MANDELBROT SET

TAE KYU KIM

1. Introduction

In this paper, I introduce and show some interesting results about the Mandelbrot Set. The Mandelbrot Set is part of complex dynamics, a relatively new field that has some of its earliest results published in the early 20th century by Pierre Fatou and Gaston Julia. Unfortunately, many results are very deep and requires complex analysis, so I am not able to provide proofs for them. However, I included views of the Mandelbrot Set that show convincing evidence for the theorems.

2. Preliminary Definitions

Definition 2.1. Define the quadratic polynomial $P_c(z) = z^2 + c$ for all complex numbers c. We define the Mandelbrot Set as follows:

$$M = \{c \in \mathbb{C} : \lim_{n\to\infty} P_c^{(n)}(0) \not\to \infty\}$$

where $P_c^{(n)}$ denotes the nth iteration of P_c. [Bra89]

For instance, we would have $0, -2 \in M$ because the sequences obtained from repeatedly applying the corresponding P_c to 0 are

$$0, 0, 0, \ldots \not\to \infty$$

and

$$0, -2, 2, 2, 2, \ldots \not\to \infty.$$

For some c-values such as $c = 1$, the sequence of values obtained by repeatedly applying P_c to 0 clearly diverges.

Proposition 2.2. *Any quadratic polynomial can be turned into the form z^2+c through some translation and stretches. That is, for any quadratic function $f(z)$, there exists a function $h(z) = az + b$ such that $P_c = h^{-1} \circ f \circ h$ for some P_c. [Bra89]*

The proposition gives us an idea as to why we might study the Mandelbrot set: By studying the set of all P_c in the form $z^2 + c$, we essentially study the dynamic behaviors of all quadratic polynomials in $\mathbb{C}$.

Definition 2.3. For a polynomial P, we call the sequence

$$z_0, z_1 = P(z_0), \ldots, z_{n+1} = P(z_n), \ldots$$

the orbit of z_0 under iteration.

Date: November 29, 2019.

We are primarily interested in z_0 such that their iterations cycle over the same set of points. Specifically, we say that z_0 is periodic with period k if

$$z_k = z_0 \quad \text{and} \quad z_j \neq z_0 \quad \text{for} \quad 0 < j < k.$$

Definition 2.4. For periodic points with period k, we can define the multiplier ρ of the cycle as the derivative of $P^{(k)}$ at z_0, and by the chain rule we have

$$(P^{(k)})'(z_0) = \prod_{j=0}^{k-1} P'(z_j),$$

so all points in a particular cycle have the same multiplier. As a result, we can talk about the multiplier of the cycle (not just one point).

We can classify the periodic cycles by their multiplier: We call a cycle

- attracting if $|\rho| < 1$
- repelling if $|\rho| > 1$
- superattracting if $|\rho| = 0$
- neutral if $|\rho| = 1$

Using the Taylor series at a point z_0 periodic with period k and multiplier ρ, we can analyze what happens near z_0 for different ρ values. We have the approximation

$$P^{(k)}(z_0 + d) = z_0 + \rho d + \dots .$$

For small enough d, we can ignore the terms after the second term. Then for $\rho < 1$, applying $P^{(k)}$ repeatedly brings $z_0 + d$ closer to z_0, and for $\rho > 1$, the iteration gets us further away from z_0. Hence, the cycles are given the respective names attracting and repelling. [Bra89]

3. Some important theorems and results

Definition 3.1. For the any polynomial P_c, we call the values ω at which $P'(\omega) = 0$ to be the critical points of P_c. In particular, $\omega = 0$ for all P_c.

Critical points are helpful because of the following two results:

Theorem 3.2. *(P. Fatou) Every attracting cycle of a polynomial (or a rational function) attracts at least one critical point. [Bra89]*

We omit the proof as it requires deep results from complex analysis, according to page 21 of this source (`https://math.la.asu.edu/~dummit/docs/dynamics_5_introduction_to_complex_dynamics.pdf`).

