MA 354: Data Analysis – Fall 2021 – Due 10/8 at 5p
Homework 2:

Complete the following opportunities to use what we've talked about in class. These questions will be graded for correctness, communication and succinctness. Ensure you show your work and explain your logic in a legible and refined submission.

The starting jobs will be applied in alphabetical order (last name) for question two.

1. **Solver:** provide a solution, if possible, and reasoning for the solution. **Due to group 10/5 or earlier.**
2. **Code Checker:** provides a first check of the solver's worked solutions and ensures they are correct with a solid interpretation.
3. **Checker** checks the solution for completeness, proposes and implements changes if agreed upon by the group. Provides a short paragraph summarizing the discussion of proposals and their reason for acceptance or non-acceptance.
4. **Double Checker** checks the solution for completeness, communication and polish. The Double Checker ensures that the solution is correct and highly polished for submission.

For subsequent questions student roles will move down one position. The rolls change as follows.

1. **Solver** $\Longrightarrow$ **Code Checker**
2. **Code Checker** $\Longrightarrow$ **Checker**
3. **Checker** $\Longrightarrow$ **Double Checker**
4. **Double Checker** $\Longrightarrow$ **Solver**

While students have assigned jobs for each question I encourage students to help each other complete the homework in collaboration.

1. Continue with the discrete distribution you selected for Question

   (a) Provide the mean, standard deviation, skewness, and kurtosis of the PMF. Ensure to interpret each.

   Bernoulli Distribution:

   PMF:

$$f(k;p) = p^k (1-p)^{1-k} \qquad \text{for } k \in \{0,1\}$$

   Mean:

$$E(X) = p$$

   The mean represents the average or typical value that we would expect to see in a data set.

   Standard Deviation:

$$\sigma = \sqrt{p(1-p)}$$

   The standard deviation tells us how much the data in our data set varies from the mean. Data sets with lower standard deviations will have their data points grouped more tightly together, while data sets with higher standard deviations will have their data points more spread out.

   Skewness:

$$\xi_X = \frac{(1-p)-p}{\sqrt{p(1-p)}}$$

   The skewness tells us how symmetrical the data will be. If our data is positively or right-skewed, that means there will be a large grouping of data towards the left side of our distribution, with a long tail extending out to the right. Conversely, negatively or left-skewed data means there will be a large grouping of data towards the right side of our distribution, with a long tail extending out to the left.

   Kurtosis:

$$\kappa_Y = 3 + \frac{1-6p(1-p)}{p(1-p)}$$

   The kurtosis is used to tell us how thick or thin the peak and tails of our data distribution are. A data set with kurtosis greater than 3 will have a distribution that has thicker tails and a thinner peak. Conversely, a data set with kurtosis less than 3 will have thinner tails and a thicker peak.

   (b) Generate a random sample of size $n = 10, 25, 100$, and 1000 for your two sets of parameter(s). Calculate the sample mean, standard deviation, skewness, and kurtosis. Interpret the results.

```r
library(e1071)

even.10 <- rbinom(n=10,          #number of observations
              size=1,            #number of trials (size=1 for a Bernoulli distribution)
              prob=.5)           #probability of success

mean(even.10)                    #mean
```

```
## [1] 0.2
```

```r
sd(even.10)                      #standard deviation
```

```
## [1] 0.421637
```

```r
skewness(even.10)                #skewness
```

```
## [1] 1.280722
```

```r
kurtosis(even.10)                #kurtosis
```

```
## [1] -0.3675
```

```r
even.25  <- rbinom(n=25,         #number of observations
              size=1,            #number of trials (size=1 for a Bernoulli distribution)
              prob=.5)           #probability of success

mean(even.25)                    #mean
```

```
## [1] 0.52
```

```r
sd(even.25)                      #standard deviation
```

```
## [1] 0.509902
```

```r
skewness(even.25)                #skewness
```

```
## [1] -0.0753086
```

```r
kurtosis(even.25)                #kurtosis
```

```
## [1] -2.072492
```

```r
even.100 <- rbinom(n=100,        #number of observations
              size=1,            #number of trials (size=1 for a Bernoulli distribution)
              prob=.5)           #probability of success

mean(even.100)                   #mean
```

```
## [1] 0.41
```

```r
sd(even.100)                     #standard deviation
```

```
## [1] 0.4943111
```

```r
skewness(even.100)               #skewness
```

```
## [1] 0.3605017
```

```r
kurtosis(even.100)               #kurtosis
```

```
## [1] -1.888626
```

```r
even.1000 <- rbinom(n=1000,       #number of observations
              size=1,             #number of trials (size=1 for a Bernoulli distribution)
              prob=.5)            #probability of success

mean(even.1000)                   #mean

## [1] 0.473

sd(even.1000)                     #standard deviation

## [1] 0.4995203

skewness(even.1000)               #skewness

## [1] 0.1079956

kurtosis(even.1000)               #kurtosis

## [1] -1.990324

uneven.10 <- rbinom(n=10,         #number of observations
                 size=1,          #number of trials (size=1 for a Bernoulli distribution)
                 prob=.7)         #probability of success

mean(uneven.10)                   #mean

## [1] 0.6

sd(uneven.10)                     #standard deviation

## [1] 0.5163978

skewness(uneven.10)               #skewness

## [1] -0.3485685

kurtosis(uneven.10)               #kurtosis

## [1] -2.055

uneven.25  <- rbinom(n=25,        #number of observations
                  size=1,         #number of trials (size=1 for a Bernoulli distribution)
                  prob=.7)        #probability of success

mean(uneven.25)                   #mean

## [1] 0.68

sd(uneven.25)                     #standard deviation

## [1] 0.4760952

skewness(uneven.25)               #skewness

## [1] -0.7259052

kurtosis(uneven.25)               #kurtosis

## [1] -1.529506
```

```r
uneven.100 <- rbinom(n=100,       #number of observations
                     size=1,      #number of trials (size=1 for a Bernoulli distribution)
                     prob=.7)     #probability of success

mean(uneven.100)                  #mean
```

```
## [1] 0.66
```

```r
sd(uneven.100)                    #standard deviation
```

```
## [1] 0.4760952
```

```r
skewness(uneven.100)              #skewness
```

```
## [1] -0.6654131
```

```r
kurtosis(uneven.100)              #kurtosis
```

```
## [1] -1.572653
```

```r
uneven.1000 <- rbinom(n=1000,     #number of observations
                      size=1,     #number of trials (size=1 for a Bernoulli distribution)
                      prob=.7)    #probability of success

mean(uneven.1000)                 #mean
```

```
## [1] 0.704
```

```r
sd(uneven.1000)                   #standard deviation
```

```
## [1] 0.4567194
```

```r
skewness(uneven.1000)             #skewness
```

```
## [1] -0.8924342
```

```r
kurtosis(uneven.1000)             #kurtosis
```

```
## [1] -1.204763
```

