(\omega_z^2 + \omega_x^2) & \omega_z\omega_y \\ \omega_x\omega_z & \omega_y\omega_z & -(\omega_x^2 + \omega_y^2) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \tag{B.7}$$

for the centrifugal accelerations. Which we can easily check for a simplified case in 2D, where we have rotation about the z-axis only, $\boldsymbol{\omega} = (0,\, 0,\, \omega_z)^{\mathrm{T}}$, and the centrifugal accelerations are $(-\omega_z^2 x,\, -\omega_z^2 y,\, 0)$.