Proposition 3.3. *The dynamical behavior of a function is dominated by the behavior of the critical points. [Bra89]*

Intuitively, this seems plausible as the derivative of the function is 0 at a critical point, so other points will be attracted to the behaviors of 0. However, it is not easy to see why all of the points must follow this pattern. The proof of this result requires extensive discussion of the Julia sets and their connections to the Mandelbrot set (See the Fundamental Dichotomy).

The proposition essentially means that we only have to follow the dynamical behavior of the critical points to know about the dynamical behaviors of the other points, since they should follow the critical points.

For the Mandelbrot set in which the only critical point is 0 for all polynomials, this result means that the behavior of all starting values in $\mathbb{C}$ follow the behavior of the seed 0. Hence, any c-value for which P_c has an attracting cycle will be in M. This gives us a very easy way to check if a particular c is in M using a computer (just apply P_c to 0 repeatedly until you eventually get to a big enough number that you know will start to diverge, or reach your upper bound for number of iteration).

We can begin to classify the bulbs of the Mandelbrot set to observe their properties.

Definition 3.4. We can let $H(M)$ denote such values of c:

$$H(M) = \{c \in \mathbb{C} | P_c \text{ has an attracting cycle.}\}$$

Then because of the above results, all points in $H(M)$ must be in M. However, it is an open problem whether $H(M)$ contains *all* of the points in the interior of M. Although it is conjectured to be true, we do not know for sure if they are equivalent.

We can further break up the set $H(M)$ into its connected components:

Definition 3.5. A connected component W of $H(M)$ is called the hyperbolic component of M.

In topology, a connected component is the smallest open set that cannot be broken up into the union of multiple disjoint open sets.

4. Results about the Mandelbrot Set bulbs

Proposition 4.1. *The main cardioid of the Mandelbrot set contains all of the points that contain an attracting fixed point. [MSC17]*

Proof. If z is an attracting fixed point of P_c, then clearly $z^2 + c = z$ and $P_c'(z) = 2z < 1$. We make the substitution $z = re^{i\theta}$. The points on the boundary of such points will have $2z = 1$, or $2r = 1$, or $z = \frac{1}{2}e^{i\theta}$. Plugging back into our first equation, we get $4c = 2e^{i\theta} - e^{2i\theta}$. As θ goes from 0 to 2π, it can be shown that our equation (precisely, the set of c) will trace out a cardioid. As this is our boundary, the actual set of points with attracting fixed points will be in the interior of the cardioid (for $0 \leq r < 1/2$). ■

Proposition 4.2. *The set of points with attracting cycles of period* 2 *is the circle centered at* $z = -1$ *with radius* $\frac{1}{4}$. *[MSC17]*