Calculated Mean, Standard Deviation, Skewness and Kurtosis:

For $p = .5$:

Mean:

$$E(X) = .5$$

Standard Deviation:

$$\sigma = \sqrt{.5\,(1 - .5)}$$

$$= .5$$

Skewness:

$$\xi_X = \frac{(1-.5)-.5}{\sqrt{.5\,(1-.5)}}$$

$$= 0$$

Kurtosis:

$$\kappa_Y = 3 + \frac{1-6\,(.5)\,(1-.5)}{.5\,(1-.5)}$$

$$= 1$$

For $p = .7$:

Mean:

$$E(X) = .7$$

Standard Deviation:

$$\sigma = \sqrt{.7\,(1-.7)}$$

$$\approx .45825$$

Skewness:

$$\xi_X = \frac{(1-.7)-.7}{\sqrt{.7\,(1-.7)}}$$

$$\approx -.87287$$

Kurtosis:

$$\kappa_Y = 3 + \frac{1-6\,(.7)\,(1-.7)}{.7\,(1-.7)}$$

$$\approx 1.76190$$

By comparing the values we received computationally to the values that we calculated we can see that as our random sample size n increases, the computational values that we received for our mean, standard deviation, skewness, and kurtosis tended to approach the actual values for the distribution that we calculated.

(c) Generate a random sample of size $n = 10$ for your two sets of parameter(s). Calculate the method of moments estimator(s) and maximum likelihood estimator(s). In a 1×2 grid, plot a histogram (with bin size 1) of each set of data with (1) the method of moments estimated distribution, (2) the maximum likelihood estimated distribution, and superimpose the true distribution in both.

```
library(tidyverse)
```

```
## -- Attaching packages --------------------------------------- tidyverse 1.3.1 --
## v ggplot2 3.3.5     v purrr   0.3.4
## v tibble  3.1.4     v dplyr   1.0.7
## v tidyr   1.1.3     v stringr 1.4.0
## v readr   2.0.1     v forcats 0.5.1
## -- Conflicts ------------------------------------------ tidyverse_conflicts() --
## x dplyr::filter() masks stats::filter()
## x dplyr::lag()    masks stats::lag()
```

```
library(patchwork)

second.moment <- function(data, x.bar){  #function to find the second moment
  temp = 0
  for(i in data){
    temp = temp + (i - x.bar)^2
  }
  return (temp/length(data))
}

MOM.bernoulli <- function(data){          #function to find the MOM Estimator for p in a Berno
  mu.hat <- mean(data)
  var.hat <- second.moment(data,mu.hat)
  p.hat <- (mu.hat-var.hat)/mu.hat
  return(p.hat)
}

# Negative of the Likelihood function
bernoulliLogLik.neg <- function(par, data){
  -sum(dbinom(x=data, prob=par, size=1, log=TRUE))} #Sum is negative, as we are looking for a

MLE.bernoulli <- function(data){
  MLE <- optim(par = .5,               # best guess for the parameter
               fn = bernoulliLogLik.neg,   # function to minimize
               data = data,       # data (an argument to fn)
               method = "Brent",       # required for univariate optimization
               lower = 0,              # lowest possible lambda
               upper = 1)              # reasonable upper bound

  # Note that our mle is hat(lambda) = xbar
  return(MLE$par)
}

bernoulli.plot <- function(data){

  dfActual <- data.frame(matrix(nrow = 2, ncol = 2))
  colnames(dfActual) <- c("Prop","Value")
  rownames(dfActual) <- c("0","1")
```

```r
ActualTab <- as.data.frame(data) %>% group_by(data) %>%
  summarize(count = n()) # Calculate the counts

dfActual[1,1] <- prop.table(ActualTab$count)[1]
dfActual[2,1] <- prop.table(ActualTab$count)[2]
dfActual[1,2] <- 0
dfActual[2,2] <- 1


dfEstimate <- data.frame(matrix(nrow = 2, ncol = 2))
colnames(dfEstimate) <- c("Prop","Value")
rownames(dfEstimate) <- c("0","1")

EstimateTab <- as.data.frame(data) %>% group_by(data) %>%
  summarize(count = n()) # Calculate the counts

dfEstimate[1,1] <- 1-MOM.bernoulli(data)
dfEstimate[2,1] <- MOM.bernoulli(data)
dfEstimate[1,2] <- 0
dfEstimate[2,2] <- 1


p1 <- ggplot(mapping = aes(y=Prop, x=Value)) +
  geom_col(data = dfActual, position = "dodge", fill = "lightblue") +
  geom_col(data = dfEstimate, aes(group = Value),
           fill = "black", width = 0.01, position = position_dodge(width = 0.9))

dfActual <- data.frame(matrix(nrow = 2, ncol = 2))
colnames(dfActual) <- c("Prop","Value")
rownames(dfActual) <- c("0","1")

ActualTab <- as.data.frame(data) %>% group_by(data) %>%
  summarize(count = n()) # Calculate the counts

dfActual[1,1] <- prop.table(ActualTab$count)[1]
dfActual[2,1] <- prop.table(ActualTab$count)[2]
dfActual[1,2] <- 0
dfActual[2,2] <- 1


dfEstimate <- data.frame(matrix(nrow = 2, ncol = 2))
colnames(dfEstimate) <- c("Prop","Value")
rownames(dfEstimate) <- c("0","1")

EstimateTab <- as.data.frame(data) %>% group_by(data) %>%
  summarize(count = n()) # Calculate the counts

dfEstimate[1,1] <- 1-MLE.bernoulli(data)
dfEstimate[2,1] <- MLE.bernoulli(data)
dfEstimate[1,2] <- 0
dfEstimate[2,2] <- 1
```

```
  p2 <- ggplot(mapping = aes(y=Prop, x=Value)) +
    geom_col(data = dfActual, position = "dodge", fill = "lightblue") +
    geom_col(data = dfEstimate, aes(group = Value),
             fill = "black", width = 0.01, position = position_dodge(width = 0.9))

  return(p1+p2)

}
```

```
even.10 <- rbinom(n=10,         #number of observations
                  size=1,           #number of trials (size=1 for a Bernoulli distribution)
                  prob=.5)          #probability of success

MOM.bernoulli(even.10)

MLE.bernoulli(even.10)

bernoulli.plot(even.10)
```

```
## [1] 0.2
## [1] 0.2
```

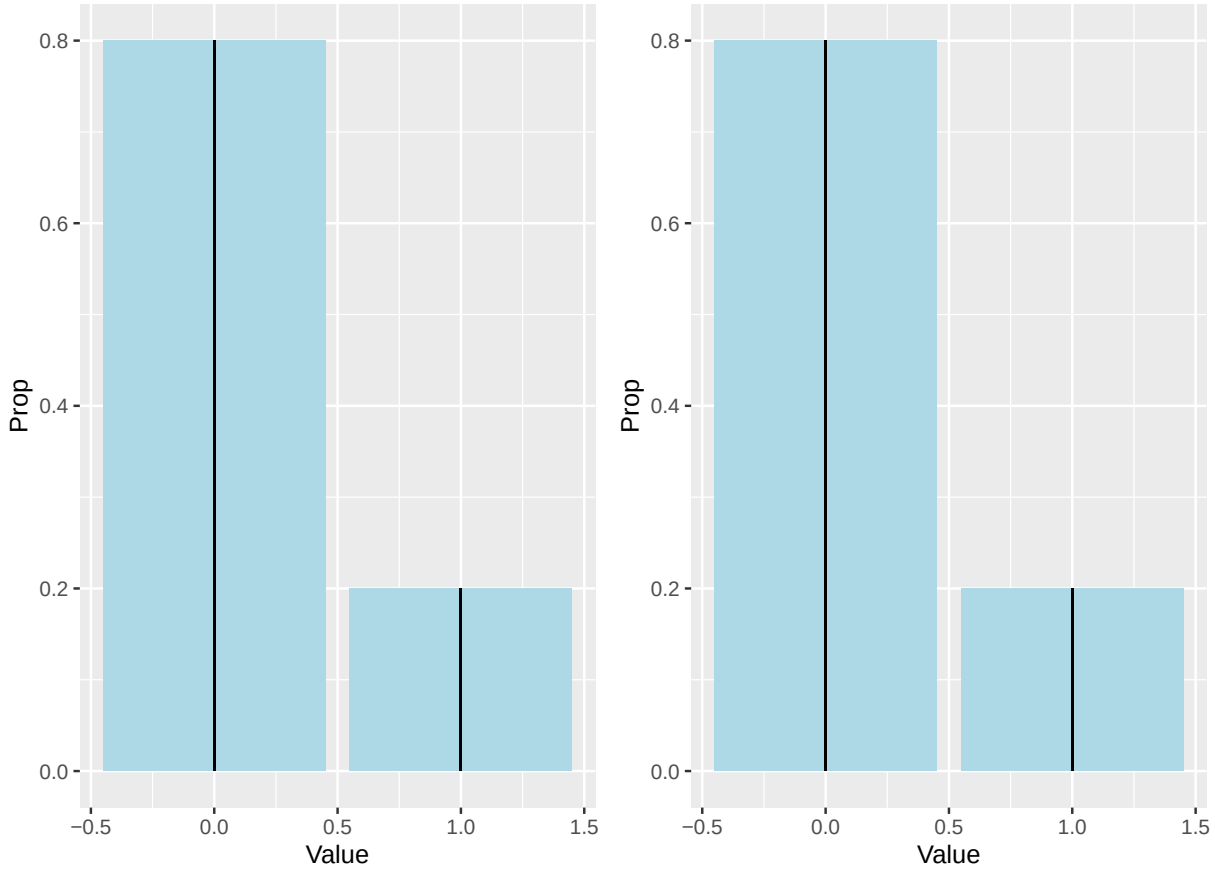
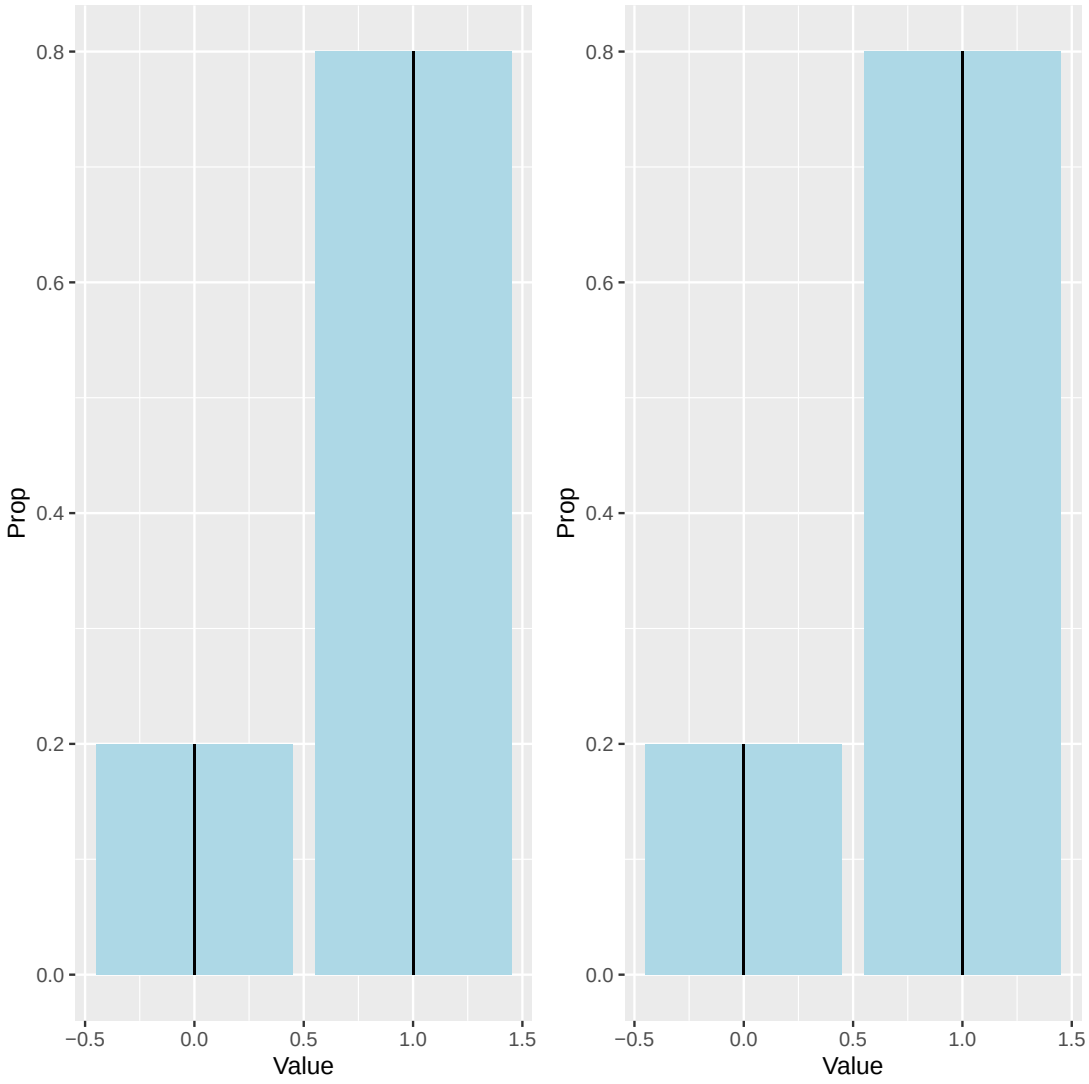