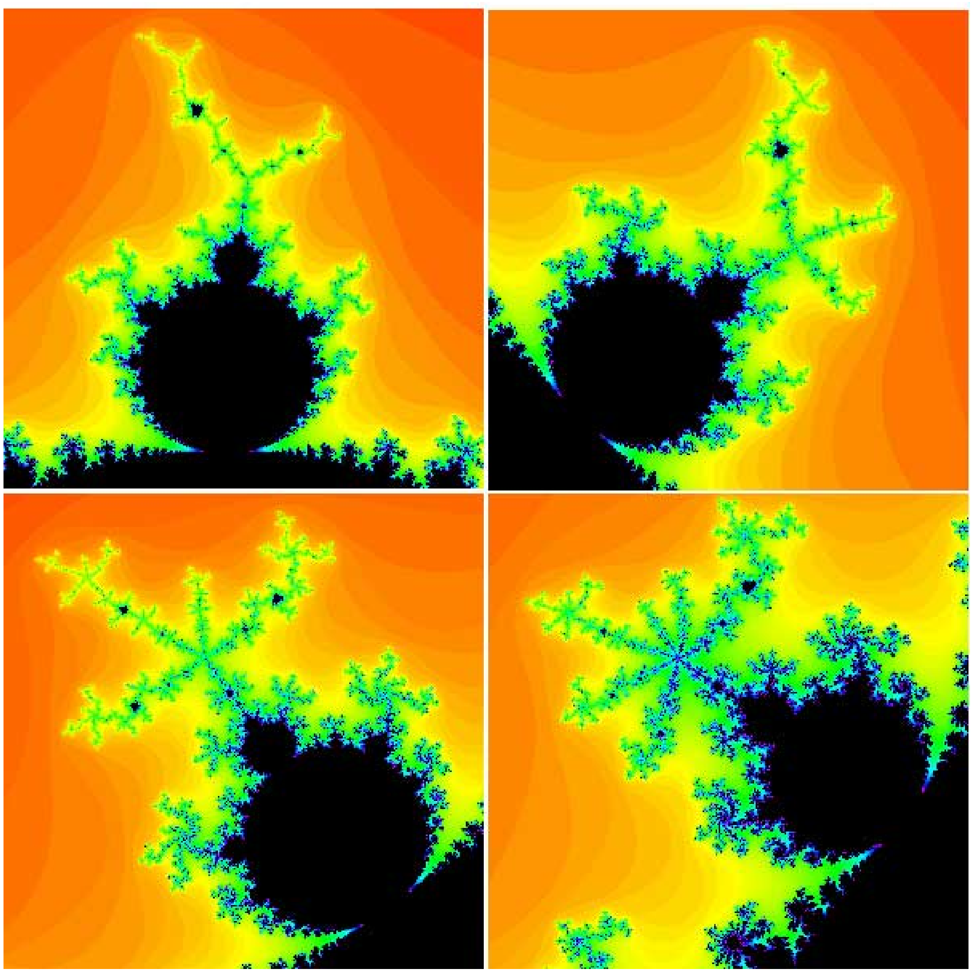
Proof. We want $P_c(P_c(z)) = z$, so $((z^2 + c)^2 + c) - z = 0$. This factors to $(z^2 + z + 1 + c)(z^2 - z + c) = 0$. The second factor just gives us c-values for which there are attracting fixed points, so we only look at the first factor, which should be equal to 0.

The quadratic will have two roots z_1, z_2. By Vieta's formula, the product of the roots is simply the constant term, so $z_1 z_2 = c + 1$. As they are periodic with period 2, we have $z_1^2 + c = z_2$. We also have that the derivative of $P_c^{(2)}(z) = 4z^3 + 4zc$, so $(P_c^{(2)})'(z_1) = 4z_1^3 + 4z_1 c = 4z_1(z_1^2 + c) = 4z_1 z_2$. Finally, the cycles are attracting, so $0 \leq (P_c^{(2)})'(z_1) < 1$. So we have $0 \leq 4z_1 z_2 = 4(c+1) < 1$. Hence, the set of points c is a circle centered at $c = -1$ and with radius $\frac{1}{4}$. ■

We can actually label every bulb directly attached to the main cardioid (commonly called decorations), assigning each decoration a unique rational number $\frac{p}{q}$. It actually turns out that the periods of the points in each bulb are the same, and so we can label the bulbs so that q denotes the period of each bulb. (For instance, we would give the period-2 bulb from Proposition 4.2 the label $\frac{1}{2}$). Then, we can assign p looking at the geometry of the Mandelbrot Set, one way is through the antennae that project out from the end of the decorations. There will be q antennas for a decoration of period q, and counting counterclockwise from the antenna containing the decoration, the pth antenna will be the smallest antenna. [Dev06]

I will not prove these more general results as they are too difficult, and I was actually unable to find proofs for some results listed above.

However, I have provided views of the Mandelbrot that convincingly illustrate the geometry of the periodic bulbs.



Top-left view shows a $\frac{1}{3}$ bulb. Top-right view shows a $\frac{1}{4}$ bulb.
Bottom-left view shows a $\frac{2}{5}$ bulb. Bottom-right view shows a $\frac{3}{7}$ bulb.

Pictures were taken from `https://plus.maths.org/content/unveiling-mandelbrot-set`.

References

[Bra89] Bodil Branner. The mandelbrot set. In *Proc. symp. appl. math*, volume 39, pages 75–105, 1989.

[Dev06] Robert L Devaney. Unveiling the mandelbrot set. *Plus [Online]*, 2006.

[MSC17] Arun Mahanta, Hemanta Sarmah, and Gautam Choudhury. Some structural and dynamical properties of mandelbrot set. *International Journal of Applied Mathematics and Statistical Science*, 6:35–58, 05 2017.