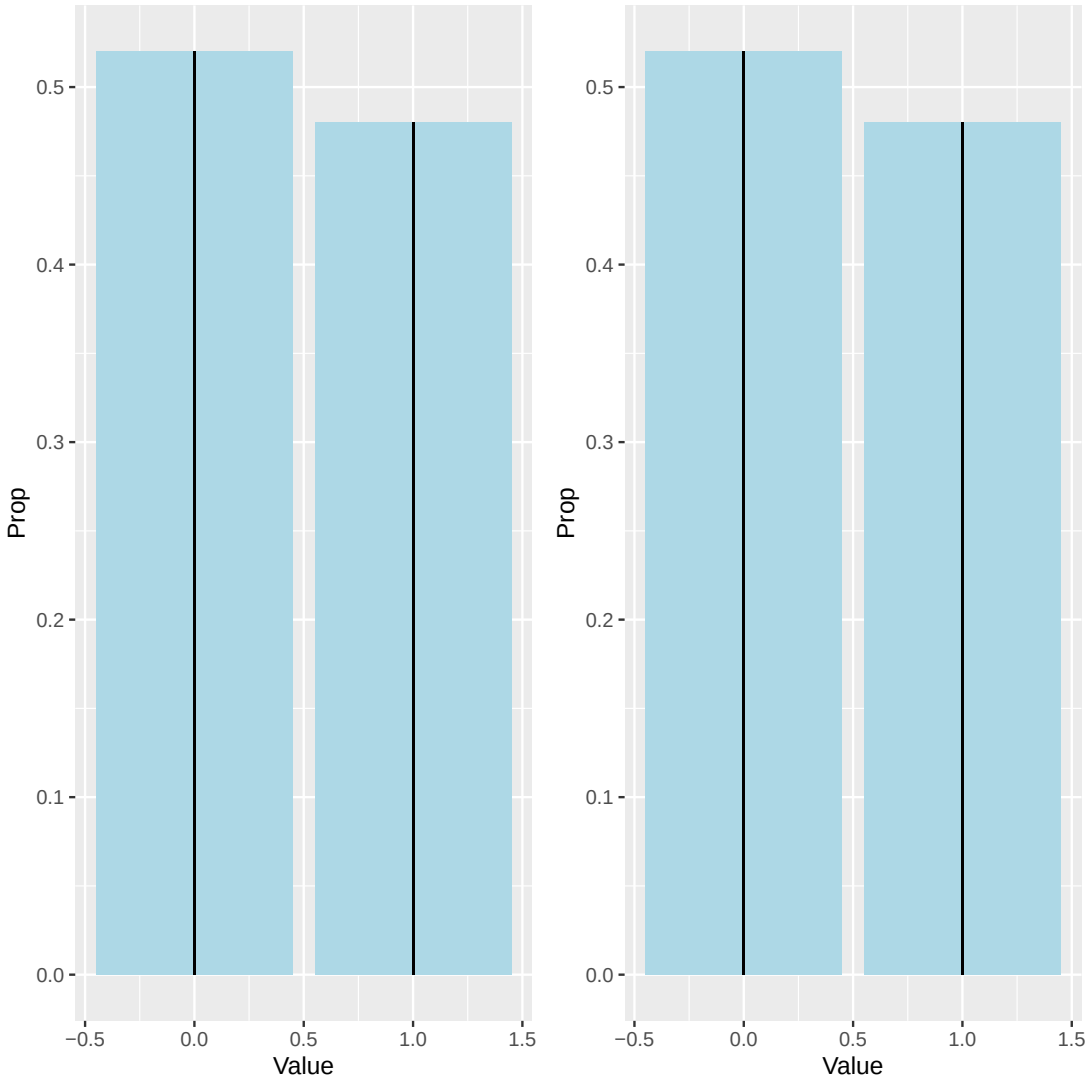
Figure 1: caption here

```
uneven.10 <- rbinom(n=10,          #number of observations
                    size=1,          #number of trials (size=1 for a Bernoulli distribution)
                    prob=.7)       #probability of success

MOM.bernoulli(uneven.10)

## [1] 0.8

MLE.bernoulli(uneven.10)

## [1] 0.8

bernoulli.plot(uneven.10)
```



(d) Generate a random sample of size $n = 25$ for your two sets of parameter(s). Calculate the method of moments estimator(s) and maximum likelihood estimator(s). In a 1×2 grid, plot a histogram (with bin size 1) of each set of data with (1) the method of moments estimated distribution, (2) the maximum likelihood estimated distribution, and superimpose the true distribution in both.

```
even.25 <- rbinom(n=25,          #number of observations
                  size=1,          #number of trials (size=1 for a Bernoulli distribution)
                  prob=.5)         #probability of success

MOM.bernoulli(even.25)

## [1] 0.48

MLE.bernoulli(even.25)

## [1] 0.48

bernoulli.plot(even.25)
```



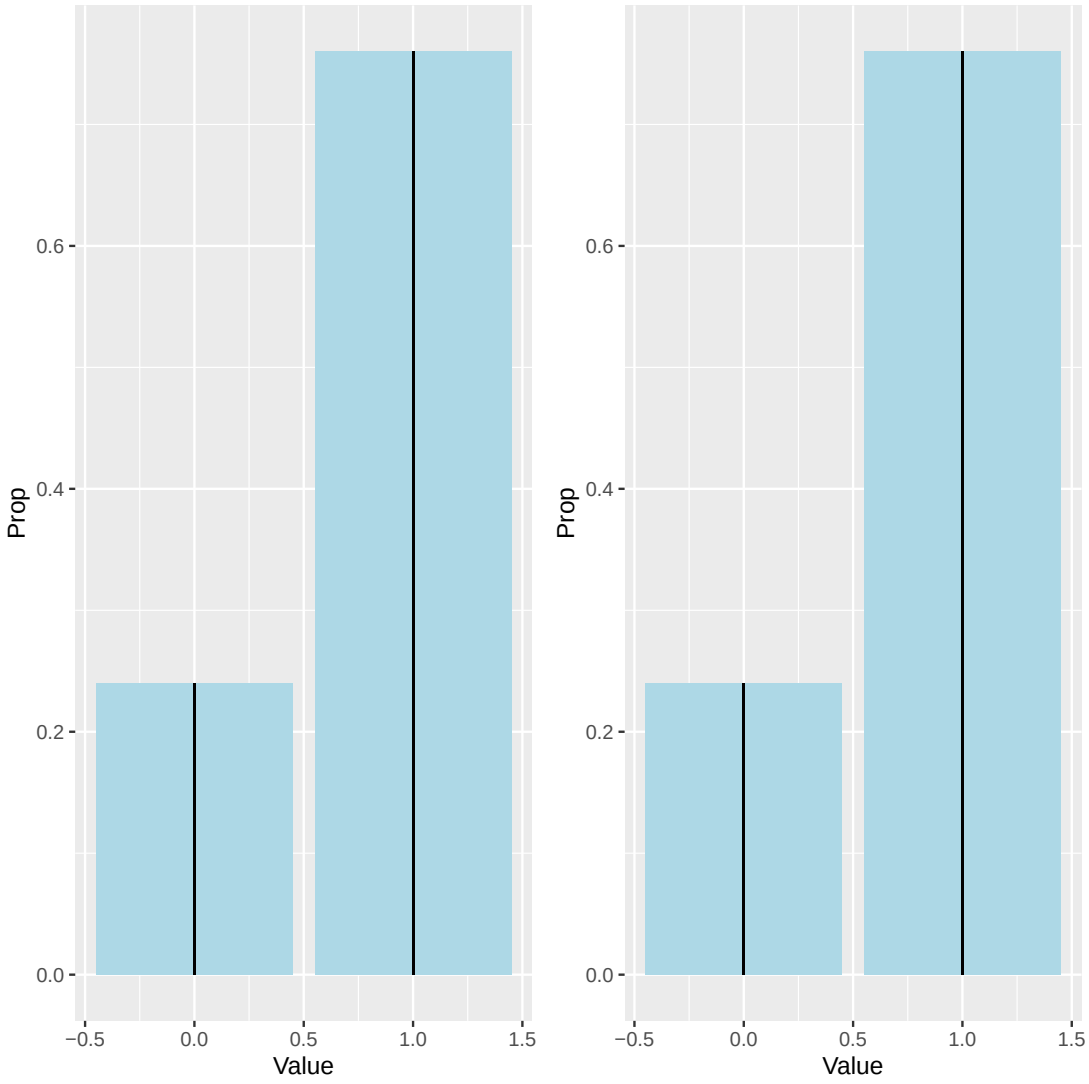

```r
uneven.25 <- rbinom(n=25,          #number of observations
                    size=1,          #number of trials (size=1 for a Bernoulli distribution)
                    prob=.7)       #probability of success

MOM.bernoulli(uneven.25)
```

```
## [1] 0.76
```

```r
MLE.bernoulli(uneven.25)
```

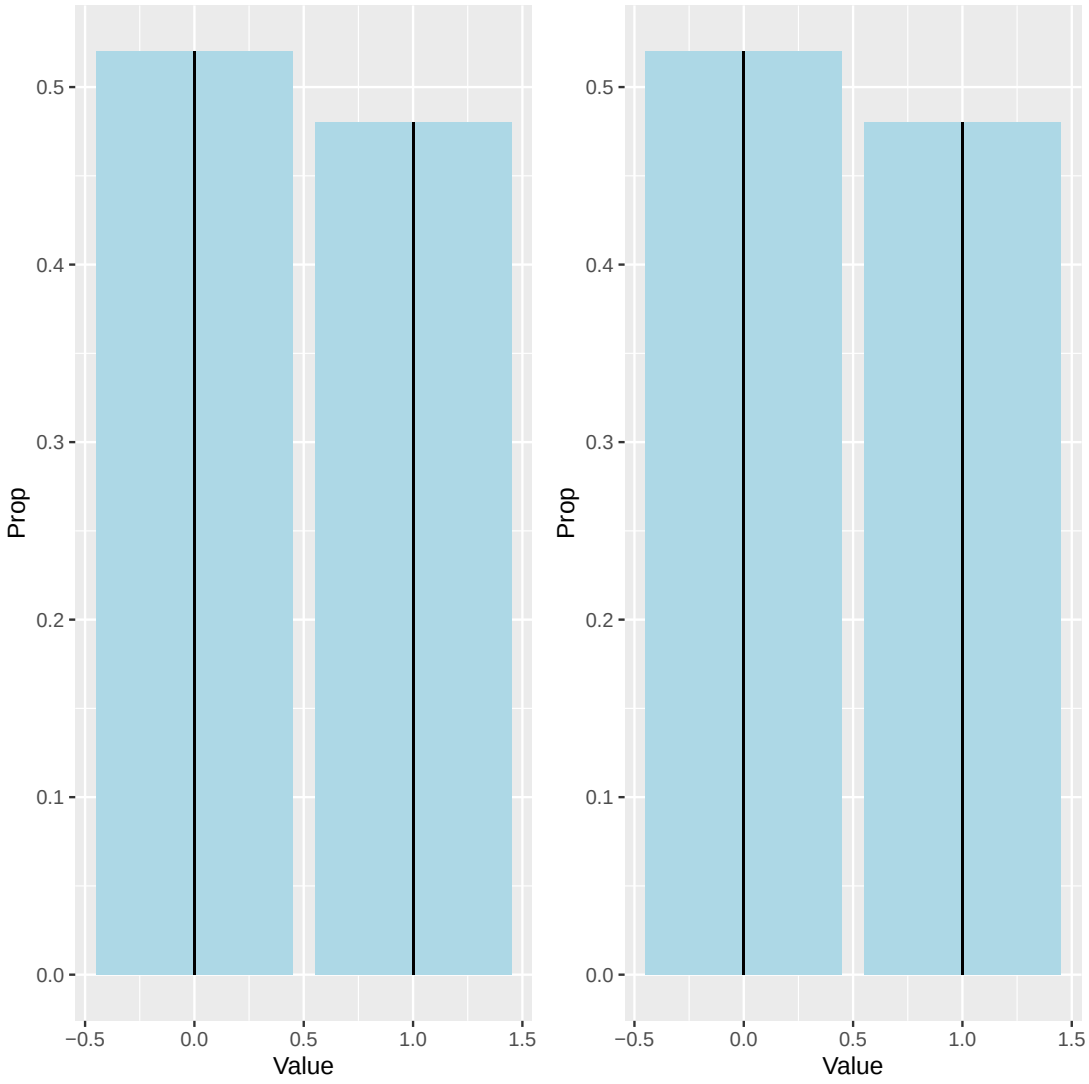
```
## [1] 0.76
```

```r
bernoulli.plot(uneven.25)
```

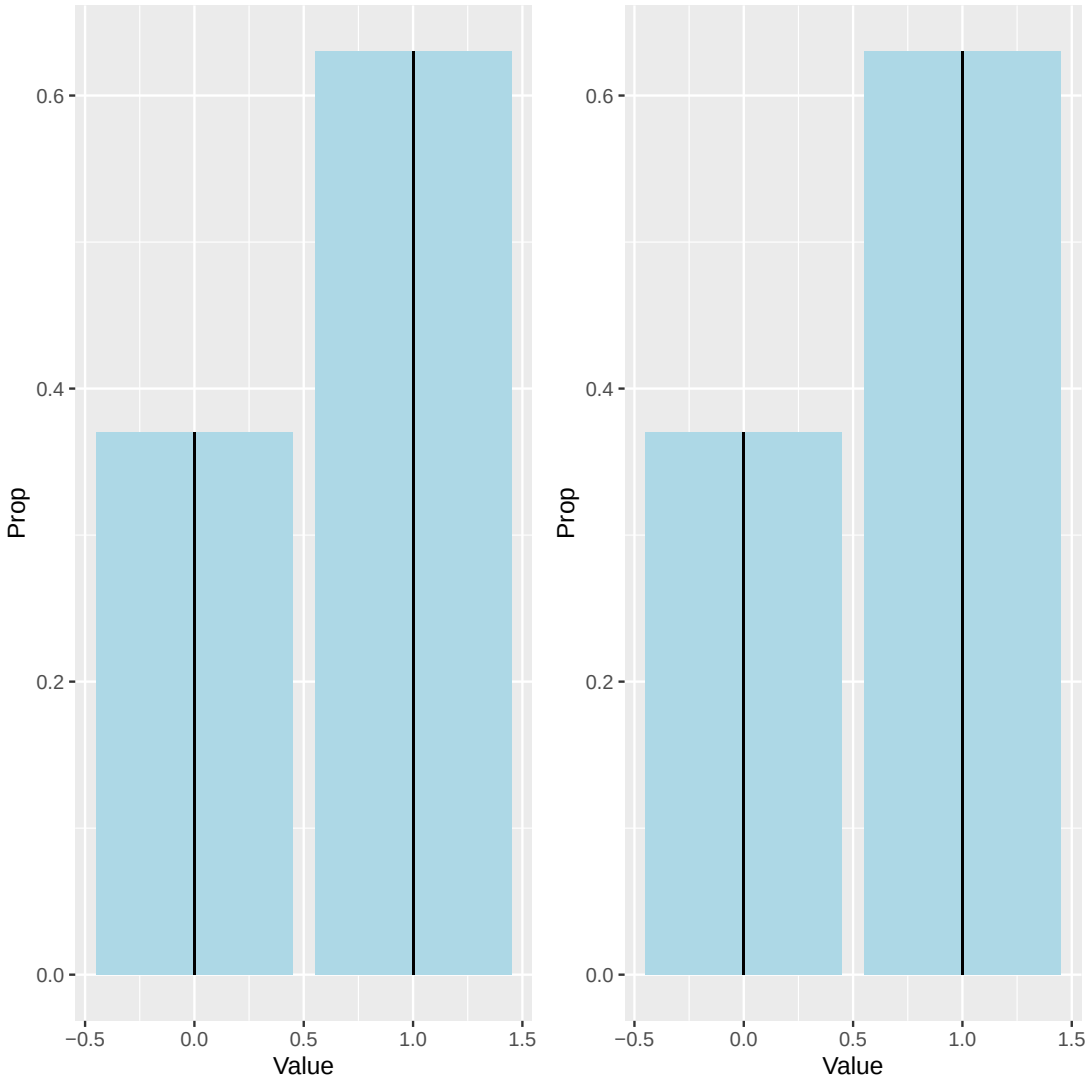


(e) Generate a random sample of size $n = 100$ for your two sets of parameter(s). Calculate the method of moments estimator(s) and maximum likelihood estimator(s). In a 1×2 grid, plot a histogram (with bin size 1) of each set of data with (1) the method of moments estimated distribution, (2) the maximum likelihood estimated distribution, and superimpose the true distribution in both.

```
even.100 <- rbinom(n=100,          #number of observations
                   size=1,         #number of trials (size=1 for a Bernoulli distribution)
                   prob=.5)        #probability of success

MOM.bernoulli(even.100)

## [1] 0.48

MLE.bernoulli(even.100)

## [1] 0.48

bernoulli.plot(even.100)
```

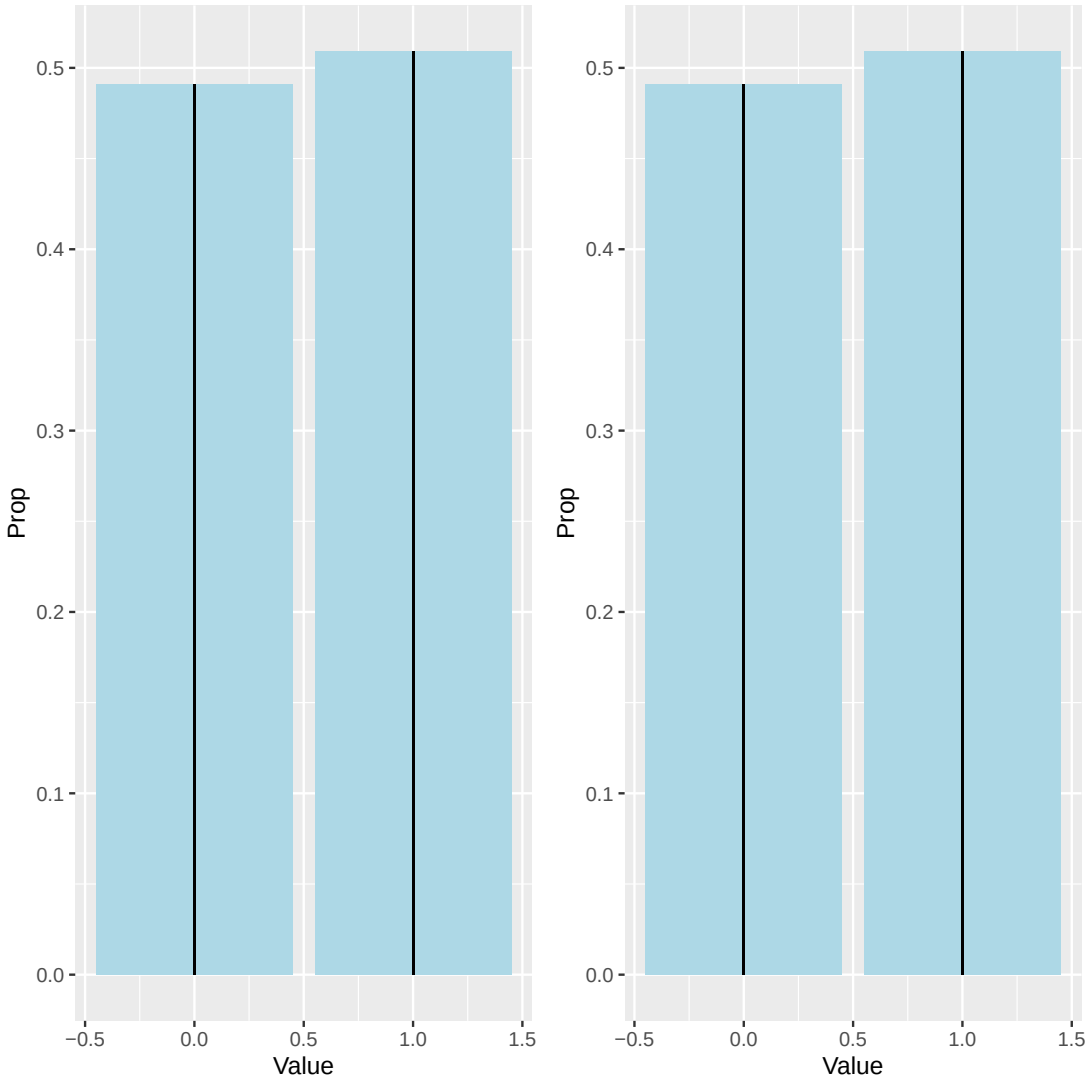



```r
uneven.100 <- rbinom(n=100,          #number of observations
                     size=1,         #number of trials (size=1 for a Bernoulli distribution)
                     prob=.7)        #probability of success

MOM.bernoulli(uneven.100)

## [1] 0.63

MLE.bernoulli(uneven.100)

## [1] 0.63

bernoulli.plot(uneven.100)
```

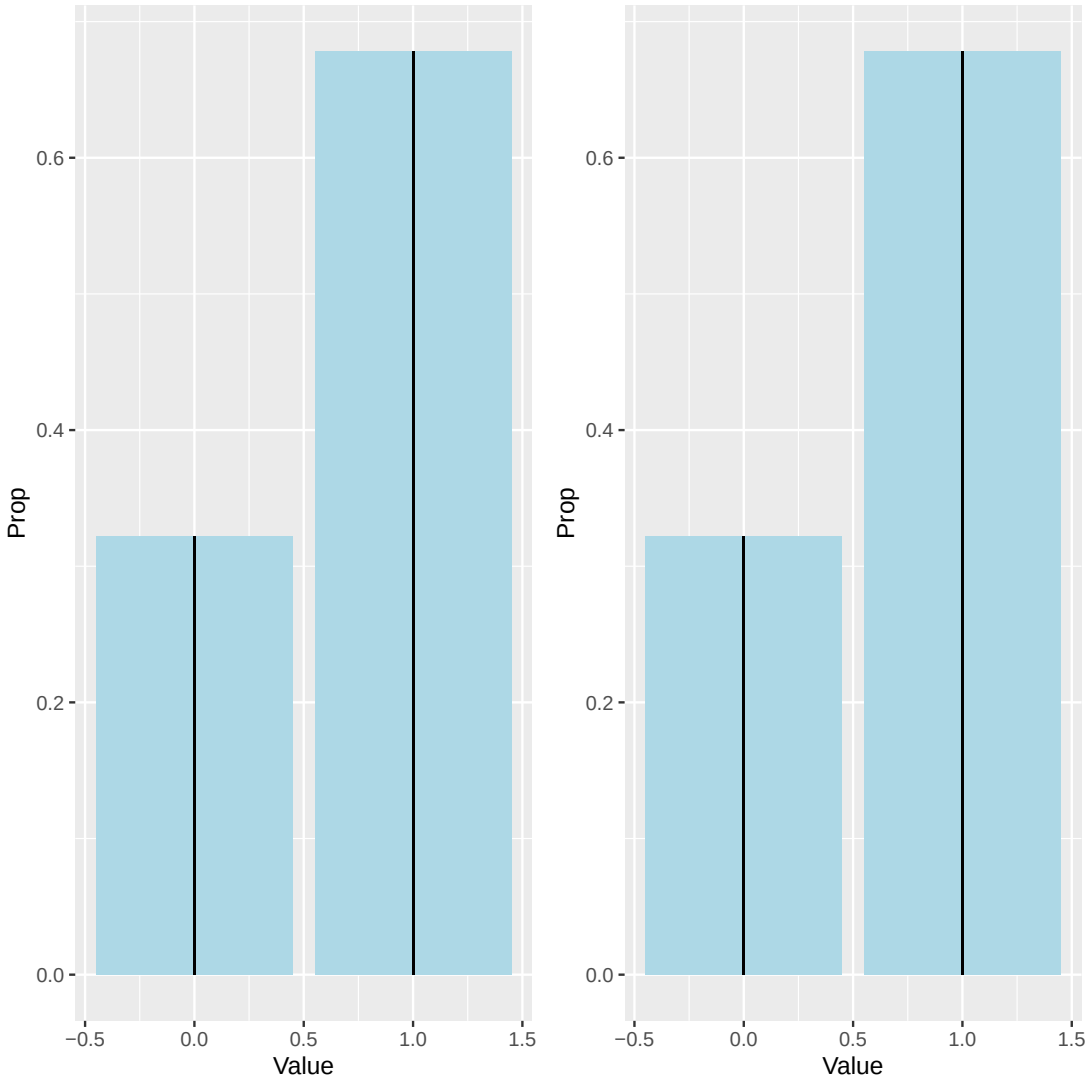


(f) Generate a random sample of size $n = 100$ for your two sets of parameter(s). Calculate the method of moments estimator(s) and maximum likelihood estimator(s). In a 1×2 grid, plot a histogram (with bin size 1) of each set of data with (1) the method of moments estimated distribution, (2) the maximum likelihood estimated distribution, and superimpose the true distribution in both.

```
even.1000 <- rbinom(n=1000,         #number of observations
                    size=1,        #number of trials (size=1 for a Bernoulli distribution)
                    prob=.5)       #probability of success

MOM.bernoulli(even.1000)

## [1] 0.509

MLE.bernoulli(even.1000)

## [1] 0.509

bernoulli.plot(even.1000)
```




```
uneven.1000 <- rbinom(n=1000,          #number of observations
                      size=1,          #number of trials (size=1 for a Bernoulli distribution)
                      prob=.7)         #probability of success

MOM.bernoulli(uneven.1000)

## [1] 0.678

MLE.bernoulli(uneven.1000)

## [1] 0.678

bernoulli.plot(uneven.1000)
```



(g) Comment on the results of parts (c)-(f).

From the plots in the results of parts (c)-(f), we can see that both the Method of Moments estimator and the Maximum Likelihood Estimator were able to provide a perfect estimate for the parameter p regardless of our random sample size n. Therefore, I feel it is reasonable to conclude that using larger random sample sizes to find the MOM and MLE of a Bernoulli distribution is unnecessary.